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Lorentzian Kleinian Groups

Thierry Barbot*

Abstract

Classical Kleinian groups are discrete subgroups of isometries of \mathbb{H}^n . The well-known theory of Kleinian groups starts with the definition of their associated limit set in the boundary of \mathbb{H}^n , and includes the geometric properties of the quotient hyperbolic space.

This approach, naively applied, fails in the Lorentzian analogue anti-de Sitter space: discrete subgroups do not act properly discontinuously, and in many cases the set of accumulation points of orbits at the conformal boundary at infinity depends on the orbit.

In this survey, we point out a way to extend this classical theory by introducing causality notions: the theory of limit sets and regularity domains extend naturally to achronal subgroups. This is closely related to the notions of globally hyperbolic spacetimes, and we present what is known about the classification of globally hyperbolic spacetimes of constant curvature. We also review the close connection revealed by G. Mess ([79]) between globally hyperbolic spacetimes of dimension $2 + 1$ and Teichmüller space. This link can be understood via the space of time-like geodesics of anti-de Sitter space, and this space has also an interesting role, presented here, in the recent works about proper group actions on spacetimes of constant curvature.

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1 Introduction

Kleinian groups, *i.e.* discrete groups of isometries of the hyperbolic space \mathbb{H}^n , are central objects of study in geometry, most often in low dimension, and particularly in dimension 3. They have several ramifications in other fields of mathematics: number theory, topology, and, of course, group theory. For nice surveys on this very rich topic, including reports on recent results in this field, let us mention the references [78, 82] in this series of handbooks dedicated to group actions.

The isometry group of \mathbb{H}^n is $\mathrm{SO}(1, n)$, and here we consider the analogous case of discrete subgroups of $\mathrm{SO}(2, n)$, that we call in this survey *Lorentzian Kleinian groups*. The group $\mathrm{SO}(2, n)$ is naturally the group of isometries of the anti-de Sitter space, denoted by $\mathrm{AdS}^{1, n}$. Very little is known on Lorentzian Kleinian groups, if compared with the venerable hyperbolic case. The purpose of this survey is to present part of this relatively young matter.

Classical Kleinian groups have several nice basic properties:

- the action on \mathbb{H}^n is properly discontinuous;
- for any point x in \mathbb{H}^n , the orbit of x under the action of the group accumulates at the conformal boundary $\partial\mathbb{H}^n \approx \mathbb{S}^{n-1}$ on a closed invariant set, called the limit set, which does not depend on x .

These properties completely fail in the AdS case: stabilizers of points are not anymore compact. Therefore, the action is proper only in particular cases, and the asymptotic behavior of orbits is not uniform: for example a Lorentzian Kleinian group may have infinite orbits and other orbits reduced to a point.

Hence, at first glance, the Lorentzian case seems radically different, and some geometers may consider it as a quite completely different field. As a matter of fact, the first works on Lorentzian geometry in the case of constant curvature were mostly devoted to the study of *compact* Lorentzian manifolds, which has very few similarity with the Riemannian case. Even the geodesic completeness of compact Lorentzian manifolds of constant curvature, even if true, is far from being trivial ([32, 71]) — and as a matter of fact, compact Lorentzian manifolds are *not* geodesically complete in general. Wolf's book has long been — and still is! — one of the main references in the field, and was mainly concerned with the compact case. One indication on the Riemannian oriented spirit of Wolf's book [98] is that it is also one of the main references for the classification of *Riemannian* crystallographic groups.

Nevertheless, there are several important common features between the Riemannian and the Lorentzian case.

- anti-de Sitter space admits a natural conformal boundary: the Einstein universe $\text{Ein}^{1,n-1}$;
- in the same way as the Euclidean space and the hyperbolic space embed conformally in the sphere, the anti-de Sitter space embeds in the Einstein space, and moreover, the Minkowski space and the de Sitter space, Lorentzian analogues of the Euclidean space and the sphere, embed in $\text{Ein}^{1,n}$ as well.

The spirit underlying this survey is that there is a fundamental framework in which Riemannian or Lorentzian Kleinian groups have the same nature, but involving a notion that is unapparent because trivial in the Riemannian case: the notion of *causality*.

It is good in this celebration's year of General Relativity¹ to recall that Lorentzian geometry is the geometry of space *and time*; in which the classical Riemannian geometry is enclosed as the *static* case, *i.e.* the case in which the space does not change with time.

This point of view, developed in Section 5, is that one should distinguish certain subgroups, the *achronal subgroups*, that have a reasonable behavior relatively to causality. Riemannian Kleinian groups are automatically achronal, whereas Lorentzian ones may not be — as a matter of fact, Lorentzian Kleinian groups acting cocompactly are *never* achronal. Achronal Kleinian groups then appear as completely similar to their Riemannian counterparts: they admit a limit set in the conformal boundary, and they do act properly, not on the entire space itself, but on a certain domain: the domain of points invisible from the limit set.

Therefore, we start this paper by a review on the causality notions (Section 2), followed by a presentation of the Lorentzian spacetimes of constant curvature (Section 3). What appear as the true analogues of compact Riemannian spaces of constant curvature are not the compact Lorentzian manifolds, but the *maximal globally hyperbolic spatially compact spacetimes* (abbreviation MGHC), a notion

¹This survey has been written in 2015.

arising from the analytic treatment of General Relativity with the tools of Partial Differential Equations. In Section 4, we present the classification of MGHC spacetimes of constant curvature. This classification, initiated by the fundamental work of G. Mess ([79, 5]), is quite recent, and still incomplete. It provides an interesting framework in which many results or interrogations concerning hyperbolic Kleinian groups may very well find in a near future a natural continuation and extension.

After Section 5 mentioned above devoted to achronal Kleinian groups (but where we also mention links with the theory of Anosov representations), Section 6 points out very interesting connections, initiated by the pioneering work of Mess, between MGHC spacetimes of constant curvature and the Teichmüller space $\text{Teich}(S)$ of a closed surface S : it generalizes the link between hyperbolic 3-manifolds and $\text{Teich}(S)$, appearing for example in the case of quasi-fuchsian manifolds. But it also provides new ones, to which we propose a brief introduction. One explanation behind this phenomenon is the fact that $\text{SO}(2, n)$ has rank 2, whereas $\text{SO}(1, n)$ has “only” rank 1. In other words, unlike $\text{SO}(1, n)$, $\text{SO}(2, n)$ has not only one, but two maximal parabolic subgroups, meaning that there is not only one geometry, like the hyperbolic geometry, associated with $\text{SO}(2, n)$, but *two* geometries: anti-de Sitter geometry, and also the geometry of the symmetric space \mathcal{T}_{2n} associated with $\text{SO}(2, n)$, which happens to be *the space of AdS timelike geodesics*. For $n = 2$, \mathcal{T}_4 is nothing but the product $\mathbb{H}^2 \times \mathbb{H}^2$. The space \mathcal{T}_{2n} will appear in this survey in two situations:

— there is an interplay between the AdS MGHC spacetime M_Γ associated with a Lorentzian Kleinian group Γ and the quotient $\Gamma \backslash \mathcal{T}_{2n}$, emerging through Cauchy hypersurfaces, *i.e.* isometric embeddings of Riemannian hypersurfaces in M_Γ . More precisely, $\Gamma \backslash \mathcal{T}_{2n}$ has a natural Kähler structure, and there is an almost correspondence between Lagrangian submanifolds in $\Gamma \backslash \mathcal{T}_{2n}$ and Cauchy hypersurfaces in M_Γ . Moreover, in the case $n = 2$, the symplectic form on $\Gamma \backslash \mathcal{T}_{2n} \approx \Gamma \backslash (\mathbb{H}^2 \times \mathbb{H}^2)$ is the difference $p_1^* \omega_0 - p_2^* \omega_0$ where ω_0 is the volume form on \mathbb{H}^2 , and p_1, p_2 the projections on the left and right factors, respectively. Therefore, typical Lagrangian submanifolds of $\Gamma \backslash \mathcal{T}_{2n} \approx \Gamma \backslash (\mathbb{H}^2 \times \mathbb{H}^2)$ are graphs of volume preserving maps between hyperbolic surfaces. This connection between AdS geometry and special volume preserving maps between hyperbolic surfaces will be developed further, but with a quite different point of view, in [45], to appear in one of the *Handbooks of group actions*.

— \mathcal{T}_{2n} is also related to questions of proper actions: at least in dimension $2+1$, a Lorentzian Kleinian group Γ acts properly discontinuously on $\text{AdS}^{1,2}$ if and only if there is an embedded surface S in $\Gamma \backslash (\mathbb{H}^2 \times \mathbb{H}^2)$ such that the restriction to S of the pseudo-Riemannian metric $g_{hyp} - g_{hyp}$ is positive definite - where g_{hyp} denotes the hyperbolic metric, and $g_{hyp} - g_{hyp}$ a simplified notation for $p_1^* g_{hyp} - p_2^* g_{hyp}$. This criterion is related to the existence of Γ -invariant foliations by timelike geodesics, and we conjecture an extension of this fact in higher dimensions (see Remark 7.4).

We conclude the survey with a quick overview on Lorentzian Kleinian groups acting properly, even cocompactly. This topic has received recent brilliant contributions by J. Danciger, F. Guéritaud and F. Kassel that we mention very briefly, as important illustrations of the role of the space of timelike geodesics. There are several more or less recent surveys on this topic in which the reader may find more

substantial information ([9, 16, 91]).

The topic is currently growing quite quickly, and this survey has no pretention to be complete. We would like to attract the attention to a very recent work, transposing to Anti-de Sitter space the notion of Patterson-Sullivan measures, and establishing in this context an inequality between the critical exponent and the Hausdorff dimension of the (acausal) limit set ([53]).

Remarks on conventions and notation. A *Lorentzian manifold* is a manifold equipped with a pseudo-Riemannian metric of signature $(1, n)$ for some $n \geq 1$. The manifold is then of dimension $n + 1$: n -dimensional in space, and 1-dimensional in time. In our convention a Lorentzian metric has signature $(-, +, \dots, +)$; the value of the metric on a tangent vector v is called the *norm* of v — even if it would be more adequate, comparing with the Riemannian case, to call it the square of the norm. An orthonormal frame is a frame (e_0, e_1, \dots, e_n) where e_0 has norm 1, every e_i ($i \geq 1$) has norm +1 and every scalar product $\langle e_i, e_j \rangle$ with $i \neq j$ is 0.

We denote by $\mathrm{SO}_0(1, n)$, $\mathrm{SO}_0(2, n)$ the identity components of respectively $\mathrm{SO}(1, n)$, $\mathrm{SO}(2, n)$ ($n \geq 2$). For any cocompact lattice Γ of $\mathrm{SO}_0(1, n)$ and any Lie group G we denote by $\mathrm{Rep}(\Gamma, G)$ the moduli space of representations of Γ into G modulo conjugacy, equipped with the usual topology as an algebraic variety (see for example [56]):

$$\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G.$$

Finally, if (M_1, g_1) and (M_2, g_2) are two pseudo-Riemannian manifolds, we denote by $g_1 - g_2$ the metric $p_1^*g_1 - p_2^*g_2$ on $M_1 \times M_2$, where p_1 and p_2 are the projections onto the first and second factor, respectively.

Index of notations. We introduce in this survey many objects and notions. We provide here for the reader's convenience an index of these objects, indicating the page in which each of them is introduced (the list does not include objects already introduced such as $\mathrm{SO}_0(2, n)$ or \mathcal{T}_{2n}).

- c, \hat{c}, \dot{c} : causal curve, extension of the causal curve, derivative (beginning of Section 2).
- $[g]$: conformal class of the pseudo Riemannian metric g ,
- I^\pm, J^\pm : future or past; causal future or past (Section 2).
- $U(p, q), \bar{U}(p, q)$: diamond, closed diamond (Section 2).
- $P(S), F(S), \mathrm{dev}(S)$: past development, future development, development of the closed edgeless achronal (CEA) subset S (Section 2).
- \mathcal{C} : a category of spacetimes (Section 2).
- M_{max} : maximal extension of the spacetime M (Theorem 2.5).
- $L(c), d_{lor}$: length of the causal curve c , Lorentzian distance (just after Remark 2.6).

- $(\mathbb{R}^{r,s}, q_{r,s})$: $r + s$ -dimensional vector space equipped with a quadratic form of signature (r, s) (Section 3).
- $\text{Ein}^{1,n}, \overline{\text{Ein}}^{1,n}$ and $p : \widetilde{\text{Ein}}^{1,n} \rightarrow \text{Ein}^{1,n}$: Einstein universes of dimension $n + 1$ and their universal covering (Section 3.1).
- δ and δ_0 : transformations on $\overline{\text{Ein}}^{1,n}$ (Section 3.1).
- $E(\tilde{\Lambda}), E(\Lambda)$: globally hyperbolic domains of $\widetilde{\text{Ein}}^{1,n}$ or $\text{Ein}^{1,n}$ associated with a CEA $\tilde{\Lambda}$ or Λ (Section 3.1).
- $\text{Fill}(\tilde{\Lambda})$: filling of the CEA $\tilde{\Lambda}$ (Section 3.1).
- $S(\mathcal{C}_{n+1})$: Klein model of $\text{Ein}^{1,n}$ (Section 3.1).
- $\text{Conv}(\Lambda)$ and $\text{Conv}^*(\Lambda)$: convex hull of an achronal subset of $\text{Ein}^{1,n}$ and its convex dual (Section 3.1).
- I_0^+ : future of the origin in Minkowski space (Section 3.2).
- $\text{Mink}^+(\tilde{x}), \mathcal{I}^\pm$: Minkowski domain in $\widetilde{\text{Ein}}^{1,n}$ associated with an element \tilde{x} of $\widetilde{\text{Ein}}^{1,n}$ and its Penrose components (Section 3.2).
- $\mathbb{H}(x), \partial\mathbb{H}(x)$: totally geodesic hypersurface of \mathbb{H}^{n+1} and its boundary defined by an element x of de Sitter space $dS^{1,n}$ (Section 3.3).
- $\mathcal{B}(\mathbb{S}^n)$: space of round disks in \mathbb{S}^n (naturally identified with $dS^{1,n}$, see Section 3.3).
- $\mathbb{DS}^{1,n}$: Klein model of de Sitter space (Section 3.3).
- $\partial_\pm dS^{1,n}$: past and future conformal boundaries of $dS^{1,n}$ (Section 3.3).
- $\mathbb{ADS}^{1,n}$: Klein model of Anti-de Sitter space (Section 3.4).
- $U(x)$: affine domain centered at an element x of $\mathbb{ADS}^{1,n}$ (Section 3.4).
- $H^\pm(x)$: past and future hyperplanes dual to an element x of $\mathbb{ADS}^{1,n}$ (Section 3.4).
- \mathcal{G}, \mathcal{K} : Lie algebras of $\text{SO}_0(2, n)$ and of its maximal subgroup K (Remark 3.3).
- $\mathcal{U}^{1,n}$: space of future oriented vectors tangent to $\text{AdS}^{1,n}$ of norm -1 (Remark 3.3).
- λ : Liouville form on $\mathcal{U}^{1,n}$ (Remark 3.3).
- J, ω : complex structure and Kähler form on \mathcal{T}_{2n} (Remark 3.3).
- ν, B, II : Gauss map, shape operator and second fundamental form of a smooth spacelike surface in $\text{AdS}^{1,n}$ (end of Section 3.4).

- $M_\Lambda(\Gamma) = \Gamma \backslash \Omega(\Lambda)$: model maximal globally hyperbolic flat spacetime (Section 4.1).
- $M(\Sigma)$: maximal globally hyperbolic de Sitter spacetime associated with the $(\mathbb{S}^n, \text{SO}_0(1, n+1))$ -manifold Σ (Section 4.2).
- $\mathcal{H}^\pm(\Lambda)$: past and future horizons of a globally hyperbolic domain $\Omega(\Lambda)$ of $\text{AdS}^{1,n}$ (Section 4.3).
- $E_0^-(\Lambda)$: Past tight region of $\Omega(\Lambda)$ (Section 4.3).
- $E(\Lambda_{k,\ell})$: Split AdS-spacetime (end of Section 4.3).
- $D(\Lambda)$: conformal boundary of the invisible domain $\Omega(\Lambda)$ of $\text{AdS}^{1,n}$ (Section 4.4).
- Λ_Γ : limit set of an achronal group of isometries (Section 5).
- $\text{eu}_b(\Gamma)$: bounded Euler class of the subgroup Γ of $\text{SO}_0(2, n)$ (Theorem 5.4).
- $\text{Rep}_{an}(\Gamma, \text{SO}_0(1, G))$: space of Anosov representations of Γ into G (Section 5.4).
- ρ_L, ρ_R : left and right representations associated with a MGHC $\text{AdS}^{1,2}$ spacetime (beginning of Section 6.2).
- λ^\pm : pleating laminations on the boundary of the convex core (Section 6.2).

2 A brief introduction to causality notions

Let (M^{n+1}, g) be a Lorentzian manifold. A tangent vector is *spacelike* if its norm is positive; *timelike* if its norm is negative; *lightlike* if it is non zero and its norm is 0. We also define *causal* vectors as tangent vectors that are timelike or lightlike. An immersed hypersurface is *spacelike* if all vectors tangent to S are spacelike; it is *nontimelike* if tangent vectors are all spacelike or lightlike. A *causal* (resp. *timelike*) curve is an immersion $c : I \subset \mathbb{R} \rightarrow M$ such that for every t in I the derivative $\dot{c}(t)$ is causal (resp. timelike). This notion extends naturally to non-differentiable curves (see below, or [18]). Such a curve is *extendible* if there is another causal curve $\hat{c} : J \rightarrow M$ and a homeomorphism $\phi : I \rightarrow K \subsetneq J$ such that c coincides with $\hat{c} \circ \phi$. The causal curve c is *inextendible* if it is not extendible.

Conformal Lorentzian manifolds. The notion of timelike, lightlike and causal vectors or curves are the same for Lorentzian metrics in the same conformal class. Therefore, all the causality notions to be presented below apply to conformally Lorentzian manifolds $(M, [g])$, where $[g]$ denotes the conformal class of the Lorentzian metric g .

Time orientation. We always assume that the manifold M is oriented. On $(M, [g])$ we have another orientability notion: a *time orientation* of $(M, [g])$ is a continuous choice, for every p in M , of one of the two connected components

of the set of timelike vectors at p . When such a choice is possible, $(M, [g])$ is *time-orientable*, and in short we will mention time orientable (conformal class of) Lorentzian manifolds as (*conformal*) *spacetimes*. Any conformal manifold is doubly covered by a time-orientable one. Once the time-orientation has been selected we have a notion of *future-oriented* or *past-oriented* causal vectors, therefore of causal curves. We also have the notion of time function: a map $t : M \rightarrow \mathbb{R}$ which is non decreasing along any causal curve. Note that a time function may be non differentiable, and that a differentiable map $f : M \rightarrow \mathbb{R}$ is a time function if and only if its differential takes non negative values on future oriented causal vectors.

