



HAL
open science

Spectral asymptotics for the Schrödinger operator with spreading and oscillating potentials

Vincent Duchêne, Nicolas Raymond

► **To cite this version:**

Vincent Duchêne, Nicolas Raymond. Spectral asymptotics for the Schrödinger operator with spreading and oscillating potentials. 2016. hal-01361768v1

HAL Id: hal-01361768

<https://hal.science/hal-01361768v1>

Preprint submitted on 7 Sep 2016 (v1), last revised 28 Jun 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SPECTRAL ASYMPTOTICS FOR THE SCHRÖDINGER OPERATOR ON THE LINE WITH SPREADING AND OSCILLATING POTENTIALS

V. DUCHÊNE AND N. RAYMOND

ABSTRACT. This study is devoted to the asymptotic spectral analysis of multiscale Schrödinger operators with oscillating and decaying electric potentials. Different regimes, related to scaling considerations, are distinguished. By means of a normal form filtrating the oscillations, a reduction to a non-oscillating effective Hamiltonian is performed.

1. THE PROBLEM

1.1. Context and motivation. In this work, we study the asymptotic behavior (as $\epsilon \rightarrow 0$) of the low-lying spectrum of the following multiscale Schrödinger operator on the line:

$$(1.1) \quad \mathcal{L}_{\epsilon,\beta} := \epsilon^\beta D_x^2 + q\left(x, \frac{x}{\epsilon}\right),$$

where $q(x, y)$ is localised in the first variable, and 1-periodic and *zero-mean* in the second variable. In other words, we are interested in the non-zero solutions of the following eigenvalue equation:

$$(1.2) \quad \mathcal{L}_{\epsilon,\beta}\varphi_{\epsilon,\beta}(x) = \lambda_{\epsilon,\beta}\varphi_{\epsilon,\beta}(x), \quad \varphi_{\epsilon,\beta} \in L^2(\mathbb{R}).$$

In addition to its intrinsic mathematical interest, the spectral investigation of (1.1) can be motivated by two reasons:

- this is a toy problem for the propagation of waves in a material with *high contrast microstructure* (see for instance [1]),
- it can also be related to the famous Anderson localization phenomenon (see for instance [11]). In the present context, we will see a discrete spectral structure emerging from the strong oscillations of the electric potential. Here, in some sense, the oscillations *play the role of randomness*.

Of course, in the case of the trivial potential $q \equiv 0$, the spectrum is purely essential, and no eigenfunction with finite energy is allowed. Adding a sufficiently localized potential $q\left(x, \frac{x}{\epsilon}\right)$ does not perturb the essential spectrum [21], so that $\text{sp}_{\text{ess}}(\mathcal{L}_{\epsilon,\beta}) = \mathbb{R}^+$. However, as we shall see, the presence of the highly oscillatory potential generates negative eigenvalues. Our aim is to describe the asymptotic behavior of these eigenvalues through non-oscillatory, effective operators of the form:

$$\mathcal{L}_{\epsilon,\beta}^{\text{eff}} := \epsilon^\beta D_x^2 + \epsilon^\gamma V(x) \quad \text{on } L^2(\mathbb{R}),$$

where V is given in terms of q and does not depend on ϵ , and $\gamma = \gamma(\beta) \in \mathbb{R}$.

The eigenvalue asymptotics may be very different, depending on the value of the parameter β , as one can see by looking at the following two cases which have been treated in the literature.

- i. The case $\beta = 0$ corresponds to the classical case of *homogenization*. Although standard homogenization arguments yield an effective potential

$$V_{\text{eff}}(\mathbf{x}) = \int_{\mathbb{S}^1} q(\mathbf{x}, \mathbf{y}) d\mathbf{y} = 0,$$

a more precise study [4, 9] shows that the low-lying spectrum is driven by a non-trivial effective potential $V_{\text{eff}}(\mathbf{x}) = \epsilon^2 V(\mathbf{x}) \leq 0$. Consistently, there exists a negative eigenvalue, $\lambda_{\epsilon,0} \sim -\epsilon^{4\frac{1}{4}} \left(\int_{\mathbb{R}} V \right)^2$. This eigenvalue is unique for ϵ sufficiently small, and the corresponding eigenfunction behaves like $\varphi_{\epsilon,0} \sim \exp(-\sqrt{-\lambda_{\epsilon,0}}|\cdot|)$.

- ii. The case $\beta = 2$ has been studied in [8], and corresponds to a *semiclassical* scaling. In particular, Dimassi shows that the number of negative eigenvalues grows as $\epsilon \rightarrow 0$ and satisfies a Weyl type asymptotics. Although the method therein relies on the use of an effective Hamiltonian, it is not clear how to relate this effective Hamiltonian to an effective potential, even in our one-dimensional setting.

1.2. Results. Here we aim at studying the case $\beta \in (0, 2)$ in a unified manner. We find it convenient to first rescale our problem: let $\alpha = \frac{\beta}{2-\beta}$, $\varepsilon = \epsilon^{1-\beta/2}$ and (abusing notations) $\varphi_{\varepsilon,\alpha} = \varphi_{\epsilon,\beta}(\epsilon^{\beta/2}x)$, $\lambda_{\varepsilon,\alpha}(x) = \lambda_{\epsilon,\beta}$. Then the eigenproblem (1.2) reads

$$(1.3) \quad \mathcal{L}_{\varepsilon,\alpha} \varphi_{\varepsilon,\alpha}(x) := (D_x^2 + q(\varepsilon^\alpha x, x/\varepsilon)) \varphi_{\varepsilon,\alpha}(x) = \lambda_{\varepsilon,\alpha} \varphi_{\varepsilon,\alpha}(x), \quad \varphi_{\varepsilon,\alpha} \in L^2(\mathbb{R}).$$

From now on, we shall only focus on the eigenproblem (1.3) rather than (1.2).

1.2.1. An effective Hamiltonian to describe the low-lying spectrum. Let us describe how the effective potential may be extracted from the knowledge of q . Consistently with homogenization techniques, we introduce the auxiliary cell operator

$$\mathcal{M}_{X,\varepsilon} := \varepsilon^{-2} D_y^2 + q(X, y) \quad \text{on } L^2(\mathbb{S}^1),$$

where $X \in \mathbb{R}$ is a fixed parameter. For ε sufficiently small, we can (and will) prove that there exists a unique negative eigenvalue, $\mu_\varepsilon(X) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^2 V(X)$ where

$$(1.4) \quad V(X) = - \int_{\mathbb{S}^1} |\partial_y \Psi_0(X, y)|^2 dy,$$

and $\Psi_0(X, \cdot)$ is the unique zero-mean solution to

$$D_y^2 \Psi_0(X, y) = -q(X, y).$$

We then show that $\varepsilon^2 V(\varepsilon^\alpha \cdot)$ acts as an *effective potential*, in the sense that the asymptotic behavior of the low-lying spectrum of our original operator, $\mathcal{L}_{\varepsilon,\alpha}$, may be described through the one of the non-oscillatory effective operator:

$$(1.5) \quad \mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} := D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x) \quad \text{on } L^2(\mathbb{R}).$$

In the *whole* paper, we work under the following assumption.

Assumption 1.1. $q \in W^{4,\infty}(\mathbb{R}; L^\infty(\mathbb{S}^1))$, $\langle \cdot \rangle V' \in L^\infty(\mathbb{R})$ and $V(X) \rightarrow 0$ as $|X| \rightarrow \infty$.

Notation 1.2. Thereafter, we denote

$$\|q\|_{W^{\ell,\infty}(\mathbb{R} \times \mathbb{S}^1)} = \sup_{l \in \{0, \dots, \ell\}} \|\partial_X^l q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}.$$

Notation 1.3. Denote $\lambda_{n,\varepsilon,\alpha}$ be the n^{th} (necessarily simple) eigenvalue of $\mathcal{L}_{\varepsilon,\alpha}$ counted increasingly, and $\lambda_{n,\varepsilon,\alpha} = 0$ afterwards. Define similarly $\lambda_{n,\varepsilon,\alpha}^{\text{eff}}$ through the eigenvalues of $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$.

We can now state one of our main results.

Theorem 1.4. *Assume $\alpha > -1$. There exist $\varepsilon_0, C > 0$, depending only on α , $\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}$ and $\|\langle \cdot \rangle V'\|_{L^\infty}$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $n \in \mathbb{N}$, one has*

$$|\lambda_{n,\varepsilon,\alpha} - \lambda_{n,\varepsilon,\alpha}^{\text{eff}}| \leq C\varepsilon^{\min\{4, 2+2\alpha\}}.$$

Remark 1.5. The benefit of the estimate of Theorem 1.4 is that the asymptotic behavior of $\lambda_{n,\varepsilon,\alpha}^{\text{eff}}$ for V sufficiently localized, depending on the value of α , is well understood. There are three different regimes at stake:

- $\alpha > 1$ The low-lying spectrum of (1.5) is dictated by a *semiclassical limit*. There is a growing number of simple negative eigenvalues accumulating below the edge of the essential spectrum, as $\varepsilon \rightarrow 0$;
- $\alpha < 1$ The low-lying spectrum of (1.5) is dictated by a *weak coupling limit*. Since the effective potential has negative mass, there exists for ε sufficiently small a unique negative eigenvalue, at a distance $\mathcal{O}(\varepsilon^{4-2\alpha})$ of the origin;
- $\alpha = 1$ In this regime, the effective problem is *self-similar*, and there exists a (non-zero) finite number of eigenvalues below the essential spectrum.

Notice $\alpha = 0$ (respectively $\alpha = +\infty$) corresponds to $\beta = 0$ and (respectively $\beta = 2$), already described, and the eigenvalue asymptotics are, of course, consistent.

1.2.2. *About the approximation of the eigenfunctions.* Theorem 1.4 shows that the asymptotic behavior of $\lambda_{n,\varepsilon,\alpha}$ follows the one of $\lambda_{n,\varepsilon,\alpha}^{\text{eff}}$ provided that the spectral gap is asymptotically smaller than $\varepsilon^{\min\{2+2\alpha, 3+\alpha\}}$. This generically occurs for $\alpha \in (\frac{1}{2}, 3)$. In that case, the effective potential also allows to describe asymptotically the behavior of the corresponding eigenfunctions as we shall see in the following propositions.

Proposition 1.6 (Semiclassical regime). *Assume $\alpha \in (1, 3)$. We also assume that $X \mapsto V(X)$ has a unique minimum at $X = 0$ and that it is non-degenerate. Let $N \in \mathbb{N}$. Then there exists $\varepsilon_0 > 0$, such that if $\varepsilon \in (0, \varepsilon_0)$, then $\mathcal{L}_{\varepsilon,\alpha}$ has at least N negative eigenvalues, $\lambda_{1,\varepsilon,\alpha} < \dots < \lambda_{N,\varepsilon,\alpha}$, satisfying*

$$\lambda_{n,\varepsilon,\alpha} = \varepsilon^2 V(0) + \varepsilon^{1+\alpha} (2n-1) \sqrt{\frac{V''(0)}{2}} + \mathcal{O}(\varepsilon^{\min\{4, 2\alpha\}}).$$

Up to changing its sign, the corresponding n^{th} L^2 -normalized eigenfunction, $\psi_{n,\varepsilon,\alpha}$, satisfies

$$\|\psi_{n,\varepsilon,\alpha} - \varphi_{n,\varepsilon,\alpha}^{\text{eff}}\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{3-\alpha}),$$

where $\varphi_{n,\varepsilon,\alpha}^{\text{eff}}$ is the n^{th} L^2 -normalized eigenfunction of the effective operator (1.5). Moreover, we have the approximation

$$\|\varphi_{n,\varepsilon,\alpha}^{\text{eff}}(x) - \varepsilon^{\frac{1+\alpha}{4}} H_n(\varepsilon^{\frac{1+\alpha}{2}} x)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{\alpha-1}{2}}),$$

where H_n is the n -th rescaled Hermite function satisfying

$$-H_n'' + \frac{V''(0)}{2} x^2 H_n = (2n-1) \sqrt{\frac{V''(0)}{2}} H_n.$$

If it exists, any other negative eigenvalue satisfies $\tilde{\lambda}_{\varepsilon,\alpha} \geq \varepsilon^2 V(0) + \varepsilon^{1+\alpha} (2N) \sqrt{\frac{1}{2} V''(0)}$.

