STochastic OPTimization library in C++

Hugo Gevret, Nicolas Langrené, Jérôme Lelong, Xavier Warin, Aditya Maheshwari

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Hugo Gevret [1] Nicolas Langrené [2] Jerome Lelong [3]
Xavier Warin [4]

[1]EDF R&D , Hugo.Gevret@edf.fr
[2]data61 CSIRO, locked bag 38004 docklands vic 8012 Australia, Nicolas.Langrene@data61.csiro.au
[3]Ensimag, Laboratoire Jean Kuntzmann, 700 avenue Centrale Domaine Universitaire - 38401 St Martin d'Hres
[4]EDF R&D & FiME, Laboratoire de Finance des Marchés de l'Energie, ANR PROJECT CAESARS, Xavier.Warin@edf.fr

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Part I

Introduction

The STochastic OPTimization library (StOpt)
`https://gitlab.com/stochastic-control/StOpt`
aims at providing tools for solving some stochastic optimization problems encountered in finance or in the industry.
In a continuous setting, the controlled state is given by a stochastic differential equation

$$\left\{ \begin{array}{lcl} dX_s^{x,t} & = & b_a(t, X_s^{x,t})ds + \sigma_a(s, X_s^{x,t})dW_s \\ X_t^{x,t} & = & x \end{array} \right.$$

where

- W_t is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the natural (completed and right-continuous) filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by W up to some fixed time horizon $T > 0$,
- σ_a is a Lipschitz continuous function of (t, x, a) defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^n$ and taking values in the set of d-dimensional square matrices,
- b_a is a Lipschitz continuous function of (t, x, a) defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^n$ and taking values in $\mathbb{R}^d$,
- a a control adapted to the filtration taking values in $\mathbb{R}^n$.

Suppose we want to minimize a cost function $J(t, x, a) = \mathbb{E}[\int_t^T f_a(s, X_s^{x,t}) e^{\int_t^s c_a(u, X_u^{x,t})du} ds + e^{\int_t^T c_a(u, X_u^{x,t})} g(X_T^{x,t})]$ with respect to the control a. It is well known [1] that the optimal value $\hat{J}(t, x) = \inf_a J(T - t, x, a)$ is a viscosity solution of the equation

$$\begin{array}{rcl} \frac{\partial v}{\partial t}(t, x) & - & \inf_{a \in A} \left(\frac{1}{2} tr(\sigma_a(t, x)\sigma_a(t, x)^T D^2 v(t, x)) + b_a(t, x) Dv(t, x) \right. \\ & & \left. + c_a(t, x)v(t, x) + f_a(t, x) \right) = 0 \text{ in } \mathbb{R}^d \\ v(0, x) & = & g(x) \text{ in } \mathbb{R}^d \end{array} \quad (1)$$

Under some classical assumptions on the coefficients [], the previous equation known as the Hamilton Jacobi Bellman equation admits an unique viscosity solution ([2]).
The resolution of the previous equation is quite hard especially in dimension greater than 3 or 4.
The library provides tools to solve this equation and simplified versions of it.

- a first method supposes that $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$ where $X_{1,s}^{x,t}$ is not controlled

$$\left\{ \begin{array}{lcl} dX_{1,s}^{x,t} & = & b(t, X_{1,s}^{x,t})ds + \sigma_(s, X_{1,s}^{x,t})dW_s \\ X_{1,t}^{x,t} & = & x \end{array} \right. \quad (2)$$

  and $X_{2,s}^{x,t}$ has no diffusion term

$$\left\{ \begin{array}{lcl} dX_{2,s}^{x,t} & = & b_a(t, X_{2,s}^{x,t})ds \\ X_{2,t}^{x,t} & = & x \end{array} \right.$$

In this case we can use Monte Carlo methods based on regression to solve the problem. The method is based on the Dynamic Programming principle and can be used even if the non controlled SDE is driven by a general Levy process. This method can be used even if the controlled state takes only some discrete values.

- The second case is a special case of the previous one when the problem to solve is linear and when the controlled state takes some values in some continuous intervals. The value function has to be convex or concave with respect to the controlled variables. This method, the SDDP method, is used when the dimension of the controlled state is high, preventing the use of the Dynamic Programming method.

  Remark 1 *The use of this method requires other assumptions that will be described the devoted chapter.*

- A third method permits to solve the problem with Monte Carlo when a process is controlled but by the mean of an uncontrolled process. This typically the case of the optimization of a portfolio :
  - The portfolio value is controlled and determistically discretized on a grid,
  - The portfolio evolution is driven by an exogenous process not controlled : the market prices.
- In the last method, we will suppose that the state takes continuous values, we will solve equation (1) using Semi Lagrangian methods discretizing the Brownian motion with two values and using some interpolations on grids.

