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# PINNING OF A RENEWAL ON A QUENCHED RENEWAL

KENNETH S. ALEXANDER AND QUENTIN BERGER

ABSTRACT. We introduce the pinning model on a quenched renewal, which is an instance of a (strongly correlated) disordered pinning model. The potential takes value 1 at the renewal times of a quenched realization of a renewal process  $\sigma$ , and 0 elsewhere, so nonzero potential values become sparse if the gaps in  $\sigma$  have infinite mean. The “polymer” – of length  $\sigma_N$  – is given by another renewal  $\tau$ , whose law is modified by the Boltzmann weight  $\exp(\beta \sum_{n=1}^N \mathbf{1}_{\{\sigma_n \in \tau\}})$ . Our assumption is that  $\tau$  and  $\sigma$  have gap distributions with power-law-decay exponents  $1 + \alpha$  and  $1 + \tilde{\alpha}$  respectively, with  $\alpha \geq 0, \tilde{\alpha} > 0$ . There is a localization phase transition: above a critical value  $\beta_c$  the free energy is positive, meaning that  $\tau$  is *pinned* on the quenched renewal  $\sigma$ . We consider the question of relevance of the disorder, that is to know when  $\beta_c$  differs from its annealed counterpart  $\beta_c^{\text{ann}}$ . We show that  $\beta_c = \beta_c^{\text{ann}}$  whenever  $\alpha + \tilde{\alpha} \geq 1$ , and  $\beta_c = 0$  if and only if the renewal  $\tau \cap \sigma$  is recurrent. On the other hand, we show  $\beta_c > \beta_c^{\text{ann}}$  when  $\alpha + \frac{3}{2} \tilde{\alpha} < 1$ . We give evidence that this should in fact be true whenever  $\alpha + \tilde{\alpha} < 1$ , providing examples for all such  $\alpha, \tilde{\alpha}$  of distributions of  $\tau, \sigma$  for which  $\beta_c > \beta_c^{\text{ann}}$ . We additionally consider two natural variants of the model: one in which the polymer and disorder are constrained to have equal numbers of renewals ( $\sigma_N = \tau_N$ ), and one in which the polymer length is  $\tau_N$  rather than  $\sigma_N$ . In both cases we show the critical point is the same as in the original model, at least when  $\alpha > 0$ .

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## 1. INTRODUCTION

1.1. **Motivations.** A variety of polymer pinning models have been studied in theoretical and mathematical physics in the past decades, see [15, 20, 21] for reviews. We introduce in this paper a new type of disordered pinning model that we call *pinning on a quenched renewal*. Before giving its definition, we recall two well-studied related models that motivate the introduction and the study of this new model.

*Pinning on an inhomogeneous defect line.* The disordered pinning model was introduced by Poland and Scheraga [30] to model DNA denaturation, and it has recently been the subject of extensive rigorous mathematical studies, cf. [20, 21]. We recall its definition.

Let  $\tau = \{\tau_0 = 0, \tau_1, \dots\}$  be a discrete recurrent renewal process of law  $\mathbf{P}$ , and let  $\omega = (\omega_n)_{n \in \mathbb{N}}$  be a sequence of IID random variables of law denoted  $\mathbb{P}$ , with finite exponential moment  $M(\lambda) = \mathbb{E}[e^{\lambda\omega}] < +\infty$ , for  $\lambda > 0$  small enough. Then, for  $\beta > 0$  and  $h \in \mathbb{R}$ , and a fixed realization of  $\omega$  (quenched disorder), the Gibbs measure is defined by

$$(1.1) \quad \frac{d\mathbf{P}_{N,h}^{\beta,\omega}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}^{\beta,\omega}} e^{\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{\{n \in \tau\}}},$$

where  $Z_{N,h}^{\beta,\omega}$  is the *partition function*, which normalizes  $\mathbf{P}_{N,h}^{\beta,\omega}$  to a probability measure. This Gibbs measure corresponds to giving a (possibly negative) reward  $\beta\omega_n+h$  to the occurrence of a renewal point at  $n$ . In this context it is natural to think of the polymer configuration as the space-time trajectory of some Markov chain, with the polymer interacting with a potential whenever it returns to some particular site, for example the origin; then  $\tau$  represents the times of these returns.

This system is known to undergo a localization phase transition when  $h$  varies, and much attention has been given to the question of disorder relevance, that is, whether the *quenched* and *annealed* systems (with respective partition functions  $Z_{N,h}^{\beta,\omega}$  and  $\mathbb{E}Z_{N,h}^{\beta,\omega}$ ) have different critical behaviors. The so-called Harris criterion [25] is used to predict disorder relevance, and its characterization in terms of the distribution of the renewal time  $\tau_1$  has now been mathematically settled completely [1, 4, 6, 14, 16, 23, 28, 31], the complete necessary and sufficient condition for critical point shift being given only recently, in [6].

An interesting approach to this problem is based on a large deviation principle for cutting words out of a sequence of letters [9], see [14]: quantities of the model, such as critical points, are expressed as variational formulas involving (quenched and annealed) rate functions  $I^{\text{que}}$  and  $I^{\text{ann}}$ . In [14], the authors consider a version of  $\tau$  truncated at a level  $T$ , and it is implicitly shown that a lower bound on the critical point shift is, for all  $\beta > 0$ ,

(1.2)

$$\lim_{T \rightarrow \infty} \frac{1}{m_T} \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma \mathbb{E} \log \mathbf{E} \left[ \exp \left( \sum_{n=1}^{\sigma_N} \left[ (\beta\omega_n - \log M(\beta)) \mathbf{1}_{\{n \in \tau\}} + \frac{1}{2} \log M(2\beta) \mathbf{1}_{\{n \in \tau \cap \sigma\}} \right] \right) \right]$$

where  $m_T$  is the mean inter-arrival time of the truncated renewal, and  $\sigma$  is a quenched trajectory of a renewal with the same (truncated) law  $\mathbf{P}$ ; thus  $\sigma_N \sim m_T N$ .

Note that the first term of the summand on the right side of (1.2) gives the quenched Hamiltonian from (1.1) calculated at the annealed critical point  $u = -\log M(\beta)$ . The second term of the summand corresponds to a quenched system in which the disorder is sparse (at least without the truncation), in the sense that it is 0 except at the sites of the quenched renewal  $\sigma$ , and these sites have a limiting density of 0 when  $\sigma_1$  has infinite mean, that is

$$(1.3) \quad \lim_{N \rightarrow \infty} \frac{|\sigma \cap \{1, \dots, N\}|}{N} = 0 \quad \mathbf{P}_\sigma - \text{a.s.}$$

Since the quenched system is not pinned at the annealed critical point, the limit (1.2) would be 0, meaning no pinning, without that second term. The question is then roughly whether the additional presence of the very sparse disorder  $\sigma$  is enough to create pinning (this oversimplifies slightly, as we are ignoring the limit  $T \rightarrow \infty$ ). It therefore suggests the question, of independent interest, of whether such very sparse disorder creates pinning for arbitrarily small  $\beta$ , in the simplified context where the first term in the summand is absent and there is no truncation. Our purpose here is to answer that question positively: under an appropriate condition on the tail exponents of the return times  $\tau_1$  and  $\sigma_1$ , there is pinning for arbitrarily small  $\beta$ .

It should also be noted that the sum in (1.2) is up to  $\sigma_N$ , not  $N$  as in (1.1). To create positive free energy,  $\tau$  must be able to hit an order- $N$  number of sites  $\sigma_i$ , which is impossible for a length- $N$  polymer with extremely sparse disorder satisfying (1.3), see [26].

*Pinning on a Random Walk.* The Random Walk Pinning Model considers a random walk  $X$  (whose law is denoted  $\mathbf{P}_X$ ), which is pinned to another independent (quenched) random walk  $Y$  with distribution  $\mathbf{P}_Y$  identical to  $\mathbf{P}_X$ , see [10]. For any  $\beta > 0$ , the Gibbs measure is given by

$$(1.4) \quad \frac{d\mathbf{P}_{N,\beta}^Y}{d\mathbf{P}_X}(X) := \frac{1}{Z_{N,\beta}^Y} e^{\beta \sum_{n=1}^N \mathbf{1}_{\{X_n=Y_n\}}}.$$

This system undergoes a localization phase transition: above some critical value  $\beta_c$ , the random walk  $X$  sticks to the quenched trajectory of  $Y$ ,  $\mathbf{P}_Y$ -a.s. Here  $X$  and  $Y$  are assumed irreducible and symmetric, with  $\log \mathbf{P}_X(X_{2n} = 0) \stackrel{n \rightarrow \infty}{\sim} -\rho \log n$ : for example, with  $\rho = d/2$  in the case of symmetric simple random walks on  $\mathbb{Z}^d$ .

One can compare this model to its annealed counterpart, in which the partition function is averaged over the possible trajectories of  $Y$ ,  $\mathbf{E}_Y[Z_{N,\beta}^Y]$ , with corresponding critical value  $\beta_c^{\text{ann}}$ . Non-coincidence of quenched and annealed critical points implies the existence of an intermediate phase for the long-time behavior of several systems (such as coupled branching processes, parabolic Anderson model with a single catalyst, directed polymer in random environment): we refer to [10, 11] for more details on the relations between these models and the random walk pinning model.

The question of non-coincidence of critical points for pinning on a random walk has only been partially answered: it is known that  $\beta_c > \beta_c^{\text{ann}}$  if  $\rho > 2$ , see [10], and in the special cases of  $d$ -dimensional simple random walks,  $\beta_c > \beta_c^{\text{ann}}$  if and only if  $d \geq 3$  [7, 11, 12] (note that the case  $d \geq 5$  was already dealt with the case  $\rho > 2$ ). It is however believed that one has  $\beta_c > \beta_c^{\text{ann}}$  whenever  $\rho > 1$ .

The model we introduce now is related to this one, in the sense that we replace the random walks by renewals: we study a renewal  $\tau$ , which interacts with an object of the same nature, that is a quenched renewal  $\sigma$ .

**1.2. Pinning on a quenched renewal.** We consider two recurrent renewal processes  $\tau$  and  $\sigma$ , with (possibly different) laws denoted respectively by  $\mathbf{P}_\tau$  and  $\mathbf{P}_\sigma$ . We assume that there exist  $\alpha \geq 0, \tilde{\alpha} > 0$ , and slowly varying functions  $\varphi(\cdot), \tilde{\varphi}(\cdot)$  (see [8] for a definition) such that for all  $n \geq 1$

$$(1.5) \quad \mathbf{P}_\tau(\tau_1 = n) = \varphi(n) n^{-(1+\alpha)} \quad \text{and} \quad \mathbf{P}_\sigma(\sigma_1 = n) = \tilde{\varphi}(n) n^{-(1+\tilde{\alpha})}.$$

Let

$$(1.6) \quad d_\tau = \min \text{Supp } \tau_1, \quad d_\sigma = \min \text{Supp } \sigma_1,$$

where  $\text{Supp } \chi$  denotes the support of the distribution of a random variable  $\chi$ . At times we will add the assumption

$$(1.7) \quad d_\tau \leq d_\sigma,$$

which ensures that certain partition functions carrying the restriction  $\sigma_N = \tau_N$  cannot be 0, for sufficiently large  $N$ .

Let us write  $|\tau|_n$  for  $|\tau \cap \{1, \dots, n\}|$ . We consider the question of the pinning of  $\tau$  by the renewal  $\sigma$ : the Hamiltonian, up to  $N$   $\sigma$ -renewals, is  $\mathcal{H}_{N,\sigma}(\tau) := |\tau \cap \sigma|_{\sigma_N}$ . For  $\beta \geq$

0 (the inverse temperature) and for a quenched trajectory of  $\sigma$ , we introduce the Gibbs transformation  $\mathcal{P}_{N,\beta}^\sigma$  of  $\mathbf{P}_\tau$  by

$$(1.8) \quad \frac{d\mathcal{P}_{N,\beta}^\sigma}{d\mathbf{P}_\tau} := \frac{1}{Z_{N,\beta}^\sigma} e^{\beta|\tau \cap \sigma|_{\sigma_N}} \mathbf{1}_{\{\sigma_N \in \tau\}},$$

where  $Z_{N,\beta}^\sigma := \mathbf{E}_\tau [\exp(\beta|\tau \cap \sigma|_{\sigma_N}) \mathbf{1}_{\{\sigma_N \in \tau\}}]$  is the partition function. Note that the resulting polymer is *constrained*, meaning  $\tau$  is required to have a renewal at the endpoint  $\sigma_N$  of the polymer.

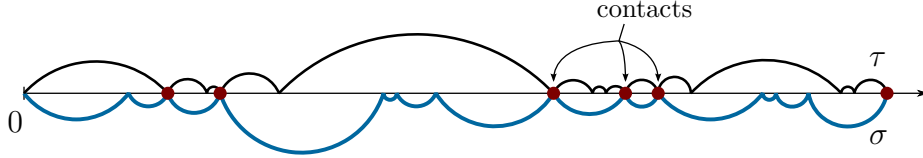


FIGURE 1. The renewal  $\sigma$  is quenched (here, we have  $N = 13$   $\sigma$ -renewals), and the renewal  $\tau$  collects a reward  $\beta$  for each contact with a  $\sigma$ -renewal (here,  $|\tau \cap \sigma|_{\sigma_N} = 6$ ).

**Proposition 1.1.** *The quenched free energy, defined by*

$$(1.9) \quad \mathcal{F}(\beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,\beta}^\sigma = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma \log Z_{N,\beta}^\sigma$$

*exists and is constant  $\mathbf{P}_\sigma$ -a.s.  $\mathcal{F}$  is non-negative, non-decreasing and convex. There exists  $\beta_c = \beta_c(\mathbf{P}_\sigma) \geq 0$  such that  $\mathcal{F}(\beta) = 0$  if  $\beta \leq \beta_c$  and  $\mathcal{F}(\beta) > 0$  if  $\beta > \beta_c$ .*

*Proof.* We have

$$Z_{N+M,\beta}^\sigma \geq \mathbf{E}_\tau [\exp(\beta|\tau \cap \sigma|_{\sigma_N}) \mathbf{1}_{\{\sigma_{N+M} \in \tau\}} \mathbf{1}_{\{\sigma_N \in \tau\}}] = Z_{N,\beta}^\sigma Z_{M,\beta}^{\theta^N \sigma},$$

where  $\theta$  is the ‘‘shift operator applied to increments’’:

$$(\theta\sigma)_i = \sigma_{i+1} - \sigma_1.$$

Therefore the sequence  $(\log Z_{N,\beta}^\sigma)_{n \in \mathbb{N}}$  is superadditive, and using Kingman’s subadditive Theorem [27], one gets the  $\mathbf{P}_\sigma$ -a.s. existence of the limit in (1.9), and moreover

$$(1.10) \quad \mathcal{F}(\beta) = \mathcal{F}(\beta, \mathbf{P}_\sigma) = \sup_{n \in \mathbb{N}} \frac{1}{N} \mathbf{E}_\sigma \log Z_{N,\beta}^\sigma.$$

The non-negativity trivially follows from the fact that  $\beta \geq 0$ , and the convexity is classical and comes from a straightforward computation.  $\square$

A standard computation gives that, when the derivative of  $\mathcal{F}(\beta)$  exists,

$$(1.11) \quad \frac{\partial}{\partial \beta} \mathcal{F}(\beta) = \lim_{N \rightarrow \infty} \mathcal{P}_{N,\beta}^\sigma \left[ \frac{1}{N} |\tau \cap \sigma|_{\sigma_N} \right],$$

where  $\mathcal{P}_{N,\beta}^\sigma$  also denotes the expectation with respect to the measure defined in (1.8). Therefore, when  $\mathcal{F}(\beta) > 0$ , a positive proportion of  $\sigma$ -renewal points are visited by  $\tau$ , and  $\beta_c$  marks the transition between a delocalized and localized phase.

**Remark 1.2.** We have chosen to work with the constrained version of the model, by including  $\mathbf{1}_{\{\sigma_N \in \tau\}}$  in the partition function. We stress that the free energy is the same as for the corresponding “free” partition function,

$$(1.12) \quad Z_{N,\beta}^{\sigma,\text{free}} := \mathbf{E}_\tau \left[ e^{\beta \sum_{n=1}^N \mathbf{1}_{\{\sigma_n \in \tau\}}} \right].$$

Indeed, let  $K$  be such that  $\mathbf{P}_\tau(\tau_1 = k) > 0$  for all  $k \geq K$ . Consider a trajectory  $\tau$  for which the last renewal in  $[0, \sigma_N - K]$  is at some point  $\sigma_N - k$ . We can map  $\tau$  to a constrained trajectory  $\hat{\tau}$  by removing all renewals of  $\tau$  in  $(\sigma_N - K, \sigma_N)$  and adding a renewal at  $\sigma_N$ . This is a  $2^K$ -to-1 map, and the Boltzmann weight of  $\hat{\tau}$  is smaller than that of  $\tau$  by a factor no smaller than  $e^{-\beta K} \mathbf{P}_\tau(\tau_1 = k) \geq e^{-\beta K} \min_{K \leq j \leq \sigma_N} \mathbf{P}_\tau(\tau_1 = j) \geq c \sigma_N^{-(2+\alpha)}$  for some constant  $c > 0$ . Hence there exists  $C > 0$  such that for all large  $N$ ,

$$(1.13) \quad Z_{N,\beta}^\sigma \leq Z_{N,\beta}^{\sigma,\text{free}} \leq C(\sigma_N)^{2+\alpha} Z_{N,\beta}^\sigma.$$

It is straightforward from (1.5) (and because we impose  $\tilde{\alpha} > 0$ ) that  $\lim_{N \rightarrow \infty} N^{-1} \log \sigma_N = 0$   $\mathbf{P}_\sigma$ -a.s., which with (1.13) gives  $\lim_{N \rightarrow \infty} N^{-1} \log Z_{N,\beta}^{\sigma,\text{free}} = \mathcal{F}(\beta)$ ,  $\mathbf{P}_\sigma$ -a.s.

**1.3. The annealed model.** We will compare the quenched-renewal pinning model to its annealed counterpart, with partition function  $\mathbf{E}_\sigma Z_{N,\beta}^\sigma = \mathbf{E}_{\tau,\sigma} [e^{\beta |\tau \cap \sigma|} \mathbf{1}_{\{\sigma_N \in \tau\}}]$ . The annealed free energy is defined by

$$(1.14) \quad \mathcal{F}^{\text{ann}}(\beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbf{E}_\sigma Z_{N,\beta}^\sigma.$$

The existence of the annealed free energy is straightforward, using the superadditivity of  $\log \mathbf{E}_\sigma Z_{N,\beta}^\sigma$ .

Note that this model does not treat  $\tau$  and  $\sigma$  symmetrically. Closely related is what we will call the *homogeneous doubled pinning model*, in which the length of the polymer is fixed, rather than the number of renewals in  $\sigma$ : the partition function and free energy are

$$Z_{n,\beta}^{\text{hom}} := \mathbf{E}_{\tau,\sigma} [e^{\beta |\tau \cap \sigma|} \mathbf{1}_{\{n \in \tau \cap \sigma\}}], \quad \mathcal{F}^{\text{hom}}(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_{n,\beta}^{\text{hom}},$$

and the corresponding critical point is denoted  $\beta_c^{\text{hom}}$ . This model is exactly solvable, see [20, Ch. 2], and in particular, its critical point is  $\beta_c^{\text{hom}} = -\log \mathbf{P}_{\tau,\sigma}((\tau \cap \sigma)_1 < \infty) = \log(1 + \mathbf{E}_{\sigma,\tau}[|\tau \cap \sigma|^{-1}])$ , with the convention that  $\frac{1}{\infty} = 0$ .

**Proposition 1.3.** *There exists some  $\beta_c^{\text{ann}} \geq 0$  such that  $\mathcal{F}^{\text{ann}}(\beta) = 0$  if  $\beta \leq \beta_c^{\text{ann}}$  and  $\mathcal{F}^{\text{ann}}(\beta) > 0$  if  $\beta > \beta_c^{\text{ann}}$ . One has  $\beta_c^{\text{ann}} = \beta_c^{\text{hom}} = \log(1 + \mathbf{E}_{\sigma,\tau}[|\tau \cap \sigma|^{-1}])$ , and in particular  $\beta_c^{\text{ann}} = 0$  if and only if the renewal  $\tau \cap \sigma$  is recurrent.*

*Proof.* Similarly to Remark 1.2, the annealed free energy is unchanged when one considers the *free* annealed partition function:  $\mathbf{E}_\sigma Z_{N,\beta}^{\sigma,\text{free}} = \mathbf{E}_{\sigma,\tau} [e^{\beta |\tau \cap \sigma|} \mathbf{1}_{\{\sigma_N \in \tau\}}]$ .

We now compare  $\mathbf{E}_\sigma Z_{N,\beta}^{\sigma,\text{free}}$  to  $Z_{N,\beta}^{\text{hom}}$ . First, since  $\sigma_N \geq N$ , we have  $\mathbf{E}_\sigma Z_{N,\beta}^{\sigma,\text{free}} \geq Z_{N,\beta}^{\text{hom}}$ , so that  $\mathcal{F}^{\text{ann}}(\beta) \geq \mathcal{F}^{\text{hom}}(\beta)$ ,  $\beta_c^{\text{ann}} \leq \beta_c^{\text{hom}}$ . We are done if  $\beta_c^{\text{hom}} = 0$ , or equivalently, if  $\sigma \cap \tau$  is recurrent, so assume it is transient. Then  $\mathbf{E}_\sigma Z_{N,\beta}^{\sigma,\text{free}} \leq \mathbf{E}_{\sigma,\tau} [e^{\beta |\tau \cap \sigma|} \mathbf{1}_{\{\sigma_N \in \tau\}}]$  for all  $N \in \mathbb{N}$ . Since  $|\tau \cap \sigma|_\infty$  is a geometric random variable of parameter  $\mathbf{P}_{\sigma,\tau}((\sigma \cap \tau)_1 = +\infty)$ , we have  $\sup_{N \in \mathbb{N}} \mathbf{E}_\sigma Z_{N,\beta}^{\sigma,\text{free}} < +\infty$  for all  $\beta < -\log \mathbf{P}_{\sigma,\tau}((\sigma \cap \tau)_1 < +\infty) = \beta_c^{\text{hom}}$ , showing that  $\beta_c^{\text{ann}} \geq \beta_c^{\text{hom}}$ .  $\square$

From Jensen's inequality, we have  $\mathcal{F}(\beta) \leq \mathcal{F}^{\text{ann}}(\beta)$ , so that  $\beta_c \geq \beta_c^{\text{ann}}$ . When  $\tau \cap \sigma$  is recurrent,  $\beta_c^{\text{ann}} = 0$ , and our main theorem will say that  $\beta_c = 0$  as well. In the transient case we have  $\beta_c^{\text{ann}} > 0$  and we can ask whether  $\beta_c = \beta_c^{\text{ann}}$  or not.

**1.4. On the recurrence/transience of  $\tau \cap \sigma$ .** The criterion for the recurrence of the renewal  $\tau \cap \sigma$  is that

$$(1.15) \quad \mathbf{E}_\sigma \mathbf{E}_\tau[|\tau \cap \sigma|] = \sum_{n \in \mathbb{N}} \mathbf{P}_\sigma(n \in \sigma) \mathbf{P}_\tau(n \in \tau) = +\infty.$$

Under the assumption (1.5), the exact asymptotic behavior of  $\mathbf{P}_\tau(n \in \tau)$  is known. We write  $a \wedge b$  for  $\min(a, b)$ . Defining the truncated mean and tail

$$\bar{m}(n) = \mathbf{E}_\tau(\tau_1 \wedge n), \quad r_n := \mathbf{P}_\tau(\tau_1 > n),$$

we have

$$(1.16) \quad \mathbf{P}_\tau(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \begin{cases} r_n^{-2} n^{-1} \varphi(n) & \text{if } \alpha = 0, \\ \frac{\alpha \sin(\pi\alpha)}{\pi} n^{-(1-\alpha)} \varphi(n)^{-1} & \text{if } 0 < \alpha < 1, \\ \bar{m}(n)^{-1} & \text{if } \alpha = 1, \mathbf{E}_\tau[\tau_1] = +\infty, \\ \mathbf{E}_\tau[\tau_1]^{-1} & \text{if } \mathbf{E}_\tau[\tau_1] < +\infty. \end{cases}$$

The first is from [29], the second is due to Doney [17, Thm. B], the third is due to Erickson [18, Eq. (2.4)], and the fourth is the classical Renewal Theorem.

Applying (1.16) to  $\tau \cap \sigma$  in (1.15), we see that  $\alpha + \tilde{\alpha} > 1$  implies that  $\tau \cap \sigma$  is recurrent, and also that  $\alpha + \tilde{\alpha} < 1$  implies that  $\tau \cap \sigma$  is transient. The case  $\alpha + \tilde{\alpha} = 1$  is marginal, depending on the slowly varying functions  $\varphi(\cdot)$  and  $\tilde{\varphi}(\cdot)$ : if also  $\alpha, \tilde{\alpha} > 0$  then  $\tau \cap \sigma$  is recurrent if and only if

$$(1.17) \quad \sum_{n \in \mathbb{N}} \frac{1}{n \varphi(n) \tilde{\varphi}(n)} = +\infty.$$

For the case  $\alpha = 0, \tilde{\alpha} = 1$ , we have recurrence if and only if

$$(1.18) \quad \sum_{n \in \mathbb{N}} \frac{\varphi(n)}{n \tilde{m}(n) \mathbf{P}_\tau(\tau_1 \geq n)^2} = +\infty,$$

where  $\tilde{m}(n) = \mathbf{E}_\sigma(\sigma_1 \wedge n)$ , which is slowly varying since  $\tilde{\alpha} = 1$ .

## 2. MAIN RESULTS

**2.1. Results and comments.** Our main result says that in the case where  $\tau \cap \sigma$  is recurrent, or transient with  $\alpha + \tilde{\alpha} = 1$ , the quenched and annealed critical points are equal (so both equal to 0 in the recurrent case). When  $\alpha + \tilde{\alpha} < 1$ , a sufficient condition for unequal critical points involves another exponent

$$\alpha^* := \frac{1 - \alpha - \tilde{\alpha}}{\tilde{\alpha}},$$

which arises naturally in that  $\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \varphi^*(n) n^{-(1+\alpha^*)}$  for some slowly varying  $\varphi^*$  (see Lemma 4.1).

**Theorem 2.1.** *Assume (1.5).*

$$(2.1) \quad \text{If } \tilde{\alpha} > 0 \text{ and } \alpha + \tilde{\alpha} \geq 1, \quad \text{then } \beta_c = \beta_c^{\text{ann}};$$

$$(2.2) \quad \text{If } \tilde{\alpha} > 0, \alpha + \tilde{\alpha} < 1 \text{ and } \alpha^* > \frac{1}{2}, \quad \text{then } \beta_c > \beta_c^{\text{ann}}.$$

Note that  $\alpha^* > 1/2$  is equivalent to  $\alpha + \frac{3}{2}\tilde{\alpha} < 1$ . We suspect that this condition can be removed in (2.2), and in that direction, our next result shows that if (as seems plausible) the equality of critical points is determined entirely by the exponents  $\alpha, \tilde{\alpha}$  and the asymptotics of  $\varphi, \tilde{\varphi}$ , then under the condition  $\alpha, \tilde{\alpha} > 0$ , it is necessary and sufficient for equality of critical points that  $\alpha + \tilde{\alpha} \geq 1$ , that is, (2.2) is true without the condition  $\alpha^* > 1/2$ .

**Theorem 2.2.** *For any  $\alpha, \tilde{\alpha} > 0$  with  $\alpha + \tilde{\alpha} < 1$  and any slowly varying functions  $\varphi(\cdot), \tilde{\varphi}(\cdot)$ , there exist distributions for  $\tau$  and  $\sigma$  satisfying  $\mathbf{P}_\tau(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} \varphi(n)n^{-(1+\alpha)}$  and  $\mathbf{P}_\sigma(\sigma_1 = n) \stackrel{n \rightarrow \infty}{\sim} \tilde{\varphi}(n)n^{-(1+\tilde{\alpha})}$ , with  $\beta_c > \beta_c^{\text{ann}}$ .*

We expect that, following the same scheme of proof, one can extend Theorem 2.2 to the case  $\alpha = 0, \tilde{\alpha} \in (0, 1)$ . However, in the interest of brevity we do not prove it here, since it would require separate estimates from [3].

Let us now make some comments about these results.

1. Since the renewal  $\tau \cap \sigma$  can be recurrent only if  $\tilde{\alpha} + \alpha \geq 1$ , one has from (2.1) that  $\beta_c = 0$  if and only if  $\beta_c^{\text{ann}} = 0$ , that is if and only if  $\tau \cap \sigma$  is recurrent. This is notable in that  $|\tau \cap \sigma| = +\infty$   $\mathbf{P}_{\sigma, \tau}$ -a.s. is enough for  $\tau$  to be pinned on a quenched trajectory of  $\sigma$  for all  $\beta > 0$ , even though a typical  $\sigma$ -trajectory is very sparse.

2. When  $\alpha + \tilde{\alpha} = 1$ ,  $\tau \cap \sigma$  can be either recurrent or transient depending on (1.17)-(1.18). Thus we have examples with transience where the critical points are positive and equal.

3. If  $\mathbf{E}_\sigma[\sigma_1] < +\infty$ , we have a system of length approximately  $\mathbf{E}_\sigma[\sigma_1]N$ . Here we already know from [26] that the renewal  $\tau$  is pinned for all  $\beta > 0$ , since there is a positive density of  $\sigma$ -renewal points in the system.

4. We note in Remark 4.4 that if  $n \mapsto \mathbf{P}(n \in \tau)$  is non-increasing, then some calculations are simplified, and we can go slightly beyond the condition  $\alpha^* > 1/2$ , to allow  $\alpha^* = 1/2$  with  $\lim_{k \rightarrow \infty} \tilde{\varphi}(k)^{(1-\alpha)/\tilde{\alpha}} \bar{\varphi}(k) = +\infty$ . Here  $\bar{\varphi}(k) = \varphi(k)^{-1}$  if  $\alpha \in (0, 1)$  and  $\bar{\varphi}(k) = \varphi(k)/\mathbf{P}_\tau(\tau_1 > k)^2$  if  $\alpha = 0$ . The monotonicity of  $n \mapsto \mathbf{P}(n \in \tau)$  follows easily if the renewal points  $j$  of  $\tau$  correspond to times  $2j$  of return to 0 of a nearest-neighbor Markov chain (birth-death process.) Indeed, one can then couple two trajectories (one non-delayed, the other one delayed by  $2k$  for some  $k \geq 1$ ) the first time they meet, and show that  $\mathbf{P}(k + n \in \tau) \leq \mathbf{P}(n \in \tau)$  for any  $n \geq 1$ .

