Supplementarity is Necessary for Quantum Diagram Reasoning *

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Abstract

The ZX-calculus is a powerful diagrammatic language for quantum mechanics and quantum information processing. We prove that its $\frac{\pi}{4}$-fragment is not complete, in other words the ZX-calculus is not complete for the so called "Clifford+T quantum mechanics". The completeness of this fragment was one of the main open problems in *categorical quantum mechanics*, a programme initiated by Abramsky and Coecke. The ZX-calculus was known to be incomplete for quantum mechanics. On the other hand, its $\frac{\pi}{2}$-fragment is known to be complete, i.e. the ZX-calculus is complete for the so called "stabilizer quantum mechanics". Deciding whether its $\frac{\pi}{4}$-fragment is complete is a crucial step in the development of the ZX-calculus since this fragment is approximately universal for quantum mechanics, contrary to the $\frac{\pi}{2}$-fragment.

To establish our incompleteness result, we consider a fairly simple property of quantum states called supplementarity. We show that supplementarity can be derived in the ZX-calculus if and only if the angles involved in this equation are multiples of $\pi/2$. In particular, the impossibility to derive supplementarity for $\pi/4$ implies the incompleteness of the ZX-calculus for Clifford+T quantum mechanics. As a consequence, we propose to add the supplementarity to the set of rules of the ZX-calculus.

We also show that if a ZX-diagram involves antiphase twins, they can be merged when the ZX-calculus is augmented with the supplementarity rule. Merging antiphase twins makes diagrammatic reasoning much easier and provides a purely graphical meaning to the supplementarity rule.

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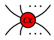
1 Introduction

The ZX-calculus has been introduced by Coecke and Duncan [7] as a graphical language for pure state qubit quantum mechanics where each diagram can be interpreted as a linear map or a matrix in a typical way (so-called *standard interpretation*). Intuitively, a ZX-diagram

* This work was partially supported by Région Lorraine.

is made of three kinds of vertices: , , and , where each green or red vertex is parameterised by an angle.

Unlike the quantum circuit notation which has no transformation rules, the ZX-calculus combines the advantages of being intuitive with a built-in system of rewrite rules. These rewrite rules make the ZX-calculus into a formal system with nontrivial equalities between diagrams. As shown in [7], the ZX-calculus can be used to express any operation in pure state qubit quantum mechanics, i.e. it is *universal*. Furthermore, any equality derived in the ZX-calculus can also be derived in the standard matrix mechanics, i.e. it is *sound*.

The converse of soundness is *completeness*. Informally put, the ZX-calculus would be complete if any equality that can be derived using matrices can also be derived graphically. It has been shown in [15] that the ZX-calculus is incomplete for the overall pure state qubit quantum mechanics, and there is no way on how to complete it by now. However, some fragments of the ZX-calculus are known to be complete. The $\frac{\pi}{2}$-fragment, which corresponds to diagrams involving angles multiple of $\pi/2$, is complete [1]. This fragment corresponds to the so called stabilizer quantum mechanics [14]. The π-fragment is also complete [12] and corresponds to real stabilizer quantum mechanics. Meanwhile, the stabilizer completeness proof in [1] carries over to a ZX-like graphical calculus for Spekkens′ toy theory [4].

While it is an important and active area of research, stabilizer quantum mechanics is only a small part of all quantum mechanics. In particular stabilizer quantum mechanics is not universal, even approximately. This fragment is even efficiently simulatable on a classical computer. On the contrary, the $\frac{\pi}{4}$-fragment, which corresponds to the so-called "Clifford+T quantum mechanics" is approximately universal [6]: any unitary transformation can be approximated with an arbitrary precision by a diagram involving angles multiple of $\pi/4$ only. The $\frac{\pi}{4}$-fragment corresponds actually to the post selected Clifford+T quantum mechanics: any diagram of the $\frac{\pi}{4}$-fragment can be interpreted as a composition of: (*i*) preparations of qubits in the computational basis; (*ii*) "Clifford+T" unitary transformations; (*iii*) post selected measurements in the computational basis. Post selected measurements, which are noting but projections, are useful to compute the probability that a given computation produces a given output. An actual quantum measurement which, roughly speaking, consists in applying with some probability a projector among a complete set of projectors, can also be represented as a ZX-diagram using formal variables as described in [11].

The completeness of the $\frac{\pi}{4}$-fragment is a crucial property and has even been stated as one of the major open question in the categorical approach to quantum mechanics [1, 2, 17]. A partial result has been proved in [2]: the fragment composed of path diagrams involving angles multiple of $\pi/4$ is complete.

Our main contribution is to prove that the $\frac{\pi}{4}$-fragment of the ZX-calculus is incomplete. In other words, we prove that the ZX-calculus is not complete for the "Clifford+T quantum mechanics". To this end, we consider a simple equation called supplementarity. This equation is inspired by a work by Coecke and Edwards [8] on the structures of quantum entanglement. We show that supplementarity can be derived in the ZX-calculus if and only if the angles involved in this equation are multiples of $\pi/2$. In particular, the impossibility to derive this equation for $\pi/4$ implies the incompleteness of the ZX-calculus for Clifford+T quantum mechanics.

We also show that in the ZX-calculus augmented with the supplementarity rule, antiphase twins can be merged. A pair of antiphase twins is a pair of vertices which have: the same colour; the same neighbourhood; and antiphase angles (the difference between their angles is π). Merging antiphase twins makes diagrammatic reasoning much easier and provides a purely graphical meaning to the supplementarity rule.