In the sequel, we suppose that a time discretization is given for the resolution of the optimization problem. We suppose the step discretization is constant and equal to h such that $t_i = ih$. First, we describe some useful tools developed in the library for stochastic control. Then, we explain how to solve some optimization problems using these developed tools.

Remark 2 *In the library, we heavily relies on the Eigen library: "ArrayXd" stands for a vector of double, "ArrayXXd" for a matrix of double and "ArrayXi" a vector of integer.*

Part II

Useful tools for stochastic control

Chapter 1

The grids and their interpolators

In this chapter we develop the tools used to interpolate a function discretized on a given grid. A grid is a set of point in $\mathbb{R}^d$ defining some meshes that can be used to interpolate a function on an open set in $\mathbb{R}^d$. These tools are used to interpolate a function given for example at some stock points, when dealing with storages. There are also useful for Semi Lagrangian methods, which need effective interpolation methods. In StOpt currently four kinds of grids are available :

- the first and second one are grids used to interpolate a function linearly on a grid,
- the third kind of grid, starting from a regular grid, permits to interpolate on a grid at the Gauss Lobatto points on each mesh.
- the last grid permits to interpolate a function in high dimension using the sparse grid method. The approximation is either linear, quadratic or cubic in each direction.

To each kind of grids are associated some iterators. An iterator on a grid permits to iterate on all points of the grids. All iterators derive from the abstract class "GridIterator"

```
1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #ifndef GRIDITERATOR_H
5  #define GRIDITERATOR_H
6  #include <Eigen/Dense>
7
8  /**  \file GridIterator.h
9   *   \brief Defines an iterator on the points of a grid
10  *   \author Xavier Warin
11  */
12 namespace StOpt
13 {
14
15 /// \class  GridIterator  GridIterator.h
16 ///    Iterator on a given grid
17 class GridIterator
18 {
19
```

```
20 public :
21
22     /// \brief Constructor
23     GridIterator() {}
24
25     /// \brief Destructor
26     virtual ~GridIterator() {}
27
28     /// \brief get current  coordinates
29     virtual  Eigen::ArrayXd  getCoordinate() const = 0 ;
30
31     /// \brief Check if the iterator is valid
32     virtual  bool isValid(void) const = 0;
33
34     /// \brief iterate on point
35     virtual void next() = 0;
36
37     /// \brief iterate jumping some point
38     /// \param p_incr  increment in the jump
39     virtual void nextInc(const int &p_incr) = 0;
40
41     /// \brief get counter  : the integer associated the current point
42     virtual int getCount() const = 0;
43
44     /// \brief Permits to  jump to a given place given the number of
           processors (permits to use MPI and openmp)
45     /// \param  p_rank    processor rank
46     /// \param  p_nbProc  number of processor
47     /// \param  p_jump    increment jump for iterator
48     virtual void jumpToAndInc(const int &p_rank, const int &p_nbProc, const
           int &p_jump) = 0;
49
50     /// \brief return relative position
51     virtual int  getRelativePosition() const = 0 ;
52
53     /// \brief return number of points treated
54     virtual  int getNbPointRelative() const = 0 ;
55
56     /// \brief Reset the interpolator
57     virtual void reset() = 0 ;
58
59 };
60 }
61 #endif /* GRIDITERATOR_H */
```

All the iterators share some common features :

- the “getCount” method permits to get the number associated to the current grid point,
- the “next” method permits to go to the next point, while the “nextInc” method permits to jump forward to the “p_incr”the point,
- the “isValid” method permits to check that we are still on a grid point,

- the “getNbPointRelative” method permits to get the number of points that a given iterator can iterate on,
- the “getRelativePosition” get the number of points already iterated by the iterator.

Besides, we can directly jump to a given point : this feature is useful for “mpi” when a treatment on the grid is split between some processor and threads. This possibility is given by the “jumpToAndInc” method.
Using a grid “regGrid” the following source code permits to iterate on the points of the grids and get coordinates. For each coordinate, a function f is used to fill in an array of values. As pointed out before, each type of grid has its own grid iterator that can be obtained by the “getGridIterator” method.

```
    ArrayXd data(regGrid.getNbPoints()); // create an array to store the
        values of the function f
    shared_ptr<GridIterator> iterRegGrid  =  regGrid.getGridIterator();
    while (iterRegGrid->isValid())
    {
        ArrayXd pointCoord = iterRegGrid->getCoordinate(); // store the
            coordinates of the point
        data(iterRegGrid->getCount()) = f(pointCoord); // the value is stored
            in data at place iterRegGrid->getCount()
        iterRegGrid->next(); // go to next point
    }
```