Causality notions. Two points in M are *causally related* if there exists a causal curve joining them; they are *strictly causally related* if moreover this curve can be chosen timelike. More generally: let E a subset of M and U an open neighborhood of E in M . E is *achronal* in U if there is no timelike curve contained in U joining two points of the subset. It is *acausal*, or *strictly achronal* in U if there is no causal curve contained in U joining two points of E . We say simply that E is (strictly) achronal if it is (strictly) achronal in $U = M$. Finally, we say that E is locally (strictly) achronal if every point p in E admits a neighborhood U in M such that $E \cap U$ is (strictly) achronal in U . Spacelike hypersurfaces are locally acausal, and nontimelike hypersurfaces are locally achronal.

Past, future. The *future* of a subset A of M is the open set $I^+(A)$ made of final points of future oriented timelike curves not reduced to one point and starting from a point of A . The *causal future* $J^+(A)$ of A is the (non necessarily closed) set of final points of future oriented causal curves, possibly reduced to one point and starting from a point of A (hence A itself belongs to its causal future). The (causal) past ($J^-(A)$) $I^-(A)$ of A is the (causal) future of A when the time-orientation of M is reversed. This induces two partial orders on M : for every p and q in M , we write $p \preceq q$ if q lies in the causal future of p , and $p \prec q$ if q lies in $I^+(p)$.

Alexandrov topology. An *open diamond* is a domain $U(p, q) = I^-(p) \cap I^+(q)$ that is the intersection between the future and the past of two points p, q . Open diamonds form the basis of some topology on M , the so-called *Alexandrov topology* (see [18]). Every $U(x, y)$ is open for the manifold topology, but the converse in general is false; when it holds, $(M, [g])$ is said *strongly causal*.

Strong causality is equivalent to the following property (Proposition 3.11 of [18]): for every point p in M , every neighborhood of p contains an open neighborhood U (for the usual manifold topology) of p which is *causally convex*, i.e. such that any causal curve in M joining two points in U is actually contained in U .

From now on, we always assume that the spacetime $(M, [g])$ is strongly causal.

For any p, q in M the *closed diamond* $\bar{U}(p, q)$ is the intersection between the causal future of p and the causal past of q .

Refined causality notions. In strongly causal spacetimes, one can extend the class of causal curves in the following way: a curve $c : I \subset \mathbb{R} \rightarrow M$ is causal and future oriented if it is locally non decreasing for the usual order on I and the partial order \preceq . It is strictly causal if it is locally increasing for the partial order

\prec . It is an easy exercise to see that acausal curves are locally Lipschitz, but in general non smooth.

One also has the important following notion among locally achronal subsets: a locally achronal subset A is *edgeless* if every point $p \in A$ admits a neighborhood U in M such that any causal curve contained in U and with extremities in respectively $I_U^+(A)$ and $I_U^-(A)$ crosses A . Then, closed edgeless achronal subsets (abbrev. CEA) are natural generalizations of (smooth) nontimelike hypersurfaces: they are locally graphs of Lipschitz maps.

Global hyperbolicity. A spacetime $(M, [g])$ is *globally hyperbolic* (abbrev. GH) if:

- it is strongly causal,
- for any p, q in M the closed diamond $\overline{U}(p, q)$ is compact or empty.

This definition makes clear that, in a globally hyperbolic spacetime $(M, [g])$, an open domain V of M is globally hyperbolic if and only if it is causally convex. Indeed, each of these notions is equivalent to the fact that for every x, y in V , the closed diamond $\overline{U}(x, y)$ in M coincides with the closed diamond in V . It follows that intersections of GH domains of $(M, [g])$ are still GH.

The notion of global hyperbolicity is closely related to the notion of Cauchy surfaces that we define now: let S be a spacelike hypersurface embedded in M (or, more generally, a CEA in M). The past development $P(S)$ (resp. the future development $F(S)$) is the set of points p in M such that every inextendible causal path containing p meets S in its future (resp. in its past). The *Cauchy development* $\text{dev}(S)$ is the union $P(S) \cup F(S)$. When $\text{dev}(S)$ is the entire M , S is a *Cauchy hypersurface*. An important fact, that can be considered as a generalisation of the Hopf-Rinow Theorem, is R. Geroch's Theorem ([51]):

Theorem 2.1. *A strongly causal spacetime $(M, [g])$ is globally hyperbolic if and only if it admits a Cauchy hypersurface. In this case, it is foliated by Cauchy hypersurfaces; more precisely, there is a smooth time function $t : M \rightarrow \mathbb{R}$ such that every level set of t is a Cauchy hypersurface.*

It follows directly from this theorem that every GH spacetime (M, g) is isometric to a product $S \times \mathbb{R}$ equipped with a metric of the form $\bar{g}_t - N dt^2$ where \bar{g}_t is a one parameter family of Riemannian metrics on S and $N : M \rightarrow]0, +\infty[$ is a positive function, called the *lapse function* (see Proposition 6.6.8 of [67]).

In particular, Cauchy hypersurfaces in a given GH spacetime are diffeomorphic to each other.

Remark 2.2. There has been some imprecision in the literature concerning the proof the smoothness of the splitting of globally hyperbolic spacetimes. See [20, 21, 22] for a survey on this question and a complete proof of the smoothness of the splitting $M \approx S \times \mathbb{R}$. See also a more recent proof with different methods, in [44].

Remark 2.3. The notion of global hyperbolicity in terms of Cauchy hypersurfaces has been introduced by J. Leray (1952). The key point is that the finite propagation property of the Einstein equations ensures that a metric solution of the

Einstein equations is completely determined by its restriction to a neighborhood of a Cauchy hypersurface S (more precisely, by the Riemannian metric obtained by restricting the Lorentzian metric to S , and by its second fundamental form). Therefore, globally hyperbolic spacetimes form a well posed problem from the viewpoint of Partial Differential Equations.

Remark 2.4. GH spacetimes are never compact. The suitable compactness notion is spatial compactness: a spacetime is *globally hyperbolic spatially compact* (abbrev. GHC) if it admits a compact Cauchy hypersurface — all Cauchy hypersurfaces are then compact.

Maximal globally hyperbolic spacetimes. An isometric embedding $f : M \rightarrow N$ is a *Cauchy embedding* if the image by f of any Cauchy hypersurface of M is a Cauchy hypersurface of N .

In this paragraph, we have to treat separately Lorentzian spacetimes and conformal Lorentzian spacetimes. If (M, g) and (N, h) are GH Lorentzian spacetimes, a map $f : M \rightarrow N$ is a *Cauchy embedding* if it is an isometric embedding such that the image by f of any Cauchy hypersurface in M is a Cauchy hypersurface in N . A *conformal Cauchy embedding* is a conformal embedding $f : M \rightarrow N$ between conformal spacetimes mapping Cauchy hypersurfaces into Cauchy hypersurfaces. Note that a conformal Cauchy embedding might be non-isometric, therefore not a Cauchy embedding in our terminology.

Let \mathcal{C} be a category of Lorentzian spacetimes, i.e. a class of Lorentzian spacetimes stable by isometries, by union, and restriction to open domains — for example, the category of C^r spacetimes, or the category of analytic spacetimes etc. . . A GH \mathcal{C} -spacetime (M, g) is \mathcal{C} -*maximal* (abbrev. \mathcal{C} -MGH) if every Cauchy embedding $f : M \rightarrow N$ in a \mathcal{C} -spacetime N is surjective (hence a global isometry).

We have a similar notion of maximal conformal GH spacetimes, but where \mathcal{C} is a category of conformal spacetimes. The two notions may differ; as we will see later, a Lorentzian spacetime may be maximal among spacetimes of constant curvature, but not maximal in the category of conformally flat spacetimes.

A Lorentzian category \mathcal{C} is *rigid* if it has the following property: if p, q are points in \mathcal{C} -spacetimes M, M' such that any isometry between $J^-(p)$ and $J^-(q)$ extends to an isometry between neighborhoods of p and q . The traditional example is the category of solutions of the Einstein equations in the void, or for us, the category of constant curvature spacetimes.

One has a similar notion of *rigid conformal categories*: the ones for which any conformal diffeomorphism between $J^-(p)$ and $J^-(q)$ extends to a conformal diffeomorphism between neighborhoods of p and q .

Theorem 2.5 ([86]). *Let \mathcal{C} be a rigid category of Lorentzian spacetimes or of conformal spacetimes. Then any GH \mathcal{C} -spacetime M admits a Cauchy embedding $f : M \rightarrow M_{max}$ in a \mathcal{C} -MGH spacetime. Moreover, M_{max} is unique up to right composition by an isometry in the case of Lorentzian categories, and up to right composition by a conformal diffeomorphism in the case of conformal categories.*

Remark 2.6. This theorem was first established in the case of solutions of the Einstein equations ([35]). There is a more recent proof, with the same ideas, but a

different order allowing to avoid in a clever way the use of the Zorn lemma ([88]). In her work [86], C. Rossi Salvemini observed that the proof applies to the more general case of rigid categories of Lorentzian spacetimes, and also in the context of conformal Lorentzian categories that she introduced. Moreover, she proposed an entirely new proof, based on the notion of *shadows* (intersections $J^\pm(p) \cap S$ between a Cauchy hypersurface S and past/future of points). She also proved the following important result: *if a GHC spacetime has nonpositive constant curvature and is maximal among spacetimes of constant curvature, then it is also maximal among conformally flat spacetimes.* This statement is false in the positive constant curvature case: a GH spacetime with constant positive curvature is *never* maximal as a conformally flat spacetime.

Lorentzian distance. Let M be a time-oriented Lorentzian spacetime. The *length-time* $L(c)$ of a causal curve $c : I \rightarrow M$ is the integral over I of the square root of $-\langle c(t)|c(t) \rangle$. Observe that this is well-defined, since causal curves are always Lipschitz. The *Lorentzian distance* $d_{lor}(p, q)$ between two points p, q is $\text{Sup}\{L(c)/c \in C(p, q)\}$ where $C(p, q)$ is the set of causal curves with extremities p, q (see for example [4]). By convention, if p, q are not causally related, $d_{lor}(p, q) = 0$: when M is globally hyperbolic, it defines a continuous function $d_{lor} : M \times M \rightarrow [0, +\infty[$ since if q lies on the boundary of $J^\pm(p)$ then there is a lightlike curve joining p to q and $d_{lor}(p, q) = 0$.

Theorem 2.7 (Corollary 4.7 and Theorem 6.1 of [18]). *If M is globally hyperbolic, then $d_{lor} : M \times M \rightarrow [0, +\infty[$ is continuous and admits only finite values. Moreover, if p is in the causal future of q , then there exists a geodesic c with extremities p, q such that $L(c) = d(x, y)$.*

It is to obtain this theorem that one does not restrict the definition of causal curves to piecewise C^1 curves.

Cosmological time. In any spacetime, we can define the notion of cosmological time ([4]): For any p in M , the cosmological time $\tau(p)$ is $\text{Sup}\{L(c)/c \in \mathcal{R}(p)\}$, where $\mathcal{R}(p)$ is the set of past-oriented causal curves starting at p . This function could have in general a bad behavior: for example, in Minkowski space, the cosmological time is everywhere infinite.

Definition 2.8. A Lorentzian spacetime (M, g) is said to have regular cosmological time if:

- M has finite existence time, i.e. $\tau(p) < \infty$ for every p in M ,
- for every past-oriented inextendible curve $c : [0, +\infty[\rightarrow M$, we have $\lim_{t \rightarrow \infty} \tau(c(t)) = 0$.

Theorem 1.2 in [4] expresses many nice properties of spacetimes with regular cosmological time functions. We need only the following statement:

Theorem 2.9. *If M has regular cosmological time, then the cosmological time is Lipschitz regular and M is globally hyperbolic.*

3 Model spacetimes

In this section, we describe the model spacetimes for every sign of the (constant) curvature. This includes a description of their causal curves and their achronal subsets. We will end by the presentation of the space \mathcal{T}_{2n} of timelike geodesics in the anti-de Sitter space that will play an important role in this survey. In the next section, we will use this material for the classification of maximal GH spacetimes of constant curvature.

An important feature is that causality notions in the model spacetimes are much easier to deal with once it is observed that they all admit a conformal embedding in the Einstein Universe. We therefore start with this central geometric model. As a reference for the content of this section, we mention [46, 85] as very complete references (in French), the papers [48, 86] extracted from these works, and also [15, 6].

On notations: all model spacetimes involve $(\mathbb{R}^{r,s}, q_{r,s})$, for some integers r, s , where $\mathbb{R}^{r,s}$ is the $r + s$ -dimensional vector space $\mathbb{R}^{r,s}$ and $q_{r,s}$ a quadratic form of signature (r, s) . More precisely, the first r coordinates of $\mathbb{R}^{r,s}$ are denoted by u_1, \dots, u_r and the other coordinates by x_1, \dots, x_s . Elements of $\mathbb{R}^{r,s}$ are denoted by x, y, \dots . The quadratic form is then:

$$q_{r,s}(x) = -u_1^2 - \dots - u_r^2 + x_1^2 + \dots + x_s^2$$

The associated scalar product is denoted by $\langle \cdot | \cdot \rangle_{r,s}$, or simply $\langle \cdot | \cdot \rangle$.

We will also denote by $(\mathbb{S}^n, \bar{g}_n)$ the sphere of dimension n equipped with its usual metric \bar{g}_n : the restriction of $q_{0,n+1}$ to the unit sphere $\{q_{0,n+1} = 1\}$ of $\mathbb{R}^{0,n+1}$. The distance on \mathbb{S}^n induced by \bar{g}_n is denoted by d_n .

3.1 Einstein universe

The four-dimensional Einstein universe was the first cosmological model for our universe proposed by A. Einstein soon after the birth of General Relativity. The $n+1$ dimensional Einstein universe $\widetilde{\text{Ein}}^{1,n}$ can be simply described as the (oriented) product $\mathbb{S}^n \times \mathbb{R}$ of the n -dimensional sphere and the real line, equipped with the metric $\bar{g}_n - dt^2$, and time-oriented so that the coordinate t is a time function.

The importance of the Einstein universe is essentially due to the following extension of the Liouville Theorem ([47]): *when $n \geq 2$, any conformal transformation between two open subsets of $\widetilde{\text{Ein}}^{1,n}$ extends to a global conformal transformation.* It follows that conformal Lorentzian spacetimes of dimension $\geq 2 + 1$ are locally modeled on the Einstein universe. Hence we don't really consider the Einstein universe as a Lorentzian manifold, but as a conformally Lorentzian spacetime.

We also consider the product $\text{Ein}^{1,n} = \mathbb{S}^n \times \mathbb{S}^1$ equipped with the $\bar{g}_n - \bar{g}_1$. We denote by $p : \widetilde{\text{Ein}}^{1,n} \rightarrow \text{Ein}^{1,n}$ the cyclic covering map. Let $\delta : \widetilde{\text{Ein}}^{1,n} \rightarrow \widetilde{\text{Ein}}^{1,n}$ be the map $(\bar{x}, t) \mapsto (\bar{x}, t + 2\pi)$: it generates the Galois group of p . We will also consider the quotient $\overline{\text{Ein}}^{1,n}$ of $\widetilde{\text{Ein}}^{1,n}$ by $\delta_0 : (\bar{x}, t) \mapsto (-\bar{x}, t + \pi)$, even if its topology is slightly more difficult to handle. This quotient, which is doubly covered by $\text{Ein}^{1,n}$, is sometimes called *Einstein universe* in the literature.

Photons. Lightlike geodesics of $\text{Ein}^{1,n}$, $\widetilde{\text{Ein}}^{1,n}$ or $\overline{\text{Ein}}^{1,n}$, considered as non-parameterized curves, do not depend on the choice of the representative of the conformal class. They are called *photons*. In $\widetilde{\text{Ein}}^{1,n}$, they are represented by the curves $t \mapsto (\bar{x}(t), t)$, where $t \mapsto \bar{x}(t)$ is a geodesic of $(\mathbb{S}^n, \bar{g}_n)$. For every \tilde{x} in $\widetilde{\text{Ein}}^{1,n}$, $\delta_0(x)$ is its first conjugate point in the future: by this, we mean that every future oriented photon exiting from \tilde{x} also contain $\delta_0(\tilde{x})$, and that there is no other such conjugate point in $J^+(\tilde{x}) \cap J^-(\delta_0(\tilde{x}))$.

The union of all photons containing \tilde{x} is the *lightcone* $C(\tilde{x})$. If we write \tilde{x} as a pair (\bar{x}, t) in $\mathbb{S}^n \times \mathbb{R}$, the lightcone $C(\tilde{x})$ is the set of pairs (\bar{y}, s) such that the difference $t - s$ is equal to $d_n(\bar{x}, \bar{y})$ modulo 2π . It is a cylinder pinched at every conjugate points $\delta_0^k(\tilde{x})$ (see for example Figure 4.3 in [46]).

The projections of lightcones in $\text{Ein}^{1,n}$ are also called lightcones; a lightcone $C(x)$ for x in $\text{Ein}^{1,n}$ has two singular points, x and its conjugate $\delta_0(x)$.

Causal curves. More generally, causal curves in $\widetilde{\text{Ein}}^{1,n}$, suitably parameterized, are the curves $t \mapsto (\bar{x}(t), t)$, where $t \mapsto \bar{x}(t)$ is a 1-Lipschitz map from an interval I into \mathbb{S}^n — it is timelike if \bar{x} is contracting, i.e.:

$$\forall s, t \in \mathbb{R} \quad d_n(\bar{x}(s), \bar{x}(t)) < |s - t|.$$

In particular, inextendible causal curves are the ones parameterized by $I = \mathbb{R}$. It clearly follows that $\widetilde{\text{Ein}}^{1,n}$ is globally hyperbolic spatially compact: every level set $\{t = \text{Const.}\}$ is a Cauchy hypersurface.

Achronal subsets. Achronal subsets of $\widetilde{\text{Ein}}^{1,n}$ are precisely the graphs of 1-Lipschitz functions $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a subset of \mathbb{S}^n . The achronal set is acausal if and only if f is 1-contracting. It is closed if and only if Λ_0 is closed, and edgeless if and only if Λ_0 is open. In particular, closed edgeless achronal subsets are exactly the graphs of the 1-Lipschitz functions $f : \mathbb{S}^n \rightarrow \mathbb{R}$: they are topological n -spheres, which are all Cauchy hypersurfaces.

Stricto sensu, there is no achronal subset in $\text{Ein}^{1,n}$ since closed timelike curves through a given point cover the entire $\text{Ein}^{1,n}$. Nevertheless, we can keep track of this notion in $\text{Ein}^{1,n}$ by defining “achronal” subsets of $\text{Ein}^{1,n}$ as projections of genuine achronal subsets of $\widetilde{\text{Ein}}^{1,n}$. This definition is justified by the following results (Lemma 2.4, Corollary 2.5 in [15]): *The restriction of p to any achronal subset of $\widetilde{\text{Ein}}^{1,n}$ is injective. Moreover, if $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ are two achronal subsets of $\widetilde{\text{Ein}}^{1,n}$ admitting the same projection in $\text{Ein}^{1,n}$, then there is an integer k such that:*

$$\tilde{\Lambda}_1 = \delta^k \tilde{\Lambda}_2$$

In this setting, closed edgeless achronal subsets of $\text{Ein}^{1,n}$ are graphs of 1-Lipschitz maps from \mathbb{S}^n into \mathbb{S}^1 .

Globally hyperbolic domains. Let $\tilde{\Lambda}$ be a closed achronal subset of $\widetilde{\text{Ein}}^{1,n}$, i.e. the graph of a 1-Lipschitz map $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a closed subset of \mathbb{S}^n .