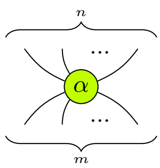
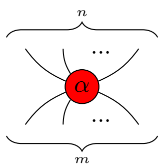
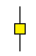
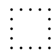
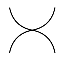
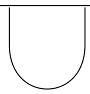
Notice that various slightly different notions of soundness/completeness have been used so far in the context of the ZX-calculus, depending on whether the rules of the language should strictly preserve the standard interpretation (as used in this paper), or up to a global phase, or even up to a (non-zero) scalar. Our result of incompleteness applies to any of these variants. However, we believe that the recent attempts to treat carefully the scalars and in particular the zero scalar are valuable, that is why we consider in this paper the strict notion of soundness and completeness. It should also be noticed that the notion of completeness used in the context of the ZX-calculus is different from a related one used in [16] to prove that finite dimensional Hilbert spaces are complete for dagger compact closed categories. The difference lies in that the concept of completeness used in the present paper is only concerned with the standard interpretation in finite dimensional Hilbert spaces, whereas, roughly speaking, in [16] it is considered for every possible interpretation (of object variables as spaces and morphism variables as linear maps).

This paper is structured as follows: the ZX-calculus (diagrams, standard interpretation, and rules) is presented in Section 2. Section 3 is dedicated to the supplementarity equation. In Section 4 we show that supplementarity involving angles which are not multiples of $\pi/2$ cannot be derived in the ZX-calculus which implies the incompleteness of the $\frac{\pi}{4}$-fragment. In Section 5, we identify an infinite family of equations which derivations require the supplementarity rule, and given a graphical interpretation of supplementarity by means of antiphase twins. Finally, in Section 6, we discuss the simplification of the ZX-calculus augmented with the supplementarity rule and its completeness.

2 ZX-calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \to l$ with k inputs and l outputs is generated by:

$R_Z^{(n,m)}(\alpha) : n \to m$		$R_X^{(n,m)}(\alpha) : n \to m$	
$H : 1 \to 1$		$e : 0 \to 0$	
$\mathbb{I} : 1 \to 1$		$\sigma : 2 \to 2$	
$\epsilon : 2 \to 0$		$\eta : 0 \to 2$	

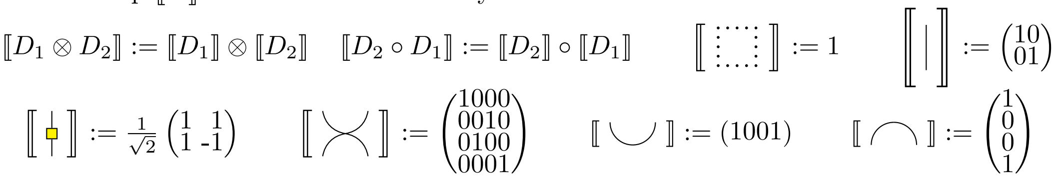
where $m, n \in \mathbb{N}$ and $\alpha \in [0, 2\pi)$.

- Spacial composition: for any $D_1 : a \to b$ and $D_2 : c \to d$, $D_1 \otimes D_2 : a + c \to b + d$ consists in placing D_1 and D_2 side-by-side, D_2 on the right of D_1.
- Sequential composition: for any $D_1 : a \to b$ and $D_2 : b \to c$, $D_2 \circ D_1 : a \to c$ consists in placing D_1 on the top of D_2, connecting the outputs of D_1 to the inputs of D_2.

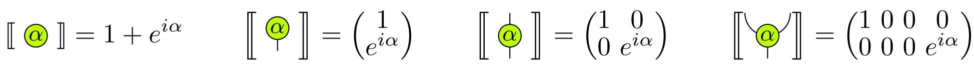
When equal to 0 modulo 2π the angles of the green and red dots are omitted:

The standard interpretation of the ZX-diagrams associates with any diagram $D : n \to m$ a linear map $[\![D]\!] : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m}$ inductively defined as follows:

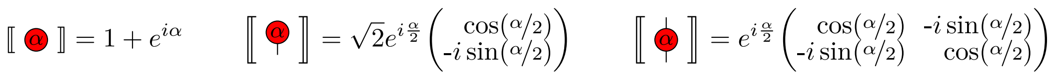
$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!]$ $\quad$ $[\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$



$[\![R_Z^{(0,0)}(\alpha)]\!] := 1+e^{i\alpha}$, and when $a+b > 0$, $[\![R_Z^{(a,b)}(\alpha)]\!]$ is a matrix with 2^a columns and 2^b rows such that all entries are 0 except the top left one which is 1 and the bottom right one which is $e^{i\alpha}$, e.g.:



For any $a, b \geq 0$, $[\![R_X^{a,b}(\alpha)]\!] := [\![H]\!]^{\otimes b} \times [\![R_Z^{a,b}(\alpha)]\!] \times [\![H]\!]^{\otimes a}$, where $M^{\otimes 0} = 1$ and for any $k > 0$, $M^{\otimes k} = M \otimes M^{\otimes k-1}$. E.g.,



ZX-diagrams are universal in the sense that for any $m, n \geq 0$ and any linear map $U : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m}$, there exists a diagram $D : n \to m$ such that $[\![D]\!] = U$ [7]. In particular, any unitary quantum evolution on a finite number of qubits can be represented by a ZX-diagram. Notice that universality implies working with a uncountable set of angles. As a consequence, the approximate version of universality, i.e. the ability to approximate with arbitrary accuracy any linear map, is generally preferred in quantum information processing. The $\frac{\pi}{4}$-fragment of language, which consists of all diagrams which angles are multiples of $\pi/4$, is approximately universal, whereas the $\frac{\pi}{2}$-fragment is not.

