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Bounded Variation in Time

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1. Introduction

Describing a motion consists in defining the *state* or *position* q of the investigated system as a function of the real variable t , the time. Commonly, q takes its values in some set Q , suitably structured for the *velocity* u to be introduced as the derivative of $t \rightarrow q$, when it exists. This, in fact, makes sense if Q is a topological linear space or, more generally, a differential manifold modelled on such a space.

For smooth situations, classical dynamics rests, in turn, on the consideration of the *acceleration*. This is the derivative of $t \rightarrow u$, if it exists in the sense of the topological linear structure of Q , or, when Q is a manifold, in the sense of some connection. But, from its early stages, classical dynamics has also had to face *shocks*, i.e. velocity jumps. For isolated shocks, one traditionally resorts to the equations of the dynamics of *percussions*. Even in the absence of impact, it has been known for a long time [1] that systems submitted to such nonsmooth effects as *dry friction* may exhibit time discontinuity of the velocity. Furthermore, *nonsmooth mechanical constraints* may also prevent $t \rightarrow u$ from admitting a derivative. In all these cases, the laws governing the motion can no longer be formulated in terms of acceleration. Incidentally, one may find in [2] an attempt at gaining a synthetic view: a formulation of classical dynamics is proposed, valid for continuous media as well as for systems of finite freedom, without reference to acceleration.

It should be understood below that $t \rightarrow u$ takes its values in some real Banach space X and is locally Lebesgue-integrable. So this function will be connected with the motion $t \rightarrow q \in X$ through

$$q(t) = q_0 + \int_{t_0}^t u(\tau) d\tau. \quad (1.1)$$

The usual problems of dynamics are governed by first-order differential conditions imposed on u and involving q through expression (1.1). These may be viewed equally as second-order differential conditions for $t \rightarrow q$, as first-order

differential equations for $t \rightarrow (q, u)$ or as first-order integro-differential conditions for $t \rightarrow u$.

We shall therefore concentrate on conditions formulated in terms of the velocity u . As long as $t \rightarrow u$ is locally absolutely continuous, these conditions can be differential equations in the classical (Carathéodory) sense. If this is not the case, then the next available analytical framework in which some object playing the role of derivative may be associated with $t \rightarrow u$ is the following. By writing $u \in \text{lbv}(I, X)$, we mean that u is a function of the real (time) interval I to the Banach space X with *locally bounded variation*, i.e. it has bounded variation over every compact subinterval of I . Then, an X -valued measure on I , which we shall denote by du , is known to be associated with the function u . Characteristically, for every compact subinterval $[\sigma, \tau]$ of I one has

$$\int_{[\sigma, \tau]} du = u^+(\tau) - u^-(\sigma), \quad (1.2)$$

where u^+ and u^- refer, respectively, to the right-limit and the left-limit of the function u at a point of I (under an ad hoc convention for the possible endpoints of I ; see Sect. 3 below). In particular, if at some $\tau \in I$ the two limits are different, the vector measure du possesses at this point an *atom*, with value $u^+(\tau) - u^-(\tau)$.

When I is open in \mathbb{R} , the vector measure du may also be seen as the derivative of $t \rightarrow u$ in the sense of distributions. But for applications it is essential not to restrict ourselves to this case.

We shall call du the *differential measure* of u . The smooth case corresponds to du admitting a *density function*, say u'_t , relatively to the Lebesgue measure on I . For reflexive X (more generally, for a Banach space possessing what is called the Radon-Nikodym property; see Sect. 13 below) the existence of such a density function is equivalent to the absolute continuity of u over every compact subinterval of I .

Assuming $u \in \text{lbv}(I, X)$ is enough for u to be involved as the unknown in a *measure differential equation*, as in the following example. Take $I = [0, +\infty[$; one looks for u , a *right-continuous* lbv function verifying some initial condition $u(0) = u_0$, and this equality of X -valued measures on I :

$$du = F(t, u(t))dt + G(t, u(t))df.$$

Here dt denotes the Lebesgue measure on I (it equals the differential measure of $t \rightarrow t$) and df a given (signed) real measure; F and G are given functions of $I \times X$ to X , regular enough for the X -valued functions $t \rightarrow F(t, u(t))$ and $t \rightarrow G(t, u(t))$ to be locally integrable relative to dt and df , respectively. The monograph [3] is devoted to problems of this sort, with $X = \mathbb{R}^n$; it contains numerous references to papers motivated by operations research, control theory or mathematical

physics, possibly also involving *stochastic* versions of measure differential equations.

Better matched to the needs of contemporary nonsmooth mechanics are *measure differential inclusions*, about which we shall give more information in the concluding section of this chapter. These consist in requiring the density function u'_μ of du , relative to some nonnegative real measure $d\mu$, to be a selection of $t \rightarrow \Gamma(t, u(t))$, where $(t, x) \rightarrow \Gamma(t, x) \subset X$ denotes a given *multifunction*. The real measure $d\mu$ may be prescribed in advance; alternatively, provided every $\Gamma(t, x)$ is a *conic* subset of X , $d\mu$ has not to be given and may indifferently equal any nonnegative real measure relative to which du happens to possess a density function. The latter setting has been introduced when extending to noncontinuous situations the abstract unilateral evolution problem called the *sweeping process* [4, 5]. In Sect. 16 we shall come back to this problem, which was initially motivated by the quasistatic motion of elastoplastic systems and which has also been studied under a stochastic version [6]. It has been shown more recently that the dynamics of systems of finite freedom in the presence of *unilateral constraints* is governed by measure differential inclusions of the same conical type [7, 8]. This holds even if the unilateral contacts involve *dry friction* [9]. The existential and numerical study of measure differential inclusions with such a *dissipative* character is currently in progress; further references on this subject will be given in Sect. 16.

Investigation is also under way of the dynamics of systems submitted to frictionless unilateral constraints, under the assumption that the possible shocks are *elastic*, i. e. energy is conserved. Various formulations have been proposed for this type of mechanical problem; they are closely connected with measure differential inclusions and naturally involve lbv velocity functions [10–12].

Apart from in mechanics, the variational problem of Lagrange in \mathbb{R}^n has been extended in [13], under a convexity assumption for the integrand, to *arcs of bounded variation*; instead of the traditional Hamilton-Jacobi system of differential equations, there appears a measure differential inclusion.

Without explicitly resorting to measure differential equations or inclusions, some authors have been induced to accept functions of bounded variations as “weak” solutions to problems primitively formulated in a stronger setting (see, e. g. [14]).

As all this currently has created an active domain of research, it is considered helpful to devote the present chapter to a systematic exposition of lbv (I, X) . In fact, the basic information about functions of bounded variation on an interval is scattered throughout various treatises on real analysis and measure theory, without sufficient emphasis having been placed on the points specifically pertinent to evolution problems. This chapter also includes some recently developed material, partly unpublished or available only from seminar reports.

We shall generally assume that X is an infinite-dimensional Banach space, because a large part of the current mathematical work on evolution problems is done in such a setting. Actually, it should be acknowledged that the most effective mechanical applications of $\text{lbv}(I, X)$ so far concern systems of finite freedom, whose treatment only requires finite dimensionality for X . In fact, mechanical systems of infinite freedom are continuous media, the dynamics of which is commonly expressed by partial differential equations in some domain of time-space. The consideration of the solutions of such partial differential equations as functions of time, with values in some function space, is only one of the possible approaches. Even when such a standpoint is adopted, it frequently happens that the velocity function $t \rightarrow u$ is defined in some indirect way, so that it does not take its values in the same space as the system state; in turn, acceleration may be conceived of as an element of yet another space..

In the context of partial differential equations, the space $BV(\Omega, \mathbb{R}^p)$ of the *functions of bounded variation in an open subset Ω of \mathbb{R}^n* should also be recalled, with the books [15] and [16] as standard references. For $n=1$, one naturally recovers the functions of bounded variation in an open real interval, with values in \mathbb{R}^p . But the virtual absence of overlap between the content of this chapter and the mentioned books above attests to the essential difference between these subjects. Significantly, defining the variation of a function on a (not necessarily open) real interval relies only on the ordering of \mathbb{R} , without any reference to Lebesgue measure. In contrast, the Lebesgue measure on the subset Ω of \mathbb{R}^n is inherent in the definition of $BV(\Omega, \mathbb{R}^p)$. Incidentally, for $n > 1$, the relevance of $BV(\Omega, \mathbb{R}^p)$ to mechanics emerges through the theory of minimal surfaces, which may be viewed as an aspect of the statics of membranes. Also connected with $BV(\Omega, \mathbb{R}^p)$ are *vector fields of bounded deformation*, primarily motivated by the field of plasticity [17, 18].

The concept of a measure with values in a Banach space naturally plays the central role in this chapter. References [19–22] are standard books on this matter. By extracting from them some key facts and subsequently relying only on basic knowledge in real integration, we have attempted to spare the reader from being obliged to refer to these texts.

Recall that two different approaches compete in the expositions of measure-theory. Most commonly, a measure on a set S is introduced as a countably additive function defined on some σ -field of subsets of S . In contrast, the duality approach, as developed by Bourbaki [20], defines a measure *on a locally compact topological space T* , with values in \mathbb{R} , \mathbb{C} or in any locally convex topological linear space, as a linear functional, say m , on the space $\mathcal{X}(T, \mathbb{R})$ of the real continuous functions with compact support in T , meeting the following continuity requirement: For every compact subset K of T , the restriction of the functional $\varphi \rightarrow m(\varphi)$ to the subspace of $\mathcal{X}(T, \mathbb{R})$ consisting of the continuous

functions with support contained in K is continuous in the norm $\sup_K |\varphi|$.

We have preferred the latter standpoint for several reasons. First this makes measures appear to be a special case of Schwartz's Distributions, a notion which is familiar in any case to the theoretically minded researcher of mechanics. The procedure readily carries over to the definition of vector or tensor distributions and measures on differential manifolds, in line with de Rham's theory of *currents*. The author [2, 23, 24] has recently drawn attention to the use of such concepts in classical mechanics. But, above all, the duality of linear spaces appears to be a cornerstone in the whole of mechanics. Researchers in mechanics may claim precedence in implementing such a duality, prior even to any attempt at axiomatizing linear algebra. We are referring to the method of virtual work (or virtual power), which consists in placing some linear space of displacements (or velocities) in duality with a linear space of efforts. Starting with mechanical instances, the duality of linear spaces has today also become a general tool in numerical analysis, at the stage of constructing finite-dimensional approximations to functional problems. From another viewpoint, the relevance of duality to nonsmooth mechanics is evident throughout this book.

In any case, what in measure theory is considered a definition from one of the two standpoints appears as a proposition from the other one. We have tried, especially in Sects. 13 and 14, to bridge the gap between the respective approaches, for the reader accustomed to the set functions setting to easily recover his familiar background.

The author is indebted to M. Valadier for helpful criticism throughout this chapter.

2. Review of Contents

Section 3 presents some elementary facts concerning the right-side limit f^+ and left-side limit f^- , for a function f of a real interval I to a regular topological space E , when such limits exist. If I includes its possible left endpoint, say t_0 , Convention 3.5, frequently used in the following, stipulates that $f^-(t_0)$ should be understood as equal to $f(t_0)$; symmetrically, $f^+ = f$ at the possible right endpoint. In connection with this, the meaning of *initial conditions* for evolution problems is discussed.

It is supposed in Sect. 4 that f is a function of the real interval I to a *metric space* E . The variation $\text{var}(f, J)$ of f over a subinterval J of I is defined. We mean by $f \in \text{lbv}(I, E)$ that f has *locally bounded variation*, i. e. it has finite variation over every compact subinterval of I . This is equivalent to the existence of $V_f : I \rightarrow \mathbb{R}$, said to be a *variation function* of f , nondecreasing and such that for every

$[a, b] \subset I$ one has $\text{var}(f, [a, b]) = V_f(b) - V_f(a)$. The connection of V_f with the possible one-sided limits of f at a point (they are sure to exist if E is complete) is investigated, and an approximation property of $f \in \text{lbv}(I, X)$ through *local step functions* is established.

Section 5 develops the effect of a monotone change of variable upon a function of locally bounded variation. This produces the basis of the concept of a *rectifiable curve* which is to play an essential role in Sect. 15.

From Sect. 6 onward, X denotes a *real Banach space*. Associated with every $f \in \text{lbv}(I, X)$ is its *differential measure*, an X -valued measure on the interval I denoted by df .

Section 7 begins with discussion, along the line of Bourbaki [20], of the concept of a *majorable* X -valued measure, say m , on some locally compact topological space; such is the case, in particular, of every X -valued measure if X has finite dimension. In alternative constructions of vector measure theory, the corresponding notion is that of an X -valued measure *with finite variation* [21]. Associated with such an m is its *modulus* $|m|$, a nonnegative real measure on the same space. It is involved in some inequalities and calculation rules concerning the integral of a real function relative to the vector measure m . A typical instance of a majorable X -valued measure is provided by defining m from a given nonnegative real measure μ , through a *density function* $m'_\mu \in L^1_{\text{loc}}(\mu, X)$. This one expresses by writing $m = m'_\mu \mu$.

When the locally compact space we have considered equals a real interval I , the differential measure df of any $f \in \text{lbv}(I, X)$ is found to be majorable. In the sense of the ordering of real measures, its modulus measure satisfies $|df| \leq dV_f$, with equality if f has “aligned jumps”.

The integral of df over a subinterval of I is expressed in Sect. 8. Typically, $\int_{[a,b]} df = f^+(b) - f^-(a)$, under Convention 3.5 if needed; similar formulas apply to open or semi-open intervals with endpoints in I . For every subinterval J of I , one defines $(df)_J$, the measure *induced* by df on J . The restriction f_J of f to J is an element of $\text{lbv}(J, X)$ whose differential measure may differ from $(df)_J$, if J is not open relative to I .

Section 9 extends to vector measures a procedure familiar in probability theory. It is shown that every majorable X -valued measure on the interval I equals the differential measure of some $f \in \text{lbv}(I, X)$. This function may be constructed left-continuous (alternatively right-continuous) in the interior of I , a requirement which permits that this function be determined uniquely, up to an additive constant.

Section 10 introduces three Banach spaces X, Y, Z and a continuous bilinear mapping $\Phi: X \times Y \rightarrow Z$. If m is a majorable Y -valued measure on a locally compact space and if $g \in L^1_{\text{loc}}(|m|; X)$, a Z -valued measure denoted by $\Phi(g, m)$ is

defined. Calculation rules are established, involving in particular the case where m is defined through a nonnegative real measure μ and a density function m'_μ .

In the same three-space setting, two functions $x \in \text{lbv}(I, X)$ and $y \in \text{lbv}(I, Y)$ are considered in Sect. 11. Then $t \rightarrow \Phi(x(t), y(t))$ belongs to $\text{lbv}(I, Z)$ and some *differentiation formulas* are established; in particular

$$d\Phi(x, y) = \Phi(dx, y^-) + \Phi(x^+, dy). \quad (2.1)$$

As a special case, this is used to show that the differential measure of an lbv vector function in an open interval equals its derivative in the sense of distributions.

In Sect. 12, the above is particularized by making $Y = X$ and $Z = \mathbb{R}$. It is assumed that $\Phi : X \times X \rightarrow \mathbb{R}$ is a symmetric continuous bilinear mapping such that the quadratic form $a \rightarrow \Phi(a, a)$ is *nonnegative*. Then, for every $x \in \text{lbv}(I, X)$, the following inequalities hold, in the sense of the ordering of real measures:

$$2\Phi(x^-, dx) \leq d\Phi(x, x) \leq 2\Phi(x^+, dx). \quad (2.2)$$

This is applied to the study of certain *integral quadratic functionals* on $\text{bv}(I, X)$: positivity and lower semicontinuity in the norm $\sup_t \|x(t)\|$. Some generalizations of (2.2) to functions more general than the quadratic form $\Phi(\cdot, \cdot)$ are reviewed. All this is motivated by *energy inequalities* in evolution problems.

Section 13 is devoted to the connection between the absolute continuity of a function $f : I \rightarrow X$ and the existence of a density $f'_t \in L^1_{\text{loc}}(I, dt; X)$ for the differential measure df , relatively to the Lebesgue measure dt . The notation $f \in W^{1,1}_{\text{loc}}(I, X)$ is used to express this existence. The *Radon-Nikodym property* of the Banach space X , which holds in particular if it is reflexive, here plays the central role. This section, as well as the next, is mainly a review without proofs of some significant facts from the literature. In particular the reader may here find some conceptual and terminological links between the functional measure theory, used throughout the chapter, and the widespread concept of a countably additive set function.

Because almost every point of I is a *Lebesgue point* of $f'_t \in L^1_{\text{loc}}(I, dt; X)$, the corresponding function $f \in W^{1,1}_{\text{loc}}(I, X)$ possesses f'_t as derivative almost everywhere. On the other hand, the differentiability almost everywhere for any $f \in \text{lbv}(I, X)$ is equivalent to the Radon-Nikodym property for X . Section 14 is a review of facts connected with this. In particular, a recent result on unilateral derivation involving a base measure other than Lebesgue is quoted. The final part of this section, limited for brevity to real functions, explains the relevance of Lebesgue's *decomposition theorem* to the questions investigated. To preclude any possible confusion, the traditional decomposition of a bv function into the sum of a *jump function* and of a continuous component is also recalled.

Section 15 introduces a new way of studying lbv functions. It is established that, among other factorization properties, for every $f \in \text{lbv}(I, X)$ there exist an increasing function w of I to some real interval Δ and a Lipschitz function δ of Δ to X such that $f = \delta \circ w$; the indefinite variation V_f is concomitantly factorized through w . The proof consists in filling the possible gaps in the graph of f in $I \times X$ by line segments and constructing from that a rectifiable curve in $\mathbb{R} \times X$. The relevance of this result will be demonstrated in the concluding section by the *graph approximation* of lbv unknowns.

Section 16 illustrates the material in this chapter by some indications on how the theory of lbv vector functions applies to evolution problems governed by measure differential inclusions. This gives the occasion to review a few other facts from the literature, in particular Helly's selection theorem.

3. One-sided Limits and Initial Conditions

Let I denote a real interval, including or not its possible ends. Let f be a mapping of I to a Hausdorff topological space E ; in all that follows, for every $t \in I$ different from the possible right end of this interval, we shall denote by $f^+(t)$ the *right-side limit*

$$f^+(t) = \lim_{s \rightarrow t, s > t} f(s)$$

if it exists; symmetrically, for t different from the possible left end of I , the *left-side limit*, if it exists, will be denoted by $f^-(t)$.

Recall that a Hausdorff topological space E is said to be *regular* (see, e. g. [25]) iff for each point x and each neighborhood U of x there exists a closed neighborhood V of x such that $V \subset U$; equivalently, the family of closed neighborhoods of each point is a base for the neighborhood system of the point. Such are, in particular, every metric space and every Hausdorff topological linear space.

Proposition 3.1: Let E be regular and let $f: I \rightarrow E$ be such that for every $t \in I$ different from the possible right end of I , there exists $f^+(t)$; then

$$\lim_{s \rightarrow t, s > t} f^+(s) = f^+(t). \quad (3.1)$$

If, in addition, for every t different from the possible left end of I , there exists $f^-(t)$, then

$$\lim_{s \rightarrow t, s > t} f^-(s) = f^-(t). \quad (3.2)$$

In short, $(f^+)^+ = f^+$, $(f^-)^+ = f^+$ and, through the symmetric property after exchanging sides, $(f^-)^- = f^-$, $(f^+)^- = f^-$.

Proof: Let $W \subset E$ be a closed neighborhood of $f^+(t)$; there exists $\eta > 0$ such that

$$s \in]t, t + \eta[\Rightarrow f(s) \in W.$$

For every s in the open interval $]t, t + \eta[$ one has

$$f^+(s) = \lim_{\tau \rightarrow s, \tau \in]s, t + \eta[} f(\tau) \in \text{cl } f(]t, t + \eta[) \subset W.$$

Since the closed neighborhoods such as W make a base of neighborhoods of $f^+(t)$, this establishes (3.1).

Similarly, for every s in $]t, t + \eta[$,

$$f^-(s) = \lim_{\tau \rightarrow s, \tau \in]t, s[} f(\tau) \in W,$$

hence (3.2). □

Remark 3.2: The situation appears especially clear when I is an *open* interval, possibly unbounded. The proposition shows that f^+ and f^- , defined by assumption at every point of I , are, respectively, *right-continuous* and *left-continuous* functions. This introduces two operations: the right-continuous and left-continuous *regularizations*. Proposition 3.1 expresses that, if these operations are applied alternately and/or iteratively, the final result is the same as if the operation performed last had been applied to f itself.

Remark 3.3: Suppose that E is a complete metric space. Then, classically [26] the existence of the right- and left-limits of f at every interior point of I , together with the existence of the appropriate one-sided limits if I includes some of its possible ends, is equivalent to f being *regulated* in the sense that, *on every compact subinterval $[a, b]$ of I , f equals the uniform limit of a sequence of step functions*. By definition, a *step function* on $[a, b]$ equals a constant over each member of a finite partition of $[a, b]$ into *subintervals* of any sort (some of them possibly reduced to singletons).

The elementary theory of differential equations rests on the definition of possible solutions as indefinite Riemann integrals of regulated functions [26].

Remark 3.4: In Dynamics, the time interval I considered commonly contains its left end t_0 , the *initial instant*. Suppose that a function u from I to some Banach space represents the velocity of the investigated mechanical system; the *initial data* usually involve some given *initial velocity* u_0 . In the most traditional situation, the system evolution is governed by differential equations, assumed to be satisfied in the interior of I . The initial conditions are then interpreted as imposing the right-limits at t_0 of the functions one is investigating. This is similar to what is more generally done with physical phenomena governed by partial differential equations in some open subset of \mathbb{R}^n : boundary conditions involve

some “trace” operators whose simplest instance is the construction of limits at boundary points. But, if the dynamical process is not smooth enough to secure continuity of the velocity function $t \rightarrow u(t)$ in the interior of the time interval I , imposing the right-limit $u^+(t_0)$ looks inconsistent: why should not a jump also occur at the very instant t_0 ? Therefore, the governing conditions of such a nonsmooth process have to place t_0 on the same footing as the other instants.

A simple way of describing things is to imagine that the investigated process is already in progress before t_0 , i. e. it takes place over a time interval \hat{I} containing I in its interior. While the evolution for $t \geq t_0$ is the object of prediction, the “initial” data are understood to convey abridged information on the system history before t_0 : we shall interpret them as the *left-limits* of the considered functions at t_0 .

It will then prove expedient, as soon as a function u is defined on I , to extend it into a function \hat{u} defined on \hat{I} with the constant value $\hat{u}(t) = u(t_0)$ for $t < t_0$; this implies $\hat{u}^-(t_0) = u(t_0)$.

Accordingly, the following convention will often be used in the following sections.

Convention 3.5: If the real interval I contains its possible left end T_l , then for a function f defined on I we shall agree that $f^-(T_l)$ represents $f(T_l)$; symmetrically, if I contains its possible right end T_r , we shall agree that $f^+(T_r) = f(T_r)$.

Warning: The above convention clearly preserves the calculation rules $f^{++} = f^+$ and $f^{--} = f^-$ but IT IS NOT CONSISTENT with $f^{-+} = f^+$ and $f^{+-} = f^-$. This reflects the fact that the extension of f on the left of T_l with the constant value $f(T_l)$ does not commute with right-continuous regularization; similarly for the extension of f on the right of T_r .

4. Functions of Locally Bounded Variation

Let I be a real interval of any sort and E a metric space, with distance denoted by δ . Let $f: I \rightarrow E$ and let J be a subinterval of I . The *variation* of f on J is, by definition, the nonnegative extended real number

$$\text{var}(f, J) = \sup \sum_{i=1}^n \delta(f(\tau_{i-1}), f(\tau_i)),$$

where the supremum is taken over all strictly increasing finite sequences $\tau_0 < \tau_1 < \dots < \tau_n$ of points of J ; this supremum refers to the ordering of $[0, +\infty]$, i. e. $\sup \emptyset = 0$. One may equivalently accept finite sequences which are only nondecreasing.

Clearly $\text{var}(f, J)$ is zero iff f equals a constant on J .