Proposition 1.7 (Weak coupling regime). *Let $\alpha \in (\frac{1}{2}, 1)$ and assume that V is not almost everywhere zero and satisfies $(1 + |\cdot|)V \in L^1(\mathbb{R})$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{L}_{\varepsilon, \alpha}$ has one negative eigenvalue, $\lambda_{\varepsilon, \alpha}$, satisfying*

$$\lambda_{\varepsilon, \alpha} = -\frac{1}{4}\varepsilon^{4-2\alpha} \left(\int_{\mathbb{R}} V \right)^2 + \mathcal{O}(\varepsilon^{\min\{2+2\alpha, 6-4\alpha\}}),$$

with L^2 -normalized corresponding eigenfunction satisfying

$$\|\psi_{\varepsilon, \alpha}(x) - \varphi_{\varepsilon, \alpha}^{\text{eff}}(x)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{4\alpha-2}),$$

where $\varphi_{\varepsilon, \alpha}^{\text{eff}}$ is the unique L^2 -normalized eigenfunction of the effective operator (1.5). Moreover, we have the approximation

$$\left\| \varphi_{\varepsilon, \alpha}^{\text{eff}}(x) - \left(\frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} |V| \right)^{\frac{1}{2}} \exp\left(|x| \frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} V\right) \right\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{4}{3}(1-\alpha)}).$$

If it exists, any other negative eigenvalue satisfies $\tilde{\lambda}_{\varepsilon, \alpha} = \mathcal{O}(\varepsilon^{2+2\alpha})$.

Proposition 1.8 (Critical regime). *Let $\alpha = 1$ and assume that V is not almost everywhere zero and satisfies $(1 + |\cdot|)V \in L^1(\mathbb{R})$. For $n = 1, \dots, N$, denote $\lambda_{n, V}$ the n^{th} negative eigenvalue of $D_x^2 + V$, and $\varphi_{n, V}$ its corresponding L^2 -normalized eigenfunction. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{L}_{\varepsilon, \alpha}$ has at least N negative eigenvalue, $\lambda_{n, \varepsilon}$, satisfying*

$$\lambda_{n, \varepsilon} = \varepsilon^2 \lambda_{n, V} + \mathcal{O}(\varepsilon^4),$$

with L^2 -normalized corresponding eigenfunction satisfying

$$\|\psi_{n, \varepsilon}(x) - \varepsilon^{1/2} \varphi_{n, V}(\varepsilon x)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^2).$$

If it exists, any other negative eigenvalue satisfies $\tilde{\lambda}_{\varepsilon} = \mathcal{O}(\varepsilon^4)$.

1.2.3. *Numerical illustration.* In Figures 2, 3 and 4, below, we plot eigenfunctions of $\mathcal{L}_{\varepsilon, \alpha}$ in the three different regimes, and compare them with the approximations involved in our results, *i.e.* the eigenfunction of the effective operator, $\mathcal{L}_{\varepsilon, \alpha}^{\text{eff}}$, and its asymptotic approximation —namely $\varphi_{n, \varepsilon, \alpha}^{\text{app}} := \varepsilon^{\frac{1+\alpha}{4}} H_n(\varepsilon^{\frac{1+\alpha}{2}} \cdot)$ in the semiclassical regime, and $\varphi_{\varepsilon, \alpha}^{\text{app}} := \left(\frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} |V| \right)^{\frac{1}{2}} \exp\left(|\cdot| \frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} V\right)$ in the weak coupling regime).

The numerical scheme used for computing the eigenfunctions is described in Appendix A. We defined the oscillatory potential by

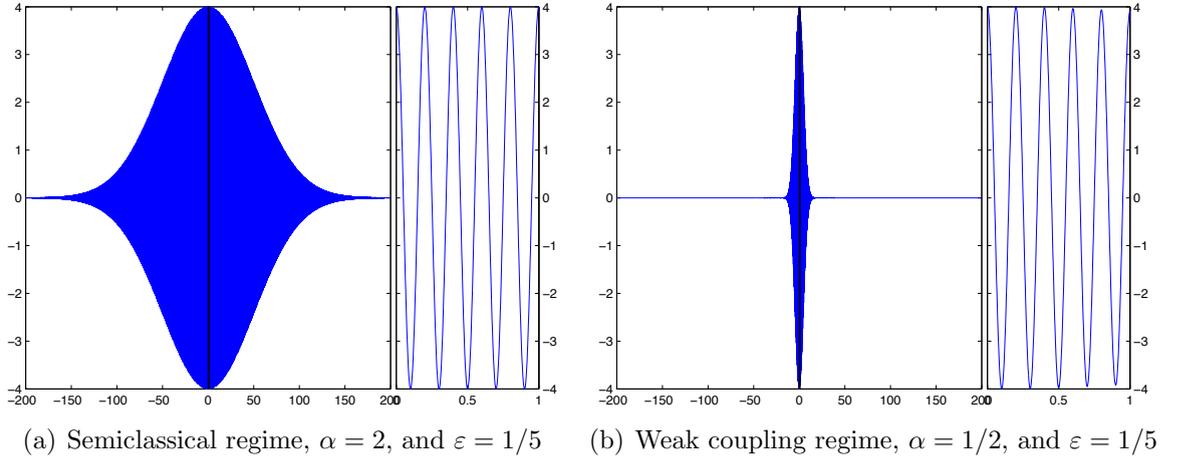
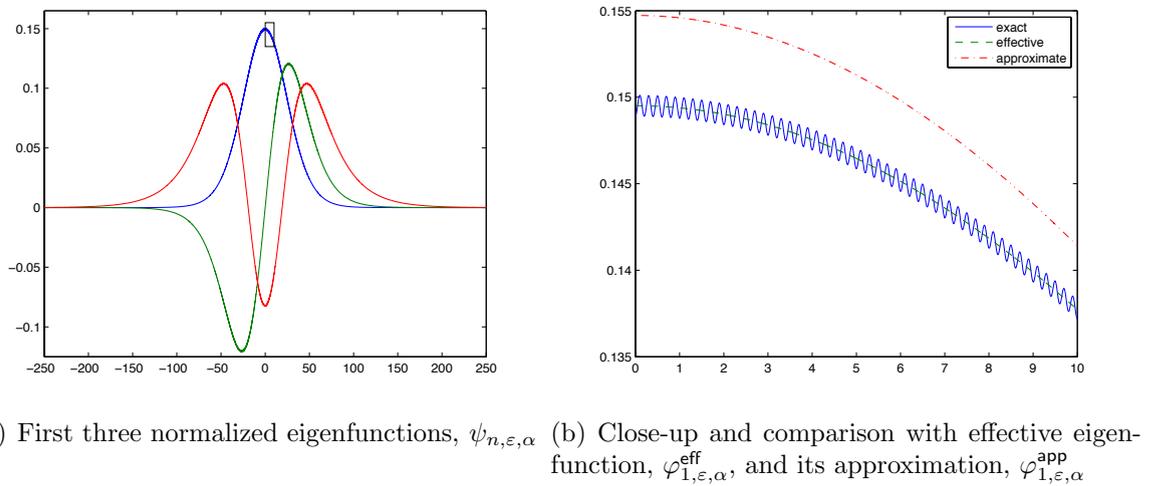
$$q(X, y) = 4 \cos(2\pi y) \times \exp(-X^2/8),$$

(see Figure 1), so that

$$\Psi_0(X, y) = -\frac{1}{\pi^2} \cos(2\pi y) \times \exp(-X^2/8), \quad V(X) = -\frac{2}{\pi^2} \exp(-X^2/4).$$

The value of the parameters are $\varepsilon = 1/5$, and $\alpha = 2$ (semiclassical regime), $\alpha = 1/2$ (weak coupling regime) and $\alpha = 1$ (critical regime).

A striking observation on these numerical experiments is that, as suggested in Propositions 1.6, 1.7 and 1.8, the main source of imprecision arises when approximating the eigenfunction of the effective problem, φ^{eff} with its semiclassical or weak coupling asymptotics, φ^{app} . Notice however that, in our results, the estimates between the exact and effective eigenfunctions is still much less precise than the eigenvalue approximation, due to the smallness of the spectral gap. As a matter of fact,

FIGURE 1. Large and small-scale behavior of the oscillatory potential, $q(\varepsilon^\alpha \cdot, \cdot/\varepsilon)$.FIGURE 2. Semiclassical regime, $\alpha = 2$.

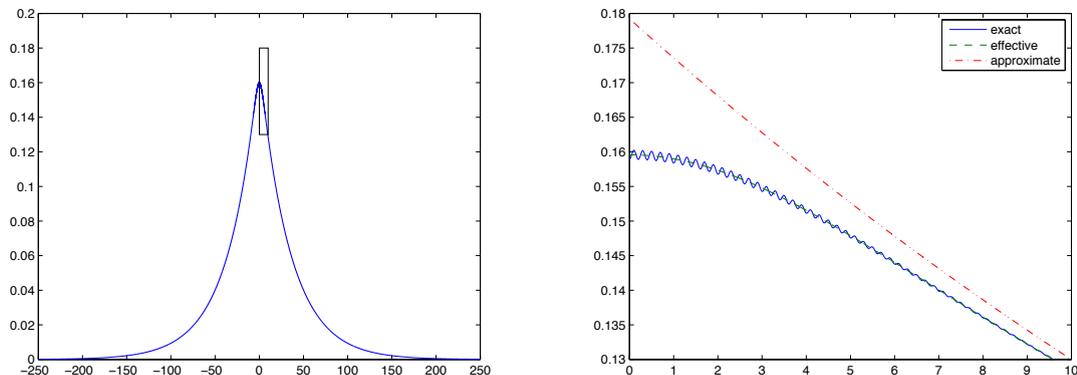
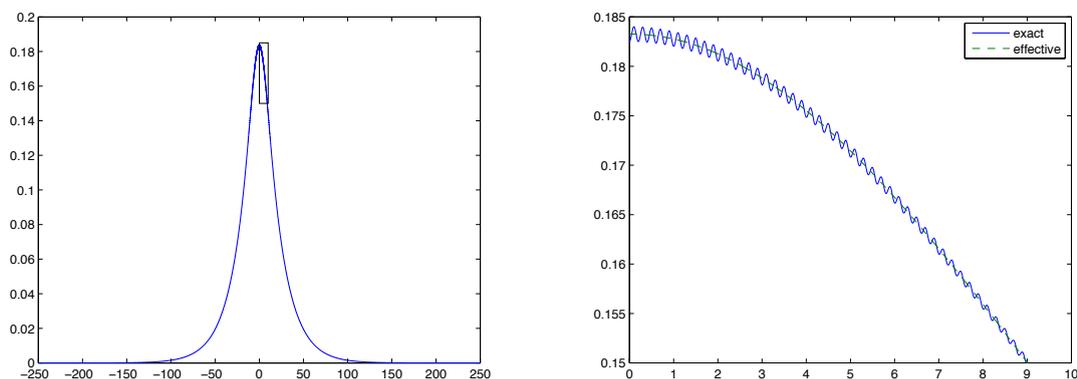
the crudeness of the former estimates hides a finer structure for the eigenfunctions, suggested by our proof: we show that

$$\psi_{\varepsilon, \alpha}(x) \approx \phi_\varepsilon(x) \times \varphi_{\varepsilon, \alpha}^{\text{eff}} \left(\int_0^x |\phi_\varepsilon|^{-2}(x') dx' \right), \quad \phi_\varepsilon(x) \approx 1 + \varepsilon^2 \Psi_0(\varepsilon^\alpha x, x/\varepsilon).$$

Notice the above shows three different scales, as the scaling involved in $\varphi_{\varepsilon, \alpha}^{\text{eff}}$ is different from the ones involved by ϕ_ε , unless $\alpha = 1$. In particular, oscillations are localized on a smaller region than the one defined by the eigenfunction in the weak coupling regime, in contrast with the semiclassical regime. Although the precision of our results is insufficient to demonstrate this three-scale structure, the latter is fully supported by our numerical simulations. Indeed, had we plotted

$$\tilde{\varphi}_{\varepsilon, \alpha}^{\text{eff}}(x) = (1 + \varepsilon^2 \Psi_0(\varepsilon^\alpha x, x/\varepsilon)) \times \varphi_{\varepsilon, \alpha}^{\text{eff}}(x),$$

in Figures 2, 3 and 4, then its graph would have been superimposed with the one of the exact eigenfunction.

