



HAL
open science

From unbalanced optimal transport to the Camassa-Holm equation

Thomas Gallouët, François-Xavier Vialard

► **To cite this version:**

Thomas Gallouët, François-Xavier Vialard. From unbalanced optimal transport to the Camassa-Holm equation. 2016. hal-01363647v1

HAL Id: hal-01363647

<https://hal.science/hal-01363647v1>

Preprint submitted on 10 Sep 2016 (v1), last revised 13 Dec 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FROM UNBALANCED OPTIMAL TRANSPORT TO THE CAMASSA-HOLM EQUATION

THOMAS GALLOUËT AND FRANÇOIS-XAVIER VIALARD

ABSTRACT. The group of diffeomorphisms of a compact manifold endowed with the L^2 metric acting on the space of probability densities gives a unifying framework for the incompressible Euler equation and the theory of optimal mass transport. Recently, several authors have extended optimal transport to the space of positive Radon measures where the Wasserstein-Fisher-Rao distance is the natural extension to the classical L^2 -Wasserstein distance. In this paper, we show a similar relation between this unbalanced optimal transport problem and the H^{div} right-invariant metric on the group of diffeomorphisms, which corresponds to the Camassa-Holm equation in one dimension. This point of view provides an isometric embedding of the group of diffeomorphisms endowed with this right-invariant metric in the automorphisms group of the fiber bundle of half densities endowed with an L^2 type of cone metric. The main application consists in writing the Camassa-Holm equation on S_1 as a particular case of the incompressible Euler equation on a group of homeomorphisms of \mathbb{R}^2 that preserve a radial density which has a singularity at 0. An other application consists in proving that smooth solutions of the Euler-Arnold equation for the H^{div} right-invariant metric are length minimizing geodesics for sufficiently short times.

1. INTRODUCTION

The seminal article of Arnold [1] showing that the incompressible Euler equation can be viewed as a geodesic flow on the group of volume preserving diffeomorphisms of a base Riemannian manifold M had an impressive impact in the mathematical literature and has led to many different works. Among others, let us emphasize two different viewpoints which have proven to be successful.

The first one has been investigated by Ebin and Marsden in [17] where an intrinsic point of view has been taken on the group of diffeomorphisms as an infinite dimensional weak Riemannian manifold. By interpreting the geodesic equation as an ordinary differential equation in a Hilbert manifold of Sobolev diffeomorphisms, they showed local well-posedness of the geodesic equation for smooth enough initial conditions. Since then, many fluid dynamic equations have been written as a geodesic flow on a group of diffeomorphisms endowed with a right-invariant metric or connection [31, 26, 42, 19, 25] and analytical properties have been proven in the direction of [17]. Note in particular that all these works assume a strong ambient topology such as H^s for s high enough and the topology given by the Riemannian metric is generically weaker, namely L^2 .

Another point of view, also motivated by the variational interpretation of geodesics as minimizers of the action functional was initiated by Brenier. He developed an extrinsic approach by considering the group of volume preserving diffeomorphisms as a Riemannian submanifold embedded in the space of maps $L^2(M, M)$ which is particularly simple when M is the Euclidean space or torus. In particular, the polar factorization theorem [4] was motivated by a numerical scheme approximating geodesics on the group of volume preserving diffeomorphisms. Optimal transport appeared as a key tool to project a map onto this group minimizing the L^2 distance and it can be interpreted as a non-linear extension of the pressure in the incompressible Euler equation. Since then, optimal transport has witnessed an impressive development and found many important applications inside and outside mathematics, see for instance the gigantic monograph of Villani [47]. Brenier also used optimal transport in order to define the notion of generalized geodesics for the incompressible Euler equation in [5].

In this article, we develop Brenier's point of view for a generalization in any dimension of the Camassa-Holm equation. Indeed, we present an isometric embedding of the group of diffeomorphisms endowed with the right-invariant H^{div} metric into a space of maps endowed with an L^2 metric. Moreover, the recently introduced Wasserstein-Fisher-Rao distance [12, 11], a generalization of optimal transport to measures that do not have the same mass, plays the role of the Wasserstein distance for the incompressible Euler equation.

Before presenting our contributions, we give a brief overview of the link between optimal transport and the incompressible Euler equation hereafter.

1.1. Optimal transport and the incompressible Euler equation. [44, 26, 27], In this section, we present the link between optimal mass transport and the incompressible Euler equation. We start from the usual static formulation of optimal transport and then present the dynamical formulation proposed by Benamou and Brenier. The link between the two formulations can be introduced via Otto's Riemannian submersion, which also provides a clear connection between incompressible Euler equation and the dynamical formulation of optimal transport. This presentation closely follows the discussion in [28, Appendix A.5].

Static formulation of optimal mass transport: The optimal mass transport problem as introduced by Monge in 1781 consists in finding, between two given probability measures ν_1 and ν_2 , a map φ such that $\varphi_*\nu_1 = \nu_2$, i.e. the image measure of ν_1 by φ is equal to ν_2 and which minimizes a cost given by

$$(1.1) \quad \int_M c(x, \varphi(x)) d\nu_1(x),$$

where c is a positive function that represents the cost of moving a particule of unit mass from location x to location y . This problem is ill-posed in the sense that solutions may not exist and the Kantorovich formulation of the problem is the correct relaxation of the Monge formulation, which can be presented as follows: On the space of probability measures on the product space $M \times M$, denoted by $\mathcal{P}(M \times M)$, find a minimizer to

$$(1.2) \quad \mathcal{I}(m) = \int_{M^2} c(x, y) dm(x, y) \quad \text{such that } p_*^1(m) = \nu_1 \text{ and } p_*^2(m) = \nu_2,$$

where $p_*^1(m), p_*^2(m)$ denote respectively the image measure of $m \in \mathcal{P}(M \times M)$ under the projections on the first and second factors on $M \times M$. Most often, the cost c will be chosen as a power of a distance. From now on, we will only discuss the case $c(x, y) = d(x, y)^2$ where d is the distance associated with a Riemannian metric. In this case, the Kantorovich minimization problem defines the so-called L^2 -Wasserstein distance on the space of probability measures. The Monge formulation can be expressed as a minimization problem as follows

$$(1.3) \quad W_2(\mu, \nu)^2 \stackrel{\text{def.}}{=} \inf_{\varphi \in \text{Diff}(M)} \left\{ \int_M d(\varphi(x), x)^2 d\nu_1(x) : \varphi_*\nu_1 = \nu_2 \right\},$$

where $\text{Diff}(M)$ denotes the group of smooth diffeomorphisms of M .

Dynamic formulation: In [2], Benamou and Brenier introduced a dynamical version of optimal transport which was inspired and motivated by the study of the incompressible Euler equation. Let $\rho \in C^\infty(M, \mathbb{R}_+)$ be a positive function, note that all the quantities will be implicitly time dependent. The dynamic formulation of the Wasserstein distance consists in minimizing

$$(1.4) \quad \mathcal{E}(v) = \int_0^1 \int_M |v(t, x)|^2 \rho(t, x) dx dt,$$

subject to the constraints $\dot{\rho} + \text{div}(v\rho) = 0$ and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$.

Equivalently, following [2], a convex reformulation using the momentum $\mathbf{m} = \rho v$ reads

$$(1.5) \quad \mathcal{E}(\mathbf{m}) = \int_0^1 \int_M \frac{|\mathbf{m}(t, x)|^2}{\rho(t, x)} dx dt,$$

subject to the constraints $\dot{\rho} + \operatorname{div}(\mathbf{m}) = 0$ and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$. Let us underline that the functional \mathcal{E} is convex in ρ, \mathbf{m} and the continuity equation is linear in (ρ, \mathbf{m}) . Due to the continuity equation, the problem is feasible if and only if the initial and final densities have the same total mass using Moser's lemma.

Otto's Riemannian submersion: The link between the static and dynamic formulations is made clear using Otto's Riemannian submersion [45] which emphasizes the idea of a group action on the space of probability densities. Let $\operatorname{Dens}_p(M)$ be the set of probability measures that have smooth positive densities with respect to a reference volume measure ν . We consider such a probability density denoted by ρ_0 . Otto showed that the map

$$\begin{aligned} \pi : \operatorname{Diff}(M) &\rightarrow \operatorname{Dens}_p(M) \\ \pi(\varphi) &= \varphi_*(\rho_0) \end{aligned}$$

is a (formal) Riemannian submersion of the metric $L^2(\rho_0)$ on $\operatorname{Diff}(M)$ to the L^2 -Wasserstein metric on $\operatorname{Dens}_p(M)$. For a brief reminder on Riemannian submersion, we refer the reader to Appendix A.1. The fiber of this Riemannian submersion at point $\rho_0 \equiv 1$ is the subgroup of diffeomorphisms preserving the measure ν , we denote it by $\operatorname{SDiff}(M)$ and we denote its tangent space at Id by $\operatorname{SVect}(M)$ the space of divergence free vector fields. The vertical space at a diffeomorphism $\varphi \in \operatorname{Diff}(M)$ for $\rho \stackrel{\text{def.}}{=} \varphi_*\rho_0$ is

$$(1.6) \quad \operatorname{Vert}_\varphi = \{v \circ \varphi; v \in \operatorname{Vect}(M) \text{ s.t. } \operatorname{div}(\rho v) = 0\},$$

In particular, if $\varphi \in \operatorname{SDiff}(M)$, $\operatorname{Vert}_\varphi = \{v \circ \varphi; v \in \operatorname{SVect}(M)\}$. The horizontal space is

$$(1.7) \quad \operatorname{Hor}_\varphi = \{\nabla p \circ \varphi; p \in C^\infty(M, \mathbb{R})\}.$$

Incompressible Euler equation: On the fiber $\operatorname{SDiff}(M) = \pi^{-1}(\{\rho_0\})$, the $L^2(\nu)$ metric is right-invariant. In Arnold's seminal work [1], it is shown that the Euler-Lagrange equation associated with this metric is the incompressible Euler equation. Arnold derived this equation as a particular case of geodesic equations on a Lie group endowed with a right-invariant metric. In its Eulerian formulation, the incompressible Euler equation is, when $M = \mathbb{T}^d$ the flat torus,

$$(1.8) \quad \begin{cases} \partial_t v(t, x) + v(t, x) \cdot \nabla v(t, x) = -\nabla p(t, x), & t > 0, x \in M, \\ v(0, x) = v_0(x), \end{cases}$$

where $v_0 \in \operatorname{SVect}(M)$ is the initial condition and p is the pressure function. On a general Riemannian manifold (M, g) compact and without boundary, the formulation is similar, omitting the time and space variables,

$$(1.9) \quad \begin{cases} \partial_t v + \nabla_v v = -\nabla p, & t > 0, x \in M, \\ v(0, x) = v_0(x), \end{cases}$$

where, in this case, the symbol ∇ denotes the Levi-Civita connection associated with the Riemannian metric on M . Another fruitful point of view consists in considering the group $\operatorname{SDiff}(M)$ as isometrically embedded in the group $\operatorname{Diff}(M)$ endowed with the $L^2(\nu)$ (non right-invariant) metric. Therefore the geodesic equations are simply geodesic equations on the Riemannian submanifold $\operatorname{SDiff}(M)$ and the geodesic equations can be written as

$$(1.10) \quad \ddot{\phi} = -\nabla p \circ \phi,$$

where $\phi \in \operatorname{SDiff}(M)$ and p is still a pressure function. Using this Riemannian submanifold approach, Brenier was able to prove that solutions for which the Hessian of p is bounded in L^∞ are length minimizing for short times and several of his analytical results were derived from this point of view [3, 5].

Inviscid Burgers equation: The geodesic equation on the group of diffeomorphisms for the L^2 metric written in Eulerian coordinates is the compressible Burgers equation. Its formulation on $M = \mathbb{T}^d$ is

$$(1.11) \quad \partial_t u(t, x) + u(t, x) \cdot \nabla u(t, x) = 0,$$

or on a general Riemannian manifold

$$(1.12) \quad \partial_t u + \nabla_u u = 0.$$

This formulation is obviously related to the incompressible Euler equation where the pressure p can be interpreted as a Lagrange multiplier associated with the incompressibility constraint, which is not present in Burgers equation. Since the map π is a Riemannian submersion, geodesics on the space of densities can be lifted horizontally to geodesics on the group. These horizontal geodesics are potential solutions of the Burgers equation, if $u_0 = \nabla q_0$, i.e. u is a potential at the initial time, then u_t stays potential for all time. The corresponding equation for the potential q is the Hamilton-Jacobi equation

$$(1.13) \quad \partial_t q(t, x) + \frac{1}{2} |\nabla q(t, x)|^2 = 0,$$

which, in this formulation, makes sense on a Riemannian manifold.

1.2. Previous works and contributions. Recently, several authors and ourselves extended optimal transport to the case of unbalanced measures, i.e. measures that do not have the same total mass. Although several works extended optimal transport to this setting, surprisingly enough, the equivalent of the L^2 -Wasserstein distance in this unbalanced setting has been introduced in 2015 simultaneously by [12, 11, 34, 35, 30]. In this paper, we show that, in the case of the Wasserstein-Fisher-Rao metric, the equivalent to the incompressible Euler equation is a generalization of the Camassa-Holm equation, namely the Euler-Arnold equation for the right-invariant metric H^{div} on the group of diffeomorphisms. In one dimension, geodesics for the right-invariant H^{div} metric is the Camassa-Holm equation introduced [10]. Since then, the Camassa-Holm equation has drawn a lot of attention since it is a bi-Hamiltonian system as well as an integrable system, it exhibits peakon solutions and it is a model for waves in shallow water [15, 13, 33, 14, 7, 16, 24]. In particular, this equation is known for its well understood blow-up in finite time and is a model for wave breaking [38]. Although the title of [8], which refers to optimal transport and the Camassa-Holm equation, is seemingly close to our article, the authors introduce a metric based on optimal transport which gives Lipschitz estimates for the solutions of the Camassa-Holm equation and it is a priori completely different to our construction. Indeed, in our article, the optimal transport metric measures the discrepancy of not being in the stabilizer of the group action defined in Section 2.3 where the solutions of the Camassa-Holm equation lie.

In this article, we rewrite the geodesic flow of the right-invariant H^{div} metric on the diffeomorphism group as a geodesic equation on a constrained submanifold of a semidirect product of group or equivalently on the automorphism group of the half-densities fibre bundle endowed with the cone metric (see Section 2.2 for its definition). This point of view has three applications: (1) We interpret solutions to the Camassa-Holm equation and one of its generalization in higher dimension as particular solutions of the incompressible Euler equation for a radial density which has a singularity at 0. This correspondence can be introduced via a sort of Madelung transform. (2) We formulate a polar decomposition on the automorphism group of the half-densities fibre bundle. (3) We generalize a result of Khesin et al. in [26] by computing the curvature of the group as a Riemannian submanifold. (4) By generalizing a result of Brenier to the case of Riemannian manifolds which states that solutions of the Euler equation are length minimizing geodesic for sufficiently short times, we prove similar results for the Camassa-Holm equation.