Define two functions $f^-, f^+ : \mathbb{S}^n \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} f^+(\bar{x}) &:= \text{Inf}_{\bar{y} \in \Lambda_0} \{f(\bar{y}) + d_n(\bar{x}, \bar{y})\}, \\ f^-(\bar{x}) &:= \text{Sup}_{\bar{y} \in \Lambda_0} \{f(\bar{y}) - d_n(\bar{x}, \bar{y})\}, \end{aligned}$$

f^+ (respectively f^-) is the maximal (respectively minimal) 1-Lipschitz map from \mathbb{S}^n into \mathbb{R} that coincides with f on Λ_0 .

Then, the set of points of $\widetilde{\text{Ein}}^{1,n}$ which are not causally related to any point of $\widetilde{\Lambda}$ is:

$$E(\widetilde{\Lambda}) = \{(\bar{x}, t) \in \mathbb{S}^n \times \mathbb{R} \mid f^-(\bar{x}) < t < f^+(\bar{x})\}.$$

Observe that if \bar{x} and \bar{y} are two points in Λ_0 such that $|f(\bar{x}) - f(\bar{y})| = d_n(\bar{x}, \bar{y})$, then the restrictions of f^+ and f^- to any minimizing d_n -geodesic segment between \bar{x} and \bar{y} coincide. Let $\text{Fill}(\Lambda_0)$ be the union of Λ_0 with the union of all minimizing d_n -geodesic segments joining two elements \bar{x}, \bar{y} of Λ_0 such that $|f(\bar{x}) - f(\bar{y})| = d_n(\bar{x}, \bar{y})$, and let $\text{Fill}(f)$ be the restriction of f^\pm to $\text{Fill}(\Lambda_0)$: the graph of $\text{Fill}(f)$ is a CEA of $\widetilde{\text{Ein}}^{1,n}$ that we denote by $\text{Fill}(\widetilde{\Lambda})$ and that we call the *filling of $\widetilde{\Lambda}$* . Then we have $E(\widetilde{\Lambda}) = E(\text{Fill}(\widetilde{\Lambda}))$. In other words, we can restrict ourselves to *filled CEA*, i.e. CEA equal to their own fillings (see [17, Remark 3.19]).

The domain $E(\widetilde{\Lambda})$ may be empty, but exactly in the case where the filling $\text{Fill}(\widetilde{\Lambda})$ is the entire sphere \mathbb{S}^n . A particular case when this happens is the case where $\widetilde{\Lambda}$ is *purely lightlike*, i.e. the case where Λ_0 contains two antipodal points \bar{x}_0 and $-\bar{x}_0$ such that the equality $f(\bar{x}_0) = f(-\bar{x}_0) + \pi$ holds (Lemma 3.6 in [15]). Purely lightlike achronal subsets are precisely the ones admitting as filling the union of lightlike geodesics joining two antipodal points of $\widetilde{\text{Ein}}_n$.

If non-empty, the invisible domain $E(\widetilde{\Lambda})$ is globally hyperbolic (indeed, it is easy to see that for any p, q in $E(\widetilde{\Lambda})$, the closed diamond $\overline{U}(p, q)$ is contained in $E(\widetilde{\Lambda})$). More precisely, the Cauchy hypersurfaces of $E(\widetilde{\Lambda})$ are precisely graphs Λ_F of 1-Lipschitz maps $F : \mathbb{S}^n \setminus \text{Fill}(\Lambda_0) \rightarrow \mathbb{R}$ such that the extension of F to $\text{Fill}(\Lambda_0)$ coincides with $\text{Fill}(f)$.

In the limit case $\Lambda_0 = \emptyset$, we have $f^+ = +\infty$ and $f^- = -\infty$, and the Cauchy development of the graph of any 1-Lipschitz map $F : \mathbb{S}^n \rightarrow \mathbb{R}$ defined on the entire sphere is the entire Einstein universe $\widetilde{\text{Ein}}^{1,n}$.

In summary, the theory of globally hyperbolic domains of $\widetilde{\text{Ein}}^{1,n}$ coincides with the theory of 1-Lipschitz maps on \mathbb{S}^n .

As for achronal subsets, even if $\text{Ein}^{1,n}$ is not strongly causal, one can abusively project the notion of globally hyperbolic domains into $\text{Ein}^{1,n}$, thanks to the following lemma:

Lemma 3.1. *For every (non-empty) closed achronal set $\widetilde{\Lambda} \subset \widetilde{\text{Ein}}^{1,n}$, the projection of $E(\widetilde{\Lambda})$ onto $E(\Lambda) = \text{p}(E(\widetilde{\Lambda}))$ is one-to-one.*

Klein model. Einstein universe can also be defined in the following way: let \mathcal{C}_{n+1} be the null-cone in $(\mathbb{R}^{2,n+1}, q_{2,n+1})$, and let $\mathbb{S}(\mathcal{C}_{n+1})$ be its projection in the space $\mathbb{S}(\mathbb{R}^{2,n+1})$ of rays in $\mathbb{R}^{2,n+1}$. $\mathbb{S}(\mathbb{R}^{2,n+1})$ is a double covering of the usual projective

space $\mathbb{P}(\mathbb{R}^{2,n+1})$, therefore we call it, slightly abusively, the projectivization of $\mathbb{R}^{2,n+1}$. Observe that the convexity is well defined in $\mathbb{S}(\mathbb{R}^{2,n+1})$: one can simply define convex subsets of $\mathbb{S}(\mathbb{R}^{2,n+1})$ as radial projections of convex cones in $\mathbb{R}^{2,n+1}$. In particular, convex hulls $\text{Conv}(B)$ of subsets B of $\mathbb{S}(\mathbb{R}^{2,n+1})$, in particular, of subsets of $\mathbb{S}(\mathcal{C}_{n+1})$, are well defined (but we don't mean that they are contained in $\mathbb{S}(\mathcal{C}_{n+1})!$).

The quadratic form $q_{2,n+1}$ induces a natural conformally Lorentzian structure on $\mathbb{S}(\mathcal{C}_{n+1})$. More precisely, for any section $\sigma : \mathbb{S}(\mathcal{C}_{n+1}) \rightarrow \mathcal{C}_{n+1}$, the pull-back $\sigma^*q_{2,n+1}$ is a Lorentzian metric g_σ , and the conformal class $[g_\sigma]$ does not depend on σ . This conformally Lorentzian metric happens to be conformally isometric to $(\text{Ein}^{1,n}, \bar{g}_n - \bar{g}_1)$.

The pair $(\mathbb{S}(\mathcal{C}_{n+1}), [g_\sigma])$ is the *Klein model* of $\text{Ein}^{1,n}$.

Isometry group. It is clear from the Klein model that the group of conformal transformations of $\text{Ein}^{1,n}$ preserving the orientation and the time orientation is $\text{SO}_0(2, n+1)$. Let $\widetilde{\text{SO}}_0(2, n+1)$ be the group of conformal transformations of $\widetilde{\text{Ein}}^{1,n}$ preserving the orientation and the time orientation. There is a natural projection from $\widetilde{\text{SO}}_0(2, n+1)$ into $\text{SO}_0(2, n+1)$ whose kernel is spanned by the transformation δ generating the Galois group of $p : \widetilde{\text{Ein}}^{1,n} \rightarrow \text{Ein}^{1,n}$ defined previously. Therefore, there is a central exact sequence:

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{SO}}_0(2, n+1) \rightarrow \text{SO}_0(2, n+1) \rightarrow 1$$

Observe that $\widetilde{\text{SO}}_0(2, n+1)$ is not the universal covering of $\text{SO}_0(2, n+1)$: there is a retraction of $\text{SO}_0(2, n+1)$ onto $\text{SO}(2) \times \text{SO}(n)$ hence the fundamental group of $\text{SO}(2, n+1)$ is not cyclic but isomorphic² to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

Remark 3.2. Concerning the notation: in the sequel, we always have in mind the identifications $\text{Ein}^{1,n} \approx \mathbb{S}(\mathcal{C}_{n+1})$, and we frequently switch from one model to the other. We denote by x elements of Ein , using the notation \mathbf{x} when we want to insist on the Klein model.

Causality notions in the Klein model. Two elements x, y of Ein are causally related if and only if $\langle x | y \rangle \geq 0$. In particular, a subset $\Lambda \subseteq \text{Ein}$ is achronal (respectively acausal) if and only if for every distinct $x, y \in \Lambda$ the scalar product $\langle x | y \rangle$ is non-positive (respectively negative).

Photons are projections on $\mathbb{S}(P)$ of isotropic 2-planes of $\mathbb{R}^{2,n+1}$. The lightcone $C(x)$ of a point x is the projection of $\mathcal{C}_{n+1} \cap x^\perp$, where x^\perp is the $q_{2,n+1}$ -orthogonal of any representative \mathbf{x} of x .

Finally, for every achronal subset Λ of $\text{Ein}^{1,n} \approx \mathbb{S}(\mathcal{C}_{n+1})$, the invisible domain $E(\Lambda)$ is:

$$E(\Lambda) = \{x \in \mathbb{S}(\mathcal{C}_{n+1}) \mid \forall y \in \Lambda \langle x | y \rangle < 0\}.$$

Recall that the dual of a convex subset $\mathbb{S}(C)$ of $\mathbb{S}(\mathbb{R}^{2,n})$ is:

$$C^* = \{x \in \mathbb{S}(\mathcal{C}_{n+1}) \mid \forall y \in C \langle x | y \rangle \leq 0\}.$$

²We thank the referee to have pointed out this fact to us.

Hence, $E(\Lambda)$ coincides with the interior of $\mathbb{S}(\mathcal{C}_{n+1}) \cap \text{Conv}^*(\Lambda)$, where $\text{Conv}^*(\Lambda)$ is the convex subset of $\mathbb{S}(\mathbb{R}^{2,n})$ dual to the convex hull $\text{Conv}(\Lambda)$: it is the intersection between a quadric and a convex subset of the projective space $\mathbb{S}(\mathbb{R}^{2,n})$ (cf. [17, section 3.3]).

3.2 Minkowski space

For this section, we indicate as references [10, 23, 11]. The **Minkowski space** is the affine space of dimension $n + 1$ equipped with the quadratic form $q_{1,n}$ on the underlying vector space $\mathbb{R}^{1,n}$ (for $n \geq 1$). We slightly abuse notations, denoting it by $\mathbb{R}^{1,n}$, whereas it should really be considered as an affine space, and not linear. We also use a coordinate system (t, x_1, \dots, x_n) such that:

$$q_{1,n}(x) = -t^2 + x_1^2 + \dots + x_n^2.$$

The causal structure of the Minkowski space is particularly simple, because of its affine structure. It is convenient to see it as the product $(\mathbb{R} \times \mathbb{R}^n, -dt^2 + |d\bar{x}|^2)$ of the line \mathbb{R} “of time” and the Euclidian plane \mathbb{R}^n , whose elements are denoted by \bar{x} . A time orientation is obtained by requiring the time coordinate t to be a time function.

Let us fix an origin 0 , identifying Minkowski space with its underlying linear space $\mathbb{R}^{1,n}$. Let I_0^+ be the set of future-oriented timelike tangent vectors at 0 . Then I_0^+ coincides with the future $I^+(0)$ through the canonical identification between $T_0\mathbb{R}^{1,n}$ and $\mathbb{R}^{1,n}$. Two elements x, y are causally related if and only if $y - x$ is timelike, and $y \in I^+(x)$ if and only if $y - x$ lies in I_0^+ .

The isometry group (as always, preserving all orientations) is the Poincaré group $\text{Isom}_0(\mathbb{R}^{1,n})$, isomorphic to $\text{SO}_0(1, n) \ltimes \mathbb{R}^{1,n}$.

Concerning the causality notions, almost all the discussion above in the case of Einstein universe applies, replacing the sphere $(\mathbb{S}^n, \bar{g}_n)$ by the Euclidean plane $(\mathbb{R}^{0,n}, q_{0,n})$: up to reparametrization, causal (resp. timelike) curves are maps $t \in I \subseteq \mathbb{R} \mapsto \bar{x}(t)$ where $\bar{x} : I \rightarrow \mathbb{R}^{0,n}$ is 1-Lipschitz (resp. 1-contracting). For inextendible curves we have $I = \mathbb{R}$. Every horizontal hyperplane $\{t = \text{Const.}\}$ is a Cauchy surface. The geodesics of Minkowski space are affine lines. The achronal (resp. acausal) subsets are graphs Λ_f of 1-Lipschitz (respectively 1-contracting) maps $f : \Lambda_0 \subseteq \mathbb{R}^{0,n} \rightarrow \mathbb{R}$, closed edgeless achronal subsets are graphs Λ_f of 1-Lipschitz maps defined on the entire $\Lambda_0 = \mathbb{R}^{0,n}$ — but as we will see later, it is not always a Cauchy hypersurface for $\mathbb{R}^{1,n}$.

Conformal model. Minkowski space admits a conformal embedding in Einstein universe; actually, it is conformally isometric to the complement in $\widehat{\text{Ein}}^{1,n}$ of any lightcone. It is actually more convenient to see it as one connected component of the complement of the lightcone $C(\tilde{x}_0)$ of some point \tilde{x}_0 in $\widehat{\text{Ein}}^{1,n}$: the “extreme diamond”, intersection $I^+(\delta_0^{-1}(\tilde{x}_0)) \cap I^-(\delta_0^{-1}(\tilde{x}_0))$, which we denote by $\text{Mink}^+(\tilde{x}_0)$. If we can take $\tilde{x}_0 = (\bar{x}_0, 0)$, then $\text{Mink}^+(\tilde{x}_0)$ is the set of points (\bar{x}, t) such that $|t| < d_n(\bar{x}, \bar{x}_0)$.

The boundary of $\text{Mink}^+(x_0)$ in $\text{Ein}^{1,n}$ can therefore be seen as a **conformal boundary** of the Minkowski space, that has already been introduced by R. Penrose ([83]). It decomposes in several parts:

- the point $\delta_0^{-1}(\tilde{x}_0)$, denoted by i^- by Penrose: $\text{Mink}^+(\tilde{x}_0)$ is entirely contained in the future of i^- ,
- the point $\delta_0(\tilde{x}_0)$, also denoted by i^+ : $\text{Mink}^+(\tilde{x}_0)$ is entirely contained in the past of i^+ ,
- the point $i_0 := \tilde{x}_0$, called the “spatial infinity”,
- the complement of these three points is the union of two lightlike cylinders $\mathbb{S}^{n-1} \times \mathbb{R}$, one in the future of $\text{Mink}^+(\tilde{x}_0)$ and denoted by \mathcal{I}^+ , and the other, \mathcal{I}^- , in the past of $\text{Mink}^+(\tilde{x}_0)$.

See Figure 1 (borrowed from [46, Figure 4.3]) where several Minkowski components $\text{Mink}^+(i_k)$ are depicted, where every i_k is a point conjugate to $i_0 = \tilde{x}_0$ (*i.e.* iterates of \tilde{x}_0 under δ_0).

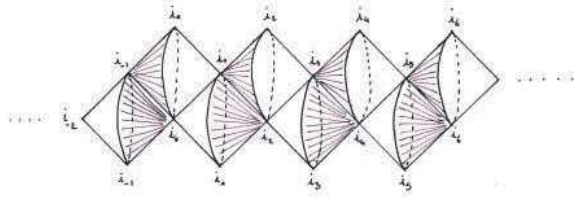


Figure 1: Several Minkowski components.

The intersection between $\text{Mink}^+(\tilde{x}_0)$ and the lightcone $C(\tilde{x})$ of any point in \mathcal{I}^\pm is an affine hyperplane $H(\tilde{x})$ in $\text{Mink}^+(\tilde{x}_0) \approx \mathbb{R}^{1,n}$. More precisely: if $\tilde{x} \in \mathcal{I}^+$, then $\text{Mink}^+(\tilde{x}_0) \cap I^-(\tilde{x})$ is the past of $H(\tilde{x})$ in $\mathbb{R}^{1,n}$, and if $\tilde{x} \in \mathcal{I}^-$, then $\text{Mink}^+(\tilde{x}_0) \cap I^+(\tilde{x})$ is the future of $H(\tilde{x})$ in $\mathbb{R}^{1,n}$. Therefore, \mathcal{I}^- can be seen as the space of half affine Minkowski spaces, that are equal to their own future, and bounded by a lightlike hyperplane.

3.3 De Sitter space

The **de Sitter space** $\text{dS}^{1,n}$ is the hypersurface $\{x \in \mathbb{R}^{1,n+1} / q_{1,n+1}(x) = +1\}$ endowed with the Lorentzian metric obtained by restriction of $q_{1,n+1}$. Hence, for the coordinates (t, x_1, \dots, x_n) we have:

$$\text{dS}^{1,n} := \{(t, x_1, \dots, x_n) \mid -t^2 + x_1^2 + \dots + x_n^2 = +1\}.$$

We equip $\text{dS}^{1,n}$ with the time orientation for which t (or, more generally, for every $u \in \mathbb{H}^{n+1}$, the map $x \rightarrow -\langle x \mid u \rangle$) is a time function. The geodesics are the intersections between $\text{dS}^{1,n}$ and 2-planes in $\mathbb{R}^{1,n+1}$. Two points x, y in $\text{dS}^{1,n}$ are causally related if and only if $\langle x \mid y \rangle \geq 1$.

Isometry group. The group of orientation preserving, time-orientation preserving isometries of $dS^{1,n}$ is $SO_0(1, n+1)$.

Duality with the hyperbolic space. For every x in $dS^{1,n}$, the intersection $x^\perp \cap \mathbb{H}^n$ is a totally geodesic hypersurface $\mathbb{H}(x)$ of hyperbolic space \mathbb{H}^{n+1} . More precisely, $\{y \in \mathbb{H}^{n+1} \mid \langle y \mid x \rangle > 0\}$ is a half-hyperbolic space bounded by $\mathbb{H}(x)$; in other words, $dS^{1,n}$ can be seen as the space of **transversely oriented** half hyperbolic spaces.

De Sitter space is the space of round disks in \mathbb{S}^n . Every $\mathbb{H}(x)$ is characterized by its conformal boundary $\partial\mathbb{H}(x)$ in $\partial\mathbb{H}^{n+1} \approx \mathbb{S}^n$. Therefore, and this will be important later, $dS^{1,n}$ can be identified with the space $\mathcal{B}(\mathbb{S}^n)$ of round disks in \mathbb{S}^n .

Klein model. The Klein model $\mathbb{DS}^{1,n}$ is the projection of $dS^{1,n}$ to $\mathbb{S}(\mathbb{R}^{1,n+1})$ *i.e.*

$$\mathbb{DS}^{1,n} := \{x \in \mathbb{S}(\mathbb{R}^{1,n+1}) \mid \langle x \mid x \rangle > 0\}.$$

This is the complement in $\mathbb{S}(\mathbb{R}^{1,n+1})$ of the closures of two Klein models \mathbb{H}_\pm^{n+1} :

$$\mathbb{H}_\pm^{n+1} := \{x \in \mathbb{S}(\mathbb{R}^{1,n+1}) \mid \langle x \mid x \rangle < 0, \pm t > 0\}.$$

Conformal model De Sitter space $dS^{1,n}$ is conformally equivalent to the domain $dS^{1,n} := \mathbb{S}^n \times]-\pi, +\pi[\subseteq \text{Ein}^{1,n}$. It immediately follows that $dS^{1,n}$ is a globally hyperbolic domain of $\text{Ein}^{1,n}$, hence globally hyperbolic. The achronal subsets are the graphs of 1-Lipschitz maps $f : \Lambda_0 \subseteq \mathbb{S}^n \rightarrow]-\pi, +\pi[$.