(a) Normalized eigenfunction, $\psi_{\varepsilon, \alpha}$ (b) Close-up and comparison with effective eigenfunction, $\varphi_{\varepsilon, \alpha}^{\text{eff}}$, and its approximation, $\varphi_{\varepsilon, \alpha}^{\text{app}}$ FIGURE 3. Weak coupling regime, $\alpha = 1/2$.(a) First normalized eigenfunction, $\psi_{1, \varepsilon}$ (b) Close-up and comparison with effective eigenfunction, $\varphi_{1, \varepsilon}^{\text{eff}} = \varepsilon \varphi_{1, V}(\varepsilon^{1/2})$ FIGURE 4. Critical regime, $\alpha = 1$.

1.3. Related results and analogies in the literature. Let us now briefly discuss the relation of our results with the existing literature.

1.3.1. Homogenization. The problem of describing the large scale behavior of partial differential equations with periodically oscillating coefficients on a small scale is often tackled by the so-called homogenization process. Our approach is closely related, as can be seen in particular from the use of an auxiliary “cell” problem for describing the effective behavior. Notice however that our problem involves three scales whereas standard works based on homogenization techniques typically involve only two scales, as in the case $\alpha = 0$ or $\alpha = 1$. This transpires for instance in [2, 3, 1] where, in order to deal with large potentials, the authors introduce a *factorization principle* which is similar to our normal form transformation (see below), although without the change of variable. We would also like to mention the recent works [18, 6, 19], where similar multiscale problem as ours are studied.

1.3.2. *Born-Oppenheimer reduction.* The derivation of the effective operator $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ from the initial operator $\mathcal{L}_{\varepsilon,\alpha}$ is reminiscent of the famous Born-Oppenheimer approximation. This method of dimensional reduction, that is a quantum averaging strategy, was initially introduced in [5] (see also [15] in relation with molecular physics) and has been developed in many different contexts (see for instance the review [13]), and in particular to derive spectral asymptotic results (see [16, 20]). Let us explain the analogy. Consider the two-dimensional operator

$$D_{x_1}^2 + D_{x_2}^2 + q(\varepsilon^\alpha x_1, \varepsilon^{-1} x_2),$$

acting on $L^2(\mathbb{R} \times \mathbb{S}^1)$ with q satisfying the same assumptions as above. This operator is formally obtained by introducing a fictitious variable x_2 and duplicating the second derivative. Using the rescaling $y = \varepsilon^{-1} x_2$, we are reduced to the unitarily equivalent operator:

$$D_{x_1}^2 + \varepsilon^{-2} D_y^2 + q(\varepsilon^\alpha x_1, y).$$

This operator is in the Born-Oppenheimer form: it is partially semiclassical with respect to the variable x_1 and $\mathcal{M}_{\varepsilon^\alpha x_1, \varepsilon} = \varepsilon^{-2} D_y^2 + q(\varepsilon^\alpha x_1, y)$ can be interpreted as an operator-valued potential. It can be proved in a rather general framework (see for instance [17] in the context of pseudo-differential operators) that, generically, the low-lying spectrum is well described by the one of the reduced operator

$$D_{x_1}^2 + \mu_\varepsilon(\varepsilon^\alpha x_1).$$

Though the context is different, this general strategy will serve as a rough guideline for our study. However, note that we will have to overcome the fact that the variables x_1 and x_2 are not independent.

1.4. **Strategy and outline.** Let us describe the key elements of the proof of our results. The main idea is a *normal form* transformation. We show that the contributions of the oscillatory potential $q(\varepsilon^\alpha x, x/\varepsilon)$ may be factorized thanks to a two-scale function $\Phi_\varepsilon(\varepsilon^\alpha x, x/\varepsilon)$, where Φ_ε is constructed as a quasimode for the two-dimensional operator

$$\varepsilon^{-2} D_y^2 + q(X, y) - 2\varepsilon^{\alpha-1} \partial_X \partial_y - \varepsilon^{2\alpha} \partial_X^2.$$

The first step of the construction of Φ_ε consists in studying the groundstate of the cell operator $\mathcal{M}_{X,\varepsilon} := \varepsilon^{-2} D_y^2 + q(X, y)$, where X is now a parameter, and the effective potential arises as the leading order of the lowest eigenvalue, μ_ε . The construction and asymptotic behavior of Φ_ε are detailed in Section 2. The normal form of the operator is made explicit and compared with the effective operator, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$, in Section 2.2.

In Section 3, we apply the normal form transformation so as to compare the Rayleigh quotients associated with $\mathcal{L}_{\varepsilon,\alpha}$ and the ones associated with $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$. Theorem 1.4 quickly follows from the min-max principle. We then deduce the asymptotic behavior of the eigenvalues in the three aforementioned regimes.

In Section 4, we use again the normal form of our operator to construct quasimodes. We detail in Section 4.1 (respectively Section 4.2 and Section 4.3) the semiclassical regime, $\alpha > 1$ (respectively weak coupling regime, $\alpha < 1$ and critical regime, $\alpha = 1$). The spectral theorem, together with the previously obtained results on the eigenvalues, allows to deduce the asymptotic behavior of the corresponding eigenfunctions, as stated above.

2. THE AUXILIARY CELL PROBLEMS

2.1. The cell eigenproblem. In this section, we study the spectrum of the following Schrödinger operator (for fixed $X \in \mathbb{R}$ and $\varepsilon > 0$)

$$\mathcal{M}_{X,\varepsilon} := \varepsilon^{-2} D_y^2 + q(X, y) \text{ on } \mathbb{S}^1.$$

Since $q(X, \cdot)$ is bounded from below and \mathbb{S}^1 is bounded, the resolvent of $\mathcal{M}_{X,\varepsilon}$ is compact and its spectrum is purely discrete, and we denote its eigenvalues

$$\mu_{\varepsilon,1}(X) < \mu_{\varepsilon,2}(X) \leq \dots,$$

counted with multiplicity. What is more, $\mu_{\varepsilon,1}(X)$ is simple and the corresponding eigenvalue does not vanish. For simplicity, we shall denote $\mu_\varepsilon(X) := \mu_{\varepsilon,1}(X)$ and Ψ_ε the positive normalized corresponding eigenvalue:

$$(2.1) \quad \mathcal{M}_{X,\varepsilon} \Psi_\varepsilon(X, y) = \mu_\varepsilon(X) \Psi_\varepsilon(X, y), \quad \|\Psi_\varepsilon(X, y)\|_{L^2(\mathbb{S}^1)} = 1, \quad \Psi_\varepsilon(X, y) > 0.$$

We shall describe the asymptotic behavior of $\mu_\varepsilon, \Psi_\varepsilon$ in the limit $\varepsilon \rightarrow 0$, through the following auxiliary functions.

Lemma 2.1. *For any $X \in \mathbb{R}$, there exist unique $\Psi_0, \Psi_1 \in L^\infty(\mathbb{R}; L^2(\mathbb{S}^1))$ satisfying*

$$(2.2) \quad D_y^2 \Psi_0(X, y) = -q(X, y), \quad \Psi_0(X, \cdot) \in L^2(\mathbb{S}^1), \quad \int_0^1 \Psi_0(X, y) dy = 0,$$

$$(2.3) \quad D_y^2 \Psi_1 = V(X) - q(X, \cdot) \Psi_0(X, \cdot), \quad \Psi_1(X, \cdot) \in L^2(\mathbb{S}^1), \quad \int_0^1 \Psi_1(X, y) dy = 0.$$

Moreover, one has

$$\sup_{X \in \mathbb{R}} \|\Psi_0(X; \cdot)\|_{H^2(\mathbb{S}^1)} \leq C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}), \quad \sup_{X \in \mathbb{R}} \|\Psi_1(X, \cdot)\|_{H^2(\mathbb{S}^1)} \leq C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

In particular,

$$\sup_{X \in \mathbb{R}} |V(X)| \leq C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}), \quad V(X) := - \int_{\mathbb{S}^1} |\partial_y \Psi_0(X, y)|^2 dy.$$

Proof. The equations (2.2) and (2.3) are uniquely solvable if the right-hand sides are zero-mean. This follows from the definition of q and from

$$\begin{aligned} & \int_{\mathbb{S}^1} V(X) - q(X, y) \Psi_0(X, y) dy \\ &= - \int_{\mathbb{S}^1} |\partial_y \Psi_0(X, y)|^2 dy + \int_{\mathbb{S}^1} (D_y^2 \Psi_0(X, y)) \Psi_0(X, y) dy = 0. \end{aligned}$$

We consider the equation (2.2) and we immediately get

$$\|D_y^2 \Psi_0\|_{L^2(\mathbb{S}^1)} = \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}$$

and

$$\|D_y \Psi_0\|_{L^2(\mathbb{S}^1)}^2 \leq \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)} \|\Psi_0\|_{L^2(\mathbb{S}^1)} \leq \frac{1}{2\pi} \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)} \|D_y \Psi_0\|_{L^2(\mathbb{S}^1)},$$

where we used Wirtinger's inequality (or the min-max principle since the mean value of Ψ_0 is zero, and the second lowest eigenvalue of D_y^2 on $L^2(\mathbb{S}^1)$ is $4\pi^2$) for the last inequality. The completion of the proof is now straightforward. \square

We are now in position to describe the asymptotic behavior of $\mu_\varepsilon, \Psi_\varepsilon$ in (2.1).

Lemma 2.2. *There exist $M, \varepsilon_0 > 0$, depending only on $\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and $X \in \mathbb{R}$,*

$$|\mu_\varepsilon(X) - \varepsilon^2 V(X)| \leq M\varepsilon^4 \quad \text{and} \quad \mu_{\varepsilon,2}(X) \geq 4\pi^2 \varepsilon^{-2} - \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}.$$

Moreover,

$$\sup_{X \in \mathbb{R}} \|\Psi_\varepsilon - \Psi_\varepsilon^{\text{app}}\|_{H^2(\mathbb{S}^1)} \leq M\varepsilon^6,$$

where

$$\Psi_\varepsilon^{\text{app}}(X, y) := \frac{1 + \varepsilon^2 \Psi_0(X, y) + \varepsilon^4 \Psi_1(X, y)}{\|1 + \varepsilon^2 \Psi_0(X, y) + \varepsilon^4 \Psi_1(X, y)\|_{L^2(\mathbb{S}^1)}}$$

and Ψ_0, Ψ_1 and V are defined in Lemma 2.1.

Proof. Let us first remark that one has $\mathcal{M}_{X,\varepsilon} \geq \varepsilon^{-2} D_y^2 - \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}$, so that the min-max principle ensures

$$(2.4) \quad \mu_{\varepsilon,2} \geq 4\pi^2 \varepsilon^{-2} - \|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}.$$

We now compute

$$\begin{aligned} \|(\mathcal{M}_{X,\varepsilon} - \varepsilon^2 V(X)) \Psi_\varepsilon^{\text{app}}\|_{L^2(\mathbb{S}^1)} &= \varepsilon^4 \frac{\| -V\Psi_0 + q\Psi_1 - \varepsilon^2 V\Psi_1 \|_{L^2(\mathbb{S}^1)}}{\|1 + \varepsilon^2 \Psi_0(X, y) + \varepsilon^4 \Psi_1(X, y)\|_{L^2(\mathbb{S}^1)}} \\ &\leq \varepsilon^4 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}), \end{aligned}$$

where the last inequality follows from Lemma 2.1, and the choice of $\varepsilon \in (0, \varepsilon_0)$ sufficiently small. By the spectral theorem, we infer that for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$|\mu_{\varepsilon, n_\varepsilon}(X) - \varepsilon^2 V(X)| \leq \varepsilon^4 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

By (2.4), we can restrict $\varepsilon \in (0, \varepsilon_0)$ in order to ensure that $n_\varepsilon \equiv 1$, *i.e.* $\mu_{\varepsilon, n_\varepsilon}(X) = \mu_\varepsilon(X)$.