It is also possible to “jump” some points and iterate to “p” points after. This possibility is useful for multithreaded tasks on points.
To each kind of grids, an interpolator is provided to interpolate a function given on a grid. Notice that the interpolator is created **for a given point** where we want to interpolate. All interpolators (not being spectral interpolators) derive from “Interpolator.h” whose source code is given below

```
// Copyright (C) 2016 EDF
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// This code is published under the GNU Lesser General Public License (GNU
    LGPL)
#ifndef INTERPOLATOR_H
#define INTERPOLATOR_H
#include <vector>
#include <Eigen/Dense>
/** \file Interpolator.h
 *  \brief Defines a  interpolator on a full grid
 * \author Xavier Warin
 */
namespace StOpt
{

/// \class Interpolator Interpolator.h
///  Interpolation  base class
class Interpolator
{
public :

```

```
21      /// \brief Default constructor
22      Interpolator() {}
23
24      /// \brief Default Destructor
25      virtual ~Interpolator() {}
26
27      /** \brief interpolate
28       * \param p_dataValues   Values of the data on the grid
29       * \return interpolated value
30       */
31      virtual double apply(const Eigen::ArrayXd &p_dataValues) const = 0;
32
33      /** \brief interpolate and use vectorization
34       * \param p_dataValues   Values of the data on the grid. Interpolation
             is achieved for all values in the first dimension
35       * \return interpolated value
36       */
37      virtual Eigen::ArrayXd applyVec(const Eigen::ArrayXXd &p_dataValues)
           const = 0;
38  };
39  }
40  #endif
```

All interpolators provide a constructor specifying the point where the interpolation is achieved and the two functions “apply” and “applyVec” interpolating either a function (and sending back a value) or an array of functions sending back an array of interpolated values.

All the grid classes derive from an abstract class “SpaceGrids.h” below permitting to get back an iterator associated to the points of the grid (with possible jumps) and to create an interpolator associated to the grid.

```
1   // Copyright (C) 2016 EDF
2   // All Rights Reserved
3   // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4   #ifndef SPACEGRID_H
5   #define SPACEGRID_H
6   #include <array>
7   #include <memory>
8   #include <Eigen/Dense>
9   #include "StOpt/core/grids/GridIterator.h"
10  #include "StOpt/core/grids/Interpolator.h"
11  #include "StOpt/core/grids/InterpolatorSpectral.h"
12
13  /** \file SpaceGrid.h
14   * \brief Defines a  base class for all the grids
15   * \author Xavier Warin
16   */
17  namespace StOpt
18  {
19
20  /// \class SpaceGrid SpaceGrid.h
21  /// Defines a  base class for grids
22  class SpaceGrid
```

```
23 {
24 public :
25     /// \brief Default constructor
26     SpaceGrid() {}
27 
28     /// \brief Default destructor
29     virtual ~SpaceGrid() {}
30 
31     /// \brief Number of points of the grid
32     virtual  size_t getNbPoints() const = 0;
33 
34     /// \brief get back iterator associated to the grid
35     virtual  std::shared_ptr< GridIterator> getGridIterator() const = 0;
36 
37     /// \brief get back iterator associated to the grid (multi thread)
38     virtual std::shared_ptr< GridIterator> getGridIteratorInc(const int &
        p_iThread) const = 0;
39 
40     /// \brief  Get back interpolator at a point Interpolate on the grid
41     /// \param  p_coord   coordinate of the point for interpolation
42     /// \return interpolator at the point coordinates on the grid
43     virtual std::shared_ptr<Interpolator> createInterpolator(const Eigen::
        ArrayXd &p_coord) const = 0;
44 
45     /// \brief Get back a spectral operator associated to a whole function
46     /// \param p_values   Function value at the grids points
47     /// \return  the whole interpolated  value function
48     virtual std::shared_ptr<InterpolatorSpectral> createInterpolatorSpectral(
        const Eigen::ArrayXd &p_values) const = 0;
49 
50     /// \brief Dimension of the grid
51     virtual  int getDimension() const = 0 ;
52 
53     /// \brief get back bounds associated to the grid
54     /// \return in each dimension give the extreme values (min, max) of the
        domain
55     virtual std::vector <std::array< double, 2>  > getExtremeValues() const =
         0;
56 
57     /// \brief test if the point is strictly inside the domain
58     /// \param   p_point point to test
59     /// \return  true if the point is strictly inside the open domain
60     virtual bool isStrictlyInside(const Eigen::ArrayXd &p_point) const = 0 ;
61 
62     /// \brief test if a point is  inside the grid (boundary include)
63     /// \param  p_point point to test
64     /// \return true if the point is inside the open domain
65     virtual bool isInside(const Eigen::ArrayXd &p_point) const = 0 ;
66 
67     /// \brief truncate a point that it stays inside the domain
68     /// \param p_point  point to truncate
69     virtual void truncatePoint(Eigen::ArrayXd &p_point) const = 0 ;
70 
71 };
```

```
72 }
73 #endif /* SPACEGRID.H */
```