The boundary of $dS^{1,n}$ has two components: the component $\mathbb{S}^n \times \{-\pi\}$ is the *past conformal boundary* $\partial_- dS^{1,n}$, and the component $\mathbb{S}^n \times \{+\pi\}$ is the *future conformal boundary* $\partial_+ dS^{1,n}$. For every x in $dS^{1,n}$, the intersection between the future (resp. the past) of x in $\text{Ein}^{1,n}$ and $\partial_+ dS^{1,n}$ (resp. $\partial_- dS^{1,n}$) is a round ball $\mathcal{B}_+(x)$ (resp. $\mathcal{B}_-(x)$). This is another way — actually, **two** other ways — to identify $dS^{1,n}$ with $\mathcal{B}(\mathbb{S}^n)$. For each of them, the causality relation in $dS^{1,n}$ corresponds to the inclusion relation in $\mathcal{B}(\mathbb{S}^n)$.

3.4 Anti-de Sitter space

Anti-de Sitter space $\text{AdS}^{1,n}$ is the hypersurface $\{x \in \mathbb{R}^{2,n} \mid q_{2,n}(x) = -1\}$ endowed with the Lorentzian metric obtained by restriction of $q_{2,n}$. We use a coordinate system (u, v, x_1, \dots, x_n) such that:

$$q_{2,n}(x) := -u^2 - v^2 + x_1^2 + \dots + x_n^2.$$

We will also consider the coordinates $(r, \theta, x_1, \dots, x_n)$ with:

$$u = r \cos(\theta), v = r \sin(\theta).$$

We equip $\text{AdS}^{1,n}$ with the time orientation defined by the vector field $\frac{\partial}{\partial\theta}$, *i.e.* the time orientation such that the timelike vector field $\frac{\partial}{\partial\theta}$ is everywhere future oriented.

Observe the analogy with the definition of Hyperbolic space \mathbb{H}^n . Moreover, for every real number θ_0 , the subset $H_{\theta_0} := \{(r, \theta, x_1, \dots, x_n) / \theta = \theta_0\} \subset \mathbb{R}^{2,n}$ is a totally geodesic copy of \mathbb{H}^n embedded in $\text{AdS}^{1,n}$. More generally, the totally geodesic subspaces of dimension k in $\text{AdS}^{1,n}$ are the connected components of the intersections of $\text{AdS}^{1,n}$ with the linear subspaces of dimension $(k+1)$ in $\mathbb{R}^{2,n}$. In particular, geodesics are intersections with 2-planes.

Conformal model Anti-de Sitter space $\text{AdS}^{1,n}$ is conformally equivalent to the domain $\mathbb{D}^n \times \mathbb{S}^1 \subseteq \text{Ein}^{1,n}$, where \mathbb{D}^n is the open upper hemisphere of \mathbb{S}^n . The boundary of this domain is $\partial\mathbb{D}^n \times \mathbb{S}^1 \approx \mathbb{S}^{n-1} \times \mathbb{S}^1$, hence conformally isometric to the Einstein universe $\text{Ein}^{1,n-1}$ with one dimension less. In other words, $\text{AdS}^{1,n}$ is one of the two connected components of $\text{Ein}^{1,n} \setminus \text{Ein}^{1,n-1}$ for the natural conformal embedding of $\text{Ein}^{1,n-1}$ in $\text{Ein}^{1,n}$. In other words, one can see $\text{Ein}^{1,n-1}$ as the conformal boundary $\partial\text{AdS}^{1,n}$.

As $\text{Ein}^{1,n}$, $\text{AdS}^{1,n}$ is not strongly causal; it contains many timelike geodesic loops. But, its universal covering $\widetilde{\text{AdS}}^{1,n}$, conformally equivalent to $\mathbb{D}^n \times \mathbb{R}$, is strongly causal. Its conformal boundary is $\widetilde{\text{Ein}}^{1,n-1}$.

However, $\widetilde{\text{AdS}}^{1,n}$ is not globally hyperbolic: for example, diamonds $J^-(\bar{x}_0, t) \cap J^+(\bar{x}_0, 0)$ are not compact as soon as $t \geq 2\pi$.

Klein model. The Klein model $\text{ADS}^{1,n}$ is the projection of $\text{AdS}^{1,n}$ to $\mathbb{S}(\mathbb{R}^{2,n})$ *i.e.* :

$$\text{ADS}^{1,n} := \{x \in \mathbb{S}(\mathbb{R}^{2,n}) / \langle x | x \rangle < 0\}.$$

The topological boundary of $\text{ADS}^{1,n}$ in $\mathbb{S}(\mathbb{R}^{2,n})$ is the Klein model $\mathbb{S}(\mathcal{C}_n)$ of $\text{Ein}^{1,n-1}$. Observe that for any subset Λ of $\text{Ein}^{1,n-1} \approx \mathbb{S}(\mathcal{C}_n)$, the convex hull $\text{Conv}(\Lambda)$ is contained in the conformal compactification $\text{AdS}^{1,n} \cup \partial\text{AdS}^{1,n}$ if and only if Λ is achronal in $\text{Ein}^{1,n-1}$.

Affine domains. For every $x = \mathbb{S}(x)$ in $\text{ADS}^{1,n}$, we define the *affine domain* (also denoted by $U(x)$):

$$U(x) := \{y \in \text{ADS}^{1,n} / \langle x | y \rangle < 0\}.$$

In other words, $U(x)$ is the connected component of $\text{ADS}^{1,n} \setminus \mathbb{S}(x^\perp)$ containing x .

The boundary $\mathbb{S}(x^\perp) \cap \text{ADS}^{1,n}$ of $U(x)$ in $\text{ADS}^{1,n}$ has two components that are totally geodesic copies of the (Klein model of) hyperbolic space \mathbb{H}^n . One distinguishes the *past component* $H^-(x)$ and the *future component* $H^+(x)$ characterized by the following property: future oriented timelike geodesics enter $U(x)$ through $H^-(x)$ and exit through $H^+(x)$. They are also called *hyperplanes dual to x*, and we distinguish the hyperplane past-dual $H^-(x)$ from the hyperplane future-dual $H^+(x)$.

Every affine domain $U(x)$, seen as a subset of $\text{Ein}^{1,n}$, and lifted in $\widetilde{\text{AdS}}^{1,n} \approx \mathbb{D}^n \times \mathbb{R}$, is isometric to a region $\mathbb{D}^n \times]-\pi/2, +\pi/2[$.

Achronal subsets. The description of achronal subsets of $\widetilde{\text{AdS}}^{1,n}$ follows easily from the description of achronal subsets of $\widetilde{\text{Ein}}^{1,n}$: they are the graphs of 1-

Lipschitz maps from \mathbb{D}^n into \mathbb{R} . They are all contained in some affine domain as defined just previously.

Globally hyperbolic domains. Let S be a closed edgeless achronal subset of $\widetilde{\text{AdS}}^{1,n}$. In the conformal model, S is the graph of a 1-Lipschitz map $f_S : \mathbb{D}^n \rightarrow \mathbb{R}$: this map uniquely extends to the boundary $\partial\mathbb{D}^n$. The graph of this extension is a closed achronal edgeless subset ∂S of $\partial\widetilde{\text{AdS}}^{1,n}$, and the Cauchy development $\text{dev}(S)$ is the intersection $\widetilde{\text{AdS}}^{1,n} \cap E(\partial S)$ where $E(\partial S)$ is the invisible domain of ∂S in $\widetilde{\text{Ein}}^{1,n}$. It follows that $\text{dev}(S)$ is contained in an affine domain too. In particular, $\text{dev}(S)$ is *never* the entire anti-de Sitter space: this is another proof that $\widetilde{\text{AdS}}^{1,n}$ is *not* globally hyperbolic. Observe that ∂S is purely lightlike if and only if $f_S(\bar{x}) = d_n(\bar{x}, \bar{x}_\infty) + t_0$ where t_0 is a real number and \bar{x}_∞ some point in $\partial\mathbb{D}^n \subset \mathbb{S}^n$.

The space of timelike geodesics. The content of this paragraph is mostly extracted from [17, Section 4.5]. Timelike geodesics in $\text{AdS}^{1,n}$ are intersections between $\text{AdS}^{1,n} \subset \mathbb{R}^{2,n}$ and 2-planes P in $\mathbb{R}^{2,n}$ such that the restriction of $q_{2,n}$ to P is negative definite. The action of $\text{SO}_0(2, n)$ on negative 2-planes is transitive, and the stabilizer of the (u, v) -plane is $\text{SO}(2) \times \text{SO}(n)$. Therefore, the space of timelike geodesics is the symmetric space:

$$\mathcal{T}_{2n} := \text{SO}_0(2, n) / \text{SO}(2) \times \text{SO}(n).$$

Remark 3.3. Let S be a closed edgeless achronal subset of $\text{AdS}^{1,n}$ such that $\partial S \subset \text{Ein}^{1,n}$ is not purely lightlike. Then, every timelike geodesic of $\text{AdS}^{1,n}$ intersects $E(\partial S)$ (cf. Lemma 3.5 in [15]), and since $E(\partial S)$ is convex, this intersection is connected, *i.e.* is a single inextendible timelike geodesic of $E(\partial S)$. In other words, one can consider \mathcal{T}_{2n} as the space of timelike geodesics of $E(\partial S)$ for any CEA S .

\mathcal{T}_{2n} has dimension $2n$. We equip it with the Riemannian metric $g_{\mathcal{T}}$ induced by the Killing form of $\text{SO}_0(2, n)$. It is well known that \mathcal{T}_{2n} has nonpositive curvature, and rank 2: the maximal flats (*i.e.* totally geodesic embedded Euclidean subspaces) have dimension 2. It is also naturally Hermitian. More precisely: let $\mathcal{G} = \mathfrak{so}(2, n)$ be the Lie algebra of $G = \text{SO}_0(2, n)$, and let \mathcal{K} be the Lie algebra of the maximal compact subgroup $K := \text{SO}(2) \times \text{SO}(n)$. We have the Cartan decomposition:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp$$

where \mathcal{K}^\perp is the orthogonal of \mathcal{K} for the Killing form. Then, \mathcal{K}^\perp is naturally identified with the tangent space at the origin of G/K . The adjoint action of the $\text{SO}(2)$ term in the stabilizer defines a K -invariant complex structure on $\mathcal{K}^\perp \approx T_K(G/K)$ that propagates through left translations to a genuine complex structure J on $\mathcal{T}_{2n} = G/K$. Therefore, \mathcal{T}_{2n} is naturally equipped with a structure of n -dimensional complex manifold, together with a J -invariant Riemannian metric, *i.e.* a Kähler structure.

Here our purpose is to give another way to define this Kähler structure, starting from the anti-de Sitter space and clarifying the geometric nature of the associated symplectic form.

Let $\mathcal{U}^{1,n}$ be the space of future oriented timelike tangent vectors of norm -1 of $\text{AdS}^{1,n}$. Since the tangent space of $\text{AdS}^{1,n}$ at a point \mathbf{x} is naturally identified with \mathbf{x}^\perp , there is a natural identification between $\mathcal{U}^{1,n}$ and the pairs (\mathbf{x}, \mathbf{y}) of elements of $\text{AdS}^{1,n}$ satisfying $\mathbf{y} \in H^+(\mathbf{x})$. The tangent bundle of $\mathcal{U}^{1,n}$ at a point (\mathbf{x}, \mathbf{y}) is naturally identified with elements $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$ of $\mathbb{R}^{2,n} \times \mathbb{R}^{2,n}$ satisfying:

$$\begin{aligned} \langle \dot{\mathbf{x}} | \mathbf{x} \rangle &= 0, \\ \langle \dot{\mathbf{y}} | \mathbf{y} \rangle &= 0, \\ \langle \dot{\mathbf{x}} | \mathbf{y} \rangle + \langle \dot{\mathbf{y}} | \mathbf{x} \rangle &= 0. \end{aligned}$$

There is a canonical $\text{SO}_0(2, n)$ -invariant pseudo-Riemannian metric $\|\cdot\|$ on $\mathcal{U}^{1,n}$:

$$\|(\dot{\mathbf{x}}, \dot{\mathbf{y}})\| := q_{2,n}(\dot{\mathbf{x}}) + q_{2,n}(\dot{\mathbf{y}}).$$

A quick computation shows that $\|\cdot\|$ is Lorentzian. Moreover, it is preserved by the geodesic flow Φ^t on $\mathcal{U}^{1,n}$ that can be defined by:

$$\Phi^t(\mathbf{x}, \mathbf{y}) = (\cos(t)\mathbf{x} + \sin(t)\mathbf{y}, -\sin(t)\mathbf{x} + \cos(t)\mathbf{y}).$$

The Killing vector field generating Φ^t is $Z(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, -\mathbf{x})$, hence of $\|\cdot\|$ -norm -2 : the $\|\cdot\|$ -orthogonal Z^\perp is therefore spacelike.

Now \mathcal{T}_{2n} is naturally identified with the orbit space of Φ^t : we write elements of \mathcal{T}_{2n} as equivalence classes $[\mathbf{x}, \mathbf{y}]$ of the orbital relation on $\mathcal{U}^{1,n}$. We have a natural way to define an $\text{SO}_0(2, n)$ -invariant Riemannian metric $\|\cdot\|$ on \mathcal{T}_{2n} : for every $[\mathbf{x}, \mathbf{y}]$ and every tangent vector ξ at $[\mathbf{x}, \mathbf{y}]$, select a representative (\mathbf{x}, \mathbf{y}) of $[\mathbf{x}, \mathbf{y}]$ and the unique vector v tangent to \mathcal{T}_{2n} , orthogonal to $Z(\mathbf{x}, \mathbf{y})$, and projecting onto ξ . Define $\|(\xi)\|$ as $\|v\|$: since Φ is isometric, this value does not depend on the choice of (\mathbf{x}, \mathbf{y}) . This defines a Riemannian metric on \mathcal{T}_{2n} that happens to be (up to a constant factor) the metric on the symmetric space $\mathcal{T}_{2n} := \text{SO}_0(2, n)/\text{SO}(2) \times \text{SO}(n)$.

The symplectic form ω associated with the Kähler-Hermitian structure of \mathcal{T}_{2n} can be described as follows: the Liouville 1-form λ on $\mathcal{U}^{1,n}$ is defined by:

$$\lambda_{(\mathbf{x}, \mathbf{y})}(\dot{\mathbf{x}}, \dot{\mathbf{y}}) = -\langle \dot{\mathbf{x}} | \mathbf{y} \rangle = \langle \dot{\mathbf{y}} | \mathbf{x} \rangle.$$

The contact hyperplane, kernel of λ , is the orthogonal Z^\perp . The geodesic flow Φ^t is actually the Reeb flow for λ : we have $\lambda(Z) = +1$ and $L_Z\lambda = 0$, so the exterior derivative $\hat{\omega} = d\lambda$ is Φ^t -invariant and for any v in Z^\perp we have $\hat{\omega}(Z, v) = i_Z d\lambda(v) = i_Z d\lambda(v) + di_Z\lambda = L_Z\lambda = 0$. Hence we can define on \mathcal{T}_{2n} the 2-form $\omega(\xi, \xi') = \hat{\omega}(v, v')$ where v, v' are the lifts of ξ, ξ' above v, v' orthogonal to Z . This 2-form is closed and non-degenerate, *i.e.* a symplectic form on \mathcal{T}_{2n} .

Finally, we define the complex structure: first observe that there is a natural complex structure on each contact hyperplane $Z(\mathbf{x}, \mathbf{y})^\perp$: this hyperplane is identified with the set of pairs $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$ with $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ both in $\mathbf{x}^\perp \cap \mathbf{y}^\perp$. Then $(\dot{\mathbf{x}}, \dot{\mathbf{y}}) \mapsto (\dot{\mathbf{y}}, -\dot{\mathbf{x}})$ is an involution in $Z(\mathbf{x}, \mathbf{y})^\perp$, commuting with the geodesic flow, and induces the complex structure J on \mathcal{T}_{2n} . Observe that J satisfies $\omega(v, J(v)) = \|v\|$: it is a calibration between ω and $\|\cdot\|$.

Gauss map. Let S be a differentiable Cauchy hypersurface in a GH domain $E(\partial S)$. The *Gauss map* of S is the map $\nu : S \rightarrow \mathcal{T}_{2n}(\rho)$ that sends every element x of S to the unique timelike geodesic orthogonal to S at x .

Since every timelike geodesic intersects S at most once, the Gauss map is always injective. The image of the Gauss map is actually the set of timelike geodesics that are orthogonal to S . Since every timelike geodesic intersects S , it follows easily that the image of the Gauss map is closed, and that the Gauss map is actually an embedding.

Assume that S is the image of a smooth spacelike embedding $x : \Sigma \rightarrow \text{AdS}^{1,n}$. Then it induces a map $y : \Sigma \rightarrow \text{AdS}^{1,n}$ where $y(p)$ is the dual of the unique totally geodesic hypersurface tangent to S at $x(p)$, and $\nu(p) = [x(p), y(p)]$. Here we point out that the linear map $dx(\dot{p}) \rightarrow dy(\dot{p})$ is the shape operator B at $x(p)$, so that the second fundamental form is $\Pi(\dot{p}) = \langle dx(\dot{p}) | B(dx(\dot{p})) \rangle = \langle dx(\dot{p}) | dy(\dot{p}) \rangle$.

We also point out the following consequence of the equality $\langle dx(\dot{p}) | y(p) \rangle = 0$: *the image of the Gauss map is a Lagrangian submanifold of ω* . Conversely, let $\varphi : \Sigma \rightarrow \mathcal{T}_{2n}$ be a Lagrangian immersion for some n -dimensional simply connected manifold Σ . One can lift φ to some immersion $\hat{\varphi} : \Sigma \rightarrow \mathcal{U}^{1,n}$ orthogonal to Z : indeed, select a base point p_0 in Σ , and for any p in Σ let $\alpha : [0, 1] \rightarrow \Sigma$ and $\beta : [0, 1] \rightarrow \Sigma$ be two path with $\alpha(0) = \beta(0) = p_0$ and $\alpha(1) = \beta(1) = p$. Then, lift them to paths $\hat{\alpha}, \hat{\beta}$ in $\mathcal{U}^{1,n}$ orthogonal to Z and such that $[\hat{\alpha}(t)] = \varphi(\alpha(t))$ and $[\hat{\beta}(t)] = \varphi(\beta(t))$. Assume $\hat{\alpha}(1)$ and $\hat{\beta}(1)$ are both above $\alpha(1) = \beta(1) = p$, hence there is a real number t such that $\hat{\beta}(1) = \Phi^t(\hat{\alpha}(1))$. The loop obtained by composing $\hat{\alpha}$, the portion of the Φ -orbit between $\hat{\alpha}(1)$ and $\hat{\beta}(1)$ and the inverse of $\hat{\beta}$ is homotopically trivial, hence the boundary of a disk D . Then, since the integral of λ along the portion of Φ -orbit is t , we have:

$$\int_D \varphi^* \omega = \int_{\hat{\alpha}} \lambda + t - \int_{\hat{\beta}} \lambda.$$

Now observe that $\int_{\hat{\alpha}} \lambda = \int_{\hat{\beta}} \lambda = 0$, hence, since $\int_D \varphi^* \omega = 0$ (because φ is Lagrangian) we get $t = 0$. The equality $\hat{\alpha}(1) = \hat{\beta}(1)$ follows: we can define $\hat{\varphi}(p) = \hat{\alpha}(1)$.

