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# Low-energy spectrum of Toeplitz operators: the case of wells

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## Abstract

In the 1980s, Helffer and Sjöstrand examined in a series of articles the concentration of the ground state of a Schrödinger operator in the semi-classical limit. In a similar spirit, and using the asymptotics for the Szegő kernel, we show a theorem about the localization properties of the ground state of a Toeplitz operator, when the minimal set of the symbol is a finite set of non-degenerate critical points. Under the same condition on the symbol, for any integer  $K$  we describe the first  $K$  eigenvalues of the operator.

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# 1 Introduction

## 1.1 Motivations

In classical mechanics, the minimum of the energy, when it exists, is a critical value, and any point in phase space achieving this minimum corresponds to a stationary trajectory. In quantum mechanics, the situation is quite different. A quantum state cannot be arbitrarily localized in phase space and occupies at least some small space, because of the uncertainty principle. Nevertheless, due to the correspondence principle, one expects the quantum states of minimal energy to concentrate, in some way, near the minimal set of the Hamiltonian, when the effective Planck constant is very small.

In a series of articles [10, 11, 12, 13, 14], Helffer and Sjöstrand considered the Schrödinger operator  $P(h) = -h^2\Delta + V$ , acting on  $\mathbb{R}^n$ , where  $V$  is a smooth function. Under reasonable assumptions on  $V$ , the infimum of the spectrum of  $P(h)$  is a simple eigenvalue. Helffer and Sjöstrand then studied the concentration properties of a sequence of associated unit eigenvectors, named a ground state, in the semi-classical limit  $h \rightarrow 0$ . It is well known that the ground state is  $O(h^\infty)$  outside any fixed neighbourhood of  $\{x \in \mathbb{R}^n, V(x) = \min(V)\}$ . If there is only one such  $x$ , then the ground state

concentrates only on  $x$ . But if there are several minima, it is not clear a priori whether the ground state is evenly distributed on them or not.

In the first articles [10, 11, 12], the potential  $V$  is supposed to reach its minimum only on a finite set of non-degenerate critical points, named “wells”. It is proven that only some of the wells are selected by the ground state, that is, the sequence of eigenvectors is  $O(h^\infty)$  outside any fixed neighbourhood of this subset of wells. The selected wells are the most flat, in a sense that we will make clear later on. Sharper estimates lead to a control, outside of the wells, of the form  $\exp(-Ch^{-1})$ , where  $C$  is expressed in terms of the Agmon distance to the selected wells. Moreover, when the potential  $V$  has two symmetric wells, the ground state “tunnels” between these wells, so that there exists another eigenvalue which is exponentially close to the minimal one.

In the two last papers [13, 14], the potential  $V$  is supposed to reach its minimal value on a submanifold of  $\mathbb{R}^n$ . Again, it is easy to prove that the ground state concentrates on this submanifold. From this fact, a formal calculus leads to the study of a Schrödinger operator, on the submanifold, with an effective potential that depends on the 2-jet behaviour of  $V$  near the submanifold. The authors treated the case of an effective potential with one non-degenerate minimum, which they call the miniwell condition. In this case, the ground state concentrates only at the miniwell. On the contrary, when the minimal submanifold corresponds to a symmetry of  $V$ , the ground state is spread out on the submanifold.

This is an instance of what is called quantum selection: not all points in phase space where the classical energy is minimal are equivalent in quantum mechanics. When there is a finite set of minimal points, only some of them are selected by the ground state. When the minimal set is a submanifold, the ground state may select only one point. In the physics literature, these effects are believed to appear in other settings than Schrödinger operators; the miniwell condition was used in [7] (without mathematical justification) to study quantum selection effects for spin systems, when the classical phase space is a product of 2-spheres. However, the arguments used by Helffer and Sjöstrand depend strongly on the fact that they deal with Schrödinger operators, when the phase space is  $T^*\mathbb{R}^n$ . Thus, it is mathematically not clear to which extent the quantum selection can be generalised to a quantization of compact phase spaces.

We propose to study the Kähler quantization, which associate to a symbol on a phase space a Toeplitz operator. In the particular case of the coordinate functions on  $\mathbb{S}^2 = \mathbb{C}\mathbb{P}^1$ , the Toeplitz operators are the spin operators [3], so that our approach contains the physical case of spin systems. In this

article we prove that quantum selection occurs in the case of wells, with  $O(h^\infty)$  remainder, extending some of the results of [10, 11, 12]. The case of miniwells is treated in our next paper. Exponential estimates, and the study of tunnelling, will be the object of a separate investigation; stronger conditions on the energy, such as analyticity, will probably be required.

## 1.2 Kähler quantization

When a compact symplectic manifold is endowed with a Kähler structure, there is a natural way to define a quantization scheme, which is compatible with abstract geometric quantization [15, 20].

**Definition 1.1.** A Kähler manifold is a complex manifold  $M$  where the tangent space at each point is endowed with a hermitian form whose imaginary part is a closed 2-form on  $M$ .

In particular, a Kähler manifold has both a symplectic and a Riemannian structure, which are respectively the imaginary part and the real part of the hermitian form.

The Kähler manifolds of interest are those that satisfy the *prequantization condition*: the symplectic form  $\omega$  has an integer Chern class [15]. This condition is equivalent [15, 22] to the existence of a hermitian holomorphic complex line bundle  $(L, h) \xrightarrow{\pi} M$ , whose curvature is exactly  $\omega$ . The hermitian structure on  $L$  and the Liouville volume form on  $M$  allow us to define  $L^2(M, L)$ , the space of square-integrable sections of  $L$ , with quadratic form

$$\langle s, s' \rangle = \int_M (\bar{s}(m), s'(m))_h d\omega^n(m).$$

For every integer  $N$ , the tensor power  $L^{\otimes N}$  inherits a hermitian structure and we can also consider  $L^2(M, L^{\otimes N})$ , defined the same way.

We are in presence of what is called in algebraic geometry an ample line bundle [9].

**Definition 1.2.** For every integer  $N$ , we denote by  $\mathcal{H}_N(M, L)$  the closed subspace of  $L^2(M, L^{\otimes N})$  consisting of holomorphic  $L^2$  sections of  $L^{\otimes N}$ .

The orthogonal projector of  $L^2(M, L^{\otimes N})$  onto  $\mathcal{H}_N(M, L)$  is denoted by  $\mathcal{S}_N$ .

We will not exactly work with the spaces  $\mathcal{H}_N(M, L)$  in the body of the article, but with equivalent spaces which we call *Hardy spaces*, see subsection 2.1. Note that the spaces  $\mathcal{H}_N(M, L)$  are finite-dimensional if  $M$  is compact.

From now on,  $M$  will always denote a compact Kähler manifold of dimension  $n$ , whose symplectic form will be denoted by  $\omega$ . We will suppose that  $\omega$  satisfies the integer cohomology class condition. Moreover  $L$  will denote a hermitian holomorphic complex line bundle over  $M$ , with curvature  $\omega$ . The projection from  $L$  to  $M$  will be denoted by  $\pi$ , when there is no confusion with the numerical constant.

**Definition 1.3** (Toeplitz quantization). Let  $f \in C^\infty(M)$  be a smooth function on  $M$ . We define the *Toeplitz operator* associated to  $f$  as

$$\begin{aligned} T_N(f) : \mathcal{H}_N(M, L) &\mapsto \mathcal{H}_N(M, L) \\ u &\mapsto \mathcal{S}_N(fu). \end{aligned}$$

When the symbol  $f$  is real-valued, for fixed  $N$ , the operator  $T_N(f)$  is obviously self-adjoint and acts on a finite-dimensional space. Thus, there is no theoretical difficulty lying in the study of the spectrum of Toeplitz operators. The spectrum is the set of the eigenvalues, each of which has a finite multiplicity. We will call the “lowest eigenvalue” the minimum of the spectrum of a Toeplitz operator.

The Toeplitz quantization has many similarities with the Weyl quantization. One can indeed find a star product on the space of formal series  $C^\infty(M)[[\nu]]$  that coincides with the composition of Toeplitz operators, that is, such that  $T_N(f \star g(N^{-1})) = T_N(f)T_N(g) + O(N^{-\infty})$  [18]. In fact, it is known since at least [4] that there is a correspondence between the symbolic calculus of Toeplitz operators and that of Weyl quantization. Thus, a possible approach to the spectral study of Toeplitz operators would be a formal conjugation, either to a pseudo-differential operator whose spectrum is known by previous work [10], or to the Toeplitz quantization of a completely integrable system, in the spirit of [21, 17]. This last approach via Birkhoff normal forms could possibly give an asymptotic expansion for eigenvalues in a larger window than the one we consider in this article. However, each of these approaches require a priori results on the concentration of eigenvectors. As our future work will focus on the case when the minimal set of the symbol is a submanifold, where a priori concentration is not known, it is unclear whether these approaches are sufficient. For this reason, and the sake of simplicity, we have chosen a more direct approach.

We slightly extend the definition of Toeplitz operators in order to deal with the Kähler manifold  $M = \mathbb{C}^n$ , which is not compact. This does not affect the definitions of  $\mathcal{H}_N(M, L)$  and  $\mathcal{S}_N$ , except that the space  $\mathcal{H}_N(\mathbb{C}^n, L)$  has infinite dimension in this case. If  $f \in C^\infty(\mathbb{C}^n)$ , one can define the Toeplitz operator  $T_N^{flat}(f)$  as an unbounded operator, and it is an essentially

self-adjoint operator when the symbol is a positive quadratic form (see section 3.3).

### 1.3 Main results

In this article, we adapt the results from [10] to the setting of Toeplitz quantization. In particular, we are only interested in the following situation:

**Definition 1.4.** A function  $h \in C^\infty(M)$  is said to *satisfy the wells condition* under the two following conditions:

- $\min(h) = 0$ ;
- Every critical point at which  $h$  vanishes is non-degenerate.