5. The random walk pinning model (1.4) is quite analogous to our model, replacing only  $\mathbf{1}_{\{\sigma_n \in \tau\}}$  by  $\mathbf{1}_{\{X_n = Y_n\}}$ . In our model, the decay exponent  $1 + \alpha^*$  of the probability  $\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)$  corresponds to the parameter  $\rho$  in [10] (the decay exponent of  $\mathbf{P}_{X, Y}(X_n = Y_n)$ .) Theorem 2.1 is in that sense stronger than its counterpart in the random walk pinning model, since  $\alpha^* > 1/2$  translates into  $\rho > 3/2$ , compared to  $\rho > 2$  in [10]. Moreover, Theorem 2.2 supports the conjecture that the quenched and annealed critical points differ whenever  $\rho > 1$ , see Conjecture 1.8 in [10].



**2.2. Variations of the main model: balanced and elastic polymers.** Pinning, in the quenched or annealed model, means that  $\tau$  hits a positive fraction of the sites in the quenched renewal  $\sigma$ , up to some  $\sigma_N$ . The number of renewals in  $\tau$  is unlimited. We can alternatively consider the *balanced polymer* with  $\tau$  constrained to satisfy  $\tau_N = \sigma_N$ .

A second alternative pinning notion asks whether a positive fraction of  $\tau$  renewals hit sites of  $\sigma$ , by considering a polymer of length  $\tau_N$  instead of  $\sigma_N$ . Physically this may be viewed as an *elastic polymer*, since  $\tau$  has a fixed number  $N$  of monomers and needs to stretch or contract to place them on renewals of  $\sigma$ .

**First variation: balanced polymer,  $\tau_N = \sigma_N$ .** We first consider  $\tau$  constrained to have  $\tau_N = \sigma_N$ , since in that case, both pinning notions are equivalent.

We introduce

$$(2.3) \quad \hat{Z}_{N,\beta}^\sigma = \mathbf{E}_\tau \left[ \exp(\beta |\tau \cap \sigma|_{\sigma_N}) \mathbf{1}_{\{\tau_N = \sigma_N\}} \right],$$

and

$$(2.4) \quad \hat{\mathcal{F}}(\beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \hat{Z}_{N,\beta}^\sigma = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma [\log \hat{Z}_{N,\beta}^\sigma] \quad \mathbf{P}_\sigma - a.s.$$

The proof of Proposition 1.9 applies here, establishing the existence and non-randomness of this limit.

For the balanced polymer we need the condition (1.7), for otherwise the partition function is 0 with positive probability (whenever  $\sigma_N = Nd_\sigma$ , see (1.6)), and therefore  $N^{-1} \mathbf{E}_\sigma [\log \hat{Z}_{N,\beta}^\sigma] = -\infty$  for all  $N \in \mathbb{N}$ .

Further, suppose  $\mathbf{E}_\sigma[\sigma_1] < \mathbf{E}_\tau[\tau_1]$ . Then  $\sigma_N \sim \mathbf{E}_\sigma[\sigma_1]N$  a.s., so  $\mathbf{P}_\tau(\tau_N = \sigma_N)$  decays exponentially in  $N$ . It is easy to see that therefore  $\hat{\mathcal{F}}(\beta) < 0$  for small  $\beta$ . In this sense the constraint  $\tau_N = \sigma_N$  dominates the partition function, which is unphysical, so we assume  $\mathbf{E}_\sigma[\sigma_1] \geq \mathbf{E}_\tau[\tau_1]$ , which ensures  $\hat{\mathcal{F}}(\beta) \geq 0$  for all  $\beta > 0$ . There then exists a critical point  $\hat{\beta}_c := \inf\{\beta : \hat{\mathcal{F}}(\beta) > 0\}$  such that  $\hat{\mathcal{F}}(\beta) > 0$  if  $\beta > \hat{\beta}_c$  and  $\hat{\mathcal{F}}(\beta) = 0$  if  $\beta \leq \hat{\beta}_c$ . The positivity of  $\hat{\mathcal{F}}(\beta)$  implies that  $\tau$  visits a positive proportion of  $\sigma$ -renewal points, and also that a positive proportion of  $\tau$ -renewals sit on a  $\sigma$ -renewal point. Clearly, one has that  $Z_{N,\beta}^\sigma \geq \hat{Z}_{N,\beta}^\sigma$ , so that  $\hat{\beta}_c \geq \beta_c$ . The following proposition establishes equality, and is proved in Section 6.

**Proposition 2.3.** *Assume (1.5) and (1.7). If  $\mathbf{E}_\sigma[\sigma_1] \geq \mathbf{E}_\tau[\tau_1]$ , and in particular if  $\mathbf{E}_\sigma[\sigma_1] = +\infty$ , then  $\hat{\beta}_c = \beta_c$ .*

Hence, here again,  $\hat{\beta}_c = 0$  if and only if  $\tau \cap \sigma$  is recurrent.

**Second variation: elastic polymer of length  $\tau_N$ .** In the elastic polymer, pinning essentially means that a positive fraction of  $\tau$ -renewals fall on  $\sigma$ -renewals. The partition function is

$$(2.5) \quad \bar{Z}_{N,\beta}^\sigma = \mathbf{E}_\tau \left[ \exp(\beta |\tau \cap \sigma|_{\tau_N}) \right].$$

Standard subadditivity methods to establish existence of  $\lim_{N \rightarrow \infty} N^{-1} \log \bar{Z}_{N,\beta}^\sigma$  do not work here, but we can consider the  $\liminf$  instead, since its positivity implies pinning in the above sense. We therefore define

$$(2.6) \quad \bar{\mathcal{F}}(\beta) := \liminf_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma \log \bar{Z}_{N,\beta}^\sigma,$$

which is non-decreasing in  $\beta$ , and the critical point  $\bar{\beta}_c := \inf\{\beta : \bar{\mathcal{F}}(\beta) > 0\}$ . Compared to the balanced polymer, here it is much less apparent *a priori* that we should have  $\bar{\beta}_c \geq \beta_c$ . It could be favorable for  $\tau$  to jump far beyond  $\sigma_N$  to reach an exceptionally favorable stretch of  $\sigma$ , before  $\tau_N$ . The original model (1.8) does not permit this type of strategy, so the question is whether this allows pinning in the elastic polymer with  $\beta < \beta_c$ . The answer is no, at least if  $\alpha > 0$ , or if  $\alpha = 0$  and  $\tilde{\alpha} \geq 1$ , as the following shows; the proof is in Section 6. We do not know whether  $\bar{\beta}_c = \beta_c$  when  $\alpha = 0$  and  $\tilde{\alpha} < 1$ .

**Proposition 2.4.** *Assume (1.5) with  $\tilde{\alpha} > 0$ :*

- (i)  $\bar{\beta}_c \leq \beta_c$  ;
- (ii)  $\bar{\beta}_c = \beta_c$  if  $\alpha > 0$  and if  $\alpha = 0$ ,  $\tilde{\alpha} \geq 1$ .

In particular,  $\bar{\beta}_c = 0$  whenever  $\tau \cap \sigma$  is recurrent.

**2.3. Organization of the rest of the paper.** We now present a brief outline of the proofs, and how they are organized.

In Section 3, we prove the first part (2.1) of Theorem 2.1, to establish pinning of the quenched system when  $\beta > \beta_c^{\text{ann}}$ , with  $\alpha + \tilde{\alpha} \geq 1$ . We use a rare-stretch strategy to get a lower bound on the partition function: the idea is to consider trajectories  $\tau$  which visit exponentially rare stretches where the renewals of  $\sigma$  have some (not-well-specified) special structure which reduces the entropy cost for  $\tau$  to hit a large number of  $\sigma$  sites. In Section 3.2, we classify trajectories of  $\sigma$  according to the size of this entropy cost reduction, then select a class which makes a large contribution to the annealed free energy  $\mathcal{F}^{\text{ann}}(\beta) > 0$  (we will fix  $\beta > \beta_c^{\text{ann}}$ ). We call trajectories in this class *accepting*, and the localization strategy of Section 3.3 consists in visiting all accepting segments of  $\sigma$ . More detailed heuristics are presented in Section 3.1.

In Section 4, we prove the second part (2.2) of Theorem 2.1. First, we rewrite the partition function as that of another type of pinning model — see Section 4.1; this reduces the problem to a system at the annealed critical point  $\beta_c^{\text{ann}} > 0$ . We then employ a fractional moment method, combined with coarse-graining, similar to one developed for the disordered pinning model [6, 16, 22, 23], and later used for the random walk pinning model [7, 11, 12]. In Sections 4.2-4.3, we show how one can reduce to proving a finite-size estimate of the fractional moment of the partition function. Then to show that the fractional moment of the partition function is small, we describe a general change of measure argument in Section 4.4. The change of measure is based on a set  $J$  of trajectories  $\sigma$ , defined in Section 4.5, which has high probability, but becomes rare under certain conditioning which, roughly speaking, makes the partition function  $Z_{N,\beta}^\sigma$  large. Identification of such an event is the key step; our choice of  $J$  only works in full generality for  $\alpha^* > 1/2$ , necessitating that hypothesis.

In Section 5, we define an alternative event in place of  $J$  which, by similar methods, allows examples of distributions of  $\tau, \sigma$  with  $\alpha + \tilde{\alpha} < 1$  and arbitrary  $\alpha^* \in (0, 1)$ , for which  $\beta_c > \beta_c^{\text{ann}}$ . In Section 5.1, we find a sufficient condition on the distributions of  $\tau, \sigma$  for the alternative event to have the required properties — primarily,  $\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)$  must be much greater than  $\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \text{ for some } n \geq 2)$ ; see Remark 5.5. Then, in Section 5.2, we construct examples of  $\tau, \sigma$  for which this condition holds.

In Section 6, we study the variants of the model introduced in Section 2.2, and prove Propositions 2.3-2.3. We mainly use some particular localization strategies to obtain lower bounds on the free energy of these two models.

The Appendix is devoted to the proof of technical lemmas on renewal processes.

**2.4. Notations.** Throughout the paper,  $c_i$  are constants which depend only on the distributions of  $\tau$  and  $\sigma$ , unless otherwise specified.

Recall the truncated means

$$\bar{m}(n) = \mathbf{E}_\tau(\tau_1 \wedge n), \quad \tilde{m}(n) = \mathbf{E}_\sigma(\sigma_1 \wedge n).$$

When  $\alpha, \tilde{\alpha} > 0$  we denote by  $a_n$  the typical size of  $\tau_n$ ,  $b_n$  the typical size of  $\sigma_n$ ,  $A_n$  the typical size of  $\max\{\tau_1, \tau_2 - \tau_1, \dots, \tau_n - \tau_{n-1}\}$ , and  $B_n$  the typical size of  $\max\{\sigma_1, \sigma_2 - \sigma_1, \dots, \sigma_n - \sigma_{n-1}\}$ . More precisely,  $B_n$  is defined (up to asymptotic equivalence) by

$$(2.7) \quad B_1 = d_\sigma, \quad B_n \geq d_\sigma, \quad B_n^{\tilde{\alpha}} \tilde{\varphi}(B_n)^{-1} \stackrel{n \rightarrow \infty}{\sim} n,$$

(see (1.6)) and  $b_n$  by

$$(2.8) \quad \begin{aligned} b_n &= B_n \text{ if } 0 < \tilde{\alpha} < 1, \quad b_n \tilde{m}(b_n)^{-1} \stackrel{n \rightarrow \infty}{\sim} n \text{ if } \tilde{\alpha} = 1 \text{ and } \mathbf{E}_\sigma[\sigma_1] = \infty, \\ b_n &= \mathbf{E}_\sigma[\sigma_1] n \text{ if } \mathbf{E}_\sigma[\sigma_1] < \infty, \end{aligned}$$

and  $A_n, a_n$  are defined analogously for the distribution of  $\tau_1$ . The following can be found in [8]. These sequences are unique up to asymptotic equivalence, and in the infinite-mean case with  $0 < \alpha, \tilde{\alpha} \leq 1$ , there exist slowly varying functions  $\psi$  and  $\tilde{\psi}$ , whose asymptotics are explicit in terms of  $\alpha, \tilde{\alpha}$  and  $\varphi, \tilde{\varphi}$ , such that

$$(2.9) \quad a_n = \psi(n) n^{1/\alpha}, \quad b_n = \tilde{\psi}(n) n^{1/\tilde{\alpha}}.$$

The sequences  $A_n^{-1} \max\{\tau_1, \tau_2 - \tau_1, \dots, \tau_n - \tau_{n-1}\}$  and  $B_n^{-1} \max\{\sigma_1, \sigma_2 - \sigma_1, \dots, \sigma_n - \sigma_{n-1}\}$  converge in distribution, and the limits have strictly positive densities on  $(0, \infty)$ . When the corresponding exponents  $\alpha, \tilde{\alpha}$  are in  $(0, 1)$ , the ratios  $\tau_n/a_n$  and  $\sigma_n/b_n$  converge to stable distributions, which have strictly positive densities on  $(0, \infty)$ . When the exponent  $\alpha, \tilde{\alpha} \in [1, \infty)$ , the same ratios converge in probability to 1. This follows from the law of large numbers when the mean is finite, and from the convergence in distribution of  $(\tau_n - n\bar{m}(n))/A_n$  to a stable law if  $\alpha = 1$  and  $\mathbf{E}_\tau[\tau_1] = +\infty$ , in which case  $n\bar{m}(n)/A_n \rightarrow \infty$ , see for example [19, XVII.5 Thm. 3].

Define

$$(2.10) \quad \bar{d}_\tau = \min\{n \geq 1 : \mathbf{P}_\tau(k \in \tau) > 0 \text{ for all } k \geq n\},$$

and  $\bar{d}_\sigma$  similarly.

### 3. RARE-STRETCH STRATEGY: PROOF OF THE FIRST PART OF THEOREM 2.1

**3.1. Sketch of the proof.** We use a rare-stretch localization strategy to show that, when  $\alpha + \tilde{\alpha} \geq 1$ , we have  $\mathcal{F}(\beta) > 0$  for all  $\beta > \beta_c^{\text{ann}}$ . The idea is to identify ‘‘accepting’’ stretches of  $\sigma$  that we then require  $\tau$  to visit; the contribution to the free energy from such trajectories  $\tau$  is shown sufficient to make the free energy positive. Here, the definition of an accepting stretch is made without describing these stretches explicitly, in the following sense: for fixed large  $\ell$ , different configurations  $\{\sigma_1, \dots, \sigma_\ell\}$  are conducive to large values of  $|\tau \cap \sigma|_\ell$  to different degrees, as measured by the probability  $\mathbf{P}_\tau(|\tau \cap \sigma|_\ell \geq \tilde{\delta}\ell)$  for some appropriate  $\tilde{\delta}$ .

We divide (most of) the possible values of this probability into a finite number of intervals, which divides configurations  $\sigma$  into corresponding classes; we select the class responsible for the largest contribution to the annealed partition function, and call the corresponding trajectories  $\sigma$  accepting. We then get a lower bound for the quenched partition function by considering trajectories of length  $N \gg \ell$  which visit all accepting stretches of  $\sigma$  of length  $\ell$ .

**3.2. Preliminaries on the annealed system.** We first define the accepting property for  $\sigma$ . Let  $I_\tau$  and  $I$  be the rate functions, for contact fractions  $\delta$ , for the renewals  $\tau$  and  $\tau \cap \sigma$ , respectively:

$$(3.1) \quad I_\tau(\delta) := -\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_\tau(|\tau|_n \geq \delta n), \quad I(\delta) := -\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \delta n).$$

It is straightforward that  $I_\tau(\delta) = \delta J_\tau(\delta^{-1})$ , where  $J_\tau$  is the usual rate function for i.i.d. sums distributed as the gap between renewals in  $\tau$ , and similarly for  $I(\delta)$ . Let  $\delta_{\max} = \sup\{\delta \in [0, 1] : I(\delta) < \infty\}$  and  $0 < \delta < \delta_{\max}$ . Then

$$(3.2) \quad \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \delta n) = e^{-I(\delta)n(1+o(1))}.$$

For technical reasons, we need the following lemma: it says that the rate function  $I(\delta)$  is unchanged when also imposing  $\sigma_n \in \tau$ , and that  $\sigma_n$  grows only polynomially.

**Lemma 3.1.** *Suppose  $\tilde{\alpha} > 0$ . For all  $\delta < \delta_{\max}$ , for  $b_n$  from (2.8),*

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n \leq 2b_n \mid |\tau \cap \sigma|_n \geq \delta n) = 0,$$

so that

$$(3.4) \quad \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \delta n, \sigma_n \in \tau, \sigma_n \leq 2b_n) = e^{-I(\delta)n(1+o(1))}.$$

*Proof.* Let  $\varepsilon > 0$  and let  $m$  be the least integer in  $[\delta n, \infty)$ . Since  $\delta < \delta_{\max}$ , there exists  $\eta > 0$  with

$$(3.5) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_{\sigma, \tau}((\tau \cap \sigma)_m \leq (1 - \eta)n \mid |\tau \cap \sigma|_n \geq \delta n) \\ & \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_{(1-\eta)n} \geq \delta n \mid |\tau \cap \sigma|_n \geq \delta n) > -\varepsilon. \end{aligned}$$

Since  $b_n > n$  and  $\sigma_n/b_n$  converges either to a stable limit or to 1, there exist  $n_0 \geq 1$  and  $\theta > 0$  as follows. For  $n$  large and  $\delta n \leq j \leq (1 - \eta)n$ ,  $m \leq k \leq j$ ,

$$(3.6) \quad \begin{aligned} & \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n \leq 2b_n \mid (\tau \cap \sigma)_m = j; \sigma_k = j) \\ & \geq \mathbf{P}_\sigma(\sigma_{n-k} \leq b_{n-k}) \min_{k \leq n+b_n} \mathbf{P}_\tau(k \in \tau) \\ & \geq \theta \min_{k \leq 2b_n} \mathbf{P}_\tau(k \in \tau), \end{aligned}$$

which is bounded below by an inverse power of  $n$ , uniformly in such  $j$ . Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n \leq 2b_n \mid (\tau \cap \sigma)_m \leq (1 - \eta)n) = 0,$$

which with (3.5) proves (3.3), since  $\varepsilon$  is arbitrary.  $\square$

Since we take  $\beta > \beta_c^{\text{ann}} = \beta_c^{\text{hom}}$ , the associated annealed and homogeneous double pinning free energies are positive:

$$(3.7) \quad \mathcal{F}^{\text{ann}}(\beta) \geq \mathcal{F}^{\text{hom}}(\beta) = \sup_{\delta > 0} (\beta\delta - I(\delta)) > 0.$$

Let  $\tilde{\delta} < \delta_{\max}$  satisfy

$$(3.8) \quad \beta\tilde{\delta} - I(\tilde{\delta}) > \frac{1}{2}\mathcal{F}^{\text{hom}}(\beta),$$

and let

$$f_n(\sigma) := \mathbf{P}_\tau(|\tau \cap \sigma|_n \geq \tilde{\delta}n) \mathbf{1}_{\{\sigma_n \in \tau\}} \mathbf{1}_{\{\sigma_n \leq 2b_n\}}.$$

Note that  $f_n$  is actually a function of the finite renewal sequence  $B := \{0, \sigma_1, \dots, \sigma_n\}$ , so we may write it as  $f_n(B)$ . We decompose the probability that appears in  $f_n$  according to what portion of the cost  $I(\tilde{\delta})n$  is borne by  $\sigma$  versus by  $\tau$ : we fix  $\varepsilon > 0$  and write

$$(3.9) \quad \begin{aligned} & \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \tilde{\delta}n, \sigma_n \in \tau, \sigma_n \leq 2b_n) \\ & \leq \sum_{0 \leq k \leq 1/\varepsilon} \mathbf{P}_\sigma \left( f_n(\sigma) \in (e^{-(k+1)\varepsilon I(\tilde{\delta})n}, e^{-k\varepsilon I(\tilde{\delta})n}] \right) e^{-k\varepsilon I(\tilde{\delta})n} + e^{-(1+\varepsilon)I(\tilde{\delta})n}. \end{aligned}$$

Then by Lemma 3.1, there exists some  $n_\varepsilon$  large enough so that for  $n \geq n_\varepsilon$ ,

$$e^{-(1+\varepsilon)I(\tilde{\delta})n} \leq \frac{1}{2} \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \tilde{\delta}n, \sigma_n \in \tau, \sigma_n \leq 2b_n).$$

Hence choosing  $k_0 = k_0(n) \in [0, 1/\varepsilon]$  to be the index of the largest term in the sum in (3.9), we have

$$(3.10) \quad \begin{aligned} & \mathbf{P}_\sigma \left( f_n(\sigma) \in (e^{-(k_0+1)\varepsilon I(\tilde{\delta})n}, e^{-k_0\varepsilon I(\tilde{\delta})n}] \right) e^{-k_0\varepsilon I(\tilde{\delta})n} \\ & \geq \frac{1}{2} \frac{1}{1 + 1/\varepsilon} \mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \tilde{\delta}n, \sigma_n \in \tau, \sigma_n \leq 2b_n). \end{aligned}$$

We let  $a := a(n) := k_0\varepsilon$ , so  $a \in [0, 1]$  represents, roughly speaking, the proportion of the cost  $I(\tilde{\delta})$  borne by  $\tau$  in the most likely cost split, i.e. in the  $k = k_0$  term. Lemma 3.1 says that if  $n \geq n_\varepsilon$  for some  $n_\varepsilon$  large enough, one has that

$$\mathbf{P}_{\sigma, \tau}(|\tau \cap \sigma|_n \geq \tilde{\delta}n, \sigma_n \in \tau, \sigma_n \leq 2b_n) \geq e^{-(1+\varepsilon/2)I(\tilde{\delta})n}.$$

Then, (3.10) gives that

$$(3.11) \quad \mathbf{P}_\sigma \left( f_n(\sigma) \in (e^{-(a+\varepsilon)I(\tilde{\delta})n}, e^{-aI(\tilde{\delta})n}] \right) \geq \frac{1}{2} \frac{\varepsilon}{1 + \varepsilon} e^{aI(\tilde{\delta})n} e^{-(1+\varepsilon/2)I(\tilde{\delta})n} \geq e^{-(1-a+\varepsilon)I(\tilde{\delta})n},$$

where the last inequality holds provided that  $n$  is large enough ( $n \geq n_\varepsilon$ ).

This leads us to define, for  $n \geq n_\varepsilon$  the event  $\mathcal{A}_n$  for  $\sigma$  (or more precisely for  $\{0, \sigma_1, \dots, \sigma_n\}$ ) of being *accepting*, by

$$(3.12) \quad \sigma \in \mathcal{A}_n \quad \text{if} \quad f_n(\sigma) \in (e^{-(a+\varepsilon)I(\tilde{\delta})n}, e^{-aI(\tilde{\delta})n}].$$

Then (3.11) gives the lower bound

$$\mathbf{P}_\sigma(\sigma \in \mathcal{A}_n) \geq e^{-(1-a+\varepsilon)I(\tilde{\delta})n} \quad \text{if} \quad n \geq n_\varepsilon.$$

Moreover we have for  $n \geq n_\varepsilon$

$$(3.13) \quad \begin{aligned} \mathbf{E}_\tau \left[ e^{\beta|\tau \cap \sigma|_{\sigma_n}} \mathbf{1}_{\{\sigma_n \in \tau\}} \right] \mathbf{1}_{\{\sigma \in \mathcal{A}_n\}} &\geq e^{\beta \tilde{\delta} n} \mathbf{P}_\tau \left( |\tau \cap \sigma|_{\sigma_n} \geq \tilde{\delta} n, \sigma_n \in \tau, \sigma_n \leq 2b_n \right) \mathbf{1}_{\{\sigma \in \mathcal{A}_n\}} \\ &\geq e^{(\beta \tilde{\delta} n - (a+\varepsilon)I(\tilde{\delta}))n} \mathbf{1}_{\{\sigma \in \mathcal{A}_n\}}. \end{aligned}$$

**3.3. Localization strategy.** Let us fix  $\beta > \beta_c^{\text{ann}} = \beta_c^{\text{hom}}$ ,  $\tilde{\delta}$  as in (3.8) and  $\ell$  large (in particular  $\ell \geq n_\varepsilon$ , so that (3.13) holds for  $n = \ell$ .) We divide  $\sigma$  into segments of  $\ell$  jumps which we denote  $Q_1, Q_2, \dots$ , that is,  $Q_i := \{\sigma_{(i-1)\ell+1} - \sigma_{(i-1)\ell}, \dots, \sigma_{i\ell} - \sigma_{(i-1)\ell}\}$ . We say that  $Q_i$  is an *accepting segment* if  $Q_i \in \mathcal{A}$ . We now write  $\mathcal{A}$  for  $\mathcal{A}_\ell$ , and  $a$  for  $a(\ell)$ . Let

$$(3.14) \quad p_{\mathcal{A}} := \mathbf{P}_\sigma(Q_i \in \mathcal{A}) \geq e^{-(1-a+\varepsilon)I(\tilde{\delta})\ell},$$

where the inequality is from (3.11). Informally, the strategy for  $\tau$  is then to visit all accepting segments, with no other restrictions on other regions. On each such segment one has an energetic gain of at least  $e^{(\beta \tilde{\delta} n - (a+\varepsilon)I(\tilde{\delta}))n}$ , thanks to (3.13).

We define  $M_0 := 0$ , and iteratively  $M_i := \inf\{j > M_{i-1}; Q_j \in \mathcal{A}\}$ ; then  $M_i - M_{i-1}$  are independent geometric random variables of parameter  $p_{\mathcal{A}}$ . Imposing visits to all the accepting segments  $Q_{M_i}$  (that is imposing  $\sigma_{M_i\ell}, \sigma_{(M_i+1)\ell} \in \tau$ ), one has

$$(3.15) \quad Z_{M_k\ell, \beta}^\sigma \geq \mathbf{E}_\tau \left( \prod_{i=1}^k \exp \left( \beta |\sigma \cap \tau \cap (\sigma_{(M_{i-1})\ell}, \sigma_{M_i\ell})| \right) \mathbf{1}_{\{\sigma_{M_i\ell} \in \tau\}} \mathbf{1}_{\{\sigma_{(M_{i-1})\ell} \in \tau\}} \right).$$

So, using (3.13), and with the convention that  $\mathbf{P}_\tau(0 \in \tau) = 1$ , we have

$$(3.16) \quad \log Z_{M_k\ell, \beta}^\sigma \geq \sum_{i=1}^k \left( \log \mathbf{P}_\tau(\sigma_{(M_{i-1})\ell} - \sigma_{M_{i-1}\ell} \in \tau) + \beta \tilde{\delta} \ell - (a + \varepsilon)I(\tilde{\delta})\ell \right).$$

Letting  $k$  go to infinity, and using the Law of Large Numbers twice, we get

$$(3.17) \quad \begin{aligned} \mathcal{F}(\beta) &\geq \liminf_{k \rightarrow \infty} \frac{1}{M_k\ell} \log Z_{M_k\ell, \beta}^\sigma \\ &\geq \frac{1}{\mathbf{E}_\sigma[M_1]} \left( \frac{1}{\ell} \mathbf{E}_\sigma \log \mathbf{P}_\tau(\sigma_{(M_1-1)\ell} \in \tau) + \beta \tilde{\delta} - (a + \varepsilon)I(\tilde{\delta}) \right), \end{aligned}$$

with  $\mathbf{E}_\sigma[M_1] = 1/p_{\mathcal{A}}$ .

We are left with estimating  $\frac{1}{\ell} \mathbf{E}_\sigma \log \mathbf{P}_\tau(\sigma_{(M_1-1)\ell} \in \tau)$ . For any  $\alpha \geq 0$  and non-random times  $n$ , (1.16) ensures that, for any  $\eta > 0$ , one can find some  $n_\eta$  such that, for all  $n \geq n_\eta$ ,

$$(3.18) \quad \log \mathbf{P}_\tau(n \in \tau) \geq -(1 - \alpha \wedge 1 + \eta) \log n.$$

Therefore if  $\ell_\eta$  is chosen large enough, and  $\ell \geq \ell_\eta$ , one has

$$(3.19) \quad \mathbf{E}_\sigma \log \mathbf{P}_\tau(\sigma_{(M_1-1)\ell} \in \tau) \geq -(1 - \alpha \wedge 1 + \eta) \mathbf{E}_\sigma[\mathbf{1}_{\{M_1 > 1\}} \log \sigma_{(M_1-1)\ell}].$$

The following is proved below.

