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On a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture

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Abstract

This paper is dedicated to studying the following question: Is it always possible to injectively assign the weights $1, \dots, |E(G)|$ to the edges of any given graph G (with no component isomorphic to K_2) so that every two adjacent vertices of G get distinguished by their sums of incident weights? One may see this question as a combination of the well-known 1-2-3 Conjecture and the Antimagic Labelling Conjecture.

Throughout this paper, we exhibit evidence that this question might be true. Benefiting from the investigations on the Antimagic Labelling Conjecture, we first point out that several classes of graphs, such as regular graphs, indeed admit such assignments. We then show that trees also do, answering a recent conjecture of Arumugam, Premalatha, Bača and Semaničová-Feňovčíková. Towards a general answer to the question above, we then prove that claimed assignments can be constructed for any graph, provided we are allowed to use some number of additional edge weights. For some classes of sparse graphs, namely 2-degenerate graphs and graphs with maximum average degree 3, we show that only a small (constant) number of such additional weights suffices.

Keywords: 1-2-3 Conjecture, Antimagic Labelling Conjecture, equitable edge-weightings, neighbour-sum-distinguishing edge-weightings.

1. Introduction

In order to present our investigations in this paper, as well as our motivations, we first need to introduce a few particular graph concepts and notions. We refer the reader to textbooks on graph theory for more details on any standard notion or terminology not introduced herein.

Given a (undirected, simple, loopless) graph G and a set W of weights, by a W -edge-weighting of G we mean an edge-weighting with weights from W . For any $k \geq 1$, a k -edge-weighting is a $\{1, \dots, k\}$ -edge-weighting. Given an edge-weighting w of G , one can compute, for every vertex v of G , the sum $\sigma(v)$ (or $\sigma_w(v)$ when more precision is needed)

of weights assigned by w to the edges incident to v . That is,

$$\sigma_w(v) := \sum_{u \in N(v)} w(vu)$$

for every vertex v of G . In case we have $\sigma_w(u) \neq \sigma_w(v)$ for every edge uv of G , we call w *neighbour-sum-distinguishing*. It can be observed that every graph with no connected component isomorphic to K_2 admits neighbour-sum-distinguishing edge-weightings using sufficiently large weights. In the context of the current investigations, when speaking of a *nice graph* we mean a graph with no connected component isomorphic to K_2 . For a nice graph G , it hence makes sense to study the smallest k such that G admits a neighbour-sum-distinguishing k -edge-weighting. We denote this chromatic parameter by $\chi_\Sigma^e(G)$.

Throughout this paper, we deal with edge-weightings that are not only neighbour-sum-distinguishing but also do not assign any edge weight more than once. We say that such edge-weightings are *edge-injective*. Still under the assumption that G is a nice graph, we denote by $\chi_\Sigma^{e,1}(G)$ the smallest k such that G admits an edge-injective neighbour-sum-distinguishing k -edge-weighting.

In this paper, we consider the following conjecture. Our motivations for studying this conjecture, as well as our evidences to suspect that it might be true, are described below.

Conjecture 1.1. *For every nice graph G , we have $\chi_\Sigma^{e,1}(G) = |E(G)|$.*

By the edge-injectivity property, we note that $|E(G)|$ is a lower bound on $\chi_\Sigma^{e,1}(G)$ for every nice graph G . Conjecture 1.1, in brief words, hence asks whether, for every nice graph G , we can bijectively assign weights $1, \dots, |E(G)|$ to the edges of G so that no two adjacent vertices of G get the same value of σ .

Conjecture 1.1 is related to the well-known **1-2-3 Conjecture**, raised in 2004 by Karoński, Łuczak and Thomason [10], which states the following.

1-2-3 Conjecture. *For every nice graph G , we have $\chi_\Sigma^e(G) \leq 3$.*

Many aspects of the 1-2-3 Conjecture have been studied in literature. For an overview of those considered aspects, we refer the interested reader to the wide survey [11] by Seamone, which is dedicated to this topic. Our investigations in this paper are mostly related to a recent equitable variant of the 1-2-3 Conjecture that was considered by Baudon, Piłśniak, Przybyło, Senhaji, Sopena and Woźniak in [3]. In this variant, the authors studied, for some families of nice graphs, the existence of neighbour-sum-distinguishing edge-weightings being *equitable*, i.e. in which any two distinct edge weights are assigned about the same number of times (being equal, or differing by 1). In particular, they introduced and studied, for any given graph G , the chromatic parameter denoted by $\overline{\chi}_\Sigma^e(G)$ being the smallest maximal weight in an equitable neighbour-sum-distinguishing edge-weighting of G . In brief words, they proved that, at least for particular common classes of nice graphs (such as complete graphs and some bipartite graphs), the two parameters χ_Σ^e and $\overline{\chi}_\Sigma^e$ are equal except for a few exceptions.

Despite their results, the authors of [3] did not dare addressing a general conjecture on how should $\overline{\chi}_\Sigma^e$ behave in general, or compared to χ_Σ^e for a given nice graph. In particular, it does not seem obvious how big $\overline{\chi}_\Sigma^e$ can be, neither whether this parameter can be arbitrarily large. This is one of our motivations for studying edge-injective neighbour-sum-distinguishing edge-weightings, as an edge-injective edge-weighting is always equitable. Thus, $\overline{\chi}_\Sigma^e(G) \leq \chi_\Sigma^{e,1}(G)$ holds for every nice graph G . Hence, attacking Conjecture 1.1 can be regarded as a way to get progress towards all those questions.

Our second motivation for considering Conjecture 1.1 is that edge-injective neighbour-sum-distinguishing edge-weightings can be regarded as a weaker notion of well-known **antimagic labellings**. Formally, using our own terminology, an *antimagic labelling* w of a graph G is an edge-injective $|E(G)|$ -edge-weighting of G for which σ_w is injective, i.e. all vertices of G get a distinct sum of incident weights by w . We say that G is *antimagic* if it admits an antimagic labelling. Many lines of research concerning antimagic labellings can be found in literature, most of which are related to the following conjecture addressed by Hartsfield and Ringel in [8].

Antimagic Labelling Conjecture. *Every nice connected graph is antimagic.*

Despite lots of efforts (refer to the dynamic survey [7] by Gallian for an in-depth summary of the vast and rich literature on this topic), the Antimagic Labelling Conjecture is still open in general, even for common classes of graphs such as nice trees. Conjecture 1.1, which is clearly much weaker than the Antimagic Labelling Conjecture, as the distinction condition here only concerns the adjacent vertices, hence sounds as a much easier challenge to us, in particular concerning classes of nice graphs that are not known to be antimagic.

Hence, every antimagic graph G agrees with Conjecture 1.1, implying, as described earlier, that

$$\overline{\chi}_{\Sigma}^e(G) \leq \chi_{\Sigma}^{e,1}(G) = |E(G)|$$

holds, thus providing an upper bound on $\overline{\chi}_{\Sigma}^e(G)$ for G . This is of interest as several classes of graphs, such as nice regular graphs and nice complete partite graphs, are known to be antimagic, see [7]. Let us here further mention the works of Bérci, Bernáth and Vizer [4], and of Cranston, Liang and Zhu [6], who led to the verification of the Antimagic Labelling Conjecture for nice regular graphs, and whose some proof techniques partly inspired some used in the current paper. Conversely, proving that a graph G verifies $\chi_{\Sigma}^{e,1}(G) = |E(G)|$ and agrees with Conjecture 1.1 is similar to proving that, in some sense, G is “locally antimagic”.

Conjecture 1.1 can essentially be considered as a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture, as the notions behind it have flavours of both conjectures. As described earlier, proving Conjecture 1.1 for some classes of graphs has, to some extent, consequences on the 1-2-3 Conjecture and the Antimagic Labelling Conjecture, or at least on variants of these conjectures.

Our work in this paper, is focused on both proving Conjecture 1.1 for particular classes of nice graphs, and providing upper bounds on $\chi_{\Sigma}^{e,1}$ for some classes of nice graphs. This paper is organized as follows. Tools and preliminary results we use throughout are introduced in Section 2. After that, we start off by providing support to Conjecture 1.1 in Section 3, essentially by showing and pointing out that the conjecture holds for some classes of graphs, such as nice trees and regular graphs. Towards Conjecture 1.1, we then provide, in Section 4, general weaker upper bounds on $\chi_{\Sigma}^{e,1}$. These bounds are then improved for some classes of nice sparse graphs in Section 5. These classes include nice graphs with maximum average degree at most 3 and nice 2-degenerate graphs. Concluding comments are gathered in Section 6.

Remark: During the review process, we have been notified that a paper introducing the notion of “locally antimagic graphs”, written by Arumugam, Premalatha, Bača and Semaničová-Feňovčíková, appeared online [2]. That paper and the current one consider different aspects of this notion. Namely, [2] is focused on the smallest number of colour sums by an edge-injective neighbour-sum-distinguishing $|E(G)|$ -edge-weighting. In particular, our Theorem 3.3 on trees answers positively to Conjecture 2.3 in [2].

2. Preliminary remarks and results

In this section, we introduce several observations that will be of some use in the next sections. Conjecture 1.1 is mainly about k -edge-weightings; however, to lighten some proofs, we will rather focus on edge-weightings assigning strictly positive weights only. The reader should keep this detail in mind.