2.2 Calculus

The representation of a matrix in this graphical language is not unique. We present in this section the rules of the ZX calculus. These rules are sound in the sense that if two diagrams D_1 and D_2 are equal according to the rules of the ZX calculus, denoted $ZX \vdash D_1 = D_2$, then $[\![D_1]\!] = [\![D_2]\!]$. The rules of the language are given in Figure 1, and detailed bellow. The colour-swapped version and upside-down version of each rule given in Figure 1 also apply.

Spider. According to the (S1) rule any two directly connected green dots can be merged. Moreover, a dot with a single input, single output and angle 0 can be removed according to the (S2) rule. These rules have their origins in the axiomatisation of orthonormal bases by means of dagger special Frobenius algebras (see [9] for details). According to the standard interpretation $[\![.]\!]$, the green dots are associated with the so-called standard basis $\{\binom{1}{0}, \binom{0}{1}\}$, whereas the red dots (which also satisfies the spider property since colour-swapped rules also apply) are associated with the so-called diagonal basis $\{\frac{1}{\sqrt{2}}\binom{1}{1}, \frac{1}{\sqrt{2}}\binom{1}{-1}\}$.

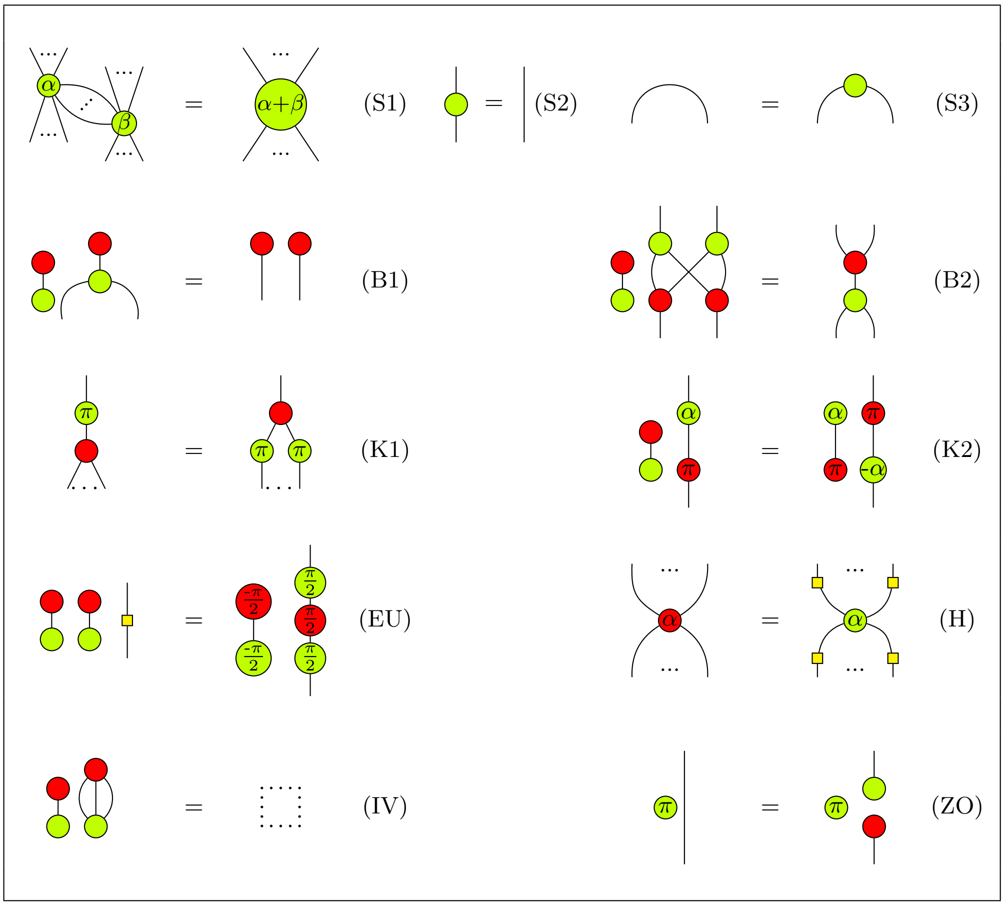
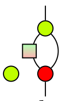
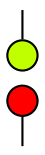
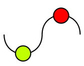
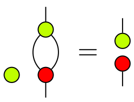


Figure 1 Rules of ZX-calculus. The colour-swapped and/or upside-down versions of each rule also applies. Horizontal dots (...) mean 'arbitrary number', whereas diagonal dots ($\cdot^{\cdot^{\cdot}}$) mean 'at least one'.

Green-Red Interactions. Monochromatic diagrams are lax: according to the (S1) rule any (green- or red-) monochromatic connected diagram is equivalent to a single dot with the appropriate number of legs and which angle is the sum of the angles. Thus the interesting structures arises when the two colours interact. The bialgebra rule (B1) and the copy rule (B2), imply that the red and the green bases are complementary, which roughly speaking capture the notion of uncertainty principle and of unbiasedness a fundamental property in quantum information (see [7] for details).