The immersion $\hat{\varphi} : \Sigma \rightarrow \mathcal{U}^{1,n}$ is spacelike for \parallel (since it is orthogonal to Z). Write $\hat{\varphi}(p) = (x(p), y(p))$. It may happen that $p \mapsto x(p)$ is not an immersion at some point p_0 , but then, near p_0 , simply replace $\hat{\varphi}$ by $\hat{\varphi}_t = \Phi^t \circ \hat{\varphi}$ by any non zero real number t (it amounts to replacing $x(p)$ by $\cos(t)x(p) + \sin(t)y(p)$): after this modification, $p \mapsto x(p)$ becomes a spacelike immersion near p_0 , whose Gauss map is the restriction of φ near p_0 . Of course, this construction is local and we may fail to find a t valid over the entire Σ .

4 The classification of MGHC spacetimes

In this section, we present the classification of MGHC spacetimes with constant curvature. This classification has been initiated by G. Mess who introduced many ideas ([79]), and continued by several authors ([10, 23, 89]). This classification is almost complete, but there are some remaining questions.

4.1 The flat case

The simplest example of MGHC flat spacetime is the quotient of the entire Minkowski space by a discrete group Γ of translations, such that the translation vectors of Γ form a lattice in $\mathbb{R}^{0,n}$. These spacetimes are geodesically complete. They are isometric to the product $(\mathbb{T}^n, \bar{h}_0) \times (\mathbb{R}, -dt^2)$, where $(\mathbb{T}^n, \bar{h}_0)$ is a flat torus. We call them *translation spacetimes*.

The most fundamental example of MGHC flat spacetime is the quotient of the cone I_0^+ in $\mathbb{R}^{1,n}$ by a cocompact lattice Γ of $\text{SO}_0(1, n)$:

$$M_0(\Gamma) = \Gamma \backslash I_0^+.$$

The restriction of $-q_{1,n}$ to I_0^+ is a time function whose levels sets are Cauchy hypersurfaces — in particular, the hyperboloid model \mathbb{H}^n of the hyperbolic space which is the 1-level set.

We call this example the *standard conformally static example*.

Identify $\mathbb{R}^{1,n}$ with the domain $\text{Mink}^+(x_0)$ of $\widetilde{\text{Ein}}^{1,n}$. The closure of \mathbb{H}^n in $\widetilde{\text{Ein}}^{1,n}$ is then the union of \mathbb{H}^n with a closed edgeless achronal set $\partial\mathbb{H}^n \subseteq \mathcal{I}^+$. Then, I_0^+ coincides with the invisible domain $E(\partial\mathbb{H}^n) = \widetilde{\text{Ein}}^{1,n} \setminus (J^-(\partial\mathbb{H}^n) \cup J^+(\partial\mathbb{H}^n))$.

More generally, let Λ be a (non edgeless) closed achronal subset of $\widetilde{\text{Ein}}^{1,n}$ contained in the conformal boundary \mathcal{I}^+ of a Minkowski domain $\text{Mink}^+(x_0) \approx \mathbb{R}^{1,n}$, invariant by a torsionfree discrete subgroup Γ of $\text{Isom}(\mathbb{R}^{1,n})$. Then, $\Omega(\Lambda) = E(\Lambda) \cap \text{Mink}^+(x_0)$ is globally hyperbolic, Γ -invariant, and the action of Γ on $\Omega(\Lambda)$ is free and proper. The quotient $M_\Lambda(\Gamma)$ is a globally hyperbolic spacetime, but not necessarily Cauchy compact. The cosmological time of $\Omega(\Lambda)$ — that is the lift of the cosmological time of $M_\Lambda(\Gamma)$ — is regular. Moreover, if Λ contains no proper Γ -invariant closed subset, then $M_\Lambda(\Gamma)$ is maximal among flat GH spacetimes. Finally, $M_\Lambda(\Gamma)$ is future complete, in the sense that any future oriented timelike ray is geodesically complete.

We have observed that elements of \mathcal{I}^+ correspond to past-half spaces in $\mathbb{R}^{1,n}$, and $\Omega(\Lambda)$ is obtained by removing all the half-spaces corresponding to elements of Λ . It follows that $\Omega(\Lambda)$ is an intersection of half-spaces, hence convex. We recover the notion of *regular domain* as defined in [23]. The boundary of $\Omega(\Lambda)$ in $\mathbb{R}^{1,n}$ is very interesting: it is a CEA, that is, the graph of a 1-Lipschitz map f whose differential has norm 1 almost everywhere.

Observe that if we replace \mathcal{I}^+ by \mathcal{I}^- , the result will be a similar spacetime, but that is past complete and not future complete.

The following Theorem was proved by Mess in dimension $2 + 1$ ([79]), by Bonsante in the case where the holonomy is assumed to have a discrete cocompact linear part in $\text{SO}_0(1, n)$ ([23]), and independently and in full generality in [10]:

Theorem 4.1. *Up to finite coverings, every MGHC flat spacetime is isometric to either a translation spacetime, or a quotient $M_\Lambda(\Gamma)$, where Γ is a discrete subgroup of $\text{Isom}(\mathbb{R}^{1,2})$.*

We want to describe further $M_\Lambda(\Gamma)$ when it is Cauchy compact. Most of the following claims are non-trivial and we refer to [10] for their proofs. Notice first

that in this case, the achronal subset Λ of $\mathcal{I}^\pm \approx \mathbb{R} \times \mathbb{S}^{n-1}$ is necessarily a topological sphere, the graph of a continuous map $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$. The invisible domain $E(\Lambda)$ is then contained in $\text{Min}_+(x_0)$, hence we have $\Omega(\Lambda) = E(\Lambda)$.

Consider first the *proper* case, *i.e.* the case where the closure of $\Omega(\Lambda)$ does not contain any affine line of $\mathbb{R}^{1,n}$: Γ is then isomorphic to a cocompact lattice of $\text{SO}_0(1, n)$. More precisely, the linear part homomorphism $\text{Isom}(\mathbb{R}^{1,n}) \rightarrow \text{SO}_0(1, n)$ is faithful and has discrete and cocompact image. In this case, we call $M_\Lambda(\Gamma)$ a *standard spacetime*.

In general, excluding the particular case of *Misner spacetimes*, up to finite coverings, a MGHC spacetime is a twisted product over a standard spacetime by flat tori. In particular, Cauchy hypersurfaces are always finite covers of products of hyperbolic closed manifolds by tori. See [10] for more details.

There is also an interesting case of flat MGH spacetime, but not Cauchy compact: the *unipotent spacetimes*. A unipotent spacetime is the quotient of either a half-space in $\mathbb{R}^{1,n}$ bounded by a lightlike hyperplane, or the region between two parallel lightlike hyperplanes, by a discrete nilpotent group Γ whose linear part is a discrete subgroup of the stabilizer in $\text{SO}_0(1, n)$ of a point of $\partial\mathbb{H}^n$. For more details, see [10].

4.2 The dS case

In some way, MGHC Cauchy compact spacetimes locally modeled on $\text{dS}^{1,n}$ first appeared in a paper by Kulkarni and Pinkall (§3.4 of [74]), but the authors did not insist on the de Sitter nature of the spaces they were considering, and, presumably, were not aware of their interpretation as globally hyperbolic spacetimes. The fact that these examples give the complete list of MGHC de Sitter spacetimes was proved by K. Scannell ([89]), involving some ideas of Mess. There is also an alternative description in [6] which is the one we use here.

Unlike the other two cases (the flat and anti-de Sitter cases), locally de Sitter MGHC spacetimes are not in general quotients of open domains in $\text{dS}^{1,n}$.

The crucial point is that locally de Sitter MGHC spacetimes are in one-to-one correspondence with *closed Möbius manifolds*, *i.e.* closed manifolds locally modeled on the conformal sphere \mathbb{S}^n . This correspondence involves the identification of $\text{dS}^{1,n}$ with the space of round discs in \mathbb{S}^n . It goes as follows: let Σ be a closed manifold of dimension n , locally modeled on $(\mathbb{S}^n, \text{SO}_0(1, n+1))$. Let $\mathcal{D} : \tilde{\Sigma} \rightarrow \mathbb{S}^n$ be the developing map of this $(\mathbb{S}^n, \text{SO}_0(1, n+1))$ -structure, and let $\rho : \Gamma = \pi_1(\Sigma) \rightarrow \text{SO}_0(1, n+1)$ be the holonomy representation. Let $\mathcal{B}(\tilde{\Sigma})$ be the space of open domains B in $\tilde{\Sigma}$ such that the restriction of \mathcal{D} to B is a homeomorphism onto a round disc of \mathbb{S}^n . Then the action of Γ on $\mathcal{B}(\tilde{\Sigma})$ is free and proper: let $\mathcal{B}(\Sigma)$ be the quotient space. The developing map \mathcal{D} induces a Γ -equivariant local homeomorphism $\hat{\mathcal{D}} : \mathcal{B}(\tilde{\Sigma}) \rightarrow \mathcal{B}(\mathbb{S}^n) \approx \text{dS}^{1,n}$, hence a locally de Sitter structure on $\mathcal{B}(\Sigma)$. As a locally de Sitter manifold, $\mathcal{B}(\Sigma)$ is maximal globally hyperbolic.

Actually, we observed that there are two ways to identify $\text{dS}^{1,n}$ with $\mathcal{B}(\mathbb{S}^n)$: for one of them the spacetime $\mathcal{B}(\Sigma)$ is geodesically complete in the future, and for the other one, $\mathcal{B}(\Sigma)$ is geodesically complete in the past.

Theorem 4.2 (K. Scannell [89]). *Every maximal globally hyperbolic Cauchy compact locally de Sitter spacetime is isometric to the spacetime $M(\Sigma)$ associated with a $(\mathbb{S}^n, \text{SO}_0(1, n+1))$ -manifold Σ .*

Thurston observed that $(\mathbb{S}^2, \text{SO}_0(1, 3))$ -manifolds are in one-to-one correspondence with *hyperbolic ends*, *i.e.* hyperbolic 3-manifolds homeomorphic to $\Sigma \times [0, +\infty[$ with a concave boundary $\Sigma \times \{0\}$ and complete at the end $\Sigma \times \{+\infty\}$ (for more details, see for example [19]). This generalizes in any dimension: every locally de Sitter MGHC spacetime $M(\Sigma)$ has an associated dual hyperbolic manifold homeomorphic to $\Sigma \times [0, +\infty[$. See [6] for more details.

We point out particular elementary cases:

- *the elliptic case*: this is the case where Σ is the round sphere \mathbb{S}^n ; $M(\Sigma)$ is then the de Sitter space;

- *the parabolic case*: this is the case where Σ is a quotient of the flat conformal Euclidean space \mathbb{R}^n , *i.e.* the once punctured sphere. Then, in the geodesically future complete case, $M(\Sigma)$ is a quotient of the complement in $d\mathbb{S}^{1,n}$ of the past of a point in the future conformal boundary. It is dual to the hyperbolic end corresponding to one hyperbolic cusp. In particular, Cauchy surfaces are finite quotients of umbilic tori.

In the remaining non-elementary *hyperbolic case*, $M(\Gamma)$ has a regular cosmological time.

4.3 The AdS case

A convenient recent reference for the content of this section is [17]. Let $\tilde{\Lambda}$ be a closed edgeless achronal subset of $\partial \widetilde{\text{AdS}}^{1,n} \approx \widetilde{\text{Ein}}^{1,n-1}$ ($n \geq 2$). Consider it as a closed achronal subset of $\widetilde{\text{Ein}}^{1,n}$. Assume that it is not purely lightlike. Then, the invisible domain $E(\tilde{\Lambda})$ in $\widetilde{\text{Ein}}^{1,n}$ has two connected components: one contained in $\widetilde{\text{AdS}}^{1,n}$ and the other in the second anti-de Sitter component of $\widetilde{\text{Ein}}^{1,n} \setminus \widetilde{\text{Ein}}^{1,n-1}$. We call $\Omega(\tilde{\Lambda})$ the first component. We can write:

$$\Omega(\tilde{\Lambda}) =: \widetilde{\text{AdS}}^{1,n} \setminus \left(J^-(\tilde{\Lambda}) \cup J^+(\tilde{\Lambda}) \right)$$

We denote by $\Omega(\Lambda)$ the projection of $\Omega(\tilde{\Lambda})$ in $\text{AdS}^{1,n}$ (cf. Lemma 3.1).

The domains $\Omega(\Lambda)$ and $\Omega(\tilde{\Lambda})$ are isometric to each other. They have regular cosmological time, in particular, they are GH.

Let Γ be a torsionfree discrete subgroup of $\text{SO}_0(2, n)$ preserving Λ . Then, the action of Γ on $\Omega(\Lambda)$ is free and properly discontinuous, and preserves the cosmological time τ . The quotient $M_\Lambda(\Gamma) := \Gamma \backslash \Omega(\Lambda)$ is a MGH spacetime locally modeled on the anti-de Sitter space. We call it a *regular MGH anti-de Sitter spacetime*.

Theorem 4.3. *Every locally anti-de Sitter MGHC spacetime is isometric to a regular MGHC AdS spacetime.*

We now give a more detailed presentation of the geometric features of these spacetimes. In the conformal model, $\Omega(\tilde{\Lambda})$ is the region in $\mathbb{D}^n \times \mathbb{R}$ between the graphs of two 1-Lipschitz maps $f^\pm : \mathbb{D}^n \rightarrow \mathbb{R}$ that are extensions of the map $f_\Lambda : \partial\mathbb{D}^n \rightarrow \mathbb{R}$ whose graph is Λ . The graph of f^- (respectively f^+) is a closed achronal subset of $\widetilde{\text{AdS}}^{1,n}$ that we call the *lifted past* (respectively *future*) *horizon* of $\Omega(\tilde{\Lambda})$, and denote by $\tilde{\mathcal{H}}^-(\Lambda)$ (respectively $\tilde{\mathcal{H}}^+(\Lambda)$). The projections in $\text{AdS}^{1,n}$ of $\tilde{\mathcal{H}}^\pm(\Lambda)$ are called *past* and *future horizons* of $\Omega(\Lambda)$, and denoted by $\mathcal{H}^\pm(\Lambda)$.

Consider now Λ as a closed subset in $\mathbb{S}(\mathcal{C}_n) \subset \mathbb{S}(\mathbb{R}^{2,n})$, boundary of the Klein model $\mathbb{A}\text{DS}^{1,n}$: since Λ is achronal, the convex hull $\text{Conv}(\Lambda)$ is contained in the closure of $\mathbb{A}\text{DS}^{1,n}$. Actually, its intersection with $\mathbb{S}(\mathcal{C}_n)$ is precisely Λ ; in particular, $E(\Lambda)$ characterizes Λ . It happens ([6]) that $E(\Lambda)$ is the interior of the dual $\text{Conv}(\Lambda)^*$ — in particular, $E(\Lambda)$ contains the interior of $\text{Conv}(\Lambda)^*$.

The Fuchsian case. There is a particular case: the case where Λ is the boundary of a totally geodesic copy of \mathbb{H}^n in $\text{AdS}^{1,n}$. This means that Λ is the graph of an affine map. Then, $\text{Conv}(\Lambda)$ has empty interior, and the complement of Λ in $\text{Conv}(\Lambda)$ is the totally geodesic subspace bounded by Λ . The group Λ is then a discrete subgroup of a conjugate of $\text{SO}_0(1, n)$ in $\text{SO}_0(2, n)$. We call this case the *Fuchsian case*.

Past tight region. From now on, we assume that Λ is not Fuchsian. Its complement in the boundary $\partial\text{Conv}(\Lambda)$ has two connected components. Both are closed achronal subsets of $\text{AdS}^{1,n}$. More precisely, in the conformal model their lifts to $\widetilde{\text{AdS}}^{1,n}$ are graphs of 1-Lipschitz maps F^+, F^- from \mathbb{D}^n into \mathbb{R} such that

$$f^- \leq F^- \leq F^+ \leq f^+. \quad (1)$$

The graph of F^- is the *past component* $S^-(\Lambda)$ and the graph of F^+ is the *future component* $S^+(\Lambda)$. The region between the past horizon $\mathcal{H}^-(\Lambda)$ and the future component $S^+(\Lambda)$ is the *past tight region* and it is denoted by $E_0^-(\Lambda)$.

Since $E(\Lambda)$ and $\text{Conv}(\Lambda)$ are convex and dual to each other, for every element x in $S^-(\Lambda)$ (respectively $S^+(\Lambda)$) there is an element p of Λ or $\mathcal{H}^+(\Lambda)$ (respectively $\mathcal{H}^-(\Lambda)$) such that $H^-(p)$ (respectively $H^+(p)$) is a support hyperplane for $S^-(\Lambda)$ (respectively $S^+(\Lambda)$) at x : these support hyperplanes are either totally geodesic copies of \mathbb{H}^n (if $p \in \text{AdS}^{1,n}$) or degenerate (if $p \in \Lambda$).

Similarly, at every element x of $\mathcal{H}^-(\Lambda)$ (respectively $\mathcal{H}^+(\Lambda)$) there is a support hyperplane $H^-(p)$ (respectively $H^+(p)$) where p is an element of $S^+(\Lambda) \cup \Lambda$ (respectively $S^-(\Lambda) \cup \Lambda$). See Figure 2.

Cosmological lines. The past tight region $E_0^-(\Lambda)$ is precisely the region where the cosmological time τ takes value $< \pi/2$:

$$E_0^-(\Lambda) = \{\tau < \pi/2\}.$$

For every x in $E_0^-(\Lambda)$ there is a unique *realizing geodesic* for x . More precisely, there is one and only one element $r(x)$ in the past horizon $\mathcal{H}^-(\Lambda)$ — called the *cosmological retract* of x — such that the segment $]r(x), x]$ is a timelike geodesic whose Lorentzian length is precisely the cosmological time $\tau(x)$.

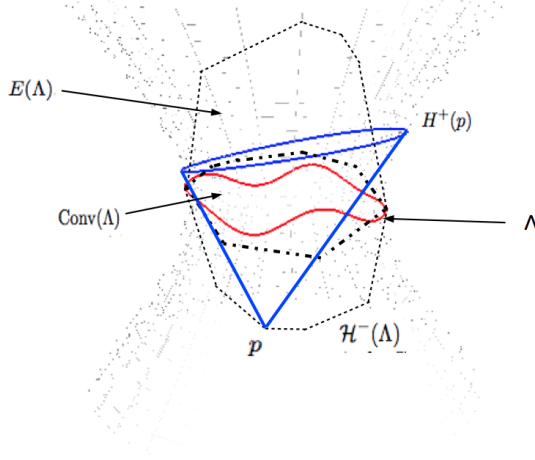


Figure 2: The global situation. The dotted hyperboloid represents the boundary of an affine domain of $\text{AdS}^{1,n}$ containing the invisible domain $E(\Lambda)$. The limset Λ is represented by a topological circle turning around the hyperboloid, and $\text{Conv}(\Lambda)^\circ$ is a convex subset inside the (dual) convex subset $E(\Lambda)$. The future-dual plane $H^+(p)$ for p in the past horizon $\mathcal{H}^-(\Lambda)$ is a support hyperplane of $S^+(\Lambda)$.