As a consequence, one has

$$(2.5) \quad \|(\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon(X)) (\Psi_\varepsilon^{\text{app}} - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon)\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^4 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

Moreover, since $\Psi_\varepsilon^{\text{app}} - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon$ is orthogonal to Ψ_ε , the function associated with the first eigenvalue of $\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon(X)$, the min-max principle and Cauchy-Schwarz inequality ensure

$$\begin{aligned} (\mu_{\varepsilon,2}(X) - \mu_\varepsilon(X)) \|\Psi_\varepsilon^{\text{app}} - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}^2 \\ \leq \varepsilon^4 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}) \times \|\Psi_\varepsilon^{\text{app}} - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}, \end{aligned}$$

and therefore

$$(2.6) \quad \|\Psi_\varepsilon^{\text{app}} - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^6 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

By the normalization and triangular inequality and since Ψ_ε and $\Psi_\varepsilon^{\text{app}}$ are positive for ε sufficiently small, we deduce

$$|1 - \langle \Psi_\varepsilon^{\text{app}}, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}| \leq \varepsilon^6 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

Plugging this estimate into (2.6) and (2.5) yields, respectively,

$$\|\Psi_\varepsilon^{\text{app}} - \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^6 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)})$$

and

$$\|(\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon(X)) (\Psi_\varepsilon^{\text{app}} - \Psi_\varepsilon)\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^4 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}).$$

The desired estimate now follows from straightforward computations. \square

Lemma 2.3. *For all $\ell \geq 1$, there exist M_ℓ , depending only on ℓ and $\|q\|_{W^{\ell,\infty}(\mathbb{R} \times \mathbb{S}^1)}$, such that under the hypothesis of Lemma 2.2, one has*

$$\sup_{X \in \mathbb{R}} \|\partial_X^\ell \Psi_\varepsilon(X, \cdot)\|_{H^2(\mathbb{S}^1)} + \sup_{X \in \mathbb{R}} |\mu_\varepsilon^{(\ell)}(X)| \leq \varepsilon^2 M_\ell.$$

Proof. The proof may be done by induction. Let us start with $\ell = 1$.

By formally differentiating the eigenvalue equation (2.1), we find

$$(2.7) \quad (\varepsilon^{-2} D_y^2 + q(X, y) - \mu_\varepsilon(X)) \partial_X \Psi_\varepsilon(X, y) = (\mu'_\varepsilon(X) - \partial_X q(X, y)) \Psi_\varepsilon(X, y).$$

By the Fredholm alternative, the Feynman-Hellmann formula holds:

$$\mu'_\varepsilon(X) = \int_{\mathbb{S}^1} \partial_X q(X, y) |\Psi_\varepsilon(X, y)|^2 dy,$$

where we used the L^2 normalization of Ψ_ε . Using the approximation on Ψ_ε given in Lemma 2.2 and since $\int_{\mathbb{S}^1} \partial_X q(X, y) dy = 0$, we deduce

$$\sup_{X \in \mathbb{R}} |\mu'_\varepsilon(X)| \leq \varepsilon^2 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}) \|\partial_X q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}.$$

By using that $\int_{\mathbb{S}^1} (\partial_X \Psi_\varepsilon) \Psi_\varepsilon = \frac{1}{2} \frac{d}{dX} \|\Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}^2 = 0$, the min-max principle yields

$$\begin{aligned} (\mu_{\varepsilon,2}(X) - \mu_\varepsilon(X)) \|\partial_X \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}^2 &\leq \int_{\mathbb{S}^1} (\partial_X \Psi_\varepsilon) (\mu'_\varepsilon(X) - \partial_X q(X, y)) \Psi_\varepsilon(X, y) \\ &\leq C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}) \|\partial_X q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)} \|\partial_X \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}. \end{aligned}$$

By Lemma 2.2, one has

$$(2.8) \quad \mu_{\varepsilon,2}(X) - \mu_\varepsilon(X) \geq c\varepsilon^{-2},$$

so that

$$\|\partial_X \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^2 C(\|q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}) \|\partial_X q\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)}.$$

The similar control of $\|D_y^2 \partial_X \Psi_\varepsilon\|_{L^2(\mathbb{S}^1)}$ immediately follows from (2.7), and the estimates of the statement, for $\ell = 1$, are proved. The above calculations are made rigorous by replacing X -derivatives with the well-defined difference quotient, and taking the limit.

Let us assume now assume that the conclusion holds for $k \in \{1, \dots, \ell\}$. When differentiating equation (2.1) $\ell + 1$ times with respect to X , one has by the Leibniz formula

$$\begin{aligned} &(\varepsilon^{-2} D_y^2 + q(X, y) - \mu_\varepsilon(X)) \partial_X^{\ell+1} \Psi_\varepsilon(X, y) \\ &= \mu_\varepsilon^{(\ell+1-k)}(X) \Psi_\varepsilon(X, y) + \sum_{k=0}^{\ell} \binom{\ell+1}{k} (\mu_\varepsilon^{(\ell+1)}(X) - \partial_X^{\ell+1-k} q(X, y)) \partial_X^k \Psi_\varepsilon(X, y). \end{aligned}$$

As above, by considering the Fredholm condition and the induction assumption, we find that

$$|\mu_\varepsilon^{(\ell+1)}(X)| \leq \varepsilon^2 C(M_\ell, \|q\|_{W^{\ell+1,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

and consequently

$$\|(\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon(X))\partial_X^{\ell+1}\Psi_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq C(M_\ell, \|q\|_{W^{\ell+1,\infty}(\mathbb{R}\times\mathbb{S}^1)}).$$

Now, differentiating the identity

$$\langle \Psi_\varepsilon, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} = 1,$$

we find

$$2\langle \Psi_\varepsilon, \partial_X^{\ell+1}\Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} = -\sum_{k=1}^{\ell} \binom{\ell+1}{k} \langle \partial_X^k \Psi_\varepsilon, \partial_X^{\ell+1-k}\Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}$$

so that, by the induction assumption and Cauchy-Schwarz inequality, we have

$$|\langle \Psi_\varepsilon, \partial_X^{\ell+1}\Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}| \leq 2\varepsilon^4 M_\ell^2.$$

It follows that

$$\|(\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon(X))(\partial_X^{\ell+1}\Psi_\varepsilon - \langle \Psi_\varepsilon, \partial_X^{\ell+1}\Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}\Psi_\varepsilon)\|_{L^2(\mathbb{S}^1)} \leq C(M_\ell, \|q\|_{W^{\ell+1,\infty}(\mathbb{R}\times\mathbb{S}^1)}).$$

As above, by the min-max principle and (2.8), we deduce

$$c\varepsilon^{-2}\|\partial_X^{\ell+1}\psi_{X,\varepsilon} - \langle \Psi_\varepsilon, \partial_X^{\ell+1}\Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}\Psi_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq C(M_\ell, \|q\|_{W^{\ell+1,\infty}(\mathbb{R}\times\mathbb{S}^1)}),$$

and the result follows. \square

2.2. The normal form. In this section, we introduce the normal form which is the key ingredient of our strategy.

2.2.1. The function ϕ_ε . The main element of the normal form is a function of the form

$$(2.9) \quad \phi_\varepsilon(x) = \Phi_\varepsilon(\varepsilon^\alpha x, x/\varepsilon)$$

which we construct so as to satisfy

$$(2.10) \quad (\mathcal{L}_{\varepsilon,\alpha}\phi_\varepsilon)(x) \sim_{\varepsilon \rightarrow 0} \mu_\varepsilon(\varepsilon^\alpha x)\phi_\varepsilon(x).$$

Notice that

$$(\mathcal{L}_{\varepsilon,\alpha}\phi_\varepsilon)(x) = (\mathcal{M}_{X,\varepsilon} - 2\varepsilon^{\alpha-1}\partial_X\partial_y - \varepsilon^{2\alpha}\partial_X^2)\Phi_\varepsilon(\varepsilon^\alpha x, x/\varepsilon).$$

Thus we define Φ_ε as

$$(2.11) \quad \Phi_\varepsilon(X, y) := \Psi_\varepsilon(X, y) + \varepsilon^{\alpha-1}\tilde{\Psi}_\varepsilon(X, y)$$

where Ψ_ε has been defined in (2.1), and $\tilde{\Psi}_\varepsilon$ is defined by

$$(2.12) \quad (\mathcal{M}_{X,\varepsilon} - \mu_\varepsilon)\tilde{\Psi}_\varepsilon = 2\partial_X\partial_y\Psi_\varepsilon + \varepsilon^{\alpha+1}\partial_X^2\Psi_\varepsilon - \langle 2\partial_X\partial_y\Psi_\varepsilon + \varepsilon^{\alpha+1}\partial_X^2\Psi_\varepsilon, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}\Psi_\varepsilon.$$

Note here that the correction $\tilde{\Psi}_\varepsilon$ is added so that the remainder in (2.10) is sufficiently small. This is crucial to get our eigenvalues and eigenfunctions asymptotic results.

Lemma 2.4. *Assume $\alpha > -1$. There exists ε_0 , depending only on α and $\|q\|_{W^{4,\infty}(\mathbb{R}\times\mathbb{S}^1)}$, such that for any $\varepsilon \in (0, \varepsilon_0)$, one has*

$$\|\phi_\varepsilon\|_{W^{2,\infty}(\mathbb{R})} \leq C(\|q\|_{W^{4,\infty}(\mathbb{R}\times\mathbb{S}^1)}),$$

$$\|\phi_\varepsilon - 1\|_{L^\infty(\mathbb{R})} \leq \varepsilon^2 C(\|q\|_{W^{4,\infty}(\mathbb{R}\times\mathbb{S}^1)}) \leq 1/2,$$

and

$$\|(\mathcal{L}_{\varepsilon,\alpha} - \mu_\varepsilon(\varepsilon^\alpha \cdot))\phi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \varepsilon^{\min\{3+\alpha, 2+2\alpha\}} C(\|q\|_{W^{4,\infty}(\mathbb{R}\times\mathbb{S}^1)}).$$

Proof. We first remark that $\tilde{\Psi}_\varepsilon$ is well-defined by (2.12) since the right-hand side, which we denote r_ε in the following, satisfies

$$r_\varepsilon \in L^2(\mathbb{S}^1) \quad \text{and} \quad \langle r_\varepsilon, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} = 0.$$

What is more, the min-max principle and (2.8) gives, for any fixed $X \in \mathbb{R}$,

$$c\varepsilon^{-2} \|\tilde{\Psi}_\varepsilon\|_{L^2(\mathbb{S}^1)}^2 \leq (\mu_{\varepsilon,2}(X) - \mu_\varepsilon(X)) \|\tilde{\Psi}_\varepsilon\|_{L^2(\mathbb{S}^1)}^2 \leq |\langle r_\varepsilon, \tilde{\Psi}_\varepsilon \rangle_{L^2(\mathbb{S}^1)}| \leq \|\tilde{\Psi}_\varepsilon\|_{L^2(\mathbb{S}^1)} \|r_\varepsilon\|_{L^2(\mathbb{S}^1)}.$$

By Lemma 2.3, we deduce

$$\sup_{X \in \mathbb{R}} \|\tilde{\Psi}_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^2 \sup_{X \in \mathbb{R}} \|r_\varepsilon\|_{L^2(\mathbb{S}^1)} \leq \varepsilon^4 C(\|q\|_{W^{2,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

Plugging this estimate in (2.12), one immediately finds

$$\sup_{X \in \mathbb{R}} \|\tilde{\Psi}_\varepsilon\|_{H^2(\mathbb{S}^1)} \leq \varepsilon^4 C(\|q\|_{W^{2,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

Differentiating (2.12) with respect to X and proceeding as above eventually yields

$$\sup_{X \in \mathbb{R}} \left(\|\tilde{\Psi}_\varepsilon\|_{H^2(\mathbb{S}^1)} + \|\partial_X \tilde{\Psi}_\varepsilon\|_{H^2(\mathbb{S}^1)} + \|\partial_X^2 \tilde{\Psi}_\varepsilon\|_{H^2(\mathbb{S}^1)} \right) \leq \varepsilon^4 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

Together with Lemmata 2.1, 2.2, and 2.3, one deduces the first and second estimates of the statement.