All the grids objects, interpolators and iterators on grids point are in

StOpt/core/grids

The grids objects are mapped with python, giving the possibility to get back the iterators and the interpolators associated to a grid. Python examples can be found in

test/python/unit/grids

1.1 Linear grids

1.1.1 Definition and C++ API

Two kinds of grids are developed:

- the first one is the "GeneralSpaceGrid.h" with constructor

```
1 GeneralSpaceGrid(const std::vector<shared_ptr<Eigen::ArrayXd> > &
    p_meshPerDimension)
```

  where $std :: vector < shared\_ptr < Eigen :: ArrayXd >>$ is a vector of (pointer of) arrays defining the grids points in each dimension. In this case the grid is not regular and the mesh size varies in space (see figure 1.1).

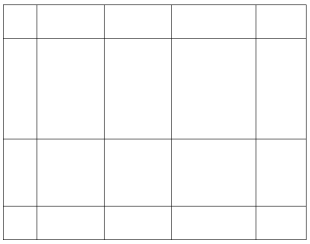

Figure 1.1: 2D general grid

- the second one is the "RegularSpaceGrid" with constructor

```
1 RegularSpaceGrid(const Eigen::ArrayXd &p_lowValues, const Eigen::ArrayXd
    &p_step, const Eigen::ArrayXi &p_nbStep)
```

  The $p\_lowValues$ correspond to the bottom of the grid, p_step the size of each mesh, p_nbStep the number of steps in each direction (see figure 1.2)

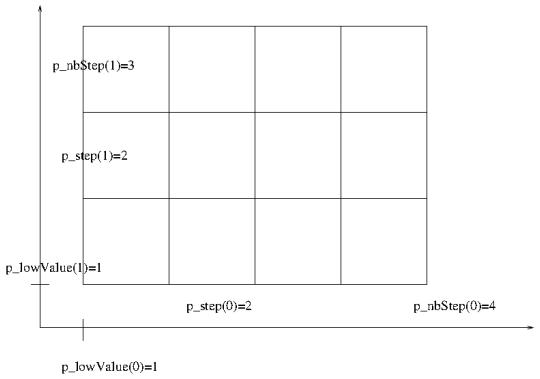

Figure 1.2: 2D regular grid

For each grid, a linear interpolator can be generated by call to the *createInterpolator* method or by creating directly the interpolator:

```
/** \brief Constructor
 *  \param p_grid   is the grid used to interpolate
 *  \param p_point  is the coordinates of the points used for interpolation
 */
LinearInterpolator( const FullGrid *   p_grid , const Eigen::ArrayXd &
    p_point):
```

Its construction from a grid (*regLin*) and an array *data* containing the values of the function at the grids points is given below (taking an example above to fill in the array *data*)

```
    ArrayXd data(regGrid.getNbPoints()); // create an array to store the
        values of the function f
    shared_ptr<GridIterator> iterRegGrid  =  regGrid.getGridIterator();
    while (iterRegGrid->isValid())
    {
        ArrayXd pointCoord = iterRegGrid->getCoordinate(); // store the
            coordinate of the point
        data(iterRegGrid->getCount()) = f(pointCoord); // the value is stored
            in data at place iterRegGrid->getCount()
        iterRegGrid->next(); // go to next point
    }
    // point where to interpolate
    ArrayXd point = ArrayXd::Constant(nDim, 1. / 3.);
    // create the interpolator
    LinearInterpolator  regLin(&regGrid, point);
    // get back the interpolated value
    double interpReg = regLin.apply(data);
```

Let $I_{1,\Delta X}$ denote the linear interpolator where the mesh size is $\Delta x = (\Delta x^1, ..., \Delta x^d)$. We get for a function f in $C^{k+1}(\mathbb{R}^d)$ with $k \leq 1$

$$||f - I_{1,\Delta x} f||_\infty \leq c \sum_{i=1}^{d} \Delta x_i^{k+1} \sup_{x \in [-1,1]^d} |\frac{\partial^{k+1} f}{\partial x_i^{k+1}}| \tag{1.1}$$