The restriction of τ to $E_0^-(\Lambda)$ is $C^{1,1}$, *i.e.* it is differentiable with locally Lipschitz derivative) (see [79, 19] for the case $n = 2$, and [17] for the general case). The realizing geodesics are orthogonal to the level sets of τ . The space of realizing geodesics is then an n -dimensional closed embedded Lipschitz submanifold. We denote it by $\Sigma(\Lambda)$ and call it the *space of cosmological lines*.

The initial singularity. For every p in $\mathcal{H}^-(\Lambda)$, $H^+(p)$ is a support hyperplane for $\text{Conv}(\Lambda)$, but it could be at a point in Λ . Elements of $\mathcal{H}^-(\Lambda)$ that are dual to support hyperplanes for $\text{Conv}(\Lambda)$ at a point inside $\text{AdS}^{1,n}$, *i.e.* in $S^+(\Lambda)$, form an interesting subset of $\mathcal{H}^-(\Lambda)$, the *initial singularity set* (cf. [19]).

Split AdS-spacetimes. Consider the sum $\mathbb{R}^{1,k} \oplus \mathbb{R}^{1,\ell}$ with $k + \ell = n$, equipped with the quadratic form $q_{1,k} + q_{1,\ell}$. It is isometric to $(\mathbb{R}^{2,n}, q_{2,n})$. It provides an embedding of $\text{SO}_0(1, k) \times \text{SO}_0(1, \ell)$ in $\text{SO}_0(2, n)$. It preserves in $\partial \text{AdS}^{1,n}$ an achronal topological sphere $\Lambda_{k,\ell}$, link of two spheres \mathbb{S}^{k-1} and $\mathbb{S}^{\ell-1}$, where every point in \mathbb{S}^{k-1} is linked to every point in $\mathbb{S}^{\ell-1}$ by the unique future oriented lightlike segment in $\partial \text{AdS}^{1,n}$ going from the point in \mathbb{S}^{k-1} towards the point in $\mathbb{S}^{\ell-1}$. The associated domain $E(\Lambda_{k,\ell})$ is then globally hyperbolic, and for every cocompact lattice Γ in $\text{SO}_0(1, k) \times \text{SO}_0(1, \ell)$ the quotient $\Gamma \backslash E(\Lambda_{k,\ell})$ is a MGHC AdS-spacetime, called a *split AdS-spacetime*. Observe that the achronal sphere $\Lambda_{k,\ell}$ is not acausal. A natural conjecture is that any MGHC AdS-spacetimes either is a split AdS-spacetime, or has an associated achronal subset Λ that is acausal. For more details, see [48] or [17].

4.4 Invisible domains

Let Λ be a non-purely lightlike closed achronal subset of $\partial \text{AdS}^{1,n} \approx \text{Ein}^{1,n-1}$, but not necessarily edgeless: Λ is the graph of a 1-Lipschitz map $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a closed subset of \mathbb{S}^{n-1} . Assume that it is preserved by a torsionfree discrete subgroup Γ of $\text{SO}_0(2, n)$. We consider it as a subset of $\text{Ein}^{1,n}$: let $E(\Lambda)$ be the invisible domain in $\text{Ein}^{1,n}$, $D(\Lambda)$ be the restriction of $E(\Lambda)$ to $\text{Ein}^{1,n-1}$ and $\Omega(\Lambda)$ be its restriction to $\text{AdS}^{1,n}$. Then, the quotient $M(\Lambda) := \Gamma \backslash \Omega(\Lambda)$ is a strongly causal spacetime, diffeomorphic to a product $S \times \mathbb{R}$, but not globally hyperbolic if $\Lambda_0 \neq \mathbb{S}^{n-1}$: some causal curves may escape from it by some point in $\Gamma \backslash D(\Lambda)$.

Actually, one can consider $\Gamma \backslash D(\Lambda)$ as the conformal boundary of $M(\Lambda)$. The past of $\Gamma \backslash D(\Lambda)$ in $M(\Lambda)$ is the quotient of $J^-(D(\Lambda)) \cap \Omega(\Lambda)$ by Γ . Its complement in $M(\Lambda)$ is globally hyperbolic: it is the quotient of $\Omega(\Lambda^+)$ by Γ , where Λ^+ is the future boundary of $D(\Lambda)$, *i.e.* the graph of the maximal 1-Lipschitz extension of f on \mathbb{S}^{n-1} .

In Section 6.3 we will give a geometrico-relativistic interpretation of this feature as idealistic models of spacetimes containing black-holes.

5 Discrete groups of isometries

In this section, we explore how the notion of global hyperbolicity may help to understand actions of groups of isometries on the model spacetimes, in a way similar to the traditional theory of Kleinian groups. We focus on the notion of *achronal* subgroups, and show how they allow to develop a theory similar to the classical theory of groups of isometries of the hyperbolic space \mathbb{H}^n , in particular, that such a subgroup has always a canonical *limit set* in the conformal boundary.

We then clarify the relation between achronal subgroups and *Anosov representations* in the context of isometry groups of spacetimes of constant curvature.

Definition 5.1. Let Γ be a group of isometries of a spacetime (M, g) . An orbit $\Gamma.p$ is *achronal* if for every non-trivial element γ of Γ the iterate γp is not in the strict future or past $I^\pm(p)$ of p . The group Γ itself is *achronal* if it admits an achronal orbit.

In some way, one may extend this notion to any pseudo-Riemannian metric, and it happens that isometry groups of Riemannian spaces are always achronal since there is no timelike tangent vectors in this case.

Remark 5.2. We do not assume Γ to be a discrete subgroup, even if it is the main case we have in mind. Discreteness is not necessary for this basic theory of limit sets.

5.1 The de Sitter case

In this case Γ is a subgroup of $\text{SO}_0(1, n + 1)$, hence can be seen as a group of isometries of \mathbb{H}^{n+1} . The key observation is that, in our framework, Γ is always achronal.

Indeed, the boundary of $dS^{1,n}$ in $\widetilde{\text{Ein}}^{1,n}$ is the union of two spacelike spheres: the past and future conformal boundaries $\partial_{\pm} dS^{1,n}$ which are achronal. Actually, every component of $\partial dS^{1,n}$ is conformally equivalent to $\mathbb{S}^n \approx \partial \mathbb{H}^{n+1}$, and the classical theory of limit sets for groups of hyperbolic isometries provides a limit set Λ_{Γ} , usually defined as the set of points in $\partial \mathbb{H}^{n+1}$ that are accumulation points of any orbit in \mathbb{H}^{n+1} .

Here, we have two copies of the limit set in $\partial dS^{1,n}$: one Λ_{Γ}^+ in $\partial_+ dS^{1,n}$, and the other one Λ_{Γ}^- in $\partial_- dS^{1,n}$. Each of them is acausal in $\widetilde{\text{Ein}}^{1,n}$, hence we can define as before the (non-empty) regions $\Omega_{\pm}(\Lambda_{\Gamma})$, intersections between $dS^{1,n}$ and the invisible domain $E(\Lambda_{\Gamma}^{\pm})$ in $\widetilde{\text{Ein}}^{1,n}$. Observe that $\Omega_{\pm}(\Lambda_{\Gamma})$ is globally hyperbolic, since $dS^{1,n}$ and $E(\Lambda_{\Gamma}^{\pm})$ are GH domains in $\widetilde{\text{Ein}}^{1,n}$. The union $\Omega(\Lambda_{\Gamma})$ of $\Omega_+(\Lambda_{\Gamma})$ and $\Omega_-(\Lambda_{\Gamma})$ can be characterized as the interior of the set of points in $dS^{1,n}$ with achronal Γ -orbit, and $\Lambda_{\Gamma} = \Lambda_{\Gamma}^+ \cup \Lambda_{\Gamma}^-$ is the set of non-trivial accumulation points of **achronal** orbits $\Gamma.p$ in $dS^{1,n}$.

$$\Lambda_{\Gamma} := \overline{\Gamma.p} \setminus \Gamma p, \text{ for any } p \text{ with achronal } \Gamma\text{-orbit.}$$

Λ_{Γ} is empty if and only if Γ is relatively compact in $\text{SO}_0(1, n)$, in which case one can take $\Omega(\emptyset)$ as the entire de Sitter space — actually, in this case Γ is conjugate to a subgroup of the maximal subgroup $\text{SO}(n)$ of $\text{SO}_0(1, n)$: it preserves a foliation by umbilical spacelike spheres: every orbit is acausal, and the action on $dS^{1,n}$ is proper.

When Γ is discrete, $\Omega(\Lambda_{\Gamma})$ can be seen as the natural domain of $dS^{1,n}$ on which Γ acts properly discontinuously.

Remark 5.3. Consider the Klein model in $\mathbb{P}(\mathbb{R}^{1,n+1})$: the convex hull $\text{Conv}(\Lambda_{\Gamma})$ is contained in the Klein model $\mathbb{P}(\{q_{1,n+1} < 0\})$ of the hyperbolic space, and $\Omega(\Lambda_{\Gamma})$ is the intersection between the Klein model $\mathbb{P}(\{q_{1,n+1} > 0\})$ of the de Sitter space and the interior of the dual $\text{Conv}(\Lambda_{\Gamma})^*$. Actually, it was already observed in Thurston's book on the geometry of topology of three-manifolds ([96]) that the action of Γ is proper not only on \mathbb{H}^n , but also in the bigger region $\text{Conv}(\Lambda_{\Gamma})^*$.

5.2 The flat case

In the flat case, the subgroup Γ is not always achronal — for example, consider the case of a cocompact lattice in the group of translations of $\mathbb{R}^{1,n}$. Nevertheless, as in the de Sitter case, one can show that achronal subgroups are subgroups of $\text{Isom}(\mathbb{R}^{1,n})$ that preserve a closed achronal subset Λ_{Γ} in the Penrose conformal boundary $\partial \text{Mink}^+(x_0) = C(x_0)$ of a point x_0 in $\text{Ein}^{1,n}$. More precisely, such a group always preserves the “spatial infinity” $i_0 = x_0$, hence we have to be more precise and distinguish several cases:

– **Case (1):** Γ is relatively compact: then Γ is achronal and no Γ -orbit accumulates at the conformal boundary; the limit set Λ_{Γ} is then the empty set, and if Γ is discrete, the action of Γ on $\mathbb{R}^{1,n} = \Omega(\emptyset)$ is proper.

– **Case (2):** Γ is not relatively compact, but its linear part is relatively compact in $\text{SO}_0(1, n)$: then, Γ preserves a flat euclidian metric in $\mathbb{R}^{1,n}$; hence a foliation by

parallel spacelike hyperplanes. Every Γ -orbit is achronal, and accumulates at the spatial infinity i_0 , that we define to be the limit set. If Γ is discrete, the action on the entire Minkowski space, which is the invisible domain for x_0 , is proper. Observe that according to the Bieberbach Theorem, if Γ is discrete, then, up to finite index, Γ is a group of spacelike translations.

– **Case (3)**: the linear part of Γ is not relatively compact in $\mathrm{SO}_0(1, n)$ and admits a limit set Λ_Γ^0 in \mathbb{S}^{n-1} . Then, Γ is achronal if and only if it preserves an achronal set in \mathcal{I}^+ . The set of accumulation points in $\mathcal{I}^\pm \approx \mathbb{S}^{n-1} \times \mathbb{R}$ is a limit set Λ_Γ^\pm , which is the graph of a map $f : \Lambda_\Gamma^0 \rightarrow \mathbb{R}$ (not necessarily Lipschitz). The interior of the set of achronal Γ -orbits in the set $\Omega(\Lambda_\Gamma)$, which is the union of the two globally hyperbolic domains $\Omega(\Lambda_\Gamma^\pm)$, on which, if it is discrete, Γ acts properly.

5.3 The anti-de Sitter case

In the anti-de Sitter case, we still have the situation that subgroups of $\mathrm{SO}_0(2, n)$ may fail to be achronal. The criterion is simpler than in the flat case: Γ is achronal if and only if it preserves a closed achronal subset in $\partial \mathrm{AdS}^{1, n}$. Then, one can define a limit set Λ_Γ as the set of accumulation points of achronal orbits. This is a closed achronal subset, contained in any Γ -invariant closed achronal subset of $\partial \mathrm{AdS}^{1, n}$. The interior of the set of achronal Γ -orbits in $\mathrm{AdS}^{1, n}$ is the invisible domain $\Omega(\Lambda) = E(\Lambda) \cap \mathrm{AdS}^{1, n}$, on which Γ acts properly, even if the quotient is not always globally hyperbolic (it is globally hyperbolic if and only if Λ_Γ is edgeless, *i.e.* is the graph of a 1-Lipschitz map defined on the entire sphere \mathbb{S}^{n-1}).

There is a very interesting criterion for achronality, involving a particular element of the bounded cohomology of $\mathrm{SO}_0(2, n)$. Recall the exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SO}}_0(2, n) \rightarrow \mathrm{SO}_0(2, n) \rightarrow 0$$

where the \mathbb{Z} is the cyclic group generated by the Galois automorphism δ . Then, there is a canonical non-algebraic section $s : \mathrm{SO}_0(2, n) \rightarrow \widetilde{\mathrm{SO}}_0(2, n)$: loosely speaking, $s(g)$ is the only lift of $g \in \mathrm{SO}_0(2, n)$ such that for every affine domain U in $\widetilde{\mathrm{AdS}}^{1, n}$, the intersection $U \cap s(g)U$ is never empty. Then, for every g_1, g_2 in $\mathrm{SO}_0(2, n)$, let $c(g_1, g_2)$ be the integer in $\{-1, 0, +1\}$ characterized by $s(g_1 g_2) = \delta^{c(g_1, g_2)} s(g_1) s(g_2)$. Then, c is a cocycle, representing an bounded cohomology class in $H_b^2(\mathrm{SO}_0(2, n), \mathbb{Z})$, called the *bounded Euler class*. Then, for every subgroup Γ of $\mathrm{SO}_0(2, n)$ the restriction of c to Γ is an element of $H_b^2(\Gamma, \mathbb{Z})$, called the bounded Euler class of Γ and denoted by $\mathrm{eu}_b(\Gamma)$.

Theorem 5.4 ([17]). *A subgroup Γ of $\mathrm{SO}_0(2, n)$ preserves a closed achronal subset of $\partial \mathrm{AdS}^{1, n}$ if and only if $\mathrm{eu}_b(\Gamma) = 0$.*

5.4 Anosov representations

Recall that for any Lie group G , we denote by $\mathrm{Rep}(\Gamma, G)$ the modular space of representations of Γ in G up to conjugacy in the target G .

$$\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G.$$

In this section, we restrict to the case where Γ is a Gromov hyperbolic group. We denote by $\partial\Gamma$ the Gromov boundary of Γ , and by $(\widetilde{U\Gamma}, \tilde{\phi}^t)$ the *geodesic flow of Γ* (see [33, 81]): there is a proper cocompact action of Γ on $\widetilde{U\Gamma}$ commuting with $\tilde{\phi}^t$ such that the induced flow ϕ^t on the quotient space $U\Gamma := \Gamma \backslash \widetilde{U\Gamma}$ has a hyperbolic behavior. In particular, there are two Γ -equivariant maps $\xi^\pm : \widetilde{U\Gamma} \rightarrow \partial\Gamma$ which are constant along the orbits of $\tilde{\phi}^t$; more precisely, one can see $\xi^+(p)$ (respectively $\xi^-(p)$) as the limit of $\tilde{\phi}^t(p)$ for $t \rightarrow +\infty$ (respectively for $t \rightarrow -\infty$). There is also a map ν^+ defined on $\widetilde{U\Gamma}$ that associates with every p a metric $\nu^+(p)$ in a neighborhood of $\xi^+(p)$, such that for every γ in Γ , the action of γ on $\partial\Gamma$ is an isometry between the metric $\nu^+(p)$ near $\xi^+(p)$ and the metric $\nu^+(\gamma.p)$ near $\xi^+(\gamma.p)$. Moreover, $\nu^+(\tilde{\phi}^t(p))$ increases exponentially with t .

Let X be a manifold on which G acts analytically, typically, a homogeneous manifold G/H . A representation $\rho : \Gamma \rightarrow G$ is (G, X) -Anosov if the properties of the geodesic flow still hold when we replace $\partial\Gamma$ by X . More precisely:

- there is a continuous equivariant map $f : \partial\Gamma \rightarrow X$, where $\partial\Gamma$ is the Gromov boundary of Γ ,
- there is a continuous Γ -equivariant family of metrics $\nu_\rho^+(p)$ parameterized by $\widetilde{U\Gamma}$, so that $\nu_\rho^+(p)$ is a metric on X near $f(\xi^+(p))$ that increases exponentially along the orbits of $\tilde{\phi}^t$.

For more details or other presentations of the notion of Anosov representations, see [65, 66, 61] or the section 2.1 of [26] or also the article [30].

The notion of Anosov representations was introduced by Labourie ([75]). It has several interesting features:

- Anosov representations form an open domain $\text{Rep}_{an}(\Gamma, G)$ of $\text{Rep}(\Gamma, G)$,
- Anosov representations are always faithful with discrete image.

Labourie proved in particular that for any $n \geq 2$, and in the case where Γ is the fundamental group of a closed surface, one connected component (the so-called *Hitchin component*) of $\text{Rep}(\Gamma, \text{PSL}(n, \mathbb{R}))$ is made of $(\text{PSL}(n, \mathbb{R}), \mathcal{F}_n)$ -Anosov representations, where \mathcal{F}_n is the variety of complete flags in \mathbb{R}^n .

The de Sitter case. Once more, de Sitter space is the dual of hyperbolic space \mathbb{H}^{n+1} . It is well-known (see [65, 30]) that for any subgroup Γ of $\text{SO}_0(1, n+1)$, the inclusion $\Gamma \subset \text{SO}_0(1, n+1)$ is $(\text{SO}_0(1, n+1), \mathbb{S}^n)$ -Anosov if and only if it is convex cocompact.

A particular interesting family of convex cocompact subgroups are the *quasi-Fuchsian subgroups*, i.e. discrete subgroups isomorphic to a cocompact lattice of $\text{SO}_0(1, n+1)$ such that the limit set Λ_Γ is a topologically embedded sphere in $\partial\mathbb{H}^{n+1}$.

We point out here an interesting characterization of convex cocompact subgroups appearing in our framework: *a non-elementary discrete subgroup Γ of $\text{SO}_0(1, n+1)$ is convex cocompact if and only if the locally de Sitter globally hyperbolic domain $\Gamma \backslash \Omega_\pm(\Lambda_\Gamma)$ is spatially compact.*

In other words, *one can characterize $(\text{SO}_0(1, n+1), \mathbb{S}^n)$ -Anosov representations as the holonomy representations of MGHC de Sitter spacetimes that are quotients of hyperbolic domains of $\text{dS}^{1,n}$* (observe that this last condition is restrictive: there

are closed Möbius manifolds whose developing map is not injective; therefore their associated MGHC de Sitter spacetimes have also a non-injective developing map).