**Lemma 3.2.** *Let  $\eta > 0$  and suppose  $\tilde{\alpha} > 0$  in (1.5). Provided  $\ell$  is sufficiently large we have*

$$\mathbf{E}_\sigma[\mathbf{1}_{\{M_1 > 1\}} \log \sigma_{(M_1-1)\ell}] \leq \frac{1 + \eta}{\tilde{\alpha} \wedge 1} \left( \log \ell + \log \frac{1}{p_{\mathcal{A}}} \right).$$

Combining (3.19) with Lemma 3.2, one gets for  $\ell \geq \ell_\eta$ , and provided that  $\ell_\eta$  is large so that  $\frac{1}{\ell} \log \ell \leq \eta$ ,

$$(3.20) \quad \frac{1}{\ell} \mathbf{E}_\sigma \log \mathbf{P}_\tau(\sigma_{(M_1-1)\ell} \in \tau) \geq -(1 - \alpha \wedge 1 + \eta) \frac{1 + \eta}{\tilde{\alpha} \wedge 1} \left( \frac{1}{\ell} \log \frac{1}{p_A} + \eta \right).$$

From (3.17) and (3.20), and assuming that  $\eta$  is sufficiently small (depending on  $\varepsilon$ ), we get

$$(3.21) \quad \mathcal{F}(\beta) \geq p_A \left( \beta \tilde{\delta} - (a + \varepsilon) I(\tilde{\delta}) - \frac{1 - \alpha \wedge 1}{\tilde{\alpha} \wedge 1} (1 + \varepsilon) \frac{1}{\ell} \log \frac{1}{p_A} - \varepsilon \right).$$

Now, the crucial point is that  $\alpha + \tilde{\alpha} \geq 1$ , so that  $(1 - \alpha \wedge 1)/\tilde{\alpha} \wedge 1 \leq 1$ , and one has

$$(3.22) \quad \begin{aligned} \mathcal{F}(\beta) &\geq p_A \left( \beta \tilde{\delta} - (a + \varepsilon) I(\tilde{\delta}) - (1 + \varepsilon) \frac{1}{\ell} \log \frac{1}{p_A} - \varepsilon \right) \\ &\geq p_A \left( \beta \tilde{\delta} - (a + \varepsilon) I(\tilde{\delta}) - (1 + \varepsilon)(1 - a + \varepsilon) I(\tilde{\delta}) - \varepsilon \right) \\ &\geq p_A \left( \frac{1}{2} \mathcal{F}^{\text{hom}}(\beta) - (3 - a + \varepsilon) \varepsilon I(\tilde{\delta}) - \varepsilon \right), \end{aligned}$$

where we used (3.14) in the second inequality, and (3.8) in the third one. If we choose  $\varepsilon$  small enough, we therefore have that  $\mathcal{F}(\beta) \geq \frac{1}{4} p_A \mathcal{F}^{\text{hom}}(\beta) > 0$ , as soon as  $\beta > \beta_c^{\text{ann}} = \beta_c^{\text{hom}}$ . This completes the proof of (2.2) Theorem 2.1.  $\square$

*Proof of Lemma 3.2.* Let us first bound  $\mathbf{E}_\sigma \log \sigma_n$  for deterministic large  $n$ . Consider first  $\tilde{\alpha} \in (0, 1)$ . Let  $\eta \in (0, 1/4)$ . From [17], whenever  $k/b_n \rightarrow \infty$ ,  $\mathbf{P}_\sigma(\sigma_n = k) \sim n \mathbf{P}_\sigma(\sigma_1 = k)$ . Hence using (1.5), uniformly in  $t > 1 + \eta$ , as  $n \rightarrow \infty$ ,

$$(3.23) \quad \mathbf{P}_\sigma(\sigma_n \geq n^{t/\tilde{\alpha}}) = (1 + o(1)) n \mathbf{P}_\sigma(\sigma_1 \geq n^{t/\tilde{\alpha}}) \leq n^{1-t+\eta}.$$

Therefore there exists some  $n_\eta$  such that if  $n \geq n_\eta$ ,

$$(3.24) \quad \begin{aligned} \mathbf{E}_\sigma \frac{\tilde{\alpha} \log \sigma_n}{\log n} &\leq 1 + \eta + \int_{1+\eta}^{\infty} \mathbf{P}_\sigma \left( \log \sigma_n \geq \frac{t}{\tilde{\alpha}} \log n \right) dt \\ &\leq 1 + \eta + n^{1+\eta} \int_{1+\eta}^{\infty} e^{-t \log n} dt \leq 1 + \eta + \frac{1}{\log n}, \end{aligned}$$

so that, provided that  $n$  is large enough,

$$(3.25) \quad \mathbf{E}_\sigma \log \sigma_n \leq \frac{1 + 2\eta}{\tilde{\alpha}} \log n.$$

For  $\tilde{\alpha} \geq 1$ , we can multiply the probabilities  $\mathbf{P}_\sigma(\sigma_1 = n)$  by the increasing function  $cn^\gamma$  for appropriate  $c, \gamma > 0$ , and thereby obtain a distribution with tail exponent in  $(1 - \eta, 1)$  which stochastically dominates the distribution of  $\sigma_1$ . This shows that

$$\mathbf{E}_\sigma \log \sigma_n \leq \frac{1 + 2\eta}{1 - \eta} \log n.$$

Thus for all  $\tilde{\alpha} > 0$  and  $\eta \in (0, 1/4)$ , for  $n$  large,

$$(3.26) \quad \mathbf{E}_\sigma \log \sigma_n \leq \frac{1 + 4\eta}{\tilde{\alpha} \wedge 1} \log n.$$

Now let  $n_{\mathcal{A}} = K/p_{\mathcal{A}}$ , with  $K$  (large) to be specified. First, provided that  $\ell$  is large enough,

$$(3.27) \quad \mathbf{E}_{\sigma} [\mathbf{1}_{\{M_1 > 1\}} \mathbf{1}_{\{M_1 \leq 2n_{\mathcal{A}}\}} \log \sigma_{(M_1-1)\ell}] \leq \mathbf{E}_{\sigma} [\log \sigma_{2n_{\mathcal{A}}\ell}] \leq \frac{1+4\eta}{\tilde{\alpha} \wedge 1} \log(2n_{\mathcal{A}}\ell),$$

where we used (3.26) in the last inequality. Second, fixing  $m \geq 3$ , we have for any  $j \in ((m-1)n_{\mathcal{A}}, mn_{\mathcal{A}}]$

$$(3.28) \quad \begin{aligned} \mathbf{E}_{\sigma} [\mathbf{1}_{\{M_1 > 1\}} \log \sigma_{(M_1-1)\ell} \mid M_1 = j] &= \mathbf{E}_{\sigma} [\log \sigma_{(j-1)\ell} \mid Q_1 \notin \mathcal{A}, \dots, Q_{j-1} \notin \mathcal{A}] \\ &\leq \mathbf{E}_{\sigma} [\log \sigma_{mn_{\mathcal{A}}\ell} \mid Q_1 \notin \mathcal{A}, \dots, Q_{mn_{\mathcal{A}}} \notin \mathcal{A}] \\ &\leq \sum_{r=1}^m \mathbf{E}_{\sigma} \left[ \log (\sigma_{rn_{\mathcal{A}}\ell} - \sigma_{(r-1)n_{\mathcal{A}}\ell}) \mid Q_i \notin \mathcal{A} \forall i \in ((r-1)n_{\mathcal{A}}, rn_{\mathcal{A}}] \right] \\ &= m \mathbf{E}_{\sigma} (\log \sigma_{n_{\mathcal{A}}\ell} \mid Q_i \notin \mathcal{A} \forall i \leq n_{\mathcal{A}}) \\ &\leq m \frac{\mathbf{E}_{\sigma} [\log \sigma_{n_{\mathcal{A}}\ell}]}{(1-p_{\mathcal{A}})^{n_{\mathcal{A}}}}. \end{aligned}$$

Hence, using (3.26), one has that if  $\ell$  is large enough,

$$(3.29) \quad \begin{aligned} \mathbf{E}_{\sigma} \left[ \mathbf{1}_{\{M_1 \in ((m-1)n_{\mathcal{A}}, mn_{\mathcal{A}}]\}} \log \sigma_{(M_1-1)\ell} \right] &\leq \mathbf{P}_{\sigma}(M_1 > (m-1)n_{\mathcal{A}}) m \frac{1+4\eta}{\tilde{\alpha} \wedge 1} \frac{\log(n_{\mathcal{A}}\ell)}{(1-p_{\mathcal{A}})^{n_{\mathcal{A}}}} \\ &\leq m(1-p_{\mathcal{A}})^{(m-2)n_{\mathcal{A}}} \frac{1+4\eta}{\tilde{\alpha} \wedge 1} \log(n_{\mathcal{A}}\ell) \\ &\leq 2me^{-(m-2)K} \frac{1}{\tilde{\alpha} \wedge 1} \log(n_{\mathcal{A}}\ell). \end{aligned}$$

Combining (3.27), and (3.29), we obtain, provided that  $K$  is large enough (depending on  $\eta$ )

$$(3.30) \quad \begin{aligned} \mathbf{E}_{\sigma} [\mathbf{1}_{\{M_1 > 1\}} \log \sigma_{(M_1-1)\ell}] &\leq \frac{1+4\eta}{\tilde{\alpha} \wedge 1} \log(2n_{\mathcal{A}}\ell) + \sum_{m=3}^{\infty} 2me^{-(m-2)K} \frac{1}{\tilde{\alpha} \wedge 1} \log(n_{\mathcal{A}}\ell) \\ &\leq \frac{1+5\eta}{\tilde{\alpha} \wedge 1} \log(2n_{\mathcal{A}}\ell) \\ &\leq \frac{1+6\eta}{\tilde{\alpha} \wedge 1} \left( \log \ell + \log \frac{1}{p_{\mathcal{A}}} \right), \end{aligned}$$

where the last inequality is valid provided that  $\ell$  is large.  $\square$

#### 4. COARSE-GRAINING AND CHANGE OF MEASURE: PROOF OF THE SECOND PART OF THEOREM 2.1

In this section and the following ones, we deal with the case  $\alpha + \tilde{\alpha} < 1$ . In particular, one has  $0 \leq \alpha < 1$ ,  $0 < \tilde{\alpha} < 1$  and  $\alpha^* > 0$ . Moreover, the renewal  $\tau \cap \sigma$  is transient, so  $\mathbf{E}_{\sigma, \tau} [|\tau \cap \sigma|] < +\infty$ , and the annealed critical point is  $\beta_c^{\text{ann}} = -\log(1 - \mathbf{E}_{\sigma, \tau} [|\tau \cap \sigma|^{-1}]) > 0$ .

**4.1. Alternative representation of the partition function.** We use a standard alternative representation of the partition function, used for example in the Random Walk Pinning Model, see [10] and [7, 11, 12], and in other various context: it is the so-called *polynomial chaos expansion*, which is the cornerstone of [13].



We write  $e^\beta = 1 + z$ , and expand  $(1 + z)^{\sum_{n=1}^N \mathbf{1}_{\{\sigma_n \in \tau\}}}$ : we get

$$(4.1) \quad Z_{N,\beta}^\sigma = \sum_{m=1}^N z^m \sum_{1 \leq i_1 < \dots < i_m = N} \mathbf{E}_\tau \left[ \prod_{k=1}^m \mathbf{1}_{\{\sigma_{i_k} \in \tau\}} \right] = \sum_{m=1}^N z^m \sum_{1 \leq i_1 < \dots < i_m = N} \prod_{k=1}^m \mathbf{E}_\tau [\mathbf{1}_{\{\sigma_{i_k} - \sigma_{i_{k-1}} \in \tau\}}],$$

where  $i_0 := 0$ . Now, let us define

$$(4.2) \quad K^*(n) := \frac{1}{\mathbf{E}_{\tau,\sigma}[\tau \cap \sigma]} \mathbf{E}_{\tau,\sigma}[\mathbf{1}_{\{\sigma_n \in \tau\}}],$$

so that  $\sum_{n \in \mathbb{N}} K^*(n) = 1$ , and we denote  $\nu$  a renewal process with law  $\mathbf{P}_\nu$  and inter-arrival distribution  $\mathbf{P}_\nu(\nu_1 = n) = K^*(n)$ . In particular,  $\nu$  is recurrent. The following lemma is proven in Appendix A.2; the asymptotics of  $\varphi^*$  are given explicitly in the proof.

**Lemma 4.1.** *Under Assumption (1.5), with  $\tilde{\alpha} > 0$  and  $\alpha + \tilde{\alpha} < 1$ , there exists a slowly varying function  $\varphi^*(\cdot)$  such that*

$$(4.3) \quad K^*(n) = \varphi^*(n) n^{-(1+\alpha^*)} \quad \text{with } \alpha^* = \frac{1 - \alpha - \tilde{\alpha}}{\tilde{\alpha}} > 0.$$

We want to interpret the expansion (4.1) as the partition function of some pinning model, with underlying renewal  $\nu$ . We define  $z_c^{\text{ann}} := e^{\beta c} - 1$ , so that  $z_c^{\text{ann}} = \mathbf{E}_{\tau,\sigma}[\tau \cap \sigma]^{-1}$ . Then we write  $z = z_c^{\text{ann}} e^u$ : thanks to (4.2), we have

$$(4.4) \quad \begin{aligned} Z_{N,\beta}^\sigma &=: \check{Z}_{N,u}^\sigma = \sum_{m=1}^N e^{um} \sum_{i_1 < \dots < i_m = N} \prod_{k=1}^m K^*(i_k - i_{k-1}) w(\sigma, i_{k-1}, i_k) \\ &= \mathbf{E}_\nu \left[ e^{u|\nu \cap (0,N)]} W(\nu, \sigma, [0, N]) \mathbf{1}_{\{N \in \nu\}} \right], \end{aligned}$$

with

$$(4.5) \quad w(\sigma, a, b) := \frac{\mathbf{E}_\tau[\mathbf{1}_{\{\sigma_b - \sigma_a \in \tau\}}]}{\mathbf{E}_{\tau,\sigma}[\mathbf{1}_{\{\sigma_b - \sigma_a \in \tau\}}]}, \quad \mathbf{E}_\sigma[w(\sigma, a, b)] = 1,$$

and

$$(4.6) \quad W(\nu, \sigma, [a, b]) := \frac{\mathbf{E}_\tau[\prod_{k:\nu_k \in [a,b]} \mathbf{1}_{\{\sigma_{\nu_k} \in \tau\}}]}{\mathbf{E}_{\tau,\sigma}[\prod_{k:\nu_k \in [a,b]} \mathbf{1}_{\{\sigma_{\nu_k} \in \tau\}}]}, \quad \mathbf{E}_\sigma[W(\nu, \sigma, [a, b])] = 1.$$

Thus  $\check{Z}_{N,u}^\sigma$  corresponds to a pinning model: an excursion  $(\nu_i, \nu_{i+1}]$  of  $\nu$  is weighted by  $e^u w(\sigma, \nu_i, \nu_{i+1})$ . Note that, since  $\mathbf{E}_\sigma[w(\sigma, a, b)] = 1$ , the annealed partition function,

$$\mathbf{E}_\sigma \check{Z}_{N,u}^\sigma = \mathbf{E}_\nu \left[ e^{u|\nu \cap (0,N)]} \mathbf{1}_{\{N \in \nu\}} \right],$$

is the partition function of a homogeneous pinning model, with an underlying recurrent renewal  $\nu$ : the annealed critical point is therefore  $u_c^{\text{ann}} = 0$ .

**4.2. Fractional moment method.** We are left with studying the new representation of the partition function,  $\check{Z}_{N,u}^\sigma$ , and in particular, we want to show that its (quenched) critical point is positive. For that purpose, it is enough to show that there exists some  $u > 0$  and some  $\zeta > 0$  such that

$$(4.7) \quad \liminf_{n \in \mathbb{N}} \mathbf{E}_\sigma \left[ (\check{Z}_{N,u}^\sigma)^\zeta \right] < +\infty.$$

Indeed, using Jensen's inequality, one has that

$$(4.8) \quad \mathcal{F}(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma \log Z_{N,\beta}^\sigma = \lim_{N \rightarrow \infty} \frac{1}{\zeta N} \mathbf{E}_\sigma \log (\check{Z}_{N,u}^\sigma)^\zeta \leq \liminf_{N \rightarrow \infty} \frac{1}{\zeta N} \log \mathbf{E}_\sigma \left[ (\check{Z}_{N,u}^\sigma)^\zeta \right],$$

so that (4.7) implies  $\mathcal{F}(\beta) = 0$ .

**4.3. The coarse-graining procedure.** Let us fix the coarse-graining length  $L = 1/u$ , and decompose the system into blocks of size  $L$ :  $\mathcal{B}_i := \{(i-1)L + 1, \dots, iL\}$ ,  $i \geq 1$ . Considering a system of length  $nL$ , for  $n \in \mathbb{N}$ , we have

$$(4.9) \quad \check{Z}_{nL,u}^\sigma = \sum_{\mathcal{I}=\{1 \leq i_1 < \dots < i_m = n\}} \sum_{\substack{d_{i_1} \leq f_{i_1} \\ d_{i_1}, f_{i_1} \in \mathcal{B}_{i_1}}} \cdots \sum_{\substack{d_{i_m} \leq f_{i_m} = nL \\ d_{i_m} \in \mathcal{B}_{i_m}}} \prod_{k=1}^m \check{Z}_{[d_{i_k}, f_{i_k}], u}^\sigma K^*(d_{i_k} - f_{i_{k-1}}) w(\sigma, f_{i_{k-1}}, d_{i_k}),$$

where  $f_0 = i_0 = 0$  and

$$\begin{aligned} \check{Z}_{[a,b],u}^\sigma &:= \mathbf{E}_\nu \left[ e^{u|\nu \cap (a,b)|} W(\nu, \sigma, [a, b]) \mathbf{1}_{\{b \in \nu\}} \mid a \in \nu \right] \\ &\leq e \mathbf{E}_\nu \left[ W(\nu, \sigma, [a, b]) \mathbf{1}_{\{b \in \nu\}} \mid a \in \nu \right] = e \check{Z}_{[a,b],0}^\sigma. \end{aligned}$$

Then, we denote by  $\mathcal{Z}_\mathcal{I}$  the partition function with  $u = 0$ , and where  $\nu$  is restricted to visit only blocks  $\mathcal{B}_i$  for  $i \in \mathcal{I}$ : if  $\mathcal{I} = \{1 \leq i_1 < \dots < i_m = n-1\}$ ,

$$(4.10) \quad \mathcal{Z}_\mathcal{I} := \sum_{\substack{d_{i_1} \leq f_{i_1} \\ d_{i_1}, f_{i_1} \in \mathcal{B}_{i_1}}} \cdots \sum_{\substack{d_{i_m} \leq f_{i_m} = nL \\ d_{i_m} \in \mathcal{B}_{i_m}}} \prod_{k=1}^m \check{Z}_{[d_{i_k}, f_{i_k}], 0}^\sigma K^*(d_{i_k} - f_{i_{k-1}}) w(\sigma, f_{i_{k-1}}, d_{i_k}).$$

For  $\zeta < 1$  we have

$$(4.11) \quad (\check{Z}_{nL,u}^\sigma)^\zeta \leq \sum_{\mathcal{I} \subset \{1, \dots, n\}} e^{\zeta |\mathcal{I}|} (\mathcal{Z}_\mathcal{I})^\zeta,$$

and we are left with estimating  $\mathbf{E}_\sigma [(\mathcal{Z}_\mathcal{I})^\zeta]$ .

We choose  $\zeta$  given by  $(1 + \alpha^*/2)\zeta = 1 + \alpha^*/4$ . We will show in the next two sections that for every  $\delta > 0$ , there exists  $L_0$  such that for any  $L \geq L_0$

$$(4.12) \quad \mathbf{E}_\sigma [(\mathcal{Z}_\mathcal{I})^\zeta] \leq \delta^{|\mathcal{I}|} \prod_{k=1}^{|\mathcal{I}|} (i_k - i_{k-1})^{-(1+\alpha^*/4)}.$$

With (4.11), this shows that

$$(4.13) \quad \mathbf{E}_\sigma [(\check{Z}_{nL,u}^\sigma)^\zeta] \leq \sum_{\mathcal{I} \subset \{1, \dots, n\}, n \in \mathcal{I}} \prod_{k=1}^{|\mathcal{I}|} \frac{e^{\zeta \delta}}{(i_k - i_{k-1})^{1+\alpha^*/4}}.$$

We choose  $\delta$  so that  $\hat{K}(n) = e^{\zeta \delta} n^{-(1+\alpha^*/4)}$  sums to 1, making it the inter-arrival probability of a recurrent renewal process  $\hat{\tau}$ . We then have  $\mathbf{E}_\sigma [(\check{Z}_{nL,u}^\sigma)^\zeta] \leq \mathbf{P}_{\hat{\tau}}(n \in \hat{\tau}) \leq 1$ , which yields (4.7) and concludes the proof of the second part of Theorem 2.1.  $\square$

**4.4. Change of measure argument.** To estimate  $\mathcal{Z}_{\mathcal{I}}$  and prove (4.12), we use a change of measure, but only on the blocks  $\mathcal{B}_i$ , for  $i \in \mathcal{I}$ . The idea is to choose an event  $J_L$  depending on  $\{0, \sigma_1, \dots, \sigma_L\}$  which has a small probability under  $\mathbf{P}_\sigma$ , but large probability under the modified measure, see Lemma 4.2 or Lemma 5.1.

With the event  $J_L$  to be specified, we define, for some (small)  $\eta > 0$ ,

$$(4.14) \quad g(\sigma) := \eta \mathbf{1}_{J_L} + \mathbf{1}_{J_L^c}, \quad g_{\mathcal{I}} := \prod_{i \in \mathcal{I}} g(\sigma_{\mathcal{B}_i}),$$

where  $\sigma_{\mathcal{B}_i} = \{0, \sigma_{iL+1} - \sigma_{iL}, \dots, \sigma_{(i+1)L} - \sigma_{iL}\}$  is the translation of  $\sigma$  by  $\sigma_{iL}$ .

Using Hölder's inequality, we have

$$(4.15) \quad \mathbf{E}_\sigma[(\mathcal{Z}_{\mathcal{I}})^\zeta] = \mathbf{E}_\sigma[g_{\mathcal{I}}^{-\zeta} (g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}})^\zeta] \leq \mathbf{E}_\sigma[(g_{\mathcal{I}})^{-\zeta/(1-\zeta)}]^{1-\zeta} \mathbf{E}_\sigma[g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}}]^\zeta.$$

The first term on the right in (4.15) is easily computed: assuming we choose  $J_L$  with  $\mathbf{P}_\sigma(J_L) \leq \eta^{\zeta/(1-\zeta)}$ , we have

$$(4.16) \quad \begin{aligned} \mathbf{E}_\sigma \left[ (g_{\mathcal{I}})^{-\zeta/(1-\zeta)} \right]^{1-\zeta} &= \mathbf{E}_\sigma \left[ (g(\sigma))^{-\zeta/(1-\zeta)} \right]^{|I|(1-\zeta)} \\ &= (\eta^{-\zeta/(1-\zeta)} \mathbf{P}_\sigma(J_L) + \mathbf{P}_\sigma(J_L^c))^{|I|(1-\zeta)} \\ &\leq 2^{|I|}. \end{aligned}$$

We are left to estimate  $\mathbf{E}_\sigma[g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}}]$ . For this it is useful to control  $\mathbf{E}_\sigma[g(\sigma) \check{Z}_{[a,b]}^\sigma]$  for  $0 \leq a < b \leq L$ . From the definition of  $g$ ,

$$(4.17) \quad \mathbf{E}_\sigma[g(\sigma) \check{Z}_{[a,b]}^\sigma] \leq \eta \mathbf{E}_\sigma[\check{Z}_{[a,b]}^\sigma] + \mathbf{E}_\sigma[\mathbf{1}_{J_L^c} \check{Z}_{[a,b]}^\sigma].$$

Then, provided that we can show  $\mathbf{E}_\sigma[\mathbf{1}_{J_L^c} \check{Z}_{[a,b]}^\sigma] \leq \eta \mathbf{P}_\nu(b-a \in \nu)$  whenever  $b-a \geq \varepsilon L$ , we conclude

$$(4.18) \quad \mathbf{E}_\sigma[g(\sigma) \check{Z}_{[a,b]}^\sigma] \leq (2\eta + \mathbf{1}_{\{b-a < \varepsilon L\}}) \mathbf{P}_\nu(b-a \in \nu).$$

The following lemma fills in the missing pieces of the preceding, so we can conclude that (4.16) and (4.18) hold in the case  $\alpha^* > 1/2$ . The proof is in section 4.5.

**Lemma 4.2.** *Suppose (1.5) holds, with  $\tilde{\alpha} > 0$  and  $\alpha + \tilde{\alpha} < 1$ . If  $\alpha^* > 1/2$ , then for any fixed  $\eta > 0$  and  $\varepsilon > 0$ , there exist events  $J_L$  determined by  $\{0, \sigma_1, \dots, \sigma_L\}$ , and  $L_0 > 0$ , such that, if  $L \geq L_0$ ,*

$$(4.19) \quad \mathbf{P}_\sigma(J_L) \leq \eta^{\zeta/(1-\zeta)}$$

and moreover, for  $0 \leq a < b \leq L$  with  $b-a \geq \varepsilon L$ ,

$$(4.20) \quad \mathbf{E}_\sigma[\mathbf{1}_{J_L^c} \check{Z}_{[a,b],0}^\sigma] \leq \eta \mathbf{P}_\nu(b-a \in \nu).$$

Additionally, for all  $L$ ,  $\mathbf{E}_\sigma[\mathbf{1}_{J_L} \check{Z}_{[a,b],0}^\sigma] \leq \mathbf{P}_\nu(b-a \in \nu)$ .

In Section 5, we show that without the restriction  $\alpha^* > 1/2$ , for arbitrary  $\alpha, \tilde{\alpha}$  with  $\alpha + \tilde{\alpha} < 1$ , there exist distributions of  $\tau$  and  $\sigma$  with these tail exponents, and suitable events  $I_L$  with the same properties as  $J_L$ , see Lemma 5.1. The same conclusion (4.18) follows similarly.

To bound  $\mathbf{E}_\sigma[g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}}]$  we need the following extension of Lemma 4.2, which concerns a single block, to cover all blocks. The proof is in section 4.5.

**Lemma 4.3.** *Suppose (1.5) holds, with  $\tilde{\alpha} > 0$  and  $\alpha + \tilde{\alpha} < 1$ . If  $\alpha^* > 1/2$  and  $\zeta < 1$ ,  $\eta > 0$ ,  $\varepsilon > 0$  are fixed, then there exist an event  $J_L$  on  $\{0, \sigma_1, \dots, \sigma_L\}$ , and  $L_0 > 0$ , such that if  $L \geq L_0$  then  $\mathbf{P}_\sigma(J_L) \leq \eta^{\zeta/(1-\zeta)}$ , and for every  $\mathcal{I} = \{1 \leq i_1 < \dots < i_m = n\}$  one has*

$$(4.21) \quad \mathbf{E}_\sigma \left[ g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}} \right] \leq \sum_{\substack{d_{i_1} \leq f_{i_1} \\ d_{i_1}, f_{i_1} \in \mathcal{B}_{i_1}}} \cdots \sum_{\substack{d_{i_m} \leq f_{i_m} = nL \\ d_{i_m} \in \mathcal{B}_{i_m}}} \prod_{k=1}^m K^*(d_{i_k} - f_{i_{k-1}}) (2\eta + \mathbf{1}_{\{f_{i_k} - d_{i_k} < \varepsilon L\}}) \mathbf{P}_\nu(f_{i_k} - d_{i_k} \in \nu).$$

For fixed  $\delta > 0$ , we claim that, by taking  $\varepsilon, \eta$  small enough in Lemma 4.3, if  $L$  is large enough,

$$(4.22) \quad \mathbf{E}_\sigma \left[ g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}} \right] \leq \prod_{k=1}^{|\mathcal{I}|} \frac{\delta}{(i_k - i_{k-1})^{1+\alpha^*/2}}.$$

This, together with (4.15) and (4.16), enables us to conclude that (4.12) holds, with  $\delta$  replaced by  $\delta' = 2\delta^{2\zeta}$ , since  $\zeta(1 + \alpha^*/2) = 1 + \alpha^*/4$ .

*Proof of (4.22).* For every  $\mathcal{I} = \{1 \leq i_1 < i_2 < \dots < i_m = n\}$  we have by Lemma 4.3

$$(4.23) \quad \mathbf{E}_\sigma \left[ g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}} \right] \leq \mathbf{E}_\nu \left[ \mathbf{1}_{E_{\mathcal{I}}} \prod_{k=1}^m (2\eta + \mathbf{1}_{\{F_{i_k} - D_{i_k} < \varepsilon L\}}) \right],$$

where we set

$$E_{\mathcal{I}} := \left\{ \nu : \{i : \nu \cap \mathcal{B}_i \neq \emptyset\} = \mathcal{I} \right\}, \quad D_i := \min\{\nu \cap \mathcal{B}_i\}, \quad F_i := \max\{\nu \cap \mathcal{B}_i\}.$$

We next show that given  $\delta > 0$ , for  $\varepsilon, \eta$  sufficiently small, for any given  $\mathcal{I}$  and  $1 \leq k \leq m$

$$(4.24) \quad \sup_{f_{i_{k-1}} \in \mathcal{B}_{i_{k-1}}} \mathbf{E}_\nu \left[ (2\eta + \mathbf{1}_{\{F_{i_k} - D_{i_k} < \varepsilon L\}}) \mathbf{1}_{\{\nu \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \nu \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] \leq \delta (i_k - i_{k-1})^{-(1+\alpha^*/2)},$$

where we used the convention that  $i_0 = 0, f_0 = 0$  and  $\mathcal{B}_{i_0} = \{0\}$ . Then, we easily get (4.22) by iteration.

First, we see that for every  $f_{i_{k-1}} \in \mathcal{B}_{i_{k-1}}$ , we have

$$(4.25) \quad \mathbf{E}_\nu \left[ \mathbf{1}_{\{\nu \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \nu \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] = \mathbf{P}_\nu(f_{i_{k-1}} + \nu_1 \in \mathcal{B}_{i_k} \mid \nu_1 \geq i_{k-1}L - f_{i_{k-1}}).$$

If  $i_{k-1} - i_k = 1$ , we bound this by 1. If  $i_k - i_{k-1} \geq 2$ , writing for simplicity  $j_k := i_k - i_{k-1} - 1 \geq 1$ , the right side of (4.25) is at most

$$(4.26) \quad \sup_{0 \leq m \leq L} \mathbf{P}_\nu(\nu_1 \in (m + j_k L, m + (j_k + 1)L] \mid \nu_1 > m) \leq \frac{L}{\mathbf{P}_\nu(\nu_1 \geq L)} \sup_{j_k L \leq x \leq (j_k + 2)L} \mathbf{P}_\nu(\nu_1 = x) \leq c_1 L \varphi^*(L)^{-1} L^{\alpha^*} \varphi^*(j_k L) (j_k L)^{-(1+\alpha^*)} \leq c_2 j_k^{-(1+\alpha^*/2)},$$

where we used the existence of a constant  $c_3$  such that, for all  $a \geq 1$  and  $L$  large,  $\varphi^*(aL)/\varphi^*(L) \leq c_3 a^{\alpha^*/2}$ . In the end, we have

$$(4.27) \quad 2\eta \mathbf{E}_\nu \left[ \mathbf{1}_{\{\tau \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \tau \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] \leq c_4 \eta (i_k - i_{k-1})^{-(1+\alpha^*/2)}.$$

It remains to bound the rest of (4.24). We decompose the expectation according to whether  $D_{i_k} \in ((i_k - 1)L, (i_k - \gamma)L]$  or not, where  $\gamma := \varepsilon^{(\alpha^* \wedge 1)/(1 + \alpha^*)} \gg \varepsilon$ .