Observe that, by definition, Morse functions satisfy the wells condition, as does the square modulus of a generic holomorphic section of  $L^{\otimes N}$  for  $N$  large. Note that a function that satisfies the wells condition has a finite cancellation set.

In subsection 2.2, we consider convenient local maps of “normal coordinates” around any point  $P \in M$ , which preserve infinitesimally the Kähler structure. If a non-negative function  $h$  vanishes with positive Hessian at  $P \in M$ , the 2-jet of  $h$  at  $P$  reads in these coordinates as a positive quadratic form  $q(P)$  on  $\mathbb{C}^n$ . The first eigenvalue  $\mu$  of the Toeplitz operator  $T_N^{flat}(q(P))$  (which we call *model quadratic operator*) does not, in fact, depend on the choice of coordinates. We define this value to be  $\mu(P)$ .

**Remark 1.5.** The value of  $\mu(P)$  is related to the first eigenvalue of the Weyl quantization of  $q$ , when  $\mathbb{C}^n$  is identified with  $T^*\mathbb{R}^n$ . This value is explicit in terms of the Jordan form of  $Jq$ , where

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix},$$

but we will not make use of an explicit expression.

We need the following definition to state our main theorems:

**Definition 1.6.** Let  $Z$  be a subset of  $M$ , and let

$$V_\delta(N) = \{m \in M, \text{dist}(m, Z) > N^{-\delta}\}.$$

A sequence  $(u_N)_{N \in \mathbb{N}}$ , with  $u_N \in \mathcal{H}_N(M, L)$  for every  $N$ , is said to *concentrate* on  $Z$  when, for every  $\delta \in ]0, \frac{1}{2}[$ , one has

$$\|u_N \mathbf{1}_{V_\delta(N)}\|_{L^2(M, L^{\otimes N})} = O(N^{-\infty}).$$

Note that concentration, in the sense of the definition above, implies microsupporting in the sense of Charles [4], that is, a  $O(N^{-\infty})$  estimate outside any fixed neighbourhood of the considered set.

**Theorem A.** *Let  $h$  be a smooth function on  $M$  that satisfies the wells condition of definition 1.4. For every  $N \in \mathbb{N}$ , let  $\lambda_N$  be the first eigenvalue of the operator  $T_N(h)$ , and  $u_N$  an associated normalized eigenfunction.*

*Then the sequence  $(u_N)_{N \in \mathbb{N}}$  concentrates on the vanishing points of  $h$  on which  $\mu$  is minimal.*

*If there is only one such point  $P_0$ , then there is a real sequence  $(a_k)_{k \geq 0}$  with  $a_0 = \mu(P_0)$  such that, for each  $K$ , one has*

$$\lambda_N = N^{-1} \sum_{k=0}^K N^{-K} a_k + O(N^{-K-2}).$$

*Moreover, if there is only one such point, then  $\lambda_N$  is simple, and there exists  $C > 0$  such that  $\lambda_N$  is the only eigenvalue of  $T_N$  in the interval  $[0, N^{-1}(\mu(P_0) + C)]$ .*

**Theorem B.** *Let  $h$  be a smooth function on  $M$  that satisfies the wells condition. Let  $C > 0$ . Then there is a bounded number of eigenvalues (counted with multiplicity) of  $T_N(h)$  in the interval  $[0, CN^{-1})$ .*

*More precisely, let  $K$  and  $(a_k)_{0 \leq k \leq K}$  be such that*

$$\{a_{0,k}, k \leq K\} = \bigcup_{\substack{P \in M \\ h(P)=0}} \text{Sp} \left( T_1^{\text{flat}}(q(P)) \right) \cap [0, C]$$

*with multiplicity.*

*Then one can find  $c > 0$  and a list of real numbers  $(b_k)_{0 \leq k \leq K}$  such that, for each  $k$ , one of the eigenvalues of  $T_N(h)$  lies in the interval*

$$[N^{-1}a_k + N^{-3/2}b_k - cN^{-2}, N^{-1}a_k + N^{-3/2}b_k + cN^{-2}].$$

*Moreover, there are at most  $K$  eigenvalues of  $T_N(h)$  in  $[0, CN^{-1})$  and each of them belongs to one of the intervals above.*

*For generic symbols, the eigenvalues in  $[0, CN^{-1})$  admit an expansion in integer powers of  $N^{-1}$ .*

The case of “miniwells”, a transposition of [13], will be treated in future work. Under analyticity conditions, we also hope to state results on exponential decay in the forbidden region and on tunnelling, like in [12].



## 1.4 Outline

We review in section 2 the definitions and semiclassical properties of the Szegő kernel. Using well-known results about the semiclassical expansion [19, 4, 6], we derive a useful lemma of universality.

In Section 3, we remind the reader of the symbolic properties of Toeplitz operators [18]. The state of the art is such that one can compose Toeplitz operators with classical symbols. We then show a new result on localization: low-energy eigenvalues concentrate (in some sense which we will make precise) where the symbol is minimal. Finally, we study in detail a particular case of Toeplitz operators, when the base manifold is  $\mathbb{C}^n$  and the symbol is a positive quadratic form.

Section 4 is devoted to the proof of Theorem A. We build an approximate eigenfunction of the Toeplitz operator and prove that the corresponding eigenvalue is the lowest one. The most important part is proposition 4.2 for which we use the same method than in [10]. For this, we consider the hessian of  $h$  at a cancellation point, as read in local coordinates; this is a function  $q$  on  $\mathbb{C}^n$ . Then we compare the Toeplitz operator  $T_N(h)$  with the Toeplitz operator  $T_N^{flat}(q)$ , which we call *model quadratic operator*.

In section 5, we modify the arguments used in section 4 to describe, under the same hypotheses on the symbol, the spectrum of a Toeplitz operator in the interval  $[0, CN^{-1})$  where  $C > 0$  is arbitrary (Theorem B).

## 2 The Szegő projector

### 2.1 Geometric context

We are mainly interested in this article with square-integrable holomorphic sections of Kähler manifolds, that is, the sequence of spaces  $\mathcal{H}_N(M, L) = L^2_{holo}(M, L^{\otimes N})$ . Each  $\mathcal{H}_N(M, L)$  is a closed subspace of  $L^2(M, L^{\otimes N})$ . To make their study easier, the following construction reformulates the spaces  $\mathcal{H}_N$  as subspaces of the *same*  $L^2$  space, following the idea of “Grauert tubes” (as presented in [9], chapter VI, section 6).

Let  $L^*$  denote the dual bundle of  $L$ , with  $h^*$  the dual metric. Define

$$\begin{aligned} D &= \{(m, v) \in L^*, h^*(v) < 1\} \\ X &= \partial D = \{(m, v) \in L^*, h^*(v) = 1\}. \end{aligned}$$

Let us consider the *Hardy space* on  $X$ , defined as follows.

**Definition 2.1.** If  $Y$  is the boundary of an open set  $U$  in a complex manifold, we call *Hardy space* on  $Y$  and denote by  $H(Y)$  the closed subspace

of  $L^2(Y)$  consisting of functions which are boundary values of holomorphic functions inside  $U$ . The orthogonal projector from  $L^2(Y)$  onto  $H(Y)$  is denoted by  $S$ , the *Szegő* projector.

In the paper,  $X$  plays the role of  $Y$ . The space  $H(X)$  can be further decomposed, because  $X$  admits an  $S^1$  action  $r_\theta : (m, v) \mapsto (m, e^{i\theta}v)$ . Thus, for  $N \in \mathbb{N}$ , we can say that an element  $f$  of  $H$  is  $N$ -equivariant when, for each  $x \in X$  and  $\theta \in S^1$ , the identity  $f(r_\theta x) = e^{iN\theta}f(x)$  holds. The space of  $N$ -equivariant functions is denoted by  $H_N(X)$ ; then  $H(X)$  is the orthogonal sum of the different spaces  $H_N(X)$  for  $N \geq 1$ . We will call  $S_N$  the orthogonal projection on  $H_N(X)$ .

The main point is that  $H_N(X)$  and  $\mathcal{H}_N(M, L)$  are isometric, the isometry being as follows: to any section  $s \in \mathcal{H}_N(M, L)$ , we can associate  $\hat{s} \in L^2(X)$  with the formula  $\hat{s}(m, v) = \langle v^{\otimes N}, s(m) \rangle$ . The function  $\hat{s}$  has the same norm as  $s$ . It belongs to  $H(X)$ , and it is the trace of the function on  $D$  defined with the same formula. Moreover,  $\hat{s}$  is  $N$ -equivariant. Reciprocally, each function in  $H_N(X)$  may be expressed this way.

As a helpful illustration (see [8] or [16] for a full review), we consider the usual  $n$ -dimensional complex space  $\mathbb{C}^n$ , with the natural Kähler structure. Strictly speaking, this is not an example as  $\mathbb{C}^n$  is not compact, but the previous definitions can be extended.

Because  $\mathbb{C}^n$  is contractile, the bundle  $L$  is isomorphic to  $\mathbb{C}^{n+1}$ , but the hermitian structure  $h$  is not the trivial one, whose curvature is zero. Indeed, one can show that  $h(m, v) = e^{-|m|^2}|v|^2$ . Here, the spaces  $H_N(\mathbb{C}^n, L)$  are called the *Bargmann spaces* and will be denoted by  $\mathcal{B}_N$ . They can be expressed as

$$\mathcal{B}_N \simeq \{f \text{ holomorphic in } \mathbb{C}^n, e^{-\frac{N}{2}|\cdot|^2}f(\cdot) \in L^2(\mathbb{C}^n)\}.$$

The projector on  $\mathcal{B}_N$  has a Schwartz kernel. Indeed, one Hilbert basis of  $\mathcal{B}_N$  is the family  $(e_\nu)_{\nu \in \mathbb{Z}^n}$  with  $e_\nu(z) = N^n \frac{N^{|\nu|/2} z^\nu}{\pi^n \sqrt{\nu!}}$ . Hence the kernel may be expressed as:

$$\Pi_N(x, y) = \frac{N^n}{\pi^n} \exp(-N/2|x|^2 - N/2|y|^2 + Nx \cdot \bar{y}). \quad (1)$$

The space  $\mathcal{B}_N$  is isometric to  $\mathcal{B}_1$  by a dilatation of factor  $N^{1/2}$ . Moreover, there is a unitary transformation between  $\mathcal{B}_1$  and  $L^2(\mathbb{R}^n)$ , called the *Bargmann transform*. The transformation  $B_1 : L^2(\mathbb{R}^n) \mapsto \mathcal{B}_1$  reads:

$$B_1 f(z) = \int \exp[-(1/2z \cdot z + 1/2x \cdot x - 2\sqrt{2}z \cdot x)] f(x) dx.$$

From  $B_1$ , one can deduce an isometry from  $\mathcal{B}_N$  to  $L^2(\mathbb{R}^n)$  by composing the scaling isometry and the Bargmann transform.