We start off by pointing out a few situations in which, for a given edge uv of any graph G , we necessarily get $\sigma(u) \neq \sigma(v)$ by an edge-injective edge-weighting of G . We omit a formal proof as it is easily seen that these claims are true. We note that the third item is more general, as it implies the other two.

Observation 2.1. *Let G be a graph, and w be an edge-injective edge-weighting of G . Then, for every edge uv of G , we have $\sigma(u) \neq \sigma(v)$ in any of the following situations:*

1. $d(u) = 1$ and $d(v) \geq 2$;
2. $d(u) = d(v) = 2$;
3. $d(u) \geq d(v)$ and

$$\min \{w(uv') : v' \in N(u) \setminus \{v\}\} \geq \max \{w(vu') : u' \in N(v) \setminus \{u\}\}.$$

We now observe that to be able to successfully extend a partial neighbour-sum-distinguishing edge-weighting to an edge, we need to have sufficiently distinct weights in hand for that purpose.

Observation 2.2. *Let G be a graph, uv be an edge of G , and w be a neighbour-sum-distinguishing edge-weighting of $G - \{uv\}$ such that $\sigma(u) \neq \sigma(v)$. Then w can be successfully extended to uv , provided we have a set W of at least $d(u) + d(v) - 1$ distinct strictly positive weights that can be assigned to uv .*

Proof. We note that w currently must satisfy $\sigma(u) \neq \sigma(v)$, as, otherwise, no matter what weight we assign to uv , we would eventually get $\sigma_w(u) = \sigma_w(v)$. Under that assumption, we note that weighting uv with any weight completely determines the value of both $\sigma_w(u)$ and $\sigma_w(v)$. The value of $\sigma_w(u)$ eventually has to be different from the sums of weights incident to the $d(u) - 1$ neighbours of u different from v . Similarly, the value of $\sigma_w(v)$ eventually has to be different from the sums of weights incident to the $d(v) - 1$ neighbours of v different from u . The neighbours of u and v hence forbid us from assigning at most $d(u) + d(v) - 2$ possible distinct weights to uv . Now, since weighting uv with distinct weights results in distinct values of $\sigma_w(u)$ and $\sigma_w(v)$, it should be clear that we can find a correct weight for uv in W , provided W includes at least $d(u) + d(v) - 1$ distinct weights. \square

Throughout this paper, several of the proofs consist in deleting two adjacent edges vu_1 and vu_2 from G , edge-weighting the remaining graph, and correctly extending the weighting to vu_1 and vu_2 . In this regard, we will often refer to the following result, which is about the number of weights that are sufficient to weight vu_1 and vu_2 .

Observation 2.3. *Let G be a graph having two adjacent edges vu_1 and vu_2 such that $G' := G - \{vu_1, vu_2\}$ admits a neighbour-sum-distinguishing edge-weighting $w_{G'}$. Assume further that $d_G(u_1) \geq d_G(u_2)$, and set*

$$\mu := (d_G(u_1) + 1) + \max \{0, d_G(v) + d_G(u_2) - d_G(u_1) - 1\}.$$

Then, assuming we have a set W of at least μ distinct strictly positive weights, we can extend $w_{G'}$ to a neighbour-sum-distinguishing edge-weighting of G by assigning two distinct weights of W to vu_1 and vu_2 .

Proof. We extend $w_{G'}$ to a neighbour-sum-distinguishing edge-weighting w_G of G by first assigning a weight of W to vu_1 , and then assigning a distinct weight to vu_2 . We determine, in this proof, the smallest number μ of weights that W should contain so that this strategy has sufficiently many weights to be successfully applied.

We note that extending $w_{G'}$ to vu_1 completely determines the value of $\sigma_{w_G}(u_1)$, while the value of $\sigma_{w_G}(v)$ is not determined until vu_2 is also weighted. Hence, when first weighting vu_1 , we mainly have to make sure that $\sigma_{w_G}(u_1)$ does not get equal to the sum of weights incident to a neighbour of u_1 different from v . Also, we should make sure that $\sigma_{w_{G'}}(v) + w_G(vu_1)$ does not get equal to $\sigma_{w_{G'}}(u_2)$, as otherwise we would necessarily get $\sigma_{w_G}(v) = \sigma_{w_G}(u_2)$ no matter how we weight vu_2 . There are hence $d_G(u_1)$ conflicts to take into account when weighting vu_1 . Provided W includes at least $d_G(u_1) + 1$ distinct weights, we can hence weight vu_1 correctly, i.e. so that we avoid all conflicts mentioned above, with one weight from W , since assigning different weights to vu_1 alters $\sigma_{w_G}(u_1)$ in distinct ways.

Now assume vu_1 has been weighted with the additional property that $\sigma_{w_{G'}}(v) + w_G(vu_1) \neq \sigma_{w_{G'}}(u_2)$. Since that property holds, Observation 2.2 tells us that we can correctly extend $w_{G'}$ to vu_2 provided $W \setminus \{w_G(vu_1)\}$ includes at least $d_G(v) + d_G(u_2) - 1$ distinct weights. We hence need $W \setminus \{w_G(vu_1)\}$ to include that many distinct weights.

As explained above, W necessarily includes at least $d_G(u_1)$ weights that were not assigned to vu_1 . Hence, to make sure, after weighting vu_1 , that W still includes at least $d_G(v) + d_G(u_2) - 1$ distinct weights, we need W to include at least

$$(d_G(v) + d_G(u_2) - 1) - d_G(u_1)$$

other weights. This quantity can be negative, as, notably, vu_1 may need a lot of weights to be weighted. Hence

$$\mu = (d_G(u_1) + 1) + \max\{0, d_G(v) + d_G(u_2) - d_G(u_1) - 1\},$$

as claimed, and, under the assumption that W has size μ , we can achieve the extension of $w_{G'}$ to G as described earlier. \square

In our proofs, we will also use the fact that, in some situations, pendant edges can easily be weighted assuming we are provided enough distinct weights.

Observation 2.4. *Let G be a graph having a pendant edge vu , where u is the degree-1 vertex, such that $G' := G - \{uv\}$ admits a neighbour-sum-distinguishing edge-weighting $w_{G'}$. Then, assuming we have a set W of at least $d_G(v)$ distinct strictly positive weights, we can extend $w_{G'}$ to a neighbour-sum-distinguishing edge-weighting of G by assigning a weight of W to vu .*

Proof. Following Observation 2.1, when extending $w_{G'}$ to vu , we do not have to care whether $\sigma(u)$ gets equal to $\sigma(v)$. We thus just have to make sure that $\sigma(v)$ does not get equal to the sum of weights incident to one of its neighbours in G' . Recall that assigning distinct weights to vu results in different sums as $\sigma(v)$. Therefore, since v has $d_G(v) - 1$ neighbours in G' while W has size at least $d_G(v)$, there is necessarily a weight in W that can be assigned to vu such that no conflict is created. An extension of $w_{G'}$ to G hence exists. \square

3. Classes of graphs agreeing with Conjecture 1.1

As mentioned in Section 1, we directly benefit, in the context of Conjecture 1.1, from the investigations on antimagic labellings, as antimagic graphs verify Conjecture 1.1. Following the survey [7] by Gallian, the following classes of nice graphs hence agree with Conjecture 1.1.

Theorem 3.1. *The classes of known antimagic graphs notably include:*

- *nice paths (Hartsfield, Ringel [8]),*
- *wheels (Hartsfield, Ringel [8]),*
- *nice regular graphs (Bérci, Bernáth, Vizer [4]),*
- *nice complete partite graphs (Alon, Kaplan, Lev, Roditty, Yuster [1]).*

Consequently, every of these graphs G verifies $\chi_{\Sigma}^{e,1}(G) = |E(G)|$.

When it comes to nice graphs with maximum degree 2, it is easily seen, as we are assigning strictly positive weights only, that any edge-injective edge-weighting is neighbour-sum-distinguishing. Disjoint unions of nice paths and cycles hence agree with Conjecture 1.1.

Observation 3.2. *Let G be a nice graph with $\Delta(G) = 2$. Then any edge-injective edge-weighting of G is neighbour-sum-distinguishing.*

One of the main lines of research concerning antimagic labellings is to determine whether nice trees are all antimagic. In the following result, we prove that this question can be answered positively when relaxed to edge-injective neighbour-sum-distinguishing edge-weightings. We actually prove a stronger statement that will be useful in the next sections.

Theorem 3.3. *Let F be a nice forest. Then, for every set W of $|E(F)|$ distinct strictly positive weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of F . In particular, we have $\chi_{\Sigma}^{e,1}(F) = |E(F)|$.*

Proof. If $\Delta(F) = 2$, then the result follows from Observation 3.2. So the claim holds whenever F has size 2. Assume now that the claim is false, and let F be a counterexample that is minimum in terms of $n_F + m_F$, where $n_F := |V(F)|$ and $m_F := |E(F)|$. By the remark above, we have $m_F \geq 3$. Let $W := \{\alpha_1, \dots, \alpha_{m_F}\}$ be a set of distinct strictly positive integers such that F does not admit an edge-injective neighbour-sum-distinguishing W -edge-weighting. Free to relabel the weights in W , we may suppose that $\alpha_1 < \dots < \alpha_{m_F}$. Due to the minimality of F , we may assume that F is a tree (as otherwise we could invoke the induction hypothesis). Furthermore, we may assume that F has maximum degree at least 3 (at otherwise Observation 3.2 would apply).