If J is a compact subinterval $[a, b]$ of I , we shall also write $\text{var}(f; a, b)$ instead of $\text{var}(f, [a, b])$; this symbol will be considered as meaningless if $a > b$. Evidently $\text{var}(f; a, a) = 0$ and, if $a \leq b \leq c$ in I , one readily finds

$$\text{var}(f; a, c) = \text{var}(f; a, b) + \text{var}(f; b, c). \quad (4.1)$$

Definition 4.1: The function $f: I \rightarrow E$ is said to be of bounded variation on I iff $\text{var}(f, I) < +\infty$; notation $f \in \text{bv}(I, E)$.

It is called function of locally bounded variation on I iff $\text{var}(f; a, b) < +\infty$ for every compact subinterval $[a, b]$ of I ; notation $f \in \text{lbv}(I, E)$.

Observe that $\text{var}(f, I)$ equals the supremum of $\text{var}(f; a, b)$ for all $[a, b] \subset I$.

The addition rule (4.1) elementarily implies that f belongs to $\text{lbv}(I, E)$ iff there exists a (nondecreasing) function $V_f: I \rightarrow \mathbb{R}$ such that

$$\forall [a, b] \subset I: \text{var}(f; a, b) = V_f(b) - V_f(a). \quad (4.2)$$

This real function, called a *variation function* or an *indefinite variation* of f , is defined up to the addition of an arbitrary constant.

One readily sees that $f: I \rightarrow E$ is *Lipschitzian* on I iff $f \in \text{lbv}(I, E)$ with $V_f: I \rightarrow \mathbb{R}$ Lipschitzian in the usual metric of \mathbb{R} ; then f and V_f admit the same Lipschitz constant.

Recall that $f: I \rightarrow E$ is said to be *absolutely continuous* on I iff, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, denoting by $]a_i, b_i[$ any finite collection of nonoverlapping open subintervals of I , one has the following implication

$$\sum_i (b_i - a_i) < \eta \Rightarrow \sum_i \delta(f(a_i), f(b_i)) < \varepsilon.$$

The function is said to be *locally absolutely continuous* on I if it is absolutely continuous on every compact subinterval of I .

It is easily proved that f is absolutely continuous (alternatively locally absolutely continuous) iff $f \in \text{lbv}(I, E)$ with variation function V_f absolutely continuous (locally absolutely continuous respectively) in the usual metric of \mathbb{R} .

We shall now investigate one-sided limits of lbv functions.

Proposition 4.2: If the metric space (E, δ) is complete and $f \in \text{lbv}(I, E)$, then f possesses a left-limit $f^-(t)$ for every $t \in I$ different from the possible left end of I (otherwise, Convention 3.5 makes $f^-(t)$ trivially meaningful); the symmetric statement holds for $f^+(t)$.

Proof: Since $V_f: I \rightarrow \mathbb{R}$ is nondecreasing, the left-limit $V_f^-(t)$ exists for every $t > \inf I$. Thus $V_f(s)$ possesses the Cauchy property for s tending to t from the left. In view of the inequality $\delta(f(s), f(s')) \leq |V_f(s') - V_f(s)|$, the same is true for f . \square

Proposition 4.3: Let $f \in \mathcal{C}^-(I, E)$; then, for every $t \in I$ different from the possible left end, there exists

$$\lim_{s \rightarrow t, s < t} \delta(f(s), f(t))$$

equal to

$$\lim_{s \rightarrow t, s < t} \text{var}(f; s, t) = V_f(t) - V_f^-(t).$$

If $f^-(t)$ exists, these expressions equal $\delta(f^-(t), f(t))$.

Symmetrical properties hold true for right-limits, as long as t is different from the possible right end of I .

Proof: Let $t_0 < t$ in I and let $\varepsilon > 0$. By the definition of $\text{var}(f; t_0, t)$ there exists a strictly increasing finite sequence $t_0 = \tau_0 < \tau_1 < \dots < \tau_{n-1} < \tau_n = t$ such that

$$\text{var}(f; t_0, t) \leq \varepsilon + \sum_{i=1}^n \delta(f(\tau_{i-1}), f(\tau_i)).$$

Let $s \in [\tau_{n-1}, t[$; through the triangle inequality the above entails

$$\begin{aligned} \text{var}(f; t_0, t) &\leq \varepsilon + \sum_{i=1}^{n-1} \delta(f(\tau_{i-1}), f(\tau_i)) + \delta(f(\tau_{n-1}), f(s)) + \delta(f(s), f(t)) \\ &\leq \varepsilon + \text{var}(f; t_0, s) + \delta(f(s), f(t)). \end{aligned}$$

In short,

$$s \in [\tau_{n-1}, t[\Rightarrow \delta(f(s), f(t)) + \varepsilon \geq \text{var}(f; s, t),$$

while, essentially, $\delta(f(s), f(t)) \leq \text{var}(f; s, t)$. Now $\text{var}(f; s, t) = V_f(t) - V_f(s)$ admits, for $s \rightarrow t, s < t$, a limit equal to $V_f(t) - V_f^-(t)$, since $V_f: I \rightarrow \mathbb{R}$ is nondecreasing; this proves the first statement. If $f^-(t)$ exists, the asserted equality follows from the continuity of δ . \square

Corollary 4.4: The function $f \in \text{lbv}(I, E)$ is left-continuous at a point of I (alternatively right-continuous), iff such is the real function V_f .

Elementarily, the set of the discontinuity points in I of the nondecreasing real function V_f is (finite or) countable; Corollary 4.4 shows that the same is true for every $f \in \text{lbv}(I, E)$.

Proposition 4.5: Let $f \in \text{lbv}(I, E)$; if f^- (alternatively f^+) exists at every point of I , under Convention 3.5 if needed, this constitutes a function belonging to $\text{lbv}(I, E)$.

Proof: Let $s < t$ in I ; if s is not the left end of I , one has, due to the continuity of δ ,

$$\delta(f^-(s), f^-(t)) = \lim_{\sigma \rightarrow s, \sigma < s} \lim_{\tau \rightarrow t, \tau < t} \delta(f(\sigma), f(\tau)).$$

Since $\delta(f(\sigma), f(\tau)) \leq |V_f(\tau) - V_f(\sigma)|$, one obtains, by going to limits, $\delta(f^-(s), f^-(t)) \leq V_f^-(t) - V_f^-(s)$. This inequality is similarly established, under Convention 3.5, if s happens to be the left end of I . It readily implies

$$\forall [a, b] \subset I: \text{var}(f^-; a, b) \leq V_f^-(b) - V_f^-(a);$$

therefore, $f^- \in \text{lbv}(I, E)$. □

Incidentally, the above does not imply that V_f^- equals a variation function of f^- .

The end of this section is devoted to the *approximation by local step functions*.

By writing $P \in \text{lfp}(I)$ we shall mean that P is a *partition of I into subintervals of any sort* (some of them possibly reduced to singletons) and that this partition is *locally finite*, i.e. every compact subset of I is covered by a finite collection of members of P .

For P and Q in $\text{lfp}(I)$, we shall write $Q \succ P$ iff Q is a *refinement* of P , i.e. every member of Q is contained in a member of P . Visibly this partial ordering makes of $\text{lfp}(I)$ a *directed set* (see, e.g. [25]) in the sense that every two elements of $\text{lfp}(I)$ possess a common refinement.

A function l of I to a set E is said to be a *local step function* iff there exists $P \in \text{lfp}(I)$ on each member of which l equals a constant.

For every $f: I \rightarrow E$ and every $Q \in \text{lfp}(I)$ let us agree to denote by f_Q a local step function $I \rightarrow E$ constructed as follows: on every member of Q , f_Q assumes the constant value $f(\tau)$, where τ is a point chosen in the said member.

Proposition 4.6: Let (E, δ) be a metric space and $f \in \text{lbv}(I, E)$. For every $\varepsilon > 0$, there exists $P \in \text{lfp}(I)$ such that, for every $Q \in \text{lfp}(I)$ which is a refinement of P , any function f_Q constructed as above satisfies

$$\forall t \in I: \delta(f(t), f_Q(t)) \leq \varepsilon, \tag{4.3}$$

$$\forall [a, b] \subset I: \text{var}(f_Q; a, b) \leq 2\varepsilon + \text{var}(f; a, b). \tag{4.4}$$

Proof: Take a locally finite partition of \mathbb{R} into intervals with lengths $\leq \varepsilon$. The inverse images of these intervals by the nondecreasing function $V_f: I \rightarrow \mathbb{R}$ are

subintervals of I ; the nonempty ones constitute a partition $P \in \text{lfp}(I)$. Let $Q \succ P$; if two points t and τ belong to the same member of Q , they belong a fortiori to the same member of P , say J , hence

$$\delta(f(t), f(\tau)) \leq \text{var}(f, J) \leq \varepsilon.$$

By taking as τ in each member of Q the point where, by construction, f_Q equals f , one obtains the inequality (4.3).

Those intersections of $[a, b]$ with members of Q which are nonempty will be indexed from left to right as J_0, J_1, \dots, J_m . If a point τ_i is chosen in each interval J_i , one trivially has

$$\text{var}(f_Q; a, b) = \sum_{i=1}^m \delta(f_Q(\tau_{i-1}), f_Q(\tau_i)).$$

In particular, let us take $\tau_0 = a, \tau_m = b$ and for $0 < i < m$ let us choose as τ_i the point at which f_Q equals f . Then

$$\text{var}(f_Q; a, b) = \delta(f_Q(a), f(\tau_1)) + \sum_{i=2}^{m-1} \delta(f(\tau_{i-1}), f(\tau_i)) + \delta(f(\tau_{m-1}), f_Q(b)).$$

Using the triangle inequality and, at points a and b , inequality (4.3), this yields

$$\text{var}(f_Q; a, b) \leq 2\varepsilon + \sum_{i=1}^m \delta(f(\tau_{i-1}), f(\tau_i)) \leq 2\varepsilon + \text{var}(f; a, b). \quad \square$$

Remark 4.7: A concept weaker than that of variation may be developed by considering, instead of the metric space (E, δ) , a space F equipped with a *nonsymmetric pseudo-metric*; this means that, for every x and y in F , some *écart* $e(x, y) \in [0, +\infty]$ verifying the triangle inequality $e(x, z) \leq e(x, y) + e(y, z)$ is defined. But possibly $e(x, y) \neq e(y, x)$ and possibly $e(x, y) = 0$ for $x \neq y$. This is done in [27], where a *multifunction* of the interval I to the metric space (E, δ) is considered, i.e. a function of I to $F = 2^E$. For any two elements A, B of F , the *écart*

$$e(A, B) = \sup_{\alpha \in A} \inf_{\beta \in B} \delta(\alpha, \beta)$$

is defined (here \sup and \inf refer to the ordering of $[0, +\infty]$, i.e. $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$). This generates the theory of *multifunctions with finite retraction*, motivated by some mechanical evolution problems of the unilateral sort [5].

5. Monotone Change of Variable

Proposition 5.1: Let H and I be two real intervals and let $c: H \rightarrow I$ be nondecreasing. If $f \in \text{lbv}(I, E)$, then the function $f \circ c: s \rightarrow f(c(s))$ belongs to $\text{lbv}(H, E)$ and, for every $[a, b] \subset H$,

$$\text{var}(f \circ c; a, b) \leq \text{var}(f; c(a), c(b)). \quad (5.1)$$

If, in addition, c is assumed continuous on $[a, b]$ (equivalently, c is surjective of $[a, b]$ to $[c(a), c(b)]$) equality holds in (5.1).

The same statements are true for nonincreasing $c : H \rightarrow I$, provided the right-hand member of (5.1) is replaced by $\text{var}(f; c(b), c(a))$.

Proof: For $\sigma_{i-1} < \sigma_i$ in $[a, b]$ one has $c(\sigma_{i-1}) \leq c(\sigma_i)$ and

$$\delta((f \circ c)(\sigma_{i-1}), (f \circ c)(\sigma_i)) = \delta(f(c(\sigma_{i-1})), f(c(\sigma_i))) \leq \text{var}(f; c(\sigma_{i-1}), c(\sigma_i)).$$

The definition of the variation of f readily yields inequality (5.1).

Let $\varepsilon > 0$; except in the trivial case where $c(a) = c(b)$, there exists a finite sequence $\tau_0 < \tau_1 < \dots < \tau_n$ in $[c(a), c(b)]$ such that

$$\text{var}(f; c(a), c(b)) \leq \varepsilon + \sum_{i=1}^n \delta(f(\tau_{i-1}), f(\tau_i)).$$

If the nondecreasing function c is surjective of $[a, b]$ to $[c(a), c(b)]$, there exist $\sigma_0 \leq \sigma_1 \leq \dots \leq \sigma_n$ in $[a, b]$ such that $\tau_i = c(\sigma_i)$; hence

$$\sum_{i=1}^n \delta(f(\tau_{i-1}), f(\tau_i)) = \sum_{i=1}^n \delta((f \circ c)(\sigma_{i-1}), (f \circ c)(\sigma_i)) \leq \text{var}(f \circ c; a, b).$$

As ε may be taken arbitrarily small, this establishes equality in (5.1). \square

Increasing changes of variables make the basis of the concept of *oriented curve* in the metric space E . Let I and I' be two real intervals; let $f : I \rightarrow E$ and $f' : I' \rightarrow E$ denote two continuous mappings. Suppose that no subinterval of their respective domains exists on which f or f' are constant. One defines an equivalence relation by writing $f \sim f'$ iff there exists $c : I' \rightarrow I$, nondecreasing and bijective (hence continuous strictly increasing, with continuous inverse) such that $f' = f \circ c$. Every equivalence class relative to \sim is called an *oriented curve* in E ; every member of such a class is called a *proper parametrization* of the oriented curve.

As a consequence of Proposition 5.1, if some member of the class is an lbv function, so are all members. In that case the class is called an *oriented curve of locally bounded variation*, or also a *rectifiable curve*, an archaic way of saying that the *length* of any of its *arcs* may be defined. In fact, let $f \in \text{lbv}(I, E)$ and $f' = f \circ c \in \text{lbv}(I', E)$ be two proper parametrizations; let t' range through some compact subinterval $[a', b']$ of I' ; equivalently $t = c(t')$ ranges through the subinterval $[a, b] = [c(a'), c(b')]$ of I . Then the point $f(t) = f'(t')$ of E is said to describe an arc of the curve, with $f(a) = f'(a')$ and $f(b) = f'(b')$ as endpoints. Proposition 5.1 entails $\text{var}(f; a, b) = \text{var}(f'; a', b')$; this is, by definition, the length of the arc, a real number independent of the parametrization used.

Since f is continuous and nowhere constant, a variation function V_f of f is a continuous and strictly increasing mapping of I onto a real interval Γ . Therefore,

$\gamma = f \circ V_f^{-1} \in \text{lbv}(\Gamma, E)$ is also a proper parametrization of the curve; if $\alpha = V_f(a)$ and $\beta = V_f(b)$, the length $\text{var}(f; a, b) = \beta - \alpha$ is found, through Proposition 5.1, to be equal to $\text{var}(\gamma; \alpha, \beta)$. This equality holds for every $[\alpha, \beta] \subset \Gamma$; in other words, γ admits as variation function the identity mapping of Γ . We shall express this property by saying that γ is a *curve-isometry*; in fact, γ maps any compact subinterval of Γ onto an arc of equal length. It is commonly said that $\tau = V_f(t) \in \Gamma$ is the *oriented arc length* (or *curvilinear abscissa*) locating the point $f(t) = \gamma(\tau)$ on the curve. If one starts from f' instead of f , the same procedure yields another curve-isometry, say γ' , and for every $\tau \in \Gamma$ it results easily that $\gamma(\tau) = \gamma'(\tau - \tau_0)$, with τ_0 a constant.

Incidentally, observe that a rectifiable curve may be generated from any continuous lbv function g of an interval J to E ; to recover the above setting, it suffices to perform on the interval J a reducing manipulation: every possible subinterval over which g is a constant will be contracted to a singleton. If such nonzero subintervals exist, g is said to be an *improper parametrization* of the corresponding curve.

6. Differential Measures

As before, I denotes a real interval of any form; in all the following the metric space considered will be a *real Banach space* X , with norm denoted by $\|\cdot\|$.

For every $f_1, f_2: I \rightarrow X$, every λ_1, λ_2 in \mathbb{R} and every subinterval J of I , one easily establishes

$$\text{var}(\lambda_1 f_1 + \lambda_2 f_2, J) \leq |\lambda_1| \text{var}(f_1, J) + |\lambda_2| \text{var}(f_2, J).$$

This shows that $\text{bv}(I, X)$ and $\text{lbv}(I, X)$ are linear subspaces of X^I .

Call \mathcal{S} the totality of the finite subsets of I ; every member S of \mathcal{S} may be uniquely enumerated as a strictly increasing finite sequence of points of I , say

$$S: \tau_0 < \tau_1 < \dots < \tau_n. \quad (6.1)$$

By an *intercalator*, we shall mean an assignment – call it θ – associating with every $S \in \mathcal{S}$, as written in the form (6.1), and every $i \in \{1, \dots, n\}$ an element denoted by θ_S^i of the compact interval $[\tau_{i-1}, \tau_i]$. By $\varphi \in \mathcal{K}(I)$ it is classically expressed that φ is a continuous real function on I whose support (relative to I) is compact.

Let $f: I \rightarrow X$; for every S, θ and φ as above let us construct the following element of X :

$$M(S, \theta, \varphi) = \sum_{i=1}^n \varphi(\theta_S^i) (f(\tau_i) - f(\tau_{i-1})). \quad (6.2)$$

The *inclusion ordering* makes out of \mathcal{S} a *directed set*, i. e. for every two elements S, S' of \mathcal{S} , there exists $S'' \in \mathcal{S}$ such that $S'' \supset S$ and $S'' \supset S'$, namely $S'' = S \cup S'$. This enables one to use the concept of the *convergence* of a mapping, say $S \rightarrow M_S$, of (S, \supset) to the topological space X (see, e. g. [25]); such a mapping is also called a *generalized sequence* of elements of X , indexed in (S, \supset) , or, in short, a *net*.

Proposition 6.1: Let $f \in \text{lbv}(I, X)$; for every $\varphi \in \mathcal{K}(I)$ and every intercalator θ , the mapping $S \rightarrow M(S, \theta, \varphi)$ of the directed set (S, \supset) to X converges to a limit independent of θ ; convergence is uniform with regard to the choice of θ .

Denote this limit by $\int \varphi df$; for every compact subinterval $[a, b]$ of I containing the support of φ the inequality

$$\left\| \int \varphi df \right\| \leq \max |\varphi| \text{ var}(f; a, b) \quad (6.3)$$

holds, showing that the linear mapping $\varphi \rightarrow \int \varphi df$ of $\mathcal{K}(I)$ to X constitutes a *vector measure* on I in the sense of Bourbaki.

Definition 6.2: The X -valued measure above will be called the *differential measure* (or Stieltjes measure) of the function $f \in \text{lbv}(I, X)$ and be denoted by df .

Proof of Proposition 6.1: Let $[a, b]$ be a compact subinterval of I containing the support of φ . As the members of \mathcal{S} which include a and b constitute a cofinal subset of (\mathcal{S}, \supset) (see, e. g. [25]), one may restrict oneself to such members. Besides, the points of \mathcal{S} which do not belong to $[a, b]$ may be neglected when introducing the enumeration (6.1) since they have zero contribution in such expressions as (6.2), due to the vanishing of φ outside $[a, b]$. Hence we shall only take S under the form

$$S: a = \tau_0 < \tau_1 < \dots < \tau_n = b. \quad (6.4)$$

Let us first consider the special intercalator $\bar{\theta}$, which consists in taking $\bar{\theta}_S^i = \tau_i$, and prove the convergence of $S \rightarrow M(S, \bar{\theta}, \varphi)$.

Let $\varepsilon > 0$; as φ is uniformly continuous on $[a, b]$, the finite subset S may be chosen such that the following holds: for every i and for every τ, τ' in $[\tau_{i-1}, \tau_i]$

$$|\varphi(\tau) - \varphi(\tau')| \leq \varepsilon. \quad (6.5)$$

Let $S' \in \mathcal{S}$ such that $S \subset S' \subset [a, b]$; with S written in the form (6.4), for every $i \in \{1, 2, \dots, n\}$ let us denote by τ_i^j the points of S' which lie in $[\tau_{i-1}, \tau_i]$, so that

$$\tau_{i-1} = \tau_i^0 < \tau_i^1 < \dots < \tau_i^p = \tau_i$$

($p = 1$ when S' possesses no point in $] \tau_{i-1}, \tau_i [$; if $i < n$ the same point of S' appears as τ_i^p and as τ_{i+1}^0).

The intervals delimited by S' and contained in $[\tau_{i-1}, \tau_i]$ contribute to $M(S', \bar{\theta}, \varphi)$ by terms whose sum equals

$$\sum_{j=1}^p \varphi(\tau_j^i) (f(\tau_j^i) - f(\tau_{j-1}^{i-1})), \quad (6.6)$$

to be compared with the following term of $M(S, \bar{\theta}, \varphi)$

$$\varphi(\tau_i) (f(\tau_i) - f(\tau_{i-1})) = \sum_{j=1}^p \varphi(\tau_i) (f(\tau_j^i) - f(\tau_{j-1}^{i-1})). \quad (6.7)$$

If S meets the requirement (6.5), the difference between expressions (6.6) and (6.7) lets itself be bounded as follows:

$$\begin{aligned} & \left| \sum_{j=1}^p (\varphi(\tau_i) - \varphi(\tau_j^i)) (f(\tau_j^i) - f(\tau_{j-1}^{i-1})) \right| \\ & \leq \varepsilon \sum_{j=1}^p \|f(\tau_j^i) - f(\tau_{j-1}^{i-1})\| \leq \varepsilon \text{var}(f; \tau_{i-1}, \tau_i). \end{aligned} \quad (6.8)$$

After summation with respect to i , this yields

$$\|M(S, \bar{\theta}, \varphi) - M(S', \bar{\theta}, \varphi)\| \leq \varepsilon \text{var}(f; a, b). \quad (6.9)$$

Therefore, if S'' denotes another member of \mathcal{S} , also contained in $[a, b]$, one has the implication

$$S' \supset S, S'' \supset S \Rightarrow \|M(S', \bar{\theta}, \varphi) - M(S'', \bar{\theta}, \varphi)\| \leq 2\varepsilon \text{var}(f; a, b),$$

meaning that the generalized sequence $S \rightarrow M(S, \bar{\theta}, \varphi) \in X$ possesses the Cauchy property; this establishes convergence.

As for the use of another intercalator θ , one observes that, if S meets the requirement (6.5), a calculation similar to that in (6.8) yields, for every $S' \in \mathcal{S}$ which is a refinement of S ,

$$\|M(S', \theta, \varphi) - M(S', \bar{\theta}, \varphi)\| \leq \varepsilon \text{var}(f; a, b).$$

This proves that, uniformly with regard to the choice of θ , the same limit is approached.

Since $\varphi \rightarrow M(S, \theta, \varphi)$ is, for fixed S and θ , a linear mapping of $\mathcal{K}(I)$ to X , it results, after going to the limit, that $\varphi \rightarrow \int \varphi df$ is linear.

Finally, the definition of M trivially implies

$$\|M(S, \theta, \varphi)\| \leq \max|\varphi| \text{var}(f; a, b),$$

yielding (6.3) at the limit.

That the linear mapping $\varphi \rightarrow \int \varphi df$ of $\mathcal{K}(I)$ to the Banach space X constitutes a vector measure on I in the sense of Bourbaki means, as recalled in Sect. 1, that it

possesses the following continuity property: for every compact subset K of I , the restriction of this mapping to the subspace consisting of the elements of $\mathcal{K}(I)$ with support contained in K is continuous in the supremum norm. Since K is necessarily contained in some $[a, b]$, this in fact follows from the inequality (6.3). \square

Remark 6.3: One similarly finds that, if S meets the requirement (6.5), then for every $S' \supset S$ in \mathcal{S}

$$\left\| \int \varphi df - M(S, \theta, \varphi) \right\| \leq \varepsilon \operatorname{var}(f; a, b). \quad (6.10)$$

Remark 6.4: The totality of the X -valued measures on I naturally constitutes a real linear space. As the first calculation rule concerning differential measures, there readily results the *linearity* of the mapping $f \rightarrow df$.