One noteworthy subspace of  $\mathcal{B}_1$  is the dense subset of functions  $f \in \mathcal{B}_1$  such that  $fP \in \mathcal{B}_1$  for any polynomial  $P$ . That space is denoted by  $\mathcal{D}$ . Any element of the previously given Hilbert basis belongs to  $\mathcal{D}$  and the Bargmann transform is a bijection from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{D}$ ; the preimage of  $e_\nu$  is the function  $x \mapsto C_\nu x^\nu e^{-|x|^2/2}$ , where  $C_\nu$  is a normalizing factor.

## 2.2 Semiclassical asymptotics

Semi-classical expansions of  $S_N$  are derived in [23, 19, 16, 4, 2], in different settings. In [23, 19], the Fourier Integral Operator approach is used to prove an asymptotic expansion of  $S_N$  in a neighbourhood of size  $N^{1/2}$  of a point. In [4, 16, 2], one derives asymptotic expansions of  $S_N$  in a neighbourhood of fixed size of a point.

The off-diagonal of the Szegő kernel is rapidly decreasing as  $N \rightarrow +\infty$ :

**Proposition 2.2** ([6], prop 4.1). *For every  $k \in \mathbb{N}$  and  $\epsilon > 0$ , there exists  $C > 0$  such that, for every  $N \in \mathbb{N}$ , for every  $x, y \in X$ , if*

$$\text{dist}(\pi(x), \pi(y)) \geq \epsilon,$$

then

$$|S_N(x, y)| \leq CN^{-k}.$$

The analysis of the Szegő kernel near the diagonal requires a convenient choice of coordinates, that we will use throughout this paper.

Let  $P_0 \in M$ . The real tangent space  $T_{P_0}M$  carries a euclidian structure and an almost complex structure coming from the Kähler structure on  $M$ . We then can (non-uniquely) identify  $\mathbb{C}^n$  with  $T_{P_0}M$ .

**Definition 2.3.** Let  $U$  be a neighbourhood of 0 in  $\mathbb{C}^n$  and  $V$  be a neighbourhood of  $P_0$  in  $M$ .

A smooth diffeomorphism  $\rho : U \times \mathbb{S}^1 \rightarrow \pi^{-1}(V)$  is said to be a *normal map* or map of *normal coordinates* under the following conditions:

- $\forall (z, v) \in U \times \mathbb{S}^1, \forall \theta \in \mathbb{R}, \rho(z, ve^{i\theta}) = r_\theta \rho(z, v)$ ;
- Identifying  $\mathbb{C}^n$  with  $T_{P_0}M$  as previously, one has:

$$\forall (z, v) \in U \times \mathbb{S}^1, \pi(\rho(z, v)) = \exp(z).$$

The following theorem states that, as  $N \rightarrow +\infty$ , in normal coordinates, the Szegő kernel has an asymptotical expansion whose first term is the flat kernel of equation (1):

**Proposition 2.4** ([6], theorem 4.18). *Let  $P_0 \in X$  and  $\rho$  a normal map on  $X$  such that  $\rho(0,0) = P_0$ . For  $z, w \in \mathbb{C}^n$  small enough and  $N \in \mathbb{N}$ , let*

$$S_N^{P_0}(z, w) := e^{-iN(\theta-\phi)} S_N(\rho(z, \theta), \rho(w, \phi)),$$

which does not depend on  $\theta$  and  $\phi$ .

There exist  $C > 0$ ,  $C' > 0$ ,  $m \in \mathbb{N}$ ,  $\epsilon > 0$  and a sequence of polynomials  $(b_j)_{j \geq 1}$ , with  $b_j$  of same parity as  $j$ , such that, for any  $N \in \mathbb{N}, K \geq 0$  and  $|z|, |w| \leq \epsilon$ , one has:

$$\left| S_N^{P_0}(z, w) - \Pi_1(z, w)^N \left( 1 + \sum_{j=1}^K N^{-j/2} b_j(\sqrt{N}z, \sqrt{N}w) \right) \right| \leq CN^{n-(K+1)/2} \left( 1 + |\sqrt{N}z| + |\sqrt{N}w| \right)^m e^{-C'\sqrt{N}|z-w|} + O(N^{-\infty}). \quad (2)$$

Hence, the typical range of interaction of  $S_N$  is of size  $N^{-1/2}$ . In particular, we are able to refine the proposition 2.2:

**Corollary 2.5.** *For every  $k \in \mathbb{N}$  and  $\delta > 0$ , there exists  $C > 0$  such that, for every  $N \in \mathbb{N}$ , for every  $x, y \in X$  with  $\text{dist}(\pi(x), \pi(y)) \geq N^{-\delta}$ , one has:*

$$|S_N(x, y)| \leq CN^{-k}.$$

In particular, if  $u \in L^2(X)$  is  $O(N^\infty)$  outside the pull-back of a ball of size  $N^{-\delta}$ , then  $S_N(u)$  is  $O(N^{-\infty})$  outside the pull-back of a ball of size  $2N^{-\delta}$ .

**Remark 2.6.** The proposition 2.4 gives asymptotics for the kernel of  $S_N$ , read in local coordinates. However, the normal maps of definition 2.3 do not preserve the volume form, except infinitesimally on the fiber over  $P_0$ .

If  $dVol$  is the volume form on  $X$  and  $dLeb$  is the Lebesgue form on  $\mathbb{C}^n$ , one has, for any normal map  $\rho$ :

$$\rho^*(dLeb \otimes d\theta) = h dVol,$$

for some function  $h$  on  $M$  with  $h(P_0) = 1$ . Hence, replacing  $S_N^{P_0}$  in equation (2) by the corresponding half-form corresponds to multiplying the other terms by  $\sqrt{h(z)h(w)}$ . On the one hand, for some  $m'$ , one has:

$$\begin{aligned} & \left| \Pi_1(z, w)^N \left( 1 + \sum_{j=1}^K N^{-j/2} b_j(\sqrt{N}z, \sqrt{N}w) \right) \left( 1 - \sqrt{h(z)h(w)} \right) \right| \\ & \leq C(|z| + |w|)^{m'} e^{-\frac{1}{2}N|z-w|^2} \\ & \leq CN^{n-(K+1)/2} \left( |\sqrt{N}z| + |\sqrt{N}w| \right)^{m'+K+1-2n} e^{-C'\sqrt{N}|z-w|}. \end{aligned}$$

On the other hand, there holds:

$$\begin{aligned} & N^{n-(K+1)/2} \left(1 + |\sqrt{N}z| + |\sqrt{N}w|\right)^m e^{-C'\sqrt{N}|z-w|} \left|1 - \sqrt{h(z)h(w)}\right| \\ & \leq N^{n-(K+1)/2} \left(1 + |\sqrt{N}z| + |\sqrt{N}w|\right)^{m+K+1-2n} e^{-C'\sqrt{N}|z-w|}. \end{aligned}$$

Hence, the effects of the volume form can be absorbed in the error terms of equation (2), and the proposition 2.4 also holds when  $S_N$  is replaced by the corresponding half-form.

### 2.3 Universality

In the previously given local expansions of the Szegő kernel (2), the dominant term is the projector on the Bargmann spaces of equation (1). Thus the Bargmann spaces appear to be a universal model for Hardy spaces, at least locally. To make this intuition more precise, we derive a useful proposition.

We can pull-back by a normal map the projector  $\Pi_N$  on the Bargmann spaces by the following formula:

$$\rho^*\Pi_N(\rho(z, \theta), \rho(w, \phi)) := e^{iN(\theta-\phi)}\Pi_N(z, w).$$

By convention,  $\rho^*\Pi_N$  is zero outside  $\pi^{-1}(V)^2$ .

**Proposition 2.7** (Universality). *Let  $\epsilon > 0$ . There exists  $\delta \in (0, 1/2)$ , a constant  $C > 0$  and an integer  $N_0$  such that, for any  $N \geq N_0$ , for any function  $u \in L^2(X)$  such that the support of  $u$  is contained in the fibres over a ball on  $M$  of radius  $N^{-\delta}$ , one has:*

$$\|\Pi_N^*u - S_N u\|_{L^2(X)} \leq CN^{-1/2+\epsilon}\|u\|_{L^2(X)}.$$

Here  $\Pi_N^*$  denotes the pull-back of  $\Pi_N$  by a normal map centred on a point in the support of  $u$ .

**Proof.** Equation (2), for  $K = 0$ , can be formulated as:

$$e^{-iN(\theta-\phi)}S_N(\rho(z, \theta), \rho(w, \phi)) = \Pi_1(z, w)^N + R(z, w) + O(N^{-\infty}),$$

with

$$|R(z, w)| \leq CN^{-1/2}(1 + |\sqrt{N}z| + |\sqrt{N}w|)^m e^{-C'\sqrt{N}|z-w|}.$$

Let  $\delta \in (0, 1/2)$  and  $u$  a function contained in the pull-back of a ball of size  $N^{-\delta}$ .