1.3. Plan of the paper. In Section 2, we introduce the Wasserstein-Fisher-Rao metric which generalizes the Wasserstein metric on the space of probability densities to the space of densities, thereby relaxing the mass constraint. Its presentation emphasizes the dynamical formulation similar to the Benamou-Brenier formulation. This dynamical formulation naturally introduces a cone metric which is detailed in Section 2.2. Then, we present the generalization of Otto's Riemannian submersion to this unbalanced case. This generalization uses a semidirect product of group which can be interestingly interpreted as the automorphism group of the principal fibre bundle of half-densities, as

explained in Section 2.3. This semidirect group has a natural left action on the space of densities and it gives the Riemannian submersion between an L^2 type of metric on the group and the Wasserstein-Fisher-Rao metric on the space of densities. The generalization of the group action point of view is a key tool to introduce in Section 2.5 the Monge formulation of the Wasserstein-Fisher-Rao metric and we also present its Kantorovich counterpart in Section 2.6. In addition in Section 2.5, we also propose a generalization of Brenier's polar factorization in this context which can be seen as a constrained version of it on the automorphism group of the half-densities principal fibre bundle.

In Section 3, we explain the Euler-Arnold derivation of the incompressible Euler equation and other fluid dynamic equations such as the Camassa-Holm equation. We then recall in Section 3.3 the Ebin-Marsden approach to show local well-posedness of the Camassa-Holm equation.

Section 4 presents the corresponding submanifold point of view corresponding to the Camassa-Holm equation (its generalization). The submanifold is the isotropy subgroup of the left action of the semidirect product of group and the ambient metric is the L^2 type of metric. As a direct consequence, it gives a generalization of a result on the sectional curvature in [26, Theorem A.2].

Two main applications are detailed in Section 5. In Section 5.1 we show that solutions of the Camassa-Holm equation (its generalization) can be seen as particular solutions of an incompressible Euler equation on the cone for a particular density on the cone which has a singularity at 0. In Section 5.2, we improve a result of Ebin and Marsden by applying Brenier's approach to show that every smooth geodesics are length minimizing on a sufficiently short time interval under mild conditions.

2. A GEOMETRIC POINT OF VIEW ON UNBALANCED OPTIMAL TRANSPORT

2.1. The dynamical formulation. The Wasserstein-Fisher-Rao metric: The continuity equation enforces the mass conservation property in the Benamou-Brenier formulation (1.4). This constraint can be relaxed for instance by introducing a source term $\mu \in C^\infty(M, \mathbb{R})$,

$$(2.1) \quad \dot{\rho} = -\operatorname{div}(\rho v) + \mu.$$

For a given variation of the density $\dot{\rho}$, there exist a priori many couples (v, μ) that reproduce this variation. Following [46], it can be determined via the minimization of the norm of (v, μ) , for a given choice of norm. The penalization of μ was chosen in [36] as the L^2 norm but a natural choice is rather the Fisher-Rao metric

$$\operatorname{FR}^2(\mu) = \int_M \frac{\mu(t, x)^2}{\rho(t, x)} dx.$$

In other words, this is the L^2 norm of the growth rate w.r.t. the density ρ since it can be written as $\int_M \alpha(t, x)^2 \rho(t, x) dx$ where α is the growth rate $\alpha(t, x) \stackrel{\text{def.}}{=} \frac{\mu(t, x)}{\rho(t, x)}$. Note in particular that this action is 1-homogeneous with respect to the couple (μ, ρ) . This point is important for convex analysis properties in order to define the model on singular measures. Obviously, there are many other choices of norms that satisfies this homogeneity property but this one can be related to the Camassa-Holm equation.

Thus, the Wasserstein-Fisher-Rao functional also known as the Hellinger-Kantorovich [34], or Kantorovich-Fisher-Rao [22] is defined as follows, with $a, b \in \mathbb{R}_+^*$ some length parameters,

$$(2.2) \quad \operatorname{WF}^2(\rho_0, \rho_1) = \inf_{\mathbf{m}, \mu} a^2 \int_0^1 \int_M \frac{|\mathbf{m}(t, x)|^2}{\rho(t, x)} dx + b^2 \int_0^1 \int_M \frac{\mu(t, x)^2}{\rho(t, x)} dx dt,$$

subject to the constraints

$$\dot{\rho} + \operatorname{div}(\mathbf{m}) = \mu$$

and initial condition $\rho(0) = \rho_0$ and final condition $\rho(1) = \rho_1$.

This dynamical formulation enjoys most of the analytical properties of the initial Benamou-Brenier formulation (1.4) and especially convexity. An important consequence is the existence of optimal paths in the space of time-dependent measures [12] by application of the Fenchel-Rockafellar theorem.

Not only analytical properties are conserved but also some interesting geometrical properties of standard optimal transport such as the Riemannian submersion of Otto as explained in the introduction. In this section, we extend this property to the generalized model. The group of diffeomorphisms will be replaced by a semi-direct product of group between $\text{Diff}(M)$ and the space $C^\infty(M, \mathbb{R})$ which is a group under pointwise multiplication. In addition, this group acts on the space of densities $\text{Dens}(M)$ and this action gives a Riemannian submersion between the group endowed with an L^2 type of metric, namely $L^2(M, \mathcal{C}(M))$ and the space of densities endowed with the Wasserstein-Fisher-Rao metric. The notation $\mathcal{C}(M)$ is the cone over M defined in the next section 2.2, it is the manifold $M \times \mathbb{R}_+^*$ endowed with the Riemannian metric given in Definition 1. Moreover, this semidirect product of groups will be identified as the automorphism group of the fibre bundle of half densities in section 2.3.

2.2. A cone metric. To motivate the introduction of the cone metric, let us first discuss informally what happens for a particle of mass $m(t)$ at a spatial position $x(t)$ in a Riemannian manifold (M, g) under the generalized continuity constraint (2.1). Let us assume the following structure for the measure $m(t)\delta_{x(t)}$ where $m(t) \in \mathbb{R}_+^*$ is the mass of the Dirac measure and $x(t) \in M$ its location, the system reads

$$(2.3) \quad \begin{cases} \dot{x}(t) = v(x(t)) \\ \dot{m}(t) = \alpha(x(t))m(t) \end{cases}$$

where $\alpha = \frac{\dot{m}}{m}$ is the growth rate. The action associated with formula (4.5) reads $\int_0^1 a^2 |v(x(t))|^2 m(t) + b^2 \frac{\dot{m}(t)^2}{m(t)} dt$. Thus, considering the particle as a point in $M \times \mathbb{R}_+^*$, the Riemannian metric seen by the particle is $a^2 mg + b^2 \frac{dm^2}{m}$. Therefore, it will be of importance to study this Riemannian metric $M \times \mathbb{R}_+^*$. Actually, this space is isometric to a Riemannian cone. Let us first define this object.

Definition 1 (Cone). Let (M, g) be a Riemannian manifold. The cone over M denoted by $\mathcal{C}(M)$ is the quotient space $(M \times \mathbb{R}_+^*) / (M \times \{0\})$. The apex of the cone $M \times \{0\}$ will be denoted by \mathcal{S} . The cone will be endowed with the metric $g_{\mathcal{C}(M)} \stackrel{\text{def.}}{=} r^2 g + dr^2$ defined on $M \times \mathbb{R}_+^*$ and r is the parameter in \mathbb{R}_+^* .

The isometry is given by the square root change of variable on the mass, as stated in the following proposition.

Proposition 1. *The space $(M \times \mathbb{R}_+^*, mg + \frac{1}{4m} dm^2)$ is isometric to $(\mathcal{C}(M), g_{\mathcal{C}(M)})$ by the change of variable $r = \sqrt{m}$. Therefore, the distance on $(M \times \mathbb{R}_+^*, a^2 mg + \frac{b^2}{m} dm^2)$ is given by*

$$(2.4) \quad d((x_1, m_1), (x_2, m_2))^2 = 4b^2 \left(m_2 + m_1 - 2\sqrt{m_1 m_2} \cos \left(\frac{a}{2b} d_M(x_1, x_2) \wedge \pi \right) \right).$$

Moreover, if c is a geodesic for the metric $\frac{a^2}{4b^2} g$, an isometry $S : \mathbb{C} \setminus \mathbb{R}_- \mapsto M \times \mathbb{R}_+^*$ defined by $S(\sqrt{m}e^{i\theta}) = (c(\theta), 2bm)$.

This implies that mass can "appear" and "disappear" at finite cost. In other words, the Riemannian cone is not complete but adding the vertex of the cone, which represents $M \times \{0\}$, to $M \times \mathbb{R}_+^*$ turns it into a complete metric space when M is complete. Importantly, the distance associated with the cone metric (2.4) is 1-homogeneous in (m_1, m_2) . We now collect known facts on the Riemannian cone.

Proposition 2. *On the cone $\mathcal{C}(M)$, we denote by e the vector field defined by $\frac{\partial}{\partial m}$. The Levi-Civita connection on (M, g) will be denoted by ∇^g . For a given vector field X on M , define its lift as a vector field on $M \times \mathbb{R}_+^*$ by $\hat{X} = (X, 0)$. The Levi-Civita connection on $\mathcal{C}(M)$ denoted by ∇ is given by*

$$\nabla_{\hat{X}} \hat{Y} = \widehat{\nabla_X^g Y} - rg(X, Y)e, \quad \nabla_e e = 0 \quad \text{and} \quad \nabla_e \hat{X} = \nabla_{\hat{X}} e = \frac{1}{r} \hat{X}.$$

The curvature tensor R on the cone satisfies the following properties,

$$(2.5) \quad R(\hat{X}, e) = 0 \text{ and } R(\hat{X}, \hat{Y})\hat{Z} = (R_g(X, Y)Z - g(Y, Z)X + g(X, Z)Y, 0)$$

where R_g denotes the curvature tensor of (M, g) . Let X, Y be two orthonormal vector fields on M ,

$$(2.6) \quad K(\hat{X}, \hat{Y}) = K_g(X, Y) - 1$$

where K and K_g denote respectively the sectional curvatures of $\mathcal{C}(M)$ and M .

Proof. Direct computations, see [21]. \square

Let us give simple comments on Riemannian cones. Usual cones, embedded in \mathbb{R}^3 are cones over a segment of length less than 2π . Although Riemannian cones over a segment in \mathbb{R} are locally flat, the curvature still concentrates at the apex of the cone. The cone over the sphere is isometric to the Euclidean space (minus the origin) and the cone over the Euclidean space has nonpositive curvature. In particular, the cone over S_1 is isometric to $\mathbb{R}^2 \setminus \{0\}$. We refer to [9] for more informations on cones from the point of view of metric geometry.

We need the explicit formulas for the geodesic equations on the cone.

Corollary 3. *The geodesic equations on the cone $\mathcal{C}(M)$ are given by*

$$(2.7a) \quad \frac{D^g}{Dt} \dot{x} + 2\frac{\dot{r}}{r}\dot{x} = 0$$

$$(2.7b) \quad \ddot{r} - rg(\dot{x}, \dot{x}) = 0,$$

where $\frac{D^g}{Dt}$ is the covariant derivative associated with (M, g) .

Alternatively, the geodesic equations on $(M \times \mathbb{R}_+, a^2mg + \frac{b^2}{m}dm^2)$ can be written w.r.t. the initial "mass" coordinate as follows

$$(2.8a) \quad \frac{D^g}{Dt} \dot{x} + \frac{\dot{m}}{m}\dot{x} = 0$$

$$(2.8b) \quad \ddot{m} - \frac{\dot{m}^2}{2m} - \frac{a^2}{2b^2}g(\dot{x}, \dot{x})m = 0.$$

2.3. The automorphism group of the bundle of half-densities. The cone can be seen as a trivial principal fibre bundle since $\mathcal{C}(M)$ is the direct product of M with the group \mathbb{R}_+^* . Let us denote $p_M : \mathcal{C}(M) \mapsto M$ the projection on the first factor. The group \mathbb{R}_+^* induces a group action on $\mathcal{C}(M)$ defined by $\lambda \cdot (x, \lambda') \stackrel{\text{def.}}{=} (x, \lambda\lambda')$, for all $x \in M$ and $\lambda, \lambda' \in \mathbb{R}_+^*$. We now identify the cone with the trivial fibre bundle of half densities.

Definition 2. Let M be a smooth manifold without boundary and (U_α, u_α) be a smooth atlas. The bundle $\text{Dens}_s(M)$ of s -densities is the line bundle given by the following cocycle

$$\Psi_{\alpha\beta} : U_\alpha \cap U_\beta \mapsto \text{GL}_1(\mathbb{R}) = \mathbb{R}^*$$

$$\Psi_{\alpha\beta}(x) = |\det(d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x)))|^s = \frac{1}{|\det(d(u_\alpha \circ u_\beta^{-1})(u_\beta(x)))|^s}.$$

We will denote by $\text{Dens}(M)$ the density bundle, instead of $\text{Dens}_1(M)$. This definition shows that the fibre bundle $\text{Dens}_s(M)$ is also a principal fibre bundle over \mathbb{R}_+^* and it will be the point of view adopted in the rest of the paper.

On any smooth manifold M , the fibre bundle of s -densities is a trivial principal bundle over \mathbb{R}_+^* since there exists a smooth positive density on M . Note that this trivialization depends on the choice of this reference positive density. If one chooses such a positive density, then the $1/2$ -density bundle can be identified to the cone $\mathcal{C}(M)$. Let us fix a reference volume form ν , that is a section of the density bundle which trivializes it.

From now on, we identify $\text{Dens}_{1/2}(M)$ with the cone $\mathcal{C}(M)$.

Thus every element of $\text{Dens}_{1/2}(M)$ is a section of the cone $\mathcal{C}(M)$. We are now interested in transformations that preserve the group structure. Namely, one can define

$$(2.9) \quad \text{Aut}(\mathcal{C}(M)) = \left\{ \Phi \in \text{Diff}(\mathcal{C}(M)); \Phi(x, \lambda) = \lambda \cdot \Phi(x, 1) \text{ for all } \lambda \in \mathbb{R}_+^* \right\},$$

which is the instantiation, in this particular case, of the definition of the automorphisms group of a principal fibre bundle. In other words, this is the subgroup of diffeomorphisms of the cone that preserve the group action on the fibers. In particular, $\text{Aut}(\mathcal{C}(M))$ is a subgroup of $\text{Diff}(\mathcal{C}(M))$. Of particular interest is the subgroup of $\text{Aut}(\mathcal{C}(M))$ which is defined as

$$(2.10) \quad \text{Gau}(\mathcal{C}(M)) = \{ \Phi \in \text{Aut}(\mathcal{C}(M)); p_M \circ \Phi = \text{id}_M \}.$$

The set $\text{Gau}(\mathcal{C}(M))$ called the gauge group and it is a normal subgroup of $\text{Aut}(\mathcal{C}(M))$. We now consider the injection $s : \text{Diff}(M) \hookrightarrow \text{Aut}(\mathcal{C}(M))$ defined by $s(\varphi) = (\varphi, \text{id}_{\mathbb{R}_+^*})$. This is the standard situation of a semidirect product of groups between $i(\text{Diff}(M))$ and $\text{Gau}(\mathcal{C}(M))$ since the following sequence is exact

$$(2.11) \quad \text{Gau}(\mathcal{C}(M)) \hookrightarrow \text{Aut}(\mathcal{C}(M)) \rightarrow \text{Diff}(M).$$

where s defined above provides an associated section of the short exact sequence. Note that we could also have chosen the natural section associated to the natural bundle of half-densities that we define hereafter since it will be used for the generalization of the polar factorization in Section 2.5.