Furthermore, if the vector function f is a constant, then $df=0$. Remark 6.6 below shows that the converse is not true in general; however, Corollary 8.4 will describe a situation free from this inconvenience.

We now calculate df in the case where f is a *local step function*, i.e. (cf. Sect. 4) there exists $P \in \operatorname{lfp}(I)$ on each interval of which f is a constant. Every point of I which equals an end of some of the intervals constituting P will be called a *node* of P , and also a node of f . Such a function is readily found to belong to $\operatorname{lbv}(I, X)$.

Proposition 6.5: If $f: I \rightarrow X$ is a local step function and $\varphi \in \mathcal{K}(I)$, one has, under Convention 3.5 if needed,

$$\int \varphi df = \sum_{j=1}^p \varphi(t_k) (f^+(t_k) - f^-(t_k)), \quad (6.11)$$

where t_1, t_2, \dots, t_p are the nodes of f contained in the support of φ .

In other words, the differential measure df equals the sum of a locally finite collection of *point measures* placed at the discontinuity points of f , the respective values of which are equal to the jumps $f^+ - f^-$.

Remark 6.6: If one of the intervals over which f is constant reduces to a singleton, say $\{t_s\}$, interior to I , the value $f(t_s)$ may be fixed independently of the values $f^+(t_s)$ and $f^-(t_s)$ that the function takes on the two intervals adjacent to t_s ; hence the measure df has no relationship with $f(t_s)$. In particular, one may take f equal to a constant throughout I , except at some isolated interior points: this gives an example of a nonconstant lbv function with zero differential measure.

Proof of Proposition 6.5: Let $[a, b] \subset I$ contain the support of φ . First suppose I open in \mathbb{R} ; then a and b may be chosen to be different from all the nodes of P . The

partition P induces a finite partition of $[a, b]$ whose nodes will be enumerated as

$$a < t_1 < t_2 < \dots < t_p < b. \quad (6.12)$$

By definition, $\int \varphi df$ equals the limit of the convergent net $S \rightarrow M(S, \theta, \varphi)$, as expressed in (6.2), with

$$S: a = \tau_0 < \tau_1 < \dots < \tau_n = b.$$

Take the same intercalator $\bar{\theta}$ as in the proof of Proposition 6.1. Using Remark 6.3, one observes that $\int \varphi df$ also equals the limit of $M(S_j, \bar{\theta}, \varphi)$, where S_j , $j \in \mathbb{N}$, denotes an infinite sequence of choices of S , agreeing with the following requirements:

- (a) There exists a sequence $\eta_j > 0$, tending to zero, such that the distance between any two successive points of S_j is less than η_j .
- (b) No point of S_j equals a node of P .
- (c) Every open interval $]\tau_{i-1}, \tau_i[$ determined by S_j contains at most one of the said nodes.

Since f is a constant between the points (6.12) the only possible nonzero terms in $M(S_j, \bar{\theta}, \varphi)$ correspond to the p values of i such that $]\tau_{i-1}, \tau_i[$ contains one of the points t_k . The term so associated with t_k equals $\varphi(\tau_i) (f(\tau_i) - f(\tau_{i-1}))$. When j tends to infinity this clearly has the limit $\varphi(t_k) (f^+(t_k) - f^-(t_k))$, hence (6.11) is proved.

So far we have supposed I open in \mathbb{R} . If I possesses, for instance, a left end T_l and includes it, the support of φ may contain this point; then necessarily $a = T_l$. Again, let $t_1 < t_2 < \dots < t_p$ denote the nodes of P contained in $[a, b]$; it may happen that $t_1 = a = T_l$, so the term corresponding to $i = 1$ in $M(S_j, \bar{\theta}, \varphi)$, namely $\varphi(\tau_1) (f(\tau_1) - f(T_l))$, possesses the limit $\varphi(T_l) (f^+(T_l) - f(T_l))$ when j tends to infinity. According to Convention 3.5, $f(T_l)$ is interpreted as $f^-(T_l)$; this agrees with (6.11). \square

Remark 6.7: We acknowledge a certain lack of consistency in the use of the notation df to represent the differential measure of f . In fact, the symbol d usually does another job when the integral of a function h , relative to some measure called μ , is written down as $\int h d\mu$ or, if a dummy variable is needed, $\int h(t) d\mu(t)$ (also $\int h(t) dt(\mu)$). So some authors are entitled to prefer using Df to mean the differential measure of f [3]. An integral relative to this measure would then logically be expressed as $\int h dDf$ or $\int h(t) dDf(t)$. If that system of notation were strictly implemented, the Lebesgue measure on an interval would stand out as Did (since it equals the differential measure of the identity function $\text{id} : I \rightarrow I$) and the traditional integral $\int h(t) dt$ relative to it would take on the form $\int h(t) dDid(t)$ or $\int h(t) Did(dt)$. Clearly, for simplicity's sake, some consistency has to be sacrificed.

7. The Modulus Measure

In the following definitions, one may take as I a locally compact topological space which, for simplicity, will be supposed “countable at infinity”, i. e. it equals the union of a countable collection of compact subsets; such is evidently the case when I is a real interval.

Definition 7.1: The vector measure m is said to be *majorable* iff there exists a nonnegative real measure μ on I such that, for every $\varphi \in \mathcal{K}_+(I)$ (the subset of $\mathcal{K}(I)$ consisting of nonnegative functions), one has

$$\|\int \varphi dm\| \leq \int \varphi d\mu. \quad (7.1)$$

This is equivalent to Bourbaki’s terminology (cf. [20], Chap. VI, § 2, No. 3, where, more generally than X , a locally convex topological linear space, with a collection of semi-norms characterizing its topology, is considered).

Note that every $\varphi \in \mathcal{K}(I)$ equals the difference of two elements φ_- and φ_+ of $\mathcal{K}_+(I)$, with $\varphi_- + \varphi_+ = |\varphi|$. Hence the above property is equivalent to

$$\|\int \varphi dm\| \leq \int |\varphi| d\mu \quad (7.2)$$

holding for every $\varphi \in \mathcal{K}(I)$.

If X has finite dimension, any X -valued measure is majorable. This easily follows from the use of a base in X and from the complete lattice property of real measures

Let us consider the dual Banach space X' of X , with duality pairing denoted by $\langle \cdot, \cdot \rangle$. Inequality (7.1) is equivalent to saying that, for every $x' \in X'$,

$$\forall \varphi \in \mathcal{K}_+(I) : \langle x', \int \varphi dm \rangle \leq \|x'\| \int \varphi d\mu. \quad (7.3)$$

For fixed x' in X' , the linear mapping $\varphi \rightarrow \langle x', \int \varphi dm \rangle$ of $\mathcal{K}(I)$ to \mathbb{R} clearly constitutes a real measure, which we shall denote by $x' \circ m$, in accordance with the customary notation for the composition of mappings (the writing $\langle x', m \rangle$ could also be used, consistent with Sect. 10 below, where more general bilinear expressions will be considered). Then (7.3) means equivalently that the collection of real measures $x' \circ m$, for x' ranging through the unit ball B' of X' , is majorized by μ in the sense of the ordering of real measures. Hence this collection possesses a supremum, majorized by μ , which we shall denote by $|m|$ (or $|dm|$ when it appears in an integral). Equivalently, $|m|$ equals the supremum, for $x' \in B'$, of the collection of nonnegative real measures

$$|x' \circ m| = \sup \{ -x' \circ m, x' \circ m \};$$

therefore $|m|$ is nonnegative and vanishes iff m vanishes. Clearly (7.1), (7.2) and (7.3) hold with $\mu = |m|$.

More generally, a similar procedure is used in [20], Chap. VI, § 2, No. 3, starting with a continuous semi-norm q on X , in order to define some real measure denoted by $q(m)$. Regarding diverse situations where a measure is also constructed by extending the symbol of a real function, see [20], Chap. V, § 5, No. 9, and [28–30].

Definition 7.2: For every majorable vector measure m , the nonnegative real measure $|m|$ introduced above is called the *modulus* (or *absolute value*) of m .

If $X = \mathbb{R}$, this is nothing but the traditional absolute value of a real signed measure.

When vector measures are treated from the standpoint of set functions, the concept corresponding to majorability is that of a *measure of finite variation* (or locally bounded variation); then $|m|$ is introduced as the *variation measure* of m , also called the modulus of m in [21], § 3, No. 4.

Counterexample 7.3: Let us show a nonmajorable vector measure. Suppose that I is a compact topological space and take as the Banach space X the function space $\mathcal{K}(I)$ equipped with the supremum norm. Then the identity mapping of $\mathcal{K}(I)$ constitutes an X -valued vector measure, say m ; in other words, for every $\varphi \in \mathcal{K}(I)$, the integral $\int \varphi dm$ equals the element φ itself. Asserting m to be majorable would mean the existence of a nonnegative real measure μ majorizing $x' \circ m$ for every x' in the unit ball of X' . Any element of X' is a real measure on I ; take in particular $x' = \delta_a$, the Dirac measure at point $a \in I$. Clearly $\|\delta_a\| = 1$ and $\delta_a \circ m$ is nothing but the real functional associating with every $\varphi \in \mathcal{K}(I)$ the value that φ takes at point a ; in short $\delta_a \circ m = \delta_a$. Now, one may easily check that, provided I contains more than a finite number of points, the collection of real measures δ_a , for a ranging through I , cannot be majorized by any real measure.

Let us come now to the *integration of a real function* with respect to a vector measure. Commonly, a function $h : I \rightarrow \mathbb{R}$ is said to be *scalarly integrable* relative to m iff h is integrable relative to all real measures $x' \circ m$, $x' \in X'$. Since the mapping $x' \rightarrow \int h d(x' \circ m)$ is linear of X' to \mathbb{R} , this leads one to define the integral $\int h dm$ as an element of X'^* , the algebraic dual of X' , in which X is naturally imbedded. Sufficient conditions may be found in [20], Chap. VI, for $\int h dm$ to lie in X ; here we shall only need the following.

Proposition 7.4: If m is majorable, for every $h \in \mathcal{L}^1(I, |m|; \mathbb{R})$ the integral $\int h dm$ is an element of X satisfying

$$\left\| \int h dm \right\| \leq \int |h| |dm|. \quad (7.4)$$

Proof: Let φ_n be a sequence of elements of $\mathcal{X}(I)$ converging to h in $\mathcal{L}^1(I, |m|; \mathbb{R})$. Using inequality (7.2) with $\mu = |m|$, one finds that φ_n is a Cauchy sequence in $\mathcal{L}(I, |x' \circ m|; \mathbb{R})$ for every $x' \in X'$; this entails the existence of $\int h dm$ as an element H of X'^* . Now, for every two integers n, p , inequality (7.2) with $\mu = |m|$ yields

$$\left\| \int \varphi_n dm - \int \varphi_p dm \right\| \leq \int |\varphi_n - \varphi_p| |dm|$$

showing that the sequence $\int \varphi_n dm$ is Cauchy in X ; its limit is readily identified with H . Finally, by applying (7.2) to φ_n and going to the limit in $\mathcal{L}^1(I, |m|; \mathbb{R})$ one obtains (7.4). \square

What precedes naturally leads to this extension of the dominated convergence theorem.

Proposition 7.5: Let h_n be a sequence of elements of $\mathcal{L}^1(I, |m|; \mathbb{R})$ satisfying:

- (a) There exists $g \in \mathcal{L}^1(I, |m|; \mathbb{R}_+)$ such that $|h_n| \leq g$ holds $|m|$ -a.e.
- (b) For $|m|$ -almost every value of t , the real sequence $h_n(t)$ possesses a limit, say $h(t)$.

Then $h \in \mathcal{L}^1(I, |m|; \mathbb{R})$ and, normwise in X ,

$$\lim_{n \rightarrow \infty} \int h_n dm = \int h dm.$$

Proof: The dominated convergence theorem implies that the sequence h_n converges to h in $\mathcal{L}^1(I, |m|; \mathbb{R})$. Then, for every two integers n and p , (7.4) yields

$$\left\| \int h_n dm - \int h_p dm \right\| \leq \int |h_n - h_p| |dm|,$$

which entails that the sequence $\int h_n dm$ is Cauchy in X ; the limit is readily found to be equal to the element $\int h dm$ of X . \square

Remark 7.6: Like many authors (e. g. those of [20, 21]) we use \mathcal{L} in the notation of the non-Hausdorff topological linear spaces, whose elements are *functions everywhere defined in I* , with some integrability properties relative to a given real measure. Taking the quotients of such spaces by the subspaces consisting of functions which are *negligible* relative to the said measure yields the corresponding Hausdorff spaces traditionally denoted by L ; the elements of the latter are not properly functions. We strongly advise paying attention to this distinction when treating nonsmooth evolution problems, in which several real measures often appear jointly.

Clearly, in Propositions 7.4 and 7.5, h and h_n could equivalently be considered as elements of $L^1(I, |m|; \mathbb{R})$.

We now come to a common way of constructing majorable vector measures. Let there be given a nonnegative real measure μ on I and let m'_μ denote an X -valued function, μ -integrable (in the sense of Bourbaki [20]; equivalently Bochner-integrable) over every compact subset of I ; notation $m'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; X)$. Then, for every $\varphi \in \mathcal{K}(I)$, the vector function $\varphi m'_\mu$ is μ -integrable and, if K denotes a compact subset of I containing the support of φ ,

$$\left\| \int_K \varphi m'_\mu d\mu \right\| \leq \int_K \|m'_\mu\| d\mu \max |\varphi|.$$

This shows that the mapping $m : \varphi \rightarrow \int \varphi m'_\mu d\mu$ constitutes a vector measure.

Definition 7.7: The X -valued measure m constructed above is called the *product of the real measure μ by the vector function $m'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; X)$* ; notation $m = m'_\mu \mu$ (and, when written in an integral, $dm = m'_\mu d\mu$). Also, m is said to admit m'_μ as *density* relative to μ .

(Observe that a weaker concept of density is developed in [20], Chap. VI, § 2, No. 4: instead of being an element of $\mathcal{L}^1_{\text{loc}}(I, \mu; X)$, the function m'_μ is only assumed scalarly locally μ -integrable).

Clearly, the same vector measure m is obtained after replacing m'_μ by any μ -equivalent X -valued function. Therefore, the density of m relative to μ may be considered as an element of $L^1_{\text{loc}}(I, \mu; X)$ and, from that standpoint, it is unique.

Due to $m'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; X)$, the nonnegative real function $t \rightarrow \|m'_\mu(t)\|$, simply denoted by $\|m'_\mu\|$, belongs to $\mathcal{L}^1_{\text{loc}}(I, \mu; \mathbb{R})$: this allows one to use the above definition with $X = \mathbb{R}$, thus obtaining the nonnegative real measure $\|m'_\mu\| \mu$.

In some contexts, it will prove safer to use the lengthier notation $\|m'_\mu(\cdot)\|$ to refer to the real function $\|m'_\mu\|$.

Proposition 7.8: For every nonnegative real measure μ on I and every $m'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; X)$, the vector measure $m = m'_\mu \mu$ is majorable; its modulus measure $|m|$ equals $\|m'_\mu\| \mu$.

Proof: Clearly, for every x' in the unit ball B' of X' , the nonnegative real measure $|x' \circ m| = |x' \circ m'_\mu| \mu$ is majorized by $\|m'_\mu\| \mu$; hence m is majorable, with modulus $|m|$ majorized by $\|m'_\mu\| \mu$. Through the Radon-Nikodym theorem, this implies that $|m|$ admits a density relative to μ , say $v'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; \mathbb{R}_+)$ and $v'_\mu(t) \leq \|m'_\mu(t)\|$ for μ -almost every value of t in I . Let us now prove the reverse inequality.

Being an element of $\mathcal{L}^1_{\text{loc}}(I, \mu; X)$ the vector function m'_μ takes μ -a.e. its values in some separable Banach subspace Y of X . The measure m is Y -valued, in the sense that, to every $|m|$ -integrable real function it assigns as integral an

element of Y . The collection of real measures $|x' \circ m| = |x' \circ m'_\mu| \mu$, for x' ranging throughout B' equals the collection of $|y' \circ m| = |y' \circ m'_\mu| \mu$, for y' ranging throughout the unit ball C' of $Y' \simeq X'/Y^\perp$. Since Y is separable, C' contains a countable subset D' , dense in the weak* topology $\sigma(Y', Y)$. By definition, $|m|$ majorizes the real measure $|y' \circ m|, y' \in C'$; hence the corresponding density function satisfies $|y' \circ m'_\mu(t)| \leq v'_\mu(t)$ with the possible exception of a μ -negligible subset $N(y')$ of values of t in I . If y' ranges throughout D' , the union of the corresponding $N(y')$ is a μ -negligible subset, say N , of I . The above inequality implies that, for $t \in I \setminus N$,

$$\sup_{y' \in D'} |y' \circ m'_\mu(t)| \leq v'_\mu(t).$$

Since D' is weak*-dense in C' , the left-hand side equals $\|m'_\mu(t)\|$. □

The above expression for $|m|$ implies that the subset of I on which the real function $\|m'_\mu\|$ vanishes is $|m|$ -negligible. Outside this subset, the vector function $m'_\mu/\|m'_\mu\|$, with values of unit norm, is defined; arbitrarily extended to the whole of I , it constitutes a density function of m relative to $|m|$.

Conversely, supposing that an arbitrary majorable X -valued measure is given, one may ask whether it is sure to admit a density relative to its modulus measure. The answer is affirmative for special choices of the Banach space X , said to have *the Radon-Nikodym property* (see [22]; we shall come back to this subject in Sect. 13). Such is in particular any *reflexive* Banach space, and hence every finite-dimensional normed space.

Example 7.9: Suppose $m = a\delta_t$, i. e. the point measure located at point $t \in I$, with value $a \in X$ (cf. Proposition 6.5). One finds $|m| = \|a\| \delta_t$; every function $h: I \rightarrow \mathbb{R}$ belongs to $\mathcal{L}^1(I, |m|; \mathbb{R})$ and the integral $\int h dm$ equals $h(t)a$. For this special X -valued measure, the existence of a density function, say m'_μ , relative to its modulus measure $\mu = \|a\| \delta_t$ is trivially secured, without any additional assumption concerning X . In fact, every function $m'_\mu: I \rightarrow X$ belongs to $\mathcal{L}^1(I, \mu; X)$; it does the job iff it takes a point t the value $a/\|a\|$; the values of m'_μ anywhere else are immaterial, since $I \setminus \{t\}$ is μ -negligible.

Let us finish this section by considering the case where I is a real interval, with m equal to the differential measure of an lbv function.

Proposition 7.10: Let I be a real interval and let $f \in \text{lbv}(I, X)$ with variation function $V_f: I \rightarrow \mathbb{R}$. Then the differential measure df is majorable and, in the sense of the ordering of real measures,

$$|df| \leq dV_f. \tag{7.5}$$

Proof: Coming back to expression (6.2) one observes that

$$\|f(\tau_i) - f(\tau_{i-1})\| \leq \text{var}(f; \tau_{i-1}, \tau_i) = V_f(\tau_i) - V_f(\tau_{i-1}).$$

Hence, for every $\varphi \in \mathcal{K}_+(I)$,

$$\left\| \sum_{i=1}^n \varphi(\theta_S^i) (f(\tau_i) - f(\tau_{i-1})) \right\| \leq \sum_{i=1}^n \varphi(\theta_S^i) (V_f(\tau_i) - V_f(\tau_{i-1})).$$

By taking the respective limits, for S ranging in the directed set (\mathcal{S}, \supset) , one gets

$$\forall \varphi \in \mathcal{K}_+(I) : \left\| \int \varphi df \right\| \leq \int \varphi dV_f,$$

which is the property involved in Definition 7.1; inequality (7.5) follows from the definition of the modulus measure of df . \square

Remark 7.11: Here is a counterexample showing that equality in (7.5) does not hold in general. Let t_1 be an interior point to I and u, v two elements of X ; define

$$f(t) = \begin{cases} u & \text{if } t \neq t_1 \\ v & \text{if } t = t_1 \end{cases}.$$

In view of Proposition 6.5 the vector measure df vanishes, and hence also $|df|$; but V_f may be constructed as

$$V_f(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \|v - u\| & \text{if } t = t_1 \\ 2\|v - u\| & \text{if } t > t_1; \end{cases}$$

this makes dV_f consist of the point measure located at t_1 , with value $2\|v - u\|$.

Recall that a normed space is said to be *strictly convex* iff

$$(\lambda \in]0, 1[, \|x\| = \|y\| = r, x \neq y) \Rightarrow \|\lambda x + (1 - \lambda)y\| < r. \quad (7.6)$$

Elementary reasoning from two-dimensional geometry shows that this property is equivalent to the following: *equality*

$$\|c - a\| = \|c - b\| + \|b - a\|$$

holds (if and) only if b belongs to the line segment with endpoints a and c .

The following proposition will be established in Sect. 8; for another proof, developed from the viewpoint of set functions, see [31].

Proposition 7.12: If f has *aligned jumps*, i.e., for every $t \in I$, the value $f(t)$ belongs to the line segment in X with endpoints $f^-(t)$, $f^+(t)$, then $|df| = dV_f$.

If, in addition, the Banach space is supposed strictly convex, equality $|df|=dV_f$ conversely implies that f has aligned jumps.

Note that, under Convention 3.5, alignment of jumps holds trivially at an endpoint of I .

If, in particular, f is right-continuous (alternatively, left-continuous) in the interior of I , then it has aligned jumps.

8. Subintervals

Let m denote an X -valued measure on the real interval I . If $[a, b] \subset I$, the notation $\int_{[a, b]} dm$ classically stands for the integral $\int \chi_{[a, b]} dm$ of the characteristic function of $[a, b]$, in other words the m -measure of this subset of I . If m is majorable, Proposition 7.4 ensures that this integral is an element of X ; in contrast, for a nonmajorable measure, Counterexample 7.3 may be used in order to exhibit an interval – for instance a singleton – whose measure, essentially an element of X'^* , cannot belong to X .

In any case, if two vector measures on I , say m_1 and m_2 , yield as integrals, on each compact subinterval of I , the same element of X'^* , then $m_1 = m_2$; this readily follows from consideration of the real measures $x' \circ m_1$ and $x' \circ m_2$, with $x' \in X'$.

Proposition 8.1: Let $f \in \text{lbv}(I, X)$; for every compact subinterval $[a, b]$ of I , under Convention 3.5 if needed, one has

$$\int_{[a, b]} df = f^+(b) - f^-(a). \quad (8.1)$$

In particular, for every $a \in I$, the df -measure of the singleton $\{a\}$ equals

$$\int_{\{a\}} df = f^+(a) - f^-(a). \quad (8.2)$$

Proof: We first consider the case where $[a, b]$ is contained in the interior of I ; then we may approach $\chi_{[a, b]}$ by the following sequence φ_p of elements of $\mathcal{K}(I)$. Denote by s_p and a_p , with $s_p < a_p < a$, two infinite sequences of points of I tending to a ; denote by t_p and b_p , with $b < b_p < t_p$, two infinite sequences of points of I tending to b . Choose the function $\varphi_p \in \mathcal{K}(I)$, with values in the interval $[0, 1]$, with support contained in $[s_p, t_p]$ and taking the value 1 throughout $[a_p, b_p]$. By establishing that, normwise in X ,

$$\lim_{p \rightarrow \infty} \int \varphi_p df = f^+(b) - f^-(a), \quad (8.3)$$

we shall prove (8.1), thanks to Proposition 7.5; in fact, the sequence φ_p converges to $\chi_{[a,b]}$ pointwise, with $|\varphi_p|$ majorized by the characteristic function of some compact interval containing all $[s_p, t_p]$.