Let  $v = S_N u - \Pi_N^* u$ . Because of Corollary 2.5,  $v$  is  $O(N^{-\infty})$  outside  $\rho(B(0, 4N^{-\delta}) \times \mathbb{S}^1)$ . Hence, it is sufficient to consider the behaviour of the kernel  $S_N - \Pi_N^*$  on  $\rho(B(0, 4N^{-\delta}) \times \mathbb{S}^1) \times \rho(B(0, 4N^{-\delta}) \times \mathbb{S}^1)$ .

Let  $S_N^* : (z, \theta, w, \phi) \mapsto e^{-iN(\theta-\phi)} S_N(\rho(z, \theta), \rho(w, \phi))$  denote the kernel  $S_N$  as read in local coordinates. For  $|z|$  and  $|w|$  small enough, one has:

$$(S_N^* - \Pi_N)(z, w) = N^n R(z, w) + O(N^{-\infty}).$$

It remains to estimate the norm of the operator with kernel  $N^n R$ , using a standard result of operator theory:

**Lemma 2.8** (Schur test). *Let  $k \in C^\infty(V \times V)$  be a smooth function of two variables in an open subset  $V$  of  $\mathbb{R}^d$ . Let  $K$  be the associated unbounded operator on  $L^2(V)$ .*

Let

$$\|k\|_{L^\infty L^1} := \max \left( \sup_{x \in V} \|k(x, \cdot)\|_{L^1(V)}, \sup_{y \in V} \|k(\cdot, y)\|_{L^1(V)} \right).$$

If  $\|k\|_{L^\infty L^1}$  is finite, then  $K$  is a bounded operator. Moreover

$$\|K\|_{L^2(V) \rightarrow L^2(V)} \leq \|k\|_{L^\infty L^1}.$$

Thus, we want to estimate the quantity:

$$\sup_{|z| \leq 4N^{-\delta}} \int_{|w| \leq 4N^{-\delta}} N^{n-1/2} (1 + |\sqrt{N}z| + |\sqrt{N}w|)^m e^{-C'|z-w|}.$$

After a change of variables and up to a multiplicative constant, it remains to estimate:

$$N^{-1/2} \sup_{|z| \leq 4N^{1/2-\delta}} \int_{|u| \leq 4N^{1/2-\delta}} (1 + |z| + |u|)^m e^{-C|u|}.$$

This quantity is  $O(N^{(m-1)\frac{1}{2}-m\delta})$ . Thus, for any  $\epsilon > 0$ , there exists  $\delta$  such that the above quantity is  $O(N^{-\frac{1}{2}+\epsilon})$ .

By the Schur test, the  $L^2$  norm of a symmetric kernel operator is controlled by the  $L^\infty L^1$  norm of the kernel. When restricted on  $B(0, 4N^{-\delta})^2$ , the kernel of  $S_N^* - \Pi_N$  has a  $L^\infty L^1$  norm of order  $N^{-\epsilon}$ , from which we can conclude.  $\square$

### 3 Toeplitz operators

#### 3.1 Calculus of Toeplitz operators

Let us reformulate the definition 1.3 as acting on  $H_N(X)$ .

**Definition 3.1.** If  $f \in C^\infty(M)$  is a smooth function, one defines the *Toeplitz operator* with symbol  $f$  as the sequence of operators  $T_N(f) = S_N f(N^{-1})$  from  $H_N(X)$  to itself.

Alternative conventions exist for the quantization (associating an operator to a symbol), though they define the same class of operators. The convention of definition 3.1 is sometimes called contravariant [4]. The reason for this choice is that we rely crucially on the positivity condition: if  $f$  is real and nonnegative, then  $T_N(f)$  is nonnegative.

The composition of two Toeplitz operators is a formal series of Toeplitz operators. The theorem 2.2 of [18] states for instance that there exists a formal star-product on  $C^\infty(M)[[\nu]]$ , written as  $f \star g = \sum_{j=0}^{+\infty} \nu^j C_j(f, g)$ , that coincides with the Toeplitz operator composition: as  $N \rightarrow +\infty$ , one has, for every integer  $K$ , that

$$T_N(f)T_N(g) - \sum_{j=0}^K N^{-j} T_N(C_j(f, g)) = O(N^{-K}).$$

The functions  $C_j$  are bilinear differential operators of degree less than  $2j$ , and  $C_0(f, g) = fg$ .

#### 3.2 A general localization result

Using the  $C^*$ -algebra structure of Toeplitz operators, one can prove a fairly general localization result:

**Proposition 3.2.** *Let  $h$  be a smooth nonnegative function on  $M$ . Let  $Z = \{h = 0\}$ , and suppose that  $h$  vanishes exactly at order 2 on  $Z$ , that is, there exists  $C > 0$  such that  $h \geq C \text{dist}(\cdot, Z)^2$ .*

*Let  $t > 0$ , and define*

$$V_N := \{(m, v) \in X, \text{dist}(m, Z) < t\}.$$

*For every  $k \in \mathbb{N}$ , there exists  $C > 0$  such that, for every  $N \in \mathbb{N}$ , for every  $t > 0$ , and for every  $u \in H_N$  such that  $T_N(h)u = \lambda u$  for some  $\lambda \in \mathbb{R}$ , one has*

$$\|u\|_{L^2(X \setminus V_t)} \leq C \left( \frac{\max(\lambda, N^{-1})}{t^2} \right)^k.$$

**Proof.** By a trivial induction, the  $k$ -th star power of a symbol  $f$  is of the form

$$f^{\star k} = f^k + N^{-1}C_{1,k}(f, \dots, f) + N^{-2}C_{2,k}(f, \dots, f) + \dots,$$

where  $C_{i,k}$  is a  $k$ -multilinear differential operator of order at most  $2i$ .

We want to study  $C_{i,k}(h, \dots, h)$  for  $i \leq k$ . The function  $h$  is smooth and nonnegative, hence  $\sqrt{h}$  is a Lipschitz function. In other terms, there exists  $C$  such that, for every  $(x, \xi) \in TM$  with  $\|\xi\| \leq 1$ , one has  $\partial_\xi h(x) \leq C\sqrt{h(x)}$ .

In local coordinates, the function  $C_{i,k}(h, \dots, h)$  is a sum of terms of the form  $a\partial^{\nu_1} h \partial^{\nu_2} h \dots \partial^{\nu_k} h$ , where  $\sum_{j=1}^k |\nu_j| = 2i$  and  $a$  is smooth.

- If  $\nu_j = 0$ , then  $\partial^{\nu_j} h = h$ .
- If  $|\nu_j| = 1$ , then  $|\partial^{\nu_j} h| \leq C\sqrt{h}$ .
- If  $|\nu_j| \geq 2$ , then  $|\partial^{\nu_j} h| \leq C$ .

Hence  $|a\partial^{\nu_1} h \partial^{\nu_2} h \dots \partial^{\nu_k} h| \leq Ch^{k-i}$ , from which we can conclude :

$$|C_{i,k}(h, \dots, h)| \leq Ch^{k-i}.$$

This means that, for every  $k \geq 0$ , the function  $h^{\star k}$  is of the form:

$$h^{\star k} = h^k + \sum_{i=1}^{k-1} \nu^i f_{i,k} + \nu^{-k} g(\nu),$$

where  $g$  is bounded independently on  $\mu$  and where, for each  $i$  and  $k$  there exists  $C$  such that  $|f_{i,k}| \leq Ch^{k-i}$ .

Using this, we can prove by induction on  $k$  that there exists  $C_k$  such that, for every  $N$  and for every eigenvector  $u$  of  $T_N(h)$  with eigenvalue  $\lambda$ , one has

$$|\langle u, h^k u \rangle| \leq C_k \max(\lambda, N^{-1})^k \|u\|^2.$$

Indeed, this is clearly true for  $k = 1$ , because  $\langle u, hu \rangle = \lambda \|u\|^2$ .

Let us suppose that, for all  $j \leq k-1$ , there holds  $\langle u_N, h^j u_N \rangle = O(N^{-j})$ . Because  $u_N$  is an eigenvector for  $T_N(h)$ , it is an eigenvector for its powers, hence

$$T_N(h^{\star k})u = T_N(h)^k u + O(N^{-\infty}) = \lambda^k u + O(N^{-\infty}).$$

Replacing  $h^{\star k}$  by its expansion and using the fact that  $h \geq 0$ , we find:

$$\langle u, h^k u \rangle \leq CN^{-K} \|u\|^2 + \sum_{i=1}^{k-1} N^{-i} \langle u, f_{i,k} u \rangle.$$



Now recall  $|f_{i,k}| \leq C_{i,k}h^{k-i}$ , and the induction hypothesis:

$$\langle u, h^{k-i}u \rangle \leq C_i \max(\lambda, N^{-1})^{k-i} \|u\|^2$$

for every  $i > 0$ . Hence

$$\langle u, h^k u \rangle \leq O(N^{-k}) + \sum_{i=1}^{k-1} C_{i,k} C_i N^{-i} \max(\lambda, N^{-1})^{k-i} \|u\|^2,$$

hence there exists  $C_k$  such that  $\langle u, h^k u \rangle \leq C_k \max(\lambda, N^{-1})^k \|u\|^2$

Now we can conclude: for every  $k$ , there exists  $C$  such that, for every  $t > 0$  one has

$$\forall z \notin V_t, h^k \geq Ct^{2k}.$$

Finally, for every  $k$  there exists  $C$  such that, for every  $N \in \mathbb{N}$ ,  $t > 0$  and  $u$  an eigenvector of  $T_N(h)$  with eigenvalue  $\lambda$ , there holds

$$\|u 1_{X \setminus V_t}\|_{L^2} \leq C \left( \frac{\max(\lambda, N^{-1})}{t^2} \right)^k.$$

□

This proposition leads to the following definition, which we will use extensively in what follows:

**Definition 3.3.** Let  $A$  be a closed set of  $M$ , and  $u_N$  be a sequence of normalized elements of  $L^2(X)$ .

We say that  $u_N$  concentrates on  $A$  if for each  $\delta \in ]0, 1/2[$ , with

$$V_N = \{(m, v) \in X, \text{dist}(m, A) < N^{-\delta}\},$$

one has

$$\|u_N 1_{X \setminus V_N}\|_{L^2} = O(N^{-\infty}).$$

In this article, we will mainly specialize the proposition 3.2 to the case  $\lambda = O(N^{-1})$  and  $t = N^{-\delta}$  for  $0 < \delta < 1/2$ :

**Corollary 3.4.** Let  $u = (u_N)_{N \in \mathbb{N}}$  be a sequence of unit eigenvectors of  $T_N(h)$ , with sequence of eigenvalues  $\lambda_N = O(N^{-1})$ . Then  $u$  concentrates on the zero set of  $h$ .