Definition 3. For a given diffeomorphism $\varphi \in \text{Diff}(M)$, its natural lift to $\text{Aut}(\mathcal{C}(M))$ is

$$(2.12) \quad \begin{aligned} (\varphi, \sqrt{\text{Jac}(\varphi)}) : \mathcal{C}(M) &\rightarrow \mathcal{C}(M) \\ (x, r) &\mapsto (\varphi(x), \sqrt{\text{Jac}(\varphi)(x)}r). \end{aligned}$$

As is well-known for a trivial principal bundle, $\text{Aut}(\mathcal{C}(M))$ is therefore equal to the semidirect product of group:

$$(2.13) \quad \text{Aut}(\mathcal{C}(M)) = \text{Diff}(M) \rtimes_{\Psi} \text{Gau}(\mathcal{C}(M)),$$

where $\Psi : \text{Diff}(M) \rightarrow \text{Aut}(\text{Gau}(\mathcal{C}(M)))$ is defined by $\Psi(\varphi)(\lambda) = \varphi^{-1}\lambda\varphi$ being the associated inner automorphism of the group $\text{Gau}(\mathcal{C}(M))$. Being a trivial principal fibre bundle, the gauge group can be identified with the space of positive functions on M . Let us denote $\Lambda_{1/2}(M) \stackrel{\text{def.}}{=} C^\infty(M, \mathbb{R}_+^*)$ which is a group under pointwise multiplication. The subscript 1/2 is a reminder of the fact that $\Lambda_{1/2}(M)$ is the gauge group of $\mathcal{C}(M)$, the bundle of 1/2-densities. Note that we do not use the standard left action but, instead, a right action for the inner automorphisms as presented in [29, Section 5.3], which fits more to our situation. The identification of $\Lambda_{1/2}$ with the gauge group $\text{Gau}(\mathcal{C}(M))$ is simply $\lambda \mapsto (\text{id}_M, \lambda)$ where $(\text{id}_M, \lambda) : (x, m) \mapsto (x, \lambda(x)m)$. Let us detail the group composition law, which is given by

$$(2.14) \quad (\varphi_1, \lambda_1) \cdot (\varphi_2, \lambda_2) = (\varphi_1 \circ \varphi_2, (\lambda_1 \circ \varphi_2)\lambda_2)$$

and the inverse is

$$(2.15) \quad (\varphi, \lambda)^{-1} = (\varphi^{-1}, \lambda^{-1} \circ \varphi^{-1}).$$

By construction, the group $\text{Aut}(\mathcal{C}(M))$ has a natural left action on the space $\text{Dens}_{1/2}(M)$ as well as on $\text{Dens}(M)$. This action on $\text{Dens}(M)$ will be denoted by the map π defined by

$$(2.16) \quad \begin{aligned} \pi : (\text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)) \times \text{Dens}(M) &\mapsto \text{Dens}(M) \\ \pi((\varphi, \lambda), \rho) &\stackrel{\text{def.}}{=} \varphi_*(\lambda^2 \rho). \end{aligned}$$

2.4. A Riemannian submersion between the automorphism group and the space of densities. The semidirect product of group $\text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)$ will be endowed with the metric $L^2(M, \mathcal{C}(M))$ with respect to the reference measure on M . This is probably the simplest type of (weak) Riemannian metrics on spaces of mappings and it has been studied in details in [20] where, in particular, the curvature is computed. Note in particular that this metric is *not* the right-invariant metric L^2 on the semidirect product of groups as in [25] or on automorphism group which would lead to an EPDiff equation on a principal fibre bundle as developed in [23].

Proposition 4. *The geodesic equations on $\text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)$ endowed with the metric $L^2(M, \mathcal{C}(M))$ with respect to the reference measure on ν are given by the geodesic equations on the cone (2.8), that is $\frac{D}{Dt}(\dot{\varphi}, \dot{\lambda}) = 0$, or more explicitly*

$$(2.17a) \quad \frac{D^g}{Dt} \dot{\varphi} + 2 \frac{\dot{\lambda}}{\lambda} \dot{\varphi} = 0$$

$$(2.17b) \quad \ddot{\lambda} - \lambda g(\dot{\varphi}, \dot{\varphi}) = 0.$$

Remark 1. *Note that the first equation (2.17a) is 0-homogeneous with respect to λ and the second equation (2.17b) is one homogeneous with respect to λ . Therefore, the automorphism group $\text{Aut}(\mathcal{C}(M))$ is totally geodesic in $\text{Diff}(\mathcal{C}(M))$ for the $L^2(\mathcal{C}(M), \mathcal{C}(M))$ metric. This is a consequence of the fact that right multiplication acts as an affine isometry on $\mathcal{C}(M)$.*

Let us first recall some useful notions. From the point of view of fluid dynamics, the next definition corresponds to the change of variable between Lagrangian and Eulerian formulations.

Definition 4 (Right-trivialization). Let H be a group and a smooth manifold at the same time, possibly of infinite dimensions, the right-trivialization of TH is the bundle isomorphism $\tau : TH \mapsto H \times T_{\text{Id}}H$ defined by $\tau(h, X_h) \stackrel{\text{def.}}{=} (h, dR_{h^{-1}}X_h)$, where X_h is a tangent vector at point h and $R_{h^{-1}} : H \rightarrow H$ is the right multiplication by h^{-1} , namely, $R_{h^{-1}}(f) = fh^{-1}$ for all $f \in H$.

In fluid dynamics, the right-trivialized tangent vector $dR_{h^{-1}}X_h$ corresponds to the spatial or Eulerian velocity and X_h is the Lagrangian velocity. Importantly, this right-trivialization map is continuous but not differentiable with respect to the variable h^{-1} , and similarly, left multiplication is not differentiable.

Example 5. *For the semi-direct product of groups defined above, we have*

$$(2.18) \quad \tau((\varphi, \lambda), (X_{\varphi}, X_{\lambda})) = ((\varphi, \lambda), (X_{\varphi} \circ \varphi^{-1}, (X_{\lambda} \lambda^{-1}) \circ \varphi^{-1})).$$

We will denote by (v, α) an element of the tangent space of $T_{(\text{Id}, 1)} \text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)$.

As an immediate consequence of Proposition 2.17, we write the geodesic equation in Eulerian coordinates.

Corollary 6 (Geodesic equations in Eulerian coordinates). *After right-trivialization, that is under the change of variable $v \stackrel{\text{def.}}{=} \dot{\varphi} \circ \varphi^{-1}$ and $\alpha \stackrel{\text{def.}}{=} \frac{\dot{\lambda}}{\lambda} \circ \varphi^{-1}$, the geodesic equations read*

$$(2.19) \quad \begin{cases} \dot{v} + \nabla_v v + v\alpha = 0 \\ \dot{\alpha} + \langle \nabla \alpha, v \rangle + \alpha^2 - g(v, v) = 0. \end{cases}$$

Recall now the infinitesimal action associated with a group action.

Definition 5 (Infinitesimal action). For a smooth left action of H on a manifold M and $q \in M$, the infinitesimal action is the map $T_{\text{Id}}H \times M \mapsto TM$ defined by

$$(2.20) \quad \xi \cdot q \stackrel{\text{def.}}{=} \left. \frac{d}{dt} \right|_{t=0} \exp(\xi t) \cdot q \in T_q M$$

where $\exp(\xi t)$ is the Lie exponential, that is the solution to $\dot{h} = dR_h(\xi)$ and $h(0) = \text{Id}$.

Example 7. For $\text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)$ acting on $\Lambda_{1/2}(M)$, the previous definition gives $(v, \alpha) \cdot \rho = -\text{div}(v\rho) + 2\alpha\rho$. Indeed, one has

$$(\varphi(t), \lambda(t)) \cdot \rho = \text{Jac}(\varphi(t)^{-1})(\lambda^2(t)\rho) \circ \varphi^{-1}(t).$$

First recall that $\partial_t \varphi(t) = v \circ \varphi(t)$ and $\partial_t \lambda = \alpha \lambda(t)$. Once evaluated at time $t = 0$ where $\varphi(0) = \text{Id}$ and $\lambda(0) = 1$, the differentiation with respect to φ gives $-\text{div}(v\rho)$ and the second term $2\alpha\rho$ is given by the differentiation with respect to λ .

We now recall the result of [40, Claim of Section 29.21] in a finite dimensional setting. This result presents a standard construction to obtain Riemannian submersions from a transitive group action.

Proposition 8. Consider a smooth left action of Lie group H on a manifold M which is transitive and such that for every $\rho \in M$, the infinitesimal action $\xi \mapsto \xi \cdot \rho$ is a surjective map. Let $\rho_0 \in M$ and a Riemannian metric G on H that can be written as:

$$(2.21) \quad G(h)(X_h, X_h) = g(h \cdot \rho_0)(dR_{h^{-1}} X_h, dR_{h^{-1}} X_h)$$

for $g(h \cdot \rho_0)$ an inner product on $T_{\text{Id}}H$. Let $X_\rho \in T_\rho M$ be a tangent vector at point $h \cdot \rho_0 = \rho \in M$, we define the Riemannian metric \bar{g} on M by

$$(2.22) \quad \bar{g}(\rho)(X_\rho, X_\rho) \stackrel{\text{def.}}{=} \min_{\xi \in T_{\text{Id}}H} g(\rho)(\xi, \xi) \text{ under the constraint } X_\rho = \xi \cdot \rho.$$

where $\xi = X_h \cdot h^{-1}$.

Then, the map $\pi_0 : H \mapsto M$ defined by $\pi_0(h) = h \cdot \rho_0$ is a Riemannian submersion of the metric G on H to the metric \bar{g} on M .

The formal application of this construction in our infinite dimensional situation leads to [11]:

Proposition 9. Let $\rho_0 \in \text{Dens}(M)$ and define the map

$$\begin{aligned} \pi_0 : \text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M) &\mapsto \text{Dens}(M) \\ \pi_0(\varphi, \lambda) &= \varphi_*(\lambda^2 \rho_0). \end{aligned}$$

Then, the map π_0 is a Riemannian submersion of the metric $L^2(M, \mathcal{C}(M))$, where $\mathcal{C}(M)$ is endowed with the cone metric, on the group $\text{Diff}(M) \rtimes_{\Psi} \Lambda_{1/2}(M)$ to the Wasserstein-Fisher-Rao on the space of densities $\text{Dens}(M)$.

Note also that the fibers of the submersion are right-cosets of the subgroup H_0 in H . The proof of the previous proposition is in fact given by the change of variables associated with right-trivialization. Let ρ_0 be a reference density, the application of Proposition 8 gives

$$\begin{aligned} G(\varphi, \lambda)((X_\varphi, X_\lambda), (X_\varphi, X_\lambda)) &= \int_M g(v, v)\rho \, dx + \int_M \alpha^2 \rho \, dx \\ &= \int_M g(X_\varphi \circ \varphi^{-1}, X_\varphi \circ \varphi^{-1})\varphi_*(\lambda^2 \rho_0) \, dx + \int_M (X_\lambda \lambda^{-1})^2 \circ \varphi^{-1} \varphi_*(\lambda^2 \rho_0) \, dx \\ &= \int_M g(X_\varphi, X_\varphi)\lambda^2 \rho_0 \, dx + \int_M X_\lambda^2 \rho_0 \, dx. \end{aligned}$$

Therefore, the metric G is the $L^2(M, \mathcal{C}(M))$ metric with respect to the density ρ_0 . This metric is a weak Riemannian metric in the sense of [17]. This is indeed a smooth Riemannian metric once restricted to $\text{Diff}^s(M) \rtimes_{\Psi} \Lambda^s(M)$ the space of Sobolev maps of order s such that $s > d/2$ essentially because these Sobolev spaces are Hilbert algebras. The same result holds for the Wasserstein-Fisher-Rao metric, as shown in [11]. Note that, in this situation, the horizontal lift (2.22) is well defined.

Proposition 10 (Horizontal lift). Let $\rho \in \text{Dens}^s(\Omega)$ be a smooth density and $X_\rho \in H^s(\Omega, \mathbb{R})$ be a tangent vector at the density ρ . The horizontal lift at $(\text{Id}, 1)$ of X_ρ is given by $(\frac{1}{2}\nabla\Phi, \Phi)$ where Φ is the solution to the elliptic partial differential equation:

$$(2.23) \quad -\text{div}(\rho\nabla\Phi) + 2\Phi\rho = X_\rho.$$

By elliptic regularity, the unique solution Φ belongs to $H^{s+1}(M)$.

Proposition 11. *The WF metric is a weak Riemannian metric on $\text{Dens}^s(M)$.*

The proof is written in [11] but let us explain it briefly. Let us denote by $L(\rho)^{-1}$ the inverse of the elliptic operator defined in Formula (2.23). The WF metric is then given by $\text{WF}(\rho)(X, X) = \int_M L(\rho)^{-1}(X)X \, dx$. Therefore the smoothness of $\text{WF}(\rho)(X, X)$ reduces to the smoothness of $L(\rho)^{-1}$ which again reduces to that of $L(\rho)$ with respect to ρ . Since composition on $\text{Diff}^{s+1}(M)$ is continuous and $H^s(M, \mathbb{R})$ is a Hilbert algebra, we have

Proposition 12. *Let $s > d/2 + 1$ and $k \in \mathbb{N}$. The following map is C^k*

$$\begin{aligned} \pi_0 : \text{Diff}^{s+k+1}(M) \times_{\Psi} \Lambda_{1/2}^{s+k}(M) &\mapsto \text{Dens}^s(M) \\ \pi_0(\varphi, \lambda) &= \varphi_*(\lambda^2 \rho_0). \end{aligned}$$

Note that, unfortunately, the map π_0 for $k = 0$ is only continuous and therefore it is not a proper Riemannian submersion in this context. To make it a proper Riemannian submersion, one can work with Fréchet spaces. Yet, the horizontal lift can be defined on C^1 curves.