We now successively show that F , because it is a counterexample to the claim, cannot contain certain structures, until we reach the point where F is shown to not exist at all, a contradiction. In particular, we focus on the length of the pendant paths of F , where a *pendant path* of F is a maximal path $v_k \dots v_1$, where $k \geq 2$, such that $d(v_k) \geq 3$, $d(v_{k-1}) = \dots = d(v_2) = 2$, and $d(v_1) = 1$. In the case where $k = 2$, we note that the pendant path is a pendant edge, in which case $v_k = v_2$ and we have $d(v_2) \geq 3$. Since $\Delta(F) \geq 3$, there are at least three pendant paths in F .

We start off by showing that the pendant paths of F all have length at most 2.

Claim 3.4. *Every pendant path of F has length at most 2.*

Proof. Assume F has a pendant path $P := v_k \dots v_1$ with $k \geq 4$, where $d(v_k) \geq 3$. In this case, let $F' := F - \{v_{k-1}v_{k-2}, \dots, v_2v_1\}$ be the tree obtained by removing, from F , all edges of P but the one incident to v_k . Clearly, F' is nice and, due to the minimality of F , there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting $w_{F'}$ of F' , where $m_{F'} := |E(F')|$. To prove that the claim holds, we have to prove that we can extend $w_{F'}$ to the edges $v_{k-1}v_{k-2}, \dots, v_2v_1$, hence to F , using weights $\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}$, so that we get an edge-injective neighbour-sum-distinguishing W -edge-weighting of F , a contradiction.

Due to the length of P , we have $|\{\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}\}| \geq 2$. When weighting the edges $v_{k-1}v_{k-2}, \dots, v_2v_1$, we note that we cannot create any sum conflicts involving any two consecutive vertices in $\{v_1, \dots, v_{k-1}\}$. That is, the incident sums of any two of these vertices can never get equal. This is according to Observation 2.1 since we are assigning weights injectively. Hence, when extending $w_{F'}$, we just have to make sure that $\sigma(v_{k-1})$ gets different from $\sigma(v_k)$, which is possible as we have at least two distinct edge weights to work with. So we can assign a weight to $v_{k-1}v_{k-2}$ which avoids that conflict, and then arbitrarily extend the weighting to the edges $v_{k-2}v_{k-3}, \dots, v_2v_1$. This yields an edge-injective neighbour-sum-distinguishing W -edge-weighting of F . \square

Now designate a vertex r with degree at least 3 of F as being the *root* of F . This naturally defines, in the usual way, an orientation of F from its root to its leaves. For every vertex v of F , the *father* $f(v)$ of v is the neighbour of v which is the closest from r (if any). Conversely, the *descendants* of v are all vertices, different from v , in the subtree of F rooted at v (if any). We note that r has no father, while the leaves of F have no descendants. The descendants of v adjacent to v (if any) are called its *children*.

A *multifather* v of F is a vertex with degree at least 3, i.e. having at least two children. In case all descendants of v have degree at most 2, we call v a *last multifather* of F . In other words, a last multifather is a vertex with at least two pendant paths attached. Since $\Delta(F) \geq 3$, there are last multifathers in F .

To further study the structure of F , we now prove properties of its last multifathers, still under the assumption that F is rooted at a vertex r with degree at least 3.

Claim 3.5. *Vertex r is not a last multifather.*

Proof. Assume the contrary. Then r is the only vertex with degree at least 3 of F . In other words, F is a subdivided star. Then it should be clear that assigning the weights $\alpha_{m_F}, \alpha_{m_F-1}, \dots, \alpha_1$, following this order, to the edges of F as they are encountered during a breadth-first search algorithm performed from r results in a neighbour-sum-distinguishing edge-weighting of F . To be convinced of this statement, one can e.g. refer to Observation 2.1. \square

Due to Claim 3.5, we may assume that the root r of F is not a last multifather. Then all last multifathers of F (there are some) are different from r , and hence have a father. We now refine Claim 3.4 to the following.

Claim 3.6. *Every pendant path attached to a last multifather of F has length 1.*

Proof. Let $v \neq r$ be a last multifather of F , and assume v is incident to pendant paths with length 2. We recall that all pendant paths attached to v have length at most 2 (Claim 3.4), and, since v is a last multifather, it is incident to at least two pendant paths.

Let F' be the tree obtained from F by removing all pendant paths attached to v . Because $m_{F'} := |E(F')|$ is smaller than m_F , there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting $w_{F'}$ of F' . For contradiction, we prove below that $w_{F'}$ can be extended correctly to the pendant paths attached to v using the weights among $\{\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}\}$ injectively.

Let $b \geq 1$ be the number of pendant paths of length 2 attached to v in F , and let vx_1y_1, \dots, vx_by_b denote those paths (so that the x_i 's have degree 2 in F , while the y_i 's have degree 1). Vertex v is also adjacent to $c \geq 0$ leaves x_{b+1}, \dots, x_{b+c} , which are, in some sense, pendant paths of length 1. Since v is a multifather, we recall that $b + c = d_F(v) - 1 \geq 2$.

We extend $w_{F'}$ to the edges of the pendant paths attached to v in the following way. First, we injectively arbitrarily assign the $d_F(v) - 1$ weights in $\{\alpha_{m_F - d_F(v) + 2}, \dots, \alpha_{m_F}\}$ to the edges vx_2, \dots, vx_{b+c} . After that, we assign to the edge vx_1 one of the weights $\alpha_{m_F - d_F(v) + 1}$ or $\alpha_{m_F - d_F(v)}$ chosen so that $\sigma_{w_F}(v)$ is different from the sum of weights incident to $f(v)$, the father of v , by $w_{F'}$. We then assign to x_1y_1 the one weight of $\alpha_{m_F - d_F(v) + 1}$ or $\alpha_{m_F - d_F(v)}$ not assigned to vx_1 . We note that no matter how we complete the extension of $w_{F'}$, eventually $\sigma_{w_F}(v)$ will be strictly bigger than $\sigma_{w_F}(x_1)$.

We finish the extension of $w_{F'}$ to F by arbitrarily injectively assigning the remaining non-used smaller weights to the edges x_1y_1, \dots, x_by_b . Because all the x_i 's have degree 2 and the y_i 's have degree 1, no conflict may arise between those vertices (Observation 2.1). Furthermore, since the degree of v is larger than the degree of the x_i 's, and the weights assigned to the vx_i 's are bigger than the weights assigned to the x_iy_i 's (with possibly the exception of vx_1 and x_1y_1 , which we have discussed above), it should be clear that no conflict may arise between v and the x_i 's (again according to Observation 2.1). So we eventually get an edge-injective neighbour-sum-distinguishing W -edge-weighting of F , a contradiction. \square

We finally study last multifathers of F being at maximum distance from r . We call these vertices the *deepest last multifathers* of F . From now on, we focus on a fixed deepest last multifather v^* of F , which we choose arbitrarily. In the upcoming proof, for any vertex v of F , we denote by F_v the subtree of F rooted at v . Recall that all children of a last multifather are leaves (Claim 3.6).

Claim 3.7. *Every last multifather v of $F_{f(v^*)}$ is a child of $f(v^*)$. In other words, v is a deepest last multifather of F .*

Proof. The claim follows from the fact that if there exists a descendant $v \neq v^*$ of $f(v^*)$ being at distance at least 2 from $f(v^*)$, then v would, in F , be at greater distance from r than v^* is. This would contradict the fact that v^* is a deepest last multifather. \square

Recall that $f(v^*)$ cannot be incident, in F , to a pendant path with length at least 3 (Claim 3.4). Hence, every child of $f(v^*)$ is either a leaf (type-1), a degree-2 vertex adjacent to a leaf (type-2, i.e. the inner vertex of a pendant path with length 2), or a deepest last multifather (type-3). See Figure 1 for an illustration. Furthermore, we know that $f(v^*)$ is adjacent to at least one type-3 vertex, which is v^* . In the following proof, we show that v^* is actually the only child of $f(v^*)$ in F .

Claim 3.8. *Vertex v^* is the only child of $f(v^*)$ in F .*

Proof. Suppose the claim is false, and let $v \neq v^*$ be another child of $f(v^*)$. Let x_1 and x_2 be two leaves adjacent to v^* , which exist since v^* is a last multifather, and all pendant paths attached to v^* have length 1 (Claim 3.6).

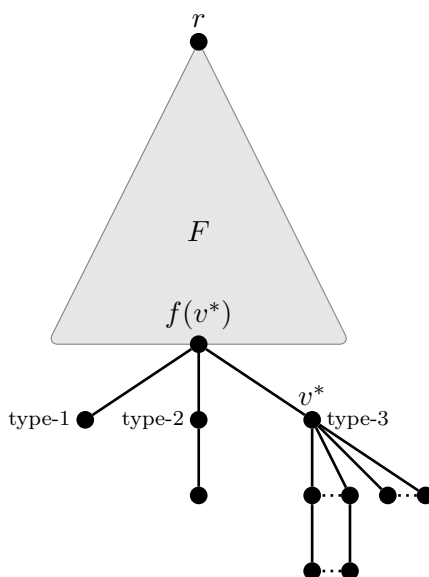


Figure 1: Illustration of the three child types mentioned in the proof of Theorem 3.3.