Let us now study the component

$$p_\varepsilon := \langle 2\partial_X \partial_y \Psi_\varepsilon + \varepsilon^{\alpha+1} \partial_X^2 \Psi_\varepsilon, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)} \Psi_\varepsilon.$$

By Lemmata 2.2 and 2.3, one has

$$\begin{aligned} \sup_{X \in \mathbb{R}} |\langle \varepsilon^{\alpha+1} \partial_X^2 \Psi_\varepsilon, \Psi_\varepsilon \rangle_{L^2(\mathbb{S}^1)}| &\leq \varepsilon^{3+\alpha} C(\|q\|_{W^{2,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \\ \sup_{X \in \mathbb{R}} |\langle \partial_X \partial_y \Psi_\varepsilon, \Psi_\varepsilon - 1 \rangle_{L^2(\mathbb{S}^1)}| &\leq \varepsilon^4 C(\|q\|_{W^{1,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \end{aligned}$$

and $\langle \partial_X \partial_y \Psi_\varepsilon, 1 \rangle_{L^2(\mathbb{S}^1)} = 0$, so that

$$\sup_{X \in \mathbb{R}} \|p_\varepsilon\|_{H^2(\mathbb{S}^1)} \leq \varepsilon^{\min\{3+\alpha, 4\}} C(\|q\|_{W^{2,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

There now only remains to compute

$$\begin{aligned} (\mathcal{L}_{\varepsilon,\alpha} - \mu_\varepsilon(\varepsilon^\alpha x)) \phi_\varepsilon(x) &= (\mathcal{M}_{X,\varepsilon} \Phi_\varepsilon - 2\varepsilon^{\alpha-1} \partial_X \partial_y - \varepsilon^{2\alpha} \partial_X^2 - \mu_\varepsilon(X)) \Phi_\varepsilon(\varepsilon^\alpha x, x/\varepsilon) \\ &= \left(-\varepsilon^{\alpha-1} p_\varepsilon - 2\varepsilon^{2(\alpha-1)} \partial_X \partial_y \tilde{\Psi}_\varepsilon - \varepsilon^{3\alpha-1} \partial_X^2 \tilde{\Psi}_\varepsilon \right) (\varepsilon^\alpha x, x/\varepsilon) \end{aligned}$$

and the second estimate of the statement follows. \square

2.2.2. The normal form. We may now introduce the normal form of our operator, $\mathcal{L}_{\varepsilon,\alpha}$, thanks to the following transformation.

Lemma 2.5. *Let $\alpha > -1$ and $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 2.4. The application*

$$T : \varphi \mapsto \psi, \quad \psi(x) := \phi_\varepsilon(x) \varphi \left(\int_0^x |\phi_\varepsilon|^{-2}(x') dx' \right)$$

defines a continuous isomorphism from $H^k(\mathbb{R})$ into $H^k(\mathbb{R})$ for $k = 0, 1, 2$, and one has

$$\begin{aligned} \|T(\varphi) - \varphi\|_{L^2(\mathbb{R})} &\leq \varepsilon^2 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \|\varphi\|_{L^2(\mathbb{R})}, \\ \|T(\varphi)\|_{H^k(\mathbb{R})} + \|T^{-1}(\varphi)\|_{H^k(\mathbb{R})} &\leq C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \|\varphi\|_{H^k(\mathbb{R})}. \end{aligned}$$

The normal form is obtained by considering $T^{-1}\mathcal{L}_{\varepsilon,\alpha}T$. More precisely, we will make use of the following identity.

Lemma 2.6. *Let $\alpha > -1$ and $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 2.4. Let $\varphi \in H^2(\mathbb{R})$ and $\psi(x) := T(\varphi)(x)$. Then*

$$(\mathcal{L}_{\varepsilon,\alpha}\psi)(x) = \phi_\varepsilon^{-3}(x)(D_x^2 + V_{\varepsilon,\alpha}^{\text{red}}(x))\varphi(\tilde{x}),$$

where we denote $\tilde{x} = \int_0^x |\phi_\varepsilon|^{-2}(x')dx'$, and $V_{\varepsilon,\alpha}^{\text{red}} := \phi_\varepsilon^3 \mathcal{L}_{\varepsilon,\alpha} \phi_\varepsilon$.

Proof. Since $V_{\varepsilon,\alpha}^{\text{red}}, \phi_\varepsilon^{-1} \in L^\infty(\mathbb{R})$, and T defines a continuous isomorphism from $H^2(\mathbb{R})$ into $H^2(\mathbb{R})$, the following identities are well-defined in $L^2(\mathbb{R})$. One has

$$\begin{aligned} (\mathcal{L}_{\varepsilon,\alpha}\psi)(x) &:= (D_x^2 + q(\varepsilon^\alpha x, x/\varepsilon))\psi(x) \\ &= q(\varepsilon^\alpha x, x/\varepsilon)\psi(x) - iD_x(\phi_\varepsilon'(x)\varphi(\tilde{x}) + \phi_\varepsilon^{-1}(x)\varphi'(\tilde{x})) \\ &= \varphi(\tilde{x})(D_x^2 + q(\varepsilon^\alpha x, x/\varepsilon))\phi_\varepsilon(x) + \phi_\varepsilon^{-3}(x)(D_x^2\varphi)(\tilde{x}) \\ &= \phi_\varepsilon^{-3}(x)(D_x^2 + V_{\varepsilon,\alpha}^{\text{red}}(x))\varphi(\tilde{x}), \end{aligned}$$

which concludes the proof. \square

It is now natural to compare the normal form of our operator with the effective operator, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} := D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x)$. Indeed, by construction of ϕ_ε , one has the following approximation.

Lemma 2.7. *Let $\alpha > -1$ and $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 2.4. One has*

$$\|V_{\varepsilon,\alpha}^{\text{red}}(x) - \varepsilon^2 V(\varepsilon^\alpha \tilde{x})\|_{L^\infty(\mathbb{R})} \leq C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}, \|\langle \cdot \rangle V'\|_{L^\infty}) \varepsilon^{\min\{4, 2+2\alpha\}},$$

where we recall the notation $\tilde{x} = \int_0^x |\phi_\varepsilon|^{-2}(x')dx'$.

Proof. By Lemma 2.4, one has for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small,

$$\|\phi_\varepsilon - 1\|_{L^\infty(\mathbb{R})} \leq \varepsilon^2 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \leq \frac{1}{2},$$

and

$$\|(\mathcal{L}_{\varepsilon,\alpha} - \mu_\varepsilon(\varepsilon^\alpha x))\phi_\varepsilon\|_{L^\infty(\mathbb{R} \times \mathbb{S}^1)} \leq \varepsilon^{\min\{3+\alpha, 2+2\alpha\}} C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}),$$

where μ_ε is defined in (2.1). By Lemmata 2.1, 2.2 and 2.4, we find

$$|\mu_\varepsilon(\varepsilon^\alpha x)\phi_\varepsilon^4(x) - \varepsilon^2 V(\varepsilon^\alpha x)| \leq \varepsilon^4 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

Collecting the above information yields

$$\|V_{\varepsilon,\alpha}^{\text{red}}(x) - \varepsilon^2 V(\varepsilon^\alpha x)\|_{L^\infty(\mathbb{R})} \leq \varepsilon^{\min\{4, 2+2\alpha\}} C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}).$$

Now, we remark that

$$|\tilde{x} - x| = \left| \int_0^x \frac{1 - |\phi_\varepsilon|^2(x')}{|\phi_\varepsilon|^2(x')} dx' \right| \leq C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \varepsilon^2 |x|.$$

By the Taylor formula, one has, for $x \neq 0$,

$$V(\varepsilon^\alpha \tilde{x}) - V(\varepsilon^\alpha x) = \frac{\tilde{x} - x}{x} \int_0^1 r_\varepsilon(x, t)^{-1} \varepsilon^\alpha x r_\varepsilon(x, t) V'(\varepsilon^\alpha x r_\varepsilon(x, t)) dt,$$

where

$$r_\varepsilon(x, t) := 1 + t \frac{\tilde{x} - x}{x}.$$

We deduce that

$$\|V(\varepsilon^\alpha \tilde{x}) - V(\varepsilon^\alpha x)\|_{L^\infty(\mathbb{R})} \leq C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \|\langle \cdot \rangle V'\|_{L^\infty} \varepsilon^2,$$

and the desired estimate follows by triangular inequality. \square

Remark 2.8. *The change of variable $\tilde{x} := \int_0^x |\phi_\varepsilon|^{-2}(x') dx'$ is used to eliminate the derivative of order 1 in Lemma 2.6. It turns out that this change of variable is crucial for the construction of quasimodes (see Section 4), but not so much for the estimate of the eigenvalues (Section 3.1, Theorem 3.1), where the simple factorization by ϕ_ε would suffice.*

3. ASYMPTOTIC ANALYSIS OF THE EIGENVALUES

3.1. Comparison of eigenvalues. In this section, we prove Theorem 1.4, recalled below.

Theorem 3.1. *Assume $\alpha > -1$. There exist $\varepsilon_0, C > 0$, depending only on α , $\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}$ and $\|\langle \cdot \rangle V'\|_{L^\infty}$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $n \in \mathbb{N}$, one has*

$$|\lambda_{n,\varepsilon,\alpha} - \lambda_{n,\varepsilon,\alpha}^{\text{eff}}| \leq C \varepsilon^{\min\{4, 2+2\alpha\}}.$$

Proof. The result is based on the min-max principle; thus we introduce the quadratic forms associated with our operators, respectively

$$\mathcal{Q}_{\varepsilon,\alpha}(\psi) := \int_{\mathbb{R}} |\psi'|^2(x) + q(\varepsilon^\alpha x, x/\varepsilon) |\psi|^2(x) dx,$$

and

$$\mathcal{Q}_{\varepsilon,\alpha}^{\text{eff}}(\psi) = \int_{\mathbb{R}} |\psi'|^2(x) + \varepsilon^2 V(\varepsilon^\alpha x) |\psi|^2(x) dx.$$

Let $f \in H^1(\mathbb{R})$. Then $\phi_\varepsilon f \in H^1(\mathbb{R})$, and one has

$$\begin{aligned} \mathcal{Q}_{\varepsilon,\alpha}(\phi_\varepsilon f) &= \int_{\mathbb{R}} (|\phi_\varepsilon f'|^2 + |\phi_\varepsilon' f|^2 + 2\phi_\varepsilon \phi_\varepsilon' f f') (x) + q(\varepsilon^\alpha x, x/\varepsilon) |\phi_\varepsilon f|^2(x) dx \\ &= \int_{\mathbb{R}} |\phi_\varepsilon|^2(x) |f'|^2(x) + |f|^2(x) \left(|\phi_\varepsilon'|^2(x) - (\phi_\varepsilon \phi_\varepsilon')'(x) + q(\varepsilon^\alpha x, x/\varepsilon) |\phi_\varepsilon|^2(x) \right) dx \\ &= \int_{\mathbb{R}} |\phi_\varepsilon|^2(x) |f'|^2(x) + |f|^2(x) \phi_\varepsilon(x) (-\phi_\varepsilon''(x) + q(\varepsilon^\alpha x, x/\varepsilon) \phi_\varepsilon(x)) dx \\ &= \int_{\mathbb{R}} |\phi_\varepsilon|^2(x) (|f'|^2(x) + \mu_\varepsilon(\varepsilon^\alpha x) |f|^2(x)) dx + r_1(f), \end{aligned}$$

with

$$r_1(f) := \int_{\mathbb{R}} |f|^2(x) \phi_\varepsilon(x) (\mathcal{L}_{\varepsilon,\alpha} - \mu_\varepsilon(\varepsilon^\alpha x)) \phi_\varepsilon(x) dx.$$

Now, we apply the near-identity change of variable

$$\tilde{x} = \int_0^x |\phi_\varepsilon|^{-2}(x') dx' \iff x = \theta(\tilde{x}),$$

and deduce

$$\mathcal{Q}_{\varepsilon,\alpha}(\phi_\varepsilon f) = \int_{\mathbb{R}} |\phi_\varepsilon|^4(\theta(\tilde{x})) (|f'|^2(\theta(\tilde{x})) + \mu_\varepsilon(\varepsilon^\alpha \theta(\tilde{x})) |f|^2(\theta(\tilde{x}))) d\tilde{x} + r_1(f).$$

Finally, denoting $\psi(\tilde{x}) = f(\theta(\tilde{x}))$, one has

$$\mathcal{Q}_{\varepsilon,\alpha}(\phi_\varepsilon f) = \int_{\mathbb{R}} |\psi'|^2(\tilde{x}) + |\phi_\varepsilon|^4(\theta(\tilde{x})) \mu_\varepsilon(\varepsilon^\alpha \theta(\tilde{x})) |\psi|^2(\tilde{x}) d\tilde{x} + r_1(f),$$

that is to say

$$\mathcal{Q}_{\varepsilon,\alpha}(T(\psi)) = \mathcal{Q}_{\varepsilon,\alpha}^{\text{eff}}(\psi) + r_1(f) + r_2(\psi),$$

with

$$r_2(\psi) := \int_{\mathbb{R}} |\psi|^2(\tilde{x}) (|\phi_\varepsilon|^4(\theta(\tilde{x})) \mu_\varepsilon(\varepsilon^\alpha \theta(\tilde{x})) - \varepsilon^2 V(\varepsilon^\alpha \tilde{x})) dx.$$