Now, under the definition (6.2) of M ,

$$\int \varphi_p df = \lim_{S \in \mathcal{S}} M(S, \theta, \varphi_p). \quad (8.4)$$

For fixed p , the elements of \mathcal{S} containing the four points s_p, a_p, b_p, t_p constitute a cofinal subset of (\mathcal{S}, \supset) ; therefore, this constraint may be imposed on S in (8.4). If an interval delimited by S , say $[\tau_{i-1}, \tau_i]$, is not contained in $[s_p, t_p]$ its contribution to $M(S, \theta, \varphi_p)$ vanishes; the total contributions of intervals contained in $[a_p, b_p]$ equals $f(b_p) - f(a_p)$, since $\varphi(\theta_S^i) = 1$ for all the corresponding values of i ; finally, because $|\varphi_p| \leq 1$, the contributions of the intervals contained in $[s_p, a_p]$ and $[b_p, t_p]$ have their norms bounded by $\text{var}(f; s_p, a_p)$ and $\text{var}(f; b_p, t_p)$, respectively. Consequently,

$$\|f(b_p) - f(a_p) - \int \varphi_p df\| \leq V_f(a_p) - V_f(s_p) + V_f(t_p) - V_f(b_p).$$

For $p \rightarrow \infty$, this tends to zero since the nondecreasing real function V_f possesses a limit on the left at point a and a limit on the right at point b ; equality (8.3) follows.

Let us now suppose that I contains its left end T_l , with $a = T_l$ while b is different from the possible right end (otherwise $I = [a, b]$, hence $\chi_{[a,b]} \in \mathcal{K}(I)$, which makes the proof of (8.1) straightforward). The sequences t_p and b_p , $b < b_p < t_p$, only have to be introduced; φ_p is constructed with support in $[T_l, t_p]$ and with the value 1 throughout $[T_l, b_p]$; thus

$$\int_{[T_l, b]} df = f^+(b) - f(T_l)$$

which, in view of Convention 3.5, is again equality (8.1). Similar reasoning applies if b is equal to the right end of I and a different from the possible left end. \square

Corollary 8.2: If a, b are two elements of I and $a \leq b$, one has, under Convention 3.5 if needed,

$$\int_{]a, b]} df = f^+(b) - f^+(a), \quad (8.5)$$

$$\int_{[a, b[} df = f^-(b) - f^-(a); \quad (8.6)$$

furthermore, if $a < b$,

$$\int_{]a, b[} df = f^-(b) - f^+(a). \quad (8.7)$$

Proof: Since $\chi_{[a,b]} = \chi_{\{a\}} + \chi_{]a,b]}$, (8.5) follows from (8.1) and similar reasoning applies to (8.6). For $a=b$ the notations $]a,b]$, $[a,b[$ and $]a,b[$ are naturally understood as representing the empty subset of I ; therefore, equality $\chi_{]a,b]} = \chi_{\{a\}} + \chi_{]a,b]}$ holds iff $a < b$, so (8.7) follows from (8.6). \square

Corollary 8.3: If $f \in \text{lbv}(I, X)$, the function f_1 equal to f^+ (respectively f^-) at every interior point of I and to f at the possible ends of I belongs to $\text{lbv}(I, X)$ and has the same differential measure as f .

Proof: That $f_1 \in \text{lbv}(I, X)$ results from Proposition 4.5. Using Propositions 3.1 and 8.1, one readily checks that df_1 and df yield the same integral over every compact subinterval of I . \square

Corollary 8.4: If two elements f_1 and f_2 of $\text{lbv}(I, X)$ have the same differential measure and are right-continuous in the interior of I (alternatively, left-continuous in the interior of I), then $f_1 - f_2$ equals a constant.

Proof: If I does not contain its possible left end T_l , this readily follows from (8.5). Otherwise one finds a constant C such that

$$t > T_l \Rightarrow f_1(t) - f_2(t) = C.$$

Therefore $f_1^+(T_l) - f_2^+(T_l) = C$; in view of (8.2) and the assumption $df_1 = df_2$, this implies $f_1(T_l) - f_2(T_l) = C$. \square

Remark 8.5: If I contains its left end T_l , the right-continuity of a function $f \in \text{lbv}(I, X)$ at this point is equivalent to $\int_{\{T_l\}} df = 0$, i.e. $\{T_l\}$ is not an *atom* of df .

A similar remark applies to left-continuity at the possible right end.

We are now able to justify a foregoing statement.

Proof of Proposition 7.12: Let us prove that, under the assumption made, the reverse inequality of (7.5) holds; we shall first show that, for every $a < b$ in I , one has

$$\int_{]a,b[} dV_f \leq \int_{]a,b[} |df|. \quad (8.8)$$

The left-hand member equals $V_f^-(b) - V_f^+(a)$, i.e. $\text{var}(f;]a,b[)$, due to Proposition 4.3; now, for every $\varepsilon > 0$ there exists an increasing finite sequence $t_1 < t_2 < \dots < t_n$ of points of $]a,b[$ such that

$$-\varepsilon + \text{var}(f;]a,b[) \leq \sum_{i=2}^n \|f(t_i) - f(t_{i-1})\|. \quad (8.9)$$

Furthermore,

$$\begin{aligned} \|f(t_i) - f(t_{i-1})\| &\leq \|f(t_i) - f^-(t_i)\| + \|f^-(t_i) - f^+(t_{i-1})\| \\ &\quad + \|f^+(t_{i-1}) - f(t_{i-1})\|. \end{aligned} \quad (8.10)$$

Define $t_0 = a$, $t_{n+1} = b$; then $]a, b[$ may be expressed as the following union of disjoint intervals:

$$]a, b[=]t_0, t_1[\cup \{t_1\} \cup]t_1, t_2[\cup \dots \cup \{t_n\} \cup]t_n, t_{n+1}[. \quad (8.11)$$

By adding to the right-hand member of (8.9) the nonnegative terms $\|f^-(t_1) - f^+(t_0)\|$, $\|f(t_1) - f^-(t_1)\|$, $\|f^+(t_n) - f(t_n)\|$ and $\|f^-(t_{n+1}) - f^+(t_n)\|$ and using (8.10), one majorizes this right-hand member by

$$\sum_{i=1}^{n+1} \|f^-(t_i) - f^+(t_{i-1})\| + \sum_{i=1}^n (\|f^-(t_i) - f(t_i)\| + \|f^+(t_i) - f(t_i)\|).$$

Now, using inequality (7.4), one has

$$\|f^-(t_i) - f^+(t_{i-1})\| = \left\| \int_{]t_{i-1}, t_i[} df \right\| \leq \int_{]t_{i-1}, t_i[} |df|.$$

Furthermore, if f has aligned jumps,

$$\|f^-(t_i) - f(t_i)\| + \|f^+(t_i) - f(t_i)\| = \|f^+(t_i) - f^-(t_i)\| = \left\| \int_{\{t_i\}} df \right\| \leq \int_{\{t_i\}} |df|.$$

In view of (8.11), since ε may be taken arbitrarily close to zero, this establishes (8.8).

Secondly, consider a singleton $\{a\} \subset I$; using Convention 3.5, if a happens to be an end of I , one has

$$\int_{\{a\}} dV_f = V_f^+(a) - V_f^-(a) = V_f^+(a) - V_f(a) + V_f(a) - V_f^-(a).$$

Through Proposition 4.3, if f has aligned jumps, this is found equal to

$$\|f^+(a) - f^-(a)\| = \left\| \int_{\{a\}} df \right\| \leq \int_{\{a\}} |df|.$$

Combining this inequality with (8.8), one concludes that, over every compact subinterval of I , the integral of $|df|$ majorizes that of dV_f : this entails the expected inequality, in the sense of the ordering of real measures.

To finish the proof of Proposition 7.12, let us assume that X is strictly convex and that equality $dV_f = |df|$ holds. Imagine a point $s \in I$ with nonaligned jumps, i.e.

$$\|f^+(a) - f^-(a)\| < \|f^+(a) - f(a)\| + \|f(a) - f^-(a)\|.$$

By the same calculation as above, this yields

$$\int_{\{a\}} dV_f > \int_{\{a\}} |df|,$$

which is contradictory. □

What concludes this section is of use, in particular, when discussing the local character of evolution problems formulated in terms of differential measures and the possible extension of their solutions.

Let J be a nonempty subinterval of I ; clearly, for every $f \in \text{lbv}(I, X)$, the restriction of f to J , denoted by f_J , is an element of $\text{lbv}(J, X)$. Then $d(f_J)$ is an X -valued measure on J ; we are to investigate its relationship with df .

In the special case where J is *open* in the topology of I (this means that, if J includes any of its possible ends, this point is also an end of I), the concept of the restriction of the measure df to J is readily available; this consists in the following. If J is open relative to I and $\varphi \in \mathcal{K}(J)$, the function $\bar{\varphi}$ extending φ to I with value zero outside J belongs to $\mathcal{K}(I)$. Clearly, the mapping $\varphi \rightarrow \int \bar{\varphi} df$ of $\mathcal{K}(J)$ to X is an X -valued measure on J , denoted by $(df)_J$ and called the restriction of df to the open subset J of I . Proposition 8.6 below implies that $(df)_J = d(f_J)$ in this special case.

For an arbitrary nonempty subinterval J of I and every $\varphi \in \mathcal{K}(J)$, the same extension $\bar{\varphi}$ as above may be considered; this function is not expected to be continuous in I but it is bounded and its support, relative to I , is compact. In addition, $\bar{\varphi}$ is measurable with regard to any real measure on I ; one way of proving this assertion is to observe that $\bar{\varphi}$ is *regulated* in the sense recalled in Sect. 3. Therefore, $\bar{\varphi}$ belongs to $\mathcal{L}^1(I, |df|; \mathbb{R})$; in view of Proposition 7.4, $\varphi \rightarrow \int \bar{\varphi} df$ is a linear mapping of $\mathcal{K}(J)$ to X meeting the continuity requirements in order to constitute a vector measure on J . We shall denote by $(df)_J$ this vector measure, called the *measure induced by df* on the subset J of I . In the previous special case of J open relative to I , this is obviously identical with the restriction of df to J .

Let us write as $x\delta_t$ the *point measure on J* , located at the point $t \in J$, with value $x \in X$, i.e. the mapping $\varphi \rightarrow x\varphi(t)$ of $\mathcal{K}(J)$ to X .

Proposition 8.6: The measure $(df)_J$ equals the sum of $d(f_J)$ and the following measures on J :

- (a) the measure $(f(a) - f^-(a))\delta_a$ if J possesses a left end, say a , contained in it;
- (b) the measure $(f^+(b) - f(b))\delta_b$ if J possesses a right end, say b , contained in it.

In the case of a or b being endpoints of I , $f^-(a)$ and $f^+(b)$ are understood in the sense of Convention 3.5.

Proof: Let us show that the above sum of measures yields the same integral as $(df)_J$ on every compact subinterval $[s, t]$ of J . Denote by h the characteristic function of $[s, t]$ in J ; the function \bar{h} extending h to I with value zero outside J equals the characteristic function of $[s, t]$ as a subinterval of I . Through similar arguments as in the proof of Proposition 8.1, h equals the limit of a dominated sequence of functions $\varphi_p \in \mathcal{K}_+(J)$. The corresponding extensions $\bar{\varphi}_p$ to I converge pointwise to \bar{h} ; then Proposition 7.5 entails

$$\int h(df)_J = \int \bar{h}df. \quad (8.12)$$

First, suppose that J possesses a left end, say a , contains it, but does not contain its possible right end; suppose $s = a$ (otherwise the expected equality is trivial). Then (8.12) yields, in view of Corollary 8.2, $\int h(df)_J = f^+(t) - f^-(a)$, while through the same corollary, applied to J , one has $\int h d(f_J) = f^+(t) - f(a)$. The difference between these two expressions, namely $f(a) - f^-(a)$, equals the integral of h relative to the measure $(f(a) - f^-(a))\delta_a$.

Similar reasoning applies if J includes its right end, say b . □

As an example, take as J a singleton; then $a = b$ and

$$(f(a) - f^-(a))\delta_a + (f^+(b) - f(b))\delta_b = (f^+(a) - f^-(a))\delta_a.$$

Such is, in this special case, the measure $(df)_J$, while $d(f_J) = 0$.

Remark 8.7: That $(df)_J$ differs from $d(f_J)$ may be seen as an inconvenience. In contrast, everything runs smoothly for the variation of the restriction: $\text{var}(f, J) = \text{var}(f_J, J)$; equivalently, for every $[a, b] \subset J$, $\text{var}(f; a, b) = \text{var}(f_J; a, b)$.

9. Cumulative Distribution Functions

As was seen in Sect. 7, for every $f \in \text{lbv}(I, X)$, the differential measure df is majorable. The following states that, conversely, every majorable X -valued measure on I equals a differential measure. Recall that, *when X has finite dimension, every X -valued measure is majorable.*

Proposition 9.1: Let m denote a majorable measure on the interval I , with values in the Banach space X . There exists $f \in \text{lbv}(I, X)$, left-continuous in the interior of I (alternatively right-continuous in the interior of I) whose differential measure equals m . Within the addition of a constant, this function is unique.

The real nondecreasing function, left-continuous (respectively right-continuous) in the interior of I , which corresponds in the same way to the modulus measure $|m|$, equals a variation function of f .

Proof: In the case when I contains its possible right end, denote this end by T_r ; choose a *reference point* $\varrho \in I$ and define, for every $t \in I$,

$$f(t) = \begin{cases} \int_{[\varrho, t]} dm & \text{if } t = T_r, \\ \int_{[\varrho, t]} dm & \text{if } \varrho \leq t < T_r, \\ - \int_{[t, \varrho]} dm & \text{if } t < \varrho. \end{cases} \quad (9.1)$$

We shall avoid uninteresting discussion by assuming $\varrho < T_r$. After examining three possible locations of ϱ with regard to a subinterval $[a, b[$ of I , one obtains that, for every $a \leq b < T_r$ in I ,

$$\int_{[a, b[} dm = f(b) - f(a). \quad (9.2)$$

Let b lie in the interior of I ; take an arbitrary sequence $a_n < b$ tending to b . Apply Proposition 7.5, with h_n equal to the characteristic function of $[a_n, b[$; replacing a by a_n in (9.2), one obtains $\lim_{n \rightarrow \infty} (f(b) - f(a_n)) = 0$, which is the left-continuity of f at point b .

Let $a \in I$, with $a < T_r$ if I contains its right end; take an arbitrary sequence $b_n > a$ tending to a . By applying Proposition 7.5 with h_n equal to the characteristic function of $[a, b_n[$, one similarly derives from (9.2) that the right-limit $f^+(a)$ exists and

$$\int_{\{a\}} dm = f^+(a) - f(a). \quad (9.3)$$

In the case of I containing its right end, similar reasoning shows the existence of $f^-(T_r)$ and

$$\int_{\{T_r\}} dm = f(T_r) - f^-(T_r). \quad (9.4)$$

Altogether, for every $[a, b] \subset I$,

$$\int_{[a, b]} dm = f^+(b) - f^-(a). \quad (9.5)$$

In order to prove that f has bounded variation on every $[s, t] \subset I$, let us consider a finite sequence $s = \tau_0 < \tau_1 < \dots < \tau_n = t$. First suppose $t < T_r$; then by making $[a, b[= [\tau_{i-1}, \tau_i[$ in (9.2) and using inequality (7.4), one obtains

$$\begin{aligned} \sum_{i=1}^n \|f(\tau_i) - f(\tau_{i-1})\| &= \sum_{i=1}^n \left\| \int \chi_{[\tau_{i-1}, \tau_i]} dm \right\| \\ &\leq \sum_{i=1}^n \int_{[\tau_{i-1}, \tau_i]} |dm| \leq \int_{[s, t]} |dm| < +\infty. \end{aligned}$$

If $t = T_r$, a similar majorization may be constructed thanks to (9.4).

Therefore, $f \in \text{lbv}(I, X)$ and the equality $m = df$ follows from (9.5).

Corollary 8.4 entails that any other function in $\text{lbv}(I, X)$, left-continuous on the interior of I and admitting m as its differential measure, can only differ from the above by a constant.

Finally, the same construction performed after replacing m by the nonnegative real measure $|m| = |df|$ yields a nondecreasing real function, left-continuous on the interior of I , whose differential measure equals $|df|$. In view of Proposition 7.12, this real function is a variation function of f , since the latter has aligned jumps. \square

Remark 9.2: Suppose that I does not contain its possible left end T_l ; in particular it may be unbounded from the left. Suppose that the nonnegative real measure $|m|$ is *bounded in the left of I* , i.e. there exists $t_0 \in I$ such that the subinterval $I^- = \{t \in I : t \leq t_0\}$ has finite $|m|$ -measure. Equivalently, a function $f \in \text{lbv}(I, X)$, with aligned jumps, admitting m as its differential measure, has bounded variation on I^- ; this implies that $f(t)$ possesses a limit when t decreases indefinitely in I . Then the additive constant in Proposition 9.1 may be adjusted for this limit to equal zero. Such is the familiar situation of probability theory; the probability law of a random variable with values in the real interval I (commonly $I = \mathbb{R}$) is defined by a nonnegative real measure m on I , with total equal to 1. One then constructs a nondecreasing real function $t \rightarrow f(t)$ as the m -measure of the interval $\{s \in I : s < t\}$. This function, left-continuous on the interior of I , is usually called the *cumulative distribution function* of m , a denomination that we propose to also use in the general situation of Proposition 9.1.

A symmetric observation applies when $|m|$ is bounded in the right of I ; then the additive constant may alternatively be adjusted for $f(t)$ to tend to zero when t increases indefinitely in I .

Remark 9.3: Let us fix ϱ in I ; the subset L_ϱ of $\text{lbv}(I, X)$ consisting of functions which are *left-continuous on the interior of I and which vanish at ϱ* is clearly a linear subspace. Denote by $M_m(I, X)$ the totality of the majorable X -valued measures on I . By adjusting the additive constant of Proposition 9.1, one associates with every $m \in M_m(I, X)$ a unique element, say f_l , of L_ϱ such that $m = df_l$ (in the situation described in Remark 9.2, one might alternatively avoid privileging a point ϱ in I by imposing on $f_l(t)$ the condition that it have zero limit when t decreases indefinitely in I). This one-to-one linear mapping is sometimes invoked to identify $M_m(I, X)$ with L_ϱ ; this would be confusing in evolution problems.

Symmetrically, Proposition 9.1 defines a one-to-one linear mapping, say $m \rightarrow f_r$, of $M_m(I, X)$ to the subspace R_ϱ of $\text{lbv}(I, X)$ consisting of functions

vanishing at ϱ , right-continuous on the interior of I . This also entails that $m \rightarrow (f_l + f_r)/2$ is a one-to-one linear mapping of $M_m(I, X)$ to the subspace of $\text{lbv}(I, X)$ consisting of functions f which vanish at ϱ and which satisfy, for every t in the interior of I ,

$$f(t) = \frac{1}{2}(f^-(t) + f^+(t)).$$

Such lbv functions are of interest when dealing with Fourier series or Fourier integrals; they also have the advantage of giving a simple form to the differentiation formula presented in Sect. 11 below.

Another way of avoiding giving preference to some of the one-sided continuities will be presented in Sect. 16.

Remark 9.4: N. Dinculeanu ([21], § 17) has presented a construction similar to what has been done above in the proof of Proposition 9.1. This gives us the opportunity to translate our assumptions into the language commonly used when vector measures are considered from the standpoint of set functions. What we call a majorable X -valued measure on the interval I is, in Dinculeanu's book, an X -valued *Borel measure on I with finite variation*. This means precisely, an X -valued countably additive function, say m , defined on the collection \mathscr{B} of the relatively compact Borel subsets of I , with the property that, for every $B \in \mathscr{B}$, one has

$$\sup \sum_i \|m(B_i)\| < +\infty ;$$

here the supremum is taken for all the finite families $\{B_i\}$ of disjoint members of \mathscr{B} contained in B (see also Remark 13.4 below).

10. Vector Measure Constructed Through a Bilinear Mapping

Let X, Y, Z be three real Banach spaces and let $\Phi : X \times Y \rightarrow Z$ denote a *continuous bilinear mapping*. The *norm* of Φ is the nonnegative real number

$$N(\Phi) = \sup \{ \|\Phi(a, b)\| : a \in X, b \in Y, \|a\| \leq 1, \|b\| \leq 1 \}.$$

The norms in X, Y or Z , indistinctly, are denoted by $\|\cdot\|$.

Here, as in Sect. 7, I may be understood as denoting any locally compact topological space, countable at infinity.

Let m be a majorable Y -valued measure on I . A proposition of Bourbaki ([20], Chap. VI, § 2, No. 7) may be transcribed with the notation we are using as follows. There exists, uniquely, a continuous linear mapping of $\mathscr{L}^1(I, |m|; X)$ to

Z , that we shall denote by $x \rightarrow \int \Phi(x, dm)$, such that for every $a \in X$ and every real function $r \in \mathcal{L}^1(I, |m|; \mathbb{R})$,

$$\int \Phi(ra, dm) = \Phi(a, \int r dm). \quad (10.1)$$

Furthermore, for every $x \in \mathcal{L}^1(I, |m|; X)$,

$$\left\| \int \Phi(x, dm) \right\| \leq N(\Phi) \int \|x\| |dm|. \quad (10.2)$$

Recall that we denote by $\|x\|$ the *real function* $t \rightarrow \|x(t)\|$; in some contexts it will prove clearer to use the more cumbersome notation $\|x(\cdot)\|$ instead. This readily entails the following proposition.

Proposition 10.1: For every $g \in \mathcal{L}_{\text{loc}}^1(I, |m|; X)$, the mapping $\varphi \rightarrow \int \Phi(\varphi g, dm)$ of $\mathcal{K}(I)$ to Z is a Z -valued measure on I that we shall denote by $\Phi(g, m)$ (or $\Phi(g, dm)$ when placed in an integral).

This measure is majorable and, in the sense of the ordering of real measures on I , its modulus measure $|\Phi(g, m)|$ satisfies

$$|\Phi(g, m)| \leq N(\Phi) \|g(\cdot)\| |m|. \quad (10.3)$$

The above system of notation results in the calculation rule

$$\forall \varphi \in \mathcal{K}(I) : \int \varphi \Phi(g, dm) = \int \Phi(\varphi g, dm). \quad (10.4)$$

This is an *associativity* property which may be generalized as follows.

Proposition 10.2: Let $g \in \mathcal{L}_{\text{loc}}^1(I, |m|; X)$; then $\|g(\cdot)\| |m|$, shortly denoted by $\|g\| |m|$, is a nonnegative real measure on I . Every function $h \in \mathcal{L}^1(I, \|g\| |m|; \mathbb{R})$ (equivalently $h \|g\| \in \mathcal{L}^1(I, |m|; \mathbb{R})$) is integrable relatively to the vector measure $\Phi(g, m)$ and

$$\int h \Phi(g, dm) = \int \Phi(hg, dm) \in Z. \quad (10.5)$$

Proof: The second member of (10.5) is a meaningful element of Z because $hg \in \mathcal{L}^1(I, |m|; X)$. In fact, under the convention $1/\|g(t)\| = 0$ for $\|g(t)\| = 0$, the real function $t \rightarrow 1/\|g(t)\|$ is $|m|$ -measurable on I ; by writing $hg = h \|g\| g / \|g\|$, one obtains that the vector function $hg : I \rightarrow X$ is $|m|$ -measurable, hence $|m|$ -integrable since $\|hg\| = |h| \|g\|$ belongs to $\mathcal{L}^1(I, |m|; \mathbb{R})$. In addition, by making $x = hg$ in (10.2), one has

$$\left\| \int \Phi(hg, dm) \right\| \leq N(\Phi) \int |h| d(\|g\| |m|).$$

The left-hand member of (10.5) is also meaningful because, in view of the measure inequality (10.3), h belongs to $\mathcal{L}^1(I, |\Phi(g, m)|; \mathbb{R})$; this inequality furthermore implies

$$\left\| \int h \Phi(g, dm) \right\| \leq N(\Phi) \int |h| d(\|g\| |m|).$$

This shows that both members of (10.5) depend in a linear continuous way on h in $\mathcal{L}^1(I, \|g\| |m|; \mathbb{R})$. Now (10.4) means that equality (10.5) holds for every h in $\mathcal{X}(I)$, which is a dense subspace. \square

Defining the *product*, denoted by hm , (or hdm , when written in an integral) of the Y -valued vector measure m by $h \in \mathcal{L}_{\text{loc}}^1(I, |m|; \mathbb{R})$ may be seen as a special case of what precedes: take $X = \mathbb{R}$, $Z = Y$ and $\Phi : (r, b) \rightarrow rb$. This gives rise to another associativity rule, in the following proposition.