Note that if  $D_{i_k} \leq (i_k - \gamma)L$  and  $F_{i_k} - D_{i_k} \leq \varepsilon L$  then  $i_k L - F_{i_k} \geq \frac{1}{2}\gamma L$ . We therefore have

$$\begin{aligned} \mathbf{E}_\nu & \left[ \mathbf{1}_{\{D_{i_k} \in ((i_k - 1)L, (i_k - \gamma)L]\}} \mathbf{1}_{\{F_{i_k} - D_{i_k} < \varepsilon L\}} \mathbf{1}_{\{\tau \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \tau \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] \\ & \leq \sup_{0 \leq m \leq L} \mathbf{P}_\nu(\nu_1 \in [m + j_k L, m + (j_k + 1 - \gamma)L] \mid \nu_1 > m) \times \sum_{\ell=0}^{\varepsilon L} \mathbf{P}_\nu(\ell \in \nu) \mathbf{P}_\nu(\nu_1 \geq \frac{1}{2}\gamma L). \end{aligned}$$

Then, using (4.26) if  $i_k - i_{k-1} \geq 2$ , or bounding the probability by 1 if  $i_k - i_{k-1} = 1$ , we bound this from above for large  $L$  by

$$(4.28) \quad c_2 (i_k - i_{k-1})^{-(1 + \alpha^*/2)} \frac{\varphi^*(\gamma L)}{(\gamma L)^{\alpha^*}} \sum_{\ell=0}^{\varepsilon L} \mathbf{P}_\nu(\ell \in \nu) \leq c_5 \varepsilon^{\alpha^* \wedge 1} \gamma^{-\alpha^*} (i_k - i_{k-1})^{-(1 + \alpha^*/2)}.$$

For the last inequality, we used (1.16) to get that  $\mathbf{P}_\nu(\ell \in \nu) \leq c_6 \min\{1, \varphi^*(\ell)^{-1} \ell^{\alpha^* - 1}\}$  for all  $\ell \geq 1$ .

We now deal with the term when  $D_{i_k} \in ((i_k - \gamma)L, i_k L]$ . We have

$$(4.29) \quad \begin{aligned} \mathbf{E}_\nu & \left[ \mathbf{1}_{\{D_{i_k} \in ((i_k - \gamma)L, (i_k + 1)L]\}} \mathbf{1}_{\{\tau \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \tau \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] \\ & \leq \sup_{0 \leq m \leq L} \mathbf{P}_\nu(\nu_1 \in (m + (j_k + 1 - \gamma)L, m + (j_k + 1)L] \mid \nu_1 > m). \end{aligned}$$

We can now repeat the argument of (4.24). If  $i_k - i_{k-1} = 1$ , we get

$$(4.30) \quad \sup_{0 \leq m \leq L} \mathbf{P}_\nu(\nu_1 \in (m + (1 - \gamma)L, m + L] \mid \nu_1 > m) \leq \frac{\gamma L}{\mathbf{P}(\nu_1 \geq L)} \sup_{(1 - \gamma)L \leq x \leq 2L} \mathbf{P}_\nu(\nu_1 = x) \leq c_7 \gamma.$$

If  $i_k - i_{k-1} \geq 2$ , we end up similarly with the bound  $c_8 \gamma (i_k - i_{k-1})^{-(1 + \alpha^*/2)}$ .

Combining the bounds (4.28) and (4.30), one gets that

$$(4.31) \quad \mathbf{E}_\nu \left[ \mathbf{1}_{\{F_{i_k} - D_{i_k} < \varepsilon L\}} \mathbf{1}_{\{\tau \cap \mathcal{B}_i = \emptyset \ \forall i_{k-1} < i < i_k, \tau \cap \mathcal{B}_{i_k} \neq \emptyset\}} \mid F_{i_{k-1}} = f_{i_{k-1}} \right] \leq c_9 \gamma (i_k - i_{k-1})^{-(1 + \alpha^*/2)},$$

where we used the fact that  $\varepsilon^{\alpha^* \wedge 1} \gamma^{-\alpha^*} = \gamma$ .

Combining (4.27) and (4.31), we obtain (4.24) with  $\delta = c_4 \eta + c_9 \gamma$ , completing the proof of (4.22) and thus of (4.12).  $\square$

**4.5. Proof of Lemma 4.2: choice of the change of measure,** We rewrite the partition function in (4.20) as follows:

$$(4.32) \quad \begin{aligned} \frac{1}{\mathbf{P}_\nu(b - a \in \nu)} \mathbf{E}_\sigma \left[ \mathbf{1}_{J_L^\varepsilon} \check{Z}_{[a,b],0}^\sigma \right] & = \mathbf{E}_\nu \mathbf{E}_\sigma \left[ \mathbf{1}_{J_L^\varepsilon} W(\sigma, \nu, [a, b]) \mid a, b \in \nu \right] \\ & = \mathbf{E}_\nu \mathbf{E}_{\sigma, \tau} \left[ \mathbf{1}_{J_L^\varepsilon} \frac{\prod_{k: \nu_k \in [a,b]} \mathbf{1}_{\{\sigma_{\nu_k} \in \tau\}}}{\mathbf{E}_{\tau, \sigma} \left[ \prod_{k: \nu_k \in [a,b]} \mathbf{1}_{\{\sigma_{\nu_k} \in \tau\}} \right]} \mid a, b \in \nu \right] \\ & = \mathbf{E}_\nu \left[ \mathbf{P}_{\sigma, \tau} (J_L^\varepsilon \mid A_L(\nu)) \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset \right], \end{aligned}$$

where, for any renewal trajectory  $\nu$ , we defined the event

$$(4.33) \quad A_L(\nu) = \{(\sigma, \tau) : \sigma_{\nu_k} \in \tau \text{ for all } k \text{ such that } \nu_k \in [0, L]\}.$$

Observe that if we find a set  $\mathcal{G}$  of (good) trajectories of  $\nu$  satisfying

$$\mathbf{P}_\nu(\mathcal{G} \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset) \geq 1 - \eta/2$$

whenever  $L \geq L_0$  and  $b - a \geq \varepsilon L$ , then

$$\mathbf{E}_\nu \left[ \mathbf{1}_{\mathcal{G}^c} \mathbf{P}_{\sigma, \tau}(J_L^c \mid A_L(\nu)) \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset \right] \leq \eta/2.$$

To get (4.20), it is therefore enough to show that provided  $L$  is large and  $b - a \geq \varepsilon L$ , we have that for all  $\nu \in \mathcal{G}$

$$(4.34) \quad \mathbf{P}_{\sigma, \tau}(J_L^c \mid A_L(\nu)) \leq \eta/2.$$

We now construct the event  $J_L$  and the set  $\mathcal{G}$  (see (4.67) and (4.58)), and show (4.19)-(4.20). Write  $L = \gamma_L k_L$  and decompose  $[0, L]$  into  $\gamma_L$  subblocks of length  $k_L$ . We take

$$(4.35) \quad k_L \equiv 1 \quad \text{if } \mathbf{E}_\nu[\nu_1] < \infty, \quad k_L, \gamma_L \rightarrow \infty \quad \text{to be specified, if } \mathbf{E}_\nu[\nu_1] = \infty.$$

Define

$$(4.36) \quad S_{L,j} = \max_{(j-1)k_L < i \leq jk_L} \{\sigma_i - \sigma_{i-1}\}.$$

It is standard that  $S_{L,1}/B_{k_L}$  has a nontrivial limiting distribution if  $k_L \rightarrow \infty$  (see Section 2.4), and since  $0 < \tilde{\alpha} < 1$  we also have  $b_n = B_n$ . Define

$$Y_L = |\{j \leq \gamma_L : S_{L,j} > r_L B_{k_L}\}| = \sum_{j=1}^{\gamma_L} \mathbf{1}_{\{S_{L,j} > r_L B_{k_L}\}},$$

where  $r_L \xrightarrow{L \rightarrow \infty} \infty$  slowly, to be specified.

Our goal is to show that the conditioning by  $A_L(\nu)$  (for  $\nu \in \mathcal{G}$  with appropriate  $\mathcal{G}$ ) induces a reduction in  $\mathbf{E}_\sigma[Y_L]$  of a size  $y_L$  which is much larger than  $\sqrt{\text{Var}(Y_L)}$  and  $\sqrt{\text{Var}(Y_L | A_L(\nu))}$ ; then, one can take  $J_L$  of the form  $\{Y_L - \mathbf{E}_\sigma[Y_L] \leq -y_L/2\}$ , and obtain  $\mathbf{P}_\sigma(J_L) \rightarrow 0$ , and  $\mathbf{P}_{\sigma, \tau}(J_L | A_L(\nu)) \rightarrow 1$  as  $L \rightarrow \infty$ .

**Step 1.** We first estimate  $\mathbf{E}[Y_L]$  and  $\text{Var}(Y_L)$ , without the conditioning on  $A_L(\nu)$ . Suppose first that  $k_L \rightarrow \infty$ . Since  $\mathbf{P}_\sigma(\sigma_1 > n) \stackrel{n \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha}} \varphi(n) n^{-\tilde{\alpha}}$ , one easily has that, for fixed  $r > 0$ , as  $L \rightarrow \infty$ ,

$$(4.37) \quad \begin{aligned} \mathbf{P}_\sigma(S_{L,1} > r B_{k_L}) &= 1 - \left(1 - \mathbf{P}_\sigma(\sigma_1 > r B_{k_L})\right)^{k_L} \sim 1 - \exp(-k_L \mathbf{P}_\sigma(\sigma_1 > r B_{k_L})) \\ &\sim 1 - \exp\left(-\frac{1}{\tilde{\alpha}} r^{-\tilde{\alpha}} k_L \tilde{\varphi}(B_{k_L}) B_{k_L}^{-\tilde{\alpha}}\right) \sim 1 - \exp\left(-\frac{1}{\tilde{\alpha}} r^{-\tilde{\alpha}}\right), \end{aligned}$$

where we used (2.8) for the last equivalence. Therefore when  $r_L \xrightarrow{L \rightarrow \infty} \infty$  slowly enough we have

$$(4.38) \quad \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) \stackrel{L \rightarrow \infty}{\sim} k_L \mathbf{P}_\sigma(\sigma_1 > r_L B_{k_L}) \stackrel{L \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha}} r_L^{-\tilde{\alpha}},$$

and hence

$$(4.39) \quad \begin{aligned} \mathbf{E}[Y_L] &= \gamma_L \mathbf{P}(S_{L,1} > r_L B_{k_L}) \stackrel{L \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha}} \gamma_L r_L^{-\tilde{\alpha}}, \\ \text{Var}(Y_L) &= \gamma_L \mathbf{P}(S_{L,1} > r_L B_{k_L}) (1 - \mathbf{P}(S_{L,1} > r_L B_{k_L})) \stackrel{L \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha}} \gamma_L r_L^{-\tilde{\alpha}}. \end{aligned}$$

In the alternative case  $k_L \equiv 1$ , we have  $B_{k_L} \equiv d_\sigma$  so whenever  $r_L \rightarrow \infty$  we have

$$\mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) = \mathbf{P}_\sigma(\sigma_1 > d_\sigma r_L) \stackrel{L \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha} d_\sigma^{\tilde{\alpha}}} r_L^{-\tilde{\alpha}} \tilde{\varphi}(r_L),$$

so similarly to (4.39),

$$(4.40) \quad \mathbf{E}[Y_L] \stackrel{L \rightarrow \infty}{\sim} \text{Var}(Y_L) \stackrel{L \rightarrow \infty}{\sim} \frac{1}{\tilde{\alpha} d_\sigma^{\tilde{\alpha}}} r_L^{-\tilde{\alpha}} \tilde{\varphi}(r_L) L.$$

**Step 2.** We now study the influence of the conditioning by  $A_L(\nu)$  on the events  $\{\sigma_i - \sigma_{i-1} > r_L B_{k_L}\}$  and  $\{S_{L,j} > r_L B_{k_L}\}$ . Heuristically one expects the probabilities to decrease, and this is readily shown to be true if  $\mathbf{P}_\tau(n \in \tau)$  is decreasing in  $n$ , but in general such monotonicity only holds asymptotically. So instead we show that, for  $L$  large enough,

$$(4.41) \quad \mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L} \mid A_L(\nu), \{S_{L,\ell}, \ell \neq j\}) \leq (1 + 2\tilde{\epsilon}_L) \mathbf{P}_\sigma(S_{L,j} > r_L B_{k_L}).$$

where  $\tilde{\epsilon}_L \rightarrow 0$  is defined below, see (4.47).

To prove this, we first show that for  $n \geq j \geq 1$  and  $t \geq 0$ ,  $\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \mid \sigma_j - \sigma_{j-1} > r_L B_{k_L}, \sigma_{j-1} = t)$  is not much more than  $\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \mid \sigma_{j-1} = t)$ . To this end, we can form a coupling  $(\rho, \rho')$  such that  $\rho$  has the distribution  $\mathbf{P}_\sigma(\sigma_1 \in \cdot)$ ,  $\rho'$  has the distribution  $\mathbf{P}_\sigma(\sigma_1 \in \cdot \mid \sigma_1 \geq r_L B_{k_L})$ , and  $\rho \leq \rho'$ . Then, thanks to the exchangeability of the  $(\sigma_i - \sigma_{i-1})_{i \geq 1}$ , we have for any  $n \geq j \geq 1$  and  $t \geq 0$ ,

$$(4.42) \quad \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \mid \sigma_j - \sigma_{j-1} > r_L B_{k_L}, \sigma_{j-1} = t) = \mathbf{P}_{\sigma,\tau,\rho'}(t + \rho' + \sigma_{n-j} \in \tau),$$

and

$$(4.43) \quad \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \mid \sigma_{j-1} = t) = \mathbf{P}_{\sigma,\tau,\rho}(t + \rho + \sigma_{n-j} \in \tau).$$

To bound (4.42) relative to (4.43), we proceed as follows. Let  $R \geq 1$  and observe that

$$(4.44) \quad \begin{aligned} &\mathbf{P}_{\sigma,\tau,\rho}(t + \rho + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} \leq R) \\ &\geq \mathbf{P}_\rho(R \leq \rho < 2R) \mathbf{P}_{\sigma,\tau,\rho}(t + \rho + \sigma_{n-j} \in \tau \mid R \leq \rho < 2R, t + \sigma_{n-j} \leq R) \\ &\geq \mathbf{P}_\rho(R \leq \rho < 2R) \times \min_{R \leq m < 3R} \mathbf{P}_\tau(m \in \tau), \end{aligned}$$

which depends only on  $R$ . On the other hand, since  $\rho' \geq r_L B_{k_L}$ ,

$$(4.45) \quad \begin{aligned} \mathbf{P}_{\sigma,\tau,\rho'}(t + \rho' + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} \leq R) &\leq \max_{m \geq r_L B_{k_L}} \mathbf{P}_\tau(m \in \tau) \\ &\leq \mathbf{P}_{\sigma,\tau,\rho}(t + \rho + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} \leq R), \end{aligned}$$

where by (4.44) the last inequality holds for any fixed sufficiently large  $R$ , provided that  $L$  is large enough.

Now from (1.16), since  $\alpha < 1$ , if  $R$  and  $L$  are large enough, we have that

$$\mathbf{P}_\tau(m \in \tau) \geq \mathbf{P}_\tau(n \in \tau) \quad \text{for all } R \leq m \leq \frac{1}{2} r_L B_{k_L} \text{ and } n \geq r_L B_{k_L},$$

which yields

$$(4.46) \quad \begin{aligned} & \mathbf{P}_{\sigma, \tau, \rho} \left( t + \rho + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} > R, t + \rho + \sigma_{n-j} \leq \frac{1}{2} r_L B_{k_L} \right) \\ & \geq \mathbf{P}_{\sigma, \tau, \rho'} \left( t + \rho' + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} > R, t + \rho + \sigma_{n-j} \leq \frac{1}{2} r_L B_{k_L} \right). \end{aligned}$$

To control the contribution when  $m \geq \frac{1}{2} r_L B_{k_L}$ , we define the quantity

$$(4.47) \quad \epsilon_k = \sup \left\{ \frac{\mathbf{P}_\tau(u \in \tau)}{\mathbf{P}_\tau(v \in \tau)} - 1 : u \geq v \geq k \right\},$$

which satisfies  $\lim_{k \rightarrow \infty} \epsilon_k = 0$ . Then, writing  $\tilde{\epsilon}_L$  for  $\epsilon_{r_L B_{k_L}/2}$ , we have

$$(4.48) \quad \begin{aligned} & (1 + \tilde{\epsilon}_L) \mathbf{P}_{\sigma, \tau, \rho} \left( t + \rho + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} > R, t + \rho + \sigma_{n-j} > \frac{1}{2} r_L B_{k_L} \right) \\ & \geq \mathbf{P}_{\sigma, \tau, \rho'} \left( t + \rho' + \sigma_{n-j} \in \tau \mid t + \sigma_{n-j} > R, t + \rho + \sigma_{n-j} > \frac{1}{2} r_L B_{k_L} \right). \end{aligned}$$

Combining (4.45), (4.46) and (4.48), we obtain

$$\mathbf{P}_{\sigma, \tau, \rho'}(t + \rho' + \sigma_{n-j} \in \tau) \leq (1 + \tilde{\epsilon}_L) \mathbf{P}_{\sigma, \tau, \rho}(t + \rho + \sigma_{n-j} \in \tau)$$

By (4.43) and (4.42) this is the same as

$$\frac{\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \mid \sigma_j - \sigma_{j-1} > r_L B_{k_L}, \sigma_{j-1} = t)}{\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \mid \sigma_{j-1} = t)} \leq 1 + \tilde{\epsilon}_L,$$

which is equivalent to

$$\frac{\mathbf{P}_{\sigma, \tau}(\sigma_j - \sigma_{j-1} > r_L B_{k_L} \mid \sigma_n \in \tau, \sigma_{j-1} = t)}{\mathbf{P}_{\sigma, \tau}(\sigma_j - \sigma_{j-1} > r_L B_{k_L} \mid \sigma_{j-1} = t)} \leq 1 + \tilde{\epsilon}_L,$$

or by independence,

$$(4.49) \quad \frac{\mathbf{P}_{\sigma, \tau}(\sigma_j - \sigma_{j-1} > r_L B_{k_L} \mid \sigma_n \in \tau, \sigma_{j-1} = t)}{\mathbf{P}_{\sigma, \tau}(\sigma_j - \sigma_{j-1} > r_L B_{k_L})} \leq 1 + \tilde{\epsilon}_L.$$

Since  $t$  and  $n$  are arbitrary, this shows that for arbitrary  $\nu$ , conditionally on  $A_L(\nu)$ , the variables  $(\mathbf{1}_{\{\sigma_i - \sigma_{i-1} > r_L B_{k_L}\}})_{i \leq L}$  are (jointly) stochastically dominated by a Bernoulli sequence of parameter  $(1 + \tilde{\epsilon}_L) \mathbf{P}_\sigma(\sigma_1 > r_L B_{k_L})$ . In fact, by exchangeability this domination holds for any given  $\sigma_\ell - \sigma_{\ell-1}$  conditionally on any information about the other variables  $\sigma_i - \sigma_{i-1}$ .

We need to quantify what this conclusion says about the variables  $S_{L,j}$ . The following is easily established: there exists  $\delta_0 > 0$  such that for all  $p \in (0, 1)$ ,  $\epsilon \in (0, 1)$  and  $k \geq 1$  with  $kp < \delta_0$ , we have

$$\frac{1 - (1 - (1 + \epsilon)p)^k}{1 - (1 - p)^k} \leq 1 + 2\epsilon.$$

Taking  $p = \mathbf{P}_\sigma(\sigma_1 > r_L B_{k_L})$ ,  $k = k_L$  so that  $pk \leq \delta_0$  provided  $r_L$  is large (see (4.38)), taking  $\epsilon = \tilde{\epsilon}_L$  and using the stochastic domination, we obtain that for  $L$  large, for all  $\nu$ , (4.41) holds.

**Step 3.** We next want to show that for certain  $\nu$  and  $j$  we can make a much stronger statement than (4.41). Specifically, with  $L$  fixed, we say an interval  $\mathcal{I}$  is *visited* (in  $\nu$ ) if  $\mathcal{I} \cap \nu \neq \emptyset$ , and (for  $3 \leq j < \gamma_L - 2$ ) we say the sub-block  $\mathcal{Q}_j = ((j-1)k_L, jk_L]$  is *capped* if  $\mathcal{Q}_{j-2} \cup \mathcal{Q}_{j-1}$  and  $\mathcal{Q}_{j+1} \cup \mathcal{Q}_{j+2}$  are both visited.



We now prove that, if  $j$  is such that  $\mathcal{Q}_j$  is capped, we have

$$(4.50) \quad \mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L} \mid A_L(\nu)) \leq 2(r_L)^{-(1-\alpha)/2} \mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L}).$$

Suppose that  $\mathcal{Q}_j$  is capped, and that  $s$  is an index such that  $(\nu_{s-1}, \nu_s] \cap \mathcal{Q}_j \neq \emptyset$ : we write  $s \prec \mathcal{Q}_j$ . Note that the events

$$H_{s,j} = \left\{ \sigma : \max_{i \in (\nu_{s-1}, \nu_s] \cap \mathcal{Q}_j} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L} \right\}$$

are conditionally independent given  $A_L(\nu)$ , and (by exchangeability) satisfy

$$(4.51) \quad \mathbf{P}_{\sigma,\tau}(H_{s,j} \mid A_L(\nu)) = \mathbf{P}_{\sigma,\tau}\left(\max_{1 \leq i \leq \ell} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L} \mid \sigma_k \in \tau\right),$$

where  $k := \nu_s - \nu_{s-1} \leq 5k_L$  (since  $\mathcal{Q}_j$  is capped), and  $\ell := |(\nu_{s-1}, \nu_s] \cap \mathcal{Q}_j| \leq k$ .

Furthermore, since  $\alpha \in [0, 1)$ , by (1.16) we have  $\mathbf{P}(m \in \tau) = \bar{\varphi}(m)m^{-(1-\alpha)}$  for some slowly varying  $\bar{\varphi}(\cdot)$ . Therefore

$$(4.52) \quad \begin{aligned} \mathbf{P}_{\sigma,\tau}\left(\sigma_k \in \tau \mid \max_{1 \leq i \leq \ell} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L}\right) &\leq \max_{m \geq r_L B_{k_L}} \mathbf{P}_\tau(m \in \tau) \\ &\leq c_{10}(r_L B_{k_L})^{-(1-\alpha)} \bar{\varphi}(r_L B_{k_L}). \end{aligned}$$

On the other hand, since  $0 < \tilde{\alpha} < 1$ ,  $\sigma_n/B_n$  has a non-degenerate limiting distribution with positive density on  $(0, \infty)$  (see Section 2.4). Using again that  $\mathbf{P}(m \in \tau) = \bar{\varphi}(m)m^{-(1-\alpha)}$ , we therefore find that there exist a constant  $c_{11}$  such that, for any  $k \leq 5k_L$ ,

$$(4.53) \quad \mathbf{P}_{\sigma,\tau}(\sigma_k \in \tau) \geq \mathbf{P}_\sigma(\sigma_k \in [\bar{d}_\tau B_k, 2\bar{d}_\tau B_k]) \min_{\bar{d}_\tau \leq m \leq 2\bar{d}_\tau B_{5k_L}} \mathbf{P}_\tau(m \in \tau) \geq c_{11} B_{k_L}^{-(1-\alpha)} \bar{\varphi}(B_{k_L}).$$

With (4.52) this shows there exists some  $c_{12}$  such that for large  $L$ , provided  $r_L \rightarrow \infty$  slowly enough,

$$(4.54) \quad \frac{\mathbf{P}_{\sigma,\tau}(\sigma_k \in \tau \mid \max_{1 \leq i \leq \ell} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L})}{\mathbf{P}_{\sigma,\tau}(\sigma_k \in \tau)} \leq (c_{12} \vee \bar{\varphi}(r_L)) r_L^{-(1-\alpha)} \leq r_L^{-(1-\alpha)/2},$$

or equivalently, using (4.51),

$$(4.55) \quad \frac{\mathbf{P}_{\sigma,\tau}(H_{s,j} \mid A_L(\nu))}{\mathbf{P}_{\sigma,\tau}(H_{s,j})} = \frac{\mathbf{P}_{\sigma,\tau}\left(\max_{1 \leq i \leq \ell} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L} \mid \sigma_k \in \tau\right)}{\mathbf{P}_{\sigma,\tau}\left(\max_{1 \leq i \leq \ell} (\sigma_i - \sigma_{i-1}) > r_L B_{k_L}\right)} \leq (r_L)^{-(1-\alpha)/2}.$$

Since the  $\{H_{s,j} : s \prec \mathcal{Q}_j\}$  are independent for fixed  $j$  (even when conditioned on  $A_L(\nu)$ ), with  $\bigcup_{s \prec \mathcal{Q}_j} H_{s,j} = \{S_{L,j} > r_L B_{k_L}\}$  and  $\mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L}) \rightarrow 0$  as  $L \rightarrow \infty$ , we have from

(4.55) that for large  $L$

$$\begin{aligned}
\mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L} \mid A_L(\nu)) &\leq \sum_{s \prec \mathcal{Q}_j} \mathbf{P}_{\sigma,\tau}(H_{s,j} \mid A_L(\nu)) \\
&\leq (r_L)^{-(1-\alpha)/2} \sum_{s \prec \mathcal{Q}_j} \mathbf{P}_{\sigma,\tau}(H_{s,j}) \\
(4.56) \qquad \qquad \qquad &\leq 2(r_L)^{-(1-\alpha)/2} \mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L}) .
\end{aligned}$$

**Step 4.** We now control the number of capped blocks  $\mathcal{Q}_j$ . Let

$$\mathcal{C}(\nu) = \{j \leq \gamma_L : \mathcal{Q}_j \text{ is capped in } \nu\}, \quad m^*(n) := \mathbf{E}_\nu[\nu_1 \wedge n],$$

and define what is heuristically a lower bound for the typical size of  $|\mathcal{C}(\nu)|$  (see (4.59) and its proof):

$$(4.57) \quad D_L := \begin{cases} \gamma_L^{\alpha^*} \varphi^*(k_L) \varphi^*(L)^{-1} & \text{if } 0 < \alpha^* < 1, \\ \gamma_L m^*(k_L) m^*(L)^{-1} & \text{if } \alpha^* = 1, \mathbf{E}_\nu[\nu_1] = +\infty, \\ L \varphi^*(k_L) k_L^{-\alpha^*} & \text{if } \mathbf{E}_\nu[\nu_1] < +\infty. \end{cases}$$

We now restrict our choice of  $\gamma_L, k_L$  as follows (with further restriction to come):

- (i) if  $\mathbf{E}_\nu[\nu_1] < +\infty$ , we choose  $k_L \equiv 1, \gamma_L = L$ , and get  $D_L = L \varphi^*(1)$ ;
- (ii) if  $\mathbf{E}_\nu[\nu_1] = +\infty$  we choose  $1 \ll k_L \ll L$  such that  $\gamma_L = L/k_L$  is slowly varying, and as  $L \rightarrow \infty$  we have  $\gamma_L \rightarrow \infty$  slowly enough so  $\varphi^*(k_L) \sim \varphi^*(L)$  if  $\alpha^* < 1$  or  $m^*(k_L) \sim m^*(L)$  if  $\alpha^* = 1$ , which is possible since for  $\alpha^* = 1$ ,  $m^*(n)$  is slowly varying. We obtain  $D_L \sim \gamma_L^{\alpha^*} \gg 1$ .

We then fix  $\kappa > 0$  and define the set  $\mathcal{G}$  of *good* trajectories by

$$(4.58) \quad \mathcal{G} = \{\nu : |\mathcal{C}(\nu)| \geq \kappa D_L\}.$$

We wish to choose  $\kappa = \kappa(\eta, \varepsilon)$  sufficiently small so that, provided that  $L$  is large enough and  $b - a \geq \varepsilon L$ , one has

$$(4.59) \quad \mathbf{P}_\nu(\mathcal{G} \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset) \geq 1 - \eta/2.$$

For a fixed  $\varepsilon$ , for large  $L$  we have  $D_M \geq \varepsilon D_L$  for all  $M \geq \varepsilon L$ , so it suffices to prove this for  $\varepsilon = 1$ , that is, to consider only  $a = 0, b = L$ . We now consider two cases.

- *Case of  $\alpha^* < 1$  or  $\mathbf{E}_\nu[\nu_1] < +\infty$ .* In that case, it is standard that there exists a constant  $c_{13}(\eta)$  such that, defining  $n_L := c_{13} \min(\varphi^*(L)^{-1} L^{\alpha^*}, L)$ , we have for  $L$  large enough

$$(4.60) \quad \mathbf{P}_\nu(|\nu \cap [0, L/2]| \leq n_L) \leq \eta.$$

Observe that each gap  $(\nu_{s-1}, \nu_s]$  of  $\nu$  of length  $\nu_s - \nu_{s-1} \in [2k_L, 3k_L)$  contains a capped sub-block. We therefore simply need to count the number of such gaps: denote

$$V_n := \sum_{s=1}^n \mathbf{1}_{\{\nu_s - \nu_{s-1} \in [2k_L, 3k_L)\}}.$$

There exists a constant  $c_{14} = c_{14}(\eta)$  such that, provided that  $n \mathbf{P}_\nu(\nu_1 \in [2k_L, 3k_L))$  is large enough, one has

$$(4.61) \quad \mathbf{P}_\nu(V_n \leq c_{14} n \mathbf{P}_\nu(\nu_1 \in [2k_L, 3k_L))) \leq \eta.$$

Using that  $\mathbf{P}_\nu(\nu_1 \in [2k_L, 3k_L])$  is of order  $\varphi^*(k_L)k_L^{-\alpha^*}$  we get that  $D_L \leq c_{15}^{-1}n_L\mathbf{P}_\nu(\nu_1 \in [2k_L, 3k_L])$  for some  $c_{15}(\eta)$ , and we obtain

$$\begin{aligned} \mathbf{P}_\nu(\mathcal{G}^c | L \in \nu) &\leq \mathbf{P}_\nu\left(\left|\{s \leq |\nu \cap [0, L]| : \nu_s - \nu_{s-1} \in [2k_L, 3k_L]\}\right| < \kappa D_L \mid L \in \nu\right) \\ &\leq \mathbf{P}_\nu\left(|\nu \cap [0, L/2]| \leq n_L \mid L \in \nu\right) + \mathbf{P}_\nu\left(|\nu \cap [0, L/2]| \geq n_L; V_{n_L} < \kappa D_L \mid L \in \nu\right) \\ &\leq c_{16}\eta + c_{16}\mathbf{P}_\nu\left(V_{n_L} < \frac{\kappa}{c_\eta}n_L\mathbf{P}_\nu(\nu_1 \in [2k_L, 3k_L])\right) \\ &\leq 2c_{16}\eta. \end{aligned}$$

In the third inequality, we used Lemma A.1 (Lemma A.2 in [23], that we recall in the Appendix since we use it several times) to remove the conditioning at the expense of the constant  $c_{16}$ . In the last inequality, we used (4.61), taking  $\kappa$  sufficiently small.