By Lemma 2.4, one has

$$|r_1(f)| \leq \varepsilon^{\min\{3+\alpha, 2+2\alpha\}} C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R})}^2.$$

By the estimates in the proof of Lemma 2.7, one has

$$|r_2(\psi)| \leq \varepsilon^4 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}, \|\langle \cdot \rangle V'\|_{L^\infty}) \|\psi\|_{L^2(\mathbb{R})}^2.$$

By the min-max principle, and using Lemma 2.5, we deduce that

$$\lambda_{n,\varepsilon,\alpha} \leq \lambda_{n,\varepsilon,\alpha}^{\text{eff}} + C\varepsilon^2 |\lambda_{n,\varepsilon,\alpha}^{\text{eff}}| + C\varepsilon^{\min\{4, 2+2\alpha\}}$$

and

$$\lambda_{n,\varepsilon,\alpha}^{\text{eff}} \leq \lambda_{n,\varepsilon,\alpha} + C\varepsilon^2 |\lambda_{n,\varepsilon,\alpha}| + C\varepsilon^{\min\{4, 2+2\alpha\}},$$

with $C = C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}, \|\langle \cdot \rangle V'\|_{L^\infty})$. Since by Lemma 2.1, one has

$$\forall X \in \mathbb{R}, \quad 0 \leq -V(X) \leq C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}),$$

we deduce $0 \leq -\lambda_{n,\varepsilon,\alpha}^{\text{eff}} \leq \varepsilon^2 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)})$, and Theorem 3.1 follows. \square

3.2. Application. In this section, we obtain the asymptotic behavior of the low-lying spectrum of the operator $\mathcal{L}_{\varepsilon,\alpha}$. We showed in Theorem 3.1 that the eigenvalues can be compared with the ones of the effective operator,

$$\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} := D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x),$$

where we recall that

$$V(X) := - \int_{\mathbb{S}^1} |\partial_y \Psi_0(X, y)|^2 dy,$$

with

$$D_y^2 \Psi_0(X, y) = -q(X, y), \quad \Psi_0(X, \cdot) \in L^2(\mathbb{S}^1), \quad \int_0^1 \Psi_0(X, y) dy = 0.$$

As previously mentioned, the asymptotic behavior of the low-lying spectrum of $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ strongly depends on the value of α , and we detail below the different regimes corresponding to different values of α .

Proposition 3.2 (Semiclassical regime). *Let $\alpha \in (1, 3)$ and $N \in \mathbb{N}$. Assume that $X \mapsto V(X)$ has a unique non-degenerate minimum at $X = 0$. Then there exists $\varepsilon_0 > 0$, such that if $\varepsilon \in (0, \varepsilon_0)$, then $\mathcal{L}_{\varepsilon,\alpha}$ has at least N negative eigenvalues, $\lambda_{1,\varepsilon,\alpha} < \dots < \lambda_{N,\varepsilon,\alpha}$, satisfying*

$$\lambda_{n,\varepsilon,\alpha} = \varepsilon^2 V(0) + \varepsilon^{1+\alpha} (2n-1) \sqrt{\frac{V''(0)}{2}} + \mathcal{O}(\varepsilon^{\min\{4, 2\alpha\}}).$$

If it exists, any other negative eigenvalue satisfies $\tilde{\lambda}_{\varepsilon,\alpha} \geq \varepsilon^2 V(0) + \varepsilon^{1+\alpha} (2N) \sqrt{\frac{1}{2} V''(0)}$.

Proof. By a rescaling argument, $\lambda_{\varepsilon,\alpha}^{\text{eff}}$ is an eigenvalue of the effective operator, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$, if and only if $\varepsilon^{-2}\lambda_{\varepsilon,\alpha}^{\text{eff}}$ is an eigenvalue of

$$\mathcal{L}_{\varepsilon,\alpha}^{\text{s.c.}} := \varepsilon^{2(\alpha-1)}D_x^2 + V.$$

Thus (see classical references [23, 7, 12] for instance), as $h = \varepsilon^{\alpha-1} \rightarrow 0$, one has

$$\varepsilon^{-2}\lambda_{\varepsilon,\alpha}^{\text{eff}} = V(0) + (2n-1)\varepsilon^{\alpha-1}\sqrt{\frac{V''(0)}{2}} + \mathcal{O}(\varepsilon^{2(\alpha-1)}).$$

The result now follows from Theorem 3.1, since the restriction $\alpha \in (1, 3)$ ensures that $1 + \alpha < \min\{4, 2 + 2\alpha, 2\alpha\} = \min\{4, 2\alpha\}$. \square

Proposition 3.3 (Weak coupling regime). *Let $\alpha \in (1/2, 1)$, and assume that V is not almost everywhere zero and such that $(1 + |\cdot|)V \in L^1$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{L}_{\varepsilon,\alpha}$ has a negative eigenvalue, $\lambda_{\varepsilon,\alpha}$, satisfying*

$$\lambda_{\varepsilon,\alpha} = -\frac{1}{4}\varepsilon^{4-2\alpha} \left(\int_{\mathbb{R}} V \right)^2 + \mathcal{O}(\varepsilon^{\min\{2+2\alpha, 6-4\alpha\}}).$$

If it exists, any other negative eigenvalue satisfies $\tilde{\lambda}_{\varepsilon,\alpha} = \mathcal{O}(\varepsilon^{2+2\alpha})$.

Proof. By a rescaling argument, $\lambda_{\varepsilon,\alpha}^{\text{eff}}$ is an eigenvalue of the effective operator, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$, if and only if $\varepsilon^{-2\alpha}\lambda_{\varepsilon,\alpha}^{\text{eff}}$ is an eigenvalue of

$$\mathcal{L}_{\varepsilon,\alpha}^{\text{s.a.}} := D_x^2 + \varepsilon^{2(1-\alpha)}V.$$

Since $\alpha < 1$, $(1 + |\cdot|)V \in L^1$, and $V \leq 0$ by (1.4), the results of [22, 14] apply. Thus for ε sufficiently small, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ has a unique negative eigenvalue, $\lambda_{\varepsilon,\alpha}^{\text{eff}}$, and

$$\varepsilon^{-2\alpha}\lambda_{\varepsilon,\alpha}^{\text{eff}} = -\frac{1}{4}\varepsilon^{4-4\alpha} \left(\int_{\mathbb{R}} V \right)^2 + \mathcal{O}(\varepsilon^{6-6\alpha}).$$

The result now follows from Theorem 3.1, since the restriction $\alpha \in (1/3, 1)$ ensures that $4 - 2\alpha < \min\{4, 2 + 2\alpha, 6 - 4\alpha\} = \min\{2 + 2\alpha, 6 - 4\alpha\}$. \square

Proposition 3.4 (Critical regime). *Let $\alpha = 1$, and assume that V is not almost everywhere zero and such that $(1 + |\cdot|)V \in L^1$. Denote*

$$\lambda_1 < \lambda_2 < \dots < \lambda_N < 0$$

the negative eigenvalues of

$$\mathcal{L}^{\text{cr}} := D_x^2 + V.$$

Then for ε sufficiently small, $\mathcal{L}_{\varepsilon,\alpha}$ has N negative eigenvalues, $\lambda_{n,\varepsilon}$, satisfying

$$\lambda_{n,\varepsilon} = \varepsilon^2\lambda_n + \mathcal{O}(\varepsilon^4).$$

If it exists, any other negative eigenvalue $\tilde{\lambda}_\varepsilon$ satisfies $\tilde{\lambda}_\varepsilon = \mathcal{O}(\varepsilon^4)$.

Proof. By a rescaling argument, $\lambda_\varepsilon^{\text{eff}}$ is an eigenvalue of the effective operator, if and only if $\varepsilon^{-2}\lambda_\varepsilon^{\text{eff}}$ is an eigenvalue of \mathcal{L}^{cr} . The result then follows from Theorem 3.1. \square

4. DESCRIPTION OF THE EIGENFUNCTIONS

This section is dedicated to the description of the eigenfunctions associated with the low-lying spectrum of our operator $\mathcal{L}_{\varepsilon,\alpha}$, as described in Theorem 3.1. The main tool is the transformation defined in Lemma 2.5 which, as seen in Lemma 2.6, allows to transform the oscillatory problem into a normal form, the latter being described at first order by the effective operator (see Lemma 2.7), $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} := D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x)$.

Consequently, the eigenmodes of the oscillatory operator define quasimodes of the effective operator. When the precision of the constructed quasimode is smaller than the spectral gap, one obtains an asymptotic description of the eigenfunctions. In the following sections, we carry out this strategy in the different regimes so as to prove Propositions 1.6, 1.7 and 1.8.

4.1. Semiclassical regime $\alpha > 1$; proof of Proposition 1.6. We shall make use of the following properties on the eigenfunctions of $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ in the semiclassical limit. This proposition is a consequence of the harmonic approximation (see the classical references [23, 7, 12]).

Proposition 4.1. *Let $\alpha > 1$ and assume that $X \mapsto V(X)$ has a unique non-degenerate minimum at $X = 0$. Then there exists $C, \varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then there exists $\lambda_{1,\varepsilon,\alpha}^{\text{eff}} < \dots < \lambda_{N,\varepsilon,\alpha}^{\text{eff}}$ eigenvalues and $\varphi_{1,\varepsilon,\alpha}^{\text{eff}}, \dots, \varphi_{N,\varepsilon,\alpha}^{\text{eff}}$ corresponding eigenfunctions of $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} = D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x)$. Moreover, $\varphi_{n,\varepsilon,\alpha}^{\text{eff}}$ is uniquely determined by*

$$\|\varphi_{n,\varepsilon,\alpha}^{\text{eff}}\|_{L^2(\mathbb{R})} = 1, \quad \int_{\mathbb{R}} \varphi_{n,\varepsilon,\alpha}^{\text{eff}}(x) H_n(\varepsilon^{\frac{1+\alpha}{2}} x) dx > 0,$$

and one has

$$(4.1) \quad \left| \lambda_{n,\varepsilon}^{\text{eff}} - \left(\varepsilon^2 V(0) + \varepsilon^{1+\alpha} (2n-1) \sqrt{\frac{V''(0)}{2}} \right) \right| \leq C \times \varepsilon^{2\alpha},$$

and

$$\varphi_{n,\varepsilon,\alpha}^{\text{eff}}(x) = \varepsilon^{\frac{1+\alpha}{4}} \left(H_n(\varepsilon^{\frac{1+\alpha}{2}} x) + r_{n,\varepsilon}(\varepsilon^{\frac{1+\alpha}{2}} x) \right)$$

with

$$(4.2) \quad \|r_{n,\varepsilon}\|_{L^2(\mathbb{R})} \leq C \times \varepsilon^{\frac{\alpha-1}{2}}.$$

Proof. Let us only sketch the main steps of the proof. Using the rescaling $\hat{x} = \varepsilon^\alpha x$ and denoting $h = \varepsilon^{\alpha-1}$ the effective semiclassical parameter, the study reduces to the spectral analysis of

$$\mathcal{L}_h = h^2 D_{\hat{x}}^2 + V(\hat{x}).$$

Since $X \mapsto V(X)$ has a unique non-degenerate minimum at $X = 0$ that is not attained at infinity (as $V(X) \rightarrow 0$ as $|X| \rightarrow \infty$), the standard harmonic approximation shows that, for all $n \geq 1$, there exist $C_n > 0$ and $h_n > 0$ such that for all $h \in (0, h_n)$, the n^{th} eigenvalue of \mathcal{L}_h , denoted $\lambda_n(h)$, satisfies

$$(4.3) \quad \left| \lambda_n(h) - V(0) - (2n-1)h \sqrt{\frac{V''(0)}{2}} \right| \leq C_n h^2,$$

and the constants C_n, h_n depend only on n and $\|V\|_{W^{3,\infty}(\mathbb{R})}$ (and thus on $\|q\|_{W^{3,\infty}(\mathbb{R} \times \mathbb{S}^1)}$). We deduce (4.1).