Proposition 10.3: If $h \in \mathcal{L}_{\text{loc}}^1(I, |m|; \mathbb{R})$ and $g \in \mathcal{L}^1(I, |h| |m|; X)$ one has

$$\int \Phi(hg, dm) = \int \Phi(g, hdm) \in Z. \quad (10.6)$$

Proof: Similarly to (10.3), the modulus measure of hm is majorized by the nonnegative real measure $|h| |m|$, since the norm of the bilinear mapping $(r, b) \rightarrow rb$ is unity. Thus, the assumptions made imply $g \in \mathcal{L}^1(I, |hm|; X)$, securing that the second member of (10.6) is a well-defined element of Z . As far as the first member is concerned, one may observe that the function $\text{sgn } h$, with values $-1, 0$ or 1 , belongs to $\mathcal{L}^\infty(I, |m|; \mathbb{R})$. Since $|h|g \in \mathcal{L}^1(I, |m|; X)$, the product $hg = |h|g \text{sgn } h$ is also an element of $\mathcal{L}^1(I, |m|; X)$; this ensures the existence of $\int \Phi(hg, dm) \in Z$.

For fixed h , the same inequalities as in the proof of Proposition 10.2 show that both members of (10.6) depend in a linear continuous way on g in $\mathcal{L}^1(I, |h| |m|; X)$. Due to the property (10.1), equality (10.6) holds when g is taken in the form ra , $r \in \mathcal{L}^1(I, |h| |m|; \mathbb{R})$, $a \in X$. Now the linear subspace of $\mathcal{L}^1(I, |h| |m|; X)$ generated by the elements of this form is dense. \square

Corollary 10.4: If $h \in \mathcal{L}_{\text{loc}}^1(I, |m|; \mathbb{R})$ and $g \in \mathcal{L}_{\text{loc}}^1(I, |h| |m|; X)$ the Z -valued measures $\Phi(hg, m)$ and $\Phi(g, hm)$ are equal.

By bringing together Propositions 10.2 and 10.3, one obtains the following corollary.

Corollary 10.5: If $h \in \mathcal{L}_{\text{loc}}^1(I, |m|; \mathbb{R})$, $g \in \mathcal{L}_{\text{loc}}^1(I, |m|; X)$ and $|h| \|g\| \in \mathcal{L}^1(I, |m|; \mathbb{R})$, all three expressions appearing in (10.5) and (10.6) are meaningful and equal.

Remark 10.6: In all the preceding, \mathcal{L} could visibly be replaced by L . Such also is the case in the following proposition, which states the calculation rule which is to apply when the Y -valued measure m is defined through some *density function* relative to a real measure, according to Definition 7.7.

Proposition 10.7: Suppose there exists a nonnegative real measure μ on I and a vector function $m'_\mu \in \mathcal{L}_{\text{loc}}^1(I, \mu; Y)$ such that $m = m'_\mu \mu$. Then, for every $g \in \mathcal{L}_{\text{loc}}^1(I, |m|; X)$ (equivalently $g \|m'_\mu\| \in \mathcal{L}_{\text{loc}}^1(I, \mu; X)$), the vector measure $\Phi(g, m)$ equals $\Phi(g, m'_\mu) \mu$.

Proof: Due to Proposition 7.8, the modulus measure of $m = m'_\mu \mu$ equals the real measure $\|m'_\mu\| \mu$. Hence the assumption $g \in \mathcal{L}_{\text{loc}}^1(I, |m|; X)$ is in fact equivalent to $g \|m'_\mu\| \in \mathcal{L}_{\text{loc}}^1(I, \mu; X)$. In view of the definition (10.4) of the measure $\Phi(g, m)$, one has to establish that, for every $\varphi \in \mathcal{X}(I)$,

$$\int \Phi(\varphi g, dm) = \int \Phi(\varphi g, m'_\mu) d\mu$$

or, more generally, that for every $f \in \mathcal{L}^1(I, m; X)$,

$$\int \Phi(f, dm) = \int \Phi(f, m'_\mu) d\mu. \quad (10.7)$$

Take f under the special form $f = ra$, with $a \in X$ and $r \in \mathcal{L}^1(I, |m|; \mathbb{R})$. Then, using (10.1), one transforms the left member of (10.7) into $\Phi(a, \int r dm)$, i.e. $\Phi(a, \int r m'_\mu d\mu)$. Because, for fixed $a \in X$, the mapping $\Phi(a, \cdot)$ is linear continuous of Y to Z , a classical calculation rule for vector integrals yields

$$\Phi(a, \int r m'_\mu d\mu) = \int \Phi(a, r m'_\mu) d\mu = \int \Phi(ra, m'_\mu) d\mu.$$

This establishes (10.7), for the said choice of f ; since the linear space generated in $\mathcal{L}^1(I, |m|; X)$ by the functions f of this form is dense, this completes the proof. \square

Example 10.8: Here is a trivial instance we shall meet again in the following. Take $m = b\delta_t$, i.e. the point measure located at the point t of I , with value $b \in Y$. Then $|m| = \|b\| \delta_t$ and every function $g: I \rightarrow X$ belongs to $\mathcal{L}^1(I, |m|; X)$ (cf. Example 7.9). Proposition 10.7 yields $\Phi(g, m) = \Phi(g(t), b) \delta_t$.

Remark 10.9: The reader preferring to treat measures from the standpoint of set functions may refer, instead of to [20], to [21], § 8; in the latter book three Banach spaces, a continuous bilinear mapping and a vector measure (with finite variation) are similarly considered.

11. Differential Measure of a Bilinear Expression

In this section, I denotes a real interval. Let X, Y, Z be three Banach spaces and $\Phi: X \times Y \rightarrow Z$ a continuous bilinear mapping.

For $x \in \text{lbv}(I, X)$ and $y \in \text{lbv}(I, Y)$ the function $t \rightarrow \Phi(x(t), y(t))$, denoted by $\Phi(x, y)$ for short, belongs to $\text{lbv}(I, Z)$. In fact, on every compact subinterval

$[a, b]$ of I , the real functions $t \rightarrow \|x(t)\|$ and $t \rightarrow \|y(t)\|$ possess some upper bounds, say A and B (e.g. $A = \|x(a)\| + \text{var}(x; a, b)$); then, for $s \leq t$ in $[a, b]$,

$$\begin{aligned} & \|\Phi(x(t), y(t)) - \Phi(x(s), y(s))\| \\ & \leq \|\Phi(x(t) - x(s), y(t))\| + \|\Phi(x(s), y(t) - y(s))\| \\ & \leq N(\Phi) [(V_x(t) - V_x(s))B + (V_y(t) - V_y(s))A]; \end{aligned}$$

this readily yields a majorization of $\text{var}(\Phi(x, y); a, b)$.

Convention 3.5 will be used when writing one-side limits at some possible end of I .

Proposition 11.1: If $x \in \text{lbv}(I, X)$ and $y \in \text{lbv}(I, Y)$, the function $\Phi(x, y): t \rightarrow \Phi(x(t), y(t))$ is an element of $\text{lbv}(I, Z)$ whose differential measure equals

$$d\Phi(x, y) = \Phi(dx, y^-) + \Phi(x^+, dy). \quad (11.1)$$

Also,

$$d\Phi(x, y) = \Phi(dx, y^+) + \Phi(x^-, dy), \quad (11.2)$$

$$d\Phi(x, y) = \Phi\left(dx, \frac{y^+ + y^-}{2}\right) + \Phi\left(\frac{x^+ + x^-}{2}, dy\right). \quad (11.3)$$

The expressions written on the right-hand sides are meaningful vector measures; in fact x^+, x^-, y^+, y^- , being lbv functions, equal, on every compact subinterval of I , the uniform limits of sequences of step functions; consequently, these functions belong to $\mathcal{L}_{\text{loc}}^1$ of any real measure on I .

We shall base the proof on the following approximation lemma, where the notation of Sect. 4 is used. In particular, for $y: I \rightarrow Y$ and $Q \in \text{lfp}(I)$, one denotes by y_Q a local step function which takes on each member of Q the constant value $y(\tau)$, where τ is a point chosen in the said member.

Lemma 11.2: Let $y \in \text{lbv}(I, Y)$ and let $u: I \rightarrow X$ be a regulated function (cf. Sect. 3). Then, for every $[a, b] \subset I$ and every $\eta > 0$, there exists $P \in \text{lfp}(I)$ such that, whatever is $Q \in \text{lfp}(I)$, a refinement of P , any local step function y_Q constructed as above verifies

$$\left\| \int_{[a, b]} \Phi(u, dy) - \int_{[a, b]} \Phi(u, dy_Q) \right\| \leq \eta. \quad (11.4)$$

Proof: Since u is regulated, there exists for every $\alpha > 0$ a local step function $u_\alpha: I \rightarrow X$ such that $\|u - u_\alpha\| \leq \alpha$ everywhere in $[a, b]$. The left-hand side of (11.4) is majorized by

$$\left\| \int_{[a, b]} \Phi(u - u_\alpha, dy) \right\| + \left\| \int_{[a, b]} \Phi(u_\alpha - u, dy_Q) \right\| + \left\| \int_{[a, b]} \Phi(u_\alpha, dy - dy_Q) \right\|. \quad (11.5)$$

Introducing the variation functions V_y and v_Q of y and y_Q , respectively, one obtains through inequalities (10.3) and (7.5),

$$\left\| \int_{[a,b]} \Phi(u - u_\alpha, dy) \right\| \leq N(\Phi) \alpha \int_{[a,b]} dV_y,$$

$$\left\| \int_{[a,b]} \Phi(u_\alpha - u, dy_Q) \right\| \leq N(\Phi) \alpha \int_{[a,b]} dv_Q.$$

Under Convention 3.5 if needed, one has

$$\int_{[a,b]} dv_Q = v_Q^+(b) - v_Q^-(a) \leq \text{var}(y_Q; a', b'), \quad (11.6)$$

where the compact interval $[a', b']$ is assumed to be a neighborhood of $[a, b]$ relative to I . In view of Proposition 4.6, a partition $P \in \text{lfp}(I)$ may be constructed in such a way that the right-hand side of (11.6) admits a fixed bound, whatever the refinement Q of P is. Therefore, α and u_α may be chosen in order that the two first terms in (11.5) are less than $\eta/4$ for every such Q . Since u_α is a local step function, there exists a partition of $[a, b]$ into a finite number m of subintervals; on each of them u_α equals a constant, which for the subinterval J is denoted by $u_\alpha(J)$. Denoting by U_α an upper bound of $\|u_\alpha\|$ throughout $[a, b]$, one has, in view of (10.1),

$$\left\| \int_J \Phi(u_\alpha, dy - dy_Q) \right\| = \left\| \Phi(u_\alpha(J), \int_J dy - dy_Q) \right\|$$

$$\leq N(\Phi) U_\alpha \left\| \int_J dy - dy_Q \right\|. \quad (11.7)$$

Using the expressions found in Sect. 8 for the integral of a differential measure on a subinterval of I , one derives from Proposition 4.6 that the last member of (11.7) is less than $\eta/2m$, provided P has been constructed to be fine enough. The expected inequality follows by addition. \square

Lemma 11.3: Equality (11.1) holds true if x and y are local step functions.

Proof: Assuming that x and y are local step functions involves the existence of partitions P_x and P_y in $\text{lfp}(I)$, on the respective members of which, these functions equal constants. A common refinement $P \in \text{lfp}(I)$ of P_x and P_y may be constructed, meeting the following requirement: every member of P is either an interval which does not contain any of its ends or a singleton, say $\{\tau\}$; points such as τ will be called the *nodes* of P .

In order to establish equality (11.1) for this special choice of x and y , we shall show that the vector measures appearing in both members yield the same integral on every compact subinterval $[a, b]$ of I .

Let $t_1 < t_2 < \dots < t_n$ be the nodes of P belonging to $[a, b]$. Since $\Phi(x, y)$ is a constant on each member of P , Proposition 6.5 implies that its differential measure equals the sum of point measures located at nodes of P . Hence, under Convention 3.5 if t_1 or t_n happen to equal an end of I , one has

$$\int_{[a,b]} d\Phi(x, y) = \sum_{i=1}^n [\Phi(x^+(t_i), y^+(t_i)) - \Phi(x^-(t_i), y^-(t_i))]. \quad (11.8)$$

Similarly, dy equals the sum of point measures placed at nodes of P such as t_i , with values $y^+(t_i) - y^-(t_i)$. Hence, in view of Example 10.8, it results that the Z -valued measure $\Phi(x^+, dy)$ equals the sum of point measures with the same locations and values $\Phi(x^+(t_i), y^+(t_i) - y^-(t_i))$. A symmetric calculation applies to $\Phi(dx, y^-)$, so

$$\begin{aligned} \int_{[a,b]} \Phi(dx, y^-) + \Phi(x^+, dy) \\ = \sum_{i=1}^n \Phi(x^+(t_i) - x^-(t_i), y^-(t_i)) + \Phi(x^+(t_i), y^+(t_i) - y^-(t_i)), \end{aligned}$$

which is visibly equal to expression (11.8). \square

Proof of Proposition 11.1: Let $\varepsilon > 0$; Lemma 11.2 implies the existence of $P \in \text{lfp}(I)$ such that for every $Q \in \text{lfp}(I)$, a refinement of P ,

$$\left\| \int_{[a,b]} \Phi(x^+, dy) - \int_{[a,b]} \Phi(x^+, dy_Q) \right\| < \frac{\varepsilon}{4}.$$

Furthermore, if P has been chosen to be fine enough, Proposition 4.6 yields, for every such Q , some upper bound ξ of $\|x - x_Q\|$ throughout I . Introducing a compact neighborhood $[a', b']$ of $[a, b]$ relatively to I , one has

$$\left\| \int_{[a,b]} \Phi(x^+ - x_Q^+, dy_Q) \right\| \leq N(\Phi) \xi \text{ var}(y_Q; a', b'),$$

an expression which can be made smaller than $\varepsilon/4$ by taking Q fine enough.

A similar approximation procedure by local step functions applies to $\int_{[a,b]} \Phi(dx, y^-)$ and also trivially to $\int_{[a,b]} d\Phi(x, y)$. Since ε may be arbitrarily small, this reduces the proof of equality (11.1) to Lemma 11.3. Equality (11.2) follows by symmetry and equality (11.3) by combining the two. \square

Hence Proposition 11.1 generates a formula of “integration by parts”, as follows.

Corollary 11.4: For every $[a, b] \subset I$, under Convention 3.5 if a or b happen to equal an end of I , one has

$$\int_{[a, b]} \Phi(dx, y^-) + \int_{[a, b]} \Phi(x^+, dy) = \Phi(x^+(b), y^+(b)) - \Phi(x^-(a), y^-(a)) \quad (11.9)$$

and analogous formulas corresponding to (11.2) and (11.3).

Remark 11.5: The results of this section were first published in [32] and [33]. At the same time, R. T. Rockafellar [13] obtained the integration by parts formula (11.9), with Φ equal to the scalar product of \mathbb{R}^n ; his proof was based on a result of E. Asplund and L. Bungart [34], involving a summation over the set of the discontinuity points of the considered pair of real lbv functions.

Remark 11.6: Let us apply Corollary 11.4 with $Y = \mathbb{R}$, $Z = X$ and the bilinear form Φ equal to the product $(x, r) \rightarrow rx$. Suppose that the function $y: I \rightarrow \mathbb{R}$ is absolutely continuous on every compact subinterval of I ; this is classically equivalent to the existence of $y'_t \in L^1_{\text{loc}}(I, dt; \mathbb{R})$, with dt denoting Lebesgue's measure, such that for every $[a, b] \subset I$

$$\int_{[a, b]} y'_t dt = y(b) - y(a).$$

This amounts to saying that $y \in \text{lbv}(I, \mathbb{R})$ with $dy = y'_t dt$ (see also Sect. 13, concerning the absolute continuity of vector functions).

In applying (11.9) to this special case, one observes that $x = x^+ = x^-$, except at the discontinuity points of x , which make a countable, hence Lebesgue-negligible, subset of I . Therefore,

$$\int_{[a, b]} y dx + \int_{[a, b]} xy'_t dt = y(b)x^+(b) - y(a)x^-(a) \quad (11.10)$$

(the same holds symmetrically after interchanging x^+ and x^-).

For simplicity, let us restrict ourselves in what follows to the case where I is an *open* real interval. More specifically than above, take $y \in \mathcal{D}^1(I, \mathbb{R})$, i.e. y is continuously differentiable with compact support in I ; the application of (11.10) with $[a, b]$ containing this support yields

$$\int_I y dx = - \int_I xy' dt, \quad (11.11)$$

where y' denotes the derivative of y in the elementary sense. This equality holding for every $y \in \mathcal{D}^1(I, \mathbb{R})$ (equivalently, every $y \in \mathcal{D}^\infty(I, \mathbb{R})$), may be expressed by saying that the X -valued measure dx constitutes the *generalized derivative* of x , or *derivative in the sense of Schwartz's distributions*.

Observe that, in the right-hand side of (11.11), x may be understood as an element of $L^1_{\text{loc}}(I, dt; X)$, i.e. an equivalence class of locally Lebesgue-integrable

functions. This could be taken as the starting point of an alternative theory of $\text{lbv}(I, X)$: an element of $L^1_{\text{loc}}(I, dt; X)$ whose derivative in the sense of distributions happens to be an X -valued measure on I would be said to have locally bounded variation. In the author's view, the policy adopted here of defining the elements of $\text{lbv}(I, X)$ as proper functions and making use only of the ordering of the real interval I , without reference to Lebesgue's measure, would seem to be better adapted to evolution problems. In contrast it should be observed that relying on Lebesgue's measure in order to define generalized derivation allows one to develop the theory of bounded variation for functions defined in an arbitrary open subset Ω of \mathbb{R}^n , with values in \mathbb{R} or \mathbb{R}^p . This concept has been extensively studied for the past thirty years, with applications to minimal hypersurfaces and various problems of mathematical physics; [15] and [16] are reference books on the subject.

Let us finally quote from [35] a proposition, which makes precise the connection between the variation of a function $I \rightarrow X$ and the integration relative to Lebesgue measure. For simplicity, the interval I is assumed compact, say $I = [0, T]$.

Proposition 11.7: Let X be a reflexive Banach space, with dual X' , and let $f \in L^1([0, T], dt; X)$. The following assertions are equivalent:

- (a) There exists $f_1 \in \text{bv}([0, T], X)$, a representative of f , such that $\text{var}(f_1, [0, T]) \leq C$;
- (b) $\forall h \in]0, T[: \int_0^{T-h} \|f(t+h) - f(t)\| dt \leq Ch$;
- (c) $\forall \varphi \in \mathcal{D}^\infty(]0, T[, X') : \left| \int_0^T \left\langle f(t), \frac{d\varphi}{dt}(t) \right\rangle dt \right| \leq C \sup_t \|\varphi(t)\|_{X'}$.

12. Quadratic Forms, Chain Rule and Inequalities

Let us apply the results of the preceding section by making $X = Y$ and by assuming that the continuous bilinear form $\Phi : X \times X \rightarrow Z$ is *symmetric*. Then the function $F : X \rightarrow Z$ defined by $F(a) = \Phi(a, a)$ constitutes, by definition, the most general Z -valued *continuous quadratic form* on X . Proposition 11.1 entails that, for every $x \in \text{lbv}(I, X)$, the function $t \rightarrow F(x(t))$, shortly denoted by $F(x)$, belongs to $\text{lbv}(I, Z)$ and that the following equality of Z -valued measures holds:

$$dF(x) = \Phi(x^+ + x^-, dx). \quad (12.1)$$

In the special case $X = Z = \mathbb{R}$, with Φ equal to the usual product, a similar formula was derived by P. J. Daniell [36].

If $x \in \text{lbv}(I, X)$ is *continuous*, (12.1) reduces to

$$dF(x) = 2\Phi(x, dx). \quad (12.2)$$

Observe that $u \rightarrow 2\Phi(a, u)$ is the tangent linear mapping, or *gradient* of F at point a . So the differentiation formula (12.2) has the familiar aspect of a *chain rule*.

Now we shall suppose that $Z = \mathbb{R}$ and that the real quadratic form F is *nonnegative*; this holds in particular if X is a real Hilbert space with Φ equal to the corresponding scalar product.

Proposition 12.1: Let $\Phi : X \times X \rightarrow \mathbb{R}$ be a real continuous symmetric bilinear form on the Banach space X , such that the quadratic form $a \rightarrow \Phi(a, a)$ is nonnegative. Then, for every $x \in \text{lbv}(I, X)$, the real measure $\Phi(x^+ - x^-, dx)$ is nonnegative.

In the sense of the ordering of real measures, one has

$$2\Phi(x^-, dx) \leq d\Phi(x, x) \leq 2\Phi(x^+, dx). \quad (12.3)$$

Proof: Let us show that $\Phi(x^+ - x^-, dx)$ yields a nonnegative integral on every compact subinterval $[a, b]$ of I .

This is true if $x : I \rightarrow X$ is a local step function; in fact, x then equals a constant on each member of some $P \in \text{lf}_p(I)$. Denote by $t_1 < t_2 < \dots < t_n$ the nodes of P belonging to $[a, b]$. Under Convention 3.5 if t_1 or t_n happen to equal an end of I , it results that, as in Example 7.9,

$$\int_{[a, b]} \Phi(x^+ - x^-, dx) = \sum_{i=1}^n \Phi(x^+(t_i) - x^-(t_i), x^+(t_i) - x^-(t_i)) \geq 0.$$

For arbitrary x in $\text{lbv}(I, X)$, an approximation procedure based on Lemma 11.2 will establish the expected nonnegativity.

Inequalities (12.3) follow from combination with (12.1). \square

The above was published in [32, 33], with a view to the study of some evolution problems in a Hilbert space (see, e.g. [5]). Physically, inequalities such as (12.3) express the *irreversibility* of processes (see, e.g. [7, 8]); they are exploited through integration over a time interval, as in the following proposition.

Proposition 12.2: Let $I = [T_l, T_r]$ be a compact interval; denote by $\text{bv}_0(I, X)$ the linear subspace of $\text{bv}(I, X)$ consisting of the functions which vanish at the “initial instant” T_l . Let Φ be as in Proposition 12.1; suppose that two nonnegative real functions on the interval I are given:

p : nonincreasing and left-continuous,

α : universally integrable (for instance a regulated function).

Then, for every $x \in \text{bv}_0(I, X)$, the real expression

$$Q(x) = \int_I p \Phi((1 + \alpha)x^+ + (1 - \alpha)x^-, dx) \quad (12.4)$$

is nonnegative.

If, in addition, it is assumed that p is strictly positive, strictly decreasing and that the quadratic form $a \rightarrow \Phi(a, a)$ is positive definite on X , then $x \rightarrow Q(x)$ constitutes a positive definite quadratic form on the linear subspace of $\text{bv}_0(I, X)$ consisting of functions with aligned jumps (cf. Proposition 7.12).

Proof: Since $t \rightarrow p(t)$ and $t \rightarrow \Phi(x(t), x(t))$ are elements of $\text{bv}(I, \mathbb{R})$, Proposition 11.1, applied to the usual product in \mathbb{R} , yields in view of (12.1),

$$d(p\Phi(x, x)) = p^- \Phi(x^+ + x^-, dx) + \Phi(x^+, x^+) dp.$$

Since p is left-continuous, one obtains through integration on I ,

$$\int p \Phi(x^+ + x^-, dx) = p(T_r) \Phi(x(T_r), x(T_r)) - \int \Phi(x^+, x^+) dp,$$

which is nonnegative by virtue of the assumptions made. Besides, as the product $p\alpha$ is a nonnegative universally integrable function, Proposition 12.1 implies

$$\int_I p \alpha \Phi(x^+ - x^-, dx) \geq 0.$$

The nonnegativity of Q in (12.4) follows, by addition.