We can reformulate the proposition 3.2 in these terms: if  $h$  is a positive smooth function on  $M$ , then any sequence of normalized eigenvectors of  $T_N(h)$  whose eigenvalues are  $O(N^{-1})$  concentrates on the zero set of  $h$ .

**Remark 3.5.** An independent work by Charles and Polterovich, that appears partially in [5], focuses on the case of a regular value of a real symbol, with similar results.

### 3.3 Quadratic symbols on the Bargmann spaces

Toeplitz operators can also be defined in the Bargmann spaces setting, but one should be careful about the domain of such operators.

This section is devoted to a full survey of the quadratic case, which is very useful as a model case for the general setting. Let  $q$  be a positive quadratic form on  $\mathbb{C}^n$ . Let

$$\mathcal{A}_N = \{f \in \mathcal{B}_N, e^{-N|\cdot|^2/2} \sqrt{q(\cdot)} f(\cdot) \in L^2(\mathbb{C}^n)\}.$$

Then  $\mathcal{A}_N$  is a dense subspace which contains  $\mathcal{D}$ . It is the domain of the positive quadratic form  $t_N : (u, v) \mapsto \int qu\bar{v}$ , and  $\mathcal{A}_N$  is closed for the norm  $\|u\|_{\mathcal{A}_N} = \|u\|_{L^2} + t_N(u, u)$ . Moreover, the injection

$$(\mathcal{A}_N, \|\cdot\|_{\mathcal{A}_N}) \rightarrow (\mathcal{B}_N, \|\cdot\|_{L^2})$$

is compact. Using the usual results of spectral theory, the associated operator  $T_N^{flat}$  is positive and has compact resolvent. The spectrum of  $T_N^{flat}$  thus consists of a sequence of nonnegative eigenvalues, whose only accumulation point is  $+\infty$ .

We now recall that the normalized scaling on  $\mathbb{C}^n$  by a factor  $N^{1/2}$  sends  $\mathcal{B}_N$  into  $\mathcal{B}$ . This conjugation sends  $T_N^{flat}$  to  $N^{-1}T_1^{flat}$ .

**Proposition 3.6.** *The first eigenvalue  $\mu_N$  of  $T_N^{flat}$  is simple.*

**Proof.** As  $q$  is positive a.e, the quadratic form  $t_N$  is strictly convex, hence the first eigenvalue is simple.  $\square$

**Proposition 3.7.** *Let  $T^*\mathbb{R}^n$  be identified with  $\mathbb{C}^n$ . Let  $Op_W(q)$  denote the Weyl quantization of  $Q_N$ , as a symbol in  $T^*\mathbb{R}^n$ , and recall that  $B_N$  is the  $N$ -th Bargmann transform. Then  $B_N T_N^{flat} B_N^{-1} = Op_W(q) + N^{-1} \text{tr}(q)$ .*

*In particular, the first eigenvalue of  $T_N^{flat}$  is positive.*

**Proof.** Let  $j, k$  be two indices in  $[[1, n]]$ .

If  $q : z \mapsto z_j z_k = (x_j + iy_j)(x_k + iy_k)$ , then  $\text{tr}(q) = 0$ , so the two operators should coincide.  $T_1$  is the operator of multiplication by  $z_j z_k$ . This operator is conjugated via  $B_1$  to the operator  $(x_j + i\partial_j)(x_k + i\partial_k) = x_j x_k - \partial_j \partial_k + ix_j \partial_k + i\partial_j x_k$ . Moreover, the Weyl quantization of  $q$  is the operator

$$Op_W(q) = x_j x_k - \partial_j \partial_k + \frac{i}{2}(\partial_k x_j + x_j \partial_k + \partial_j x_k + x_k \partial_j).$$

These two operators coincide whether  $j = k$  or not.

If  $q : z \mapsto \overline{z_j z_k} = (x_j - iy_j)(x_k - iy_k)$ , then again  $\text{tr}(q) = 0$ . Moreover,  $T_N^{flat}$  is the operator of holomorphic differentiation  $\partial_{z_j} \partial_{z_k}$ . This operator is conjugated via  $B_1$  to the operator  $(x_j - i\partial_j)(x_k - i\partial_k) = x_j x_k - \partial_j \partial_k - ix_j \partial_k - i\partial_j x_k$ . Moreover, the Weyl quantization of  $q$  is

$$Op_W(q) = x_j x_k - \partial_j \partial_k - \frac{i}{2}(\partial_k x_j + x_j \partial_k + \partial_j x_k + x_k \partial_j).$$

Again the two operators coincide.

If  $q : z \mapsto z_j \overline{z_k} = (x_j + iy_j)(x_k - iy_k)$ , then  $\text{tr}(q) = 2\delta_k^j$ . In that case,  $T_N^{flat} = \partial_{z_k} z_j$ . This operator is conjugated to  $(x_k - i\partial_k)(x_j + i\partial_j)$ . The Weyl quantization of  $q$  is

$$Op_W(q) = x_j x_k + \partial_j \partial_k + \frac{i}{2}(-\partial_k x_j - x_j \partial_k + \partial_j x_k + x_k \partial_j).$$

The two operators coincide when  $k \neq j$ , and when  $k = j$  the difference is 2.

From the conjugation, it is clear that the first eigenvalue of  $T_N^{flat}$  is positive, because the Weyl quantization of  $q$  is nonnegative and  $\text{tr}(q) > 0$ .  $\square$

Because  $T_N^{flat}$  is conjugated to  $N^{-1}T_1$ , one has  $\mu_N = N^{-1}\mu_1$ , and for some  $C > 0$ ,

$$\text{dist}(\mu_N, Sp(T_N^{flat}) \setminus \{\mu_N\}) = CN^{-1}.$$

The first eigenvalue  $\mu_1$  of  $T_1$  depends on  $q$ , but is invariant under an unitary change of variables on  $\mathbb{C}^n$ . From now on we will use the notation  $\mu(q)$  to denote  $\mu_1$ .

**Remark 3.8.** The computation of  $\mu(q)$  is non-trivial. As explained in [1], the first eigenvalue of  $Op_1^W(q)$  can be obtained the following way: let  $M \in S_{2n}^{++}(R)$  denote the symmetric matrix associated with  $q$  in the canonical coordinates. Let  $J$  be the matrix of the symplectic structure:

$$J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

Let  $A$  denote a Jordan form of  $JM$ . The diagonal of  $A$  is purely imaginary, and the blocks of  $A$  appear by pairs of blocks of the same size, one with diagonal value  $i\lambda$ , and one with diagonal value  $-i\lambda$ . Then  $\mu$  is the sum of the imaginary parts of the elements on the diagonal of  $A$  whose imaginary part is positive.

## 4 The first eigenvalue

### 4.1 Statement of the main result

If at a point  $P_0 \in M$  a non-negative function  $h$  vanishes with positive Hessian, one can build a convenient local map from a neighbourhood of  $P_0$  to a neighbourhood of 0 in  $\mathbb{C}^n$ . Thus the 2-jet of  $h$  at  $P_0$  maps to a positive quadratic form  $q$  on  $\mathbb{C}^n$ , up to a  $U(n)$  change of variables. Hence, the map associating to  $P_0$  the first eigenvalue  $\mu$  of the model quadratic operator  $T_N^{flat}(q)$  is well-defined. From now on, we will also call  $\mu$  this map.

**Theorem A.** *Let  $h$  be a smooth function on  $M$  that satisfies the wells condition of definition 1.4. For every  $N \in \mathbb{N}$ , let  $\lambda_N$  be the first eigenvalue of the operator  $T_N(h)$ , and  $u_N$  an associated normalized eigenfunction.*

*Then the sequence  $(u_N)_{N \in \mathbb{N}}$  concentrates on the vanishing points of  $h$  on which  $\mu$  is minimal.*

*If there is only one such point  $P_0$ , then there is a real sequence  $(a_k)_{k \geq 0}$  with  $a_0 = \mu(P_0)$  such that, for each  $K$ , one has*

$$\lambda_N = N^{-1} \sum_{k=0}^K N^{-K} a_k + O(N^{-K-2}).$$

*Moreover, if there is only one such point, then  $\lambda_N$  is simple, and there exists  $C > 0$  such that  $\lambda_N$  is the only eigenvalue of  $T_N$  in the interval  $[0, N^{-1}(\mu(P_0) + C)]$ .*

The method of proof is the following: for each vanishing point  $P_0$ , we construct a sequence of functions which concentrates on  $P_0$ , which is almost an eigenstate, and whose associated eigenvalue is equivalent to  $N^{-1}\mu(P_0)$ . We then show a positivity estimate for eigenfunctions concentrating on a single well. The uniqueness and the spectral gap property follow from a similar argument. At every step, we compare  $T_N(h)$  with the operator on  $\mathcal{B}_N$  whose symbol is the Hessian of  $h$  at the point of interest.