Assume first that v is type-2 or type-3, or, in other words, that $d_F(v) \geq 2$. In that case, v is adjacent to at least one leaf, say y . We here consider $F' := F - \{vy, v^*x_1, v^*x_2\}$. Note that F' remains nice and has fewer edges than F . Due to the minimality of F , there hence exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting $w_{F'}$ of F' , where $m_{F'} := |E(F')|$. We show below that $w_{F'}$ can be extended to the three removed edges with injectively using the three edge weights $\alpha_{m_{F'}-2}, \alpha_{m_{F'}-1}, \alpha_{m_{F'}}$, yielding an edge-injective neighbour-sum-distinguishing W -edge-weighting w_F of F , a contradiction.

We first assign a weight to v^*x_1 based on the conflicts that may happen when weighting vy . When assigning any of the three weights to vy , the only problem which may occur, recall Observation 2.1, is that $\sigma_{w_F}(v)$ gets equal to $\sigma_{w_{F'}}(f(v^*))$. If assigning one of the three weights $\alpha_{m_{F'}-2}, \alpha_{m_{F'}-1}, \alpha_{m_{F'}}$ to vy indeed results in that conflict, we assign that weight to v^*x_1 . Otherwise, we assign any of the three weights to v^*x_1 . In any case, no conflict may arise as $\sigma_{w_F}(v^*)$ is still not determined.

We are now left with two weights, which we must assign to v^*x_2 and vy . Due to the choice of the weight assigned to v^*x_1 , we note that no problem may occur when weighting vy . Hence, we just have to weight v^*x_2 correctly and assign the remaining weight to vy . When weighting v^*x_2 , the only problem which may occur, according to Observation 2.1, is that $\sigma_{w_F}(v^*)$ gets equal to $\sigma_{w_{F'}}(f(v^*))$. But, since we have two distinct weights to work with, one of them can be assigned to v^*x_2 so that this conflict is avoided. Thus we can weight v^*x_2 correctly and eventually weight vy with the remaining weight, resulting in the claimed w_F .

We may now assume that all children, including v , of $f(v^*)$ different from v^* are type-1, i.e. leaves. The contradiction can then be obtained quite similarly as in the previous case but with setting $F' := F - \{f(v^*)v, v^*x_1, v^*x_2\}$. When weighting $f(v^*)v$, we have to make sure, if $f(v^*) \neq r$, that $\sigma_{w_F}(f(v^*))$ does not get equal to $\sigma_{w_{F'}}(f(f(v^*)))$. Note that if $f(v^*) = r$, then the situation is actually easier as there is one less conflict to consider. If one of the three available weights $\alpha_{m_{F'}-2}, \alpha_{m_{F'}-1}, \alpha_{m_{F'}}$, when assigned to $f(v^*)v$, yields a conflict involving $f(v^*)$ and $f(f(v^*))$, then we assign that weight to v^*x_1 . Otherwise, we assign any weight to v^*x_1 . This ensures that, when assigning any of the two remaining weights to $f(v^*)v$, no conflict may involve $f(v^*)$ and $f(f(v^*))$. We finally arbitrarily assign the two

remaining weights to v^*x_2 and $f(v^*)v$. If this results in a neighbour-sum-distinguishing edge-weighting w_F of F , then we are done. Otherwise, it means that $\sigma_{w_F}(v^*) = \sigma_{w_F}(f(v^*))$. In that case, note that, because all assigned edge weights are distinct, when swapping the values assigned to v^*x_2 and $f(v^*)v$ by w_F that conflict cannot remain. Furthermore, according to the remarks above, we still do not create any sum conflict involving $f(v^*)$ and $f(f(v^*))$. After the swapping operation w_F hence gets neighbour-sum-distinguishing. \square

We are now ready to finish off the proof by showing that, under all information we have obtained, F actually admits an edge-injective neighbour-sum-distinguishing W -edge-weighting, a contradiction.

From Claim 3.8, we get that $d_F(f(v^*)) = 2$, as v^* is not the root of F , so $f(f(v^*))$ exists. Let x_1, \dots, x_k be the $k \geq 2$ leaves attached to v^* in F , which exist since v^* is a type-3 vertex. Now consider the tree $F' := F - \{v^*x_1, \dots, v^*x_k\}$ with size $m_{F'} := |E(F')|$. Due to the minimality of F , there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting $w_{F'}$ of F' . We extend $w_{F'}$ to the k removed edges so that an edge-injective neighbour-sum-distinguishing W -edge-weighting w_F of F is obtained, a contradiction. To that aim, we arbitrarily injectively assign the weights $\alpha_{m_{F'}-k+1}, \dots, \alpha_{m_{F'}}$ to the pendant edges v^*v_1, \dots, v^*v_k attached to v^* . Recall that we cannot get sum conflicts involving v^* and the v_i 's according to Observation 2.1. Furthermore, we have $d_F(v^*) \geq 3$ while $d_F(f(v^*)) = 2$ (Claim 3.8), and we have used the k biggest weights of W to weight the edges incident to v^* . From this and Observation 2.1, we get that, necessarily, $\sigma_{w_F}(v^*) > \sigma_{w_{F'}}(f(v^*))$. So w_F is neighbour-sum-distinguishing. \square

4. General upper bounds

Towards Conjecture 1.1, we start off by exhibiting, for any nice graph G , an upper bound on $\chi_{\Sigma}^{e,1}(G)$ of the form $k \cdot |E(G)|$, where k is a fixed constant.

It turns out, first, that some results towards the 1-2-3 Conjecture can be extended to the edge-injective context, hence yielding bounds to our context. This is in particular the case of the weighting algorithm by Kalkowski, Karoński and Pfender from [9], which was designed to prove that $\chi_{\Sigma}^e(G) \leq 5$ holds for every nice graph G . In very brief words, this algorithm initially assigns the list of weights $\{1, 2, 3, 4, 5\}$ to every edge of G , which contains the possible weights that any edge can be assigned at any moment of the algorithm. The algorithm then linearly processes the vertices of G with possibly adjusting some incident edges weights (but staying in the list $\{1, 2, 3, 4, 5\}$) so that sum conflicts are avoided around any vertex considered during the course.

It is easy to check that this algorithm also works under the assumption that every edge of G is assigned a (possibly unique) list of five allowed consecutive weights $\{\alpha - 2, \alpha - 1, \alpha, \alpha + 1, \alpha + 2\}$. In particular, when applied with non-intersecting such lists assigned to the edges, the algorithm yields an edge-injective neighbour-sum-distinguishing edge-weighting, as every edge weight can be assigned to at most one edge. So, applying the algorithm on a nice graph G with edges e_0, \dots, e_{m-1} where each edge e_i is assigned the list $\{5i + 1, 5i + 2, 5i + 3, 5i + 4, 5i + 5\}$ results in an edge-injective neighbour-sum-distinguishing $(5 \cdot |E(G)|)$ -edge-weighting of G . From this, we get that $\chi_{\Sigma}^{e,1}(G) \leq 5 \cdot |E(G)|$ holds for every nice graph G .

The $5 \cdot |E(G)|$ bound on $\chi_{\Sigma}^{e,1}(G)$ above can actually be improved down to $2 \cdot |E(G)|$ by means of a careful inductive proof scheme, which we describe in the following proof. We actually prove (here and further) a stronger statement to get rid of the non-connected cases.

Theorem 4.1. *Let G be a nice graph. Then, for every set W of $2 \cdot |E(G)|$ distinct strictly positive weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . In particular, we have $\chi_{\Sigma}^{e,1}(G) \leq 2 \cdot |E(G)|$.*

Proof. The proof is by induction on $n_G + m_G$, where $n_G := |V(G)|$ and $m_G := |E(G)|$. As it can easily be checked that the claim is true for small values of $n_G + m_G$, we proceed to the induction step. Consider hence a value of $n_G + m_G$ such that the claim is true for smaller values of this sum.

We may assume that G is connected, as otherwise induction could be invoked on the different connected components of G . Set $\Delta := \Delta(G)$. Since we may assume that $m_G \geq 4$ and G is nice, we clearly have $\Delta \geq 2$. We may even assume that $\Delta \geq 3$, as otherwise G would admit an edge-injective neighbour-sum-distinguishing W -edge-weighting according to Observation 3.2. Consider any vertex v^* of G verifying $d_G(v^*) = \Delta$ and denote by u_1, \dots, u_{Δ} the neighbours of v^* in G .

Set $G' := G - v^*$. Note that G' may include connected components isomorphic to K_2 , and thus be not nice. In this context, we say that a component of G' is *empty* if it has no edge, *bad* if it is isomorphic to K_2 , and *good* otherwise. Basically, a bad component of G' is an edge to which v^* is joined in G : either v^* is adjacent to the two ends of that edge, or v^* is adjacent to only one of the two ends.

If G' does not have good components, then G is a connected graph whose only vertex with degree at least 3 is v^* such that G' consists of isolated vertices and isolated edges only. In particular, all vertices of G but v^* have degree at most 2, and every degree-2 vertex u_i adjacent to v^* is either adjacent to another degree-2 vertex u_j adjacent to v^* , or adjacent to a degree-1 vertex. In such a situation, assuming $W := \{\alpha_1, \dots, \alpha_{2m_G}\}$ where $\alpha_1 < \dots < \alpha_{2m_G}$, it can easily be seen that assigning decreasing weights $\alpha_{2m_G}, \dots, \alpha_1$, following this order, to the edges of G as they are encountered while performing a breadth-first search algorithm from v^* , results in an edge-injective neighbour-sum-distinguishing edge-weighting of G . This notably follows as a consequence of Observation 2.1.