Parallel wires and Hopf law. (B1) and (B2) rules imply the following Hopf law [7, 10]:  =  where  :=  is the called the antipode. The (S3) rule trivialises the antipode and simplifies the Hopf law:

 **(Hopf Law)**

Hopf law has then a simple graphical meaning: two parallel wires between dots of distinct colours can be removed (up to the scalar ). Notice that any pair of complementary basis in arbitrary finite dimension satisfies the rules (S1), (S2), (B1) and (B2). However the (S3) rule implies that the dimension of the corresponding Hilbert space is a power of two. As a consequence the ZX-calculus is a language dedicated to *qubit quantum mechanics*.

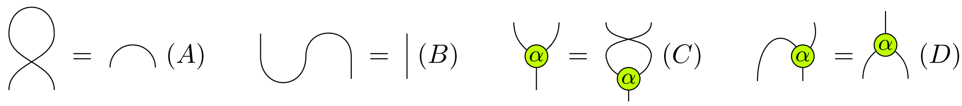
Classical point. In the context of complementary basis, the rules (K1) and (K2) imply that π is a classical point. Intuitively, it means that π together with are two elements of the red basis, so in dimension 2 they form an orthogonal basis.

Colour change. According to the (H) rule, can be used to change the colour of a dot. The (EU) rule corresponds to the Euler decomposition of the Hadamard matrix into three elementary rotations.

Scalar and zero. A scalar is a diagram with no input and no output. The standard interpretation of such a diagram is a complex number. While for simplicity, scalars have been ignored in several versions of the ZX calculus [7, 1], recently several rules have been introduced for scalars [3] and then simplified in [5], leading to the two rules (IV) and (ZO) presented in Figure 1. As the interpretation of the empty diagram is 1, the (IV) rule implies that is the inverse of . The interpretation of π is 0, as a consequence for any diagrams D_1 and D_2, $[\![\pi \otimes D_1]\!] = [\![\pi \otimes D_2]\!]$. This absorbing property is captured by the (ZO) rule.

Context. The axioms of the language presented in Figure 1 can be applied to any subdiagram. In other word, if $ZX \vdash D_1 = D_2$ then, for any D (with the appropriate number of inputs/ouputs), $ZX \vdash D \otimes D_1 = D \otimes D_2$; $ZX \vdash D_1 \otimes D = D_2 \otimes D$; $ZX \vdash D \circ D_1 = D \circ D_2$; and $ZX \vdash D_1 \circ D = D_2 \circ D$.

Only topology matters. A ZX-diagram can be deformed without changing its interpretation. This property is known as "only topology matters" in [7]. E.g.



"Only topology matters" is a consequence of the underlying dagger compact closed structure (e.g. Eq. A and B), together with the ability to interchange any two legs (Eq. C) and to turn inputs into outputs (Eq D) and vice-versa. Equations C and D are non standard in dagger compact closed categories, and are consequences of the other rules of the ZX-calculus [5].

2.3 Soundness and Completeness

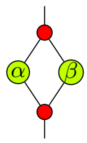
(In-)Completeness. All the rules of the ZX calculus are sound with respect to the standard interpretation, i.e. if $ZX \vdash D_1 = D_2$ then $[\![D_1]\!] = [\![D_2]\!]$. The converse of soundness is completeness: the language would be complete if $[\![D_1]\!] = [\![D_2]\!]$ implies $ZX \vdash D_1 = D_2$. The completeness would imply that one can forget matrices and do graphical reasoning only. Completeness would also imply that all the fundamental properties of qubit quantum mechanics are graphically captured by the rules of the ZX-calculus. This desirable property is one of the main open questions in categorical quantum mechanics. In the following , we review the known results about the completeness of the ZX-calculus, which are essentially depending on the considered fragment (restriction on the angles) of the language.

The very first result of incompleteness was about the original ZX-calculus in which the Euler decomposition[1] of H, the (EU) rule in Figure 1 was not derivable. This equation is now part of the language. Backens [1] proved that the $\frac{\pi}{2}$ fragment is complete. Schröder and Zamdzhiev proved that the language is not complete in general. Their argument is also based on some Euler decomposition, but contrary to the previous case this decomposition involves non rational multiples of π. The most natural – and actually the only known way – to bypass this incompleteness result is to consider a fragment of the language. Indeed, irrational multiple of π are not necessary for approximate universality. As the $\frac{\pi}{2}$-fragment is not approximately universal, the most interesting candidate for completeness is the $\frac{\pi}{4}$-fragment which is approximately universal. The completeness for the $\frac{\pi}{4}$-fragment has been conjectured in [2] and actually proved in the single qubit case, i.e. for path diagrams. The use of path diagrams (diagrams with all dots of degree two) is rather restrictive, but the completeness for this class of diagrams is not trivial and is sufficient to show that any argument based on some Euler decomposition cannot be applied in the $\pi/4$ case. However, we disprove the conjecture: the $\frac{\pi}{4}$-fragment of the ZX-calculus is not complete (Corollary 3), using a novel approach not based on Euler decompositions.