### 4.2 Existence

We let  $h$  denote a smooth function satisfying the wells condition. For every cancellation point of  $h$ , one can find a candidate for the ground state of  $T_N(h)$ . Instead of finding exact eigenfunctions, we search for approximate eigenfunctions. This is motivated by the following lemma:

**Lemma 4.1.** *Let  $T$  be a self-adjoint operator on a Hilbert space  $H$ ,  $\lambda \in \mathbb{R}$ , and  $u \in D(T)$  with norm 1.*

*Then  $\text{dist}(\lambda, \text{Sp}(H)) \leq \|T(u) - \lambda u\|$ .*

Let  $P_0 \in M$  be a point where  $h$  vanishes. Let  $\rho$  be a local map of normal coordinates in a neighbourhood of  $\pi^{-1}(P_0)$ . Recall from equation (2) that, for every  $N \in \mathbb{N}$  and every  $z, w \in \mathbb{C}^n$  such that  $(z/\sqrt{N}, 0, w\sqrt{N}, 0)$  belongs to the domain of  $\rho$ , one has

$$\begin{aligned} & N^{-n} \pi^n e^{i(\phi-\theta)} S_N \left( \rho \left( \frac{z}{\sqrt{N}}, \frac{\theta}{N} \right), \rho \left( \frac{w}{\sqrt{N}}, \frac{\phi}{N} \right) \right) \\ &= \Pi_1(z, w) \left( 1 + \sum_{k=1}^K N^{-k/2} b_k(z, w) \right) + R(z, w, N) + O(N^{-\infty}). \end{aligned} \quad (3)$$

Here the  $b_j$ 's are polynomials of the same parity as  $j$ , and

$$|r(z, w, N)| \leq C N^{n-(K+1)/2} e^{-C'|z-w|} (1 + |z|^m + |w|^m).$$

The functions that we will consider are supported on the pull-back of contractible open set of  $M$ , so we drop the fibre variable  $\theta$  and  $\phi$ .

The main proposition is

**Proposition 4.2.** *There exists a sequence  $(u_j)_{j \geq 0}$  of Schwartz functions in  $\mathbb{C}^n$ , with  $\langle u_0, u_k \rangle = \delta_k^0$ , and a sequence  $(\lambda_j)_{j \geq 0}$  of real numbers, with  $\lambda_0 = \mu(P_0)$  and  $\lambda_j = 0$  for  $j$  odd, such that, for each  $K$  and  $N$ , if  $u^K(N) \in L^2(X)$  and  $\lambda^K(N) \in \mathbb{R}$  are defined as:*

$$u^K(N)(\rho(z, \theta)) := e^{iN\theta} N^n \sum_{j=0}^K N^{-j/2} u_j(\sqrt{N}z),$$

$u^K(N)$  is supported in the image of  $\rho$ ,

$$\lambda^K(N) = N^{-1} \sum_{j=0}^K N^{-j/2} \lambda_j,$$

there holds, as  $N \rightarrow +\infty$ ,

$$\|S_N h S_N u^K(N) - \lambda^K(N) u^K(N)\| = O(N^{-(K+3)/2}).$$

**Remark 4.3.** The functions  $u^K(N)$  do not lie inside  $H_N(X)$ , because they are identically zero on an open set. Nevertheless, the operator  $S_N h S_N$  on  $L^2(X)$  decomposes orthogonally into the desired operator on  $H_N$ , and 0 on its orthogonal. Hence a nonzero eigenvalue of  $S_N h S_N$  must correspond to an eigenvalue of  $T_N(h)$  with same eigenspace. The same holds for almost eigenvalues.

Considering  $\lambda^K$  as a polynomial in  $N^{-1/2}$  whose odd terms vanish may seem surprising. However, in the proof, we construct  $\lambda^K$  as a polynomial in  $N^{-1/2}$ , as we do for  $u^K$ . The fact that it is a polynomial in  $N^{-1}$  is due to parity properties.

**Proof.** Let us try to solve the successive orders of

$$(S_N h S_N - \lambda^K(N)) u^K(N) \approx 0.$$

We write the Taylor expansion of  $h$  around  $P_0$  as

$$h(x) = q(x) + \sum_{j=3}^K r_j(x) + E(x).$$

Because of equation (3), the kernel of  $S_N h S_N$ , read in the map  $\rho$ , is:

$$\begin{aligned} S_N h S_N & \left( \rho \left( \frac{z}{\sqrt{N}} \right), \rho \left( \frac{w}{\sqrt{N}} \right) \right) \\ & = N^{-1} N^n \int \left( q(y) + \sum_{k=1}^K N^{-k/2} r_k(y) + N E(y/\sqrt{N}) \right) \\ & \quad \times \left[ \Pi_1(z, y) \left( 1 + \sum_{j=1}^K N^{-j} b_j(z, y, N) \right) + R(z, w, N) \right] \\ & \quad \times \left[ \Pi_1(y, z) \left( 1 + \sum_{l=1}^K N^{-l} b_l(y, w, N) \right) + R(y, z, N) \right] dy \\ & \quad + O(N^{-\infty}). \end{aligned}$$

The dominant order is simply

$$z, w \mapsto N^{n-1} \int_{\mathbb{C}^n} \Pi_1(z, y) q(y) \Pi_1(y, w) dy.$$

It is the kernel of the Toeplitz operator  $Q = T_1^{flat}(q)$  on  $B_1$  associated to the quadratic symbol  $q$ , which we studied in subsection 3.3. Its resolvent is compact, the first eigenvalue  $\mu(P_0)$  is simple, and if  $u_0$  is an associated eigenvector, the operator  $Q - \mu(P_0)$  has a continuous inverse on  $u_0^\perp$  which sends  $\mathcal{D}$  into itself. Moreover  $u_0$  is an even function.

This determines  $u_0$  and  $\lambda_0 = \mu(P_0)$ . Here  $u_0 \in \mathcal{D}$ , so we can truncate the function  $z, \theta \mapsto e^{iN\theta} N^n u_0(N^{1/2}z)$  to a function supported on the domain of  $\rho$ , with only  $O(N^{-\infty})$  error. The push-forward by  $\rho$  of this truncation, extended by zero outside the image of  $\rho$ , is denoted by  $u^0(N)$ .

The error is thus:

$$\begin{aligned} \int |S_N h S_N u^0(N) - N^{-1} \lambda_0 u^0(N)|^2 & \leq C N^{-2} \int A(z, y, w, N)^2 |u_0(w)|^2 dy dw \\ & \quad + O(N^{-\infty}), \end{aligned}$$

where

$$A(z, y, w, N) = N|E(y/\sqrt{N})\Pi_1(z, y)\Pi_1(y, w)| \\ + h(y) \left( |R(z, y, N)|\Pi_1(y, w) + |R(y, w, N)|\Pi_1(z, y) + |R(z, y, N)R(y, w, N)| \right).$$

Here,  $E$  is a Taylor remainder of order 3 on a compact set, so

$$|NE(y/\sqrt{N})| \leq C|y|N^{-1/2}.$$

Moreover, recall that

$$|R(z, y, N)| \leq CN^{-1/2}e^{-C'|z-y|}(1 + |z|^m + |y|^m).$$

Hence

$$|A(z, y, w, N)| \leq CN^{-1/2}e^{-C'|z-y|-C'|y-w|}(1 + |z|^m + |y|^m + |w|^m).$$

Because  $u_0 \in \mathcal{D}$ , there holds:

$$N^3 \int |S_N h S_N u_0 - N^{-1} \lambda_0 u_0|^2 \\ \leq C \iiint e^{-2C'|z-y|-2C'|y-w|}(1 + |z|^{2m} + |y|^{2m} + |w|^{2m}) |u_0(w)|^2 dy dz dx \\ + O(N^{-\infty}) \\ \leq C \left( \int |u|^{2m} e^{-C'|u|} du \right)^2 \int |w|^{2m} |u_0(w)|^2 dw + O(N^{-\infty}) \\ \leq C.$$

From there we deduce that  $u_0$  is an approximate eigenvector:

$$S_N h S_N u_0 - N^{-1} \lambda_0 u_0 = O(N^{-3/2}).$$

This proves the proposition for  $K = 0$ .

We construct by induction on  $K$  the following terms of the expansion.

For  $j \in \mathbb{N}$ , we let  $J_j : B_1 \mapsto L^2(e^{-|\cdot|^2} Leb)$ , unbounded and symmetric, whose kernel is

$$J_j(x, z) = \int \Pi_1(x, y)\Pi_1(y, z) \left( \sum_{k+l+m=j} b_k(x, y)r_{2+l}(z)b_m(y, z) \right) dy.$$

The dense subspace  $\mathcal{D}$  is included in the domain of  $J_j$ , moreover  $J_j(\mathcal{D}) \subset \mathcal{D}$ .

Let  $K \in \mathbb{N}$ , and suppose we found functions  $(u_k)_{k \leq K} \in \mathcal{D}$ , orthogonal to  $u_0$ , and of the same parity as  $k$ , and real numbers  $\lambda_k$  that vanish when  $k$  is odd, and such that, for each  $k \leq K$ , there holds:

$$(Q - \lambda_0)u_k = \sum_{j=1}^k J_j u_{k-j} + \lambda_k u_0 + \sum_{j=1}^{k-1} \lambda_j u_{k-j}. \quad (4)$$

Let us find  $u_{K+1}$ , orthogonal to  $u_0$ , and  $\lambda_{K+1}$  so that equation (4) also holds for  $k = K + 1$ : take the scalar product with  $u_0$ . As  $Q$  is symmetric, the left hand side vanishes, so  $\lambda_{K+1}$  is determined. Moreover if  $K + 1$  is odd, this equation boils down to  $\lambda_{K+1} = 0$ .

We now are able to find  $u_{K+1}$  because we can inverse  $Q - \lambda_0$  on the orthogonal set of  $u_0$ . Finally,  $u_{K+1}$  is of the same parity as  $K + 1$ .

It remains to show that to this sequence of functions  $u$  corresponds an approximate eigenvector of  $S_N h S_N$ .

Let  $K \geq 0$ , fixed in what follows. For each  $N \in \mathbb{N}$ , we can build a function  $u^K(N)$  on  $X$ , supported in the image of  $\rho$  and such that, for  $x$  in the image of  $\rho$ , one has  $u^K(N)(\rho(z, \theta)) = e^{iN\theta} N^n \sum_{k=0}^K N^{-k} u_k(\sqrt{N}z)$ . Note that  $u^K(N)$  concentrates on  $P_0$ .