Hence we may assume that G' has good connected components C_1, C_2, \dots . Let H denote the union of the C_i 's, and set $m_H := |E(H)|$. Since the C_i 's are nice, so is H . Furthermore, we have that $m_H < m_G$. According to the induction hypothesis, there hence exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{2m_H}\}$ -edge-weighting w_H of H . In order to get an edge-injective neighbour-sum-distinguishing W -edge-weighting w_G of G , we eventually need to extend w_H to the remaining edges of G , i.e. to the v^*u_i 's and the edges of the bad components of G' .

To that aim, we restrict ourselves to injectively using weights among $\{\alpha_{2m_H+1}, \dots, \alpha_{2m_G}\}$, i.e. we do not use non-used weights among $\{\alpha_1, \dots, \alpha_{2m_H}\}$. Let u_1, \dots, u_k denote the neighbours of v^* belonging to good components of G . We start by injectively assigning weights to the edges v^*u_1, \dots, v^*u_k using $\Delta + k$ of the weights in $\{\alpha_{2m_G-(\Delta+k)+1}, \dots, \alpha_{2m_G}\}$, without raising any sum conflict. This is possible for every considered edge v^*u_i , since each u_i has degree at most $\Delta - 1$ in H and we have at least $\Delta + k - (i - 1) \geq \Delta + 1$ different available weights.

We are now left with weighting the edges of G belonging to the bad components, or being incident to the bad components (i.e. being incident to v^*). Assume there are m' of them. Then we have $m_G = m_H + k + m'$, and, since $k + m' \geq \Delta$, we have

$$2m_G - (\Delta + k) - 2m_H = k + 2m' - \Delta \geq m'.$$

The set $\{\alpha_{2m_H+1}, \dots, \alpha_{2m_G-(\Delta+k)}\}$ hence contains sufficiently many weights for weighting all of the m' remaining edges. To that aim, we assign the weights $\alpha_{2m_G-(\Delta+k)}, \dots, \alpha_{2m_H+1}$,

following this order (i.e. in decreasing order of magnitude), to these m' remaining edges as they are encountered during a breadth-first search algorithm performed from v^* .

It can easily be checked that, by the weighting scheme described above, the weights on the edges incident to v^* are greater than all the weights on the edges incident to the neighbours of v^* . Hence, by Observation 2.1, vertex v^* is distinguished from all its neighbours. By similar arguments, it can be checked that no sum conflicts can involve vertices of $G - H$, thus that the resulting edge-injective edge-weighting is neighbour-sum-distinguishing. \square

We now provide a second upper bound on $\chi_{\Sigma}^{e,1}(G)$ of the form $|E(G)| + k$ for every nice graph G . Here, our k is a small linear function of $\Delta(G)$, making the bound 1) mostly interesting in the context of nice graphs with bounded maximum degree, and 2) generally better than the bound in Theorem 4.1 (except in some cases to be discussed later). The proof scheme we employ here is different from the one used to prove Theorem 4.1.

Theorem 4.2. *Let G be a nice graph. Then, for every set W of $|E(G)| + 2\Delta(G)$ distinct strictly positive weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . In particular, we have $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 2\Delta(G)$.*

Proof. We may assume that G is connected. Set $\Delta := \Delta(G)$, and let $n := |V(G)|$ and $m := |E(G)|$ denote the order and size, respectively, of G . Also, set $W := \{\alpha_1, \dots, \alpha_{m+2\Delta}\}$ where $\alpha_1 < \dots < \alpha_{m+2\Delta}$. First choose a vertex v^* with degree Δ in G , and let T be a spanning tree of G including all edges incident to v^* . From T , we deduce a partition $V_0 \cup \dots \cup V_k$ of $V(G)$, where each part V_i includes the vertices of G being at distance i from v^* in T . In particular, $V_0 = \{v^*\}$, and, for every vertex u in a part V_i with $i \neq 0$, there is exactly one edge from u to V_{i-1} in T . We call this edge the *private edge* of u .

We now describe how to obtain an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . We start by assigning the edge weights $\alpha_1, \dots, \alpha_{m-(n-1)}$ to the edges of $E(G) \setminus E(T)$ in an arbitrary way. This leaves us with all edges of T to be weighted, which includes at least one incident (private) edge for every vertex different from v^* , and all edges incident to v^* . To weight these edges without creating any conflict, we will first consider all vertices of V_k and weight their private edges carefully, then do the same for all vertices of V_{k-1} , and so on layer by layer until all edges of T are weighted. Fixing any ordering over the vertices of V_k, \dots, V_1 , this weighting scheme yields an ordering u_1, \dots, u_{n-1} in which the vertices are considered (i.e. the $|V_k|$ first u_i 's belong to V_k , the $|V_{k-1}|$ next u_i 's belong to V_{k-1} , and so on; the $|V_1|$ last u_i 's belong to V_1). We note that the private edges of the $|V_1|$ last u_i 's go to v^* .

To extend the edge-injective neighbour-sum-distinguishing edge-weighting to the edges of T correctly, we consider the u_i 's in order, and for each of these vertices, we weight its private edge in such a way that no sum conflict arises. Assume we are currently dealing with vertex u_i , meaning that all previous u_i 's have been correctly treated. If $u_i \notin V_1$, then we assign to the private edge of u_i a non-used weight among $\{\alpha_{m-(n-1)+1}, \dots, \alpha_m\}$ in such a way that $\sigma(u_i)$ gets different from the sums of the at most $\Delta - 1$ already treated neighbours of u_i . Note that, even for the last u_i not in V_1 to be considered, the number of remaining non-used weights in $\{\alpha_{m-(n-1)+1}, \dots, \alpha_m\}$ is at least $\Delta + 1$, so this weighting extension can be applied to every vertex.

Now, if $u_i \in V_1$, then we apply the same strategy but with the weights among $\{\alpha_{m+1}, \dots, \alpha_{m+2\Delta}\}$. Again, even for u_{n-1} , note that this set includes at least $\Delta + 1$ non-used weights, so we can correctly choose a weight for $u_{n-1}v^*$ so that $\sigma(u_{n-1})$ gets different from the sums of the previously-treated vertices. To finish off the proof, we note that, by that strategy, all

edges incident to v^* have been weighted with weights among $\{\alpha_{m+1}, \dots, \alpha_{m+2\Delta}\}$. Since $d(v^*) = \Delta$, by Observation 2.1 we get that $\sigma(v^*)$ is eventually strictly bigger than the sums incident to its neighbours. \square

As a concluding remark, we would like to point out that the $2 \cdot |E(G)|$ bound from Theorem 4.1, can, in several situations, be better than the $|E(G)| + 2\Delta(G)$ bound from Theorem 4.2. To be convinced of that statement, consider the class of graphs obtained by starting from any star with Δ leaves u_1, \dots, u_Δ and adding no more than $\Delta - 1$ edges joining pairs of vertices among $\{u_1, \dots, u_\Delta\}$.

5. Refined bounds for particular classes of sparse graphs

We now improve the bounds in Section 4 to bounds of the form $|E(G)| + k$, where k is a small constant, for several classes of nice graphs G . Our weighting strategy here relies on removing some edges from G , then deducing a correct edge-weighting of the remaining graph, and extending that weighting to G . So that this weighting strategy applies, we focus on rather sparse graph classes with particular properties inherited by their subgraphs. In that respect, we give a special focus to nice 2-degenerate graphs, and nice graphs with maximum average degree at most 3. It is worth recalling that these graphs may have arbitrarily large maximum degree, so Theorem 4.2 does not provide the kind of bound we are here interested in.

Throughout this section, when speaking of a k -vertex, we mean a degree- k vertex. By a k^- -vertex (resp. k^+ -vertex), we refer to a vertex with degree at most (resp. at least) k .

5.1. 2-degenerate graphs

A graph G is said to be k -degenerate if every subgraph of G has a k^- -vertex. In the next result, we focus on nice 2-degenerate graphs, and exhibit an upper bound on their value of $\chi_\Sigma^{e,1}$.

Theorem 5.1. *Let G be a nice 2-degenerate graph. Then, for every set W of $|E(G)| + 4$ distinct strictly positive weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . In particular, we have $\chi_\Sigma^{e,1}(G) \leq |E(G)| + 4$.*

Proof. Assume the claim is false, and let G be a counterexample that is minimal in terms of $n_G + m_G$, where $n_G := |V(G)|$ and $m_G := |E(G)|$. Set $W := \{\alpha_1, \dots, \alpha_{m_G+4}\}$. We show below that G cannot be a counterexample, and thereby get a contradiction. This is done by showing that we can always remove some edges from G while keeping the graph nice, then deduce an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting $w_{G'}$ of the remaining graph G' , where $m_{G'} := |E(G')|$, and finally extend $w_{G'}$ to get an edge-injective neighbour-sum-distinguishing W -edge-weighting w_G of G .

We start by pointing out properties of G we may assume. Clearly, we may suppose that G is connected. According to Observation 3.2, we may also assume that $\Delta(G) \geq 3$, and, therefore, that $m_G \geq 4$, as otherwise G would be a tree, in which case a weighting exists according to Theorem 3.3. We note as well that the 1-vertices of G must be adjacent to vertices with sufficiently large degree.