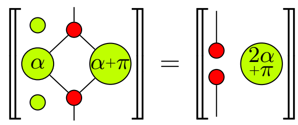
Scalars and completeness. In several versions of the ZX-calculus scalars are ignored, leading to a slightly different notion of soundness and completeness involving proportionality. Roughly speaking, ignoring the scalars consists in an additional rule which allows one to freely add or remove diagram with no input/output. A particular attention has to be paid to 'zero' diagrams, i.e. diagrams whose interpretations are zero, like π. When scalars are ignored, the notion of soundness is modified as follows: if $D_1 = D_2$ then $[\![D_1]\!]$ and $[\![D_2]\!]$ are proportional. The definition of completeness is modified likewise. Notice that in [15] yet another notion of soundness is considered where scalars are not ignored in general but global phases are, i.e. if $D_1 = D_2$ then $\exists\theta, [\![D_1]\!] = e^{i\theta}[\![D_2]\!]$. Our main result of incompleteness (Theorem 2) applies for any of these variants of soundness/completeness. However, we believe that the recent attempts to treat carefully the scalars and in particular the zero scalar are valuable that is why we consider in this paper the strict notion of soundness and completeness.

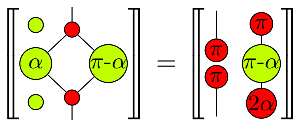
3 Supplementarity

In [8], Coecke and Edwards introduced the notion of *supplementarity* by pointing out that when $\alpha \neq 0 \bmod \pi$ the standard interpretation of the following diagram is proportional to the projector $\binom{10}{00}$ if $\alpha - \beta = \pi$ and to the projector $\binom{00}{01}$ if $\alpha + \beta = \pi$.



Putting back the scalars, one gets the following equations, which are true for any angle α, even when $\alpha = 0$:



and


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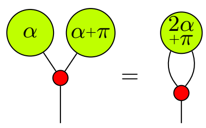
[1] By Euler decomposition we mean the existence, for any 1-qubit unitary U, of 4 angles $\alpha, \beta, \gamma, \delta$ s.t. $U = e^{i\alpha} R_x(\beta) R_z(\gamma) R_x(\delta)$ where $R_x(.)$ and $R_z(.)$ are elementary rotations about orthogonal axis.

Coecke and Edwards showed that the concept of supplementarity is related to the entanglement of quantum states. Up to stochastic local operations and classical communications (SLOCC), there are only two kinds of three-qubit states with genuine tripartite entanglement: those which are SLOCC-equivalent to a GHZ state, and those which are SLOCC-equivalent to a W state. A GHZ-state is a particular instance of a graph state which can be easily represented with a ZX-diagram [13]. On the other hand this is more involved to represent a W-type entangled states. The concept of supplementarity allowed Coecke and Edwards to characterise inhabitants of the W-class.

Albeit Coecke and Edwards did not address explicitly the question of proving whether the above equations can be derived in the ZX-calculus or not, these equations were known to be candidates for proving the incompleteness of the language[2]. We prove in Section 4 that these equations can be derived in the ZX-calculus only when $\alpha = 0 \bmod \frac{\pi}{2}$.

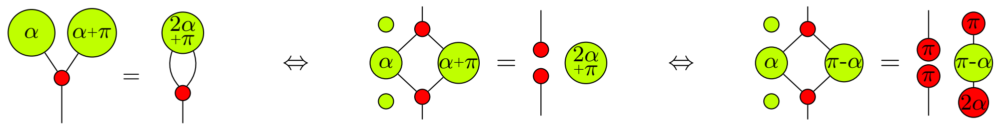
Inspired by the property pointed out by Coecke and Edwards we introduce the following equation that we call *supplementarity*:



(1)

Supplementarity is sound in the sense that both diagrams of (Eq. 1) have the same standard interpretation $\frac{1}{\sqrt{2}}\binom{1-e^{2i\alpha}}{0}$. It is provable in the ZX-calculus that supplementarity (Eq. 1) is equivalent to the equations pointed out by Coecke and Edwards:

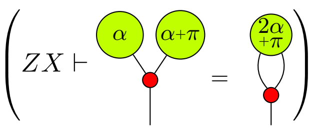
▶ **Lemma 1.** *In the ZX calculus, for any $\alpha \in [0, 2\pi)$:*



4 Supplementarity is necessary

In this section, we prove the main result of the paper: supplementarity involving angles which are not multiples of $\frac{\pi}{2}$ cannot be derived using the rules of the ZX-calculus, and as a corollary the $\frac{\pi}{4}$-fragment of ZX-calculus is incomplete.

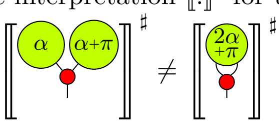
▶ **Theorem 2.** *Supplementarity can be derived in the ZX-calculus only for multiples of $\pi/2$:*



$$\Leftrightarrow \qquad \alpha = 0 \bmod \frac{\pi}{2}$$

▶ **Corollary 3.** *The $\frac{\pi}{4}$-fragment of ZX-calculus is not complete. In other words, ZX-calculus is not complete for the so-called "Clifford+T quantum mechanics".*

The rest of the section is dedicated to the proof of Theorem 2. To do so, we introduce an alternative interpretation $[\![.]\!]^\sharp$ for the diagrams, that we prove to be sound (Lemma 5) but for which