• *Case of  $\alpha^* = 1$ ,  $\mathbf{E}_\nu[\nu_1] = +\infty$ .* First, we claim that with our choice of  $k_L$ , we have  $\mathbf{P}_\nu(\mathcal{Q}_j \text{ is capped}) \xrightarrow{L \rightarrow \infty} 1$ . Indeed, summing over possible locations of a last visit to  $\mathcal{Q}_j$  gives that for  $1 \leq j \leq \gamma_L$  and  $i \leq (j-1)k_L$ ,

$$\begin{aligned} \mathbf{P}_\nu\left(\mathcal{Q}_j \text{ is visited} \mid i \in \nu\right) &\geq \sum_{m=0}^{k_L-1} \mathbf{P}_\nu(jk_L - m - i \in \nu)\mathbf{P}_\nu(\nu_1 > m) \\ &\geq \inf_{\ell \leq L} \mathbf{P}_\nu(\ell \in \nu)m^*(k_L) \\ (4.62) \quad &\sim \frac{m^*(k_L)}{m^*(L)} \sim 1 \quad \text{as } L \rightarrow \infty, \end{aligned}$$

where we used  $\sum_{m=0}^{k_L-1} \mathbf{P}_\nu(\nu_1 > m) = m^*(k_L)$  in the second inequality, and (1.16) in the last line. Note that the lower bound in (4.62) is uniform in the specified  $i, j$ . Therefore, applying (4.62) three times, we obtain uniformly in  $j \leq \gamma_L$

$$(4.63) \quad \mathbf{P}_\nu(\mathcal{Q}_j \text{ is capped}) \geq \mathbf{P}_\nu(\mathcal{Q}_{j-1}, \mathcal{Q}_j, \mathcal{Q}_{j+1} \text{ are visited}) \xrightarrow{L \rightarrow \infty} 1.$$

Recall  $D_L \sim \gamma_L$ , and denote  $W_L := |\{j \leq \gamma_L/2; \mathcal{Q}_j \text{ is not capped}\}|$ , so that  $\mathbf{E}_\nu[W_L] = o(\gamma_L)$ , by (4.63). Taking  $\kappa = 1/4$ , thanks to Markov's inequality we then have for large  $L$

$$(4.64) \quad \mathbf{P}_\nu(\mathcal{G}^c | L \in \nu) \leq \mathbf{P}_\nu(W_L \geq \frac{1}{8}\gamma_L | L \in \nu) \leq c_{16}\mathbf{P}_\nu(W_L \geq \frac{1}{8}\gamma_L) \rightarrow 0 \quad \text{as } L \rightarrow \infty,$$

where in the second inequality, we used Lemma A.1 to remove the conditioning at the expense of the constant  $c_{16}$ .

**Step 5.** We now have the ingredients to control how the conditioning by  $A_L(\nu)$  shifts  $\mathbf{E}_\sigma[Y_L] = \gamma_L\mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L})$ , when  $\nu \in \mathcal{G}$ .

We wish to choose  $\gamma_L, k_L$  so that, for  $L$  sufficiently large,

$$(4.65) \quad \gamma_L \tilde{\varepsilon}_L \leq \frac{1}{8}\kappa D_L \leq \frac{1}{8}|\mathcal{C}(\nu)| \quad \text{for all } \nu \in \mathcal{G}.$$

The second inequality is just the definition of  $\mathcal{G}$ . When  $\alpha^* \geq 1$ ,  $\gamma_L$  and  $D_L$  are of the same order, so the first inequality follows from  $\tilde{\varepsilon}_L \rightarrow 0$ . So consider the case  $\alpha^* < 1$ , where  $D_L \sim \gamma_L^{\alpha^*}$  by (ii) after (4.57). Here the first inequality in (4.65) follows if we have  $\gamma_L^{1-\alpha^*} \ll \tilde{\varepsilon}_L^{-1}$ . Since  $\varepsilon_n \searrow 0$  and  $L/k_L$  is slowly varying, we have  $B_{k_L} \gg L$  so  $\tilde{\varepsilon}_L \leq \varepsilon_L$  so a

sufficient condition is  $\gamma_L^{1-\alpha^*} \ll \varepsilon_L^{-1}$ . But our only restriction so far is from (ii) after (4.57), that  $\gamma_L \rightarrow \infty$  slowly enough, so we may choose  $\gamma_L$  to satisfy this sufficient condition also.

Thanks to (4.41), (4.55) and (4.65), we have for  $\nu \in \mathcal{G}$  and large  $L$

$$\begin{aligned}
& \mathbf{E}_{\sigma,\tau}[Y_L \mid A_L(\nu)] \\
&= \sum_{j=1}^{\gamma_L} \mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L b_{k_L} \mid A_L(\nu)) \\
&\leq 2|\mathcal{C}(\nu)| (r_L)^{-(1-\alpha)/2} \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) + (\gamma_L - |\mathcal{C}(\nu)|)(1 + 2\tilde{\varepsilon}_L) \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) \\
&\leq \left[ \gamma_L - |\mathcal{C}(\nu)| + 2|\mathcal{C}(\nu)| (r_L)^{-(1-\alpha)/2} + 2\gamma_L \tilde{\varepsilon}_L \right] \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) \\
(4.66) \quad &\leq \left[ \gamma_L - \frac{1}{2}\kappa D_L \right] \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}).
\end{aligned}$$

**Step 6.** We now define

$$\begin{aligned}
(4.67) \quad J_L &:= \left\{ \sigma : \sum_{j \leq \gamma_L} \mathbf{1}_{\{S_{L,j} > r_L B_{k_L}\}} \leq \left[ \gamma_L - \frac{1}{4}\kappa D_L \right] \mathbf{P}_\sigma(S_{L,1} > r_L b_{k_L}) \right\} \\
&= \left\{ Y_L - \mathbf{E}[Y_L] \leq -\frac{\kappa}{4} D_L \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L}) \right\}.
\end{aligned}$$

Let us compare  $\mathbf{P}(J_L)$  and  $\mathbf{P}(J_L^c \mid A_L(\nu))$  for  $\nu \in \mathcal{G}$ . Since  $\alpha^* > 1/2$ ,  $\gamma_L D_L^{-2}$  is of order  $L^{-(2\alpha^*-1)\wedge 1} \rightarrow 0$ , so we can choose  $r_L$  to satisfy  $\gamma_L D_L^{-2} r_L^{\tilde{\alpha}} \rightarrow 0$ , which is compatible with our previous requirement on  $r_L$  involving (4.54). Using (4.39) and Chebyshev's inequality we then get that

$$(4.68) \quad \mathbf{P}(J_L) \leq \frac{16\gamma_L}{\kappa^2 D_L^2 \mathbf{P}_\sigma(S_{L,1} > r_L b_{k_L})} \stackrel{n \rightarrow \infty}{\sim} \frac{16\tilde{\alpha}}{\kappa^2} \gamma_L D_L^{-2} r_L^{\tilde{\alpha}} \rightarrow 0 \quad \text{as } L \rightarrow \infty.$$

On the other hand, by (4.41), conditionally on  $A(\nu)$  with  $\nu \in \mathcal{G}$ , the variables  $\mathbf{1}_{\{S_{L,j} > r_L b_{k_L}\}}$ ,  $1 \leq j \leq \gamma_L$ , are (jointly) stochastically dominated by a collection of independent Bernoulli variables with parameter  $(1 + 2\tilde{\varepsilon}_L) \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L})$ . Hence,  $\text{Var}(Y_L \mid A_L(\nu)) \leq 2\gamma_L \mathbf{P}_\sigma(S_{L,1} > r_L B_{k_L})$  and as in (4.68) we obtain

$$(4.69) \quad \mathbf{P}_{\sigma,\tau}(J_L^c \mid A_L(\nu)) \leq \mathbf{P}\left( Y_L - \mathbf{E}_{\sigma,\tau}[Y_L \mid A_L(\nu)] \geq \frac{\kappa}{4} D_L \mathbf{P}_\sigma(S_{L,1} > r_L b_{k_L}) \right) \xrightarrow{L \rightarrow \infty} 0.$$

We have thus proved (4.19) and (4.34), and hence also (4.20).  $\square$

**Remark 4.4.** If  $n \mapsto \mathbf{P}(n \in \tau)$  is non-increasing, then we have that  $\epsilon_k = 0$  for all  $k$ , and we can replace (4.41) with

$$\mathbf{P}_{\sigma,\tau}(S_{L,j} > r_L B_{k_L} \mid A_L(\nu), \{S_{L,\ell}, \ell \neq j\}) \leq \mathbf{P}_\sigma(S_{L,j} > r_L B_{k_L}).$$

The term  $\gamma_L \tilde{\varepsilon}_L$  does not appear in the computation in (4.66), and we can drop the condition (4.65). We can therefore choose  $\gamma_L = L, k_L = 1$  in all cases, not just when  $\mathbf{E}_\nu[\nu_1] < \infty$ .

Then for  $\alpha^* < 1$ , we have  $D_L$  of order  $L^{\alpha^*} \varphi^*(L)^{-1}$ , and the condition  $\gamma_L D_L^{-2} \rightarrow 0$  in Step 6 becomes

$$L^{2\alpha^*-1} \varphi^*(L)^{-2} \gg 1.$$

We therefore need  $\alpha^* > 1/2$ , or  $\alpha^* = 1/2$  and  $\varphi^*(L) \xrightarrow{L \rightarrow \infty} 0$ , going slightly beyond the condition  $\alpha^* > 1/2$ . In Appendix A.2, we show that  $\varphi^*(n) \sim c \tilde{\varphi}(b_n)^{-(1-\alpha)/\tilde{\alpha}} \varphi_0(b_n)$ , with

$\varphi_0(k) = \varphi(k)^{-1}$  if  $\alpha \in (0, 1)$ , and  $\bar{\varphi}(k) = \varphi(k)/\mathbf{P}_\tau(\tau_1 > k)^2$  if  $\alpha = 0$ , cf. (1.16). We have that  $\lim_{n \rightarrow \infty} \varphi^*(n) = 0$  if and only if  $\lim_{k \rightarrow \infty} \tilde{\varphi}(k)^{(1-\alpha)/\tilde{\alpha}} \varphi_0(k) = +\infty$ .

*Proof of Lemma 4.3.* The proof of Lemma 4.2 is for a single interval  $[0, L]$ , and we now adapt it to the whole system: we take the same definition for  $J_L$ . Then, recalling  $g_{\mathcal{I}}$  from (4.14), we have similarly to (4.32)

$$(4.70) \quad \mathbf{E}_\sigma \left[ g_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}} \right] = \sum_{\substack{d_{i_1} \leq f_{i_1} \\ d_{i_1}, f_{i_1} \in \mathcal{B}_{i_1}}} \cdots \sum_{\substack{d_{i_m} \leq f_{i_m} = nL \\ d_{i_m} \in \mathcal{B}_{i_m}}} \prod_{k=1}^m K^*(d_{i_k} - f_{i_{k-1}}) \mathbf{P}(f_{i_k} - d_{i_k} \in \nu) \\ \times \mathbf{E}_\nu \left[ \mathbf{E}_{\sigma, \tau} \left( \prod_{k=1}^m (\eta + \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}}) \mid A(\nu) \right) \Big| \mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} \right],$$

where, setting  $\nu_{\mathcal{B}_i} = \nu \cap \mathcal{B}_i$ ,

$$\mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} := \left\{ \nu : \forall 1 \leq k \leq m, \min(\nu_{\mathcal{B}_{i_k}}) = d_{i_k}, \max(\nu_{\mathcal{B}_{i_k}}) = f_{i_k}; \nu_{\mathcal{B}_i} = \emptyset \text{ if } i \notin \mathcal{I} \right\}.$$

Now, for any  $1 \leq k \leq m$  such that  $f_{i_k} - d_{i_k} \leq \varepsilon L$  or  $\nu_{\mathcal{B}_{i_k}} \notin \mathcal{G}$ , we bound  $\mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}}$  by 1. We get the bound

$$(4.71) \quad \eta + \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}} \leq \eta + \mathbf{1}_{\{f_{i_k} - d_{i_k} \leq \varepsilon L\}} + \mathbf{1}_{\{f_{i_k} - d_{i_k} > \varepsilon L\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \notin \mathcal{G}\}} + \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \in \mathcal{G}\}}.$$

By expanding the product over  $k \in \{1, \dots, m\}$ , we obtain

$$(4.72) \quad \mathbf{E}_\nu \left[ \mathbf{E}_{\sigma, \tau} \left( \prod_{k=1}^m (\eta + \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}}) \mid A(\nu) \right) \Big| \mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} \right] \\ \leq \sum_{\substack{K_1, K_2, K_3, K_4 \text{ disjoint} \\ \cup_{j=1}^4 K_j = \{1, \dots, m\}}} \eta^{|K_1|} \prod_{k \in K_2} \mathbf{1}_{\{f_{i_k} - d_{i_k} \leq \varepsilon L\}} \\ \mathbf{E}_\nu \left[ \prod_{k \in K_3} \mathbf{1}_{\{f_{i_k} - d_{i_k} > \varepsilon L\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \notin \mathcal{G}\}} \mathbf{E}_{\sigma, \tau} \left( \prod_{k \in K_4} \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \in \mathcal{G}\}} \mid A(\nu) \right) \Big| \mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} \right].$$

The argument in Step 6 of the proof of Lemma 4.2, using domination of the variables  $(\mathbf{1}_{S_{L,j} > r_L B_{k_L}})$  in some  $\mathcal{B}_{i_k}$  by independent Bernoullis, remains valid if we also condition on any information about the other  $\mathcal{B}_i, i \neq i_k$ . This means we can ignore the dependencies between different  $i_k \in \mathcal{I}$ , and if  $L$  is sufficiently large we get that, since  $\nu_{\mathcal{B}_{i_k}} \in \mathcal{G}$  for all  $k \in K_4$

$$(4.73) \quad \mathbf{E}_{\sigma, \tau} \left[ \prod_{k \in K_4} \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \in \mathcal{G}\}} \mid A(\nu) \right] \leq (\eta/2)^{|K_4|},$$

where the bound comes from (4.69).

Then, by independence of the  $\nu_{\mathcal{B}_{i_k}} \cap [d_{i_k}, f_{i_k}]$  conditionally on  $\mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}}$ , we get that

$$(4.74) \quad \begin{aligned} & \mathbf{E}_\nu \left[ \prod_{k \in K_3} \mathbf{1}_{\{f_{i_k} - d_{i_k} > \varepsilon L\}} \mathbf{1}_{\{\nu_{\mathcal{B}_{i_k}} \notin \mathcal{G}\}} \mid \mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} \right] \\ &= \prod_{k \in K_3} \mathbf{1}_{\{f_{i_k} - d_{i_k} > \varepsilon L\}} \mathbf{P}_\nu(\nu \notin \mathcal{G} \mid \nu_0 = 0; f_{i_k} - d_{i_k} \in \nu; \nu \cap (f_{i_k} - d_{i_k}, L] = \emptyset) \leq (\eta/2)^{|K_3|}, \end{aligned}$$

where we used (4.59) in the last inequality.

Therefore, plugging (4.73)-(4.74) in (4.72), we get that

$$\begin{aligned} & \mathbf{E}_\nu \left[ \mathbf{E}_{\sigma, \tau} \left( \prod_{k=1}^m (\eta + \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in J_L^c\}}) \mid A(\nu) \right) \mid \mathcal{E}_{(d_{i_k}, f_{i_k})_{1 \leq k \leq m}} \right] \\ & \leq \sum_{\substack{K_1, K_2, K_3, K_4 \text{ disjoint} \\ \bigcup_{j=1}^4 K_j = \{1, \dots, m\}}} \eta^{|K_1|} (\eta/2)^{|K_3|} (\eta/2)^{|K_4|} \prod_{k \in K_2} \mathbf{1}_{\{f_{i_k} - d_{i_k} \leq \varepsilon L\}} = \prod_{k=1}^m (2\eta + \mathbf{1}_{\{f_{i_k} - d_{i_k} \leq \varepsilon L\}}). \end{aligned}$$

□

## 5. PROOF OF THEOREM 2.2: EXAMPLES WITH UNEQUAL CRITICAL POINTS

**5.1. Adaptation of the change of measure.** Our proof for  $\alpha^* > 1/2$  made use of an event  $J_L$ , see Lemma 4.2; here we will define an event  $I_L$  which will play the same role. For this it needs to satisfy the following lemma. Recall that  $A_L(\nu) = \{(\sigma, \tau) : \sigma_{\nu_k} \in \tau \text{ for all } k \text{ such that } \nu_k \in [0, L]\}$ , and  $\zeta$ , from (4.12), is given by  $(1 + \alpha^*/2)\zeta = 1 + \alpha^*/4$ .

**Lemma 5.1.** *There exist events  $I_L$  determined by  $\{0, \sigma_1, \dots, \sigma_L\}$  as follows. For any  $\eta > 0$ ,  $\varepsilon > 0$ , there exists some  $L_0$  such that, if  $L \geq L_0$ ,*

$$(5.1) \quad \mathbf{P}_\sigma(I_L) \leq \eta^{\zeta/(1-\zeta)}.$$

Moreover, for  $0 \leq a < b \leq L$ , with  $b - a \geq \varepsilon L$

$$(5.2) \quad \mathbf{E}_\sigma[\mathbf{1}_{\{I_L^c\}} \check{Z}_{[a, b], 0}^\sigma] \leq \eta \mathbf{P}_\nu(b - a \in \nu).$$

*Proof.* From (4.32) applied to  $I_L$ , for (5.1) and (5.2) it suffices to have that, as  $L \rightarrow \infty$

$$(5.3) \quad \mathbf{P}_\sigma(I_L) \rightarrow 0, \quad \mathbf{E}_\nu[\mathbf{P}_{\sigma, \tau}(I_L^c \mid A(\nu)) \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset] \rightarrow 0.$$

Let us introduce or recall some notations

$$(5.4) \quad \begin{aligned} n_i &= \nu_i - \nu_{i-1}, \quad K^*(n) = \mathbf{P}_\nu(\nu_1 = n) = \varphi^*(n)n^{-(1+\alpha^*)}, \quad m^*(n) = \mathbf{E}_\nu[\nu_1 \wedge n], \\ q_L &:= \begin{cases} L^{\alpha^*} \varphi^*(L)^{-1} & \text{if } 0 < \alpha^* < 1, \\ L/m^*(L) & \text{if } \alpha^* \geq 1. \end{cases} \end{aligned}$$

Here  $q_L$  represents the typical order of  $|\nu \cap [0, L]|$ .

The event  $I_L$  will be given in terms of an expectation over trajectories  $\nu$ , so to avoid ambiguous notation e.g. in (5.3), we define  $\nu'$  to be a renewal with the same distribution as

$\nu$ . We can then define, for  $\kappa > 0$  to be specified,

$$(5.5) \quad I_L = \bigcup_{\substack{0 \leq a \leq b \leq L \\ b-a \geq \varepsilon L}} \{ \sigma : \log \mathbf{E}_{\nu'} [W(\sigma, \nu', [a, b]) \mathbf{1}_{b \in \nu'} | a \in \nu'] \geq \kappa q_L \}.$$

Then the first half of (5.3) is immediate from a union bound and Markov's inequality:

$$(5.6) \quad \mathbf{P}_\sigma(I_L) \leq \sum_{\substack{0 \leq a \leq b \leq L \\ b-a \geq \varepsilon L}} e^{-\kappa q_L} \mathbf{E}_\sigma \mathbf{E}_{\nu'} [W(\sigma, \nu', [a, b]) \mathbf{1}_{b \in \nu'} | a \in \nu'] \leq L^2 e^{-\kappa q_L} \rightarrow 0$$

as  $L \rightarrow \infty$ , where we used that  $\mathbf{E}_\sigma[W(\sigma, \nu', [a, b])] = 1$ .

For the second bound in (5.3), let

$$(5.7) \quad U_i = U_i(\sigma) = \log \frac{\mathbf{P}_\tau(\sigma_{\nu_i} - \sigma_{\nu_{i-1}} \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_{n_i} \in \tau)}.$$

Observe that for any fixed  $\nu$  with  $a, b \in \nu$ ,

$$(5.8) \quad \mathbf{E}_{\nu'} [W(\sigma, \nu', [a, b]) \mathbf{1}_{b \in \nu'} | a \in \nu'] \geq \mathbf{P}_{\nu'}(\nu' \cap [a, b] = \nu \cap [a, b] | a \in \nu') W(\sigma, \nu, [a, b])$$

$$(5.9) \quad = \exp \left( \sum_{i=1}^{|\nu \cap [a, b]|} \log K^*(n_i) \right) \exp \left( \sum_{i=1}^{|\nu \cap [a, b]|} U_i \right).$$

Hence, for any  $\nu$  with  $a, b \in \nu$

$$(5.10) \quad \begin{aligned} \mathbf{P}_{\sigma, \tau}(I_L^c | A_L(\nu)) &\leq \mathbf{P}_{\sigma, \tau} \left( \log (\mathbf{E}_{\nu'} [W(\sigma, \nu', [a, b]) \mathbf{1}_{b \in \nu'} | a \in \nu']) < \kappa q_L \mid A_L(\nu) \right) \\ &\leq \mathbf{P}_{\sigma, \tau} \left( \sum_{i=1}^{|\nu \cap [a, b]|} U_i < \kappa q_L - \sum_{i=1}^{|\nu \cap [a, b]|} \log K^*(n_i) \mid A_L(\nu) \right). \end{aligned}$$

Now, we denote

$$(5.11) \quad \xi(n) := \mathbf{E}_{\sigma, \tau} \left[ \log \frac{\mathbf{P}_\tau(\sigma_n \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)} \mid \sigma_n \in \tau \right] = \mathbf{E}_\sigma \left[ \frac{\mathbf{P}_\tau(\sigma_n \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)} \log \frac{\mathbf{P}_\tau(\sigma_n \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)} \right].$$

Note that  $\mathbf{E}_{\sigma, \tau}[U_i | A(\nu)] = \xi(\nu_i - \nu_{i-1}) = \xi(n_i)$ , and that  $\xi(n) > 0$  thanks to Jensen's inequality.

We denote

$$M_\xi := \mathbf{E}_\nu[\xi(\nu_1)] = \sum_{n \geq 1} K^*(n) \xi(n),$$

and the entropy of  $K^*$

$$H^* := \mathbf{E}_\nu[-\log K^*(\nu_1)] = \sum_{n=1}^{\infty} K^*(n) \log \frac{1}{K^*(n)}.$$

Now, for fixed  $a, b$  such that  $b - a \geq \varepsilon L$ , let us fix  $\kappa$  to be specified and define the event  $\hat{\mathcal{G}}$  of  $\nu$  to be *good* (on  $[a, b]$ ) by

$$(5.12) \quad \nu \in \hat{\mathcal{G}} \quad \text{if} \quad \begin{aligned} (i) \quad & -\sum_{i=1}^{|\nu \cap [a, b]|} \log K^*(n_i) \leq 2H^* |\nu \cap [a, b]|, \\ (ii) \quad & \sum_{i=1}^{|\nu \cap [a, b]|} \xi(n_i) > \frac{1}{2} M_\xi |\nu \cap [a, b]|, \\ (iii) \quad & |\nu \cap [a, b]| \geq \frac{\kappa}{H^*} q_L. \end{aligned}$$

The following lemmas are proven in Section 5.3; the first shows that good  $\nu$ 's are typical.

**Lemma 5.2.** *Suppose  $\alpha^* > 0$ . For any  $\varepsilon, \eta > 0$ , there exists some  $\kappa_0 > 0$  such that for any  $0 < \kappa \leq \kappa_0$ , and any  $a, b$  with  $b - a \geq \varepsilon L$ ,*

$$\limsup_{L \rightarrow \infty} \sup_{\substack{0 \leq a < b \leq L \\ b - a \geq \varepsilon L}} \mathbf{P}_\nu(\nu \notin \hat{\mathcal{G}} \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset) \leq \eta.$$

**Lemma 5.3.** *Suppose  $\alpha, \tilde{\alpha} > 0$  with  $\alpha + \tilde{\alpha} < 1$ . For any  $\nu$ , the  $U_i$ 's are independent given  $A_L(\nu)$ , and we have  $\mathbf{E}_{\sigma, \tau}[U_i \mid A_L(\nu)] = \xi(n_i)$ . Moreover, there exists a constant  $c_{17}$  such that, for all  $i \geq 1$ ,*

$$\mathbf{E}_{\sigma, \tau}[U_i^2 \mid A_L(\nu)] = \mathbf{E}_\sigma \left[ \frac{\mathbf{P}_\tau(\sigma_{n_i} \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_{n_i} \in \tau)} \log^2 \frac{\mathbf{P}_\tau(\sigma_{n_i} \in \tau)}{\mathbf{P}_{\sigma, \tau}(\sigma_{n_i} \in \tau)} \right] \leq c_{17} \quad \text{for all } i \geq 1.$$

For any  $\nu \in \hat{\mathcal{G}}$ , thanks to (5.10),

$$(5.13) \quad \mathbf{P}_{\sigma, \tau}(I_L^c \mid A_L(\nu)) \leq \mathbf{P}_{\sigma, \tau} \left( \sum_{i=1}^{|\nu \cap [a, b]|} U_i < 3H^* |\nu \cap [a, b]| \mid A_L(\nu) \right).$$

In Lemma 5.6 we will show that there are distributions of  $\tau, \sigma$  for which  $H^* \leq M_\xi/12$ . For such distributions we have from (5.13), Chebychev's inequality, and Lemma 5.3 that for any  $\nu \in \hat{\mathcal{G}}$ ,

$$(5.14) \quad \begin{aligned} & \mathbf{P}_{\sigma, \tau} \left( \sum_{i=1}^{|\nu \cap [a, b]|} U_i < 3H^* |\nu \cap [a, b]| \mid A_L(\nu) \right) \\ & \leq \mathbf{P}_{\sigma, \tau} \left( \sum_{i=1}^{|\nu \cap [a, b]|} [U_i - \xi(n_i)] < -\frac{M_\xi}{4} |\nu \cap [a, b]| \mid A_L(\nu) \right) \\ & \leq \frac{16c_{17}}{M_\xi^2 |\nu \cap [a, b]|} \leq \frac{16c_{17}H^*}{M_\xi^2 \kappa q_L}. \end{aligned}$$

Combining (5.14) with Lemma 5.2, we obtain that provided that  $H^* \leq M_\xi/12$ , for all  $\eta > 0, \varepsilon > 0$  there exist  $\kappa > 0$  and  $L_0$  such that, for any  $L \geq L_0$  and any  $a, b$  with  $b - a \geq \varepsilon L$ ,

$$\mathbf{E}_\nu \left[ \mathbf{P}_{\sigma, \tau}(I_L^c \mid A(\nu)) \mid \nu_0 = a; b \in \nu; \nu \cap (b, L] = \emptyset \right] \leq \frac{16c_{17}H^*}{M_\xi^2 \kappa q_L} + \eta.$$

Since  $q_L \rightarrow +\infty$  as  $L \rightarrow \infty$  and  $\eta$  is arbitrary, this proves the second half of (5.3), completing the proof of Lemma 5.1.  $\square$

We need an analog of Lemma 4.3, to deal with all blocks simultaneously. Let  $\hat{g}_{\mathcal{I}}$  be similar to  $g_{\mathcal{I}}$  in (4.14) but with the event  $J_L$  replaced by  $I_L$ .



**Lemma 5.4.** *Suppose the distributions of  $\tau, \sigma$  satisfy  $H^* \leq M_\xi/12$ , and let  $\zeta < 1$ ,  $\eta > 0$ ,  $\varepsilon > 0$  be fixed. Then there exists  $L_0 > 0$  such that, if  $L \geq L_0$ , then  $\mathbf{P}_\sigma(J_L) \leq \eta^{\zeta/(1-\zeta)}$ , and for every  $\mathcal{I} \subset \{1, \dots, m\}$  we have*

$$(5.15) \quad \mathbf{E}_\sigma \left[ \hat{g}_{\mathcal{I}} \mathcal{Z}_{\mathcal{I}} \right] \leq \sum_{\substack{d_{i_1} \leq f_{i_1} \\ d_{i_1}, f_{i_1} \in \mathcal{B}_{i_1}}} \cdots \sum_{\substack{d_{i_m} \leq f_{i_m} = nL \\ d_{i_m} \in \mathcal{B}_{i_m}}} \prod_{k=1}^m K^*(d_{i_k} - f_{i_{k-1}}) (2\eta + \mathbf{1}_{\{f_{i_k} - d_{i_k} < \varepsilon L\}}) \mathbf{P}_\nu(f_{i_k} - d_{i_k} \in \nu).$$

*Proof.* As in Lemma 4.3, the key is the analog of (4.73). To that end, conditionally on  $A(\nu)$ , the  $Y_j$  are independent, so by (5.13) and (5.14), if  $\nu_{\mathcal{B}_{i_k}} \in \hat{\mathcal{G}}$  for all  $k \in \mathcal{K}$  then

$$(5.16) \quad \mathbf{E}_{\sigma, \tau} \left( \prod_{k \in \mathcal{K}} \mathbf{1}_{\{\sigma_{\mathcal{B}_{i_k}} \in I_L^c\}} \mid A(\nu) \right) \leq \prod_{k \in \mathcal{K}} \mathbf{P}_{\sigma, \tau} \left( \sum_{j=1}^{|\nu \cap [d_{i_k}, f_{i_k}]|} U_j < 3H^* \mid \nu \cap [d_{i_k}, f_{i_k}] \mid A_L(\nu) \right) \leq \gamma_L^{|\mathcal{K}|},$$

where  $\gamma_L \rightarrow 0$  as  $L \rightarrow \infty$ . The rest of the proof is similar to Lemma 4.3.  $\square$

Given Lemma 5.4, the proof of Theorem 2.2 proceeds exactly as in Section 4. So the only remaining step is to show there are distributions satisfying  $H^* \leq M_\xi/12$ .