Let

$$\lambda^K(N) = N^{-1} \sum_{k=0}^K N^{-k} \lambda_k.$$

We evaluate  $(S_N M_h S_N - \lambda^K(N))u^K(N) =: f^K(N)$ . Consider an open set  $V_1$ , containing  $P_0$ , and compactly included in the image of  $\rho$ . One has

$$\|f^K(N)\|_{L^\infty(cV_1)} = O(N^{-\infty})$$

because  $u^K(N)$  concentrates on  $P_0$ .

To compute  $f^K(N)$  in  $V_1$ , we use the equation (3) at order  $K$ . A change of variables leads to:

$$\begin{aligned} & N^{-n-1} e^{-iN\theta} f^K(N) \left( \rho(x/\sqrt{N}, \theta) \right) \\ &= \sum_{k=0}^K N^{-\frac{k}{2}} \left[ (Q - \lambda_0)u_k(x) - \sum_{j=1}^k J_j u_{k-j}(x) - \lambda_k u_0(x) - \sum_{j=1}^{k-1} \lambda_j u_{k-j}(x) \right] \\ &\quad + \sum_{k=K+1}^{2K} N^{-\frac{k}{2}} \left[ - \sum_{j=k-K}^K (J_j - \lambda_j) u_{k-j}(x) \right] \\ &+ \sum_{k,j,l=0}^K N^{-\frac{k+j+l}{2}} E_{j,l,N} u_k(x) + \sum_{k,j=0}^K N^{-\frac{k+j}{2}} E'_{j,N} u_k(x) + \sum_{k=0}^K N^{-\frac{k}{2}} E''_N u_k(x). \end{aligned}$$



By construction, the right-hand term of the first line vanishes. The second line  $O(N^{-(K+1)/2})$ .

There are three error terms in the last line.  $E_{j,l,N}$  is the operator with kernel:

$$E_{j,l,N}(x, z) = \int \Pi_1(x, y)\Pi_1(y, z)b_j(x, y)b_l(y, z)E(N^{-1/2}y)dy.$$

By dominated convergence, for each function  $u \in \mathcal{D}$ , one has

$$\|E_{j,l,N}(u)\|_{L^2} = O(N^{-(K+1)/2}).$$

In particular it is true of the functions  $u_k$ .

$E'_{j,N}$  is the operator with kernel:

$$\begin{aligned} E'_{j,N}(x, z) &= \int \Pi_1(x, y)b_j(x, y)h(N^{-1/2}y)R_K(y, z, N)dy \\ &\quad + \int \Pi_1(y, z)b_j(y, z)R_K(x, y, N)h(N^{-1/2}y)dy. \end{aligned}$$

For  $u \in \mathcal{D}$ , there holds, for  $m$  large enough:

$$\begin{aligned} &\int |E'_{j,N}u(x)|^2 dx \\ &\leq 2 \int \left| \iint \Pi_1(x, y)\Pi_1(y, z)b_j(x, y)h(N^{-1/2}y)r_K(y, z, N)u(z)dydz \right|^2 dx \\ &\leq CN^{-(K+1)} \iiint e^{-2C'|x-y|-2C'|y-z|}(1 + |x| + |y| + |z|)^m |u(z)|^2 dx dy dz \\ &\leq CN^{-(K+1)} \iiint e^{-C'|v|-C'|w|}(1 + |v|^m + |w|^m + |z|^m) |u(z)|^2 du dv dz \\ &\leq CN^{-(K+1)}. \end{aligned}$$

$E''_N$  is the operator with kernel

$$E''_N(x, z) = \int r_K(x, y, N)h(N^{-1/2}y)r_K(y, z, N)dy.$$

For  $u \in B_0$ , there holds, for  $m$  large enough:

$$\begin{aligned}
& N^{2(K+1)} \int |E_N''(x, z)u(x)|^2 dx \\
& \leq N^{2(K-1)} \int \left| \iint r_K(x, y, N)h(N^{-1/2}y)r_K(y, z, N)u(z)dydz \right|^2 dx \\
& \leq C \iiint e^{-C'|x-y|-C'|y-z|}(1 + |x-y|^m + |y-z|^m + |z|^m)|u(z)|^2 dx dy dz \\
& \leq C \iiint e^{-C'|v|-C'|w|}(1 + |u|^m + |v|^m + |w|^m)|u(z)|^2 dv dw dz \\
& \leq C.
\end{aligned}$$

This concludes the proof.  $\square$

From this proposition we conclude that, for every well  $P$ , *there exists* an eigenvalue of  $T_N(h)$  which has an asymptotical expansion in inverse powers of  $N$ , the dominant term being  $N^{-1}\mu(P)$ . In particular, the first eigenvalue of  $T_N(h)$  is  $O(N^{-1})$ .

### 4.3 Positivity

The following proposition implies that the first eigenfunctions only concentrate on the wells that are minimal:

**Proposition 4.4.** *Let  $(v_N)_{N \in \mathbb{N}}$  a sequence of normalized functions in  $L^2(X)$ . Suppose  $v$  localizes on a point  $P_0$ , on which  $h$  cancels. Then for each  $\epsilon > 0$  there exists  $N_0$  and  $C$  such that, if  $N > N_0$ ,*

$$\langle v_N, hv_N \rangle \geq N^{-1}\mu(P_0) - CN^{-3/2+\epsilon}.$$

**Proof.** Let  $\delta \leq 1/2$  be close to  $1/2$ . Let  $(z, \theta)$  denote normal coordinates around  $P_0$ , and  $\rho$  the associated map. Then the sequence  $(w_N)_{N \geq 0} = (\rho^*v_N)_{N > 0}$  is such that  $\|w_N\|_{L^2(c_B(0, N^{-\delta}))} = O(N^{-\infty})$ . Then

$$\begin{aligned}
\|\Pi_N w_N\|_{L^2(c_B(0, 2N^{-\delta}))} &= O(N^{-\infty}) \\
\|S_N^* w_N\|_{L^2(c_B(0, 2N^{-\delta}))} &= O(N^{-\infty})
\end{aligned}$$

as well.

Using the Proposition 2.7, for  $\delta$  close enough to  $\frac{1}{2}$ , if  $\Pi_N^*$  is a push-forward of  $\Pi_N$  by  $\rho$ , one has  $\|(S_N - \Pi_N^*)v_N\| \leq CN^{-\frac{1}{2}+\epsilon}$ . Hence, if  $S_N$  is a pull-back of  $S_N$  by  $\rho$ , one has  $\|(S_N^* - \Pi_N)w_N\| \leq CN^{-\frac{1}{2}+\epsilon}$ .

If  $q$  is the Hessian of  $h$  at  $P_0$  read in the chosen coordinates, the spectrum of the model quadratic operator  $\Pi_N q \Pi_N$  is known: one has

$$\langle w_N, \Pi_N q \Pi_N w_N \rangle \geq N^{-1} \mu(P_0) \|\Pi_N w_N\|^2.$$

Moreover, on  $B(0, 2N^{-\delta})$  the following holds :  $CN^{-2\delta} \geq h \geq q - CN^{-3\delta}$ .

Now, if  $\delta$  is close enough to  $\frac{1}{2}$ , one has:

$$\begin{aligned} & \langle w_N, S_N^* h S_N^* w_N \rangle \\ & \geq \langle w_N, S_N^* q S_N^* w_N \rangle - CN^{-3\delta} \\ & = \langle w_N, S_N^* q \Pi_N w_N \rangle + \langle w_N, S_N^* q (S_N^* - \Pi_N) w_N \rangle - CN^{-3\delta} \\ & \geq \langle w_N, S_N^* q \Pi_N w_N \rangle - CN^{-2\delta - \min(\delta, \frac{1}{2} - \epsilon)} \\ & = \langle w_N, \Pi_N q \Pi_N w_N \rangle + \langle w_N, (S_N^* - \Pi_N) q \Pi_N w_N \rangle - CN^{-2\delta - \min(\delta, \frac{1}{2} - \epsilon)} \\ & \geq \langle w_N, \Pi_N q \Pi_N w_N \rangle - CN^{-2\delta - \min(\delta, \frac{1}{2} - \epsilon)} \\ & \geq N^{-1} \mu(P_0) - CN^{-2\delta - \min(\delta, \frac{1}{2} - \epsilon)}. \end{aligned}$$

This concludes the proof.  $\square$

**Remark 4.5.** In the proof, we used not fully the fact that  $v$  localizes on  $P_0$ , but only the fact that, for some  $\delta$  determined by the geometry of  $M$ , one has

$$\|v_N 1_{\pi(x) \notin B(P_0, N^{-\frac{1}{2} + \delta})}\|_{L^2} = O(N^{-\infty}).$$

Thus, this proposition could be used in a more general context.

#### 4.4 Uniqueness and spectral gap

**Proposition 4.6.** *Suppose  $h$  satisfies the wells condition, and that there is only one well with minimal  $\mu$ . Then the approximate eigenvalue of proposition 4.2 associated to this well corresponds to the first eigenvalue  $\lambda_N$  of  $T_N(h)$ . This eigenvalue is simple; moreover there exists  $C > 0$  such that, for  $N$  large enough*

$$\text{dist}(\lambda_N, \text{Sp}(T_N(h)) \setminus \lambda_N) \geq CN^{-1}.$$

**Proof.** The proposition is equivalent to the following claim: let  $u_K(N)$  denote the approximate eigenvector of order  $K$  associated to the well with minimal  $\mu$ . Let  $F_N$  be the orthogonal complement of  $u_K(N)$  in  $H_N(X)$ , and  $P_N$  be the orthogonal projection from  $H_N(X)$  to  $F_N$ . Then the operator

$T_N^\sharp(h) : F_N \rightarrow F_N$ , defined as  $T_N^\sharp(h) = P_N T_N(h)$ , is bounded from below by  $\lambda_N + CN^{-1}$ .