Proposition 13. *Let $c : [0, 1] \rightarrow \text{Dens}^s(M)$ a C^1 curve then any horizontal lift $\tilde{c} : [0, 1] \rightarrow \text{Diff}^{s+1}(M) \times_{\Psi} \Lambda^s(M)$ is C^1 .*

Proof. The horizontal lift is given by the curve on the group $\text{Diff}^{s+1}(M) \times_{\Psi} \Lambda^s(M)$ defined by

$$(2.24) \quad \begin{cases} (\varphi, \lambda) = (\varphi_0, \lambda_0), \\ (\dot{\varphi}, \dot{\lambda}) = L(c(t))^{-1}(\dot{c}) \circ (\varphi, \lambda). \end{cases}$$

Since the operator $L(\rho)^{-1}(\dot{c})$ is smooth with respect to c , the result follows since composition is continuous on $\text{Diff}^{s+1}(M) \times_{\Psi} \Lambda^s(M)$. \square

Let us now detail the horizontal spaces and vertical spaces at $(\varphi, \lambda) \in \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ such that $\varphi_*(\lambda \rho_0) = \rho$,

$$(2.25) \quad \text{Vert}_{(\varphi, \lambda)} = \{(v, \alpha) \circ (\varphi, \lambda); (v, \alpha) \in \text{Vect}(M) \times C^\infty(M, \mathbb{R}) \text{ s.t. } \text{div}(\rho v) = \alpha \rho\},$$

and the horizontal space is

$$(2.26) \quad \text{Hor}_{(\varphi, \lambda)} = \left\{ \left(\frac{1}{2} \nabla p, p \right) \circ (\varphi, \lambda); p \in C^\infty(M, \mathbb{R}) \right\}.$$

A direct application of this result is the formal computation of the sectional curvature of the Wasserstein-Fisher-Rao in this smooth setting by applying O'Neill's formula recalled in appendix, see [11]. To recall it hereafter, we need the Lie bracket of right-invariant vector fields on $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$.

Proposition 14. *Let (v_1, α_1) and (v_2, α_2) be two tangent vectors at identity in $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$. Then,*

$$(2.27) \quad [(v_1, \alpha_1), (v_2, \alpha_2)] = ([v_1, v_2], \nabla \alpha_1 \cdot v_2 - \nabla \alpha_2 \cdot v_1),$$

where $[v_1, v_2]$ denotes the Lie bracket on vector fields.

Note that the application of this formula to horizontal vector fields gives $[(\frac{1}{2} \nabla \Phi_1, \Phi_1), (\frac{1}{2} \nabla \Phi_2, \Phi_2)] = (\frac{1}{4} [\nabla \Phi_1, \nabla \Phi_2], 0)$.

Proposition 15. *Let ρ be a smooth positive density on M and X_1, X_2 be two orthonormal tangent vectors at ρ and $\xi_{\Phi_1}, \xi_{\Phi_2}$ be their corresponding right-invariant horizontal lifts on the group. If O'Neill's formula can be applied, the sectional curvature of $\text{Dens}(\Omega)$ at point ρ is non-negative and is given by,*

$$(2.28) \quad K(\rho)(X_1, X_2) = \int_{\Omega} k(x, 1)(\xi_1(x), \xi_2(x)) w(\xi_1(x), \xi_2(x)) \rho(x) \, d\nu(x) + \frac{3}{4} \|\xi_1, \xi_2\|^2$$

where

$$w(\xi_1(x), \xi_2(x)) = g_{\mathcal{C}(M)}(x, 1)(\xi_1(x), \xi_1(x)) g_{\mathcal{C}(M)}(x)(\xi_2(x), \xi_2(x)) - (g_{\mathcal{C}(M)}(x, 1)(\xi_1(x), \xi_2(x)))^2$$

and $[\xi_{\Phi_1}, \xi_{\Phi_2}]^V$ denotes the vertical projection of $[\xi_{\Phi_1}, \xi_{\Phi_2}]$ at identity, $\|\cdot\|$ denotes the norm at identity and $k(x, 1)$ is the sectional curvature of the cone at point $(x, 1)$ in the directions $(\xi_1(x), \xi_2(x))$.

2.5. The corresponding Monge formulation, Monge-Ampère equation and polar decomposition. In this section, we only state formal derivations of the Monge-Ampère equation and the polar decomposition based on the Riemannian submersion obtained above. We do not address the variational analysis of the problem in this article.

The first important consequence of the L^2 metric on the group and the Riemannian submersion is that one can define a Monge formulation of the Wasserstein-Fisher-Rao metric as follows:

$$(2.29) \quad \text{WF}(\rho_0, \rho_1) = \inf_{(\varphi, \lambda)} \{ \|(\varphi, \lambda) - (\text{Id}, 1)\|_{L^2(\rho_0)} : \varphi_*(\lambda^2 \rho_0) = \rho_1 \}$$

It is then possible to derive an equivalent formulation to the classical Monge-Ampère equation.

Proposition 16. *Under the assumption that there exists a smooth minimizer (φ, λ) of (2.29), there exists a function $p \in C^\infty(M, \mathbb{R})$ such that*

$$(2.30) \quad (\varphi(x), \lambda(x)) = \exp_x^{C(M)} \left(\frac{1}{2} \nabla p(x), p(x) \right),$$

where $\exp_{(x,1)}$ denotes the Riemannian exponential map on $\mathcal{C}(M)$.

$$(2.31) \quad (|1 + p(x)|^2 + \frac{1}{4} |\nabla p(x)|^2) \rho_0(x) = \text{Jac}(\varphi)(x) \rho_1(\varphi(x))$$

where

$$\varphi(x) = \exp_x^M \left(\arctan \left(\frac{|\nabla p(x)|}{2(1 + p(x))} \right) \frac{\nabla p(x)}{|\nabla p(x)|} \right).$$

Under the change of variable $z \stackrel{\text{def.}}{=} \log(1 + p)$, the previous equations become

$$(2.32) \quad (1 + |\nabla z|^2) e^{2z} \rho_0 = \det(D\varphi) \rho_1 \circ \varphi$$

and

$$\varphi(x) = \exp_x^M \left(\arctan \left(\frac{1}{2} |\nabla z| \right) \frac{\nabla z(x)}{|\nabla z(x)|} \right).$$

Note that the above result has been established in the Euclidean case in [34, Theorem 6.7] under mild assumptions on the densities. Their result is based on a detailed study of the equivalent Kantorovich formulation which is presented in 2.6. This equivalent formulation is only proven in [34] and [11] in the Euclidean case and it is expected to be true [34, Section 8.5] in the Riemannian case. Moreover, once this equivalence is proven in the Riemannian case, an adaptation of [34, Theorem 6.7] is still needed.

One can state a similar version to Brenier's polar factorization theorem [4] in this context. This can actually be understood as a constrained version of the standard polar factorization, since the diffeomorphism is restricted to be an automorphism. We reformulate the previous proposition by noticing that the decomposition can be extended to the whole cone $\mathcal{C}(M)$.

Proposition 17. *Let $(\phi, \lambda) \in \text{Aut}(\mathcal{C}(M))$ be an element of the automorphism group of the half-densities bundle such that there exists a smooth minimizer to the Monge formulation (2.29) between $\pi_0(\phi, \lambda)$ and ρ_0 . Denote by $C^\infty(\mathcal{C}(M))^{\mathbb{R}_+^*}$ the space of functions $f : \mathcal{C}(M) \mapsto \mathbb{R}$ that satisfy $f(x, \lambda r) = \lambda^2 f(x, r)$ for any $\lambda \in \mathbb{R}_+^*$.*

There exists a couple $(\varphi, \Psi_P) \in \text{Diff}(M) \times (C^\infty(\mathcal{C}(M)))^{\mathbb{R}_+^}$ such that*

$$(2.33) \quad (\phi, \lambda) = \exp^{C(M)}(\nabla \Psi_P) \circ (\varphi, \sqrt{\text{Jac}(\varphi)}),$$

where $(\varphi, \sqrt{\text{Jac}(\varphi)})$ is the natural lift (see Definition 3) of φ in $\text{Aut}(\mathcal{C}(M))$.

The actual polar factorization theorem in the Euclidean or Riemannian cases states a general result about existence and uniqueness. In the Euclidean case, an equivalent result follows from [34, Theorem 6.7]. Moreover, the existence and uniqueness is guaranteed under mild conditions on the densities: for instance, it is the case if the two densities are absolutely continuous with respect to

the Lebesgue measure. The extension of this result to the present setting is out of the scope of this paper. Although it appears to be an expected result, to the best of our knowledge, this is not a direct consequence of [37] or [34]. This result will be studied elsewhere.

2.6. The Kantorovich Formulation. From a variational point of view, it is important to derive a relaxation of the Monge formulation. It is of interest to understand first the simple situation when the source and target measures are single Dirac masses and when M is a convex and compact domain in the Euclidean space [12]. This also applies to the case of a Riemannian manifold.

Proposition 18. *Let M be a convex and compact domain in \mathbb{R}^d with the Euclidean metric. Let $m_1\delta_{x_1}$ and $m_2\delta_{x_2}$ be two Dirac masses with $x_1, x_2 \in M$ and $m_1, m_2 \in \mathbb{R}_+^*$.*

If $\frac{1}{2}d(x_1, x_2) < \pi/2$, there exists a unique geodesic which is $m(t)\delta_{x(t)}$ where $(x(t), m(t))$ is the geodesic in $M \times \mathbb{R}_+^$ with the cone metric between (x_1, m_1) and (x_2, m_2) .*

If $\frac{1}{2}d(x_1, x_2) > \pi/2$, there exists a unique geodesic which is $m_1(t)\delta_{x_1} + m_2(t)\delta_{x_2}$ where $m_1(t) = m_1(1-t)^2$ and $m_2(t) = m_2t^2$ which describe the geodesics between (x_i, m_i) and the vertex of the cone denoted by S for $i = 1, 2$.

If $\frac{1}{2}d(x_1, x_2) = \pi/2$, there exists an infinite number of geodesics which are convex combinations of the two first types defined above.

The important point is that passing to the case of measures the angle of the cone has been divided by 2. This is because we are not looking for geodesics on $M \times \mathbb{R}_+^*$ but on the space of measures on M and thus the cost between Dirac masses has to be convex. The generalization to any positive Radon measures gives a Kantorovich relaxation: For two given positive Radon measures ρ_1, ρ_2 , we define, for $\mathcal{M}(M^2)$ the space of positive Radon measures on M^2 ,

$$(2.34) \quad \Gamma(\rho_1, \rho_2) \stackrel{\text{def.}}{=} \left\{ (\gamma_1, \gamma_2) \in (\mathcal{M}_+(M^2))^2 : p_*^1 \gamma_1 = \rho_1, p_*^2 \gamma_2 = \rho_2 \right\},$$

where p^1 and p^2 denote the projection on the first and second factors of the product M^2 . The variational problem associated with the Wasserstein-Fisher-Rao distance is

$$(2.35) \quad \text{WF}(\rho_1, \rho_2)^2 = \inf_{(\gamma_1, \gamma_2) \in \Gamma(\rho_1, \rho_2)} \int_{M^2} d^2 \left(\left(x, \frac{d\gamma_1}{d\gamma} \right), \left(y, \frac{d\gamma_2}{d\gamma} \right) \right) d\gamma(x, y),$$

where d^2 is the square of the cone distance given in definition 1 and γ is any measure that dominates ρ_1 and ρ_2 . The fact that the integration result does not depend on this choice is due to the 1-homogeneity of d^2 in function of the mass. We also state the dual formulation:

Proposition 19. *It holds*

$$(2.36) \quad \text{WF}^2(\rho_0, \rho_1) = \sup_{(\phi, \psi) \in C(M)^2} \int_M \phi(x) d\rho_0 + \int_M \psi(y) d\rho_1$$

subject to $\forall(x, y) \in M^2$,

$$(2.37) \quad \begin{cases} \phi(x) \leq 1, & \psi(y) \leq 1, \\ (1 - \phi(x))(1 - \psi(y)) \geq \cos^2(|x - y|/2). \end{cases}$$

For numerical purposes, this formulation can be further reduced with a change of variable, by taking the logarithm of the multiplicative constraint (2.37) and using duality again,

$$\begin{aligned} \text{WF}^2(\rho_1, \rho_2) = \inf_{\gamma \in \mathcal{M}_+(M)} & \text{KL}(\text{Proj}_*^1 \gamma, \rho_1) + \text{KL}(\text{Proj}_*^2 \gamma, \rho_2) \\ & - \int_{M^2} \gamma(x, y) \log(\cos^2(d(x, y)/2 \wedge \pi/2)) dx dy \end{aligned}$$

with

$$\text{KL}(\mu, \nu) = \int \frac{d\mu}{d\nu} \log \left(\frac{d\mu}{d\nu} \right) d\nu + |\nu| - |\mu|,$$

which is known as the Kullback-Leibler divergence. This particular formulation of unbalanced optimal transport and its generalizations have been intensively developed in [34], where generalizations

of this metric are studied in spaces such as Hausdorff topological spaces endowed with a (pseudo) distance satisfying mild conditions. More interestingly in our situation where the underlying is a finite dimensional Riemannian manifold is the existence of solutions to the dual problem which are proven in [34].

3. THE EULER-ARNOLD EQUATION AND THE H^{div} RIGHT-INVARIANT METRIC ON THE DIFFEOMORPHISM GROUP

A prototypical example of the situation we are interested in is the case of the incompressible Euler equation. As shown by Arnold [1], the incompressible Euler equation is the Euler-Lagrange equation of geodesics on the group of volume preserving diffeomorphisms for the L^2 right-invariant metric. Let us motivate this section with the following simple proposition whose proof is omitted.

Proposition 20. *Consider a Riemannian submersion built as in Proposition 8. Let H_0 be the isotropy subgroup of ρ_0 , then, considering H_0 as a Riemannian submanifold of H and denoting G_{H_0} its induced metric, G_{H_0} is a right-invariant metric on H_0 .*

It is therefore interesting to start with this point of view, a right-invariant metric on a group of diffeomorphisms and to write the corresponding geodesic equations. The right-invariance implies that the geodesic equation can be written on the Lie algebra or the tangent space at identity ($T_{\text{Id}}G$ for a Lie group G). This is the case of the usual formulation incompressible Euler equation as in Equation 3.11 and this is the point of view taken in [1]. Actually, it is a particular case of Lagrangians that can be written by a change of variable only at the tangent space of identity $\mathfrak{g} \stackrel{\text{def.}}{=} T_{\text{Id}}G$, the Lie algebra under the constraint of the flow equation. This class of Lagrangians leads to the so-called Euler-Poincaré or Euler-Arnold equation when the Euler-Lagrange equation is written on $T_{\text{Id}}G$. We describe the derivation of this Euler-Lagrange equation in the next paragraph.

3.1. The Euler-Arnold equation. A short proof of the derivation of this equation is given in [43, Theorem 3.2] in the case of a kinetic energy but let us underline that the same equation holds true for general Lagrangians that are right-invariant. We will need the definition of the adjoint and co-adjoint operators:

Definition 6. Let G be a Lie group and $h \in G$, the adjoint operator $\text{Ad}_h : G \times \mathfrak{g} \mapsto \mathfrak{g}$ is defined by

$$(3.1) \quad \text{Ad}_h(v) \stackrel{\text{def.}}{=} dL_h \cdot dR_{h^{-1}}(v).$$

Then, Ad_h^* is the adjoint of Ad_h defined by duality on \mathfrak{g} .

Their corresponding differential map at Id are respectively denoted by ad and ad^* .