From the estimate (4.3), it is possible to deduce an approximation of the corresponding eigenfunctions by observing that

$$\left\| \left(\mathcal{L}_h - V(0) - (2n-1)h\sqrt{\frac{V''(0)}{2}} \right) h^{-\frac{1}{4}} H_n(h^{-\frac{1}{2}} \cdot) \right\|_{L^2(\mathbb{R})} \leq C_n h^{\frac{3}{2}}.$$

Since the n^{th} eigenspace is one-dimensional (and the spectral gap of order h), we get that the n^{th} normalized eigenfunction is at a distance, in L^2 -norm, at most $C_n h^{\frac{1}{2}}$ of the normalized quasimode $h^{-\frac{1}{4}} H_n(h^{-\frac{1}{2}} \cdot)$. In other words, if \hat{r}_h is the difference between the quasimode and the normalized eigenfunction, we have $\|\hat{r}_h\|_{L^2(\mathbb{R})} \leq Ch^{\frac{1}{2}}$. After rescaling, we deduce (4.2). \square

We can now prove Proposition 1.6. Let $\psi_{n,\varepsilon,\alpha}$ be the normalized eigenfunction associated with $\lambda_{n,\varepsilon,\alpha}$, eigenvalue of $\mathcal{L}_{\varepsilon,\alpha}$, as defined by Proposition 3.2. By Lemma 2.6, it follows that $\varphi_{n,\varepsilon,\alpha} := T^{-1}(\psi_{n,\varepsilon,\alpha})$ satisfies

$$(D_x^2 + V_{\varepsilon,\alpha}^{\text{red}}(x))\varphi_{n,\varepsilon,\alpha}(\tilde{x}) = \lambda_{n,\varepsilon,\alpha}\phi_\varepsilon^4(x)\varphi_{n,\varepsilon,\alpha}(\tilde{x}).$$

By Lemma 2.7, Theorem 3.1 and Lemma 2.4 and since $4 < 2 + 2\alpha$, one deduces

$$\|(\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} - \lambda_{n,\varepsilon,\alpha}^{\text{eff}})\varphi_{n,\varepsilon,\alpha}\|_{L^2(\mathbb{R})} \leq \varepsilon^4 C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}, \|\langle \cdot \rangle V'\|_{L^\infty(\mathbb{R})}) \|\varphi_{n,\varepsilon,\alpha}\|_{L^2(\mathbb{R})}.$$

The spectral gap is of order $\varepsilon^{\alpha+1}$ and thus, for $\alpha \in (1, 3)$, the spectral theorem yields

$$\|\varphi_{n,\varepsilon,\alpha} - \varphi_{n,\varepsilon,\alpha}^{\text{eff}}\|_{L^2(\mathbb{R})} \leq \varepsilon^{3-\alpha} C \|\varphi_{n,\varepsilon,\alpha}\|_{L^2(\mathbb{R})}.$$

Proposition 1.6 now follows from Lemma 2.5 and Proposition 4.1.

4.2. Weak coupling regime $\alpha < 1$; proof of Proposition 1.7. We shall make use of the following properties on the eigenfunctions of $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ in the weak coupling limit.

Proposition 4.2. *Let $\alpha < 1$, and assume that $X \mapsto V(X)$ is not almost everywhere zero and satisfies the integrability condition $(1 + |\cdot|)V \in L^1(\mathbb{R})$. Then there exists $C, \varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} = D_x^2 + \varepsilon^2 V(\varepsilon^\alpha x)$ has a unique eigenvalue denoted $\lambda_{\varepsilon,\alpha}^{\text{eff}} < 0$. The corresponding eigenfunction, $\varphi_{\varepsilon,\alpha}^{\text{eff}}$, is uniquely determined by*

$$\|\varphi_{\varepsilon,\alpha}^{\text{eff}}\|_{L^2(\mathbb{R})} = 1, \quad \int_{\mathbb{R}} \varphi_{\varepsilon,\alpha}^{\text{eff}}(x) dx > 0,$$

and one has

$$(4.4) \quad \left| \lambda_{\varepsilon,\alpha}^{\text{eff}} + \frac{1}{4}\varepsilon^{4-2\alpha} \left(\int_{\mathbb{R}} V \right)^2 \right| \leq C \times \varepsilon^{6-4\alpha},$$

and

$$\varphi_{\varepsilon,\alpha}^{\text{eff}}(x) = \left(\frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} |V| \right)^{\frac{1}{2}} \left(\exp\left(|x| \frac{\varepsilon^{2-\alpha}}{2} \int_{\mathbb{R}} V\right) + r_{\varepsilon,\alpha}(\varepsilon^{2-\alpha}x) \right)$$

with

$$(4.5) \quad \|r_{\varepsilon,\alpha}\|_{L^2(\mathbb{R})} \leq C \times \varepsilon^{\frac{4}{3}(1-\alpha)}.$$

Proof. By rescaling, $(\lambda_{\varepsilon,\alpha}^{\text{eff}}, \varphi_{\varepsilon,\alpha}^{\text{eff}})$ is an eigenmode of the operator $\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}}$ if and only if $(\varepsilon^{-2\alpha}\lambda_{\varepsilon,\alpha}^{\text{eff}}, \varphi_{\varepsilon,\alpha}^{\text{eff}}(\varepsilon^{-\alpha}\cdot))$ is an eigenmode of

$$\mathcal{L}_{\varepsilon,\alpha}^{\text{s.a.}} := D_x^2 + \varepsilon^{2(1-\alpha)}V.$$

The existence and uniqueness for $\varepsilon^{2(1-\alpha)}$ sufficiently small of a negative eigenvalue (since V is real-valued, has negative mass and satisfies the integrability condition) as well as its asymptotic behavior as $\varepsilon^{2(1-\alpha)} \rightarrow 0$, yielding (4.4), is a classical result of Simon [22] and Klaus [14]. As far as we know, the corresponding eigenfunction asymptotic has been first described in [24], but their result is restricted to smooth, compactly supported potentials. A less precise estimate was given in [10, Theorem 3.1], namely

$$\sup_{x \in \mathbb{R}} \left| \varphi_{\varepsilon,\alpha}^{\text{eff}}(\varepsilon^{-\alpha}x) - K \exp\left(|x| \frac{\varepsilon^{2(1-\alpha)}}{2} \int_{\mathbb{R}} V\right) \right| = \mathcal{O}(\varepsilon^{1-\alpha}),$$

with renormalization constant $K \in \mathbb{R}$. We prove below a variant of this estimate, which allows to control the L^2 -norm.

Define $\underline{x} = \varepsilon^{2-\alpha}x$, $\lambda_{\varepsilon,\alpha}^{\text{eff}} =: -\varepsilon^{4-2\alpha}\theta^2$ and $\varphi_{\varepsilon,\alpha}^{\text{eff}} =: \varepsilon^{1-\alpha/2}\varphi(\varepsilon^{2-\alpha}x)$, so that

$$(D_{\underline{x}}^2 + \theta^2)\varphi(\underline{x}) = -\delta^{-1}V(\delta^{-1}\underline{x})\varphi(\underline{x}),$$

where $\delta = \varepsilon^{2-2\alpha}$ is a small parameter. Applying the Fourier transform, we find

$$(4.6) \quad (4\pi^2|\cdot|^2 + \theta^2)\widehat{\varphi} = -\widehat{V}(\delta\cdot) \star \widehat{\varphi},$$

where the Fourier transform of a function f is defined by the formula

$$\forall \xi \in \mathbb{R}, \quad \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi x} f(x) dx.$$

Then, we decompose the solution of (4.6) in terms of small and large frequencies

$$\widehat{\varphi} = \widehat{\varphi}_{\text{small}} + \widehat{\varphi}_{\text{large}} = \chi(|\cdot| \leq \delta^{-r}) \times \widehat{\varphi} + \chi(|\cdot| > \delta^{-r}) \times \widehat{\varphi},$$

with $\chi(S)$ the characteristic function of the set S , and $r > 0$ is a parameter, to be determined. With these notations, (4.6) implies that

$$(4.7) \quad \widehat{\varphi}_{\text{large}} = \frac{\chi(|\cdot| > \delta^{-r})}{4\pi^2|\cdot|^2 + \theta^2} \widehat{V}(\delta\cdot) \star (\widehat{\varphi}_{\text{small}} + \widehat{\varphi}_{\text{large}}),$$

$$(4.8) \quad \widehat{\varphi}_{\text{small}} = \frac{\chi(|\cdot| \leq \delta^{-r})}{4\pi^2|\cdot|^2 + \theta^2} \widehat{V}(\delta\cdot) \star (\widehat{\varphi}_{\text{small}} + \widehat{\varphi}_{\text{large}}).$$

One easily checks that the operator

$$\mathcal{T} : \widehat{f} \mapsto \frac{\chi(|\cdot| > \delta^{-r})}{4\pi^2|\cdot|^2 + \theta^2} \widehat{V}(\delta\cdot) \star \widehat{f}$$

is bounded as an operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$. Moreover, one has

$$\|\mathcal{T}\widehat{f}\|_{L^1(\mathbb{R})} \leq \left\| \frac{\chi(|\cdot| > \delta^{-r})}{4\pi^2|\cdot|^2 + \theta^2} \right\|_{L^1(\mathbb{R})} \|\widehat{V}(\delta\cdot) \star \widehat{f}\|_{L^\infty(\mathbb{R})} \leq \delta^r C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) \|\widehat{f}\|_{L^1(\mathbb{R})}.$$

It follows that, provided $r > 0$ and δ is chosen sufficiently small, (4.7) defines uniquely $\widehat{\varphi}_{\text{large}}$, and we get the following rough microlocalization estimate:

$$(4.9) \quad \|\widehat{\varphi}_{\text{large}}\|_{L^1(\mathbb{R})} \leq \delta^r C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) \|\widehat{\varphi}_{\text{small}}\|_{L^1(\mathbb{R})}.$$

Now, by (4.8), we get

$$(4.10) \quad (4\pi^2|\cdot|^2 + \theta^2)\widehat{\varphi}_{\text{small}} = \chi(|\cdot| \leq \delta^{-r})\widehat{V}(0) \int_{\mathbb{R}} \chi(|\eta| \leq \delta^{-r})\widehat{\varphi}_{\text{small}}(\eta) d\eta + R_I + R_{II},$$

where

$$R_I := \chi(|\cdot| \leq \delta^{-r})\widehat{V}(\delta\cdot) \star \widehat{\varphi}_{\text{large}}, \quad R_{II} := \chi(|\cdot| \leq \delta^{-r})(\widehat{V}(\delta\cdot) - \widehat{V}(0)) \star \widehat{\varphi}_{\text{small}}.$$

We estimate below the two remainders. By Young's inequality, one gets

$$\|(1+|\cdot|)^{-1}R_I\|_{L^2(\mathbb{R})} \leq \left\| \frac{\chi(|\cdot| \leq \delta^{-r})}{1+|\cdot|} \right\|_{L^2(\mathbb{R})} \|\widehat{V}(\delta\cdot) \star \widehat{\varphi}_{\text{large}}\|_{L^\infty(\mathbb{R})} \leq C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) \|\widehat{\varphi}_{\text{large}}\|_{L^1(\mathbb{R})},$$

and thus, with (4.9),

$$\|(1+|\cdot|)^{-1}R_I\|_{L^2(\mathbb{R})} \leq \delta^r C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) \|\widehat{\varphi}_{\text{small}}\|_{L^1(\mathbb{R})},$$

and by Cauchy-Schwarz inequality,

$$(4.11) \quad \|(1+|\cdot|)^{-1}R_I\|_{L^2(\mathbb{R})} \leq \delta^r C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) \|(1+|\cdot|)\widehat{\varphi}_{\text{small}}\|_{L^2(\mathbb{R})}.$$

By the Taylor formula and the fact that $(1+|\cdot|)V \in L^1(\mathbb{R})$, we can write

$$\begin{aligned} & \|(1+|\cdot|)^{-1}R_{II}\|_{L^2(\mathbb{R})}^2 \\ & \leq \int_{\mathbb{R}_\xi} \frac{\chi(|\cdot| \leq \delta^{-r})}{1+\xi^2} \left(\int_{\mathbb{R}_\eta} |\widehat{V}(\delta\xi - \delta\eta) - \widehat{V}(0)| \widehat{\varphi}_{\text{small}}(\eta) d\eta \right)^2 d\xi \\ & \leq \|\widehat{V}'\|_{L^\infty(\mathbb{R})}^2 \|(1+|\cdot|)\widehat{\varphi}_{\text{small}}\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}^2} \frac{\delta^2|\xi - \eta|^2}{(1+\xi^2)(1+\eta^2)} \chi(|\xi| \leq \delta^{-r}) \chi(|\eta| \leq \delta^{-r}) d\eta d\xi. \end{aligned}$$

From elementary considerations to estimate the last integral, we deduce that

$$(4.12) \quad \|(1+|\cdot|)^{-1}R_{II}\|_{L^2(\mathbb{R})} \leq \delta^{1-\frac{r}{2}} C(\|\widehat{V}'\|_{L^\infty(\mathbb{R})}) \|(1+|\cdot|)\widehat{\varphi}_{\text{small}}\|_{L^2(\mathbb{R})}.$$

Combining (4.11) and (4.12), we are led to take $r = \frac{2}{3}$ and we get

$$\|(1+|\cdot|)^{-1}(R_I + R_{II})\|_{L^2(\mathbb{R})} \lesssim \delta^{\frac{2}{3}} C(\|\widehat{V}\|_{W^{1,\infty}(\mathbb{R})}) \|(1+|\cdot|)\widehat{\varphi}_{\text{small}}\|_{L^2(\mathbb{R})}.$$

Coming back to (4.10) and using [10, Lemma 4.4], we find that there exists $K > 0$ such that

$$(4.13) \quad \left\| (1+|\cdot|) \left(\widehat{\varphi}_{\text{small}} - K \frac{\chi(|\cdot| \leq \delta^{-r})}{4\pi^2|\cdot|^2 + \theta_0^2} \right) \right\|_{L^2(\mathbb{R})} \leq \delta^{\frac{2}{3}} C(\|\widehat{V}\|_{W^{1,\infty}(\mathbb{R})}),$$

where we denote $\theta_0 = \frac{1}{2}|\widehat{V}(0)|$.