Remark 5.5. Unlike the other cases (see below), $\text{Rep}_{an}(\Gamma, \text{SO}_0(1, n+1))$ is **not** a closed subset of $\text{Rep}(\Gamma, \text{SO}_0(1, n+1))$. Indeed, quasi-Fuchsian subgroups can be continuously deformed to the trivial subgroup. It follows that the holonomy representation of a MGHC de Sitter spacetime, even if it is Gromov hyperbolic, is not necessarily $(\text{SO}_0(1, n+1), \mathbb{S}^n)$ -Anosov.

The anti-de Sitter case. The de Sitter case has been fully treated in [15]:

Theorem 5.6 (Theorem 1.2 in [15]). *Let Γ be the fundamental group of a closed manifold of dimension n . Assume that Γ is Gromov hyperbolic. Then, a representation $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$ is $(\text{SO}_0(2, n), \text{Ein}^{1, n-1})$ -Anosov if and only if it preserves a closed edgeless **acausal** subset Λ of $\text{Ein}^{1, n-1}$.*

Remark 5.7. If Γ is a discrete subgroup acting on $\text{AdS}^{1, n}$ and preserving a closed edgeless acausal subset Λ of $\text{Ein}^{1, n-1}$ such that the quotient $\Gamma \backslash \Omega(\Lambda)$ is spatially compact, then it is Gromov hyperbolic (see Section 8.3.2 in [15]). Therefore, $(\text{SO}_0(2, n), \text{Ein}^{1, n-1})$ -Anosov representations are precisely holonomy representations of MGHC AdS-spacetimes whose limit set is acausal, and not simply achronal.

Remark 5.8. Split AdS-spacetimes have a fundamental group isomorphic to a lattice of $\text{SO}_0(1, k) \times \text{SO}_0(1, \ell)$, hence are not Gromov hyperbolic. We already mentioned that they may be exactly the MGHC AdS-spacetimes with non-acausal limit set.

Finally, we have the following:

Theorem 5.9 ([17]). *Let Γ be a Gromov hyperbolic group, isomorphic to the fundamental group of a closed n -dimensional manifold. Then, if not empty, the space $\text{Rep}_{an}(\Gamma, \text{SO}_0(2, n))$ of Anosov representations is open and closed in the space $\text{Rep}(\Gamma, \text{SO}_0(2, n))$. In particular, it is a union of components of $\text{Rep}(\Gamma, \text{SO}_0(2, n))$.*

One of the main intermediate steps in the proof of this theorem is the following result: if $\Gamma \subset \text{SO}_0(2, n)$ is a Gromov hyperbolic group and the holonomy group of a MGHC $\text{AdS}^{1, n}$ -spacetime, then any achronal Γ -invariant closed edgeless achronal subset of $\partial \text{AdS}^{1, n}$ is automatically acausal.

The flat case. This is the only case that has not been yet completely studied. The only published result is the work of S. Ghosh ([52]): let Γ be a discrete subgroup of $\text{Isom}(\mathbb{R}^{1, 2})$, acting properly discontinuously on $\mathbb{R}^{1, 2}$, and admitting as linear part a convex cocompact subgroup of $\text{SO}_0(1, 2)$. Then, the inclusion $\Gamma \subset \text{Isom}(\mathbb{R}^{1, 2})$ is $(\text{Isom}(\mathbb{R}^{1, 2}), \mathcal{I}^+)$ -Anosov. This result is the starting point for the construction of a certain metric on the space of such representations, involving the thermodynamical formalism, the *pressure metric*.

We propose here the following conjecture: *let Γ be a Gromov hyperbolic group. Then a representation $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{R}^{1, n})$ is $(\text{Isom}(\mathbb{R}^{1, n}), \mathcal{I}^+)$ -Anosov if and only if it is achronal and the associated globally hyperbolic domain $\Omega(\Lambda)$ is proper.*³

³Recall that $\Omega(\Lambda)$ is proper if its closure contains no affine line.

These representations should also be characterized by the following property: *their linear part $L_\rho : \Gamma \rightarrow \mathrm{SO}_0(1, n)$ is faithful and convex cocompact.*

6 The three-dimensional case: links with Teichmüller space

In this section, we describe the several connections between the Teichmüller space and MGHC $\mathrm{AdS}^{1,2}$ -spacetimes revealed in Mess's work ([79, 5]) and followers. There are also similar links with MGHC spacetimes of constant curvature 0 or +1, but we have decided to focus on the AdS case. For more information on this topic, see [31, 19, 90].

6.1 Anti-de Sitter space as a space of matrices

An alternative description of $\mathrm{AdS}^{1,2}$. Consider the linear space $\mathrm{Mat}(2, \mathbb{R})$ of two-by-two matrices with real coefficients, equipped with the quadratic form $-\det$. This space is isometric to $(\mathbb{R}^{2,2}, q_{2,2})$, hence $\mathrm{AdS}^{1,2}$ is naturally identified with the space of matrices of determinant 1, *i.e.* $\mathrm{SL}(2, \mathbb{R})$.

We have also the formula $-\det(A)\mathrm{Id} = A\hat{A} = \hat{A}A$ where Id is the identity matrix and \hat{A} the transpose of the matrix of cofactors. Therefore the bilinear form associated with $-\det$ is:

$$\langle X | Y \rangle = \mathrm{Tr}(\hat{X}Y)/2 = \mathrm{Tr}(X\hat{Y})/2$$

where Tr denotes the trace.

The isometry group. The actions of $\mathrm{SL}(2, \mathbb{R})$ on itself by left and right translations are isometric, and they provide a natural identification between $\mathrm{SO}_0(2, 2)$ and the quotient of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ by the group of order two generated by $(-\mathrm{Id}, -\mathrm{Id})$. Therefore, a representation $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, 2)$ splits in two representations ρ_L, ρ_R from Γ into $\mathrm{PSL}(2, \mathbb{R})$.

The Minkowski space and the hyperbolic plane. Let M be the 3-dimensional linear subspace of $\mathrm{Mat}(2, \mathbb{R})$ consisting of matrices with null trace. This is the tangent space of $\mathrm{SL}(2, \mathbb{R})$ at the identity map. The restriction of $-\det$ to M has index $(1, 2)$, hence $(M, -\det)$ is a model for $\mathbb{R}^{1,2}$. Observe also that for X in M , $-\det(X)$ coincides with $\mathrm{Tr}(X^2)/2$ in M . In particular, the hyperbolic plane \mathbb{H}^2 is naturally identified with the space of matrices of determinant 1 and trace 0, therefore, with the space of complex structures J on \mathbb{R}^2 (since $J^2 + \mathrm{Id} = 0$). In this model, the isometric action of an element A of $\mathrm{SL}(2, \mathbb{R})$ is the action by conjugacy $J \mapsto AJA^{-1}$.

The Einstein spacetime $\mathrm{Ein}^{1,1}$. The Klein model of the boundary of $\mathrm{AdS}^{1,2}$ is the projectivization of the $-\det$ null cone in $\mathrm{Mat}(2, \mathbb{R})$, *i.e.* of non zero non-invertible matrices. Such a matrix determines two lines in \mathbb{R}^2 : its image and its kernel; moreover the left action $\mathrm{PSL}(2, \mathbb{R})$ preserves the kernel and the right action $\mathrm{PSL}(2, \mathbb{R})$ preserves the image. Therefore, there is a canonical identification

of $\overline{\text{Ein}}^{1,1}$ with $\mathbb{RP}^1 \times \mathbb{RP}^1$, where the left factor of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ acts trivially on the left factor \mathbb{RP}^1 , and the right factor acts trivially on the right factor \mathbb{RP}^1 . The lightlike geodesics of $\overline{\text{Ein}}^{1,2}$ are the factors $\{*\} \times \mathbb{RP}^1$ and the factors $\mathbb{RP}^1 \times \{*\}$. The image of the isotropic cone $\mathbb{P}(\mathcal{C}_2)$ in $\mathbb{P}(\mathbb{R}^{2,2})$ is also ruled by two families of lines: they are these two families of lightlike geodesics.

This description of $\overline{\text{Ein}}^{1,1}$ as a product of two circles is different from the analogous decomposition for its double cover $\text{Ein}^{1,1}$, which was defined as the product $\mathbb{S}^1 \times \mathbb{S}^1$ equipped with the metric $\bar{g}_1 - \bar{g}_1$. The factors $\{*\} \times \mathbb{RP}^1$ and $\mathbb{RP}^1 \times \{*\}$ can be seen, locally, as the diagonal and anti-diagonal in $\mathbb{S}^1 \times \mathbb{S}^1$.

Achronal subsets of $\text{Ein}^{1,1}$. Every closed acausal subset of $\overline{\text{Ein}}^{1,1}$ intersects every circle $\mathbb{RP}^1 \times \{*\}$ and $\{*\} \times \mathbb{RP}^1$ in at most one point. Therefore, these sets are graphs of increasing maps from \mathbb{RP}^1 into \mathbb{RP}^1 . In particular, closed edgeless acausal subsets are precisely the graphs of the homeomorphisms of \mathbb{RP}^1 .

At the limit, every achronal subset Λ is a *generalized graph of a semi-conjugacy*. By this we mean that Λ is the union of the graph of a non-decreasing map $f : \Lambda_0 \rightarrow \mathbb{RP}^1$ (Λ_0 is a closed subset of \mathbb{RP}^1) and some vertical segments $\{*\} \times I$ where I is a closed segment in \mathbb{RP}^1 (there is also the limiting case of purely lightlike subsets, where $\Lambda_0 = \emptyset$ and where Λ is one factor $\{*\} \times \mathbb{RP}^1$).

We hope that the reader will easily agree with the idea that the less awkward definition of “generalized graph of generalized maps from \mathbb{RP}^1 onto \mathbb{RP}^1 ” is the first definition given here, as closed achronal subsets of $\overline{\text{Ein}}^{1,1}$.

Timelike geodesics. We invite the reader to keep in mind the content of the end of Section 3.4. See also the reference [12, Section 7.3]. In this low dimension, the space of timelike geodesics $\mathcal{T}_4 := \text{SO}_0(2, 2)/(\text{SO}(2) \times \text{SO}(2)) \approx (\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}))/(\text{SO}(2) \times \text{SO}(2))$ is isometric (up to a constant factor) to $\mathbb{H}^2 \times \mathbb{H}^2$ equipped with the product metric. This correspondence is expressed in the following way: the timelike geodesic corresponding to an element (x, y) of $\mathbb{H}^2 \times \mathbb{H}^2$ is the unique timelike geodesic preserved by the subgroup of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ fixing (x, y) , namely made of pairs (g, h) where g is a rotation at x and h a rotation at y . In particular, for every g in $\text{PSL}(2, \mathbb{R}) \approx \text{AdS}^{1,2}$, the set of timelike geodesics containing g is the graph of g in $\mathbb{H}^2 \times \mathbb{H}^2$.

The symplectic form ω is $\omega_0 - \omega_0$ where ω_0 is the volume form on \mathbb{H}^2 .

The following observation will be useful later: Let (x_1, y_1) and (x_2, y_2) be two elements of $\mathbb{H}^2 \times \mathbb{H}^2$. Then, the associated timelike geodesics of $\text{AdS}^{1,2}$ have a common point if and only if the hyperbolic distance between x_1 and x_2 is equal to the distance between y_1 and y_2 — common points are then the elements of $\text{PSL}(2, \mathbb{R})$ mapping x_1 to y_1 and x_2 to y_2 .

Timelike geodesics 2: more explicit computations. Recall that $\mathcal{U}^{1,2}$ is the space of future oriented timelike vectors of $\text{AdS}^{1,2}$. In our matrix model, $\mathcal{U}^{1,2}$ is the space of pairs of matrices (A, C) satisfying:

- $\det(A) = \det(C) = 1$,
- A and C are orthogonal for $-\det$.

The last condition is equivalent to $\text{Tr}(AC^{-1}) = 0$. But recall that the space of matrices of trace zero and determinant 1 is a model for \mathbb{H}^2 . Therefore, $(A, C) \mapsto$

(J_L, J_R) , where $J_L := AC^{-1}$ and $J_R := C^{-1}A$, defines a map between $\mathcal{U}^{1,2}$ and $\mathbb{H}^2 \times \mathbb{H}^2$. This map is constant along the geodesic flow, and induces (up to a scalar constant) the $\text{SO}_0(2,2)$ -equivariant isometry between \mathcal{T}_4 and $\mathbb{H}^2 \times \mathbb{H}^2$ - it is clear that J_L (respectively J_R) is invariant by right translations (respectively left translations).

6.2 $\text{AdS}^{1,2}$ globally hyperbolic spacetimes

References adapted to the content of this section are [19] and more recently [90].

Mess's parametrization by $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$. Let $M_\Lambda(\Gamma) = \Gamma \backslash \Omega(\Lambda)$ be a MGHC spacetime locally modeled on $\text{AdS}^{1,2}$. Then Λ is the generalized graph of a semi-conjugacy between the projection Γ_L of Γ in the left factor $\text{PSL}(2, \mathbb{R})$ and the projection Γ_R of the projection in the right factor. It follows that these projections are injective and that Γ_L and Γ_R are both discrete subgroups of $\text{PSL}(2, \mathbb{R})$. Moreover, these groups are isomorphic to the fundamental group of Cauchy surfaces of $M_\Lambda(\Gamma)$, hence of a closed surface. It follows that Γ_L and Γ_R are cocompact lattices, and that Λ is the graph of the unique homeomorphism conjugating Γ_L and Γ_R — in particular, it is acausal.

Conversely, given any pair ρ_L, ρ_R of faithful and discrete representations into $\text{PSL}(2, \mathbb{R})$ of the fundamental group Γ of a closed surface, the quotient of $\Omega(\Lambda)$ (where Λ is the graph of the unique conjugacy between ρ_L and ρ_R) by the image of the representation (ρ_L, ρ_R) from Γ into $(\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})) / (-\text{Id}, -\text{Id}) \approx \text{SO}_0(2, 2)$ is MGHC.

Therefore, for every closed surface Σ , there is a canonical one-to-one correspondence between the space of AdS MGHC spacetimes diffeomorphic to $\Sigma \times \mathbb{R}$ up to isometry, and the product $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$.

Diallo's parametrization by $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$. The boundary of the convex hull $\text{Conv}(\Lambda)$ in $\text{AdS}^{1,2}$ is the union of two spacelike surfaces $S^\pm(\Lambda)$ (its past and future components). Even if they are not smooth, the metric induced on each of them is isometric to \mathbb{H}^2 , therefore the quotient surfaces $\Gamma \backslash S^\pm(\Lambda)$ represents two hyperbolic closed surfaces, *i.e.* a point in $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$ (not to be confused with the Mess parameters). In his Ph D. thesis ([40]) B. Diallo proved that any element of $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$ can be obtained in this way, but the uniqueness of the $\text{AdS}^{1,2}$ spacetime realizing this pair of metrics is still an open question. This result is the Lorentzian analogue of Epstein-Marden's Theorem establishing the realization of any element of $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$ as the metric induced on the boundary of the convex core of quasi-Fuchsian hyperbolic 3-manifolds ([43]).

Mess parametrization by measured geodesic laminations. As in the classical situation of quasi-Fuchsian hyperbolic manifolds, the future and past components $S^\pm(\Lambda)$ are *pleated* surfaces, embedded isometric copies of \mathbb{H}^2 bended in $\text{AdS}^{1,2}$ along some measured geodesic lamination λ^\pm . The pair of laminations (λ^+, λ^-) is *filling* *i.e.* any closed curve c in S which is not homotopically trivial has non-zero intersection with either λ^+ or λ^- . Mess proved that the map associating with every $\text{AdS}^{1,2}$ MGHC spacetime the measured geodesic lamination $(\Gamma \backslash S^+(\Lambda), \lambda^+)$ (or $(\Gamma \backslash S^-(\Lambda), \lambda^-)$) realizes a one-to-one correspondence.

One can also forget the hyperbolic metric on $\Gamma \backslash S^\pm(\Lambda)$. In [25] Bonsante and Schlenker proved that any filling pair of measured laminations (λ^+, λ^-) (but forgetting the hyperbolic metrics) is realized as a pair of pleated laminations on the convex core of a MGHC AdS-spacetime.

AdS^{1,2}-spacetimes and earthquakes. In [79] Mess pointed out a connection between AdS spacetimes and the notion of *earthquake* introduced by Thurston (see [70]). Let λ be a measured geodesic lamination on \mathbb{H}^2 . A convenient way to see λ is to see it as a stratification of \mathbb{H}^2 , with one-dimensional strata (the geodesics in λ) and 2-dimensional strata (the closure of the components of $\mathbb{H}^2 \setminus \text{Supp}(\lambda)$, where $\text{Supp}(\lambda)$ is the support of λ). The (left) earthquake defined by λ is a non-continuous map $E_\lambda : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ that is an isometry on each 2-dimensional stratum, but for which each 1-dimensional stratum is a *rift* on which E_λ may be non continuous. When λ is a rational lamination, *i.e.* a locally finite collection of weighted geodesic, we can give a more precise definition: if C_1 and C_2 are two 2-dimensional strata bounding the same leaf ℓ of λ , and if g_1 and g_2 are the isometries of \mathbb{H}^2 coinciding with E_λ on C_1 and C_2 , respectively, then $g_2 = h_a \circ g_1$ where h_a is the element of $\text{PSL}(2, \mathbb{R})$ defined as follows:

- orient the geodesic ℓ so that C_1 is on the left of ℓ , and C_2 on the right of ℓ ,
- let a be the weight of ℓ for λ ,

then h_a is the unique hyperbolic element preserving ℓ so that for every x in ℓ the image $h_a(x)$ is the unique element of ℓ in the direction of the orientation of ℓ and at distance a . The right earthquake is obtained with the other orientation of ℓ .

Earthquakes for general measured laminations are then defined by a limiting process, at least in the case of measured geodesic laminations invariant by a cocompact lattice of $\text{PSL}(2, \mathbb{R})$, involving the density of rational laminations.

Earthquakes have then a natural extension to the boundary $\partial\mathbb{H}^2$. When λ is preserved by a cocompact lattice, this extension is a homeomorphism, but if not, this extension in general is merely a “generalized semi-conjugacy”, *i.e.* in the point of view adopted in this survey, a closed edgeless achronal subset of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. A celebrated theorem by Thurston is that *any homeomorphism of the circle is realized by a left earthquake*, and our purpose here is to point out that this theorem is well explained in our AdS-background. Moreover, our presentation has the advantage of providing a simpler and more direct definition of earthquakes avoiding the technical difficulties associated with the step between rational measured laminations and general ones.