**Remark 5.5.** Our argument can be tightened to show that if

$$(5.17) \quad H^* < M_\xi,$$

then the critical points are distinct. The condition (5.17) is however not necessary (there are distributions of  $\sigma, \tau$  for which  $H^* > M_\xi$  with  $\alpha^* > 1/2$ ), so we include the factor of 12 to reduce technicalities.

**5.2. Distributions of  $\tau$  and  $\sigma$  with  $H^* \leq M_\xi/12$ .** Recall  $\xi(n)$  from (5.11). As noted above, the proof of Theorem 2.2 is complete when we show the following.

**Lemma 5.6.** *Given any  $\alpha, \tilde{\alpha} > 0$  with  $\alpha + \tilde{\alpha} < 1$ , and any slowly varying functions  $\varphi, \tilde{\varphi}$ , there exist distributions for  $\tau, \sigma$  verifying*

$$\mathbf{P}_\tau(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} \varphi(n) n^{-(1+\alpha)}, \quad \mathbf{P}_\sigma(\sigma_1 = n) \stackrel{n \rightarrow \infty}{\sim} \tilde{\varphi}(n) n^{-(1+\tilde{\alpha})},$$

for which

$$(5.18) \quad H^* = \sum_{n=1}^{\infty} K^*(n) \log \frac{1}{K^*(n)} \leq \frac{1}{12} \sum_{n=1}^{\infty} K^*(n) \xi(n) = \frac{1}{12} M_\xi.$$

The idea is that we can make the entropy  $H^*$  small mainly by making  $K^*(1)$  close to 1. This means we need

$$(5.19) \quad \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau) \gg \sum_{n=2}^{\infty} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau).$$

A lower bound for  $M_\xi$  is  $K^*(1)\xi(1)$ , so also  $\xi(1)$  should not be too small.

We choose the inter-arrival distribution for  $\tau$  and  $\sigma$  (satisfying (1.5)) in the following way. Let  $\rho$  and  $\tilde{\rho}$  be two recurrent renewals with inter-arrival distributions  $f_\rho(n) := \mathbf{P}_\rho(\rho_1 = n) = \varphi(n)n^{-(1+\alpha)}$  and  $f_{\tilde{\rho}}(n) := \mathbf{P}_{\tilde{\rho}}(\tilde{\rho} = n) = \tilde{\varphi}(n)n^{-(1+\tilde{\alpha})}$ . For some (large) integer  $a_0$ , and

setting  $A_0 = e^{\theta a_0}$  with  $\theta > 0$  to be specified (chosen so  $A_0$  is also an integer), we define the inter-arrival distributions of  $\tau$  and  $\sigma$ :

$$(5.20) \quad \begin{cases} \mathbf{P}_\sigma(\sigma_1 = a_0) = \frac{1}{2}, \\ \mathbf{P}_\sigma(\sigma_1 = 2a_0 + m) = \frac{1}{2}f_{\tilde{\rho}}(m), \end{cases} \quad \begin{cases} \mathbf{P}_\tau(\tau_1 = 1) = \frac{1}{2}, \\ \mathbf{P}_\tau(\tau_1 = A_0 + m) = \frac{1}{2}f_\rho(m) \end{cases} \quad \text{for } m \geq 1.$$

To prove Lemma 5.6 we will need the following lemma, which we prove later in this section.

**Lemma 5.7.** *Let  $\alpha, \tilde{\alpha} > 0$  with  $\alpha + \tilde{\alpha} < 1$ . For  $\sigma, \tau$  as in (5.20), we can choose  $a_0$  and  $\theta$  large enough so that*

$$(5.21) \quad \frac{1}{2}2^{-a_0} \leq \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau) \leq \frac{3}{4}2^{-a_0},$$

$$(5.22) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \text{ for some } n \geq 2) \leq 2^{-a_0+2} \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau),$$

$$(5.23) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) \leq n^{-(1+\alpha^*/2)} \quad \text{for all } n \geq \exp(2^{a_0/2}).$$

*Proof of Lemma 5.6.* First we use (5.21) and (5.22) to show that  $K^*(1)$  is close to 1. Define  $M = |\{n \geq 2 : \sigma_n \in \tau\}|$ . From (5.21) and (5.22), assuming  $a_0 \geq 2$ ,

$$\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \text{ for some } n \geq 1) \leq (1 + 2^{-a_0+2}) \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau) \leq \frac{1}{2}.$$

Hence provided  $a_0$  is large, using (5.21),

$$(5.24) \quad \begin{aligned} \sum_{n=2}^{\infty} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) &= \mathbf{E}_{\sigma, \tau}[M] = \mathbf{P}_{\sigma, \tau}(M \geq 1) + \sum_{k \geq 2} \mathbf{P}_{\sigma, \tau}(M \geq k) \\ &\leq \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \text{ for some } n \geq 2) + \sum_{k \geq 2} (1 + 2^{-a_0+2})^k \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau)^k \\ &\leq 2^{-a_0+2} \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau) + 2(1 + 2^{-a_0+2})^2 \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau)^2 \\ &\leq 2^{-a_0+3} \mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau). \end{aligned}$$

Therefore

$$(5.25) \quad K^*(1) = \frac{\mathbf{P}_{\sigma, \tau}(\sigma_1 \in \tau)}{\sum_{n \geq 1} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)} \geq \frac{1}{1 + 2^{3-a_0}}.$$

To show that  $H^*$  is small, we divide it into three parts:

$$(5.26) \quad H^* = K^*(1) \log \frac{1}{K^*(1)} + \sum_{n=2}^{\exp(2^{a_0/2})} K^*(n) \log \frac{1}{K^*(n)} + \sum_{n > \exp(2^{a_0/2})} K^*(n) \log \frac{1}{K^*(n)}.$$

By (5.25), the first term is smaller than  $2^{3-a_0}$ . For the middle term, denote

$$\Sigma^* := \sum_{n \geq 1} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau), \quad \hat{\Sigma}^* := \sum_{n=2}^{\exp(2^{a_0/2})} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)$$

so that the first sum in (5.26) is

$$\begin{aligned}
& \sum_{n=2}^{\exp(2^{a_0/2})} \frac{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)}{\Sigma^*} \log \frac{\Sigma^*}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \\
& \leq \frac{\hat{\Sigma}^*}{\Sigma^*} \log \frac{\hat{\Sigma}^*}{\Sigma^*} + \frac{\hat{\Sigma}^*}{\Sigma^*} \sum_{n=2}^{\exp(2^{a_0/2})} \frac{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)}{\hat{\Sigma}^*} \log \frac{\hat{\Sigma}^*}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \\
& \leq \frac{\hat{\Sigma}^*}{\Sigma^*} \log \frac{\hat{\Sigma}^*}{\Sigma^*} + \frac{\hat{\Sigma}^*}{\Sigma^*} 2^{a_0/2} \leq 2^{4-a_0/2}.
\end{aligned}$$

Here in the second inequality we used that the sum is maximized when all the terms are equal, by Jensen's inequality. For the last inequality we used (5.24) to get  $\hat{\Sigma}^* \leq 2^{3-a_0}\Sigma^*$  and took  $a_0$  large enough.

For the last sum in (5.26), we use (5.23) in Lemma 5.7, together with the fact that  $1 \geq \Sigma^* \geq 2^{-a_0-1}$  (see (5.21) and (5.24)), to bound that sum by

$$\begin{aligned}
\frac{1}{\Sigma^*} \sum_{n > \exp(2^{a_0/2})} \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau) \log \frac{1}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} & \leq 2^{a_0+1} \sum_{n > \exp(2^{a_0/2})} (1 + \alpha^*/2) n^{-(1+\alpha^*/2)} \log n \\
& \leq 2^{a_0+1} \frac{1 + \alpha^*/2}{\alpha^*/2} \exp(2^{a_0/2})^{-\alpha^*/2} 2^{a_0/2} \\
& \leq 2^{-a_0/2},
\end{aligned}$$

where we took  $a_0$  large enough in the last line.

Combining these bounds, we have for  $a_0$  large

$$(5.27) \quad H^* \leq 2^{3-a_0} + 2^{4-a_0/2} + 2^{-a_0/2} \leq 2^{5-a_0/2},$$

which can be made arbitrarily small by choosing  $a_0$  large.

In view of (5.27), we are left with showing that  $M_\xi$  is uniformly bounded away from 0. Thanks to (5.25), we have

$$M_\xi = \mathbf{E}_\nu[\xi(\nu_1)] \geq \frac{1}{2}\xi(1) = \frac{1}{2}\mathbf{E}_\sigma \left[ \frac{\mathbf{P}_\tau(\sigma_1 \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)} \log \frac{\mathbf{P}_\tau(\sigma_1 \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)} \right].$$

From (5.20) we have that  $\mathbf{P}_\tau(\sigma_1 \in \tau) = 2^{-a_0}$  if  $\sigma_1 = a_0$ . Hence, we get thanks to (5.21) that

$$\frac{\mathbf{P}_\tau(\sigma_1 \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)} \geq \frac{4}{3} \quad \text{if } \sigma_1 = a_0.$$

Since  $x \log x \geq -1/e$  for all  $x \geq 0$  we have that

$$(5.28) \quad \frac{\mathbf{P}_\tau(\sigma_1 \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)} \log \frac{\mathbf{P}_\tau(\sigma_1 \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau)} \geq \begin{cases} \frac{4}{3} \log \frac{4}{3} & \text{if } \sigma_1 = a_0, \\ -e^{-1} & \text{otherwise.} \end{cases}$$

It follows that  $\xi(1) \geq \frac{1}{2}(\frac{4}{3} \log \frac{4}{3} - e^{-1}) > 0$ , so that by (5.27),

$$\frac{1}{12}M_\xi \geq \frac{1}{48} \left( \frac{4}{3} \log \frac{4}{3} - e^{-1} \right) \geq 2^{5-a_0/2} \geq H^*,$$

provided that  $a_0$  is large. This completes the proof of Lemma 5.6.  $\square$

*Proof of Lemma 5.7.* From the definition (5.20) we have

$$(5.29) \quad \mathbf{P}_\tau(\sigma_1 \in \tau) \begin{cases} = 2^{-a_0} & \text{if } \sigma_1 = a_0, \\ \leq 2^{-2a_0} & \text{if } \sigma_1 \in [2a_0, A_0], \\ \leq 1 & \text{if } \sigma_1 > A_0. \end{cases}$$

Therefore, we already have the lower bound

$$\mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau) \geq \mathbf{P}_\sigma(\sigma_1 = a_0)2^{-a_0} = \frac{1}{2}2^{-a_0}.$$

For the upper bound, we use that

$$\mathbf{P}_\sigma(\sigma_1 > A_0) = \frac{1}{2}\mathbf{P}_\sigma(\tilde{\rho} > A_0 - 2a_0) \leq c_{18}\tilde{\varphi}(A_0)A_0^{-\tilde{\alpha}} \leq 2^{-2a_0},$$

where we recall that  $A_0 = e^{\theta a_0}$ , and took  $\theta > (2 \log 2)/\tilde{\alpha}$  and  $a_0$  large enough. Hence, we get

$$(5.30) \quad \begin{aligned} \mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau) &\leq 2^{-a_0}\mathbf{P}_\sigma(\sigma_1 = a_0) + 2^{-2a_0}\mathbf{P}_\sigma(\sigma_1 \in [2a_0, A_0]) + \mathbf{P}_\sigma(\sigma_1 > A_0) \\ &\leq 2^{-a_0-1} + 2^{-2a_0} \leq \frac{3}{4}2^{-a_0}, \end{aligned}$$

for  $a_0$  large enough.

We also have the lower bound

$$(5.31) \quad \mathbf{P}_{\sigma,\tau}(\sigma_1 \in \tau) \geq 2^{-a_0}\mathbf{P}_{\sigma,\tau}(\sigma_1 = a_0) = 2^{-(a_0+1)},$$

and (5.21) is proven.

**Proof of (5.22).** In view of (5.31), we need to show that if  $a_0$  is large enough

$$(5.32) \quad \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \text{ for some } n \geq 2) \leq 2^{1-2a_0}.$$

For that purpose, we reformulate the definition (5.20) in a more tractable way. Let  $\tilde{\xi}_1, \tilde{\xi}_2, \dots$  and  $\xi_1, \xi_2, \dots$  be i.i.d. sequences of variables taking values 0, 1 with probability 1/2 each, and let

$$M_k = \sum_{i=1}^k \xi_i, \quad \tilde{M}_k = \sum_{i=1}^k \tilde{\xi}_i.$$

Then, according to the definition (5.20), we can write

$$(5.33) \quad \tau_k \stackrel{(law)}{=} A_0 M_k + k - M_k + \rho_{M_k}, \quad \sigma_k \stackrel{(law)}{=} (k + \tilde{M}_k)a_0 + \tilde{\rho}_{\tilde{M}_k}.$$

Since  $\alpha + \tilde{\alpha} < 1$ , we can fix  $\gamma \in (0, 1)$  small, satisfying

$$(5.34) \quad \tilde{\alpha}(1 + 3\gamma) < 1, \quad \frac{(1 - \gamma)(1 - \tilde{\alpha}(1 + 3\gamma))}{\alpha} > 1 \quad \text{and} \quad \frac{\tilde{\alpha}(1 - \gamma)(1 + 3\gamma)}{1 - \alpha - \gamma} < 1.$$

We now decompose

$$(5.35) \quad \begin{aligned} \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau \text{ for some } n \geq 2) \\ \leq \mathbf{P}_{\sigma,\tau}(\exists n \geq 2; \sigma_n \in \tau \cap [1, A_0]) + \mathbf{P}_{\sigma,\tau}(\exists k \geq 1; \tau_k \in \sigma \cap (A_0, \infty)). \end{aligned}$$

In view of (5.20), the first term is bounded by

$$(5.36) \quad \mathbf{P}_\tau(\tau \cap [2a_0, A_0] \neq \emptyset) \leq 2^{-2a_0}.$$

We fix  $\kappa > 2(1 - \alpha)/\gamma$ ,  $\kappa > 1$  and decompose the second term on the right in (5.35) as

$$(5.37) \quad \mathbf{P}_{\sigma,\tau}(\exists k < a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)) + \mathbf{P}_{\sigma,\tau}(\exists k \geq a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)).$$

We begin with the first term in (5.37). We have

$$(5.38) \quad \mathbf{P}_{\sigma,\tau}(\exists k < a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)) \leq \sum_{i=1}^{a_0^\kappa} \mathbf{P}_{\sigma,\tau}(\tau_i \in \sigma \cap (A_0, \infty)) \leq a_0^\kappa \sup_{j \geq A_0} \mathbf{P}_\sigma(j \in \sigma).$$

To bound this, we write

$$(5.39) \quad \mathbf{P}_\sigma(j \in \sigma) \leq \sum_{k \leq j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\sigma_k = j) + \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\widetilde{M}_k < \frac{k}{4}) + \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\sigma_k = j, \widetilde{M}_k \geq \frac{k}{4}).$$

For the first sum in (A.12), we use the representation (5.33) to write

$$\sum_{k \leq j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\sigma_k = j) \leq \sum_{k \leq j^{\tilde{\alpha}(1+\gamma)}} \sum_{m=1}^k \mathbf{P}_\sigma(\tilde{\rho}_m = j - (k+m)a_0) \mathbf{P}_\sigma(\widetilde{M}_k = m).$$

Now, observe that for  $j \geq A_0$  we have  $2j^{\tilde{\alpha}(1+\gamma)}a_0 < j/2$  assuming  $a_0$  is large, so that in the above sum,  $j - (k+m)a_0 \geq j/2$ . Hence, thanks to Lemma A.2 applied to  $\tilde{\rho}$ , there is a constant  $c_{19}$  such that

$$(5.40) \quad \begin{aligned} \sum_{k \leq j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\sigma_k = j) &\leq \sum_{k \leq j^{\tilde{\alpha}(1+\gamma)}} \sum_{m=1}^k c_{19} m \tilde{\varphi}(j) j^{-(1+\tilde{\alpha})} \mathbf{P}_\sigma(\widetilde{M}_k = m) \\ &\leq c_{19} \tilde{\varphi}(j) j^{-(1+\tilde{\alpha})} \sum_{k=1}^{j^{\tilde{\alpha}(1+\gamma)}} \frac{k}{2} \\ &\leq c_{19} \tilde{\varphi}(j) j^{-(1-\tilde{\alpha}(1+2\gamma))}. \end{aligned}$$

The second sum in (5.39) is easily controlled: there exists a constant  $c_{20}$  such that for all  $k \geq 1$ ,  $\mathbf{P}_\tau(M_k < k/4) \leq e^{-c_{20}k}$ . Therefore, for any  $j \geq A_0$  with  $a_0$  large enough, we have

$$(5.41) \quad \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\widetilde{M}_k < k/4) \leq \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} e^{-c_{20}k} \leq e^{-j^{\tilde{\alpha}}}.$$

For the last sum in (5.39), from (5.33) and Lemma A.3 (applied to  $\tilde{\rho}$ ), there is a constant  $c_{21}$  such that

$$(5.42) \quad \begin{aligned} &\sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\sigma_k = j, \widetilde{M}_k \geq k/4) \\ &\leq \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \mathbf{P}_\sigma(\tilde{\rho}_{k/4} \leq j) \leq \sum_{k > j^{\tilde{\alpha}(1+\gamma)}} \exp(-c_{21} k^{\gamma/2(1-\tilde{\alpha})}) \leq e^{-j^{\tilde{\alpha}/2}}, \end{aligned}$$

where the last inequality holds for any  $j \geq A_0$  with  $A_0$  large. Combining (5.39)—(5.42) we obtain that there exist constant  $c_{22}, c_{23} > 0$  such that, for any  $j \geq A_0$ ,

$$(5.43) \quad \mathbf{P}_\sigma(j \in \sigma) \leq c_{22} \tilde{\varphi}(j) j^{-(1-\tilde{\alpha}(1+2\gamma))} \leq c_{23} j^{-(1-\tilde{\alpha}(1+3\gamma))}.$$

Now, recalling that  $A_0 = e^{\theta a_0}$ , provided that we take  $\theta > 2(1 - \tilde{\alpha}(1 + 3\gamma))^{-1}$  and then  $a_0$  large, it follows from (5.38) and (5.43) that

$$(5.44) \quad \mathbf{P}_{\sigma,\tau}(\exists k < a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)) \leq c_{23} a_0^\kappa A_0^{-(1-\tilde{\alpha}(1+3\gamma))} \leq e^{-2a_0}.$$

We now consider the second term in (5.37). We bound

$$(5.45) \quad \mathbf{P}_{\sigma, \tau}(\exists k \geq a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)) \leq (I) + (II) + (III),$$

where we set

$$\begin{aligned} (I) &:= \mathbf{P}_\tau \left( M_k < \frac{k}{4} \text{ for some } k \geq a_0^\kappa \right), \\ (II) &:= \mathbf{P}_\tau \left( M_k \geq \frac{k}{4}, \rho_{M_k} < k^{(1-\gamma)/\alpha} \text{ for some } k \geq a_0^\kappa \right), \\ (III) &:= \mathbf{P}_{\sigma, \tau} \left( \tau_k \in \sigma, M_k \geq \frac{k}{4}, \rho_{M_k} \geq k^{(1-\gamma)/\alpha} \text{ for some } k \geq a_0^\kappa \right). \end{aligned}$$

Similarly to (5.44),

$$(5.46) \quad (I) \leq \sum_{k \geq a_0^\kappa} \mathbf{P}_\sigma(\widetilde{M}_k < k/4) \leq \sum_{k \geq a_0^\kappa} e^{-c_{20}k} \leq e^{-2a_0},$$

where the last inequality holds if  $a_0$  is large enough, since  $\kappa > 1$ .

For (II), from Lemma A.3 (applied to  $\rho$ ), there exists a constant  $c_{22}$  such that, provided  $a_0$  is large,

$$(5.47) \quad (II) \leq \sum_{m \geq a_0^\kappa/4} \mathbf{P}_\tau(\rho_m < (4m)^{(1-\gamma)/\alpha}) \leq \sum_{m \geq a_0^\kappa/4} \exp\left(-c_{22}m^{\frac{\gamma}{2(1-\alpha)}}\right) \leq e^{-2a_0},$$

where the last inequality is valid provided that  $a_0$  is large enough, since  $\kappa > 2(1-\alpha)/\gamma$ .

As for (III), using the representation (5.33), we obtain that

$$(5.48) \quad (III) \leq \sum_{k \geq a_0^\kappa} \mathbf{P}_{\sigma, \tau}(\tau_k \in \sigma, \tau_k \geq \max\{k^{(1-\gamma)/\alpha}, \frac{k}{4}A_0\}).$$

Thanks to (5.43), there is a constant  $c_{23}$  such that

$$(5.49) \quad \sup_{j \geq \max\{k^{(1-\gamma)/\alpha}, kA_0/4\}} \mathbf{P}_\sigma(j \in \sigma) \leq c_{23} \min\{(kA_0)^{-(1-\tilde{\alpha}(1+3\gamma))} : k^{-(1-\gamma)(1-\tilde{\alpha}(1+3\gamma))/\alpha}\}.$$

Using (5.34), we therefore have that

$$\begin{aligned} (III) &\leq c_{23} \sum_{k=a_0^\kappa}^{A_0^{\alpha/(1-\alpha-\gamma)}} (kA_0)^{-(1-\tilde{\alpha}(1+3\gamma))} + c_{23} \sum_{k \geq A_0^{\alpha/(1-\alpha-\gamma)}} k^{-(1-\gamma)(1-\tilde{\alpha}(1+3\gamma))/\alpha} \\ &\leq c_{24} A_0^{-(1-\tilde{\alpha}(1+3\gamma))} A_0^{\tilde{\alpha}(1+3\gamma)\alpha/(1-\alpha-\gamma)} + c_{24} A_0^{[1-(1-\gamma)(1-\tilde{\alpha}(1+3\gamma))/\alpha]\alpha/(1-\alpha-\gamma)} \\ &\leq 2c_{24} A_0^{-1+[\tilde{\alpha}(1+3\gamma)(1-\gamma)/(1-\alpha-\gamma)]} \\ (5.50) \quad &\leq e^{-2a_0}, \end{aligned}$$

where the last inequality comes if we choose  $\theta > 2\left(\frac{\tilde{\alpha}(1+3\gamma)(1-\gamma)}{(1-\alpha-\gamma)} - 1\right)^{-1}$  (recall  $A_0 = e^{\theta a_0}$ ), and then take  $a_0$  large enough. Hence (5.46), (5.47), and (5.50) yield

$$(5.51) \quad \mathbf{P}_{\sigma, \tau}(\exists k \geq a_0^\kappa; \tau_k \in \sigma \cap (A_0, \infty)) \leq 3e^{-2a_0}.$$

Collecting the estimates (5.36), (5.44) and (5.51), we end up with

$$\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau \text{ for some } n \geq 2) \leq 2^{-2a_0} + 4e^{-2a_0},$$

provided that  $a_0$  is large enough, so we have proved (5.32).

**Proof of (5.23).** It now remains to prove (5.23). We take  $n \geq \exp(2^{a_0/2})$ , and write

$$(5.52) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) \leq \mathbf{P}_\sigma(\sigma_n \leq n^{(1-\gamma)/\tilde{\alpha}}) + \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n > n^{(1-\gamma)/\tilde{\alpha}}).$$

The first term in (5.52) is bounded easily: we use Lemma A.3 (applied to  $\tilde{\rho}$ ) to get a constant  $c_{25}$  such that

$$(5.53) \quad \begin{aligned} \mathbf{P}_\sigma(\sigma_n \leq n^{(1-\gamma)/\tilde{\alpha}}) &\leq \mathbf{P}_\sigma(\tilde{M}_n \leq n/4) + \mathbf{P}_\sigma(\tilde{\rho}_{n/4} \leq n^{(1-\gamma)/\tilde{\alpha}}) \\ &\leq \exp(-c_{20}n) + \exp(-c_{25}n^{\gamma/2(1-\tilde{\alpha})}) \leq \exp(-n^{\gamma/2}), \end{aligned}$$

where the last inequality is valid provided that  $n$  is large (ensured by having  $a_0$  large) and  $c_{20}$  is from (5.41).

To bound the second term in (5.52), we write

$$(5.54) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n > n^{(1-\gamma)/\tilde{\alpha}}) = \sum_{r > n^{(1-\gamma)/\tilde{\alpha}}} \mathbf{P}_\sigma(\sigma_n = r) \sum_{k=1}^{+\infty} \mathbf{P}_\tau(\tau_k = r).$$

Then, we control  $\mathbf{P}_\tau(\tau_k = r)$  according to whether  $k > r^{\alpha/(1-\gamma)}$  or not.

If  $k > r^{\alpha/(1-\gamma)}$ , then  $k$  is large provided  $a_0$  is large, so we have similarly to (5.53)

$$\mathbf{P}_\tau(\tau_k = r) \leq \mathbf{P}_\tau(\tau_k \leq k^{(1-\gamma)/\alpha}) \leq \exp(-k^{\gamma/2}),$$

so that if  $n$  is large enough,

$$\begin{aligned} \sum_{r > n^{(1-\gamma)/\tilde{\alpha}}} \mathbf{P}_\sigma(\sigma_n = r) \sum_{k > r^{\alpha/(1-\gamma)}} \mathbf{P}_\tau(\tau_k = r) &\leq \sum_{r > n^{(1-\gamma)/\tilde{\alpha}}} \exp(-r^{\alpha\gamma/2}) \\ &\leq \exp(-n^{\alpha\gamma(1-\gamma)/2}). \end{aligned}$$

To handle  $k \leq r^{\alpha/(1-\gamma)}$  and  $r > n^{(1-\gamma)/\tilde{\alpha}}$ , we write

$$\mathbf{P}_\tau(\tau_k = r) = \sum_{m=0}^k \mathbf{P}_\tau(M_k = m) \mathbf{P}_\tau(A_0 m + k - m + \rho_m = r).$$

Since  $n \geq \exp(2^{a_0/2})$ , we have  $A_0 r^{\alpha/(1-\gamma)} \leq r/2$ , provided that  $\alpha/(1-\gamma) < 1$ , cf. (5.34), and that  $a_0$  is large enough ( $\theta$  being fixed). In the end, we get that for such  $k, r$ , and for  $m \leq k$ , we have  $A_0 m + k - m \leq A_0 k \leq r/2$ , so that by Lemma A.2 applied to  $\rho$ , there is a constant  $c_{26}$  such that

$$\mathbf{P}_\tau(\tau_k = r) \leq \sum_{m=0}^k \mathbf{P}_\tau(M_k = m) c_{26} m r^{-(1+\alpha-\gamma)} = c_{26} \frac{k}{2} r^{-(1+\alpha-\gamma)}.$$

We thus obtain

$$(5.55) \quad \begin{aligned} \sum_{r > n^{(1-\gamma)/\tilde{\alpha}}} \mathbf{P}_\sigma(\sigma_n = r) \sum_{k \leq r^{\alpha/(1-\gamma)}} \mathbf{P}_\tau(\tau_k = r) &\leq c_{26} \sum_{r > n^{(1-\gamma)/\tilde{\alpha}}} \mathbf{P}_\sigma(\sigma_n = r) r^{-(1+\alpha-\gamma)} r^{2\alpha/(1-\gamma)} \\ &\leq c_{26} \max_{r > n^{(1-\gamma)/\tilde{\alpha}}} r^{-(1+\alpha-\gamma)} r^{2\alpha/(1-\gamma)} \\ &\leq c_{26} n^{-(1+\alpha^*)+\gamma(2+\alpha)/\tilde{\alpha}}. \end{aligned}$$

In the end, if  $\gamma$  is chosen small enough so that  $\gamma(2+\alpha)/\tilde{\alpha} < \alpha^*/4$  (and (5.34) holds), then (5.54) yields

$$(5.56) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau, \sigma_n > n^{(1-\gamma)/\tilde{\alpha}}) \leq \exp(-n^{\alpha\gamma(1-\gamma)/2}) + c_{26} n^{-(1+3\alpha^*/4)}.$$

Together with (5.53) and (5.52), this proves (5.23) provided that  $a_0$  is large enough. This completes the proof of Lemma 5.7.  $\square$

### 5.3. Proof of Lemmas 5.2 and 5.3.

*Proof of Lemma 5.3.* For  $b_n$  from (2.8),  $\sigma_n/b_n$  converges to a stable distribution (see [8]), so that thanks to (1.16), there exists a constant  $c_{29}$  such that

$$(5.57) \quad \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau) \geq \mathbf{P}_\sigma(\sigma_n \in [b_n, 2b_n]) \min_{m \in [b_n, 2b_n]} \mathbf{P}_\tau(m \in \tau) \geq c_{29} b_n^{-(1-\alpha)} \varphi(b_n)^{-1}$$

Hence, using (1.16) again, there exists  $c_{30}$  such that  $\mathbf{P}_\tau(m \in \tau) \leq c_{30} \mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)$  for any  $m \geq \frac{1}{2}b_n$ . This gives that

$$(5.58) \quad \mathbf{E}_\sigma \left[ \mathbf{1}_{\{\sigma_n \geq \frac{1}{2}b_n\}} \frac{\mathbf{P}_\tau(\sigma_n \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \log^2 \frac{\mathbf{P}_\tau(\sigma_n \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \right] \leq c_{30} (\log c_{30})^2,$$

and we now turn to the case  $\sigma_n \leq \frac{1}{2}b_n$ .