Let us notice that, by (4.7) and Young's inequality for the convolution,

$$\|\widehat{\varphi}_{\text{large}}\|_{L^2(\mathbb{R})} \leq \delta^{\frac{3r}{2}} C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}) (\|\widehat{\varphi}_{\text{small}}\|_{L^1(\mathbb{R})} + \|\widehat{\varphi}_{\text{large}}\|_{L^1(\mathbb{R})}).$$

Then, we notice, from the definition of $\widehat{\varphi}_{\text{small}}$, Cauchy-Schwarz inequality and Plancherel's theorem, that

$$\|\widehat{\varphi}_{\text{small}}\|_{L^1(\mathbb{R})} \leq \delta^{-\frac{r}{2}} \|\widehat{\varphi}_{\text{small}}\|_{L^2(\mathbb{R})} \leq \delta^{-\frac{r}{2}}.$$

Thus by the above and (4.9), one obtains

$$\|\widehat{\varphi}_{\text{large}}\|_{L^2(\mathbb{R})} \leq \delta^r C(\|\widehat{V}\|_{L^\infty(\mathbb{R})}).$$

It is now easy to deduce from (4.13) that φ , the solution to (4.6), satisfies

$$\left\| \widehat{\varphi} - K \frac{1}{4\pi^2 |\cdot|^2 + \theta_0^2} \right\|_{L^2(\mathbb{R})} \lesssim \delta^{\frac{2}{3}} = \varepsilon^{\frac{4}{3}(1-\alpha)}.$$

Estimate (4.5) follows by using the inverse Fourier transform, while the value of the constant, K , is determined by the normalization of $\varphi_{\varepsilon,\alpha}^{\text{eff}}$. Proposition 4.2 is proved. \square

We prove Proposition 1.7 as in the previous section. Let $(\lambda_{\varepsilon,\alpha}, \psi_{n,\varepsilon,\alpha})$ be the eigenmode of $\mathcal{L}_{\varepsilon,\alpha}$ uniquely defined by Proposition 3.3. By Lemma 2.6, Lemma 2.7, Theorem 3.1 and Lemma 2.4, and since $2 + 2\alpha < 4$, $\varphi_{\varepsilon,\alpha} := T^{-1}(\psi_{\varepsilon,\alpha})$ satisfies

$$\|(\mathcal{L}_{\varepsilon,\alpha}^{\text{eff}} - \lambda_{\varepsilon,\alpha}^{\text{eff}})\varphi_{\varepsilon,\alpha}\|_{L^2(\mathbb{R})} \leq \varepsilon^{2+2\alpha} C(\|q\|_{W^{4,\infty}(\mathbb{R} \times \mathbb{S}^1)}, \|\langle \cdot \rangle V'\|_{L^\infty(\mathbb{R})}) \|\varphi_{\varepsilon,\alpha}\|_{L^2(\mathbb{R})}.$$

The spectral gap is of order $\varepsilon^{4-2\alpha}$ and thus, for $\alpha \in (1/3, 1)$,

$$\|\varphi_{\varepsilon,\alpha} - \varphi_{\varepsilon,\alpha}^{\text{eff}}\|_{L^2(\mathbb{R})} \leq C\varepsilon^{4\alpha-2} \|\varphi_{\varepsilon,\alpha}\|_{L^2(\mathbb{R})}.$$

Proposition 1.7 now follows from Lemma 2.5 and Proposition 4.2.

4.3. Critical regime $\alpha = 1$; proof of Proposition 1.8. The proof in the case $\alpha = 1$ is that same as in the previous two sections. The eigenmodes of the effective operator correspond to the ones of the operator $D_x^2 + V$ after a straightforward rescaling. Proposition 3.4 allows to compare the corresponding eigenfunction to the ones of our original operator, $\mathcal{L}_{\varepsilon,\alpha}$, as above. We leave the details to the reader.

APPENDIX A. NUMERICAL SCHEME

In this section, we present the numerical scheme used in Figures 2, 3 and 4. Since our potential, q , and the expected solutions decay exponentially at infinity, it is convenient to truncate the infinite spatial domain to a periodic interval $\mathbb{S}^1(-L, L)$, and turn to Fourier spectral methods. However, because of the several scales of our problem, it is too costly to approximate the solution to the eigenvalue problem

$$\mathcal{L}_{\varepsilon,\alpha}\psi_{\varepsilon,\alpha}(x) := (D_x^2 + q(\varepsilon^\alpha x, x/\varepsilon))\psi_{\varepsilon,\alpha}(x) = \lambda_{\varepsilon,\alpha}\psi_{\varepsilon,\alpha}(x), \quad \psi_{\varepsilon,\alpha} \in L^2(\mathbb{R}).$$

with a complete set of Fourier modes:

$$\psi_{\varepsilon,\alpha}(x) \approx \sum_{k=-N}^N a_k e^{ik\frac{x}{L}}.$$

Thus we restrict to a limited number of well-chosen Fourier modes. Motivated by our results, we define

$$\mathbb{K}_n := \bigcup_{j \in \{-2, -1, 0, 1, 2\}} \left\{ k \in \mathbb{Z}, \quad \left| \frac{k\pi}{L} - \frac{2\pi j}{\varepsilon} \right| \leq \frac{n\pi}{L} \right\}.$$

and seek

$$\psi_{\varepsilon,\alpha}(x) \approx \sum_{k \in \mathbb{K}_n} a_k e^{ik\frac{x}{L}}.$$

In other words, defining the orthogonal projections

$$\begin{aligned}\Pi_N &:= f \mapsto \sum_{k=-N}^N \frac{\langle e^{ik\frac{\pi}{L}\cdot}, f \rangle_{L^2(-L,L)}}{\langle e^{ik\frac{\pi}{L}\cdot}, e^{ik\frac{\pi}{L}\cdot} \rangle_{L^2(-L,L)}} e^{ik\frac{\pi}{L}\cdot}, \\ \Pi_{\mathbb{K}_n} &:= f \mapsto \sum_{k \in \mathbb{K}_n} \frac{\langle e^{ik\frac{\pi}{L}\cdot}, f \rangle_{L^2(-L,L)}}{\langle e^{ik\frac{\pi}{L}\cdot}, e^{ik\frac{\pi}{L}\cdot} \rangle_{L^2(-L,L)}} e^{ik\frac{\pi}{L}\cdot},\end{aligned}$$

we numerically solve

$$(D_x^2 + \Pi_{\mathbb{K}_n}(\Pi_N q(\varepsilon^\alpha \cdot, \cdot/\varepsilon))) \tilde{\psi}_{\varepsilon,\alpha} = \tilde{\lambda}_{\varepsilon,\alpha} \tilde{\psi}_{\varepsilon,\alpha}, \quad \tilde{\psi}_{\varepsilon,\alpha} = \Pi_{\mathbb{K}_n} \tilde{\psi}_{\varepsilon,\alpha},$$

as an eigenvalue problem for a matrix of size $5(2n+1) \times 5(2n+1)$.

For Figures 2 3 and 4, we set $L = 500$, $N = 2^{22}$, and $n = 300$.

REFERENCES

- [1] G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess, and M. Vanninathan. Homogenization of periodic systems with large potentials. *Arch. Ration. Mech. Anal.*, 174(2):179–220, 2004.
- [2] G. Allaire and F. Malige. Analyse asymptotique spectrale d’un problème de diffusion neutronique. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(8):939–944, 1997.
- [3] G. Allaire and A. Piatnitski. Uniform spectral asymptotics for singularly perturbed locally periodic operators. *Comm. Partial Differential Equations*, 27(3-4):705–725, 2002.
- [4] D. Borisov and R. Gadyl’shin. On the spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential. *Teoret. Mat. Fiz.*, 147(1):58–63, 2006.
- [5] M. Born and R. Oppenheimer. Zur Quantentheorie der Molekeln. *Ann. Phys.*, 84:457–484, 1927.
- [6] A. Chechkina, I. Pankratova, and K. Pettersson. Spectral asymptotics for a singularly perturbed fourth order locally periodic elliptic operator. *Asymptot. Anal.*, 93(1-2):141–160, 2015.
- [7] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.
- [8] M. Dimassi. Semi-classical asymptotics for Schrödinger operator with oscillating decaying potential. *Canad. Math. Bull.*, 2016.
- [9] V. Duchêne, I. Vukićević, and M. I. Weinstein. Scattering and localization properties of highly oscillatory potentials. *Comm. Pure Appl. Math.*, 67(1):83–128, 2014.
- [10] V. Duchêne, I. Vukićević, and M. I. Weinstein. Homogenized description of defect modes in periodic structures with localized defects. *Commun. Math. Sci.*, 13(3):777–823, 2015.
- [11] A. Figotin, F. Germinet, A. Klein, and P. Müller. Persistence of Anderson localization in Schrödinger operators with decaying random potentials. *Ark. Mat.*, 45(1):15–30, 2007.
- [12] B. Helffer. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [13] T. Jecko. On the mathematical treatment of the Born-Oppenheimer approximation. *J. Math. Phys.*, 55(5):053504, 26, 2014.
- [14] M. Klaus. On the bound state of Schrödinger operators in one dimension. *Ann. Physics*, 108:288–300, 1977.
- [15] M. Klein, A. Martinez, R. Seiler, and X. P. Wang. On the Born-Oppenheimer expansion for polyatomic molecules. *Comm. Math. Phys.*, 143(3):607–639, 1992.
- [16] J. Lampart and S. Teufel. The adiabatic limit of the Laplacian on thin fibre bundles. In *Microlocal methods in mathematical physics and global analysis*, Trends Math., pages 33–36. Birkhäuser/Springer, Basel, 2013.
- [17] A. Martinez. A general effective Hamiltonian method. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 18(3):269–277, 2007.
- [18] I. Pankratova and K. Pettersson. Spectral asymptotics for an elliptic operator in a locally periodic perforated domain. *Appl. Anal.*, 94(6):1207–1234, 2015.

- [19] K. Pettersson. Subcritical perturbation of a locally periodic elliptic operator. Arxiv preprint:1505.06852.
- [20] N. Raymond. *Bound States of the Magnetic Schrödinger Operator*. To appear in EMS Tracts, 2017.
- [21] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [22] B. Simon. The bound state of weakly coupled Schrödinger operators in one and two dimensions. *Ann. Physics*, 97(2):279–288, 1976.
- [23] B. Simon. Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 38(3):295–308, 1983.
- [24] P. Zhevandrov and A. Merzon. Asymptotics of eigenfunctions in shallow potential wells and related problems. In *Asymptotic methods for wave and quantum problems*, volume 208 of *Amer. Math. Soc. Transl. Ser. 2*, pages 235–284. Amer. Math. Soc., Providence, RI, 2003.

(V. Duchêne) IRMAR, CNRS, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX, FRANCE

E-mail address: `vincent.duchene@univ-rennes1.fr`

(N. Raymond) IRMAR, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX, FRANCE

E-mail address: `nicolas.raymond@univ-rennes1.fr`