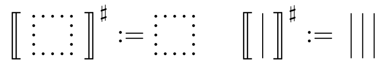


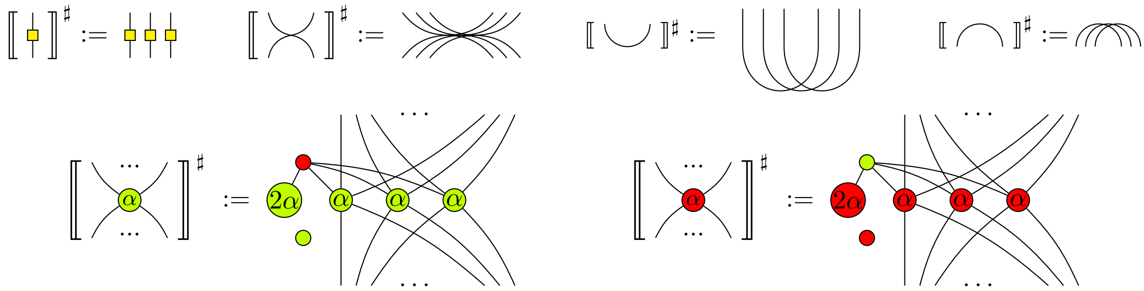
when $\alpha \neq 0 \bmod \frac{\pi}{2}$.

[2] Personal communications with Miriam Backens and Aleks Kissinger.

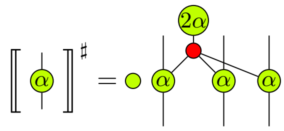
▶ **Definition 4.** For any diagram $D : n \to m$, let $[\![D]\!]^\sharp : 3n \to 3m$ be a diagram defined as follows:

$[\![D_1 \otimes D_2]\!]^\sharp := [\![D_1]\!]^\sharp \otimes [\![D_2]\!]^\sharp \qquad [\![D_2 \circ D_1]\!]^\sharp := [\![D_2]\!]^\sharp \circ [\![D_1]\!]^\sharp$





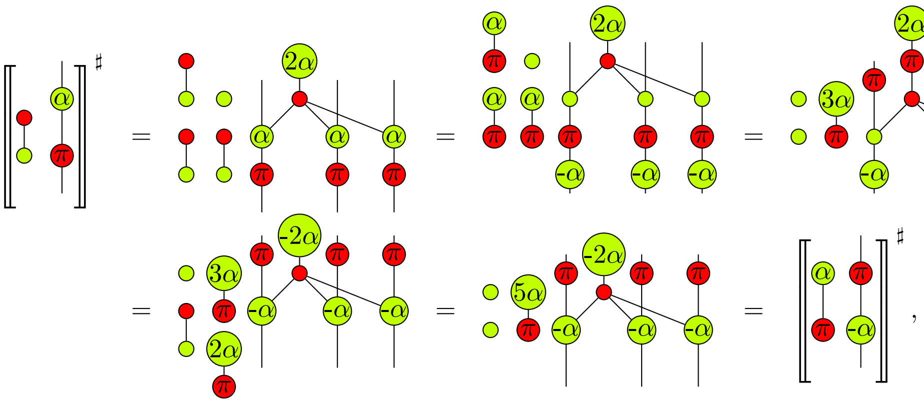
Roughly speaking, $[\![D]\!]^\sharp$ consists of three copies of D together with, for each dot of angle α, a *gadget* parameterized by the angle 2α connecting the three copies of the dot. E.g.



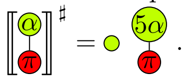
Simple calculations show that the gadget disappears when $\alpha = 0 \bmod \pi$, e.g.:

▶ **Lemma 5** (Soundness). *$[\![.]\!]^\sharp$ is a sound interpretation: if $ZX \vdash D_1 = D_2$ then $ZX \vdash [\![D_1]\!]^\sharp = [\![D_2]\!]^\sharp$.*

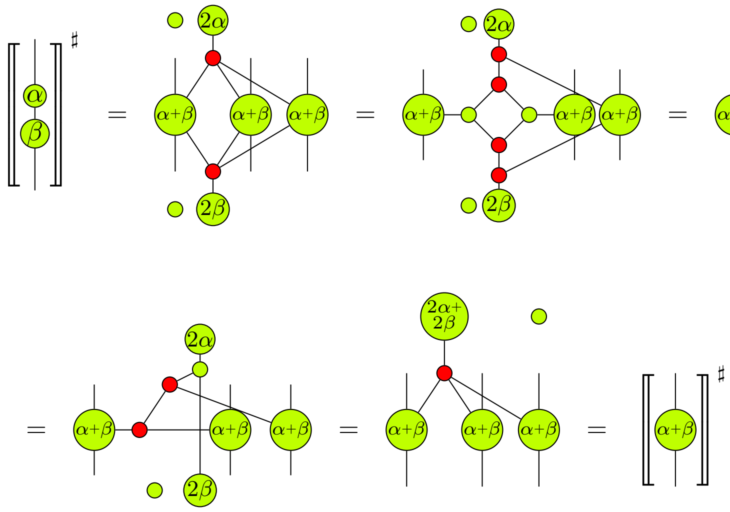
Proof. Soundness is trivial for the π-fragment of the language (i.e. when angles are multiples of π). Thus, it remains the four rules $(S1)$, $(K2)$, (EU), and (H) to complete the proof. We give the proof of $(K2)$ and a particular case of $(S1)$ to illustrate the proof, the other cases are omitted.
[(K2)]



The first equality is nothing but the definition of $[\![.]\!]^\sharp$. The second step is based on the (K2) rule. The third step consists in (i) grouping the 3 scalars depending on α into a single one, to do so rules (B1), (K1) and finally (S1) are combined; (ii) applying the (K1) rule on the non scalar part of the diagram. Fourth step consists in applying the (K2) rule on the gadget. The fifth step is combining the scalars depending on α. Finally for the last step we use