In particular if f is only Lipschitz

$$||f - I_{1,\Delta x} f||_\infty \leq K \sup_i \Delta x_i.$$

1.1.2 The python API

The python API makes it possible to use the grids with a similar syntax to the C++ API. We give here an example with a regular grid

```
# Copyright (C) 2016 EDF
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# This code is published under the GNU Lesser General Public License (GNU LGPL)
import numpy as np
import unittest
import random
import math
import StOptGrids

# unit test for regular grids
################################

class testGrids(unittest.TestCase):

    # 3 dimensional test for linear interpolation on regular grids
    def testRegularGrids(self):
        # low value for the meshes
        lowValues =np.array([1.,2.,3.],dtype=np.float)
        # size of the meshes
        step = np.array([0.7,2.3,1.9],dtype=np.float)
        # number of steps
        nbStep = np.array([4,5,6], dtype=np.int32)
        # create the regular grid
        grid = StOptGrids.RegularSpaceGrid(lowValues,step,nbStep)
        iterGrid = grid.getGridIterator()
        # array to store
        data = np.empty(grid.getNbPoints())
        # iterates on points and store values
        while( iterGrid.isValid()):
          #get coordinates of the point
          pointCoord = iterGrid.getCoordinate()
          data[iterGrid.getCount()] = math.log(1. + pointCoord.sum())
          iterGrid.next()
        # get back an interpolator
        ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
        interpol = grid.createInterpolator(ptInterp)
        # calculate interpolated value
        interpValue = interpol.apply(data)
        print(("Interpolated value" , interpValue))
        # test grids function
        iDim = grid.getDimension()
        pt = grid.getExtremeValues()

if __name__ == '__main__':
    unittest.main()
```

A similar example can be given for general grid with linear interpolation

```
# Copyright (C) 2017 EDF
# All Rights Reserved
```

```
# This code is published under the GNU Lesser General Public License (GNU LGPL)
import numpy as np
import unittest
import random
import math
import StOptGrids

# unit test for general grids
################################

class testGrids(unittest.TestCase):


    # test general grids
    def testGeneralGrids(self):
        # low value for the mesh
        lowValues =np.array([1.,2.,3.],dtype=np.float)
        # size of the mesh
        step = np.array([0.7,2.3,1.9],dtype=np.float)
        # number of step
        nbStep = np.array([4,5,6], dtype=np.int32)
        # degree of the polynomial in each direction
        degree =  np.array([2,1,3], dtype=np.int32)

        # list of mesh
        mesh1= np.array([1. + 0.7*i for  i in np.arange(5)] ,dtype=np.float)
        mesh2= np.array([2.+2.3*i for i in np.arange(6)],dtype=np.float)
        mesh3= np.array([3.+1.9*i for i in np.arange(7)],dtype=np.float)

        # create the general  grid
        grid = StOptGrids.GeneralSpaceGrid([mesh1,mesh2,mesh3] )

        iterGrid = grid.getGridIterator()
        # array to store
        data = np.empty(grid.getNbPoints())
        # iterates on point
        while( iterGrid.isValid()):
          #get coordinates of the point
          pointCoord = iterGrid.getCoordinate()
          data[iterGrid.getCount()] = math.log(1. + pointCoord.sum())
          iterGrid.next()
        # get back an interpolator
        ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
        interpol = grid.createInterpolator(ptInterp)
        # calculate interpolated value
        interpValue = interpol.apply(data)
        print(("Interpolated value Legendre" , interpValue))
         # test grids function
        iDim = grid.getDimension()
        pt = grid.getExtremeValues()


if __name__ == '__main__':
```

```
56      unittest.main()
```

1.2 Legendre grids

With linear interpolation, in order to get an accurate solution, it is needed to refine the mesh so that Δx go to zero. Another approach consists in trying to fit on each mesh a polynomial by using a high degree interpolator.