For  $k \geq 1$  and  $m = 2^{-k}b_n$ , by Lemma A.3,

$$(5.59) \quad \mathbf{P}_\sigma(\sigma_n \in (m/2, m]) \leq \mathbf{P}_\sigma(\sigma_n \leq 2^{-k}b_n) \leq \exp\left(-c_{31} 2^{k\tilde{\alpha}/2(1-\tilde{\alpha})}\right).$$

On the other hand, thanks to (1.16) there exists a constant  $c_{32}$  such that for all  $m \geq 1$ ,

$$(5.60) \quad \max_{j \in (m/2, m]} \mathbf{P}_\tau(j \in \tau) \leq c_{32} m^{-(1-\alpha)} \varphi(m)^{-1}.$$

Combining with (5.57), we get for  $m = 2^{-k}b_n$ :

$$(5.61) \quad \max_{j \in (m/2, m]} \frac{\mathbf{P}_\tau(j \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \leq c_{33} m^{-(1-\alpha)} \varphi(m)^{-1} b_n^{(1-\alpha)} \varphi(b_n) = c_{33} 2^{k(1-\alpha)} \frac{\varphi(b_n)}{\varphi(2^{-k}b_n)} \leq 2^{2k(1-\alpha)}$$

where the last inequality is valid for all  $k$  provided that  $n$  is large enough, since  $\varphi$  is slowly varying.

Now (5.59) and (5.61) give that

$$(5.62) \quad \sum_{j < \frac{1}{2}b_n} \mathbf{P}_\sigma(\sigma_n = j) \frac{\mathbf{P}_\tau(j \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \log^2 \frac{\mathbf{P}_\tau(j \in \tau)}{\mathbf{P}_{\sigma,\tau}(\sigma_n \in \tau)} \leq \sum_{k \geq 1} \exp\left(-c_{31} 2^{k\tilde{\alpha}/2(1-\tilde{\alpha})}\right) 2^{2k(1-\alpha)} (\log 2^{2k(1-\alpha)})^2 < +\infty,$$

which completes the proof.  $\square$

*Proof of Lemma 5.2.* It is enough to show that, for  $\xi(n)$  from (5.11) and  $\eta > 0$ ,

$$(5.63) \quad \lim_{n \rightarrow \infty} \mathbf{P}_\nu \left( \frac{1}{|\nu \cap [0, n]|} \sum_{i=1}^{|\nu \cap [0, n]|} \log \frac{1}{K^*(\nu_i - \nu_{i-1})} \geq 2 \mathbf{E}_\nu \left( \log \frac{1}{K^*(\nu_1)} \right) \mid n \in \nu \right) = 0;$$

$$(5.64) \quad \lim_{n \rightarrow \infty} \mathbf{P}_\nu \left( \frac{1}{|\nu \cap [0, n]|} \sum_{i=1}^{|\nu \cap [0, n]|} \xi(\nu_i - \nu_{i-1}) \leq \frac{1}{2} \mathbf{E}_\nu(\xi(\nu_1)) \mid n \in \nu \right) = 0,$$



and, for  $\kappa$  small enough,

$$(5.65) \quad \limsup_{L \rightarrow \infty} \sup_{L \geq n \geq \varepsilon L} \mathbf{P}_\nu \left( |\nu \cap [0, n]| < \frac{\kappa}{H^*} q_L \mid n \in \nu \right) \leq \eta.$$

We first prove (5.63)-(5.64). We can apply Lemma A.1 to remove the conditioning by  $n \in \nu$ , at the cost of a constant  $c_{16}$ . This and the weak law of large numbers give

$$(5.66) \quad \mathbf{P}_\nu \left( \frac{1}{|\nu \cap [0, n/2]|} \sum_{i=1}^{|\nu \cap [0, n/2]|} \log \frac{1}{K^*(\nu_i - \nu_{i-1})} \geq 2\mathbf{E}_\nu \left( \log \frac{1}{K^*(\nu_1)} \right) \mid n \in \nu \right) \\ \leq c_{16} \mathbf{P}_\nu \left( \frac{1}{|\nu \cap [0, n/2]|} \sum_{i=1}^{|\nu \cap [0, n/2]|} \log \frac{1}{K^*(\nu_i - \nu_{i-1})} \geq 2\mathbf{E}_\nu \left( \log \frac{1}{K^*(\nu_1)} \right) \right) \xrightarrow{n \rightarrow \infty} 0.$$

A symmetric result with  $[0, n/2]$  replaced with  $[n/2, n]$  holds, which in turns give (5.63). The same proof gives (5.64).

For (5.65), we apply again Lemma A.1 to get rid of the conditioning, using that  $n \geq \varepsilon L$ :

$$(5.67) \quad \mathbf{P}_\nu \left( |\nu \cap [0, n]| < \frac{\kappa}{H^*} q_L \mid n \in \nu \right) \leq \mathbf{P}_\nu \left( |\nu \cap [0, \varepsilon L/2]| < \frac{\kappa}{H^*} q_L \mid n \in \nu \right) \\ \leq c_{16} \mathbf{P}_\nu \left( |\nu \cap [0, \varepsilon L/2]| < \frac{\kappa}{H^*} q_L \right) \\ = c_{16} \mathbf{P}_\nu \left( \nu_{\kappa q_L/H^*} > \frac{\varepsilon}{2} L \right)$$

If  $\alpha^* \in (0, 1)$ , then  $q_L = L^{\alpha^*} \varphi^*(L)^{-1}$ , and Lemma A.2 gives that there exist  $\kappa$  and a constant  $c_{34}$  such that for  $L$  large,

$$\mathbf{P}_\nu \left( \nu_{\kappa q_L/H^*} > \frac{\varepsilon}{2} L \right) \leq c_{34} \frac{\kappa q_L}{H^*} (\varepsilon L)^{-\alpha^*} \varphi^*(L) = \frac{c_{34} \varepsilon^{-\alpha^*} \kappa}{H^*} < \frac{\eta}{c_{16}}.$$

If  $\alpha^* \geq 1$ , then as in (2.8) and the discussion following, for  $h_n$  satisfying  $h_n/m^*(h_n) \sim n$  we have  $\nu_n/h_n \rightarrow 1$  in probability, and for  $\delta > 0$ ,

$$\frac{h_{\delta q_L}}{m^*(h_{\delta q_L})} \sim \delta q_L \sim \frac{\delta L}{m^*(L)} \sim \frac{\delta L}{m^*(\delta L)},$$

so  $h_{\delta q_L} \sim \delta L$ . Hence the last probability in (5.67) approaches 0 as  $L \rightarrow \infty$ , whenever  $\kappa/H^* < \varepsilon/2$ .

In all cases, then, (5.65) follows from (5.67).  $\square$

## 6. VARIATIONS OF THE MODEL: PROOFS OF PROPOSITION 2.3 AND PROPOSITION 2.4

We start with the proof of Proposition 2.4, since we adapt it for the proof of Proposition 2.3, in which we need to control additionally  $|\tau|_{\sigma_N}$ .

**6.1. Proof of Proposition 2.4(i).** We fix  $\beta > \beta_c$ , and find a lower bound on  $\bar{Z}_{N,\beta}^\sigma = \mathbf{E}_\tau[\exp(\beta|\tau \cap \sigma|_{\tau_N})]$  by restricting  $\tau$  to follow a particular strategy.

As in Section 3, we divide the system into blocks  $\mathcal{B}_i := \{0, \sigma_{(i-1)L+1} - \sigma_{(i-1)L}, \dots, \sigma_{iL} - \sigma_{(i-1)L}\}$ , with  $L$  to be specified. For  $b_N$  from (2.8), there exists  $v_0 > 0$  such that  $\mathbf{P}_\sigma(\sigma_N > v_0 b_N) \leq 1/4$  for  $N$  large. Define the event of being *good* by

$$(6.1) \quad G_L := \left\{ (\sigma_1, \dots, \sigma_L) : \frac{1}{L} \log Z_{L,\beta}^\sigma \geq \frac{1}{2} \mathcal{F}(\beta) \quad \text{and} \quad \sigma_L \leq v_0 b_L \right\}.$$

Since  $\beta > \beta_c$ , there exists  $L_0$  such that, for  $L \geq L_0$

$$(6.2) \quad \mathbf{P}_\sigma \left( \frac{1}{L} \log Z_{L,\beta}^\sigma \geq \frac{1}{2} \mathcal{F}(\beta) \right) \geq \frac{3}{4},$$

so that

$$(6.3) \quad \mathbf{P}_\sigma(G_L) \geq \frac{1}{2}.$$

We now set  $\mathcal{I} = \mathcal{I}(\sigma) = \{i : \mathcal{B}_i \in G_L\} = \{i_1, i_2, \dots\}$ , the set of indices of the *good* blocks, and set  $i_0 = 0$ . There can be at most  $v_0 b_L$   $\tau$ -renewals per block, so restricting trajectories to visit only blocks with index in  $\mathcal{I}$ , we get that for all  $m \in \mathbb{N}$

$$(6.4) \quad \bar{Z}_{mv_0 b_L, \beta}^\sigma \geq \prod_{k=1}^m \mathbf{P}_\tau(\tau_1 = \sigma_{(i_k-1)L} - \sigma_{i_{k-1}L}) e^{\mathcal{F}(\beta)L/2},$$

with the convention that  $\mathbf{P}_\tau(\tau_1 = 0)$  means 1. Then, letting  $m \rightarrow \infty$ , we get

$$(6.5) \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \log \bar{Z}_{N,\beta}^\sigma \geq \frac{1}{v_0 b_L} \frac{1}{2} \mathcal{F}(\beta) L + \frac{1}{v_0 b_L} \mathbf{E}_\sigma[\log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_1-1)L})] \quad \mathbf{P}_\sigma - a.s.$$

Let us estimate the last term. Thanks to (1.5), we have that

$$\log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_1-1)L}) \geq -(2 + \alpha) \mathbf{1}_{\{i_1 > 1\}} \log \sigma_{(i_1-1)L},$$

provided  $L$  is large. Then, Lemma 3.2 applies: for  $L$  large, since  $\mathbf{P}_\sigma(G_L) \geq 1/2$  we have that

$$\mathbf{E}_\sigma[\mathbf{1}_{\{i_1 > 1\}} \log \sigma_{(i_1-1)L}] \leq \frac{2}{\tilde{\alpha} \wedge 1} \left( \log L + \log \frac{1}{\mathbf{P}_\sigma(G_L)} \right) \leq \frac{4}{\tilde{\alpha} \wedge 1} \log L.$$

Hence for some  $L_0$ , for  $L \geq L_0$ ,

$$(6.6) \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \log \bar{Z}_{N,\beta}^\sigma \geq \frac{1}{v_0 b_L} \left( \frac{1}{2} \mathcal{F}(\beta) L - \frac{4(2 + \alpha)}{\tilde{\alpha} \wedge 1} \log L \right).$$

Hence, provided that  $L$  is large enough, we have that  $\bar{\mathcal{F}}(\beta) > 0$  for any  $\beta > \beta_c$ , meaning  $\beta_c \geq \bar{\beta}_c$ .

**6.2. Proof of Proposition 2.4(ii).** Observe that the annealed systems for the original and elastic polymers have the same critical point, by Remark 1.2. Therefore  $\beta_c^{\text{ann}} \leq \bar{\beta}_c$ . When  $\alpha = 0, \tilde{\alpha} \geq 1$ , it then follows from Theorem 2.1 that  $\beta_c \leq \bar{\beta}_c$ , so equality holds.

Hence it remains to prove  $\beta_c \leq \bar{\beta}_c$  assuming  $\alpha > 0$ , by showing that pinning in the elastic polymer (length- $\tau_N$  system) implies pinning in the original polymer (length- $\sigma_N$  system.)

In the recurrent case,  $\beta_c = 0$  so there is nothing to prove. So we assume transience, which here implies  $\alpha, \tilde{\alpha} \in (0, 1)$ . Let  $v_N = e^{2\tilde{\alpha}\beta N/\alpha}$  and  $\tau_{[0,N]} = \{\tau_1, \dots, \tau_N\}$ . Define

$$X_{N,j} = X_{N,j}(\sigma) := \mathbf{E}_\tau \left( e^{\beta|\sigma \cap \tau_{[0,N]}|_{\sigma_j}} \mathbf{1}_{\{\sigma_j \in \tau_{[0,N]}\}} \right),$$

and recall

$$(6.7) \quad \begin{aligned} \bar{Z}_{N,\beta}^\sigma &= \mathbf{E}_\tau \left[ \exp(\beta |\tau \cap \sigma|_{\tau_N}) \right] \\ &= \sum_{j=1}^{\infty} X_{N,j}(\sigma) \mathbf{P}_\tau \left( \tau_{[0,N]} \cap \{\sigma_{j+1}, \sigma_{j+2}, \dots\} = \phi \mid \sigma_j \in \tau_{[0,N]} \right). \end{aligned}$$

Let  $\bar{\mathcal{F}}(\beta) = \liminf_{N \rightarrow \infty} \frac{1}{N} \log \bar{Z}_{N,\beta}^\sigma$  (which is a tail random variable of  $\{\sigma_i - \sigma_{i-1}, i \geq 1\}$ , so is nonrandom, up to a null set) and assume that  $\bar{\mathcal{F}}(\beta) > 0$ . We define the truncated sums

$$(6.8) \quad \begin{aligned} \bar{Z}_{N,\beta}^{\sigma,T(v)} &= \sum_{j=1}^v X_{N,j}(\sigma) \mathbf{P}_\tau \left( \tau_{[0,N]} \cap \{\sigma_{j+1}, \sigma_{j+2}, \dots\} = \phi \mid \sigma_j \in \tau_{[0,N]} \right), \\ \bar{Z}_{N,\beta,0}^{\sigma,T(v)} &= \sum_{j=1}^v X_{N,j}(\sigma). \end{aligned}$$

Then it is not hard to show that  $\mathbf{P}_\sigma$ -a.s., for large  $N$ ,

$$\bar{Z}_{N,\beta}^\sigma - \bar{Z}_{N,\beta}^{\sigma,T(v_N-1)} \leq e^{\beta N} \mathbf{P}_\tau(\tau_N > \sigma_{v_N-1}) \leq 1,$$

so

$$(6.9) \quad \bar{\mathcal{F}}(\beta) = \liminf_{N \rightarrow \infty} \frac{1}{N} \log \bar{Z}_{N,\beta}^{\sigma,T(v_N-1)} \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \bar{Z}_{N,\beta,0}^{\sigma,T(v_N-1)}.$$

Below, we show the following

**Lemma 6.1.** *For every  $\beta > \bar{\beta}_c$ , there exists some  $N$  large enough, so that*

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log \bar{Z}_{mN,\beta,0}^{\sigma,T(mv_N)} \geq \frac{1}{12} N \bar{\mathcal{F}}(\beta).$$

It is essential here that the truncation in the partition function be at  $mv_N$ , not at the much larger value  $v_{mN}$ , as we want the allowed length of trajectories to grow essentially only linearly in  $m$ . But we need to know that with this length restriction, the log of the partition function is still of order  $mN$ .

With this lemma in hand, we easily have that

$$\bar{Z}_{mN,\beta,0}^{\sigma,T(mv_N)} \leq \sum_{j=1}^{m(v_N+1)} \mathbf{E}_\tau \left[ e^{\beta |\sigma \cap \tau|_{\sigma_j}} \right] \leq mv_N Z_{mv_N,\beta}^{\sigma,\text{free}}.$$

Then thanks to Remark 1.2 (and because  $\frac{1}{m} \log(mv_N) \rightarrow 0$ ), we have

$$\begin{aligned} \mathcal{F}(\beta) &= \lim_{m \rightarrow \infty} \frac{1}{mv_N} \log Z_{mv_N,\beta}^{\sigma,\text{free}} \\ &\geq \liminf_{m \rightarrow +\infty} \frac{1}{mv_N} \log \bar{Z}_{mN,\beta,0}^{\sigma,T(mv_N)} \geq \frac{N}{12v_N} \bar{\mathcal{F}}(\beta), \end{aligned}$$

which gives that  $\mathcal{F}(\beta) > 0$  for any  $\beta > \bar{\beta}_c$ , that is  $\beta_c \leq \bar{\beta}_c$ . This concludes the proof of Proposition 2.4(i).

*Proof of Lemma 6.1.* Let  $U = (U_i)_{i \geq 1}$  be an i.i.d. sequence of variables independent of  $\sigma$ , and let  $(\mathcal{G}_n)_{n \geq 1}$  be the  $\sigma$ -field generated by  $\{\sigma_1, U_1, \dots, \sigma_n, U_n\}$ . We will define below a stopping time  $T_1 = T_1(\sigma, U)$  for the filtration  $(\mathcal{G}_n)_{n \geq 1}$  satisfying  $T_1 \leq v_N$ , and denote  $\hat{Q}_{N,j}(\sigma) := \mathbf{P}_T(T_1 = j \mid \sigma)$ . The stopping time property means that  $\hat{Q}_{N,j}(\sigma)$  depends only on  $\sigma_1, \dots, \sigma_j$ .

Write  $\sigma^{(j)}$  for the shifted sequence  $(\sigma_k - \sigma_j, U_k), k \geq j$ . We then define iteratively the stopping times  $(J_i)_{i \geq 1}$  by

$$J_i = J_1(\sigma^{(J_{i-1})}).$$

For any fixed  $j_1 < \dots < j_m$  with  $|j_i - j_{i-1}| \leq v_N$  (representing possible values of the  $J_i$ ), we can decompose a product of  $X$  variables as follows:

$$\prod_{i=1}^m X_{N, j_i - j_{i-1}}(\sigma^{(j_{i-1})}) = \sum_{\substack{0 = \ell_0 < \ell_1 < \dots < \ell_m \\ |\ell_i - \ell_{i-1}| \leq v_N \quad \forall i \leq m}} \mathbf{E} \left[ e^{\beta |\sigma \cap \tau_{[0, mN]}|} \prod_{i=1}^m \mathbf{1}_{\{\tau_{\ell_i} = \sigma_{j_i}\}} \right].$$

Hence, summing over all such  $j_1 < \dots < j_m$ , we get the bound

$$\sum_{\substack{0 = j_0 < j_1 < \dots < j_m \leq mv_N \\ |j_i - j_{i-1}| \leq v_N}} \prod_{i=1}^m X_{N, j_i - j_{i-1}}(\sigma^{(j_{i-1})}) \leq \sum_{\substack{0 = \ell_0 < \ell_1 < \dots < \ell_m \\ |\ell_i - \ell_{i-1}| \leq v_N \quad \forall i \leq m}} \bar{Z}_{mN, \beta, 0}^{\sigma, T(mv_N)} \leq N^m \bar{Z}_{mN, \beta, 0}^{\sigma, T(mv_N)}.$$

which can be rewritten

$$\begin{aligned} \bar{Z}_{mN, \beta, 0}^{\sigma, T(mv_N)} &\geq N^{-m} \sum_{\substack{0 = j_0 < j_1 < \dots < j_m \\ |j_i - j_{i-1}| \leq v_N}} \prod_{i=1}^m \hat{Q}_{N, j_i - j_{i-1}}(\sigma^{(j_{i-1})}) \exp \left( \sum_{i=1}^m \log \frac{X_{N, j_i - j_{i-1}}(\sigma^{(j_{i-1})})}{\hat{Q}_{N, j_i - j_{i-1}}(\sigma^{(j_{i-1})})} \right) \\ (6.10) \quad &= N^{-m} \mathbf{E}_J \left[ \exp \left( \sum_{i=1}^m \log \frac{X_{N, J_i - J_{i-1}}(\sigma^{(J_{i-1})})}{\hat{Q}_{N, J_i - J_{i-1}}(\sigma^{(J_{i-1})})} \right) \mid \sigma \right]. \end{aligned}$$

Here in a mild abuse of notation we write  $\mathbf{E}_J[\cdot \mid \sigma]$  for the expectation over  $U$ , and we will write  $\mathbf{E}_{\sigma, J}$  for the expectation over  $(\sigma, U)$ . Since  $J_1$  is a stopping time, the summands on the right side of (6.10) are i.i.d. functions of  $(\sigma, U)$ . By (6.10) and Jensen's inequality, we have

$$(6.11) \quad \log \bar{Z}_{mN, \beta, 0}^{\sigma, T(mv_N)} \geq -m \log N + \sum_{i=1}^m \mathbf{E}_J \left[ \log \frac{X_{N, J_i - J_{i-1}}(\sigma^{(J_{i-1})})}{\hat{Q}_{N, J_i - J_{i-1}}(\sigma^{(J_{i-1})})} \mid \sigma \right].$$

Since  $X_{N, J_1}(\sigma) \geq \mathbf{P}_\tau(\tau_1 = \sigma_{J_1})$  and  $J_1 \leq v_N$ , we have for some  $c_i$

$$(6.12) \quad \log \frac{X_{N, J_1}(\sigma)}{\hat{Q}_{N, J_1}(\sigma)} \geq \log \mathbf{P}_\tau(\tau_1 = \sigma_{J_1}) \geq -c_{36} \log \sigma_{v_N}$$

and hence, writing  $x_-$  for the negative part of  $x$  and using (3.25),

$$(6.13) \quad \mathbf{E}_{\sigma, J} \left[ \left( \log \frac{X_{N, J_1}(\sigma)}{\hat{Q}_{N, J_1}(\sigma)} \right)_- \right] \leq c_{36} \mathbf{E}_\sigma[\log \sigma_{v_N}] \leq c_{37} v_N < \infty.$$

It then follows from (6.10) that

$$(6.14) \quad \liminf_{m \rightarrow \infty} \frac{1}{m} \log \bar{Z}_{mN, \beta, 0}^{\sigma, T(mv_N)} \geq \mathbf{E}_{\sigma, J} \left[ \log \frac{X_{N, J_1}(\sigma)}{\hat{Q}_{N, J_1}(\sigma)} \right] - \log N, \quad \mathbf{P}_{\sigma, J} - \text{a.s.}$$

We will show that there is a choice of  $N$ ,  $J_1$ , and  $\hat{Q}_{N,J_1}(\sigma)$  satisfying

$$(6.15) \quad \mathbf{E}_{\sigma,J} \left[ \log \frac{X_{N,J_1}(\sigma)}{\hat{Q}_{N,J_1}(\sigma)} \right] \geq \frac{1}{6} \bar{\mathcal{F}}(\beta) N, \quad \log N \leq \frac{1}{12} \bar{\mathcal{F}}(\beta) N.$$

The lemma will then follow from (6.14) and (6.15).

Fix  $K$  to be specified and define

$$R_N = R_N(\sigma) := \min \left\{ v : \bar{Z}_{N,\beta,0}^{\sigma,T(v)} = \sum_{j=1}^v X_{N,j}(\sigma) \geq K e^{N\bar{\mathcal{F}}(\beta)/2} \right\} \wedge v_N.$$

From (6.9), we then have

$$(6.16) \quad \mathbf{P}_\sigma(R_N = v_N) = \mathbf{P}_\sigma \left( \bar{Z}_{N,\beta,0}^{\sigma,T(v_N-1)} < K e^{N\bar{\mathcal{F}}(\beta)/2} \right) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

We define the marking probabilities

$$Q_{N,j} = Q_{N,j}(\sigma) := 1 - \exp \left( -X_{N,j} e^{-N\bar{\mathcal{F}}(\beta)/2} \right),$$

which only depends on  $\sigma_1, \dots, \sigma_j$ , and

$$\hat{Q}_{N,j}(\sigma) = \begin{cases} \prod_{k=1}^{j-1} (1 - Q_{N,k}) \times Q_{N,j} & \text{if } 1 \leq j < R_N(\sigma), \\ \prod_{k=1}^{R_N} (1 - Q_{N,k}) & \text{if } j = R_N, \\ 0 & \text{if } j > R_N, \end{cases}$$

so that  $\sum_{j=1}^{v_N} \hat{Q}_{N,j}(\sigma) = 1$ . Then set

$$J_1 = \min \{ j \geq 1 : U_j \leq Q_{N,j} \} \wedge R_N.$$

We may view this as follows: for each  $j < R_N$  we mark  $\sigma_j$  with probability  $Q_{N,j}$ , independently, and we mark  $\sigma_{R_N}$  with probability 1;  $\sigma_{J_1}$  is the first  $\sigma_j$  to be marked. As a result we have  $\mathbf{P}_J(J_1 = j \mid \sigma) = \hat{Q}_{N,j}(\sigma)$ , and  $J_1 \leq v_N$ . Note that this weights  $J_1$  toward values  $j$  for which  $X_{N,j}$  is large, which, heuristically, occurs when  $\sigma_j$  follows a favorable stretch of the disorder  $\sigma$ .

We now consider (6.15), and write

$$(6.17) \quad \mathbf{E}_J \left[ \log \frac{X_{N,J_1}}{\hat{Q}_{N,J_1}} \mid \sigma \right] = \sum_{j=1}^{R_N-1} \hat{Q}_{N,j} \log \frac{X_{N,j}}{\hat{Q}_{N,J_1}} + \hat{Q}_{N,R_N} \log \frac{X_{N,R_N}}{\hat{Q}_{N,R_N}}.$$

For the sum in (6.17), using  $1 - e^{-x} \leq x$  we get  $X_{N,j} \geq e^{N\bar{\mathcal{F}}(\beta)/2} Q_{N,j} \geq e^{N\bar{\mathcal{F}}(\beta)/2} \hat{Q}_{N,j}$  for all  $j < R_N$ , and therefore

$$\sum_{j=1}^{R_N-1} \hat{Q}_{N,j} \log \frac{X_{N,j}}{\hat{Q}_{N,J_1}} \geq \frac{1}{2} \mathbf{P}_J(J_1 < R_N \mid \sigma) \bar{\mathcal{F}}(\beta) N.$$

For the last term in (6.17), from (6.12) we have

$$\log X_{N,R_N} \geq \log \mathbf{P}(\tau_1 = \sigma_{R_N}) \geq -c_{36} \log \sigma_{v_N}$$

and therefore

$$\hat{Q}_{N,R_N} \log \frac{X_{N,R_N}}{\hat{Q}_{N,R_N}} \geq -c_{36} \hat{Q}_{N,R_N} \log \sigma_{v_N}.$$

Combining these bounds and averaging over  $\sigma$ , we get from (6.17) that

$$(6.18) \quad \mathbf{E}_{\sigma,J} \left[ \log \frac{X_{N,J_1}}{Q_{N,J_1}} \right] \geq \frac{1}{2} \mathbf{P}_{\sigma,J}(J_1 < R_N) \bar{\mathcal{F}}(\beta) N - c_{36} \mathbf{E}_{\sigma,J} [\mathbf{P}_J(J_1 = R_N | \sigma) \log \sigma_{v_N}].$$

For the first term on the right side of (6.18), when  $R_N(\sigma) < v_N$  we have from the definition of  $R_N$  that

$$(6.19) \quad \mathbf{P}_J(J_1 < R_N | \sigma) = 1 - \prod_{k=1}^{R_N} (1 - Q_{N,k}) = 1 - \exp \left( - e^{-\bar{\mathcal{F}}(\beta)N/2} \sum_{j=1}^{R_N} X_{N,j} \right) \geq 1 - e^{-K}.$$

We therefore get using (6.16) that

$$(6.20) \quad \mathbf{P}_{\sigma,J}(J_1 < R_N) \geq \mathbf{P}_{\sigma}(R_N < v_N)(1 - e^{-K}) \geq \frac{2}{3},$$

provided that  $K$  and then  $N$  are chosen large enough.

For the last term in (6.18), we have from (6.19) and (6.13) that

$$(6.21) \quad \mathbf{E}_{\sigma,J} [\mathbf{1}_{\{R_N < v_N\}} \mathbf{P}_J(J_1 = R_N | \sigma) \log \sigma_{v_N}] \leq e^{-K} \mathbf{E}_{\sigma} [\log \sigma_{v_N}] \leq c_{38} e^{-K} \log v_N.$$

In addition, it is routine that there exists a constant  $c_{39}$  such that

$$\mathbf{E}_{\sigma} \left[ \left( \frac{\log \sigma_n}{\log n} \right)^2 \right] \leq c_{39} \quad \text{for all } n \geq 2.$$

Hence using (6.16) and the Cauchy-Schwarz inequality

$$(6.22) \quad \begin{aligned} \mathbf{E}_{\sigma} [\mathbf{1}_{\{R_N = v_N\}} \log \sigma_{v_N}] &\leq (\log v_N) \mathbf{P}_{\sigma}(R_N = v_N)^{1/2} \mathbf{E}_{\sigma} \left[ \left( \frac{\log \sigma_{v_N}}{\log v_N} \right)^2 \right] \\ &= o(\log v_N) \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Combining (6.21) and (6.22) we get that if we take  $K$  and then  $N$  is large enough,

$$(6.23) \quad c_{36} \mathbf{E}_{\sigma,J} [\mathbf{P}_J(J_1 = R_N | \sigma) \log \sigma_{v_N}] \leq 2c_{36}c_{38}e^{-K} \log v_N \leq \frac{1}{6} \bar{\mathcal{F}}(\beta) N.$$

Plugging (6.20) and (6.23) into (6.18), we finally get (6.15). □

**6.3. Proof of Proposition 2.3.** As noted in Section 2.2, we only need prove that  $\hat{\beta}_c \leq \beta_c$ , so let us fix  $\beta > \beta_c$ , and show that  $\hat{\mathcal{F}}(\beta) > 0$ . We write  $Z_{N,\beta}^{\sigma,f}(H)$  for  $\mathbf{E}_{\tau} [e^{\beta|\tau \cap \sigma|} \mathbf{1}_H]$ , for an event  $H$ . (Note this partition function may involve trajectories not tied down, that is, with  $\sigma_N \notin \tau$ .) A first observation is that for fixed  $q \geq 1$ ,

$$(6.24) \quad \log \hat{Z}_{N+q,\beta}^{\sigma} \geq Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) \min_{d_{\sigma m} \leq k \leq \sigma_{N+q}} \mathbf{P}_{\tau}(\tau_q = k).$$

By (1.7), if we fix  $q$  large enough and then take  $N$  large, the minimum here is achieved by  $k \geq \sigma_N/2$ , so by (A.3), it is at least  $\frac{1}{2}q \mathbf{P}_{\tau}(\tau_1 = \sigma_{N+q}) \geq \sigma_{N+q}^{-(2+\alpha)}$ . Using (3.25) we therefore

get for large  $N$

$$(6.25) \quad \begin{aligned} \mathbf{E}_\sigma \log \hat{Z}_{N+q,\beta}^\sigma &\geq \mathbf{E}_\sigma \log Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) - (2 + \alpha) \mathbf{E}_\sigma [\log \sigma_{N+m}] \\ &\geq \mathbf{E}_\sigma \log Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) - \frac{2(2 + \alpha)}{\tilde{\alpha}} \log N. \end{aligned}$$

Since  $Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) \geq \hat{Z}_{N,\beta}^\sigma$ , (6.25) proves that

$$\hat{\mathcal{F}}(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \hat{Z}_{N,\beta}^\sigma = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\sigma \log Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) \quad \mathbf{P}_\sigma - a.s.,$$

and we therefore work with  $Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N)$  instead of  $\hat{Z}_{N,\beta}^\sigma$ .