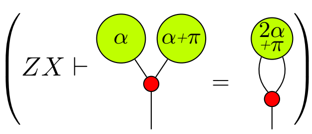
[(S1)] In the following we consider a particular case of the (S1) rule where the two dots are of degree 2. The following derivation essentially consists in applying the bialgebra rule (B2) twice:



◀

▶ **Remark (1).** The interpretation $[\![.]\!]^\sharp$ can be naturally extended to an interpretation $[\![.]\!]^\sharp_{k,\ell}$ which associates with every diagram $D : n \to m$ a diagram $[\![D]\!]^\sharp_{k,\ell} : k \times n \to k \times m$ which consists in k copies of D where the k copies of each dot are connected by a "gadget" parameterized by an angle ℓ times larger than the angle of the original dot. Moreover $[\![D]\!]^\sharp_{k,\ell}$ has additional scalars, namely $k-1$ times  per dot in D. Notice that the interpretation $[\![.]\!]^\sharp$ used in this section is nothing but $[\![.]\!]^\sharp_{3,2}$. The interpretation $[\![.]\!]^\sharp_{k,\ell}$ is sound if and only if $k = 1 \bmod 2$ and $\ell = 0 \bmod 2$, indeed $(K1)$ forces k to be odd while (EU) and (ZO) force $\ell = 0 \bmod 2$. All the other rules are sound for any k, ℓ. When $k = 1$, $[\![.]\!]^\sharp_{1,\ell}$ is nothing but an interpretation which multiplies the angles by $\ell + 1$, without changing the structure of the diagrams: $[\![.]\!]^\sharp_{1,0}$ is the identity, while $[\![.]\!]^\sharp_{1,-1}$ has been used to prove that the (EU) rule is necessary [13] and $[\![.]\!]^\sharp_{1,-2}$ has been used to prove that the ZX-calculus is incomplete [15].

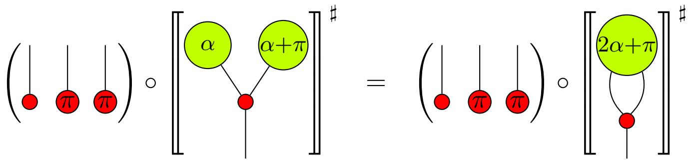
Proof of Theorem 2. In the following we prove that supplementarity can be derived in the ZX-calculus if and only if the involved angles are multiples of $\pi/2$:



$$\Leftrightarrow \qquad \alpha = 0 \bmod \frac{\pi}{2}$$

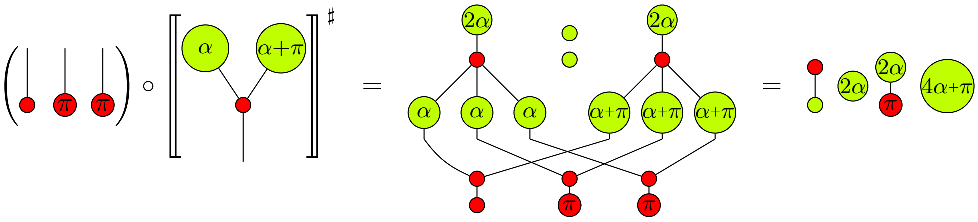
[$\Leftarrow$] Since both diagrams of the supplementarity equation have the same standard interpretation $\frac{1}{\sqrt{2}}\binom{1-e^{2i\alpha}}{0}$, by completeness of the $\frac{\pi}{2}$-fragment of the ZX-calculus, supplementarity can be derived when α is a multiple of $\frac{\pi}{2}$.

[⇒] Let $\alpha \in [0, 2\pi)$, and assume that supplementarity (1) can be derived in the ZX-calculus. Since $[\![.]\!]^\sharp$ is sound, the following equation must be derivable in the ZX-calculus:

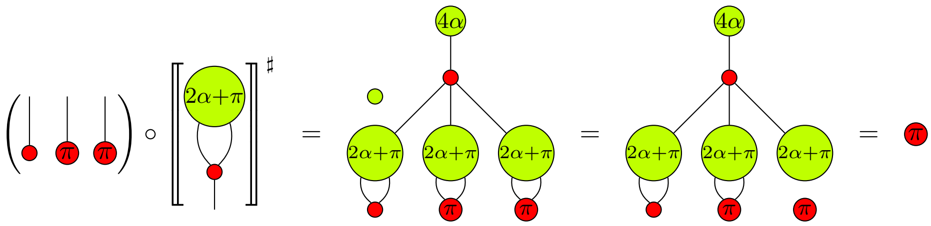


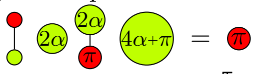
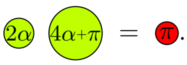
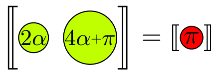
(2)

The LHS diagram is as follows.