1.2.1 Approximation of a function in 1 dimension.

From now, by re-scaling we suppose that we want to interpolate a function f on $[-1, 1]$. All the following results can be extended by tensorization in dimension greater than 1. P_N is the set of the polynomials of total degree below or equal to N. The minmax approximation of f of degree N is the polynomial $P_N^*(f)$ such that:

$$||f - P_N^*(f)||_\infty = \min_{p \in P_N} ||f - p||_\infty$$

We call I_N^X interpolator from f on a grid of $N + 1$ points of $[-1, 1]$ $X = (x_0, .., x_N)$, the unique polynomial of degree N such that

$$I_N^X(f)(x_i) = f(x_i), 0 \leq i \leq N$$

This polynomial can be expressed in terms of the Lagrange polynomial $l_i^X, 0 \leq i \leq N$ associated to the grid (l_i^X is the unique polynomial of degree N taking value equal to 1 at point i and 0 at the other interpolation points).

$$I_N^X(f)(x) = \sum_{i=0}^{N} f(x_i) l_i^X(x)$$

The interpolation error can be expressed in terms of the interpolation points:

$$||I_N^X(f)(x) - f||_\infty \leq (1 + \lambda_N(X))||f - P_N^*(f)||_\infty$$

where $\lambda_N(X)$ is the Lebesgue constant associated to Lagrange quadrature on the grid:

$$\lambda_N(X) = max_{x \in [-1,1]} \sum_{i=0}^{N} |l_i^X(x)|.$$

We have the following bound

$$||I_N^X(f)(x)||_\infty \leq \lambda_N(X) sup_{x_i \in X} |f(x_i)| \leq \lambda_N(X)||f||_\infty$$

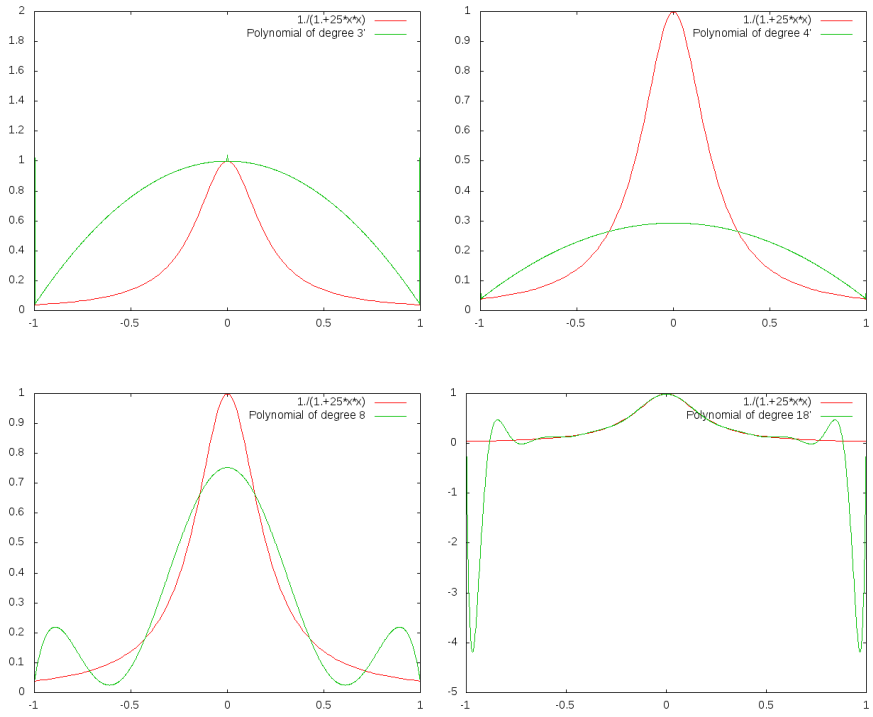
and the Erdös theorem states that

$$\lambda_N(X) > \frac{2}{\Pi} log(N + 1) - C$$

It is well–known that the use of a uniform grid X_u is not optimal, because as $N \longrightarrow \infty$, the Lebesgue constant satisfies

$$\lambda_N(X_u) \simeq \frac{2^{N+1}}{eNlnN}$$

and the quadrature error in L_∞ increases a lot with N. Its use brings some oscillations giving the Runge effect. On figure **??**, **??**, **??**, **??**, we plot the Runge function $\frac{1}{1+25x^2}$ against its interpolation with polynomial with equidistant interpolation.



So we are interested in having quadrature with an "optimal" Lebesgue constant. For example Gauss-Chebyshev interpolation points (corresponding to the 0 of the polynomial $T_{N+1}(x) = cos((N+1)arcos(x))$ give a Lebesgue constant $\lambda_N(X_{GC})$ equal to

$$\lambda_N(X_{GC}) \simeq \frac{2}{\Pi} ln(N+1)$$

For our problem, we want to interpolate a function on meshes with high accuracy on the mesh while respecting the continuity of the function between the meshes. In order to ensure this continuity we want the extreme points on the re-scaled mesh $[-1, -1]$ (so -1, 1) to be on the interpolation grid. This leads to the Gauss Lobatto Chebyshev interpolation grid.
In the library we choose to use the Gauss Lobatto Legendre interpolation grids which is as efficient as the Gauss Lobatto Chebyshev grids (in term of the Lebesgue constant) but computationally less costly due to absence of trigonometric function. We recall that the Legendre polynomial satisfies the recurrence