Claim 5.2. *Every 1-vertex of G is adjacent to a 6^+ -vertex.*

Proof. Assume for contradiction that G has a 1-vertex u adjacent to a 5^- -vertex v . Let $G' := G - \{uv\}$. Then G' is 2-degenerate, and nice as otherwise G would be a path of

length 2 (in which case Theorem 3.3 applies). Thus G' admits an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting $w_{G'}$, where $m_{G'} := m_G - 1$. According to Observation 2.4, we can correctly extend $w_{G'}$ to uv , hence to G , since we have at least five distinct weights available for that. This is a contradiction. \square

From Claim 5.2, we also deduce the following as a corollary.

Claim 5.3. $G - \{uv\}$ is nice for every edge uv .

Proof. Let uv be an edge of G , and set $G' := G - \{uv\}$. If $d_G(u) \geq 3$ and $d_G(v) \geq 3$, then G' is clearly nice. Furthermore, if $d_G(u) = 1$ or $d_G(v) = 1$, then G' is nice by Claim 5.2.

Now assume that at least one of u and v has degree 2 in G . Without loss of generality, assume that $d_G(u) = 2$, and let u' be the neighbour of u different from v . By Claim 5.2 we have $d_G(v) \geq 2$ and $d_G(u') \geq 2$. If $d_G(v) \geq 3$, then clearly G' is nice. So assume $d_G(v) = 2$, and let v' be the neighbour of v different from u . Then, again by Claim 5.2, we have $d_G(v') \geq 2$, and G' is nice. \square

As a consequence of Claim 5.3 and Observation 2.2, we immediately get the following.

Claim 5.4. G has no edge uv with $d_G(u) + d_G(v) \leq 6$.

We are now ready to start off the proof. Let S_1 denote the set of 2^- -vertices of G , and set $G_1 := G - S_1$. Since $\Delta(G) \geq 3$, graph G_1 has vertices. In particular, since G_1 is 2-degenerate, it has a 2^- -vertex v . Let us denote as $d^+(v)$ the number of neighbours, in G , of v in S_1 . Then $d_G(v) = d^+(v) + d_{G_1}(v)$.

First assume that $d^+(v) \geq 3$, and let v_1, v_2, v_3 be three neighbours of v in S_1 . We here consider $G' := G - \{vv_1, vv_2, vv_3\}$. Note that G' has to be nice, as otherwise G would have an edge violating Claim 5.4. Due to the minimality of G , and because G' is a nice 2-degenerate graph, there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting $w_{G'}$ of G' . We extend $w_{G'}$ to vv_1, vv_2, vv_3 , thus to G , assigning weights among a set of seven weights including those among $\{\alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\}$ in the following way.

We first assign a weight β_1 from $\{\alpha_{m_G+3}, \alpha_{m_G+4}\}$ to the edge vv_1 so that we do not create a sum conflict involving v_1 and its neighbour different from v (if any), which is clearly possible with two distinct weights. Similarly, we then assign a weight β_2 from $\{\alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\} \setminus \{\beta_1\}$ to vv_2 so that we do not create a sum conflict involving v_2 and its neighbour different from v (if any). Note that due to the choice of β_1 and β_2 , which are strictly bigger than the weights among $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$, no matter how we extend the weighting to vv_3 it cannot occur that $\sigma_{w_G}(v)$ gets equal to the sum of weights incident to a 2^- -vertex neighbouring v . Hence, when extending $w_{G'}$ to vv_3 , we just have to make sure that $\sigma_{w_G}(v_3)$ does not get equal to the sum of weights incident to the neighbour of v_3 different from v (if any), and that $\sigma_{w_G}(v)$ does not get equal to the sums of weights incident to its at most two neighbours in G_1 . So there are at most three conflicts to take into account while we have five weights in hand to weight vv_3 . Clearly, this is sufficient to extend the weighting.

Assume now that $d^+(v) = 2$ and let $v_1, v_2 \in S_1$ denote the two neighbours of v with degree at most 2 in S_1 . We here consider $G' := G - \{vv_1, vv_2\}$, which is 2-degenerate, and nice by Claim 5.4, and hence admits an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting $w_{G'}$, where $m_{G'} := m_G - 2$. Recall that $d_G(v_1), d_G(v_2) \leq 2$, and that $d_G(v) \leq 4$. According to Observation 2.3, we can correctly extend $w_{G'}$ to vv_1 and

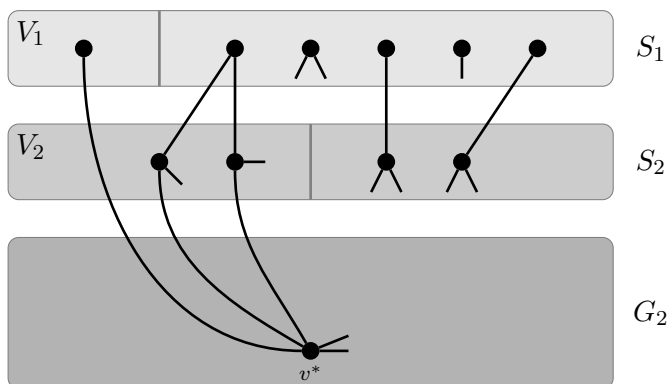


Figure 2: Illustration of the sets V_1 and V_2 introduced in the proof of Theorem 5.1.

vv_2 provided we have at least six distinct weights in hand. Since this is precisely the case here, an extension of $w_{G'}$ to G exists.

The last case to consider is when $d^+(v) = 1$, which we cannot directly treat using similar arguments as above. We may however assume that all 2^- -vertices v of G_1 verify $d^+(v) = 1$ as otherwise one of the previous situations would apply. Furthermore, these vertices have degree exactly 3, i.e. they each have exactly two neighbours in G_1 , as otherwise they would belong to S_1 . Now let S_2 denote the set of all 2^- -vertices of G_1 and set $G_2 := G - \{S_1, S_2\}$. We fix a vertex v^* for the rest of the proof, chosen as follows. If G_2 has vertices, then we choose, as v^* , a vertex of G_2 verifying $d_{G_2}(v^*) \leq 2$ (which exists, as G_2 is 2-degenerated). Otherwise, we choose as v^* one vertex verifying $0 < d_{G_1}(v^*) \leq 2$. In the latter case, note that v^* belongs to S_2 .

Now, consider the following sets (see Figure 2 for an illustration)

$$V_1 := \{v \in V(G) \mid v \in S_1 \cap N_G(v^*)\} \quad \text{and} \quad V_2 = \{v \in V(G) \mid v \in S_2 \cap N_G(v^*)\},$$

and set $d_1^+ := |V_1|$ and $d_2^+ := |V_2|$. Due to our choice of v^* , we have $d_2^+ \geq 1$. Furthermore, all vertices in V_2 are 3-vertices adjacent to v^* and to a 2^- -vertex in S_1 , while all vertices in V_1 are 2^- -vertices adjacent to v^* . Also, we have $d_1^+ + d_2^+ \leq d_G(v^*) \leq d_1^+ + d_2^+ + 2$.

First assume that $d_1^+ + d_2^+ \geq 4$, and let v_1, v_2, v_3, v_4 be any four distinct neighbours of v^* in $V_1 \cup V_2$. We here set $G' = G - \{v^*v_1, v^*v_2, v^*v_3, v^*v_4\}$. Since the v_i 's are 3^- -vertices in G , it should be clear, according to Claim 5.4, that G' is nice. As it is also 2-degenerated, by minimality of G there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting $w_{G'}$, where $m_{G'} := m_G - 4$.

We now have to prove that we can extend $w_{G'}$ to w_G using at most eight distinct weights including those among $\{\alpha_{m_G+1}, \alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\}$. Since $d_2^+ \geq 1$, some of the v_i 's belong to V_2 ; assume v_1 is one such vertex. We first assign a weight β_1 to v^*v_1 from the set $\{\alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\}$ so that no conflict involving v_1 and one of its two neighbours different from v^* arises. This is clearly possible with at least three distinct weights. Similarly, we assign two weights β_2 and β_3 from the set $\{\alpha_{m_G+1}, \alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\} \setminus \{\beta_1\}$ to v^*v_2 and v^*v_3 , respectively, so that no conflict involving v_2 or v_3 and one of their at most two neighbours different from v^* arises. We note that this is possible since, though v_2 and v_3 might be 3-vertices, they are adjacent to a 2-vertex in that case. Under the assumption that we assign a weight among $\{\alpha_{m_G+1}, \alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\}$ to v^*v_2 and v^*v_3 , we cannot create any sum conflict involving v_2 or v_3 and a neighbouring 2-vertex. In other words, only one conflict involving v_2 or v_3 may arise here.

We finally have to extend $w_{G'}$ to v^*v_4 . Note that due to the choice of $\beta_1, \beta_2, \beta_3$, and because v_4 is a 3^- -vertex in G , it cannot be that, currently, the sum of weights incident to v^* is exactly the sum of weights incident to v_4 . Furthermore, for the same reasons, no matter how we weight v^*v_4 it cannot happen that, eventually, $\sigma_{w_G}(v^*)$ gets equal to the sum of weights incident to any vertex in $V_1 \cup V_2$. Hence, when weighting v^*v_4 , we just have to make sure that $\sigma_{w_G}(v^*)$ does not get equal to the sums of weights incident to the at most two other neighbours of v^* (i.e. those not in $V_1 \cup V_2$, unless G_2 is empty in which case all neighbours of v^* belong to $V_1 \cup V_2$), and that $\sigma_{w_G}(v_4)$ does not get equal to the sums of weights incident to the at most two neighbours of v_4 different from v^* . Since we have five distinct weights left to weight v^*v_4 , necessarily one of these weights respect these conditions. The claimed extension of $w_{G'}$ hence exists.