We continue notations from Section 6.1: we take  $v_0$  such that  $\mathbf{P}_\sigma(\sigma_n \geq v_0 b_n) \leq 1/4$ , use  $G_L$  from (6.1), take  $L \geq L_0$  large so that (6.3) holds, and consider the set of good blocks  $\mathcal{I} = \{i : \mathcal{B}_i \in G_L\}$ . The idea of the proof is similar to that of Proposition 2.4, but in addition, we need to control the size of  $\tau_L$  on good blocks.

**Case 1:**  $\mathbf{E}_\sigma[\sigma_1] = +\infty$ . Here  $b_n = \tilde{\psi}(n)n^{1/\tilde{\alpha}}$ , see (2.9). We need the following technical lemma, whose proof is postponed to Section 6.4.

**Lemma 6.2.** *Assume (1.5) and (1.7) with  $\tilde{\alpha} \neq 0$ . If  $\mathbf{E}[\sigma_1] = +\infty$ , then, uniformly for  $x \geq 1/10$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{xn} \geq d_\tau n\}} \log \mathbf{P}_\tau(\tau_n \leq \sigma_{xn})] = 0.$$

We consider  $m$  blocks of length  $L$ , so  $N = mL$ , and define the events

$$E_1 := \{\sigma : |\mathcal{I} \cap [1, \frac{7}{8}m]| \geq \frac{1}{4}m\}, \quad E_2 := \{\sigma : \sigma_{mL} - \sigma_{7mL/8} \geq d_\tau mL\},$$

satisfying  $\mathbf{P}_\sigma(E_1) \rightarrow 1$  as  $m \rightarrow \infty$ , by (6.3). Using that  $\log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL}) \geq \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})$ , we get

$$(6.26) \quad \mathbf{E}_\sigma [\log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL})] \geq \mathbf{E}_\sigma [\mathbf{1}_{E_1 \cap E_2} \log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL})] + \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})].$$

According to Lemma 6.2, and because of (1.7), we have

$$\lim_{m \rightarrow \infty} \frac{1}{mL} \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})] = 0,$$

so we consider the first term on the right in (6.26).

On the event  $E_1 \cap E_2$ , as in the proof of Proposition 2.4 we restrict to  $\tau$  visiting only good blocks  $\mathcal{B}_i$ , including visits to the endpoints  $\sigma_{(i-1)L}, \sigma_{iL}$ . Since (by definition of  $G_L$ ) on each good block  $\mathcal{B}_i$  there are at most  $v_0 b_L$   $\tau$ -renewals, up to  $\tau_m$  such  $\tau$  visit at least  $\ell_m := \lfloor m/(v_0 b_L) \rfloor$  good blocks. We also choose  $L$  large so that  $\ell_m \leq \frac{1}{8}m$ : on the event  $E_1$ , it ensures that  $i_{\ell_m} \leq 7m/8$ .

We denote  $k_0 \leq mL$  the index such that  $\tau_{k_0} = \sigma_{i_{\ell_m}L}$ : since  $\tau_{mL} - \tau_{k_0} \leq \tau_{mL}$ , we have on  $E_1$ , restricting to  $\tau$  visiting the first  $\ell_m$  good blocks,

$$\begin{aligned} &\log Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N) \\ &\geq \sum_{k=1}^{\ell_m} \log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_k-1)L} - \sigma_{i_{k-1}L}) + \ell_m \frac{1}{2} \mathcal{F}(\beta)L + \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL} - \sigma_{i_{\ell_m}L}). \end{aligned}$$

On the event  $E_1 \cap E_2$  we have  $\sigma_{mL} - \sigma_{i_{\ell_m}L} \geq \sigma_{mL} - \sigma_{7mL/8} \geq d_\tau mL$ , so we then get

$$\begin{aligned}
& \mathbf{E}_\sigma [\mathbf{1}_{E_1 \cap E_2} \log Z_{N,\beta}^{\sigma,f}(\tau_N \leq \sigma_N)] \\
& \geq \sum_{k=1}^{\ell_m} \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_k-1)L} - \sigma_{i_k-1L})] \\
& \quad + \frac{1}{2} \ell_m \mathcal{F}(\beta) L \mathbf{P}_\sigma(E_1 \cap E_2) + \mathbf{E}_\sigma [\mathbf{1}_{E_2} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL} - \sigma_{i_{\ell_m}L})] \\
(6.27) \quad & \geq \ell_m \left( \frac{1}{4} \mathcal{F}(\beta) L + \mathbf{E}_\sigma \log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_1-1)L}) \right) + \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{mL/8} \geq d_\tau mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL/8})],
\end{aligned}$$

where we used that for  $L$  large enough,  $\mathbf{P}_\sigma(E_1 \cap E_2) \geq 1/2$ . Therefore dividing by  $mL$  and letting  $m \rightarrow \infty$  gives

$$(6.28) \quad L\hat{\mathcal{F}}(\beta) \geq \frac{1}{4v_0b_L} \mathcal{F}(\beta)L + \frac{1}{v_0b_L} \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_1 = \sigma_{(i_1-1)L})],$$

where we used Lemma 6.2 to get that

$$\lim_{m \rightarrow \infty} \frac{1}{mL} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{mL/8} \geq d_\tau mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL/8})] = 0.$$

Then we finish as in the proof of Proposition 2.4: there exists some  $L_0$  such that, for  $L \geq L_0$ , one has as in (6.6)

$$(6.29) \quad L\hat{\mathcal{F}}(\beta) \geq \frac{1}{v_0b_L} \left( \frac{1}{4} L\mathcal{F}(\beta) - \frac{4(2+\alpha)}{\tilde{\alpha} \wedge 1} \log L \right),$$

and then taking  $L$  sufficiently large shows  $\hat{\mathcal{F}}(\beta) > 0$ .

**Case 2.** We now deal with the case when  $\mu_\sigma := \mathbf{E}_\sigma[\sigma_1] < +\infty$  and  $\mathbf{E}_\tau[\tau_1] \leq \mathbf{E}_\sigma[\sigma_1]$ . Here  $b_n = \mu_\sigma n$ , see (2.8). Let us fix  $m \in \mathbb{N}$  large, and consider a system consisting in  $m$  blocks of length  $L$ . Decomposing according to whether the first block is good or not we have, recalling  $G_L$  from (6.1),

$$(6.30) \quad \mathbf{E}_\sigma [\log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL})] \geq \mathbf{E}_\sigma [\mathbf{1}_{\{\mathcal{B}_1 \in G_L\}} \mathbf{1}_{\{\sigma_{mL} - \sigma_L \geq d_\tau mL\}} \log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL})] + \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})].$$

Recalling the indicator  $\mathbf{1}_{\{\sigma_L \in \tau\}}$  in the definition of  $Z_{L,\beta}^\sigma$ , if  $\tau_{k_0} = \sigma_L$  for some  $k_0$ , then the relation  $\tau_{mL} - \tau_{k_0} \leq \tau_{mL}$  guarantees that

$$(6.31) \quad Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL}) \geq Z_{L,\beta}^\sigma \times \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL} - \sigma_L).$$

Provided  $m$  exceeds some  $m_0$  we have  $d_\tau m / (m-1) < \mathbf{E}[\tau_1] \leq \mu_\sigma$ , and therefore for sufficiently large  $L$  we have  $\mathbf{P}_\sigma(\sigma_{(m-1)L} \geq d_\tau mL) \geq 1/2$ . Since  $\mathcal{B}_1$  is independent of all other blocks and  $\mathbf{P}_\sigma(G_L) \geq 1/2$ , it then follows from (6.31) that for  $\sigma \in G_L$ ,

$$(6.32) \quad \mathbf{E}_\sigma [\mathbf{1}_{\{\mathcal{B}_1 \in G_L\}} \mathbf{1}_{\{\sigma_{mL} - \sigma_L \geq d_\tau mL\}} \log Z_{mL,\beta}^{\sigma,f}(\tau_{mL} \leq \sigma_{mL})] \geq \frac{1}{8} \mathcal{F}(\beta)L + \frac{1}{2} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{(m-1)L} \geq d_\tau mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{(m-1)L})].$$

It remains only to control  $\mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{(m-1)L} \geq d_\tau mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{(m-1)L})]$ . To that end we have the following lemma which we prove in Section 6.4.



**Lemma 6.3.** *If  $\mathbf{E}[\tau_1] \leq \mathbf{E}[\sigma_1] < +\infty$ , then, for any  $\varepsilon > 0$ , for sufficiently large  $m$  we have*

$$(6.33) \quad \lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{(m-1)L} \geq d_\tau m L\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{(m-1)L})] \geq -\varepsilon,$$

$$\text{and} \quad \lim_{L \rightarrow \infty} \frac{1}{L} \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})] \geq -\varepsilon.$$

We choose  $\varepsilon = \frac{1}{40} \mathcal{F}(\beta)$ , and fix  $m_1 \geq m_0$  such that one can apply Lemma 6.3. Then, we take  $L$  large enough such that  $\frac{1}{L} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{(m_1-1)L} \geq d_\tau m_1 L\}} \log \mathbf{P}_\tau(\tau_{m_1 L} \leq \sigma_{(m_1-1)L})] \geq -\frac{1}{32} \mathcal{F}(\beta)$  and also  $\frac{1}{L} \mathbf{E}_\sigma [\log \mathbf{P}_\tau(\tau_{m_1 L} \leq \sigma_{m_1 L})] \geq -\frac{1}{32} \mathcal{F}(\beta)$ . Combining this with (6.30) and (6.32), we obtain for sufficiently large  $L$

$$(6.34) \quad \mathbf{E}_\sigma [\log Z_{m_1 L, \beta}^{\sigma, f}(\tau_{m_1 L} \leq \sigma_{m_1 L})] \geq \frac{1}{16} \mathcal{F}(\beta) L.$$

To conclude the proof of Proposition 2.3, we use that  $(\log \hat{Z}_{N, \beta}^\sigma)_{N \in \mathbb{N}}$  is an ergodic super-additive sequence, so that  $\hat{\mathcal{F}}(\beta) \geq \sup_{N \in \mathbb{N}} \frac{1}{N} \mathbf{E}_\sigma [\log \hat{Z}_{N, \beta}^\sigma]$ . In view of (6.25), we therefore have that, if  $L$  is large enough, for  $q$  as specified after (6.24),

$$(6.35) \quad \hat{\mathcal{F}}(\beta) \geq \frac{1}{m_1 L + q} \mathbf{E}_\sigma [\log Z_{m_1 L, \beta}^{\sigma, f}(\tau_{m_1 L} \leq \sigma_{m_1 L})] - \frac{3(2 + \alpha) \log(m_1 L)}{\tilde{\alpha} (m_1 L + q)}.$$

Then we can use (6.34) to obtain, by taking  $L$  large enough,

$$(6.36) \quad \hat{\mathcal{F}}(\beta) \geq \frac{1}{m_1 L + q} \frac{1}{20} \mathcal{F}(\beta) L \geq \frac{1}{30 m_1} \mathcal{F}(\beta) > 0.$$

#### 6.4. Proof of Lemmas 6.2-6.3.

*Proof of Lemma 6.2.* We have the following crude bound: there exists a constant  $c_{40} > 0$  such that, for every  $k \geq d_\tau n$ ,

$$(6.37) \quad \mathbf{P}_\tau(\tau_n \leq k) \geq \mathbf{P}_\tau\left(\max_{1 \leq i \leq n} \{\tau_i - \tau_{i-1}\} \leq k/n\right) = \left(1 - \mathbf{P}_\tau(\tau_1 > \frac{k}{n})\right)^n \geq e^{-c_{40} \mathbf{P}_\tau(\tau_1 > \frac{k}{n}) n},$$

where the last inequality follows from the fact that  $\mathbf{P}_\tau(\tau_1 > \frac{k}{n})$  is bounded away from 1, for all  $k \geq d_\tau n$ .

Since  $\mathbf{E}_\sigma[\sigma_1] = +\infty$ , we may choose a sequence  $\alpha_n$  with  $\alpha_n/n \rightarrow +\infty$ , and uniformly in  $x \geq 1/10$ ,  $\mathbf{P}_\sigma(\sigma_{xn} \leq \alpha_n) \xrightarrow{n \rightarrow \infty} 0$ . We get

$$(6.38) \quad \begin{aligned} 0 &\geq \frac{1}{n} \mathbf{E}_\sigma [\mathbf{1}_{\{\sigma_{xn} \geq d_\tau n\}} \log \mathbf{P}_\tau(\tau_n \leq \sigma_{xn})] \\ &\geq \frac{1}{n} \mathbf{P}_\sigma(\sigma_{xn} > \alpha_n) \log \mathbf{P}_\tau(\tau_n \leq \alpha_n) + \frac{1}{n} \mathbf{P}_\sigma(\sigma_{xn} \leq \alpha_n) \log \mathbf{P}_\tau(\tau_1 = d_\tau)^n \\ &\geq -c_{41} \mathbf{P}(\tau_1 > \alpha_n/n) + \mathbf{P}_\sigma(\sigma_{xn} \leq \alpha_n) \log \mathbf{P}_\tau(\tau_1 = d_\tau), \end{aligned}$$

where we used (6.37) in the second inequality. Letting  $n \rightarrow \infty$ , we see that the limit is 0 thanks to our choice of  $\alpha_n$ .  $\square$

*Proof of Lemma 6.3.* From the standard large deviation principle for i.i.d. sums, we can define the rate function

$$(6.39) \quad \bar{J}_\tau(t) := J_\tau(\mu_\tau - t) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbf{P}_\tau(\tau_n \leq \mu_\tau n - tn),$$

with  $X := \mu_\tau - \tau_1$ , and  $J_\tau$  is defined after (3.1). We define  $\bar{J}_\sigma$  analogously.

It is standard that, since  $E[X] = 0$ , we have  $\bar{J}_\tau(t) = o(t)$  as  $t \searrow 0$ . Therefore for fixed  $\varepsilon > 0$ , for large  $m$  we have  $\bar{J}_\tau(2\mu_\tau/m) \leq \varepsilon/m$ . We also have, using that  $\mu_\tau \leq \mu_\sigma$ , that for any  $m \geq 2$

$$\mathbf{P}_\sigma(\sigma_{(m-1)L} \leq (m-2)L\mu_\tau) \rightarrow 0 \quad \text{as } L \rightarrow \infty.$$

Hence, observing that  $(m-2)\mu_\tau \geq d_\tau m$  provided that  $m$  has been fixed large enough, we can get

$$\begin{aligned} & \mathbf{E}_\sigma[\mathbf{1}_{\{\sigma_{(m-1)L} \geq d_\tau mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{(m-1)L})] \\ & \geq \log \mathbf{P}_\tau(\tau_{mL} \leq (m-2)L\mu_\tau) + \mathbf{P}_\sigma(\sigma_{(m-1)L} \leq (m-2)L\mu_\tau) \times mL \log \mathbf{P}_\tau(\tau_1 = d_\tau). \end{aligned}$$

Hence we obtain, using (6.39)

$$(6.40) \quad \liminf_{n \rightarrow \infty} \frac{1}{L} \mathbf{E}_\sigma[\mathbf{1}_{\{\sigma_{(m-1)L} \geq mL\}} \log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{(m-1)L})] \geq -m \bar{J}_\tau\left(\frac{2\mu_\tau}{m}\right) \geq -\varepsilon,$$

which gives the first line in (6.33).

Similarly, decomposing according to whether  $\sigma_{mL} \leq (m-2)L\mu_\tau$  or not, we have the lower bound

$$\begin{aligned} & \mathbf{E}_\sigma[\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})] \\ & \geq \log \mathbf{P}_\tau(\tau_{mL} \leq (m-2)L\mu_\tau) + \mathbf{P}_\sigma(\sigma_{mL} \leq (m-2)L\mu_\tau) \times mL \log \mathbf{P}_\tau(\tau_1 = d_\tau), \end{aligned}$$

which in turn also gives that  $\liminf_{n \rightarrow \infty} \frac{1}{L} \mathbf{E}_\sigma[\log \mathbf{P}_\tau(\tau_{mL} \leq \sigma_{mL})] \geq -\varepsilon$ .  $\square$

## APPENDIX A. TECHNICAL RESULTS ON RENEWAL PROCESSES

**A.1. Some estimates on renewal processes.** First of all, we state a result that we use throughout the paper, which is Lemma A.2 in [23] (that was slightly generalized in [2] to cover the case  $\alpha = 0$ ).

**Lemma A.1.** *Assume that  $\mathbf{P}(\tau_1 = k) = \varphi(k)k^{-(1+\alpha)}$  for some  $\alpha \geq 0$  and some slowly varying function  $\varphi(\cdot)$ . Then, there exists a constant  $C_0 > 0$  such that, for all sufficiently large  $n$ , for any non-negative function  $f_n(\tau)$  depending only on  $\tau \cap \{0, \dots, n\}$ , we have*

$$\mathbf{E}[f_n(\tau) \mid 2n \in \tau] \leq C_0 \mathbf{E}[f_n(\tau)].$$

In the rest of this section, we consider a renewal  $\tau$  satisfying (1.5) with  $\alpha \in (0, 1)$ . Similar results exist for  $\alpha = 0$  (see [3]) and for  $\alpha > 1$  (see Appendix A in [5]), but we do not need them here.

As noted in Section 2.4, for  $a_n$  as in (2.9),  $\tau_n/a_n$  converges to an  $\alpha$ -stable distribution with some density  $h$ , which is bounded and satisfies

$$(A.1) \quad h(x) \sim C_1 x^{-(1+\alpha)} \quad \text{as } x \rightarrow \infty, \quad h(x) \leq C_2 x^{-(1+\alpha)} \quad \text{for all } x > 0.$$

Further, by the local limit theorem for such convergence, see [24, §50], and the fact that  $a_k^\alpha \sim k\varphi(a_k)$ , for any given  $0 < \theta < K < \infty$  we have as  $k \rightarrow \infty$

$$(A.2) \quad \mathbf{P}_\tau(\tau_k = m) \sim \frac{1}{a_k} h\left(\frac{m}{a_k}\right) \leq C_2 a_k^\alpha m^{-(1+\alpha)} \leq 2C_2 k m^{-(1+\alpha)} \varphi(m)$$

uniformly over  $m \in [\theta a_k, K a_k]$ .

Moreover, Doney [17, Thm. A] gives that, uniformly in  $m \gg a_k$ ,

$$(A.3) \quad \mathbf{P}_\tau(\tau_k = m) = (1 + o(1)) k \mathbf{P}_\tau(\tau_1 = m) \quad \text{as } k \rightarrow \infty.$$

Together with (A.2) and Lemma A.3 below, which deals with the case  $m \ll a_k$ , we thereby obtain the following uniform bound.

**Lemma A.2.** *Assume  $\alpha \in (0, 1)$ . There exists a constant  $c_{42} > 0$  such that, for  $k$  large enough and all  $m \geq k$ ,*

$$\mathbf{P}_\tau(\tau_k = m) \leq c_{42} k \mathbf{P}_\tau(\tau_1 = m),$$

and

$$\mathbf{P}_\tau(\tau_k \geq m) \leq c_{42} k \mathbf{P}_\tau(\tau_1 \geq m).$$

For the lower tail we have the following.

**Lemma A.3.** *Assume  $\alpha \in (0, 1)$ . There exist a constant  $c_{43}$  such that, for all  $\epsilon < 1/2$  and  $n \geq 1$*

$$(A.4) \quad \mathbf{P}_\tau(\tau_n \leq \epsilon a_n) \leq \exp\left(-c_{43} \epsilon^{-\frac{3}{4} \frac{\alpha}{1-\alpha}}\right).$$

*In particular, for any  $\gamma \in (0, 1)$ , for all  $n \geq 1$ ,*

$$\mathbf{P}_\tau(\tau_n \leq n^{(1-\gamma)/\alpha}) \leq \exp\left(-c_{44} n^{\frac{\gamma}{2(1-\alpha)}}\right).$$

Note that this lemma implies that the density  $h$  also satisfies

$$(A.5) \quad h(x) \leq \exp\left(-c_{45} x^{-\frac{\alpha}{2(1-\alpha)}}\right),$$

for some constant  $c_{45} > 0$  and sufficiently small  $x$ .

*Proof of Lemma A.3.* Let  $\Lambda(t) = \log \mathbf{E}_\tau[e^{t\tau_1}]$ , so that for all  $t > 0$ ,

$$(A.6) \quad \mathbf{P}_\tau(\tau_n \leq \epsilon a_n) \leq \exp(\epsilon a_n t + \Lambda(-t) n).$$

By (1.5) and standard properties of the Laplace transform, there is a constant  $c_\alpha$  such that

$$(A.7) \quad \Lambda(-t) \sim -c_\alpha t^\alpha \varphi\left(\frac{1}{t}\right) \quad \text{as } t \searrow 0.$$

We have  $\Lambda(-t) < -d_\tau t$  for all  $t \in (0, +\infty)$ , and  $\Lambda(-t) \sim -d_\tau t$  as  $t \rightarrow +\infty$ , for  $d_\tau$  from (1.6). In the rest of the proof, we assume  $\epsilon a_n \geq d_\tau n$ , since otherwise the probability is 0.

We can approximately optimize (A.6) by taking  $t = t_n = t_n(\epsilon)$  given by

$$(A.8) \quad \frac{\Lambda(-t_n)}{t_n} = \frac{-2\epsilon a_n}{n}.$$

Such a solution exists since  $t^{-1}\Lambda(-t) \rightarrow -\infty$  as  $t \rightarrow 0$ ,  $t^{-1}\Lambda(-t) \rightarrow -d_\tau$  as  $t \rightarrow \infty$ , and  $2\epsilon a_n/n \geq 2d_\tau$ . We therefore end up with

$$(A.9) \quad \mathbf{P}_\tau(\tau_n \leq \epsilon a_n) \leq \exp(-\epsilon a_n t_n),$$

so we need a lower bound on  $a_n t_n$ .

Let  $c_{45}$  be given by  $c_{45}^{-1}\Lambda(-c_{45}) = -2d_\tau$ ; then  $t_n \in (0, c_{45}]$ , since  $\epsilon a_n \geq d_\tau n$ . Letting  $c_{46}$  large enough so that  $\varphi(x) > 0$  for all  $x \geq c_{46}/c_{45}$ , we then have that  $\varphi(c_{46}/t_n) > 0$ . Let

$\lambda = (1 - \alpha/4)/(1 - \alpha)$ . By (2.8), since  $a_n \rightarrow \infty$  and  $\epsilon a_n \geq d_\tau n$ , there exists  $n_0$  such that for  $n \geq n_0$ ,

$$(A.10) \quad 4\epsilon a_n^{1-\alpha} \varphi(a_n) \geq \frac{2\epsilon a_n}{n} \geq 2d_\tau \quad \text{and} \quad \frac{d_\tau}{2a_n^{1-\alpha} \varphi(a_n)} \geq \left(\frac{\bar{d}_\tau}{a_n}\right)^{1/\lambda},$$

which together show that  $\epsilon^\lambda a_n \geq \bar{d}_\tau$ . Therefore there exists  $c_{47}$  such that

$$\frac{\varphi(\epsilon^\lambda a_n)}{\varphi(a_n)} \geq c_{47} \epsilon^{\frac{1}{4}\alpha},$$

which with (A.7), (A.8), and the first inequality in (A.10) shows that for some  $c_{48}$ ,

$$(\epsilon^\lambda a_n)^{1-\alpha} \varphi(\epsilon^\lambda a_n) \geq c_{47} \epsilon a_n^{1-\alpha} \varphi(a_n) \geq \frac{c_{47} \epsilon a_n}{2n} \geq c_{48} \left(\frac{c_{46}}{t_n}\right)^{1-\alpha} \varphi\left(\frac{c_{46}}{t_n}\right).$$

We emphasize that the constants  $c_i$  and  $n_0$  do not depend on  $\epsilon$ . It follows that  $\epsilon^\lambda a_n \geq c_{49}/t_n$ , or equivalently  $\epsilon a_n t_n \geq c_{49} \epsilon^{-(\lambda-1)}$ , which with (A.9) completes the proof for  $n \geq n_0$ .

We finish by observing that for  $q = \max_{n < n_0} a_n$ , for all  $n < n_0$  and  $\epsilon \leq 1/2$  we have

$$\mathbf{P}_\tau(\tau_n \leq \epsilon a_n) \leq \mathbf{P}_\tau(\tau_1 \leq q) < 1,$$

and  $\mathbf{P}_\tau(\tau_n \leq \epsilon a_n) = 0$  if  $\epsilon < 1/q$ . Therefore after reducing  $c_{43}$  if necessary, (A.4) also holds for  $n < n_0$ .  $\square$

**A.2. Proof of Lemma 4.1.** Since  $\alpha + \tilde{\alpha} < 1$ , we have  $0 \leq \alpha < 1$  and  $0 < \tilde{\alpha} < 1$ . From (1.16),

$$(A.11) \quad \mathbf{P}_\tau(k \in \tau) \stackrel{n \rightarrow \infty}{\sim} \varphi_0(k) k^{-(1-\alpha)}, \quad \text{with } \varphi_0(x) = \begin{cases} \frac{\alpha \sin(\pi\alpha)}{\varphi(x)} \varphi(x)^{-1} & \text{if } \alpha \in (0, 1), \\ \frac{\alpha}{\mathbf{P}(\tau_1 \geq x)^2} & \text{if } \alpha = 0. \end{cases}$$

We now fix  $0 < \theta < K < \infty$  and  $\delta > 0$ , and split  $\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)$  into three sums:

$$(A.12) \quad \begin{aligned} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) &= \sum_{k=n}^{\infty} \mathbf{P}_\sigma(\sigma_n = k) \mathbf{P}_\tau(k \in \tau) \\ &\leq \sum_{0 \leq j \leq \log(\theta b_n)} \mathbf{P}_\sigma(\sigma_n \in (e^{-(j+1)} \theta b_n, e^{-j} \theta b_n]) \max_{k \in (e^{-(j+1)} \theta b_n, e^{-j} \theta b_n]} \mathbf{P}_\tau(k \in \tau) \\ &\quad + \sum_{\theta b_n < k < K b_n} \mathbf{P}_\sigma(\sigma_n = k) \mathbf{P}_\tau(k \in \tau) + \mathbf{P}_\sigma(\sigma_n \geq K b_n) \max_{k \geq K b_n} \mathbf{P}_\tau(k \in \tau). \end{aligned}$$

Let us first focus on the second sum, which gives the main contribution (and is also a lower bound). Using (A.2) and (A.11), it is asymptotic to

$$(A.13) \quad \varphi_0(b_n) b_n^{-(1-\alpha)} \sum_{\theta b_n < k < K b_n} \frac{1}{b_n} h\left(\frac{k}{b_n}\right) \left(\frac{k}{b_n}\right)^{-(1-\alpha)} \stackrel{n \rightarrow \infty}{\sim} \varphi_0(b_n) b_n^{-(1-\alpha)} \int_\theta^K x^{-(1-\alpha)} h(x) dx.$$

If  $\theta$  is sufficiently small (depending on  $\delta$ ), then  $\phi(e^j/\theta) \geq (e^j/\theta)^{-\tilde{\alpha}/2(1-\tilde{\alpha})}$ , and thanks to Lemma A.3 and (A.11), the first sum on the right in (A.12) can be bounded by

$$(A.14) \quad c_{50} \sum_{0 \leq j \leq \log(\theta b_n)} \exp\left(-c_{51} \left(\frac{e^j}{\theta}\right)^{\frac{\tilde{\alpha}}{2(1-\tilde{\alpha})}}\right) \times \left(\frac{\theta b_n}{e^j}\right)^{-(1-\alpha)} \varphi_0\left(\frac{\theta b_n}{e^j}\right) \leq \delta b_n^{-(1-\alpha)} \varphi_0(b_n).$$

If  $K$  is sufficiently large (depending on  $\delta$ ), then thanks to (A.11), the third sum on the right in (A.12) is bounded by

$$c_{52} (K b_n)^{-(1-\alpha)} \varphi_0(b_n) \leq \delta b_n^{-(1-\alpha)} \varphi_0(b_n).$$

By (A.1), the integral in (A.13) remains bounded as  $K \rightarrow \infty$  (using that  $\alpha + \tilde{\alpha} < 1$ ), and it also remains bounded as  $\theta \rightarrow 0$  thanks to (A.5). Because  $\delta$  is arbitrary and the second sum (A.13) is also a lower bound for  $\mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau)$ , it follows that

$$(A.15) \quad \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \left( \int_0^{+\infty} x^{-(1-\alpha)} h(x) dx \right) \times b_n^{-(1-\alpha)} \varphi_0(b_n).$$

We therefore conclude, thanks to (2.7), that

$$K^*(n) := \frac{1}{\mathbf{E}_{\sigma, \tau}[|\tau \cap \sigma|]} \mathbf{P}_{\sigma, \tau}(\sigma_n \in \tau) = \varphi^*(n) n^{-(1+\alpha^*)},$$

for  $\alpha^* = (1 - \alpha - \tilde{\alpha})/\tilde{\alpha}$  and for some slowly varying  $\varphi^*$ , given asymptotically via (A.15):

$$\varphi^*(n) \stackrel{n \rightarrow \infty}{\sim} c_{53} \tilde{\varphi}(b_n)^{-(1-\alpha)/\tilde{\alpha}} \varphi_0(b_n) \quad \text{with} \quad c_{53} = \frac{1}{\mathbf{E}_{\sigma, \tau}[|\tau \cap \sigma|]} \int_0^{+\infty} x^{-(1-\alpha)} h(x) dx.$$

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