Let  $v_N$  be a sequence of normalized eigenvectors of  $T_N^\sharp$ , and  $\mu_N$  the sequence of associated eigenvalues. One has  $T_N(h)v_N = \mu_N v_N + C_N u_K(N)$ . Because  $u_K$  is a sequence of normalized functions and  $S_N$  is bounded, the sequence  $C_N$  is bounded.

Assume  $\mu = O(N^{-1})$ . In this slightly different setting, we can repeat the proof of the proposition 3.2 using the fact that  $u_K(N)$  is itself an eigenfunction of  $T_N(h)$ . There holds:

$$T_N(h^{*k})v_N = \mu_N^k v_N + C_N \sum_{j=1}^k \mu_N^j \lambda_N^{k-j} u_N.$$

Hence  $v_N$  concentrates on the vanishing points of  $h$ .

If  $P_0, P_1, \dots, P_d$  are the vanishing points of  $h$ , one can decompose  $v_N = v_{0,N} + v_{1,N} + \dots + v_{d,N} + O(N^{-\infty})$ , where each sequence  $v_{i,N}$  concentrates on  $P_i$ . The proposition 4.4 gives estimates for  $v_{i,N}$  if  $i \neq 0$ . Namely, if  $C$  is such that  $N\lambda_N + C < \mu(P_i)$  for all  $i$  and for  $N$  large enough, then

$$\langle v_{i,N}, S_N h S_N v_{i,N} \rangle \geq (\lambda_N + CN^{-1}) \|v_{i,N}\|_2^2.$$

Recall that  $u_K(N)$  has an asymptotic expansion whose first term  $u_0$  is the pull-backed ground state of the operator on the Bargmann space with quadratic symbol. This operator has a spectral gap of order  $N^{-1}$ . Moreover  $\langle v_{0,N}, u_0 \rangle = o(1)$ , then for  $C$  strictly smaller than the spectral gap of the quadratic operator, one has for  $N$  large

$$\langle v_{0,N}, S_N h S_N v_{0,N} \rangle \geq (\lambda_N + CN^{-1}) \|v_{0,N}\|_2^2.$$

The functions  $v_{i,N}$  are orthogonal to each other with disjoint support, so that  $\langle v_{i,N}, S_N h S_N v_{j,N} \rangle = O(N^{-\infty})$  whenever  $i \neq j$ , and  $\|v_N\|_2^2 = \sum_j \|v_{j,N}\|_2^2 + O(N^{-\infty})$ . Thus the two inequalities allow us to conclude.  $\square$

## 4.5 End of the proof

It remains to show that, in the case where only one well  $P_0$  has minimal  $\mu$ , then the ground state is  $O(N^{-\infty})$  in a fixed neighbourhood of the other wells.

Let  $K$  an integer. We have constructed in subsection 4.2 a sequence  $(U_K(N))_{N \in \mathbb{N}}$  which vanishes outside a fixed neighbourhood of  $P_0$ , and

which is a sequence of approximate unit eigenvectors of  $T_N(h)$ , with approximate eigenvalue  $\lambda_K(N)$ . One has

$$\lambda_K(N) = N^{-1}\mu(P_0) + O(N^{-3/2}),$$

and

$$\text{dist}(\lambda_K(N), \text{Sp}(T_N(h))) = O(N^{-K}).$$

Moreover we proved in subsection 4.4 that there can be only one eigenvalue of  $T_N(h)$  in  $[0, N^{-1}(\mu(P_0) + C)]$  for some  $C$ , and that this eigenvalue is simple. Hence, denoting  $\lambda_\infty(N)$  this sequence of eigenvalues, one has

$$\lambda_\infty(N) = \min \text{Sp}(T_N(h)),$$

and

$$|\lambda_\infty(N) - \lambda_K(N)| = O(N^{-K}).$$

Let  $U_\infty(N)$  denote a sequence of unit eigenvectors associated to  $\lambda_\infty(N)$ , and decompose  $U_K(N) = c(N)U_\infty(N) + w_K(N)$ , where  $w_K(N) \perp U_\infty(N)$ . Then

$$(T_N(h) - \lambda_\infty(N))w_K(N) = O(N^{-K}).$$

The operator  $T_N(h) - \lambda_\infty(N)$  is invertible on  $U_\infty(N)^\perp$  and its inverse has a norm bounded by  $N$ , so  $w_K(N) = O(N^{-K+1})$ . Because both  $U_K$  and  $U_\infty$  are normalized, one has  $c(N) \rightarrow 1$ .

Finally, if  $V$  is a neighbourhood of another well, then  $U_K(N)$  is zero on  $V$ , so that

$$\|U_\infty(N)\|_{L^2(V)} = \|w_K(N)\|_{L^2(V)} = O(N^{-K+1}).$$

This concludes the proof.

## 5 Eigenvalues in a scaled window

### 5.1 Statement of the result

The last section is devoted to the proof of the following theorem :

**Theorem B.** *Let  $h$  be a smooth function on  $M$  that satisfies the wells condition. Let  $C > 0$ . Then there is a bounded number of eigenvalues (counted with multiplicity) of  $T_N(h)$  in the interval  $[0, CN^{-1}]$ .*

*More precisely, let  $K$  and  $(a_k)_{0 \leq k \leq K}$  be such that*

$$\{a_{0,k}, k \leq K\} = \bigcup_{\substack{P \in M \\ h(P)=0}} \text{Sp}\left(T_1^{\text{flat}}(q(P))\right) \cap [0, C]$$

with multiplicity.

Then one can find  $c > 0$  and a list of real numbers  $(b_k)_{0 \leq k \leq K}$  such that, for each  $k$ , one of the eigenvalues of  $T_N(h)$  lies in the interval

$$[N^{-1}a_k + N^{-3/2}b_k - cN^{-2}, N^{-1}a_k + N^{-3/2}b_k + cN^{-2}].$$

Moreover, there are at most  $K$  eigenvalues of  $T_N(h)$  in  $[0, CN^{-1})$  and each of them belongs to one of the intervals above.

For generic symbols, the eigenvalues in  $[0, CN^{-1})$  admit an expansion in integer powers of  $N^{-1}$ .

## 5.2 Approximate eigenvectors

In the proof of the proposition 4.2, the first guess for an approximate eigenvector of  $T_N(h)$  was the first eigenvector of the model quadratic operator at one of the wells.

If, instead of the first eigenvector, we start from any eigenvector of the model, we can proceed the same way; however the recursion stops after one step, in general.

**Proposition 5.1.** *Let  $P \in M$  on which  $h$  cancels, and  $Q$  be a model quadratic operator in some normal coordinates around  $P$ . Let  $\lambda$  be an eigenvalue of  $Q$ .*

*Then one can find a suitable orthonormal basis of the eigenspace  $E_\lambda$  so that, from each base vector, one can build a sequence of approximate eigenvectors of  $T_N(h)$ , modulo  $O(N^{-2})$ .*

*Moreover, if  $\dim E_\lambda = 1$ , then from a unit eigenvector of  $Q$ , one can build a sequence of approximate eigenvectors of  $T_N(h)$ , modulo  $O(N^{-\infty})$ .*

**Proof.** Recall from proposition 4.2 that one can find an approximate eigenvector at any order, starting from the ground state  $u_0$  of  $Q$ .

Let us replace  $u_0$  by an arbitrary eigenfunction of  $Q$ , which still belongs to  $\mathcal{D}$ . Let  $\lambda$  be the associated eigenvalue. When  $\lambda$  is a simple eigenvalue, one can solve equation (4) at any order. Observe that  $u_0$  is either even or odd, so that only negative integer powers of  $N$  remain in the expansion of the eigenvalue.

If  $Q - \lambda$  is not invertible on  $u_0^\perp$ , the equation (4) can still be solved for  $k = 1$ . Consider an orthonormal basis  $(v_1, \dots, v_L)$  of the eigenspace  $E_\lambda$ . Suppose  $u_0 = v_1$ . The equation (4) reads :

$$(Q - \lambda)u_1 = J_1 u_0 + \lambda_1 u_0.$$

Taking the scalar product with  $u_0$  yields  $\lambda_1 = -\langle v_1, J_1 v_1 \rangle$ . But we also need to check that  $0 = \langle v_l, J_1 v_1 \rangle$  for  $l \neq 1$ . This is done by choosing an

orthogonal basis in which  $J_1$  is diagonal. One can then find  $u_1$  in  $E_\lambda^\perp$ . The proof of the error estimate is the same.  $\square$

### 5.3 Uniqueness properties

Now we prove that each eigenspace of  $T_N(h)$ , with eigenvalue less than  $CN^{-1}$ , is spanned by eigenfunctions that we have constructed in the previous proposition.

**Proposition 5.2.** *Let  $C > 0$ . Consider the set  $E$  of approximate eigenvectors that we can construct with the proposition 5.1, starting from eigenvectors of model quadratic operators with eigenvalue less than  $C$ . Then, for all  $\epsilon > 0$ , the operator  $T_N(h)$ , corestricted on  $E^\perp$ , is bounded from below by  $(C - \epsilon)N^{-1}$ .*

**Proof.** We simply mimic the proof of the uniqueness proposition in the last section.

Let  $v_N$  be a sequence of eigenvalues of  $T_N(h)$ . Then  $v_N$  is  $O(N^{-\infty})$  outside any fixed neighbourhood of  $\{h = 0\}$ . Decomposing  $v_N = \sum v_{i,N} + O(N^{-\infty})$ , where  $v_{i,N}$  is supported in a small ball around  $P_i$ , the pull-back of  $v_{i,N}$  is approximatively orthogonal (with  $O(N^{-1/2})$  error) to all eigenfunctions of the local quadratic operator whose eigenvalues are less than  $C$ . One then has

$$\langle v_{i,N}, \Pi_N^* q_i \Pi_N^* v_{i,N} \rangle \geq (C - \epsilon/2)N^{-1} \|v_{i,N}\|^2,$$

from which one can deduce

$$\langle v_{i,N}, S_N h S_N v_{i,N} \rangle \geq (C - \epsilon)N^{-1} \|v_{i,N}\|^2.$$

$\square$

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