To complete the proof, we have to consider the cases where $d_1^+ + d_2^+ \leq 3$. Denote by v_1 one neighbour of v^* in V_2 , which exists since $d_2^+ \geq 1$. Since v_1 belongs to V_2 , we know that v_1 is a 3-vertex adjacent to a 2-vertex, say u_1 , in S_1 . Set $G' := G - \{v^*v_1, v_1u_1\}$. Again, G' is 2-degenerate and nice by Claim 5.4. So let $w_{G'}$ be an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting of G' , which exists due to the minimality of G , where $m_{G'} := m_G - 2$. For contradiction, we show that $w_{G'}$ can be extended to G and that we can do it with six distinct weights.

The degree properties here are that $d_G(v^*) \leq 5$, $d_G(v_1) = 3$ and $d_G(u_1) = 2$. It can be observed, under those assumptions, that the quantity

$$\mu := (d_G(v^*) + 1) + \max\{0, d_G(v_1) + d_G(u_1) - d_G(v^*) - 1\}$$

is bounded above by 6. From Observation 2.3, we hence know that $w_{G'}$ can be extended to v^*v_1 and v_1u_1 , as claimed. This completes the proof. \square

5.2. Graphs with maximum average degree at most 3

We recall that, for any given graph G , the *maximum average degree* of G , denoted $\text{mad}(G)$, is defined as the maximum average degree of a subgraph of G . That is

$$\text{mad}(G) := \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \text{ is a non-empty subgraph of } G \right\}.$$

In the next result, we prove an upper bound on $\chi_{\Sigma}^{e,1}$ for every nice graph with maximum average degree at most 3.

Theorem 5.5. *Let G be a nice graph with $\text{mad}(G) \leq 3$. Then, for every set W of $|E(G)| + 6$ distinct strictly positive weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . In particular, we have $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$.*

Proof. Assume there exists a counterexample to the claim, that is, there exists a nice graph G for which we have $\text{mad}(G) \leq 3$ but, for a particular set W including $|E(G)| + 6$ weights, there is no edge-injective neighbour-sum-distinguishing W -edge-weighting of G . We consider G minimum in terms of $n_G + m_G$, where $n_G := |V(G)|$ and $m_G := |E(G)|$. Set $W := \{\alpha_1, \dots, \alpha_{m_G+6}\}$, where $\alpha_1 < \dots < \alpha_{m_G+6}$. Our ultimate goal in this proof is to show that G cannot exist. The strategy we employ to this end is essentially to show that G has a nice subgraph H , with order n_H and size m_H , such that H has an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_H+6}\}$ -edge-weighting w_H that can be extended to an edge-injective neighbour-sum-distinguishing W -edge-weighting w_G of G , contradicting the fact that G is a counterexample. The main tool we want to use, in order to show that H has such an edge-weighting, is Theorem 4.2. Since G is a counterexample to the claim,

note that Theorem 4.2 already implies that $\Delta(G) \geq 4$. Furthermore, we may assume that G is connected, and is not a tree as otherwise Theorem 3.3 would apply.

The subgraph H we consider is obtained by removing all 1-vertices from G . Of course, we have $\text{mad}(H) \leq 3$ and it may happen that $G = H$. We may as well assume that H remains nice, as, if it is not the case, then G would be a tree (a bistar, i.e. a tree having exactly two 2^+ -nodes, being adjacent), which is not possible as pointed out above.

In the following result, we observe that, by showing that H verifies $\Delta(H) \leq 3$, then we will get our conclusion.

Proposition 5.6. *If $\Delta(H) \leq 3$, then G is not a counterexample.*

Proof. If $G = H$, then G admits an edge-injective neighbour-sum-distinguishing W -edge-weighting according to Theorem 4.2 since we would have $\Delta(G) \leq 3$. So assume that G has 1-vertices. Since we assume that $\Delta(H) \leq 3$, there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_H+6}\}$ -edge-weighting w_H of H , still according to Theorem 4.2.

We now extend w_H to the pendant edges of G . We successively consider every vertex v of H incident to a pendant edge. We start by assigning an arbitrary non-used weight to every pendant edge incident to v , but one, say vu .

We claim that we can find a correct weight for vu . First, we note, according to Observation 2.1, that only the neighbours of v in H can eventually cause sum conflicts. Hence, when extending w_H to vu , we just have to make sure, since vu is the last non-weighted pendant edge incident to v , that $\sigma(v)$ does not meet any of the determined sums of the vertices adjacent to v in H . By our assumption on $\Delta(H)$, there are at most three such vertices, while we have at least seven ways to weight vu (among $\{\alpha_1, \dots, \alpha_{m_G+6}\}$), each determining a distinct value for $\sigma(v)$. We can hence find a correct non-used weight for vu .

Since the process above can be applied for all vertices of H incident to a pendant edge in G , weighting w_H can hence be extended to all pendant edges of G . Thus w_H can be extended to an edge-injective neighbour-sum-distinguishing W -edge-weighting of G , as claimed. \square

It remains to show that $\Delta(H) \leq 3$. This is proved by getting successive information concerning the structure of H so that classical discharging arguments can eventually be employed.

Claim 5.7. *If $v \in V(H)$ is adjacent to 1-vertices in G , then $d_H(v) \geq 7$.*

Proof. This follows from Observation 2.4, as, when removing a pendant edge from G , applying induction, and putting the edge back, we then have seven distinct weights to achieve the extension to G . \square

Claim 5.8. *We have $\delta(H) \geq 2$.*

Proof. If $\delta(H) = 0$, then G is a star, contradicting one of our initial assumptions. Now, if $\delta(H) = 1$, then G includes a vertex v such that $d_H(v) = 1$ and v is incident to pendant edges in G . But this is impossible as such a v would not meet the condition in Claim 5.7. So $\delta(H) \geq 2$. \square

Claim 5.9. *Graph H has no two adjacent 2-vertices.*

Proof. Suppose that H has an edge uv such that $d_H(u) = d_H(v) = 2$. Recall that, according to Claim 5.7, we have $d_G(u) = d_G(v) = 2$. In this case, we consider the graph $G' := G - \{uv\}$ with size $m_{G'} := |E(G')|$. Clearly G' remains nice (otherwise Claim 5.7 would be violated), has $\text{mad}(G') \leq 3$, and, due to the minimality of G , graph G' admits an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting $w_{G'}$.

In G' , we have $d_{G'}(u) = d_{G'}(v) = 1$. Let u' and v' be the neighbours of u and v , respectively, in G' . Since $w_{G'}$ is edge-injective, we have $w_{G'}(uu') \neq w_{G'}(vv')$. We now note that, under all those assumptions, weighting $w_{G'}$ can easily be extended to an edge-injective neighbour-sum-distinguishing W -edge-weighting w_G of G , i.e. to the edge uv , a contradiction. We note that, because $w_{G'}(uu') \neq w_{G'}(vv')$ and $d_G(u) = d_G(v) = 2$, we cannot get $\sigma_{w_G}(u) = \sigma_{w_G}(v)$ when assigning any weight to uv , recall Observation 2.1. So the only constraints we have are that $\sigma_{w_G}(u)$ has to be different from $\sigma_{w_G}(u')$ (which is exactly $\sigma_{w_{G'}}(u')$) and $\sigma_{w_G}(v)$ must be different from $\sigma_{w_G}(v')$ (which is exactly $\sigma_{w_{G'}}(v')$). These constraints forbid us from assigning, to uv , at most two of the seven weights that have not been used yet. So we can extend $w_{G'}$ to w_G . \square

Claim 5.10. *Graph H has no 2-vertex adjacent to two 3-vertices.*

Proof. Assume H has such a vertex v with $d_H(v) = 2$, and v has two neighbours u_1 and u_2 verifying $d_H(u_1) = d_H(u_2) = 3$. According to Claim 5.7, we have $d_G(v) = 2$, $d_G(u_1) = 3$ and $d_G(u_2) = 3$. Let $G' := G - \{vu_1, vu_2\}$ and $m_{G'} := |E(G')|$. Clearly, G' remains nice with $\text{mad}(G') \leq 3$, and, by the minimality of G , there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting $w_{G'}$. According to Observation 2.3, weighting $w_{G'}$ can be extended to an edge-injective neighbour-sum-distinguishing W -edge-weighting of G provided we have at least five distinct edge weights in hand. Since we here have eight non-used edge weights dedicated to weighting vu_1 and vu_2 , the extension of $w_{G'}$ to G hence exists. \square

Claim 5.11. *Graph H has no 3-vertex adjacent to two 3^- -vertices.*

Proof. The proof is similar to that of the previous claim. Assume H has such a 3-vertex v being adjacent to at least two 3^- -vertices u_1 and u_2 . Again, we set $G' := G - \{vu_1, vu_2\}$, and let $w_{G'}$ be an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting of G' , where $m_{G'} := |E(G')|$. Still according to Observation 2.3, we know that an extension exists provided we have at least six weights available. So $w_{G'}$ can correctly be extended to vu_1 and vu_2 , as eight edge weights can be used in the present context. \square

Before getting our conclusion, we prove two last claims which are a bit more general than what we actually need.