In order to prove the second assertion, let us suppose that the left-hand member of (12.4) vanishes. Then both nonnegative expressions $p(T_r) \Phi(x(T_r), x(T_r))$ and $-\int \Phi(x^+, x^+) dp$ vanish. This first entails $x(T_r) = 0$. Denote by t_1 an interior point of I and assume that x has aligned jumps. If $x(t_1)$ were different from zero, then at least one of the two limits $x^+(t_1)$ and $x^-(t_1)$ would be nonzero; therefore there would exist on one side of t_1 some nonempty open interval J throughout which $\Phi(x(t), x(t))$ would be greater than some $m > 0$; hence

$$-\int_J \Phi(x^+, x^+) dp \geq -m \int_J dp.$$

Since p is strictly decreasing, one has $-\int_J dp > 0$; this contradicts the vanishing of $-\int \Phi(x^+, x^+) dp$. \square

Remark 12.3: After dividing by 2, one may write equivalently (12.4) in the form

$$\int_I p \Phi(\lambda x^+ + (1 - \lambda)x^-, dx) \geq 0,$$

where λ is a universally integrable real function on I , satisfying $\lambda \geq 1/2$. As soon as $x^+ \neq x^-$, one may characterize $\lambda x^+ + (1 - \lambda)x^-$ as a point of the straight line

drawn in X through x^+ and x^- , lying on the same side as x^+ relative to the midpoint $(x^+ + x^-)/2$.

Proposition 12.2 was motivated by the treatment of *energy inequalities* in some discontinuous evolution problems. In the same context one needs some semi-continuity properties of the quadratic functional Q . To this end, we shall equip the linear space $\text{bv}_0(I, X)$ with the norm of uniform convergence,

$$\|x\|_\infty = \sup_{t \in I} \|x(t)\|_X.$$

Proposition 12.4: With the same assumptions as in the first part of Proposition 12.2, one additionally supposes that the nonnegative function α is left-continuous, with bounded variation; then $Q: \text{bv}_0(I, X) \rightarrow \mathbb{R}$ is lower semi-continuous in the $\|\cdot\|_\infty$ norm.

If A denotes a subset of $\text{bv}_0(I, X)$ whose elements have variations bounded by some fixed $M \geq 0$, then the restriction of Q to A is continuous in the $\|\cdot\|_\infty$ norm.

Proof: Concerning any nonnegative quadratic form, such as Q , standard calculation yields that, for every y and u , one has

$$Q(y+u) = Q(y) + L_y(u) + Q(u) \leq Q(y) + L_y(u),$$

where L_y denotes a linear form. Hence Q equals the supremum of the collection of affine functions

$$x \rightarrow Q(y) + L_y(x - y), \tag{12.5}$$

with y ranging throughout $\text{bv}_0(I, X)$. We shall establish the expected lower semi-continuity by showing that for fixed y , the linear form L_y is continuous in the $\|\cdot\|_\infty$ norm. In fact $L_y(u)$ equals the sum of the following two expressions:

$$L_1(u) = \int_I p \Phi((1 + \alpha)u^+ + (1 - \alpha)u^-, dy),$$

$$L_2(u) = \int_I p \Phi((1 + \alpha)y^+ + (1 - \alpha)y^-, du).$$

For every $t \in I$, $\|u^+(t)\|_X$ and $\|u^-(t)\|_X$ are majorized by $\|u\|_\infty$; then, through inequality (10.2) and Proposition 10.2, one obtains

$$|L_1(u)| \leq N(\Phi) \|u\|_\infty \int_I (|1 + \alpha| + |1 - \alpha|) p |dy| \tag{12.6}$$

which proves the continuity of L_1 .

As for L_2 , one performs some integrations by parts:

$$\Phi(y^+, du) = d\Phi(y, u) - \Phi(dy, u^-),$$

and, since $q = (1 + \alpha)p$ is left-continuous, with bounded variation, one may again apply Proposition 11.1 so as to obtain:

$$q\Phi(y^+, du) = d(q\Phi(y, u)) - \Phi(y^+, u^+)dq - q\Phi(dy, u^-).$$

Consequently,

$$\left| \int_I q\Phi(y^+, du) \right| \leq N(\Phi) [q(T_r) \|y\|_\infty + \|y\|_\infty \text{var}(q, I) + \sup_I |q| \text{var}(y, I)] \|u\|_\infty. \quad (12.7)$$

Similar majorizations apply to the terms involving $(1 - \alpha)y^-$ in L_2 , so the continuity of $u \rightarrow L_2(u)$ in the norm $\|u\|_\infty$ is established.

Now, if x is restricted to the subset A , one may express $Q(x)$ as the supremum of the affine functions (12.5), for y ranging through A only. Then $\text{var}(y, I) \leq M$, so (12.6) and (12.7) show that this collection of affine functions is equicontinuous in the $\|\cdot\|_\infty$ norm. This proves the continuity of the restriction $Q|_A$. \square

We shall finish this section with a review of some generalizations of what precedes, referring to [37] for the proofs; an arbitrary real interval is denoted by I .

The first line of generalization consists in replacing the nonnegative real quadratic form $a \rightarrow \Phi(a, a)$ of Proposition 12.1 by a *convex continuous real function* γ , defined on an open convex subset Ω of the Banach space X . Then $\partial\gamma(a)$ denotes the *subdifferential* of γ at point $a \in \Omega$, in the sense of convex analysis, i.e. the closed convex subset of the dual space X' of X (the bilinear duality pairing will be denoted by $\langle \cdot, \cdot \rangle$) consisting of the elements a' such that the affine function $x \rightarrow \langle a', x - a \rangle + \gamma(a)$ minorizes γ . Because γ is convex and continuous on Ω , it is known that, for every $a \in \Omega$, the subdifferential $\partial\gamma(a)$ is nonempty and that this subset of X' reduces to a singleton iff γ is Gâteaux-differentiable at point a (a sufficient condition for this is the strict convexity of the conjugate convex function $\gamma^* : X' \rightarrow]-\infty, +\infty]$). Observe also that, if a function $\gamma : \Omega \rightarrow \mathbb{R}$ is Gâteaux-differentiable at every point of Ω , then the convexity of this function ensures its continuity (in fact, in that case, γ equals the supremum of a collection of continuous affine functions; hence it is l.s.c. and therefore continuous throughout the open convex set Ω , because X is a barrelled space [38]).

Proposition 12.5: Let $f \in \text{lbv}(I, X)$ with differential measure df admitting a density $f'_\mu \in \mathcal{L}_{\text{loc}}^1(I, \mu; X)$ relative to some nonnegative real measure μ . Let Ω be an open convex subset of X and $\gamma : \Omega \rightarrow \mathbb{R}$ be continuous and convex. Assume

that, for every compact subinterval $[a, b]$ of I , the closure of $f([a, b])$ is contained in Ω .

Then the real function $\gamma \circ f: t \rightarrow \gamma(f(t))$ belongs to $\text{lbv}(I, \mathbb{R})$ and its differential measure possesses, relative to μ , a density $(\gamma \circ f)'_{\mu} \in \mathcal{L}_{\text{loc}}^1(I, \mu; \mathbb{R})$.

The real functions

$$\begin{aligned}\varphi_* &: t \rightarrow \sup \{ \langle g, f'_{\mu}(t) \rangle : g \in \partial\gamma(f^-(t)) \}, \\ \varphi^* &: t \rightarrow \inf \{ \langle g, f'_{\mu}(t) \rangle : g \in \partial\gamma(f^+(t)) \}\end{aligned}$$

belong to $\mathcal{L}_{\text{loc}}^1(I, \mu; \mathbb{R})$, and the inequalities

$$\varphi_* \leq (\gamma \circ f)'_{\mu} \leq \varphi^*$$

hold μ -a.e. in I .

In the set $I_c = \{t \in I : f^+(t) = f^-(t)\}$, equality $\varphi_* = \varphi^*$ holds μ -a.e.

When the above proposition is specialized to the case of a convex Gâteaux-differentiable function γ , with gradient at point a denoted by $\nabla\gamma(a)$, it yields that a chain rule of the familiar form

$$(\gamma \circ f)'_{\mu} = \langle \nabla\gamma(f), f'_{\mu} \rangle \quad (12.8)$$

holds μ -a.e. in I_c (and, a fortiori, in the set of the continuity points of f); generally, μ -a.e. in I one has

$$\langle \nabla\gamma(f^-), f'_{\mu} \rangle \leq (\gamma \circ f)'_{\mu} \leq \langle \nabla\gamma(f^+), f'_{\mu} \rangle.$$

These inequalities extend (12.3).

H. Brézis ([35], Lemma 3.3, p. 73) has obtained a formula of the same sort as (12.4) by taking X equal to a Hilbert space H and assuming $f \in W_{\text{loc}}^{1,2}(I, H)$, but requiring only of $\gamma: H \rightarrow]-\infty, +\infty]$ that it be convex and l.s.c.: if there exists $g \in \mathcal{L}_{\text{loc}}^2(I, H)$ such that $g(t) \in \partial\gamma(f(t))$ holds Lebesgue-a.e., then, with μ equal to the Lebesgue measure on I , he proved that $\gamma \circ f$ is locally absolutely continuous and that, for almost every t ,

$$\forall h \in \partial\gamma(f(t)) : (\gamma \circ f)'(t) = \langle h, f'(t) \rangle.$$

For the second line of generalization of (12.1) and (12.3), the convexity assumption of $\gamma: \Omega \rightarrow \mathbb{R}$ is dropped; instead, this real function is supposed Lipschitz on every bounded subset of Ω . Then by $\partial\gamma(a)$ is denoted *Clarke's generalized gradient* [39] of γ at point a , a convex compact subset of X'_s (i.e. the dual space of X , equipped with the weak* topology). It is known [40] that $\partial\gamma(a)$ is a singleton for every a in the open subset Ω of X iff γ is Gâteaux-differentiable in Ω with gradient mapping $a \rightarrow \nabla\gamma(a)$ continuous from Ω to X'_s .

Again, it is assumed that for every $[a, b] \subset I$, the closure of $f([a, b])$ is contained in Ω .

Proposition 12.6: With $X, \Omega, I, f, df, \mu, f'_\mu$ as in Proposition 12.5, suppose that $\gamma: \Omega \rightarrow \mathbb{R}$ is Lipschitz on every bounded subset of Ω . Then $\gamma \circ f$ belongs to $\text{lbv}(I, \mathbb{R})$ and $d(\gamma \circ f)$ admits, relative to μ , a density $(\gamma \circ f)'_\mu \in \mathcal{L}^1_{\text{loc}}(I, \mu; \mathbb{R})$.

For μ -almost every t in I , there exists x in the closed line segment $[f^-(t), f^+(t)]$ of X and x' in Clarke's generalized gradient $\partial\gamma(x)$ such that

$$(\gamma \circ f)'_\mu(t) = \langle x', f'_\mu(t) \rangle. \quad (12.9)$$

If $f^+ = f^-$, at μ -almost every point of I , the convexity of Ω is required no more; if, in addition, γ is supposed Gâteaux-differentiable, (12.9) turns into a chain rule of the same familiar form as (12.8).

In [37] the more special case is also studied where $\gamma: \Omega \rightarrow \mathbb{R}$ is continuously Frechet-differentiable; then an expression for $(\gamma \circ f)'_\mu$ is obtained, which may be seen as a generalization of (12.2).

13. Densities and the Radon-Nikodym Property

The question of deciding whether a real function of a real variable equals an indefinite integral of its derivative has been posed since the beginnings of Lebesgue's theory of integration. The extension to more general situations of the results obtained along this line has led to the word "derivative" being used in a sometimes confusing way. The purpose of this section and of the next is to review some facts connected with our subject matter, using the most common terminology.

As an immediate consequence of Sects. 7 and 8, one has the following.

Lemma 13.1: Let X be a Banach space and I a real interval, with Lebesgue measure denoted by dt . Denote by f a function of I to X and by g an element of $L^1_{\text{loc}}(I, dt; X)$. Then the following statements are equivalent:

$$(a) \quad \forall [\sigma, \tau] \subset I: \int_{[\sigma, \tau]} g dt = f(\tau) - f(\sigma), \quad (13.1)$$

$$(b) \quad f \in \text{lbv}(I, X) \quad \text{and} \quad df = g dt, \quad (13.2)$$

i.e., according to Definition 7.7, df admits g as *density* relative to the Lebesgue measure.

Notation 13.2: One expresses by $f \in W_{\text{loc}}^{1,1}(I, X)$ the existence of $g \in L_{\text{loc}}^1(I, dt; X)$ such that the above statements are true. In accordance with Definition 7.7, g will then be denoted by f'_t .

Clearly $W_{\text{loc}}^{1,1}(I, X)$ is a linear space, the elements of which are continuous functions of I to X , since the Lebesgue measure has no atom.

For $p \in [1, +\infty]$, one denotes by $W_{\text{loc}}^{1,p}(I, X)$ the linear subspace of $W_{\text{loc}}^{1,1}$ consisting of the functions f such that $f'_t \in L_{\text{loc}}^p(I, dt; X)$.

If the interval I is compact, the subscript *loc* becomes immaterial. For I not necessarily compact there is also a natural definition for some spaces denoted by $W^{n,p}(I, X)$, equipped with Banach norms. They make an essential tool in studying some evolution problems; a review of important facts about them may be found in [35]. They do not fall into the scope of the present chapter, which is mainly aimed at the treatment of less smooth functions. Recall that the notation W is more generally used, with I replaced by an open subset of \mathbb{R}^n , to denote *Sobolev spaces*.

Let us reformulate in the normed space setting the classical definition already met in Sect. 4. A function $f: I \rightarrow X$ is said to be *locally absolutely continuous* if, whatever the compact subinterval $[a, b]$ of I is, one has the following property: for every $\varepsilon > 0$, there exists $\eta > 0$ such that, denoting by $]a_i, b_i[$ any finite collection of nonoverlapping open subintervals of $[a, b]$, the implication

$$\sum_i (b_i - a_i) < \eta \Rightarrow \sum_i \|f(b_i) - f(a_i)\| < \varepsilon \quad (13.3)$$

holds. Equivalently $f \in \text{lbv}(I, X)$ with variation function V_f locally absolutely continuous of I to \mathbb{R} .

Proposition 13.3: For any Banach space X , every element of $W_{\text{loc}}^{1,1}(I, X)$ is a locally absolutely continuous function.

Proof: Let $f \in W_{\text{loc}}^{1,1}(I, X)$; with the corresponding $f'_t \in L_{\text{loc}}^1(I, dt; X)$ is associated the element $\|f'_t\|$ of $L_{\text{loc}}^1(I, dt; \mathbb{R})$ (the class of dt -equivalent real functions, a representative of which is $t \rightarrow \|\gamma(t)\|_X$, where γ denotes a representative of f'_t). For every $\sigma \leq \tau$ in I , (13.1) yields

$$\|f(\tau) - f(\sigma)\| = \left\| \int_{] \sigma, \tau [} f'_t dt \right\| \leq \int_{] \sigma, \tau [} \|f'_t\| dt. \quad (13.4)$$

Denote by A any dt -measurable subset of $[a, b]$; it is a basic fact of integration theory that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\int_A dt < \eta \Rightarrow \int_A \|f'_t\| dt \leq \varepsilon. \quad (13.5)$$

Therefore, implication (13.3) is established by taking as A the union of the collection $]a_i, b_i[$ and by using (13.4). \square

Remark 13.4: Let us review the standard background of implication (13.5), referring to [19], Chap. III, as the main source. Let Σ be a σ -field of subset of a set S and let μ be a countably additive set function defined on Σ , with values in \mathbb{R} , in $\mathbb{R} \cup \{+\infty\}$, in $\mathbb{R} \cup \{-\infty\}$, or in a Banach space X ; in the latter case, the Banach norm will be denoted here by the same symbol $|\cdot|$ as the absolute value of extended real numbers. For every $E \in \Sigma$, the *variation* (also called total variation) of μ on E is, by definition,

$$v(\mu, E) = \sup \sum_i |\mu(E_i)|$$

where the supremum is taken over all finite collections $\{E_i\}$ of pairwise disjoint members of Σ , with $E_i \subset E$. Clearly, when μ takes only nonnegative values in \mathbb{R} or $\mathbb{R} \cup \{+\infty\}$, then $v(\mu, E) = \mu(E)$.

If the countably additive set function μ takes its values in \mathbb{R} or X , it is shown ([19], Sect. III.4) that the set of these values is bounded. In such a case, μ is said to be *finite* and the set function $E \rightarrow v(\mu, E)$, with values in $\mathbb{R} \cup \{+\infty\}$, is found to be countably additive.

Let λ denote another countably additive set function defined on the same σ -field Σ as μ . Then λ is said to be *continuous with respect to μ* or simply *μ -continuous* if

$$\lim_{v(\mu, E) \rightarrow 0} \lambda(E) = 0. \quad (13.6)$$

An interesting fact is that, if λ is finite, μ -continuity is equivalent to the implication $v(\mu, E) = 0 \Rightarrow \lambda(E) = 0$.

The classical *Radon-Nikodym theorem* states that, if the countably additive set functions λ and μ are real-valued, with $\mu \geq 0$, then λ is μ -continuous if and only if there exists (uniquely) an element g of $L^1(S, \mu; \mathbb{R})$ such that

$$\forall E \in \Sigma : \lambda(E) = \int_E g d\mu. \quad (13.7)$$

Under these conditions $v(\lambda, S)$ equals the $L^1(S, \mu; \mathbb{R})$ -norm of g .

Let us now attempt to connect the above and the functional theory of measures used in the foregoing sections. Of course, one may also establish the Radon-Nikodym theorem within the functional theory (see [20], Chap. V, § 5). Suppose that T is a locally compact topological space, and denote by B the collection of the *Borel subsets* of T (by definition, B is the smallest σ -field containing all the closed subsets of T). For every compact subset K of T , the members of B contained in K constitute a σ -field, say B_K . Let μ denote a real-

valued measure on T , in the sense of the functional theory. It is found that for every $E \in \mathcal{B}_K$, its characteristic function χ_E belongs to $L^1(T, \mu; \mathbb{R})$. Its integral is denoted as usual by $\int_E d\mu$ or $\mu(E)$ and it results that $E \rightarrow \int_E d\mu$ is a countably additive set function on the σ -field \mathcal{B}_K . The Radon-Nikodym theorem may be applied in that context. Through the use of a base in X , this yields a partial converse to Proposition 13.3.

Proposition 13.5: If X is finite dimensional, every locally absolutely continuous function of I to X belongs to $W_{\text{loc}}^{1,1}(I, X)$.

For an infinite-dimensional Banach space X the same is readily found true if and only if X meets the following requirement.

Definition 13.6: A Banach space X is said to have the Radon-Nikodym property if, for every absolutely continuous $f: [0, 1] \rightarrow X$, the differential measure df admits a density relative to Lebesgue's measure.

Counterexample 13.7: It is to be shown that $X = \mathcal{K}(I)$, equipped with the supremum norm, does not possess the Radon-Nikodym property. Let $I = [0, 1]$ and let $f: I \rightarrow X$ be defined as follows; for every $t \in I$, the element $f(t)$ of X is the continuous function

$$\sigma \rightarrow \begin{cases} t - \sigma & \text{if } \sigma \leq t \\ 0 & \text{otherwise.} \end{cases}$$

If $[a, b] \subset I$, one finds $\|f(b) - f(a)\| = b - a$; hence f is absolutely continuous on I .

Admit that $f'_i \in L^1(I, dt; X)$, the density of df relative to Lebesgue's measure, exists. Then, for every element x' of the dual of X , i.e. x' is a real measure on I ,

$$\langle x', f(s) \rangle = \left\langle x', \int_0^s f'_i(t) dt \right\rangle = \int_0^s \langle x', f'_i(t) \rangle dt, \quad (13.8)$$

since f vanishes at the origin. Take, in particular, $x' = \delta_r$, the Dirac measure at point $r \in I$. By definition $\langle \delta_r, f(s) \rangle$ is the value that the element $f(s)$ of $\mathcal{K}(I)$ takes at point r , namely $s - r$ if $r \leq s$ and zero otherwise. One sees that the real function $s \rightarrow \langle \delta_r, f(s) \rangle$ possesses a derivative equal to 1 for $s > r$, and zero for $s < r$. If (13.8) were true, this derivative would equal $\langle \delta_r, f'_i(s) \rangle$ for almost every $s \neq r$ in I . Since $f'_i(s)$ is, by assumption, an element of $X = \mathcal{K}(I)$, this real expression, namely the value of $f'_i(s)$ at point r , should depend continuously on r ; this is a contradiction.

A large part of the book [22] by J. Diestel and J. J. Uhl is devoted to discussing the Radon-Nikodym property (from a starting point other than our Definition 13.6). In a summary (Chap. VII, Sect. 6), there are listed no fewer than 23 properties equivalent to this one (plus 6 others concerning the special case where X is a dual). Several items on this list attest to the geometrical nature of the Radon-Nikodym property, not essentially connected with the Lebesgue measure on $[0, 1]$, for instance: if D is a nonempty closed bounded subset of X , then a bounded linear functional on X assumes a maximum value on D .

More important with respect to our subject matter is the following, which *removes the prominence of Lebesgue measure*.

Let us come back to the setting of Remark 13.4. Let us suppose that, on the σ -field Σ of subsets of S , two countably additive set functions are defined: μ with nonnegative real values and λ with values in the Banach space X . Suppose that X has the Radon-Nikodym property; then it is established that λ is μ -continuous, with $v(\lambda, S) < +\infty$, if and only if there exists $g \in L^1(S, \mu; X)$ ensuring (13.7).

As in Remark 13.4, this may be transferred into the functional theory of measures, by considering the σ -field B_K of the Borel subsets of every compact $K \subset T$. If the X -valued measure m on T is majorable, it results from Proposition 7.4 that the integral $\int_E dm$, denoted by $m(E)$, is an element of X for every $E \in B_K$. This defines on B_K a countably additive X -valued set function, with finite variation. Take as μ the modulus measure $|m|$ (as a set function, the latter is equivalently the variation $E \rightarrow v(m, E)$). Then inequality (7.4), with h equal to the characteristic function of E , implies the μ -continuity of m . Hence, on account of Proposition 7.8, one has the following proposition.

Proposition 13.8: If X possesses the Radon-Nikodym property, every majorable X -valued vector measure m possesses, relative to its modulus $\mu = |m|$, a density $m'_\mu \in L^\infty(I, \mu; X)$. Every representative of m'_μ is a function taking values of unit norm at μ -almost every point of I .

Also established in [22] is the equivalence of the Radon-Nikodym property of X with the *Riesz representation property*: for each bounded linear operator A of $L^1(S, \mu; \mathbb{R})$ to X , there exists $g \in L^\infty(S, \mu; X)$ such that

$$\forall f \in L^1(S, \mu; \mathbb{R}) : Af = \int_S fg d\mu.$$

Let us conclude this section by quoting some items from Diestel and Uhl's lists.

Banach spaces that have the Radon-Nikodym property

- Reflexive spaces (in particular finite-dimensional spaces, Hilbert spaces, etc.)

- Separable duals
- $L^p(S, \mu; X)$, $1 < p < +\infty$, if X has the Radon-Nikodym property
- Every closed linear subspace of a Banach space having the Radon-Nikodym property

Banach spaces that do not have the Radon-Nikodym property

- $L^1(S, \mu; \mathbb{R})$ if μ is not purely atomic
- $C(K, \mathbb{R})$, i.e. the space of the continuous real functions on an infinite (Hausdorff) compact space
- L^∞ of the Lebesgue measure on $[0, 1]$

14. Derivative at a Point

Let X be an arbitrary Banach space and $f \in W_{loc}^{1,1}(I, X)$. The element f'_t of $L_{loc}^1(I, dt; X)$ involved in Notation 13.2 is connected with the possible derivative of f at a point of I through the concept of a *Lebesgue point* of a locally integrable function.

Let $g \in L_{loc}^1(I, dt; X)$. Classically (see, e.g. [19], Chap. III, Sect. 12, Theorem 8, concerning, more generally, a function of an open subset of \mathbb{R}^n), every point τ in the complement of a Lebesgue-negligible subset of I is a Lebesgue point of g , i.e. there exists an element $\gamma(\tau) \in X$ such that

$$\lim_{l(C) \rightarrow 0} \frac{1}{l(C)} \int_C g(t) dt = \gamma(\tau), \quad (14.1)$$

where C denotes any closed subinterval of I containing τ , with length $l(C)$. The function γ , after arbitrary extension to the whole of I , makes a representative of the element g of $L_{loc}^1(I, dt; X)$.