Let G be a Lie group, and $\mathcal{L} : TG \mapsto \mathbb{R}$ be a Lagrangian which satisfies the following property,

$$(3.2) \quad \mathcal{L}(g, \dot{g}) = \mathcal{L}(\text{Id}, dR_{g^{-1}}(\dot{g})),$$

The reduced Lagrangian is $\ell : \mathfrak{g} \mapsto \mathbb{R}$ defined by $\ell(v) = \mathcal{L}(\text{Id}, v)$ for $v \in \mathfrak{g}$.

Thus, the variational problem for a reduced Lagrangian reads

$$(3.3) \quad \inf \int_0^1 \ell(v) dt \quad \text{subject to} \quad \begin{cases} \dot{g} = dR_g(v) \\ g(0) = g_0 \in G \text{ and } g(1) = g_1 \in G. \end{cases}$$

In order to compute the Euler-Lagrange equation for (3.3), one needs to compute the variation of v in terms of the variation of g . It is given by $\dot{w} - \text{ad}_v w$ for any path $w(t) \in T_{\text{Id}}G$, therefore, the Euler-Lagrange equation reads

$$(3.4) \quad \left(\frac{d}{dt} + \text{ad}_v^* \right) \frac{\delta \ell}{\delta v} = 0.$$

3.2. The particular case of H^{div} and the Camassa-Holm equation. When the Lagrangian is a kinetic energy, $\ell(v) = \frac{1}{2}\langle v, Lv \rangle$, which will be also denoted by $\frac{1}{2}\|v\|_{\mathfrak{g}}^2$, where $L : \mathfrak{g} \mapsto \mathfrak{g}$ is a quadratic form and $\langle \cdot, \cdot \rangle$ denotes the dual pairing, one has $\frac{\delta \ell}{\delta v} = Lv$ and Lv is the so-called momentum. Then, the critical curves are determined by their initial conditions $(g(0), \dot{g}(0))$ and the Euler-Poincaré equation (3.4). In the context of infinite dimensional Riemannian manifolds enjoying a group structure, this equation is called the Euler-Arnold equation. Let us compute more explicitly the Euler-Arnold equation and detail the expression of the adjoint Ad_h^* which acts on 1-forms. Let \mathfrak{m} be a 1-form density, then $\text{Ad}_\varphi^*(\mathfrak{m}) = D\varphi^T(\mathfrak{m} \circ \varphi) \text{Jac}(\varphi)$ and therefore the differentiation w.r.t. φ gives

$$(3.5) \quad \text{ad}_u^*(\mathfrak{m}) = \text{div}(u)\mathfrak{m} + Du^T \cdot \mathfrak{m} + D\mathfrak{m} \cdot u.$$

Thus, the Euler-Arnold equation reads

$$(3.6) \quad \begin{cases} \partial_t \mathfrak{m}_t + \text{div}(u_t)\mathfrak{m}_t + Du_t^T \cdot \mathfrak{m}_t + D\mathfrak{m}_t \cdot u_t = 0 \\ Lu_t = \mathfrak{m}_t, \end{cases}$$

where L is the differential operator defining the metric. A more geometrical way of writing this equation is the following,

$$(3.7) \quad \partial_t \mathfrak{m}_t + \mathcal{L}_{u_t} \mathfrak{m}_t + \text{div}(u_t)\mathfrak{m}_t = 0,$$

or alternatively

$$(3.8) \quad (\partial_t + \mathcal{L}_{u_t})(\mathfrak{m}_t \otimes \mu) = 0,$$

together with the relation $Lu_t = \mathfrak{m}_t$.

Let us present some important examples in fluid dynamics of the Euler-Arnold equation: For the L^2 metric in one dimension, $Lu = u$, one has

$$(3.9) \quad \partial_t u + 3\partial_x u u = 0,$$

which is the inviscid Burgers equation.

For the H^{div} metric in one dimension, $Lu = u - \partial_{xx}u$, one has the Camassa-Holm equation (actually when $a = b = 1$)

$$(3.10) \quad a^2 \partial_t u - b^2 \partial_{txx} u u + 3a^2 \partial_x u u - 2b^2 \partial_{xx} u \partial_x u - b^2 \partial_{xxx} u u = 0.$$

The Korteweg-de Vries equation can also be understood in this setting on a central extension of the group $\text{Diff}(S_1)$. In the case where $G = \text{Diff}_\mu(M)$ is the group of volume preserving diffeomorphisms, the Euler-Arnold equation is the incompressible Euler equation

$$(3.11) \quad \partial_t u + \nabla_u u = -\nabla p, \text{div}(u) = 0.$$

Let us detail the case of the $H^{\text{div}}(\mathbb{T}_d)$ where $\mathbb{T}^d \stackrel{\text{def.}}{=} \mathbb{R}^d / \mathbb{Z}^d$ metric which is detailed in [27, Theorem A.1]. The differential operator takes the form $Lu = a^2 u + b^2 \nabla \text{div}(u)$ which gives

$$(3.12) \quad \partial_t Lu + a^2 \left(\text{div}(u)u + \frac{1}{2} \nabla \langle u, u \rangle + Du \cdot u \right) + b^2 \left(\text{div}(u) \nabla \text{div}(u) + Du^T \cdot \nabla \text{div}(u) + D[\nabla \text{div}(u)] \cdot u \right) = 0.$$

More geometrically on a Riemannian manifold (M, g) , this equation can be written as

$$(3.13) \quad \partial_t Lu + a^2 \left(\text{div}(u)u^\flat + d\langle u, u \rangle + \iota_u du^\flat \right) + b^2 \left(\text{div}(u) d\delta u^\flat + d\iota_u d\delta u^\flat \right) = 0.$$

where the notations \flat correspond to lowering the indices. More precisely, if $u \in \chi(M)$ then u^\flat is the 1-form defined by $v \mapsto g(u, v)$. The notation δ is the formal adjoint to the exterior derivative d and ι is the insertion of vector fields which applies to forms.

In Section 5, we rewrite the Camassa-Holm equation (5.6) as an incompressible Euler equation formulated as (3.11).

3.3. Smoothness of the flow and metric properties. For the sake of completeness, we recall in this section some previous works concerning the Camassa-Holm equation as a geodesic equation on the group of diffeomorphisms for the H^{div} right-invariant metric. For instance, the reader can refer to [42] or [31] where much more results are proven. Using the Ebin and Marsden approach in [17], the geodesic equation can be interpreted as an ODE on a Hilbert space. For that purpose, one needs to consider the geodesic equation on a sufficiently regular Sobolev space H^s , for $s > d/2 + 1$. The key point consists in switching from Eulerian to Lagrangian coordinates which enables to avoid the loss of regularity of the geodesic equation. The same strategy applies for the H^{div} metric. Geodesic completeness does not hold since there exists smooth initial conditions for the Camassa-Holm equation such that the solutions blow up in finite time. As a consequence, metric completeness does not hold either (since it would imply geodesic completeness).

Theorem 21 (Ebin and Marsden). *Let M be a compact manifold without boundary. On $\text{Diff}^s(M)$ for $s > d/2 + 1$, the H^{div} right-invariant metric is a smooth and weak Riemannian metric. The exponential map is smooth and locally defined on $T\text{Diff}^s(M)$.*

Remark 2. *Although this theorem is stated in a smooth Sobolev setting, at least H^s for $s > d/2 + 1$, the result is not trivial since the composition $\text{Diff}^s(M) \times \text{Diff}^s(M) \mapsto \text{Diff}^s(M)$ defined by $(\varphi, \psi) \mapsto \varphi \circ \psi$ is smooth w.r.t. φ (because linear) but it is not smooth w.r.t. ψ . Therefore the fact that the metric defined below in (3.14) is smooth on $\text{Diff}^s(M)$ is not directly given by working in a smooth enough Sobolev setting.*

Sketch of proof. First recall that $H^s(M)$ is embedded in $C^1(M)$ for $s > d/2 + 1$ and it is a Hilbert algebra if $s > d/2$ which means that the product of two functions is a bounded bilinear operation. The idea consists in writing the H^{div} metric in Lagrangian coordinates for $X_\varphi \in H^s(M)$ a tangent vector at $\varphi \in \text{Diff}^s(M)$,

$$(3.14) \quad G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi \circ \varphi^{-1}|^2 + b^2 \text{div}(X_\varphi \circ \varphi^{-1})^2 d\mu.$$

Note that the differentiation of the composition can be written as

$$(3.15) \quad D(X_\varphi \circ \varphi^{-1}) = (DX_\varphi \cdot [D\varphi]^{-1}) \circ \varphi^{-1},$$

where the symbol \cdot denotes matrix multiplication. By the change of variable by φ , one has

$$(3.16) \quad G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi|^2 \text{Jac}(\varphi) + b^2 (\text{Tr}(DX_\varphi \cdot [D\varphi]^{-1}))^2 \text{Jac}(\varphi) d\mu.$$

Therefore, the metric only involves scalar multiplication, matrix inversion, matrix multiplications with values in $H^{s-1}(M)$ which are smooth operations since $H^{s-1}(M)$ is a Hilbert algebra for $s > d/2 + 1$. Thus, the metric is smooth as well as the exponential map and one can apply general results from Riemannian geometry in infinite dimensions [32]. \square

Consequently, the geodesic equation can be interpreted as an ODE in $H^s(M, M)$. The Gauss lemma is valid in this strong H^s topology which ensures in particular that geodesics are length minimizing among all curves that stay in a (small enough) H^s neighborhood. However, this is *not* enough to prove that the associated geodesic distance is non degenerate since the minimizing geodesic can escape this neighborhood for arbitrarily small energy. This is what happens for the right-invariant metric $H^{1/2}$ on the circle S_1 where the metric is degenerate although there exists a smooth exponential map similarly to our case (see [18]).

In [41], Michor and Mumford proved that the right-invariant metric L^2 on the group of diffeomorphisms leads to a degenerate distance, i.e. between any two diffeomorphisms, the infimum of the path lengths joining them is zero. This is not the case for the H^{div} right-invariant metric, the following theorem was also proven in their article.

Theorem 22 (Michor and Mumford). *The distance on $\text{Diff}(M)$ induced by the H^{div} right-invariant metric is non-degenerate. Namely, between two distinct diffeomorphisms the infimum of the lengths of the paths joining them is strictly positive.*

4. A RIEMANNIAN SUBMANIFOLD POINT OF VIEW ON THE H^{div} RIGHT-INVARIANT METRIC: THE WASSERSTEIN-FISHER-RAO METRIC

The Riemannian submersion $\pi_0 : \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M) \mapsto \text{Dens}(M)$ defined in Proposition 9 enables to look at the equivalent problem to the incompressible Euler equation. The fiber of the Riemannian submersion at ρ_0 is

$$(4.1) \quad \pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \lambda) \in \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M) : \varphi_*(\lambda^2 \rho_0) = \rho_0\}$$

The constraint $\varphi_*(\lambda^2 \rho_0) = \rho_0$ can be made explicit and we get

$$(4.2) \quad \pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \sqrt{\text{Jac}(\varphi)}) \in \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M) : \varphi \in \text{Diff}(M)\}.$$

This is the space of all natural lifts of diffeomorphisms of M as introduced in Definition 3. Now, the vertical space at point $(\varphi, \sqrt{\text{Jac}(\varphi)}) \in \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ is

$$(4.3) \quad \text{Ker}(d\pi_0) = \{(v, \alpha) \cdot (\varphi, \sqrt{\text{Jac}(\varphi)}) : \text{div } v = 2\alpha\},$$

and equivalently

$$(4.4) \quad \text{Ker}(d\pi_0) = \left\{ \left(v, \frac{1}{2} \text{div } v \right) \cdot (\varphi, \sqrt{\text{Jac}(\varphi)}) : v \in \text{Vect}(M) \right\}.$$

The metric on $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ restricted to $\text{Ker}(d\pi_0)$ reads

$$(4.5) \quad G(v, \text{div } v) = \int_M |v|^2 d\mu + \frac{1}{4} \int_M |\text{div } v|^2 d\mu.$$

Therefore, on $\text{Diff}(M) \simeq \text{Ker}(d\pi_0)$, the induced metric is a right-invariant H^{div} metric. As an immediate application, we retrieve theorem 22.

Corollary 23. *The distance on $\text{Diff}(M)$ with the right-invariant metric H^{div} is non degenerate.*

Proof. Let $\varphi_0, \varphi_1 \in \text{Diff}(M)$ be two diffeomorphisms and a path c joining them. The length of the path c for the right-invariant metric H^{div} is equal to the length of the lifted path \tilde{c} in $\text{Aut}(\mathcal{C}(M))$. Since $L^2(M, \mathcal{C}(M))$ is a Hilbert manifold, the length of the path \tilde{c} is bounded below by the length of the geodesic joining the natural lifts of φ_0 and φ_1 in $L^2(M, \mathcal{C}(M))$. Therefore, it leads to

$$(4.6) \quad d_{H^{\text{div}}}(\varphi_0, \varphi_1) \geq d_{L^2(M, \mathcal{C}(M))} \left((\varphi_0, \sqrt{\text{Jac}(\varphi_0)}), (\varphi_1, \sqrt{\text{Jac}(\varphi_1)}) \right).$$

If $d_{H^{\text{div}}}(\varphi_0, \varphi_1) = 0$ then $d_{L^2(M, \mathcal{C}(M))} \left((\varphi_0, \sqrt{\text{Jac}(\varphi_0)}), (\varphi_1, \sqrt{\text{Jac}(\varphi_1)}) \right) = 0$ which implies $\varphi_0 = \varphi_1$. \square

Remark 3 (The Fisher-Rao metric). *In [27], it is shown that the \dot{H}^1 right-invariant metric descends to the Fisher-Rao metric on space of densities. Let us explain why the situation differs from our: It is well known that a left action of a group endowed with a right-invariant metric will induce on the orbit a Riemannian metric for which the action is a Riemannian submersion. However, Khesin et al. do not consider a left action, but a right action on the space of densities: More precisely, if a reference density ρ is chosen, the map they considered is*

$$\begin{aligned} \text{Diff}(M) &\rightarrow \text{Dens}(M) \\ \varphi &\mapsto \varphi^* \rho. \end{aligned}$$

Using the inverse map, the situation is equivalent to a left action of a group of diffeomorphisms endowed with a left invariant metric. In such a situation, the descending metric property has to be checked [27, Proposition 2.3].

Their result can be read from our point of view: The \dot{H}^1 metric is $\frac{1}{4} \int_M |\text{div } v|^2 d\mu$ and it corresponds to the case where $a = 0$. It thus leads to a degenerate metric on the group. Viewed in the ambient space $L^2(M, \mathcal{C}(M))$, the projection on the bundle component is a (pseudo-) isometry from $L^2(M, \mathcal{C}(M))$ (endowed with this pseudo-metric) to the space of densities since $a = 0$. Moreover, on the space of densities which lie in the image of the projection, that is, the set of probability densities, the projected metric is the Fisher-Rao metric.

We now consider $\text{Diff}(M)$ endowed with the right-invariant H^{div} metric as a submanifold of $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ and write the geodesic equations in this setting. As is standard for the incompressible Euler equation, the constraint is written in Eulerian coordinates and the corresponding geodesic are written hereafter.