$$(N+1)L_{N+1}(x) = (2N+1)xL_N(x) - NL_{N-1}(x)$$

with $L_0 = 1$, $L_1(x) = x$.
These polynomials are orthogonal with the scalar product $(f, g) = \int_{-1}^{1} f(x)g(x)dx$. We are interested in the derivatives of these polynomials $L_N^{'}$ that satisfy the recurrence

$$NL_{N+1}^{'}(x) = (2N+1)xL_N^{'}(x) - (N+1)L_{N-1}^{'}(x)$$

these polynomials are orthogonal with the scalar product $(f, g) = \int_{-1}^{1} f(x)g(x)(1-x^2)dx$. The Gauss Lobatto Legendre grids points for a grids with $N+1$ points are $\eta_1 = -1, \eta_{N+1} = 1$ and the η_i $(i = 2, ..., N)$ zeros of $L_N^{'}$. The η_i $(i = 2, ..., N)$ are eigenvalues of the matrix P

$$\begin{aligned} P &= \begin{pmatrix} 0 & \gamma_1 & ... & 0 & 0 \\ \gamma_1 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & \gamma_{N-2} \\ 0 & 0 & ... & \gamma_{N-2} & 0 \end{pmatrix}, \\ \gamma_n &= \frac{1}{2}\sqrt{\frac{n(n+2)}{(n+\frac{1}{2})(n+\frac{3}{2})}}, 1 \leq n \leq N-2, \end{aligned}$$

The interpolation $I_N(f)$ is expressed in term of the Legendre polynomials by

$$\begin{aligned} I_N(f) &= \sum_{k=0}^{N} \tilde{f}_k L_k(x), \\ \tilde{f}_k &= \frac{1}{\gamma_k} \sum_{i=0}^{N} \rho_i f(\eta_i) L_k(\eta_i), \\ \gamma_k &= \sum_{i=0}^{N} L_k(\eta_i)^2 \rho_i, \end{aligned}$$

and the weights satisfies

$$\rho_i = \frac{2.}{(M+1)ML_M^2(\eta_i)}, 1 \leq i \leq N+1.$$

More details can be found in [4]. In figure 1.4, we give the interpolation obtained with the Gauss Lobatto Legendre quadrature with two degrees of approximation.

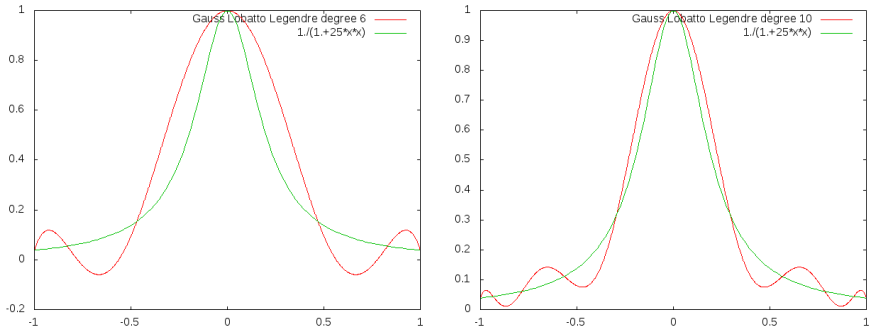


Figure 1.4: Interpolation with Gauss Legendre Lobatto grids

- When the function is not regular we introduce a notion weaker than the notion of derivatives. We note $w(f,\delta)$ the modulus of continuity on $[-1,1]$ of a function f as

$$w(f,\delta) = \sup_{\substack{x_1,x_2 \in [-1,1] \\ |x_1-x_2|<\delta}} |f(x_1)-f(x_2)|$$

The modulus of continuity permits to express the best approximation of a function by a polynomial with the Jackson theorem:

Theorem 1 *For a continuous function f on $[-1,1]$*

$$||f - P_N^*(f)||_\infty \leq K w(f, \frac{1}{N})$$

and we deduce that for a grid of interpolation X

$$\begin{aligned} ||I_N^X(f)(x) - f||_\infty &\leq M(N) \\ M(N) &\simeq K w(f,\frac{1}{N})\lambda_N(X) \end{aligned}$$

a function is Dini–Lispchitz continuous if $w(f,\delta)log(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. It is clear that Lipschitz functions are Dini–Lipschitz continuous because $w(f,\delta)log(\delta) \leq Klog(\delta)\delta$.