Indeed, let Λ be a closed achronal edgeless subset of $\text{Ein}^{1,1}$ (for example, the graph of a homeomorphism). Consider the globally hyperbolic domain $E(\Lambda)$. Then, the space $\Sigma(\Lambda)$ of cosmological lines is a closed embedded locally Lipschitz disk in $\mathcal{T}_4 \approx \mathbb{H}^2 \times \mathbb{H}^2$. This is not the graph of a map from \mathbb{H}^2 into \mathbb{H}^2 but almost: this is directly the left earthquake. Indeed, this is the set of timelike geodesics orthogonal to support hyperplanes of the future component $S^+(\Lambda)$ of the convex hull of Λ , *i.e.* the level set $\{\tau = \pi/2\}$ of the cosmological time. The geodesics orthogonal to a totally geodesic face of the pleated surface $S^+(\Lambda)$ form a region of $\Sigma(\Lambda)$ which is the graph of the restriction of an element of $\text{PSL}(2, \mathbb{R})$ to a region of \mathbb{H}^2 bounded by geodesics of \mathbb{H}^2 , whereas the timelike geodesics orthogonal to

$S^+(\Lambda)$ at the bending locus λ^+ form some “vertical bands” made of points (x, y) where x describes a geodesic ℓ_1 of \mathbb{H}^2 and y some segment I in a geodesic ℓ_2 of \mathbb{H}^2 . The hyperbolic length of I is the measure of the associated leaf of λ^+ .

In summary, in this vision, a left earthquake is the set of timelike geodesics orthogonal to a pleated surface, and the main idea of Thurston’s Earthquake Theorem reduces to the fact that the boundary of the convex hull of a closed edgeless achronal subset is a pleated surface.

Cauchy surfaces and volume preserving maps. Anti-de Sitter geometry also proposes a new vision on area preserving maps between hyperbolic surfaces. Indeed, let $M_\Lambda(\Gamma)$ be a MGHC AdS^{1,2} spacetime, defined by two Fuchsian representations $\rho_L, \rho_R : \Gamma \rightarrow \text{PSL}(2, \mathbb{R})$. For every Cauchy surface S of $M_\Lambda(\Gamma)$, the Gauss map of S provides a Γ -equivariant map $\nu : \tilde{S} \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$ whose image is ω -Lagrangian. Since $\omega = \omega_0 - \omega_0$, in the region where $\nu(\tilde{S})$ is locally the graph of a map f from \mathbb{H}^2 into \mathbb{H}^2 , f is a volume preserving map.

We can be actually more precise: recall that the map $\mathcal{U}^{1,2} \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$ is given, in term of two-by-two matrices, by $(A, C) \mapsto (J_L = AC^{-1}, J_R = C^{-1}A)$. The derivative of this map at a point (A, C) is given by:

$$\begin{aligned} \dot{J}_L &= \dot{A}C^{-1} - AC^{-1}\dot{C}C^{-1} = [\dot{A} - J_L\dot{C}]C^{-1} \\ \dot{J}_R &= C^{-1}[\dot{A} - \dot{C}J_R]. \end{aligned}$$

Hence the hyperbolic norm $-\det(\dot{J}_L)$ is $-\det(\dot{A} - J_L\dot{C})$ and the norm of $-\det(\dot{J}_L)$ is $-\det(\dot{A} - \dot{C}J_R)$.

Now let $p \mapsto (A(p), C(p))$ be the spacelike immersion induced by a smooth spacelike immersion from a surface \tilde{S} into AdS^{1,2}: $A(p)^\perp \cap C(p)^\perp$ is the plane tangent to $A(S)$ at $A(p)$. Then:

– The map $X \mapsto J_L X$ preserves $A(p)^\perp \cap C(p)^\perp$: for $X \in A(p)^\perp \cap C(p)^\perp$, we have:

$$\text{Tr}(\widehat{A}J_L X) = \text{Tr}(A^{-1}AC^{-1}X) = \text{Tr}(C^{-1}X) = 0,$$

and, since $J_L^2 = -I_2$:

$$\text{Tr}(\widehat{C}J_L X) = \text{Tr}(C^{-1}AC^{-1}X) = \text{Tr}(A^{-1}AC^{-1}AC^{-1}X) = \text{Tr}(A^{-1}X) = 0.$$

– For $X \in A(p)^\perp \cap C(p)^\perp$ we have:

$$\langle J_L X \mid X \rangle = \text{Tr}(J_L X \widehat{X}) = \det(X) \text{Tr}(J_L) = 0.$$

It follows that J_L is the complex structure on the tangent space associated with the metric I on \tilde{S} induced by the AdS-metric.

– Similar computations show that $X \mapsto -XJ_R$ is also this complex structure, therefore:

$$\forall X \in A(p)^\perp \cap C(p)^\perp \quad XJ_R = -J_L X.$$

Therefore:

$$\begin{aligned} \dot{J}_L &= [\dot{A} - J_L\dot{C}]C^{-1} \\ \dot{J}_R &= C^{-1}[\dot{A} + J_L\dot{C}]. \end{aligned}$$

We deduce the formula relating the metric I , the associated complex structure J_L , the shape operator B and the left or right hyperbolic metrics g_L, g_R induced by the composition of the Gauss map with the projection $\mathcal{T}_4 \approx \mathbb{H}^2 \times \mathbb{H}^2$ on the left factor (compare with [90, Lemma 2.9]):

$$\begin{aligned} g_L(v) &= I(v - J_L B(v)) \\ g_R(v) &= I(v + J_L B(v)). \end{aligned}$$

In particular, $\nu(\tilde{S})$ fails to be transverse to the fiber of the left or right factor if and only if there is a non zero tangent vector v at p satisfying $B(v) = \pm J_L(v)$. If this happens, then, since B is self-adjoint and this basis is orthogonal, the symmetric matrix expressing B in the basis $(v, J_L(v))$ is:

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In other words, $\nu(\tilde{S})$ is not locally the graph of a map from \mathbb{H}^2 into \mathbb{H}^2 near $p \in \tilde{S}$ if and only if its mean curvature at p vanishes and its scalar curvature is -1 . This condition can also be formulated in another way: it means that \tilde{S} is tangent at order two to the space of timelike geodesics orthogonal to a given spacelike geodesic (compare with the previous study of Gauss maps of pleated surfaces).

In particular, if the shape operator B satisfies $I(B(v)) < I(v)$ everywhere, *i.e.* if the absolute values of the principal curvatures of \tilde{S} are all < 1 , then $\nu(\tilde{S})$ is the graph of a volume preserving function $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$.

Conversely, let $f : \rho_L(\Gamma) \backslash \mathbb{H}^2 \rightarrow \rho_R(\Gamma) \backslash \mathbb{H}^2$ be any volume preserving smooth map, lifting to a conjugacy $\tilde{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ between ρ_L and ρ_R . The graph of \tilde{f} is a Γ -invariant embedded Lagrangian submanifold. At the end of Section 3 we have seen that \tilde{f} provides a map $\varphi : \mathbb{H}^2 \rightarrow \mathcal{U}^{1,2}$, where $\mathcal{U}^{1,2}$ is the space of future oriented timelike vectors of $\text{AdS}^{1,2}$. We claim that furthermore, this map is Γ -equivariant. This is not completely obvious since what is immediate is that for every γ in Γ , we have $\varphi \circ \gamma = \Phi^{t(\gamma)} \circ \varphi$ for some real number $t(\gamma)$ which *a priori* might be non-trivial. But $t : \Gamma \rightarrow \mathbb{R}$ is then a homomorphism, and if it is not trivial, then it would have arbitrary big values, and this would contradict the fact that Γ preserves a closed edgeless achronal subset of $\widetilde{\text{AdS}}^{1,n}$.

Therefore, the composition of φ with the bundle map $\mathcal{U}^{1,2} \rightarrow \text{AdS}^{1,2}$ is a Γ -equivariant map from \mathbb{H}^2 into $\text{AdS}^{1,2}$, which is “almost spacelike” as explained at the end of Section 3.

For a more complete treatment (but with a different presentation) of the link between Cauchy surfaces in MGHC $\text{AdS}^{1,2}$ -spacetimes and volume preserving maps between hyperbolic surfaces, with a list of special types of spacelike surfaces with remarkable associated volume preserving maps between hyperbolic surfaces, see [24, 90] and also [45] to appear in this *Handbook of Group actions*.

6.3 BTZ-multi black holes

In this section, we summarize the content of [12, 13], to which we refer for further details (see also [1, 2, 3, 7, 8, 27, 28] in the physics literature). Consider the case

of domains $\Omega(\Lambda)$ of $\text{AdS}^{1,2}$ associated with closed achronal subsets Λ of $\text{Ein}^{1,1}$, but not necessarily edgeless. Then, the invisible domain $E_0(\Lambda)$ in $\text{Ein}^{1,1}$ can be interpreted as the conformal boundary of $\Omega(\Lambda)$. When Λ is preserved by a discrete subgroup Γ of $\text{SO}_0(2,2)$, the quotient of $E_0(\Lambda)$ by Γ is a conformal boundary for $M_\Lambda(\Gamma)$.

In this case, Γ is the image of a representation $(\rho_L, \rho_R) : \Gamma \rightarrow \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \approx \text{SO}_0(2,2)$, and Λ is the (generalized) graph of a (semi-)conjugacy between ρ_L and ρ_R .

Let Λ^+ , Λ^- be the future (respectively past) boundary of $E_0(\Lambda)$. Then, $\Omega(\Lambda^+)$ and $\Omega(\Lambda^-)$ are globally hyperbolic domains in $\Omega(\Lambda)$ that can be respectively interpreted as *(multi)black holes* and *(multi)white holes*. Indeed, $\Omega(\Lambda^+)$ is the region in $\Omega(\Lambda)$ which is invisible from the conformal boundary where the observer is assumed to be located.

Despite the fact that it has not been done anywhere in the literature (as far as we know), this vision can be extended in higher dimensions, where an $n + 1$ -dimensional AdS multi black hole could be defined as a locally $\text{AdS}^{1,n}$ -spacetime admitting a compact convex core, and as conformal boundary a finite union of MGHC conformally flat spacetimes. Presumably, this theory coincides with the theory of convex cocompact subgroups of $\text{SO}_0(2, n)$. Such a theory would be a natural analogue of the theory of Schottky groups, where convex cocompact means $(\text{SO}_0(2, n), \text{Ein}^{1, n-1})$ -Anosov.

7 Proper actions

In this last section, we review what is known about discrete isometry groups acting properly on the entire model spacetime. We will briefly mention results in this direction, once more focusing on the connection with the space of timelike geodesics.

7.1 Cocompact actions

This section is essentially a concise extract from a previous survey [9]. One of the most important results is the completeness of Lorentzian manifolds of constant curvature. The following result is much more difficult to prove than in the well-known Riemannian case:

Theorem 7.1 ([32, 71]). *Every closed Lorentzian manifold of constant curvature is geodesically complete.*

This theorem implies that closed Lorentzian manifolds are quotients by discrete groups of isometries of the simply connected model spacetimes $\mathbb{R}^{1,n}$, $\text{dS}^{1,n}$ or $\widetilde{\text{AdS}}^{1,n}$.

The classification of closed Lorentzian manifolds of constant curvature therefore reduces to the classification of groups acting properly and cocompactly. It is essentially solved, except in the AdS case.

The de Sitter case. The de Sitter case is essentially trivial, due to the *Calabi-Markus phenomenon*: a group acting properly discontinuously on $dS^{1,n}$ is necessarily finite. Indeed, let S be the umbilical sphere, that is the intersection between $dS^{1,n} \subset \mathbb{R}^{1,n+1}$ and a spacelike hyperplane H of $\mathbb{R}^{1,n}$. Then for any g in $SO_0(1, n+1)$, the iterate gH is a hyperplane, hence it intersects non-trivially H , and $H \cap gH \cap dS^{1,n}$ is non-empty.

Therefore, since a finite group cannot act cocompactly on $dS^{1,n}$, there is no closed Lorentzian manifold of positive constant curvature.

The flat case.

Theorem 7.2 ([49, 55, 58]). *Let $M = \Gamma \backslash \mathbb{R}^{1,n}$ be a closed Lorentzian flat manifold. Then, up to finite covers, Γ is a lattice in a solvable subgroup G of $Isom(\mathbb{R}^{1,n})$ acting simply transitively on $\mathbb{R}^{1,n}$.*

There are many possibilities for the solvable Lie group G , see [43] for concrete constructions in the case of the 3-dimensional Heisenberg and SOL groups, and [58, 60] for a general study.

The anti-de Sitter case $AdS^{1,n}$ for $n \geq 3$. According to the Chern-Gauss-Bonnet formula ([34]), for even-dimensional anti de Sitter manifolds, the Euler number equals the volume, up to a non-trivial multiplicative constant. But any compact Lorentz manifold has a vanishing Euler number, since it possesses a direction field. Therefore, $AdS^{1,n}$ may admit a compact quotient only when n is even.

Conversely, for any even integer $n = 2d$, we have a natural inclusion of $U(1, d)$ in $SO_0(2, 2d)$, and $U(1, d)$ acts transitively and properly discontinuously on $AdS^{1,2d}$. Therefore, for any cocompact lattice Γ of $U(1, d)$, the quotient $\Gamma \backslash AdS^{1,2d}$ is a compact Lorentzian manifold of dimension $1 + 2d$. In [99] Zeghib conjectured that any closed Lorentzian manifold of constant negative curvature and of odd dimension ≥ 5 is of this form (observe that by the inclusion $\Gamma \subset U(1, d) \subset SO_0(2, 2d)$ is rigid ([84, 97]), *i.e.* any deformation of Γ in $SO_0(2, 2d)$ is contained in a conjugate of $U(1, d)$ in $SO_0(2, 2d)$).

The case of $AdS^{1,2}$. Lattices of $U(1, 1) \approx SO_0(2, 2)$ still provide compact quotients of $AdS^{1,2}$, but there are other examples. This was first observed by Goldman ([54]). Kulkarni and Raymond proved the following important structure theorem: let Γ be a subgroup of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ acting properly discontinuously on $PSL(2, \mathbb{R})$. Then, up to finite coverings, Γ is isomorphic to the fundamental group Γ_g of a closed surface, and more precisely, the image of some faithful representation $(\rho, r) : \Gamma_g \rightarrow PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, where (up to swapping the factors) ρ is a Fuchsian representation. In his thesis Salein ([87]) studied these representations and found some new examples, based on the following criterion: a representation $(\rho, r) : \Gamma_g \rightarrow PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ where ρ is Fuchsian acts properly discontinuously on $PSL(2, \mathbb{R})$ if and only if there is a **1-contracting map** $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that:

$$\forall \gamma \in \Gamma_g \quad f \circ \rho(\gamma) = r(\gamma) \circ f.$$

In her thesis ([68, 69]) F. Kassel proved that this criterion is equivalent to the *strict domination* of r by ρ , *i.e.* to the requirement that for every γ in Γ_g , the

translation length of $r(g)$ is strictly dominated by the translation length of $\rho(\gamma)$ (with the convention that elliptic and parabolic elements have zero translation length). Recently, Guéritaud, Kassel and Wolf proved that every connected component of $\text{Rep}(\Gamma_g, \text{PSL}(2, \mathbb{R}))$, except the two Teichmüller components, contains an element r strictly dominated by some Fuchsian representation ρ , providing many new examples ([62], see also [39, 95]).

Remark 7.3. Salein’s criterion has a very nice interpretation in terms of timelike geodesics, pointed out in [36], and that we can state in the following way: *there is a Γ -equivariant 1-contracting map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ if and only there is a Γ -invariant foliation of $\text{AdS}^{1,2}$ by timelike geodesics.* Indeed, the graph of f in $\mathbb{H}^2 \times \mathbb{H}^2 \approx \mathcal{T}_4$ is a family of timelike geodesics such that for every $(x_1, y_1), (x_2, y_2)$ in this family we have $d(y_1, y_2) \neq d(x_1, x_2)$. We have seen at the end of Section 6.1 that this means precisely that the corresponding timelike geodesics are pairwise disjoint.

Observe that this criterion is very different from the one concerning Gauss maps of Cauchy surfaces: a contracting map f cannot preserve the volume, which was the condition to be the Gauss image of a Cauchy surface.

Remark 7.4. The preceding remark may very well have an extension in the higher dimensional case. It seems related to Zeghib’s conjecture about compact quotients of $\text{SO}_0(2, 2d)$: indeed, $U(1, d)$ can be characterized as the subgroup of $\text{SO}_0(2, 2d)$ preserving a foliation of $\text{AdS}^{1,2d}$ by special timelike geodesics: the intersections between $\text{AdS}^{1,2d}$ and J -complex lines in $\mathbb{R}^{2,2d} \approx \mathbb{C}^{1,d}$ for some complex structure J calibrated with $q_{2,2d}$. Hence it sounds reasonable to split Zeghib’s conjectures in two, let us say, half-conjectures:

- a discrete subgroup Γ of $\text{SO}_0(2, n)$ acts properly discontinuously and cocompactly on $\text{AdS}^{1,2d}$ if and only if it preserves a foliation by timelike geodesics;
- if a discrete subgroup Γ of $\text{SO}_0(2, n)$ preserves a foliation by timelike geodesics, then it preserves the foliation by timelike geodesics associated with a complex structure calibrated with $q_{2,2d}$.

7.2 Margulis spacetimes

For a very nice recent review of most of this section (in French), see Schlenker’s text for the Séminaire Bourbaki ([91]).

As mentioned above, compact complete flat manifolds have (virtually) solvable fundamental groups. In [80] Milnor asked the following question: *does the free group admit a proper action on $\mathbb{R}^{1,2}$?* In [76, 77], following a very partial hint in [80], Margulis gave a positive answer to this question. Since then, quotients of $\mathbb{R}^{1,2}$ by torsion-free discrete groups of isometries are called *Margulis spacetimes*.

Afterwards, Drumm introduced the notion of *crooked planes*, giving a more intuitive geometric vision on these spacetimes ([41]), and extended considerably the list of Margulis spacetimes by proving that every discrete free subgroup of $\text{SO}_0(1, 2)$ is the linear part of the holonomy of a Margulis spacetime ([42]).

In [57], Goldman, Labourie and Margulis provided a necessary and sufficient criterion on the so-called Margulis invariant for the action of a discrete subgroup of $\text{Isom}(\mathbb{R}^{1,2})$ to be properly discontinuous.

In their remarkable recent work [36, 37] Danciger, Guéritaud and Kassel elucidated the most important remaining questions on Margulis spacetimes, at least in the case where the linear part of the group is convex cocompact (*i.e.* in the case where the inclusion $\Gamma \subset \text{Isom}(\mathbb{R}^{1,2})$ is Anosov, even if this remark has no fundamental importance in their work):

- these spacetimes are foliated by timelike geodesics, in particular, they are diffeomorphic to the interior of a handlebody,
- they all admit a fundamental domain delimited by crooked planes,
- they can be seen as infinitesimal versions of *Margulis AdS-spacetimes* in a very precise geometric way.

Concerning the last item, let us note that Margulis AdS-spacetimes are quotients of $\text{AdS}^{1,2}$ by discrete subgroups of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$, images of faithful representations $(\rho, r) : \mathbb{F} \rightarrow \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$, where \mathbb{F} is a free group and $\rho : \mathbb{F} \rightarrow \text{PSL}(2, \mathbb{R})$ a convex cocompact representation — hence, once more, (ρ, r) is $(\text{SO}_0(2, 2), \text{Ein}^{1,1})$ -Anosov. Let us mention the last paper ([38]) where the authors prove that, unlike flat Margulis spacetimes, Margulis AdS spacetimes do not necessarily admit “crooked fundamental domains”.

Finally, for extensions of these results to proper affine actions on flat affine spaces in higher dimensions (but then escaping from the Lorentzian framework), see [92, 93, 94].

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