Theorem 24. *The geodesic equations on the fiber $\pi_0^{-1}(\{\rho_0\})$ as a Riemannian submanifold of $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ endowed with the metric $L^2(M, \mathcal{C}(M))$ can be written in Lagrangian coordinates*

$$(4.7) \quad \begin{cases} \frac{D}{Dt} \dot{\varphi} + 2\frac{\dot{\lambda}}{\lambda} \dot{\varphi} = -\nabla^g P \circ \varphi \\ \ddot{\lambda} - \lambda g(\dot{\varphi}, \dot{\varphi}) = -2\lambda P \circ \varphi, \end{cases}$$

with a function $P : M \mapsto \mathbb{R}$.

In Eulerian coordinates, the geodesic equations read

$$(4.8) \quad \begin{cases} \dot{v} + \nabla_v^g v + 2v\alpha = -\nabla^g P \\ \dot{\alpha} + \langle \nabla \alpha, v \rangle + \alpha^2 - g(v, v) = -2P, \end{cases}$$

where $\alpha = \frac{\dot{\lambda}}{\lambda} \circ \varphi^{-1}$ and $v = \partial_t \varphi \circ \varphi^{-1}$.

This submanifold point of view leads to a generalization of [26, Theorem A.2] on the sectional curvature of $\text{Diff}(M)$ which has been computed and studied in [26]. The authors show that the curvature of $\text{Diff}(S_1)$ can be written using the Gauss-Codazzi formula and they show the explicit embedding in a semi-direct product of groups similar to our situation.

As mentioned above, we consider $\text{Diff}(M)$ as a submanifold of $L^2(M, \mathcal{C}(M))$. The second fundamental form can be computed as in the case of the incompressible Euler equation.

Proposition 25. *Let U, V be two smooth right-invariant vector fields on $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ that can be written as $U(\varphi, \lambda) = (u, \alpha) \circ (\varphi, \lambda)$ and $V(\varphi, \lambda) = (v, \beta) \circ (\varphi, \lambda)$. The second fundamental form for the isometric embedding $\text{Diff}(M) \hookrightarrow L^2(M, \mathcal{C}(M))$ is*

$$(4.9) \quad \text{II}(U, V) = (\nabla P \circ \varphi, 2\lambda P \circ \varphi),$$

where $P = (2\text{Id} - \Delta)^{-1} A(\nabla_{(u, \alpha)}(v, \beta))$ is the unique solution of the elliptic PDE (2.23)

$$(4.10) \quad (2\text{Id} - \Delta)(P) = A(\nabla_{(u, \alpha)}(v, \beta)).$$

with $A(w, \gamma) \stackrel{\text{def}}{=} -\text{div}(w) + \gamma$. Using the explicit expression of $\nabla_{(u, \alpha)}(v, \beta)$ the elliptic PDE reads

$$(4.11) \quad (2\text{Id} - \Delta)(P) = -\text{div}(\nabla_u v + \beta u + \alpha v) + 2\langle \nabla \beta, u \rangle - 2g(u, v) + 2\alpha\beta.$$

Proof. By right-invariance of the metric, it suffices to treat the case where $(\varphi, \lambda) = \text{Id}$. The orthogonal projection is the horizontal lift defined in Proposition 10. Therefore, we compute the infinitesimal action of $\nabla_{(u, \alpha)}(v, \beta)$ on the volume form which is given by the linear operator A and we consider its horizontal lift $(\nabla P, 2P)$ given by Proposition 10. Therefore, the orthogonal part of $\nabla_{(u, \alpha)}(v, \beta)$ to the tangent space of $\text{Diff}(M)$ at Id is given by $(\nabla P, 2\lambda P)$. By right-invariance, the orthogonal projection at (φ, λ) is given by $(\nabla P \circ \varphi, 2\lambda P \circ \varphi)$.

From Proposition 2, one has

$$(4.12) \quad \nabla_{(u, \alpha)}(v, \beta) = (\nabla_u v + \beta u + \alpha v, \langle \nabla \beta, u \rangle - g(u, v) + \alpha\beta),$$

and Formula (4.11) follows. \square

We can then state the Gauss-Codazzi formula applied in our context.

Proposition 26. *Let U, V be two smooth right-invariant vector fields on $\text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ written as $U(\varphi, \lambda) = (u, \alpha) \circ (\varphi, \lambda)$ and $V(\varphi, \lambda) = (v, \beta) \circ (\varphi, \lambda)$. The sectional curvature of $\text{Diff}(M)$ endowed with the right-invariant H^{div} metric is*

$$(4.13) \quad \langle R_{\text{Diff}(M)}(U, V)V, U \rangle = \langle R_{L^2(M, \mathcal{C}(M))}(U, V)V, U \rangle + \langle \text{II}(U, U), \text{II}(V, V) \rangle - \langle \text{II}(U, V), \text{II}(U, V) \rangle.$$

where II is the second fundamental form (4.9) and

$$(4.14) \quad \langle R_{L^2(M, \mathcal{C}(M))}(U, V)V, U \rangle = \int_M \langle R_{\mathcal{C}(M)}(u, v)v, u \rangle \circ (\varphi, \lambda) \, d\mu,$$

where $(\varphi, \lambda) \in \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$.

Proof. The only remaining point is the computation of the sectional curvature of $L^2(M, \mathcal{C}(M))$ which is done in Freed and Groisser's article [20]. \square

Note that the sectional curvature of $L^2(M, \mathcal{C}(M))$ vanishes if $M = S_n$ since $\mathcal{C}(M) = \mathbb{R}^{n+1}$, which is the case for the one-dimensional Camassa-Holm equation. However, for $M = T_n$, the flat torus, the sectional curvature of $\mathcal{C}(M)$ is non-positive and bounded below by -1 and thus the sectional curvature of $L^2(M, \mathcal{C}(M))$ is non-positive.

5. APPLICATIONS

The point of view developed above provides an example of an isometric embedding of the group of diffeomorphisms endowed with the right-invariant H^{div} metric in an L^2 space such as $L^2(M, N)$, here with $N = \mathcal{C}(M)$. This may bring additional information to the understanding of the corresponding fluid dynamic equation. Let us detail the one-dimensional situation, that is when $M = S_1$. In such a case, $\mathcal{C}(M)$ is flat. Note that, when the diameter of S_1 is less than π , the cone is a usual cone embedded in \mathbb{R}^3 . If S_1 is the Euclidean circle of radius 1 in \mathbb{R}^2 , $\mathcal{C}(M)$ is isometric to \mathbb{R}^2 and let us detail this particular case. We use this isometry to define, following Proposition 1

$$(5.1) \quad \mathcal{M} : \text{Diff}(S_1) \rightarrow \text{Diff}(S_1) \times_{\Psi} \Lambda_{1/2}(S_1) \subset L^2(S_1, \mathbb{R}^2)$$

$$(5.2) \quad \varphi \mapsto (\varphi, \sqrt{\varphi'}) = \sqrt{\varphi'} e^{i\varphi}.$$

Then, the solutions of the Camassa-Holm are geodesics on the isotropy subgroup explicitly written in (4.2). This submanifold can be also written as $\{(z, r) \in L^2(S_1, \mathbb{R}^2) : z' - r^2 = 0\}$ and one can write the geodesic equations in this coordinate system, which is done in Appendix B. Note that the map \mathcal{M} is very similar to a Madelung transform which maps solutions of the Schrödinger equation to solutions of a compressible Euler type of hydrodynamical equation. This is actually very similar to the map \mathcal{M} which will be shown to map solutions of the Camassa-Holm equation to solutions of the incompressible Euler equation for a density which has a singularity at 0. Note that this discussion generalizes directly to the case $M = S_n$ since $\mathcal{C}(M)$ can be identified to \mathbb{R}^{n+1} . In the general case, we are left with the geometry of the cone, and therefore, the map \mathcal{M} maps solutions of the geodesic equations on the diffeomorphisms group for the right-invariant H^{div} metric to solutions of the incompressible Euler equation on the $\mathcal{C}(M)$ for a density which has a singularity at the apex of the cone.

In this section, we state the correspondence given by this Madelung transform and we apply this Riemannian submanifold point of view to derive a similar result to Brenier, namely that smooth geodesics are length minimizing for short times.

5.1. The Camassa-Holm equation as an Euler equation on the cone. Formula (5.4) is close to the incompressible Euler equation in Lagrangian coordinates. However, the geodesic equation (5.4) is apparently written on the space of maps $M \mapsto \mathcal{C}(M)$. Since $\text{Aut}(\mathcal{C}(M)) \subset \text{Diff}(\mathcal{C}(M))$, this geodesic equation can be expected to be a geodesic equation on the group of diffeomorphism of the cone. This is indeed the case The second equation in (4.7) being linear in λ and the first equation being 0 homogeneous in λ , the geodesic equation can be rewritten as

$$(5.3) \quad \begin{cases} \frac{D}{Dt} \dot{\varphi} + 2 \frac{\dot{\lambda}}{\lambda} \dot{\varphi} = -\nabla^g P \circ \varphi \\ \dot{\lambda} r - \lambda r g(\dot{\varphi}, \dot{\varphi}) = -2\lambda r P \circ \varphi. \end{cases}$$

However, the diffeomorphisms $(\varphi, \lambda) \in \text{Diff}(\mathcal{C}(M))$ do not preserve the Riemannian volume measure on $\mathcal{C}(M)$ but another density which has a singularity at $\mathcal{S} \in \mathcal{C}(M)$ the apex of the cone. This amounts to rewrite the left action defined by π in (2.3) as a pushforward of a density on the cone.

Corollary 27. *On the group of diffeomorphisms of the cone, the geodesic equation can be written*

$$(5.4) \quad \frac{D}{Dt}(\dot{\varphi}, \dot{\lambda} r) = -\nabla \Psi_P \circ (\varphi, \lambda r),$$

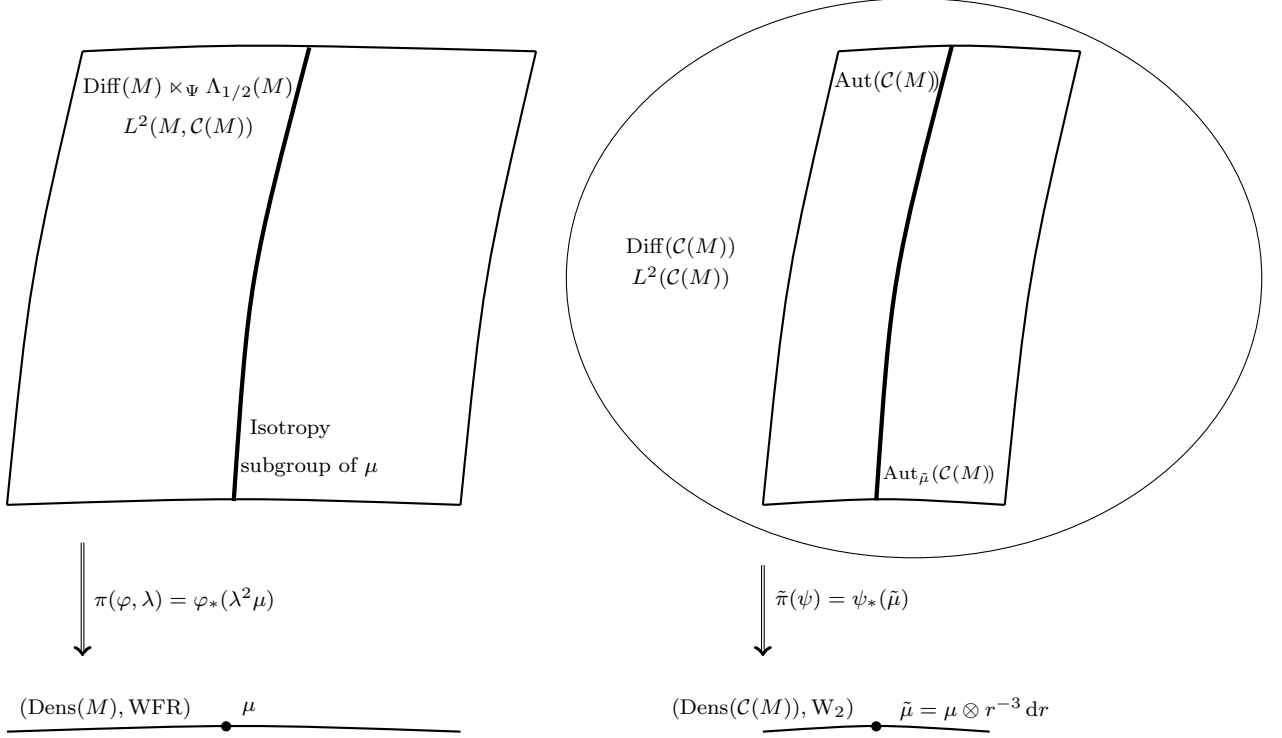


FIGURE 1. The group $\text{Aut}(\mathcal{C}(M)) = \text{Diff}(M) \times_{\Psi} \Lambda_{1/2}(M)$ is totally geodesic in $(\text{Diff}(\mathcal{C}(M)), L^2(\mathcal{C}(M)))$ and there is a Riemannian submersion of the automorphism group of the cone endowed with the L^2 metric to the space of positive densities on M endowed with the Wasserstein-Fisher-Rao metric.

where $\Psi_P(x, r) \stackrel{\text{def.}}{=} r^2 P(x)$. Moreover, the diffeomorphisms (φ, λ) preserve the measure $\tilde{\mu} \stackrel{\text{def.}}{=} r^{-3} dr \otimes d\mu$ where μ denotes the volume form on M .

In other words, a solution (φ, λ) of (5.4) is a solution of the incompressible Euler equation for the density $r^{-3-d} \text{dvol}$ where dvol is the volume form on the cone $\mathcal{C}(M)$ and d is the dimension of M .

Proof. The geodesic equations (5.3) can be rewritten in the form (5.4) since a direct computation gives $\nabla \Psi_P = (\nabla^g P, 2rP)$.

The only remaining point is that (φ, λ) preserves the measure $r^{-3} d\mu \otimes dr$ where $d\mu \otimes dr$ is the product measure on $\mathcal{C}(M)$, if the relation $\lambda = \sqrt{\text{Jac}(\varphi)}$ holds. Indeed, the volume form $r^\alpha d\mu \otimes dr$ is preserved by (φ, λ) if and only if the following equality is satisfied $(\lambda r)^\alpha \lambda \text{Jac}(\varphi) = r^\alpha$, equivalently $\lambda^{\alpha+3} = 1$. It is the case if and only if $\alpha = -3$. \square

In particular, this corollary shows that $\text{Stab}_\mu(\text{Aut}(\mathcal{C}(M))) = \text{Aut}(\mathcal{C}(M)) \cap \text{SDiff}_{\tilde{\mu}}(\mathcal{C}(M))$. In remark 1, we mentioned that $\text{Aut}(\mathcal{C}(M))$ is a totally geodesic subspace of $\text{Diff}(\mathcal{C}(M))$, which explains the fact that the geodesic equation on $\text{Stab}_\mu(\text{Aut}(\mathcal{C}(M)))$ is actually a geodesic equation on $\text{SDiff}_{\tilde{\mu}}(\mathcal{C}(M))$. We illustrate this situation in Figure 1.