The RHS diagram of Eq. 2 is:

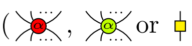


which is obtained first by applying the Hopf law and then thanks to the absorbing property of . Thus, Eq. 2 is equivalent to  which can be simplified, leading to . Finally, since $[\![.]\!]$ is sound, it implies , thus $(1 + e^{2i\alpha})(1 - e^{4i\alpha}) = 0$ which is equivalent to $\alpha = 0 \mod \frac{\pi}{2}$. ◀

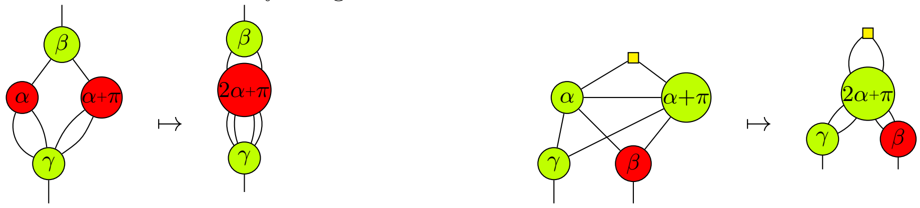
5 Supplementarity as an axiom

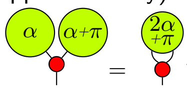
As supplementarity cannot be derive from the other rules of the language, we propose to add this equation as an axiom, a rule of the ZX-calculus. We identify an infinite class of equations that cannot be derived without the supplementarity rule. This class of equations also provides a graphical meaning to the supplementary equation. Graphically, the supplementarity equation (Eq. 1) can be interpreted as merging two dots in a particular configuration: they are antiphase (i.e. same colour and the difference between the two angles is π); of degree one; and they have the same neighbour. While antiphase is a necessary condition, the other conditions can be relaxed to any "twins" as follows:

▶ **Definition 6** (Antiphase Twins). Two dots u and v in a ZX-diagram are antiphase twins if:

- they have the same colour;
- the difference between their angles is π;
- they have the same neighbourhood: for any other vertex () w, the number of wires connecting u to w, and v to w are the same.

Notice that antiphase twins might be directly connected or not. Here two examples of antiphase twins and how they merge:

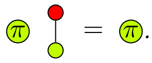


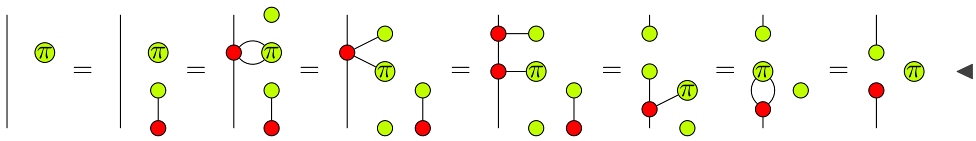
▶ **Theorem 7** (Antiphase Twins and Supplementarity). *In ZX-calculus, any pair of antiphase twins can be merged if and only if* $\forall \alpha$, .

▶ **Corollary 8.** *In the ZX-calculus augmented with the supplementarity rule, any pair of antiphase twins can be merged.*

6 Discussions

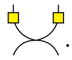
Simplified ZX-calculus. Adding a new rule to the language may lead to a simplification of the other rules of the language. Indeed the (ZO) rule can be replaced by a simpler rule:

▶ **Lemma 9.** *In the ZX-calculus augmented with the supplementarity rule, the (ZO) rule can be replaced by the following (ZO') rule:* .

Proof.  ◀

However, it seems that the language cannot be simplified much. Actually, the two most interesting candidates for simplification are the (EU) rule – which is the only one which is specific to the $\pi/2$ angle and thus may lack generality – and the (K2) rule. Even in the presence of the supplementarity rule, the (EU) rule cannot be derived from the other rules:

▶ **Lemma 10.** *In the ZX-calculus augmented with the supplementary rule, the (EU) rule cannot be derived from the other rules.*

Proof. Let $[\![.]\!]^\flat$ be defined as $[\![.]\!]^\sharp_{2,0}$ (see Remark 1) for all generators but ; and $[\![\cdot]\!]^\flat =$ . So intuitively, $[\![.]\!]^\flat$ 'doubles' the diagram, and each 'swaps' the two copies. This interpretation is sound for all rules, including the supplementarity rule, but is not sound for the (EU) rule. ◀

Regarding the (K2) rule, it has been shown recently that (K2) instantiated with an angle multiple of $\frac{\pi}{2}$ can be derived from the other rules [5], without using the supplementarity rule. We leave its necessity for arbitrary angles and in the presence of the supplementarity rule as an open question.

Completeness. Even augmented with the supplementarity rule the ZX-calculus is incomplete in general since the argument of [15] still applies. Indeed, the cornerstone of the incompleteness argument is the soundness of the interpretation which consists in multiplying the angles by -3 ($[\![.]\!]^\sharp_{1,-4}$ according to the notation of Remark 1). This interpretation is also sound with respect to the supplementarity rule, and thus the ZX-calculus is still incomplete, even augmented with the supplementarity rule. However, the second ingredient of the incompleteness result of [15] is based on the Euler decomposition of some unitary transformation which diagrammatic representation involves irrational multiples of π. As a consequence, the completeness of the ZX-calculus augmented with the supplementarity is an open question for any fragment which does not contain rational multiples of π. In particular, the completeness of the $\frac{\pi}{4}$-fragment – i.e. for Clifford+T quantum mechanics – is open.

7 Conclusion

We have proved that the ZX-calculus is not complete, even for Clifford+T quantum mechanics, which corresponds to the $\frac{\pi}{4}$-fragment of the language. We have identified an infinite set of equations that cannot be derived in the language. Moreover, we have shown that a single simple rule, called supplementarity, is sufficient to derive these equations. Supplementarity has been introduced as fundamental structure of multipartite entanglement by Coecke and Edwards. In addition to this physical interpretation, we provide a graphical meaning to the supplementarity rule by means of antiphase twins.

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