In view of Lemma 13.1 and Notation 13.2, this readily implies the following proposition.

Proposition 14.1: A function $f \in W_{loc}^{1,1}(I, X)$ possesses, at Lebesgue – almost every point τ of I , a derivative, say $\dot{f}(\tau)$; the function \dot{f} , arbitrary extended to the whole of I , constitutes a representative of the element $f'_t \in L_{loc}^1(I, dt; X)$.

In that sense, every function belonging to $W_{loc}^{1,1}(I, X)$ equals an indefinite integral of its derivative.

When a connection with absolute continuity is wished, the Radon-Nikodym property has to be assumed for X . In fact, an item on Diestel and Uhl's list asserts that X possesses this property *if and only if* every absolutely continuous $f: [0, 1] \rightarrow X$ is differentiable almost everywhere.

Another item on the list states that the Radon-Nikodym property of X is equivalent to asserting that every $f: [0, 1] \rightarrow X$ with bounded variation possesses a derivative almost everywhere. Since every compact interval may be reduced to $[0, 1]$ through some affine change of variable, we shall use this to establish the following.

Proposition 14.2: Let X possess the Radon-Nikodym property and let $f \in \text{lbv}(I, X)$. The derivative $\dot{f}(\tau)$ exists for Lebesgue – almost every point τ and, after arbitrary extension to the whole of I , it constitutes an element of $\mathcal{L}_{\text{loc}}^1(I, dt; X)$. Lebesgue – almost everywhere in I one has

$$\|\dot{f}(t)\| \leq \frac{d}{dt} V_f(t), \quad (14.2)$$

and this holds as an equality iff $f \in W_{\text{loc}}^{1,1}(I, X)$.

Suppose I to be compact. Then

$$\int_I \|\dot{f}(t)\| dt \leq \text{var}(f, I), \quad ((14.3)$$

with equality iff $f \in W^{1,1}(I, X)$.

Proof: It is enough to assume from the start that I is a compact interval, say $[0, T]$, and that the variation function V_f vanishes at 0. Then, for every $h \in]0, T[$,

$$\int_0^{T-h} \|f(t+h) - f(t)\| dt \leq h \text{var}(f; 0, T). \quad (14.4)$$

In fact, the left-hand member is majorized by

$$\int_0^{T-h} (V_f(t+h) - V_f(t)) dt \leq \int_{T-h}^T V_f(t) dt \leq h V_f(T).$$

Since the existence of

$$\dot{f}(t) = \lim_{h \rightarrow 0} \frac{1}{h} (f(t+h) - f(t))$$

is granted for almost every t , let us consider a sequence of positive values of h tending to zero. Thanks to (14.4), Fatou's lemma yields that the restriction of \dot{f} to every interval of the form $[0, T-h_0]$, $h_0 \in]0, T[$, is Lebesgue integrable and, finally, that $\dot{f} \in \mathcal{L}^1(I, dt; X)$. As an element of $\text{bv}(I, \mathbb{R})$, the nondecreasing real function V_f possesses a derivative almost everywhere. Inequality (14.2) is obtained by going to the limit from

$$\left\| \frac{1}{h} (f(t+h) - f(t)) \right\| \leq \frac{1}{h} (V_f(t+h) - V_f(t)). \quad (14.5)$$

Inequality (14.3) follows through integration.

If $f \in W^{1,1}(I, X)$, then in view of Propositions 14.1 and 7.12, equality holds in (14.2) and (14.3).

Conversely, suppose that (14.2) is verified as an equality. For every $s \in [0, T]$, this implies

$$\int_{[0,s]} \|\dot{f}(t)\| dt + \int_{[s,T]} \|\dot{f}(t)\| dt = \text{var}(f; 0, s) + \text{var}(f; s, T).$$

Since inequality (14.3) may be invoked for the intervals $[0, s]$ and $[s, T]$ as well, one concludes

$$\int_{[0,s]} \|\dot{f}(t)\| dt = \text{var}(f; 0, s)$$

for every s . This shows that the function V_f equals an indefinite integral of an element of $L^1(I, dt; \mathbb{R})$. It is therefore absolutely continuous and so is f . As X has been supposed to possess the Radon-Nikodym property, one concludes that $f \in W^{1,1}(I, X)$. \square

Corollary 14.3: If X has the Radon-Nikodym property, every $f \in \text{lbv}(I, X)$ possesses a decomposition into the sum of an element of $W_{\text{loc}}^{1,1}(I, X)$, unique up to the addition of a constant, and an element of $\text{lbv}(I, X)$ with zero derivative Lebesgue – a. e.

This demonstrates that, for a function $f \in \text{lbv}(I, X)$, the derivative \dot{f} , though it exists Lebesgue – a. e., conveys deficient information about f . In particular, an evolution problem with f as unknown cannot be posed well by asserting only that the said derivative verifies a certain condition Lebesgue – a. e., unless it is specifically stipulated that f belongs to $W_{\text{loc}}^{1,1}$. Evolution problems concerning lbv unknowns are the motivation of the following proposition, which involves a density function of df relative to a measure other than dt .

Proposition 14.4: Let X be an arbitrary Banach space and let an X -valued measure m on the interval I admit, relatively to some nonnegative real measure μ , a density, say $g \in \mathcal{L}_{\text{loc}}^1(I, \mu; X)$. Then, for μ -almost every t in I ,

$$g(t) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{m([t, t + \varepsilon])}{\mu([t, t + \varepsilon])} = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{m([t - \varepsilon, t])}{\mu([t - \varepsilon, t])}.$$

Here, by $m([t, t + \varepsilon])$ the integral of m over $I \cap [t, t + \varepsilon]$ is meant; a similar convention applies to μ and to $[t - \varepsilon, t]$.

This generalizes a theorem of Jeffery [41, 42] concerning real-valued measures on an interval. A proof, based on Jeffery's result, may be found in [37].

Another proof, valid if X has the Radon-Nikodym property (hence applicable to Jeffery's original case) was developed in [43]: there one notes the use of the technique of factorization (see Sect. 15 below) which reduces general lbv functions to Lipschitz functions. Proposition 14.1 corresponds to the special case where μ equals the Lebesgue measure.

The above was used in [37] in order to establish Proposition 12.5. Other applications concern some evolution problems [44, 45] of *irreversible* character. Unilaterality in time is a dominant feature of these problems, so derivation properties involving the interval $[t - \varepsilon, t + \varepsilon]$, as previously proposed by Daniell [36], will not do the job.

Remark 14.5: The decomposition property stated in Corollary 14.3 may be connected with the *Lebesgue decomposition* of a measure. Let us return to the setting of Remark 13.4, restricting ourselves for simplicity's sake to a pair of real-valued countably additive set functions λ and μ , defined on the σ -field Σ . Then λ is said to be μ -singular if there exists a set $E_0 \in \Sigma$, with $v(\mu, E_0) = 0$, on which λ is concentrated, i.e.

$$\forall E \in \Sigma : \lambda(E) = \lambda(E \cap E_0) ;$$

equivalently, μ is λ -singular. Lebesgue's decomposition theorem states (cf. [19], Sect. III.4) that, for given μ , every countably additive real-valued set function defined on Σ is uniquely representable as a sum $\alpha + \beta$, where α is μ -continuous and β is μ -singular.

This may be applied with μ equal to the Lebesgue measure dt on some compact interval I . To that end, take as Σ the σ -field of the Borel subsets of I . The above decomposition property yields that, for every $f \in \text{bv}(I, \mathbb{R})$, the differential measure df lets itself be expressed in a single way as the sum of a dt -continuous measure, say dw , and of a dt -singular measure, say ds . Now it is known that a function $s \in \text{bv}(I, \mathbb{R})$ has zero derivative dt -a.e. iff ds is dt -singular (see [46], Chap. 10, Sect. 4). Therefore, dw equals the differential measure of the "absolutely continuous component" $w \in W^{1,1}(I, \mathbb{R})$ of f , as it is introduced (up to an arbitrary additive constant) by Corollary 14.3.

Since w is continuous, the measure dw is *diffuse*, or *continuous*, in the sense that its integral over any singleton is zero. The reader should take care not to confuse the above with another decomposition property, in which the Lebesgue measure dt plays no part.

We still assume, for simplicity's sake, that I is a compact interval and consider only real measures in the sense of the functional theory. In Bourbaki's terminology [20] a real measure is *atomic* iff it is singular with regard to every continuous real measure (some authors, e.g. Dinculeanu [21], use this word with another meaning, namely the existence of at least one atom). Such a measure

may be characterized as concentrated on a countable I . Also observe that the real measures on a compact interval form a Banach space under the norm defined by the variation: a measure is found to be atomic iff, with regard to this norm, it equals the sum of a series of point measures. It is shown that *every real measure lets itself be uniquely decomposed into the sum of an atomic measure and a diffuse measure.*

Coming now to bv functions, one may say that an element of $\text{bv}(I, \mathbb{R})$ is a *jump function* iff its differential measure is atomic. Accordingly, any $f \in \text{bv}(I, \mathbb{R})$ equals the sum of a continuous bv function, unique up to an additive constant, and a jump function.

15. Isometric and Lipschitz Factorizations

Proposition 15.1: Let $f \in \text{lbv}(I, X)$, with variation function $v : I \rightarrow \mathbb{R}$. Denote by Γ the smallest real interval containing the range $v(I)$, i.e. the convex hull of this range in \mathbb{R} .

There exists a factorization $f = \gamma \circ v$, where $\gamma : \Gamma \rightarrow X$ is a curve-isometry, i.e. for every $\alpha \leq \beta$ in Γ

$$\text{var}(\gamma; \alpha, \beta) = \beta - \alpha. \quad (15.1)$$

If the Banach space X is strictly convex (see Remark 7.11), the function γ is uniquely determined.

Remark 15.2: If f is discontinuous, gaps are expected in its range $f(I)$. For every discontinuity point of f , say $s \in I$, one may fill the possible gaps between $f^-(s)$ and $f(s)$ and between $f(s)$ and $f^+(s)$ by rectifiable arcs, in particular by *line segments*. The union of $f(I)$ with this collection of line segments will be called the *filled-in range* of f . The following proof consists in showing that the filled-in range equals the range of a rectifiable curve and that, for every $t \in I$, the oriented arc length locating the point $f(t)$ on this curve equals $v(t)$ (cf. Sect. 5).

Proof of Proposition 15.1: Since $v : I \rightarrow \mathbb{R}$ is nondecreasing, for every σ in $v(I)$, the set $v^=(\sigma) = \{t \in I : v(t) = \sigma\}$ is a nonempty subinterval of I , possibly reduced to a singleton. For every $[a, b] \subset v^=(\sigma)$, the variation $\text{var}(f; a, b)$ is zero, hence f assumes a constant value throughout $v^=(\sigma)$; denote this value by $\gamma_v(\sigma)$, thereby defining the unique mapping γ_v of $v(I)$ to X such that

$$\forall t \in I : f(t) = \gamma_v(v(t)). \quad (15.2)$$

We now have to define γ as an extension of γ_v to the whole of Γ .

The structure of $\Gamma \setminus v(I)$ may be investigated as follows. Let $\sigma \in \Gamma \setminus v(I)$; due to the definition of Γ , the two sets $v^<(\sigma) = \{t \in I : v(t) < \sigma\}$ and $v^>(\sigma)$

$= \{t \in I : v(t) > \sigma\}$ are nonempty; they are disjoint, with union equal to I . Because v is nondecreasing these two sets are subintervals of I with a common endpoint, say $s \in I$. Necessarily, s is a discontinuity point of v ; the largest real interval containing σ and contained in $\Gamma \setminus v(I)$ is either $[v^-(s), v(s)[$ or $]v^-(s), v(s)[$ or $]v(s), v^+(s)]$ or $]v(s), v^+(s)[$. In short, $\Gamma \setminus v(I)$ equals the union of connected components which are intervals of some of the four forms above, with s ranging through the set of the discontinuity points of v .

In any nonempty interval of the form $[v^-(s), v(s)[$, let us define γ as

$$\gamma(\sigma) = f^-(s) + \frac{\sigma - v^-(s)}{v(s) - v^-(s)} (f(s) - f^-(s)) \quad (15.3)$$

and, in any nonempty interval of the form $]v(s), v^+(s)]$, as

$$\gamma(\sigma) = f(s) + \frac{\sigma - v(s)}{v^+(s) - v(s)} (f^+(s) - f(s)). \quad (15.4)$$

The definition (15.3) of γ raises the question of consistency with γ_v if $v^-(s) \in v(I)$; then the nondecreasing function v actually equals a constant in some interval $[s - \varepsilon, s[$ and so does f . Hence $v^-(s) = v(s - \varepsilon)$ and $\gamma_v(v^-(s)) = \gamma_v(v(s - \varepsilon)) = f(s - \varepsilon) = f^-(s)$, which agrees with $\gamma(v^-(s))$ as expressed in (15.3). The consistency of (15.4) with γ_v in the case $v^+(s) \in v(I)$ is similarly checked.

Convention 3.5 will be used if s equals an endpoint of I .

In view of Proposition 4.3, the elements $(f(s) - f^-(s))/(v(s) - v^-(s))$ and $(f^+(s) - f(s))/(v^+(s) - v(s))$ of X have unit norm. Therefore, the definitions (15.3) and (15.4) of γ secure equality (15.1) as soon as $[\alpha, \beta]$ is contained in the interval concerned.

If, in addition, the Banach space X is assumed to be strictly convex, one may check that (15.3) and (15.4) provide the only expression of γ possible for (15.1) to hold; the argument is detailed, for a similar situation, in the proof of Proposition 15.3 below. This is the uniqueness statement.

With a view to establishing (15.1) for every $[\alpha, \beta] \subset \Gamma$, let us first show that $\text{var}(\gamma; \alpha, \beta) \leq \beta - \alpha$, i. e. equivalently

$$\forall [\alpha, \beta] \subset \Gamma : \|\gamma(\beta) - \gamma(\alpha)\| \leq \beta - \alpha. \quad (15.5)$$

If α and β lie in $v(I)$, there exist a and b in I such that $\alpha = v(a)$, $\beta = v(b)$ and

$$\|\gamma(\beta) - \gamma(\alpha)\| = \|f(b) - f(a)\| \leq v(b) - v(a)$$

which is (15.5).

Suppose now $\alpha = v(a)$, but $\beta \in \Gamma \setminus v(I)$; then, for some $s > a$ in I , either $\beta \in]v(s), v^+(s)]$ or $\beta \in [v^-(s), v(s)[$. In the first case, (15.4) yields

$$\|\gamma(\beta) - \gamma(v(s))\| = \beta - v(s)$$

while $\|\gamma(v(s)) - \gamma(\alpha)\| \leq v(s) - \alpha$ as above; this implies (15.5) through the triangle inequality. In the second case, there exists in $]a, s[$ a sequence t_n tending to s ; then $\lim v(t_n) = v^-(s)$. Since $v(t_n) \in v(I)$, one has

$$\|\gamma(v(t_n)) - \gamma(\alpha)\| \leq v(t_n) - \alpha.$$

Hence, by the triangle inequality, due to $\gamma(v(t_n)) = f(t_n)$,

$$\|\gamma(\beta) - \gamma(\alpha)\| \leq \|\gamma(\beta) - f(t_n)\| + v(t_n) - \alpha.$$

When t_n tends to s , the right-hand member tends to $\|\gamma(\beta) - f^-(s)\| + v^-(s) - \alpha$; in view of (15.3), this equals $\beta - \alpha$.

The case $\alpha \in \Gamma \setminus v(I)$, $\beta \in v(I)$ is treated similarly.

If α and β belong to different connected components of $\Gamma \setminus v(I)$, there exists some point of $v(I)$ in between and one goes back to the preceding situations through the triangle inequality.

The end of the proof will consist in establishing $\text{var}(\gamma; \alpha, \beta) \geq \beta - \alpha$.

First suppose $\alpha = v(a)$ and $\beta = v(b)$. Choose $\varepsilon > 0$; there exists a finite sequence $a = t_0 < t_1 < \dots < t_n = b$ in I such that

$$\varepsilon + \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| \geq \text{var}(f; a, b) = v(b) - v(a).$$

Therefore, after putting $\tau_i = v(t_i)$, one has

$$\beta - \alpha \leq \varepsilon + \sum_{i=1}^n \|\gamma(\tau_i) - \gamma(\tau_{i-1})\| \leq \varepsilon + \text{var}(\gamma; \alpha, \beta)$$

whatever ε is; this yields the expected inequality.

Suppose now $\alpha = v(a)$, but $\beta \in \Gamma \setminus v(I)$; there exists $s > a$ in I and either $\beta \in]v(s), v^+(s)[$ or $\beta \in [v^-(s), v(s)[$. In the first case one has, thanks to the above result,

$$\text{var}(\gamma; \alpha, v(s)) \geq v(s) - \alpha$$

while, in view of (15.4),

$$\text{var}(\gamma; v(s), \beta) = \beta - v(s),$$

which yields the expected inequality by addition. In the second case, one considers a sequence t_n in $]a, s[$ tending to s and

$$\begin{aligned} \text{var}(\gamma; \alpha, \beta) &= \text{var}(\gamma; \alpha, v(t_n)) + \text{var}(\gamma; v(t_n), \beta) \\ &\geq v(t_n) - \alpha + \text{var}(\gamma; v(t_n), \beta). \end{aligned}$$

Inequality (15.5) shows that γ has locally bounded and continuous variation; hence the above right-hand side tends to

$$v^-(s) - \alpha + \text{var}(\gamma; v^-(s), \beta) = \beta - \alpha.$$

Finally, if α and β belong to different connected components of $\Gamma \setminus v(I)$, there exists a point of $v(I)$ in between, and one is reduced, by addition, to the preceding situations. \square

We shall now define a Banach norm on the product space $\mathbb{R} \times X$, by choosing a strictly positive constant κ and putting, for every element (r, x) of this product space,

$$\|(r, x)\| = |r| + \kappa \|x\|_X. \quad (15.6)$$

From the standpoint of analysis, little would be lost by restricting κ to equal 1. We introduce this coefficient mainly to recall, in applications to mechanical evolution problems, that the real numbers r and $\|x\|_X$ measure physical quantities of different natures – usually time and speed; the presence of κ allows one to develop calculations independent of the choice of physical units.

The mapping $(\text{id}, f) : t \rightarrow (t, f(t))$ of I to $\mathbb{R} \times X$ possesses as range a subset of $\mathbb{R} \times X$ which is nothing but *the graph of f* ; it is disconnected iff f is discontinuous (one may check that, for a regulated function, thus in particular for an lbv function, continuity is equivalent to the connectedness of its graph). By using line segments in order to fill in the gaps of this set, in the way we have described in Remark 15.2 above, one constructs the *filled-in graph* of f . This equals the union of $\text{gr } f$ with the collection of “vertical” line segments $[(t, f^-(t)), (t, f(t))]$ and $[(t, f(t)), (t, f^+(t))]$, for t ranging through the set of discontinuity points of f (or, equivalently, through the whole of I). It is easily verified that (id, f) belongs to $\text{lbv}(I, \mathbb{R} \times X)$; thus Proposition 15.1 shows that the filled-in graph of f equals the range of a rectifiable curve in $\mathbb{R} \times X$. Proposition 15.4 below will make this more precise, through the use of the following preliminary result.

Proposition 15.3: Let us equip \mathbb{R}^2 with a norm by putting, for every pair (r, u) ,

$$\|(r, u)\| = |r| + \kappa |u|. \quad (15.7)$$

Let $v : I \rightarrow \mathbb{R}$ be a nondecreasing real function; denote by Γ the smallest interval containing $v(I)$.

Then the mapping $(\text{id}, v) : t \rightarrow (t, v(t))$ belongs to $\text{lbv}(I, \mathbb{R}^2)$ and admits as variation function

$$t \rightarrow w(t) = t + \kappa v(t). \quad (15.8)$$

The smallest interval containing $w(I)$ is $\Delta = I + \kappa \Gamma$.

In this special case (though strict convexity does not hold for the norm defined in (15.7)), the factorization of (id, v) through its variation function and a curve-isometry, asserted in general by Proposition 15.1, is unique and may be described as follows. There exist, uniquely, two nondecreasing real functions, say p of Δ

onto I and q of Δ onto Γ such that $(\text{id}, v) = (p, q) \circ w$, i. e.

$$\forall t \in I: t = p(w(t)), \quad v(t) = q(w(t)), \quad (15.9)$$

and that the mapping $(p, q): \Delta \rightarrow \mathbb{R}^2$ is a curve-isometry with regard to the norm (15.7); hence p is Lipschitz with ratio 1, and q with ratio κ^{-1} .

Summing up, one has the following relations between the functions involved

$$w = \text{id}_I + \kappa v: I \rightarrow \Delta \quad (15.10)$$

$$p + \kappa q = \text{id}_\Delta: \Delta \rightarrow \Delta \quad (15.11)$$

$$p \circ w = \text{id}_I: I \rightarrow I \quad (15.12)$$

$$q \circ w = v: I \rightarrow \Gamma. \quad (15.13)$$

Proof: That w equals a variation function of (id, v) is easily checked, after observing that $t \rightarrow t$ and $t \rightarrow v(t)$ are nondecreasing real functions; (15.10) is only a repetition of (15.8).

According to Remark 15.2, the factorization amounts to the bridging of the possible gaps in the range of (id, v) (i. e. the graph of v) by rectifiable arcs, so as to obtain the range of a rectifiable curve along which every point of the special form $(t, v(t))$, $t \in I$, is located by an arc length θ equal to $w(t)$. Each of these gaps arises from the discontinuity of v at some point of I , say s ; its extremities have either the form $(s, v^-(s)), (s, v(s))$ or the form $(s, v(s)), (s, v^+(s))$. Take a gap of the first form; the isometry condition requires that the length of the bridging arc A be $w(s) - w^-(s) = \kappa(v(s) - v^-(s))$. Let us show that A is necessarily a line segment. In fact, for $(x, y) \in A$; the length of A is minorized by

$$\begin{aligned} & \| (s, v(s)) - (x, y) \| + \| (x, y) - (s, v^-(s)) \| \\ & = 2 |s - x| + \kappa(|v(s) - y| + |y - v^-(s)|) \end{aligned}$$

which shall be strictly greater than $\kappa(v(s) - v^-(s))$, unless $x = s$ and $v^-(s) \leq y \leq v(s)$; this means that (x, y) belongs to the line segment admitting the prescribed endpoints. Along this line segment, the correspondence between the location of (x, y) and the arc length θ consists in affine interpolation; hence the definition of the mapping $(p, q): \theta \rightarrow (x, y)$ in the present case is identical to the construction of γ in the proof of Proposition 15.1. Similar reasoning applies to a gap of the second form; so the uniqueness of the factorization is proved.

Formulas (15.12) and (15.13) are only a rewriting of (15.9). As for (15.11), it is clear that $p + \kappa q$ equals identity at every point of $w(I)$; now this equality is preserved under the affine interpolation procedure used to define p and q in every

gap. Since $\theta \rightarrow (p(\theta), q(\theta))$ is, by construction, a curve-isometry of Δ to \mathbb{R}^2 , one has, for every θ and θ' in Δ ,

$$|p(\theta') - p(\theta)| + \kappa |q(\theta') - q(\theta)| \leq |\theta' - \theta|,$$

which implies the Lipschitz properties of p and q .

That p and q are nondecreasing real functions easily follows from the fact that the filled-in graph of v is a monotone increasing subset of \mathbb{R}^2 , i. e. for every (x, y) and (x', y') in this subset, one has $(x - x')(y - y') \geq 0$. \square

It is also useful to visualize the mappings p and q as follows. For every $\theta \in \Delta$, the line $\{(r, u) \in \mathbb{R}^2 : r + \kappa u = \theta\}$ intersects the filled-in graph of v at exactly one point; this is the elementary one-dimensional version of the general theorem of Minty and Kachurovski, concerning maximal monotone correspondences in Hilbert spaces. If $\theta = w(t)$, $t \in I$, this point coincides with $(t, v(t))$; hence it equals $(p(\theta), q(\theta))$ as previously defined. If, on the contrary, θ belongs to some gap of $w(I)$, the intersection point lies on some of the (vertical) line segments used to fill in the corresponding gap of $\text{gr } v$; its dependence on θ clearly agrees with the affine interpolation procedure used above in constructing (p, q) .