The Camassa-Holm equation in complex coordinates: Since the map \mathcal{M} is a transformation that uses the complex exponential, it may be interesting to rewrite the equation in complex coordinates.

Theorem 28. *The Camassa-Holm equation*

$$(5.5) \quad a^2 \partial_t u - b^2 \partial_{txx} u u + 3a^2 \partial_x u u - 2b^2 \partial_{xx} u \partial_x u - b^2 \partial_{xxx} u u = 0.$$

for $a = 1$ and $b = \frac{1}{2}$ reads

$$(5.6) \quad \begin{cases} \partial_t u - \frac{1}{4} \partial_{txx} u u + 3 \partial_x u u - \frac{1}{2} \partial_{xx} u \partial_x u - \frac{1}{4} \partial_{xxx} u u = 0 \\ \partial_t \varphi(t, x) = u(t, \varphi(t, x)). \end{cases}$$

It has the following equivalent formulation: Let $Z \in L^2(S_1, \mathcal{S})$ be written in polar coordinates $Z = r e^{i\psi}$,

$$(5.7) \quad \ddot{Z} = \frac{1}{2} P Z + \frac{i}{4Z} P'.$$

where P is given by the unique solution to

$$(5.8) \quad \left(\frac{1}{r^2} P' \right)' - r^2 P - \frac{4}{r} \psi' \dot{r} = 0,$$

which is an elliptic variational problem. The corresponding change of variable is given by

$$(5.9) \quad Z = \sqrt{\text{Jac}(\varphi)} e^{i\varphi}.$$

Proof. This result is a rewriting of the previous equation given (5.3). However, we give an alternative proof in Appendix B. \square

5.2. Length minimizing geodesics on H^{div} . We now show that every smooth geodesics are length minimizing on a sufficiently short time interval. This is actually a straightforward generalization of Brenier's proof in the case of Euler equation to a Riemannian setting. Note that the Ebin and Marsden's point of view do not give such precise results since they require a strong ambient topology for the Gauss lemma to apply, for instance an H^s topology for $s > d/2 + 1$. In the worst case of our theorem, we will require only an L^∞ bound on the Jacobian and on the diffeomorphism in the worst case.

Theorem 29. *Let $(\varphi(t), r(t))$ be a smooth solution to the geodesic equations (5.4) on the time interval $[t_0, t_1]$. If $(t_1 - t_0)^2 \langle w, \nabla^2 \Psi_{P(t)}(x, r) w \rangle < \pi^2 \|w\|^2$ holds for all $t \in [t_0, t_1]$ and $(x, r) \in \mathcal{C}(M)$ and $w \in T_{(x,r)} \mathcal{C}(M)$, then for every smooth curve $(\varphi_0(t), r_0(t)) \in \pi_0^{-1}(\{\mu\})$ satisfying $(\varphi_0(t_i), r_0(t_i)) = (\varphi(t_i), r_0(t_i))$ for $i = 0, 1$ and the condition (*), one has*

$$(5.10) \quad \int_{t_0}^{t_1} \|(\dot{\varphi}, \dot{r})\|^2 dt \leq \int_{t_0}^{t_1} \|(\dot{\varphi}_0, \dot{r}_0)\|^2 dt,$$

with equality if and only if the two paths coincide on $[t_0, t_1]$.

The condition (*) is:

- (1) If the sectional curvature of $\mathcal{C}(M)$ can assume both signs, there exists $\delta > 0$ such that the curve $(\varphi_0(t), r_0(t))$ has to belong to a δ -neighborhood of $(\varphi(t), r(t))$, namely

$$d_{\mathcal{C}(M)}((\varphi_0(t, x), r_0(t, x)), (\varphi(t, x), r(t, x))) \leq \delta$$

for all $(x, t) \in M \times [t_0, t_1]$ where $d_{\mathcal{C}(M)}$ is the distance on the cone.

- (2) If $\mathcal{C}(M)$ has non positive sectional curvature, then, for every δ as above, there exists a short enough time interval on which the geodesic will be length minimizing.
- (3) If $M = S_n(r)$ the Euclidean sphere in \mathbb{R}^{n+1} of radius r , then if $r \leq 1$, the result is valid for every path $(\dot{\varphi}, \dot{r})$.

Note that the condition on the Hessian is not empty, i.e. it is fulfilled in our case of interest: Indeed, when P is a C^2 function on M , the Hessian of $\Psi_P(x, r) = r^2 P(x)$ is, in the orthonormal basis $\partial_r, \frac{1}{r} e_1, \dots, \frac{1}{r} e_d$ where e_1, \dots, e_d is an orthonormal basis of $T_x M$

$$(5.11) \quad \nabla^2 \Psi_P(x, r) = \begin{pmatrix} \nabla^2 P(x) & 2\nabla P(x) \\ 2\nabla P^T(x) & 2P(x) \end{pmatrix},$$

which is bounded if P is smooth on M compact.

Proof. To alleviate notations, we will denote $g_t = (\varphi(t), r(t))$ and $h_t = (\varphi_0(t), r_0(t))$. Since $g_t = (\varphi(t), \sqrt{\text{Jac}(\varphi(t))})$, by direct integration, for every $t \in [t_0, t_1]$

$$(5.12) \quad \int_M \Psi_P(g_t(s)) \, ds = 0,$$

and the same equality holds for h_t .

Let us consider $c(t, s, x)$ be a two parameters ($t \in [t_0, t_1]$ and $x \in M$) family of geodesics on $\mathcal{C}(M)$ such that $c(t, 0, x) = g_t(x)$ and $c(t, 1, x) = h_t(x)$. This family of geodesics is uniquely defined if one considers balls which do not intersect the cut locus. Uniformity of the radius of the balls can be obtained since $[t_0, t_1] \times M$ is compact. Consequently, the family of curves $c(t, s, x)$ is a smooth family of geodesics, at least as smooth as $g_t(x)$ and $h_t(x)$ are with respect to the parameters t, x . Since $\partial_t c(t, s, x)$ is a variation of geodesics, it is a Jacobi field as a function of s . Thus, we will use the notation $J(t, s, x) = \partial_t c(t, s, x)$. Consequently, we have

$$(5.13) \quad J(t, 0, x) = \partial_t g_t(x) \text{ and } J(t, 1, x) = \partial_t h_t(x).$$

Now, the result we want to prove can be reformulated as,

$$(5.14) \quad \int_{t_0}^{t_1} \int_M \|J(t, 0, x)\|^2 \, dt \, dx \leq \int_{t_0}^{t_1} \int_M \|J(t, 1, x)\|^2 \, dt \, dx$$

with equality if and only if for almost every x , it holds $g_t(x) = h_t(x)$ for all $t \in [t_1, t_2]$. We now use a second-order Taylor expansion of $\Psi_P(c(t, s, x))$ with respect to s at $s = 0$. Denoting by $M \stackrel{\text{def.}}{=} \sup_{t \in [t_0, t_1]} \sup_{x \in M} |\nabla^2 \Psi_{P_t}(x)|$, we have, writing $c(s)$ for $c(t, s, x)$,

$$\Psi_P(h_t(x)) - \Psi_P(g_t(x)) - \langle \nabla \Psi_P(c(0)), \partial_s c(0) \rangle \leq \frac{M}{2} \int_0^1 \|\partial_s c(s)\|^2 \, ds.$$

Now, one has that $\partial_s c(t, s, x)$ vanishes at $t = 0$ and $t = 1$. We can therefore apply Poincaré inequality to $\|\partial_s c(s)\|$ to obtain

$$(5.15) \quad \int_{t_0}^{t_1} \|\partial_s c(s)\|^2 \, ds \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} |\partial_t \|\partial_s c(s)\||^2 \, ds.$$

Since $\partial_t \|\partial_s c(s)\| = \frac{1}{\|\partial_s c\|} \langle \nabla_t \partial_s c, \partial_s c \rangle$, we have the inequality $|\partial_t \|\partial_s c(s)\|| \leq \|\nabla_t \partial_s c\|$ and we get, exchanging derivatives,

$$(5.16) \quad \int_{t_0}^{t_1} \|\partial_s c(s)\|^2 \, ds \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} \|\dot{J}(s)\|^2 \, ds,$$

where \dot{J} is the covariant derivative of J with respect to s . We thus have

$$\int_{t_0}^{t_1} \Psi_P(c(t, 1, x)) - \Psi_P(c(t, 0, x)) - \langle \nabla \Psi_P(c(t, 0, x)), \partial_s c(0) \rangle \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} \|\dot{J}(s)\|^2 \, ds.$$

However, $g_t(x) = c(t, 0, x)$ is a solution of $\nabla_t \partial_t c(t, 0, x) = -\nabla \Psi_P(t, 0, x)$, therefore, an integration by part w.r.t. t leads to

$$\int_{t_0}^{t_1} \Psi_P(c(t, 1, x)) - \Psi_P(c(t, 0, x)) - \langle \partial_t c(t, 0, x), \nabla_t \partial_s c(0) \rangle \, dt \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} \|\dot{J}(s)\|^2 \, ds.$$

Last, integrating over M and exchanging once again covariant derivatives gives

$$\int_{t_0}^{t_1} \int_M -\langle J(t, 0, x), \dot{J}(t, 0, x) \rangle \, dx \, dt \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}(t, s, x)\|^2 \, ds \, dx \, dt.$$

Writing $f(s) = \int_{t_0}^{t_1} \int_M \|J(t, s, x)\|^2 \, dt$, we want to prove $f(1) \geq f(0)$ and we have

$$-f'(0) \leq \frac{M(t_1 - t_0)^2}{2\pi^2} \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}(t, s, x)\|^2 \, ds \, dx \, dt.$$

Therefore, the result is proven if we can show

$$(5.17) \quad f(1) - f(0) - f'(0) \geq \varepsilon \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}(t, s, x)\|^2 ds dx dt.$$

The left hand side can be reformulated using $f(1) - f(0) - f'(0) = \int_0^1 (1-s)f''(s) ds$ as

$$(5.18) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s)(\|\dot{J}\|^2 - \langle R(\partial_s c, J)J, \partial_s c \rangle) ds dx dt \geq \varepsilon \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt,$$

with $\varepsilon = \frac{M(t_1-t_0)^2}{2\pi^2}$.

We now need to distinguish between two cases, the first one being when $\int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt \geq 1$. In this case, we use the inequality

$$(5.19) \quad \|J(t)\|^2 \leq 2\|J(0)\|^2 + 2 \int_0^1 \|\dot{J}(s)\|^2 ds$$

in order to get

$$(5.20) \quad - \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt \leq \delta^2 \int_{t_0}^{t_1} \int_M \int_0^1 K_{\text{sup}}(2\|J(0)\|^2 + 2\|\dot{J}(s)\|^2) ds dx dt,$$

where $\delta = \sup_{(x,t) \in M \times [t_0, t_1]} \|\partial_s c(t, 0, x)\|$ and K_{sup} is a bound on $\max(K(y), 0)$ with $K(y)$ is the maximum of the sectional curvatures at $y \in \mathcal{C}(M)$ for y in a bounded neighborhood of $\bigcup_{t \in [t_0, t_1]} g_t(M)$ which is compact. Then, there exists δ sufficiently small such that for every $(x, t) \in M \times [t_0, t_1]$,

$$(5.21) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt \leq 1 \leq \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt.$$

Now we study the second case, that is when $\int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt \leq 1$. Applying once again inequality (5.16), we obtain by the Cauchy-Schwarz inequality,

$$(5.22) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt \leq \varepsilon K_{\text{sup}} \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 \|J\|^2 ds dx dt \\ \leq \varepsilon K_{\text{sup}} \left(\int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^4 ds dx dt \right)^{1/2} \left(\int_{t_0}^{t_1} \int_M \int_0^1 \|J\|^4 ds dx dt \right)^{1/2}.$$

We now remark that for each t, x , the space of Jacobi fields is finite dimensional and consequently, norms are equivalent so that there exists a positive constant m that depends on t, x such that

$$(5.23) \quad \left(\int_0^1 \|\dot{J}\|^4 ds \right)^{1/2} \leq m \int_0^1 \|\dot{J}\|^2 ds$$

and

$$(5.24) \quad \left(\int_0^1 \|J\|^4 ds \right)^{1/2} \leq m \int_0^1 \|J\|^2 ds.$$

By compactness of $M \times [t_0, t_1]$, the constant m can be chosen independently of t, x and therefore, there exists a constant m' such that

$$(5.25) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt \leq \\ \varepsilon K_{\text{sup}} m' \left(\int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt \right) \left(\int_{t_0}^{t_1} \int_M \int_0^1 \|J\|^2 ds dx dt \right).$$

Then, inequality (5.19) leads to

$$(5.26) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt \leq \varepsilon K_{\text{sup}} M m' \left(\int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt \right).$$

with $M = \left(\int_{t_0}^{t_1} \int_M 2\|J(0)\|^2 + 2 \int_0^1 \|\dot{J}(s)\|^2 ds dx dt \right)$.

Let us recall that our goal is to prove the existence of $\varepsilon > 0$ such that

$$(5.27) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \|\dot{J}\|^2 ds dx dt \geq \varepsilon \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 + (1-s) \langle R(\partial_s c, J)J, \partial_s c \rangle ds dx dt,$$

which, in the first case, reads

$$(5.28) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \|\dot{J}\|^2 ds dx dt \geq 2\varepsilon \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt,$$

and in the second case

$$(5.29) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \|\dot{J}\|^2 ds dx dt \geq \varepsilon (1 + K_{\text{sup}} M m') \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt.$$

The existence of ε follows from the fact that the space of Jacobi fields is finite dimensional and the fact $M \times [t_0, t_1]$ is compact. It thus proves the result in the general case.

When the cone $\mathcal{C}(M)$ has non-positive sectional curvature, $K_{\text{sup}} = 0$ therefore, we only have to prove the existence of ε such that

$$(5.30) \quad \int_{t_0}^{t_1} \int_M \int_0^1 (1-s) \|\dot{J}\|^2 ds dx dt \geq \varepsilon \int_{t_0}^{t_1} \int_M \int_0^1 \|\dot{J}\|^2 ds dx dt,$$

which does not require an a priori bound on the neighborhood.

When $\mathcal{C}(M)$ is flat, Jacobi fields are constant and the constant ε does not depend on the neighborhood and is equal to $1/2$ as in Brenier's proof. \square

This generalization of Brenier's proof is not completely satisfactory in positive curvature or in the case where the diameter of the Riemannian manifold M is greater than π . In the former case, the constructed interpolating paths have to pass through the apex of the cone and therefore these paths $c(t, s, x)$ are not smooth any longer w.r.t. s and thus Jacobi fields are smooth a priori. Of course, as stated in the theorem, the result still holds when we restrict the paths to evolve inside a ball of radius less than the injectivity radius of the cone. Let us insist on the flat case, that contains the Camassa-Holm equation on S_1 :

Corollary 30. *When $M = S_n$ and $a \leq 2b$, smooth solutions to the Camassa-Holm equation (3.13) are length minimizing for short times.*

As said above, the family of geodesics is not smooth any more if $a > 2b$, and our proof does not apply. When $M = \mathbb{R}^n$, the cone $\mathcal{C}(M)$ has non-positive curvature and the second condition in (*) applies. In general, to the best of our knowledge, the proof presented above is the first generalization to Riemannian manifolds of Brenier's proof and it might be possible to improve on this result, especially to get rid of the boundedness assumption.