Claim 5.12. *Graph H has no 6-vertex adjacent to two 2-vertices.*

Proof. Assume H has such a 6-vertex v , and let u_1 and u_2 denote any two of its neighbouring 2-vertices. Recall that $d_H(v) = d_G(v)$, $d_H(u_1) = d_G(u_1)$ and $d_H(u_2) = d_G(u_2)$ according to Claim 5.7. Let $G' := G - \{vu_1, vu_2\}$ and set $n_{G'} := |V(G')|$ and $m_{G'} := |E(G')|$. Clearly G' is nice (Claims 5.7 and 5.9) with $\text{mad}(G') \leq 3$, and, since $n_{G'} + m_{G'} < n_G + m_G$, there exists an edge-injective neighbour-sum-distinguishing $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting $w_{G'}$ of G' . Again according to Observation 2.3, under these conditions, we know that $w_{G'}$ can be extended to vu_1 and vu_2 provided we have at least eight weights available. Since this is precisely the case, we are done. \square

Claim 5.13. *Graph H has no 4- or 5-vertex adjacent to at least two 3^- -vertices.*

Proof. The proof is very similar to that of Claim 5.12, and can be mimicked by letting u_1 and u_2 be two 3^- -vertices adjacent to v . We then get the same conclusion from Observation 2.3. \square

We are now ready to prove that H has maximum degree 3.

Claim 5.14. *We have $\Delta(H) \leq 3$.*

Proof. Assume the contrary, namely that $\Delta(H) \geq 4$. We prove the claim by means of the so-called discharging method, through a discharging procedure, based on the following rules.

To every vertex v of H , we assign an initial charge $\omega(v)$ being $d_H(v) - 3$. Since $\text{mad}(H) \leq 3$, we have

$$\sum_{v \in V(H)} d_H(v) \leq 3 \cdot n_H,$$

which implies that

$$\sum_{v \in V(H)} \omega(v) \leq 0.$$

Without creating or deleting any amount of charge assigned to the vertices, we now transfer a part of the assigned charges from neighbours to neighbours, through three discharging rules applied in two successive steps.

In the sequel, by a *weak 3-vertex* of H we refer to a 3-vertex neighbouring a 2-vertex (recall that a 3-vertex of H is adjacent to at most one 2-vertex according to Claim 5.11). The first discharging step consists in applying the following rule:

(R1) Every 4^+ -vertex transfers $\frac{1}{4}$ to every adjacent weak 3-vertex.

Once the first discharging step has been performed, we then apply the second step, which consists in applying the following two discharging rules:

(R2) Every weak 3-vertex transfers $\frac{1}{2}$ to every adjacent 2-vertex.

(R3) Every 4^+ -vertex transfers $\frac{1}{2}$ to every adjacent 2-vertex.

We now compute the final charge $\omega^*(v)$ that every vertex v of H gets once the two steps above have been performed. Recall that $\delta(H) \geq 2$ according to Claim 5.8.

1. If v is a 2-vertex, then v is adjacent to a 4^+ -vertex, and either a weak 3-vertex or a 4^+ -vertex according to Claims 5.9 and 5.10. Through Rules (R2) and (R3), the two neighbours of v both transfer $\frac{1}{2}$ to v . Hence, $\omega^*(v) = \omega(v) + 2 \times \frac{1}{2} = 0$.
2. If v is a 3-vertex, then v is either weak, or not. If v is not weak, it is not concerned by any of Rules (R1), (R2) and (R3), so $\omega^*(v) = \omega(v) = 0$. Now assume v is a weak 3-vertex. According to Claim 5.11, vertex v is adjacent to a 2-vertex u , and two 4^+ -vertices z_1 and z_2 . Through Rule (R1), vertex v receives $\frac{1}{4}$ from each of z_1 and z_2 , while, through Rule (R2), vertex v then transfers $\frac{1}{2}$ to u . Therefore, $\omega^*(v) = \omega(v) + 2 \times \frac{1}{4} - \frac{1}{2} = 0$.

3. If v is a 4- or 5-vertex, then v is adjacent to at most one vertex being either a 2-vertex or weak 3-vertex u according to Claim 5.13. The case where $\omega^*(v)$ is minimum is when v is a 4-vertex and u is a 2-vertex, in which case v transfers $\frac{1}{2}$ to u . In that case, through Rule (R3), we get $\omega^*(v) = \omega(v) - \frac{1}{2} = \frac{1}{2}$. So, whenever v is a 4- or 5-vertex, we get $\omega^*(v) > 0$.
4. If v is a 6-vertex, then v is adjacent to at most one 2-vertex according to Claim 5.12. The case where $\omega^*(v)$ gets minimum is essentially when v neighbours one 2-vertex and five weak 3-vertices. In that case, following Rules (R1) and (R3), we get $\omega^*(v) = \omega(v) - 5 \times \frac{1}{4} - \frac{1}{2} = \frac{5}{4}$. Hence, we always get $\omega^*(v) > 0$ in that case.
5. If v is a 7^+ -vertex, then v transfers most charge when v is adjacent to $d_H(v)$ 2-vertices. In that case, following Rule (R3) we deduce that $\omega^*(v) = \omega(v) - d_H(v) \times \frac{1}{2}$. Under the assumption that $d_H(v) \geq 7$, observe that $\omega(v) > d_H(v) \times \frac{1}{2}$. So, again, we always have $\omega^*(v) > 0$ in this case.

From the analysis above, we get, because $\Delta(H) \geq 4$, that

$$\sum_{v \in V(H)} \omega(v) \leq 0 < \sum_{v \in V(H)} \omega^*(v),$$

which is impossible as we did not create any new amount of charge when applying the discharging procedure. Hence, we have $\Delta(H) \leq 3$. \square

The result now follows from Proposition 5.6. \square

Theorem 5.5 applies to all nice graphs with maximum average degree at most 3. Among the classes of such graphs, we would like to highlight the class of nice planar graphs with girth at least 6, where the *girth* $g(G)$ of a graph G is the length of its smallest cycles. We refer the reader to e.g. [5], wherein the authors noticed that, for every planar graph G , we have

$$\text{mad}(G) < \frac{2g(G)}{g(G) - 2}.$$

This gives that every planar graph G with $g(G) \geq 6$ has $\text{mad}(G) \leq 3$.

Corollary 5.15. *Let G be a nice planar graph G with $g(G) \geq 6$. Then, for every set W of $|E(G)| + 6$ distinct weights, there exists an edge-injective neighbour-sum-distinguishing W -edge-weighting of G . In particular, we have $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$.*

6. Discussion

In this work, we have introduced and studied Conjecture 1.1 which stands, in some sense, as a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture. In particular, as a support to Conjecture 1.1, we have pointed out that some families of nice graphs agree with it, or sometimes almost agree with it, i.e. up to an additive constant term. Although these results can be regarded as a first step towards Conjecture 1.1, it is worth emphasizing that our work does not bring anything new towards attacking the 1-2-3 Conjecture and the Antimagic Labelling Conjecture but rather concerns some side aspects of these two conjectures.

As further work towards Conjecture 1.1, it would be interesting exhibiting, for all nice graphs G , bounds on $\chi_{\Sigma}^{e,1}(G)$ of the form $|E(G)| + k$ for a fixed constant k . One could

as well try to get a better bound of the form $k \cdot |E(G)|$ for some k in between 1 and 2. Obtaining one such of these two bounds would already improve the ones we have exhibited in Section 4. It is worth mentioning that our bounds in that section can slightly be improved by making some choices in a more clever way. But these improvements would allow us to save a small constant number of weights only, which is far from the desired improvement we have mentioned earlier.

As another direction, we would also be interested in knowing other classes of nice graphs agreeing with Conjecture 1.1 and being not known to be antimagic yet. Among such classes, let us mention the case of nice bipartite graphs G , for which we did not manage to come up with an $|E(G)| + k$ bound on $\chi_{\Sigma}^{e,1}(G)$, for any constant k . Another such class that would be interesting investigating is the one of nice subcubic graphs. We already know that cubic graphs agree with Conjecture 1.1, recall Theorem 3.1. Furthermore, we also know that nice subcubic graphs G , in general, verify $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$, recall Theorems 4.2 and 5.5. It nevertheless does not seem obvious how these results can be used in order to show that nice subcubic graphs agree with Conjecture 1.1. Such a result, though, would be one natural step following Observation 3.2. Nice planar graphs would also be interesting candidates to investigate, as we have been mostly successful with sparse classes of nice graphs. Our result in Corollary 5.15 may be regarded as a first step towards that direction.

Our results in this paper may also be subject to further investigations. In particular, there is still a gap for nice 2-degenerate graphs and graphs with maximum average degree at most 3 between our bounds in Section 5 and the bound in Conjecture 1.1. One could as well wonder how to generalize our results to nice k -degenerate graphs and graphs with maximum average degree at most k for larger fixed values of k . In particular, it could be interesting to exhibit, for these graphs G , a general upper bound on $\chi_{\Sigma}^{e,1}(G)$ of the form $|E(G)| + \mathcal{O}(k)$ involving a small function of k .

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