Having completed the proof, let us additionally observe the following. Knowledge of the mapping $(p, q) : \Delta \rightarrow \mathbb{R}^2$ involves knowledge of the filled-in graph of v , namely the range of (p, q) , but not knowledge of $\text{gr } v$. In other words, the function v cannot be reconstructed exactly from p and q ; but v^- and v^+ can. In fact, for $t \in I$, the limits $v^-(t)$ and $v^+(t)$ equal the values of q at the ends of the interval $p^-(t)$.

Proposition 15.4: The notations of Propositions 15.1 and 15.3 are used jointly, i. e. v equals a variation function of $f \in \text{lbv}(I, X)$; the space $\mathbb{R} \times X$ is equipped with the Banach norm (15.6).

Then (id, f) belongs to $\text{lbv}(I, \mathbb{R} \times X)$, with variation function w , as expressed in (15.8).

Let

$$(\text{id}, f) = (\pi, \delta) \circ w \tag{15.14}$$

with (π, δ) a curve-isometry of Δ to $\mathbb{R} \times X$ (Proposition 15.1 secures the existence of such a factorization); then π equals the function p involved in Proposition 15.3.

In any factorization of the form (15.14), the function $\delta : \Delta \rightarrow X$ may be modified on the subset $\Delta \setminus w(I)$ of its domain so that $\delta = \gamma \circ q$, with q defined by Proposition 15.3 and $\gamma : \Gamma \rightarrow X$.

Every $\gamma : \Gamma \rightarrow X$ is a curve-isometry iff $(p, \gamma \circ q) : \Delta \rightarrow \mathbb{R} \times X$ is a curve-isometry; in such a case $\gamma \circ q$ is Lipschitz with ratio κ^{-1} .

A necessary and sufficient condition for the factorization

$$(\text{id}, f) = (p, \gamma \circ q) \circ w \quad (15.15)$$

to hold, with $(p, \gamma \circ q)$ a curve-isometry, is that γ satisfy the conditions of Proposition 15.1.

Proof: Let $[a, b] \subset I$; the variation of (id, f) on $[a, b]$ is the supremum, over all sequences $a = \tau_0 < \tau_1 < \dots < \tau_n = b$, of the expression

$$\begin{aligned} \sum_{i=1}^n (|\tau_i - \tau_{i-1}| + \kappa \|f(\tau_i) - f(\tau_{i-1})\|_X) \\ = b - a + \kappa \sum_{i=1}^n \|f(\tau_i) - f(\tau_{i-1})\|_X. \end{aligned}$$

This supremum equals $b - a + \kappa(v(b) - v(a))$, so w is a variation function of (id, f) .

Suppose that (15.14) holds and that (π, δ) is a curve-isometry of Δ to $\mathbb{R} \times X$, relatively to the norm (15.6); let us prove that $\pi = p$. Since $\pi \circ w = \text{id}_I = p \circ w$, one has $\pi(\theta) = p(\theta)$ for every $\theta \in w(I)$. Let us now study the restriction of (π, δ) to any connected component of $\Delta \setminus w(I)$, say $[w^-(s), w(s)[$, $s \in I$ (the case of a connected component of the form $]w(s), w^+(s)[$ will be treated similarly; observe that, due to the fact that w is strictly increasing, the situation is simpler than in the proof of Proposition 15.1, involving the function v which is only nondecreasing). Since (π, δ) is a curve-isometry, its restriction to $[w^-(s), w(s)[$ is a rectifiable arc A in $\mathbb{R} \times X$, with endpoints $(s, f^-(s))$ and $(s, f(s))$, and with length equal to $w(s) - w^-(s) = \kappa(v(s) - v^-(s))$. Let $(\xi, x) \in A$; the length of A is minorized by

$$\begin{aligned} \|(s, f(s)) - (\xi, x)\|_{\mathbb{R} \times X} + \|(\xi, x) - (s, f^-(s))\|_{\mathbb{R} \times X} \\ = 2|s - \xi| + \kappa(\|f(s) - x\|_X + \|x - f^-(s)\|_X) \\ \geq 2|s - \xi| + \kappa\|f(s) - f^-(s)\|_X. \end{aligned}$$

In view of Proposition 4.3, the last term equals $\kappa(v(s) - v^-(s))$; hence the length of A can have this value only if $\xi = s$. This proves that $\pi(\theta)$ equals s , namely $p(\theta)$, for every $\theta \in [w^-(s), w(s)[$. The same holds for all connected components of $\Delta \setminus w(I)$, hence $\pi = p$.

The formula (15.14) is equivalent to $\pi \circ w = \text{id}_I$ and $\delta \circ w = f$. Since f equals a constant on every level set of v , there exists, uniquely, $\gamma_v : v(I) \rightarrow X$ such that $f = \gamma_v \circ v$. Now $v = q \circ w$, so

$$\forall \theta \in w(I) : \delta(\theta) = \gamma_v(q(\theta)).$$

The values that δ takes on $\Delta \setminus w(I)$ are immaterial regarding the factorization (15.14); hence, if γ denotes any extension of γ_v to the whole of Γ , (15.14) is preserved by taking $\delta = \gamma \circ q$. (Incidentally, observe that if $\gamma : \Gamma \rightarrow X$ is required to be a curve-isometry and if X is strictly convex, Proposition 15.1 implies the uniqueness of this extension of γ_v .)

Let $\gamma : \Gamma \rightarrow X$ and $[\alpha, \beta] \subset \Delta$. The variation of $(p, \gamma \circ q)$ on $[\alpha, \beta]$ is the supremum, over all sequences $\alpha = \theta_0 < \theta_1 < \dots < \theta_n = \beta$, of

$$\begin{aligned} \sum_{i=1}^n |p(\theta_i) - p(\theta_{i-1})| + \kappa \left\| (\gamma \circ q)(\theta_i) - (\gamma \circ q)(\theta_{i-1}) \right\|_X \\ = p(\beta) - p(\alpha) + \kappa \sum_{i=1}^n \left\| \gamma(q(\theta_i)) - \gamma(q(\theta_{i-1})) \right\|_X \end{aligned}$$

(recall that p is nondecreasing). Since q is nondecreasing and surjective of Δ to Γ , this supremum equals

$$p(\beta) - p(\alpha) + \kappa \operatorname{var}(\gamma; q(\alpha), q(\beta)) = \beta - \kappa q(\beta) - \alpha + \kappa q(\alpha) + \kappa \operatorname{var}(\gamma; q(\alpha), q(\beta)).$$

The mapping $(p, \gamma \circ q)$ is a curve-isometry iff this equals $\beta - \alpha$, i.e. equivalently

$$\operatorname{var}(\gamma; q(\alpha), q(\beta)) = q(\beta) - q(\alpha).$$

As q is surjective the latter means that γ is a curve-isometry of Γ to X .

If $(p, \gamma \circ q)$ is a curve-isometry the definition of the norm in $\mathbb{R} \times X$ yields

$$|p(\beta) - p(\alpha)| + \kappa \left\| (\gamma \circ q)(\beta) - (\gamma \circ q)(\alpha) \right\|_X \leq \beta - \alpha$$

which clearly implies that $\gamma \circ q$ is Lipschitz with ratio κ^{-1} .

Observe finally that, due to (15.12) and (15.13), (15.15) is equivalent to $f = \gamma \circ v$; this is precisely the requirement imposed on γ in Proposition 15.1. \square

Remark 15.5: The factorization pattern which is the object of this section was first introduced in [47] in the framework of multifunctions with finite retraction (see Remark 4.7); it has also been exploited in [43].

16. Conclusion

We shall conclude this chapter by giving examples of how the material presented applies to the investigation of some *measure differential inclusions*.

In the notation of Sect. 13, a differential inclusion of the ordinary sort, with unknown $u \in W_{\text{loc}}^{1,1}(I, X)$, is a condition of the form

$$u'_i \in \Gamma(t, u(t)), \tag{16.1}$$

to be satisfied for Lebesgue-almost every t in the interval I by some representative of the element $u'_t \in L^1_{\text{loc}}(I, dt; X)$. Here $(t, x) \rightarrow \Gamma(t, x) \subset X$ is a given *multifunction* (i. e. a set-valued mapping) of $I \times X$ to X . A general reference on this subject is [48], a book motivated by applications to economics and control theory.

The existence of solutions to (16.1) is naturally conditioned by topological assumptions about Γ . In common cases, these topological assumptions entail that any function u which is a solution in the above sense verifies $\Gamma(t, u(t)) \neq \emptyset$ for every t ; in some other problems, the latter condition is additionally imposed. Then the definition of a solution u becomes equivalent to requiring the existence of a representative of u'_t satisfying (16.1) for every t in I .

A natural generalization of (16.1) consists in replacing the Lebesgue measure dt by some prescribed nonnegative real measure μ on the interval I . The unknown u is then assumed to be an element of $\text{lbv}(I, X)$, with differential measure $du = u'_\mu \mu$, and the element u'_μ of $L^1_{\text{loc}}(I, \mu; X)$ is required to possess a representative satisfying for every t in I the condition

$$u'_\mu \in \Gamma(t, u(t)); \quad (16.2)$$

this implies that, for every t , the right-hand member is nonempty.

Of primary interest in that connection is the case where the given multifunction Γ takes only *conical values*, i. e. for every $(t, x) \in I \times X$ and every $\lambda \in \mathbb{R}_+ = [0, +\infty[$ one has $\lambda\Gamma(t, x) \subset \Gamma(t, x)$. Equivalently, we can assume that the latter holds for every $\lambda \in]0, +\infty[$ and additionally that

$$\Gamma(t, x) \neq \emptyset \Rightarrow 0 \in \Gamma(t, x). \quad (16.3)$$

In this special case, there is no need to specify the measure μ in advance. In fact, suppose that $u \in \text{lbv}(I, X)$, associated with a certain nonnegative measure μ , is a solution to (16.2). Let ν denote any other nonnegative measure relative to which du happens to possess a density function, say $u'_\nu \in \mathcal{L}^1_{\text{loc}}(I, \nu; X)$. Put $\beta = \mu + \nu$; the Radon-Nikodym theorem shows the existence of functions μ'_β and ν'_β , belonging to $\mathcal{L}^\infty(I, \beta; \mathbb{R}_+)$, such that $\mu = \mu'_\beta \beta$ and $\nu = \nu'_\beta \beta$. Then $du = u'_\mu \mu'_\beta \beta = u'_\nu \nu'_\beta \beta$; hence the X -valued functions $u'_\mu \mu'_\beta$ and $u'_\nu \nu'_\beta$ are equal, except in some β -negligible subset B of I . The set $N = \{t \in I : \nu'_\beta(t) = 0\}$ is ν -negligible; outside $B \cup N$ one has $u'_\nu = u'_\mu \mu'_\beta / \nu'_\beta$. Since $\mu'_\beta / \nu'_\beta \geq 0$ (16.2) implies, in view of Γ being a cone,

$$u'_\nu \in \Gamma(t, u(t)), \quad (16.4)$$

except for t in the ν -negligible set $B \cup N$. It has been observed that the right-hand member is nonempty for every t . So (16.3) entails that, after replacing by zero the values that u'_ν may take in $B \cup N$, one obtains a density function of du relative to ν which satisfies (16.4) everywhere in I .

If X possesses the Radon-Nikodym property, it may prove convenient to use as ν the *modulus measure* $|du|$.

In any event, provided that the multifunction Γ has conical values, we shall formulate the equivalent conditions (16.2) and (16.4) by writing simply

$$du \in \Gamma(t, u(t)). \quad (16.5)$$

Let us assume throughout the following that I contains its left end t_0 , the “initial instant”. Leaving aside now the question of the existence of solutions, let us come back to the differential inclusion of the ordinary sort (16.1). Because Γ is not in general a singleton, there is no reason to expect the uniqueness of $u \in W_{loc}^{1,1}(I, X)$ verifying (16.1) and some *initial condition* $u(t_0) = u_0$. A common problem of control theory is to minimize some cost functional over the set of such solutions.

There is, however, a celebrated uniqueness case. Suppose that $X = H$, a *real Hilbert space*, and that, for every $t \in I$, the multifunction $x \rightarrow A(t, x) = -\Gamma(t, x)$ is *monotone* in the sense of Kachurovski-Minty, i.e.

$$y \in A(t, x), \quad y' \in A(t, x') \Rightarrow (x - x') \cdot (y - y') \geq 0; \quad (16.6)$$

here the dot refers to the scalar product in H . Then, by calculating elementarily the derivative of a squared norm, one finds that, for every two solutions u_1 and u_2 of

$$-u_i' \in A(t, u_i(t)), \quad (16.7)$$

the distance $\|u_1(t) - u_2(t)\|$ is a nonincreasing function of t . This implies in particular that at most one such solution can agree with some initial condition. The book [35] remains the standard reference on the differential inclusion (16.7), although many papers on the subject have been published since; see also [48], Chap. 3. In applications, condition (16.6) usually reflects the *dissipative* character of the process investigated.

With a view to transposing (16.7) into the setting of measure differential inclusions, we shall assume, in addition to (16.6), that, for every t and x , the subset $A(t, x)$ of H is a cone. Then the following may be established [32]: for every t there exists $C(t)$, a closed convex subset of H , such that

$$\forall t \in I, \quad \forall x \in H: A(t, x) \subset \partial\psi_{C(t)}(x). \quad (16.8)$$

Here $\psi_{C(t)}$ denotes the *indicator function* of $C(t)$ (the function which takes the value zero in $C(t)$ and $+\infty$ outside), a convex lower semi-continuous extended real-valued function on H . The subdifferential $\partial\psi_{C(t)}(x)$ of this function at point x is known to equal the *normal cone* to $C(t)$ at x in the sense of convex analysis (it is empty iff $x \notin C(t)$; otherwise this is a nonempty closed convex cone, possibly reduced to the zero of H).

Classically, on the set of the monotone multifunctions $H \rightarrow H$, a partial ordering is induced by the inclusion of the corresponding graphs, as subsets of $H \times H$. Multifunctions such as $\partial\psi$ are well known to be monotone and *maximal elements* relative to the above ordering. It is usual in the study of the differential inclusion (16.7) to assume that, for every t , the monotone multifunction $A(t, \cdot)$ is maximal. If this is supposed here, (16.8) implies that $A(t, \cdot) = \partial\psi_{C(t)}$.

The corresponding measure differential inclusion

$$-du \in \partial\psi_{C(t)}(u(t)) \quad (16.9)$$

defines what is called a *sweeping process* [4, 5]. This is the simplest instance of an evolution problem of the *unilateral* type. Let us visualize here $t \rightarrow u$ as a moving point in H ; the derivative du/dt , if it exists, will then be seen as the velocity of u . The given multifunction $t \rightarrow C(t)$ may be visualized as a moving (closed convex) subset of H . In the special case where this set has a nonempty interior and where $t \rightarrow u$ is continuously differentiable, one may interpret condition (16.9) as follows. As long as the moving point happens to lie in the interior of $C(t)$, the normal cone $\partial\psi_{C(t)}(u)$ reduces to the zero of H ; hence u has zero velocity. It is only when “caught up with” by the boundary of $C(t)$ that u takes on a motion, in an inward normal direction, as if pushed by this boundary, so as to go on belonging to the moving set.

When the given motion of $C(t)$ is not smooth enough, the continuous differentiability of a possible solution to (16.9) cannot be expected. It is however desirable that, for two possible lbv solutions u_1 and u_2 to (16.9), the same property as for the $W_{loc}^{1,1}$ solutions of (16.7) hold: the distance $\|u_1(t) - u_2(t)\|$ should be a nonincreasing function of t . This property which ensures uniqueness for the possible solution to the initial value problem, will easily be derived from Proposition 12.1 above (see also [5]), provided that, in addition to (16.9), it is stipulated that the unknown $u \in \text{lbv}(I, H)$ should be *right-continuous*. In fact, the right-continuity of solutions has been generally prescribed from the very beginning of the theory of measure differential equations [3].

In previous sections, we have stressed that the values taken by an lbv function at possible discontinuity points are immaterial with regard to its differential measure. So, instead of requiring right-continuity, an equivalent way of dealing with the present evolution problem would be to replace in the right-hand member of (16.9) the value $u(t)$ by $u^+(t)$.

The existence of solutions was established in [5] under the assumption that $t \rightarrow C(t)$ has bounded “retraction” [27]. Another class of existence results has been initiated in [49] and [50]; along this line, [44] and [45] assume that $C(t)$ has *nonempty interior*.

Generally, the existence proofs of bv solutions to evolution problems, on a time interval I which, at the first stage, is assumed *compact*, rest on the

construction of sequences (or of nets) of approximate solutions. These may be defined as the solutions to ordinary differential equations resulting from some *regularization* of the investigated problem. They may also be *step functions* similar to what is obtained in numerical algorithms of time discretization. In any case, the crucial stage consists in proving that the constructed collection of approximate solution has uniformly bounded variation on I . After that, various extraction procedures will play a role similar to that of Ascoli's theorem in the standard theory of differential equations. The proof will end with checking that the function obtained at the limit is in fact a solution to the problem investigated; at this stage, the key often lies in integral formulae, of which our Proposition 12.4 displays an example, or in Fatou's lemma.

For finite-dimensional H , the Ascoli-like procedure of extraction is frequently provided by *Helly's selection theorem* (the same idea was also present in S. Banach's thesis [51]) an exposition of which may be found, e. g. in [52, 53]. Let us quote from [54] the following infinite-dimensional version of this theorem.

Proposition 16.1: Let X be a reflexive separable Banach space with separable dual. Let (w_n) be a sequence of elements of $\text{bv}([a, b], X)$ such that $\|w_n(t)\| \leq C$ for $t \in [a, b]$ and $\text{var}(w_n; a, b) \leq C$ for all n . Then there exists a subsequence (w_k) of the above and a function $w \in \text{bv}([a, b], X)$ such that

$$\forall t \in [a, b]: w_k(t) \rightarrow w(t) \quad \text{weakly in } X,$$

$$\forall \varphi \in \mathcal{K}([a, b]): \int_a^b \varphi dw_k \rightarrow \int_a^b \varphi dw \quad \text{weakly in } X.$$

In (16.9) only a very special example of a measure differential inclusion involving a multifunction with conical values is displayed. The motion $t \rightarrow q(t) \in \mathbb{R}^n$ of a mechanical system with a finite number n of degrees of freedom, submitted to frictionless unilateral constraints, under the assumption of *inelasticity* of the possible impacts, has been shown [7, 8] to be governed by

$$q(t) = q(t_0) + \int_{t_0}^t u(\tau) d\tau, \quad (16.10)$$

$$-du + Q(t, q, u) dt \in \partial \psi_{V(q)}(u^+). \quad (16.11)$$

Here $Q: I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given continuous function and dt refers to the Lebesgue measure on the interval I . By $V(q)$ one denotes the *tangent cone* at point $q(t)$ to the region \mathcal{R} of \mathbb{R}^n permitted to q by the given (time-independent) unilateral constraints. This cone is easily defined if the unilateral constraints may be described by a finite system of inequalities $f_\alpha(q) \leq 0$, $\alpha = 1, \dots, l$, and if the functions f_α are C^1 with nonzero gradients; in particular $V(q)$ equals the whole

of \mathbb{R}^n when q is interior to \mathcal{R} (concerning the concept of a tangent cone in more general situations, see [55], Chap. 4).

Analogous formulations also hold if *dry friction* is taken into account at the possible contacts. One source of interest is to suggest a numerical algorithm for the treatment of the corresponding evolution problems [9, 56].

The existential study of solutions to such problems is currently in progress [57], following the line described above. This question is essentially different from that of the *elastic bounce* problem treated, e.g. in [10–12].

Let us conclude by suggesting an alternative approach to problems with lbv unknowns. Since the set of the discontinuity points of u is Lebesgue-negligible, the values that u may take at these points are immaterial with regard to (16.10); the same is true for (16.11), since Q is assumed continuous.

Consequently, there is no loss of generality in assuming that u has aligned jumps (cf. Sect. 7). After that, one may introduce G , the *filled-in graph* of u . According to an observation made in Sect. 15, the function u cannot be completely reconstructed from the knowledge of this subset of $I \times X$; but u^+ and u^- can. So G conveys the information needed to make (16.10) and (16.11) meaningful.

Therefore, a consistent policy will be to look at G as the proper unknown in a measure differential inclusion of the form (16.11) or (16.9) (recall that the latter was understood with $u = u^+$). The initial condition $u(t_0) = u_0$ is equivalent to G possessing (t_0, u_0) as its starting point.

Also, in all that concerns the *approximation* of lbv evolutions, the filled-in graphs appear to be the right objects to consider. In fact, let G and G' be filled-in graphs of two elements u and u' of lbv (I, X) . Using in $\mathbb{R} \times X$ the distance δ associated with the Banach norm (15.6), let us introduce the *Hausdorff distance* of these two sets

$$h(G, G') = \max \{e(G, G'), e(G', G)\},$$

with the same notation as in Sect. 4, namely

$$e(G, G') = \sup_{g \in G} \inf_{g' \in G'} \delta(g, g').$$

The nonnegative number $h(G, G')$ makes a realistic measurement of the discrepancy between u and u' . The numerical handling of a function $u: I \rightarrow X$ commonly involves some uncertainty about the point $t \in I$ at which u is evaluated, concurrent with the uncertainty affecting the value $u(t)$. The use of h provides a way of controlling both errors jointly.

Suppose, in particular, that u has to be approximated by a sequence of functions $u_i: I \rightarrow X$. For discontinuous u , uniform approximation usually cannot be expected. In fact, suppose that u possesses, at some unknown point t , a discontinuity with oscillation $\geq \varepsilon$. Then any function u' approximating u with an

error uniformly less than ε necessarily has a discontinuity at this point; thus the construction of u' will in practice require the exact knowledge of t . In contrast, the approximation of u in the sense of h is feasible, even by means of continuous functions. More details may be found in [58] (but with another choice of the distance δ in the product space $I \times X$). In [59], the convergence of the filled-in graphs for the Yosida approximants of a sweeping process is established.

The convergence of filled-in graphs may also be used in existence proofs. Assume here that I is compact and that a sequence of presumably approximate solutions $u_i \in \text{vb}(I, X)$ to the problem investigated has been constructed. According to Proposition 15.4, for every i there exists a real interval Δ_i (compact in this case), with two nonexpanding mappings $p_i: \Delta_i \rightarrow I$ (nonincreasing) and $\delta_i: \Delta_i \rightarrow X$, such that the filled-in graph G_i of u_i equals the range of $(p_i, \delta_i): \Delta_i \rightarrow I \times X$.

Suppose now that the crucial research of $M \in \mathbb{R}$, an upper bound of all $\text{var}(u_i, I)$, has been successful. Then, the increasing affine bijections $a_i: [0, 1] \rightarrow \Delta_i$, which exist in any case, are equicontinuous. Consequently, the mappings $P_i = p_i \circ a_i: [0, 1] \rightarrow I$ (non-decreasing) and $\Pi_i = \delta_i \circ a_i: [0, 1] \rightarrow X$ are uniformly Lipschitz and so are the mappings $(P_i, \Pi_i): [0, 1] \rightarrow I \times X$. The latter possess the G_i as respective ranges. In the case where X has finite dimension, Ascoli's theorem may then be applied, yielding a subsequence (P_k, Π_k) converging uniformly to $(P, \Pi): [0, 1] \rightarrow I \times X$, a Lipschitz mapping. The range G of (P, Π) equals the filled-in graph of some $u \in \text{vb}(I, X)$ and G equals the limit of the sequence G_k in the sense of Hausdorff distance.

More refinement of this approximation procedure could be made by also observing that the derivatives of the mappings (P_k, Π_k) are elements of a ball in $L^\infty([0, 1], \text{Lebesgue}; \mathbb{R} \times X)$. Then some standard arguments of weak* compactness may be used, with a view to proving that a solution of the formulated problem is obtained at the limit.

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