6. FUTURE DIRECTIONS

In this article, we have presented an isometric embedding of the group of diffeomorphism group endowed with the right-invariant H^{div} metric in the space $L^2(M, \mathcal{C}(M))$. This isometric embedding enables to rewrite the Camassa-Holm equation, via a Madelung transform, as an incompressible Euler equation on the cone. As an application, this has also led to a result on the minimizing property of geodesics. Very few papers have been interested with the actual variational problem of minimizing geodesics for the H^{div} metric in the sense of Brenier [5, 6] which can be addressed from the point of view developed in our article. Following Brenier, we will investigate elsewhere the

uniqueness of the pressure as in [3]. This also opens the way to design new numerical simulations of Camassa-Holm equation following [39].

Following the point of view developed in this paper, we plan to rewrite other fluid dynamic equations as geodesic equations on a submanifold of a space of maps endowed with an L^2 norm. The result may have, as shown for the Camassa-Holm equation, interesting analytical consequences.

APPENDIX A. GROUP ACTION AND RIEMANNIAN SUBMERSIONS

A.1. Riemannian submersion. Let (M, g_M) and (N, g_N) be two Riemannian manifolds and $f : M \mapsto N$ a differentiable mapping.

Definition 7. The map f is a Riemannian submersion if f is a submersion and for any $x \in M$, the map $df_x : \text{Ker}(df_x)^\perp \mapsto T_{f(x)}N$ is an isometry.

In such a case, $\text{Vert}_{f(x)} \stackrel{\text{def.}}{=} \text{Ker}(df(x))$ is called the vertical space and $\text{Hor}_{f(x)} \stackrel{\text{def.}}{=} \text{Ker}(df(x))^\perp$ is called the horizontal space. The horizontal spaces can be used to lift a vector field Y on N onto a vector field \tilde{Y} on M which is horizontal. More precisely, \tilde{Y} is the unique horizontal vector field such that $df_x(\tilde{Y}(x)) = Y(f(x))$. The first immediate property is that Riemannian submersions are length decreasing.

Proposition 31. Let $c_0 : [0, 1] \mapsto M$ be a smooth curve. It then defines a smooth curve on N by $c_1 \stackrel{\text{def.}}{=} f \circ c_0$. Then,

$$(A.1) \quad \int_0^1 g_M(\dot{c}_0(t), \dot{c}_0(t)) dt \geq \int_0^1 g_N(\dot{c}_1(t), \dot{c}_1(t)) dt$$

In other words, Riemannian submersions are length decreasing.

Another property of Riemannian submersion is the following: any geodesic $\gamma(t)$ on M which is horizontal at a given time, i.e. $\gamma'(t) \in \text{Hor}_{\gamma(t)}$, is horizontal for all time.

An important property is the computation of the curvature that can be done via O'Neill's formula detailed below.

Theorem 32 (O'Neill's formula). Let f be a Riemannian submersion as defined above and X, Y be two orthonormal vector fields on M with horizontal lifts \tilde{X} and \tilde{Y} , then

$$(A.2) \quad K_N(X, Y) = K_M(\tilde{X}, \tilde{Y}) + \frac{3}{4} \|\text{vert}([\tilde{X}, \tilde{Y}])\|_M^2,$$

where K denotes the sectional curvature and vert the orthogonal projection on the vertical space.

APPENDIX B. OTHER PROOFS

Proof of Theorem 28. We treat the particular case of the Camassa-Holm equation with $a = 1, b = 1/2$ coefficients, as written in Equation (3.12). The isotropy subgroup can be alternatively described as the constrained submanifold $\mathcal{S} \subset L^2(S_1, \mathbb{C})$ defined by the differential relation

$$(B.1) \quad \mathcal{S} \stackrel{\text{def.}}{=} \{(r, \psi) : x \mapsto r(x)e^{i\psi(x)} : r^2(x) = \psi'(x) \forall x \in S_1\}.$$

The tangent space at a given map $(r, \psi) : x \mapsto r(x)e^{i\psi(x)}$ is then given by

$$(B.2) \quad T_{(r, \psi)}\mathcal{S} \stackrel{\text{def.}}{=} \{(X_r, X_\psi) : r(x)X_r(x) = X'_\psi(x) \forall x \in S_1\}.$$

As is well known, the geodesic equation on a Riemannian submanifold $N \hookrightarrow M$ (N is isometrically embedded in M) is given by

$$(B.3) \quad \frac{D}{Dt}\dot{q} = \text{II}(\dot{q}, \dot{q}),$$

where II is the second fundamental form, q is a curve on the submanifold N and $\frac{D}{Dt}$ is the covariant derivative associated with the metric on M . In our case, $M = L^2(S_1, \mathbb{C})$ is a Hilbert space and

therefore it is flat, which implies $\frac{D}{Dt}\dot{q} = \ddot{q}$. Rewriting the covariant derivative of the curve in polar coordinates:

$$(B.4) \quad \frac{D}{Dt}(X_r, X_\psi) = \left(\dot{X}_r - r\dot{X}_\psi^2, \dot{X}_\psi + 2X_\psi \frac{X_r}{r} \right).$$

We now need to compute the second fundamental form of \mathcal{S} which reduces to the computation of the orthogonal component of a curve on \mathcal{S} . If a curve (r, ψ) is drawn on \mathcal{S} , this curve can be differentiated to give $(X_r, X_\psi) = (\frac{1}{2r}X'_\psi, X_\psi)$ and differentiating once more gives

$$(B.5) \quad \begin{cases} \dot{X}_r = \frac{1}{2r}\dot{X}'_\psi - \frac{1}{2r^2}X'_\psi X_r \\ \dot{X}_\psi = \dot{X}_\psi. \end{cases}$$

Therefore the second fundamental form is given by the orthogonal component P of $-\frac{1}{2r^2}X'_\psi X_r$ to the tangent space of \mathcal{S} . It is a straightforward computation to see that the orthogonal space $T_{(r,\psi)}\mathcal{S}^\perp$ is given by

$$(B.6) \quad T_{(r,\psi)}\mathcal{S}^\perp = \left\{ \left(\frac{1}{2}rP, \frac{1}{4r^2}P' \right) : P \in H^1(S_1, \mathbb{R}) \right\}.$$

The projection of $-\frac{4b^3}{ar^2}X'_\psi X_r$ is now given by the following minimization problem

$$(B.7) \quad P \stackrel{\text{def.}}{=} \arg \min_Z \int_{S_1} \left(\frac{1}{2}rZ + \frac{1}{2r^2}X'_\psi X_r \right)^2 + \frac{1}{16r^2}Z'^2 \, dx.$$

The solution to this minimization problem exists provided that $r, \psi, X_r, X_\psi \in H^1(S_1)$ and r does not vanish. This problem is actually a (strictly if $r(x) > 0$) convex minimization problem and optimal solutions are characterized by the Euler-Lagrange equation

$$(B.8) \quad \left(\frac{1}{r^2}P' \right)' - r^2P - \frac{1}{2r}X'_\psi X_r = 0.$$

The Camassa-Holm in polar coordinates is thus given by

$$(B.9) \quad \begin{cases} \dot{X}_r - r\dot{X}_\psi^2 = \frac{1}{2}rP \\ \dot{X}_\psi + 2X_\psi \frac{X_r}{r} = \frac{1}{4r^2}P', \end{cases}$$

and coming back to Euclidean coordinates gives

$$(B.10) \quad \ddot{Z} = \left(\frac{1}{2}P + \frac{i}{4|Z|^2}P' \right) Z,$$

which can be alternatively written as

$$(B.11) \quad \ddot{Z} = \frac{1}{2}PZ + \frac{i}{4Z}P'.$$

□

REFERENCES

- [1] Vladimir Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.
- [2] J-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.
- [3] Y. Brenier. The dual least action problem for an ideal, incompressible fluid. *Archive for Rational Mechanics and Analysis*, 122(4):323–351, 1993.
- [4] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.
- [5] Yann Brenier. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math.*, 52(4):411–452, 1999.
- [6] Yann Brenier. Remarks on the minimizing geodesic problem in inviscid incompressible fluid mechanics. *Calc. Var. Partial Differential Equations*, 47(1-2):55–64, 2013.
- [7] Alberto Bressan and Adrian Constantin. Global conservative solutions of the Camassa-Holm equation. *Arch. Ration. Mech. Anal.*, 183(2):215–239, 2007.

- [8] Alberto Bressan and Massimo Fonte. An optimal transportation metric for solutions of the Camassa-Holm equation. *Methods Appl. Anal.*, 12(2):191–219, 2005.
- [9] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry. *American Mathematical Soc.*, 2001.
- [10] Roberto Camassa and Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.
- [11] L. Chizat, G. Peyré, B. Schmitzer, and F.-X. Vialard. Unbalanced Optimal Transport: Geometry and Kantorovich Formulation. *ArXiv e-prints*, August 2015.
- [12] L. Chizat, B. Schmitzer, G. Peyré, and F.-X. Vialard. An Interpolating Distance between Optimal Transport and Fisher-Rao. *ArXiv e-prints*, June 2015.
- [13] Adrian Constantin. On the scattering problem for the Camassa-Holm equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457(2008):953–970, 2001.
- [14] Adrian Constantin and Joachim Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181(2):229–243, 1998.
- [15] Adrian Constantin and David Lannes. The hydrodynamical relevance of the camassa–holm and degasperis–procesi equations. *Archive for Rational Mechanics and Analysis*, 192(1):165–186, 2008.
- [16] Raphaël Danchin. A few remarks on the Camassa-Holm equation. *Differential Integral Equations*, 14(8):953–988, 2001.
- [17] David G. Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.
- [18] J. Escher and B. Kolev. Right-invariant Sobolev metrics of fractional order on the diffeomorphism group of the circle. *ArXiv e-prints*, February 2012.
- [19] Joachim Escher and Boris Kolev. The degasperis–procesi equation as a non-metric euler equation. *Mathematische Zeitschrift*, 269(3):1137–1153, 2011.
- [20] D. S. Freed and D. Groisser. The basic geometry of the manifold of riemannian metrics and of its quotient by the diffeomorphism group. *Michigan Math. J.*, 36(3):323–344, 1989.
- [21] S. Gallot. Équations différentielles caractéristiques de la sphère. *Annales scientifiques de l’Ecole Normale Supérieure*, 12(2):235–267, 1979.
- [22] T. O. Gallouët and L. Monsaingeon. A jko splitting scheme for kantorovich-fisher-rao gradient flows, 2016.
- [23] F. Gay-Balmaz, C. Tronci, and C. Vizman. Geometric dynamics on the automorphism group of principal bundles: geodesic flows, dual pairs and chromomorphism groups. *Journal of Geometric Mechanics*, 5:39–84, 2013.
- [24] Katrin Grunert, Helge Holden, and Xavier Raynaud. Lipschitz metric for the periodic camassa–holm equation. *Journal of Differential Equations*, 250(3):1460 – 1492, 2011.
- [25] D. D. Holm, J. E. Marsden, and T. S. Ratiu. The Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.*, 137:1–81, 1998.
- [26] B. Khesin, J. Lenells, G. Misiolek, and S. C. Preston. Curvatures of Sobolev metrics on diffeomorphism groups. *ArXiv e-prints*, September 2011.
- [27] B. Khesin, J. Lenells, G. Misiolek, and S. C. Preston. Geometry of diffeomorphism groups, complete integrability and optimal transport. *ArXiv e-prints*, May 2011.
- [28] B. Khesin and R. Wendt. *The geometry of infinite-dimensional groups*, volume 51. Springer Science & Business Media, 2008.
- [29] I. Kolář, P. W. Michor, and J. Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.
- [30] S. Kondratyev, L. Monsaingeon, and D. Vorotnikov. A new optimal transport distance on the space of finite Radon measures. Technical report, Pre-print, 2015.
- [31] Shinar Kouranbaeva. The Camassa-Holm equation as a geodesic flow on the diffeomorphism group. *J. Math. Phys.*, 40(2):857–868, 1999.
- [32] Serge Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
- [33] Jonatan Lenells. Conservation laws of the Camassa-Holm equation. *J. Phys. A, Math. Gen.*, 38(4):869–880, 2005.
- [34] M. Liero, A. Mielke, and G. Savaré. Optimal Entropy-Transport problems and a new Hellinger-Kantorovich distance between positive measures. *ArXiv e-prints*, August 2015.
- [35] M. Liero, A. Mielke, and G. Savaré. Optimal transport in competition with reaction: the Hellinger-Kantorovich distance and geodesic curves. *ArXiv e-prints*, August 2015.
- [36] J. Maas, M. Rumpf, C. Schönlieb, and S. Simon. A generalized model for optimal transport of images including dissipation and density modulation. arXiv:1504.01988, 2015.
- [37] R.J. McCann. Polar factorization of maps on riemannian manifolds. *Geometric & Functional Analysis GFA*, 11(3):589–608, 2001.
- [38] Henry P. McKean. Breakdown of the Camassa-Holm equation. *Comm. Pure Appl. Math.*, 57(3):416–418, 2004.
- [39] Q. Mérigot and J.-M. Mirebeau. Minimal geodesics along volume preserving maps, through semi-discrete optimal transport. *ArXiv e-prints*, May 2015.
- [40] P. W. Michor. *Topics in Differential Geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [41] Peter W. Michor and David Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Doc. Math.*, 10:217–245, 2005.

- [42] G. Misiolek. Classical solutions of the periodic camassa—holm equation. *Geometric & Functional Analysis GFAA*, 12(5):1080–1104, 2002.
- [43] Gerard Misiolek and Stephen C. Preston. Fredholm properties of riemannian exponential maps on diffeomorphism groups. *Inventiones mathematicae*, 179(1):191–227, 2010.
- [44] K. Modin. Generalised Hunter-Saxton equations, optimal information transport, and factorisation of diffeomorphisms. *ArXiv e-prints*, March 2012.
- [45] Felix Otto. The geometry of dissipative evolution equations: The porous medium equation. *Communications in Partial Differential Equations*, 26(1-2):101–174, 2001.
- [46] A. Trounev and L. Younes. Metamorphoses through lie group action. *Foundations of Computational Mathematics*, 5(2):173–198, 2005.
- [47] C. Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.

CMLS, UMR 7640, ÉCOLE POLYTECHNIQUE, FR-91128 PALAISEAU CEDEX.
E-mail address: thomas.gallouet@polytechnique.edu

UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, CEREMADE, INRIA, PROJECT TEAM MOKAPLAN
E-mail address: fxvialard@normalesup.org