- When the solution is more regular we can express the interpolation error as a function of its derivatives and we get the following Cauchy theorem for an interpolation grid X (see [3])

Theorem 2 *If f is C^{N+1}, and X an interpolation grid with $N+1$ points, then the interpolation error verifies*

$$E(x) = f(x) - I_N^X(f)(x) = \frac{f^{N+1}(\eta)}{(N+1)!} W_{N+1}^X(x) \tag{1.2}$$

where $\eta \in [-1,1]$ and $W_{N+1}^X(x)$ is the nodal polynomial of degree $N+1$ (the polynomial with the monomial of the highest degree with coefficient 1 being null at all the $N+1$ points of X)

If we partition a domain $I=[a,b]$ in some meshes of size h and we use a Lagrange interpolator for the function $f \in C^{k+1}$, $k \leq N$ we obtain

$$||f - I_{N,\Delta x}^X f||_\infty \leq c h^{k+1} ||f^{(k+1)}||_\infty$$

1.2.2 Extension in dimension d

In dimension d, we note P_N^* the best multivariate polynomial approximation of f of total degree lesser than N on $[-1,1]^d$. On a d multidimensional grid $X = X_N^d$, we define the multivariate interpolator as the composition of one dimensional interpolator $I_N^X(f)(x) =$

$I_N^{X_N,1} \times I_N^{X_N,2} ... \times I_N^{X_N,d}(f)(x)$ where $I_N^{X_N,i}$ stands for the interpolator in dimension i. We get the following interpolation error

$$||I_N^X(f) - f||_\infty \leq (1 + \lambda_N(X_N))^d ||f - P_N^*(f)||_\infty,$$

The error associated to the min max approximation is given by Feinerman and Newman [5], Soardi [6]

$$||f - P_N^*(f)||_\infty \quad \leq \quad (1 + \frac{\pi^2}{4}\sqrt{d}) w(f, \frac{1}{N+2})$$

We deduce that if f is only Lipschitz

$$\boxed{||I_N^X(f)(x) - f||_\infty \leq C\sqrt{d}\frac{(1 + \lambda_N(X))^d}{N+2}}$$

If the function is regular (in $C^{k+1}([-1,1]^d)$, $k < N$) we get

$$||f - P_N^*(f)||_\infty \quad \leq \quad \frac{C_k}{N^k} \sum_{i=1}^{d} \sup_{x \in [-1,1]^d} |\frac{\partial^{k+1} f}{\partial x_i^{k+1}}|$$

If we partition the domain $I = [a_1, b_1] \times .. \times [a_d, b_d]$ in meshes of size $\Delta x = (\Delta x_1, \Delta x_2 .., \Delta x_d)$ such such we use a Lagrange interpolation on each mesh we obtain

$$\boxed{||f - I_{N,\Delta x}^X f||_\infty \leq c\frac{(1 + \lambda_N(X))^d}{N^k} \sum_{i=1}^{d} \Delta x_i^{k+1} \sup_{x \in [-1,1]^d} |\frac{\partial^{k+1} f}{\partial x_i^{k+1}}|}$$

On figure 1.5 we give the Gauss Legendre Lobatto points in 2D for 2×2 meshes and a polynomial of degree 8 in each direction

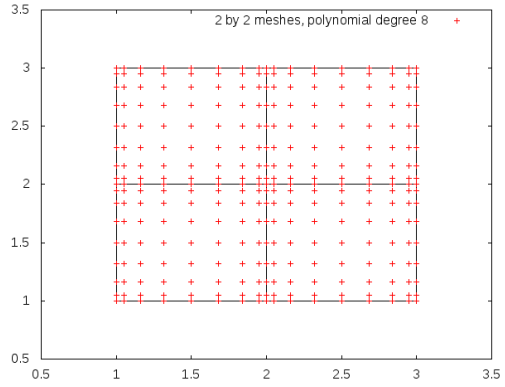


Figure 1.5: Gauss Legendre Lobatto points on 2×2 meshes.

1.2.3 Troncature

In order to avoid oscillations while interpolating, a troncature is used on each mesh such that the modified interpolator $\hat{I}^X_{N,\Delta x}$ satisfies :

$$\hat{I}^X_{N,\Delta x} f(x) = \min_{x_i \in M} f(x_i) \wedge I^X_{N,\Delta x} f(x) \vee \max_{x_i \in M} f(x_i) \tag{1.3}$$

where the x_i are the interpolation points on the mesh M containing the point x. For all caracteristics of theis modified operator, one can see [24].

1.2.4 The C++ API

The grid using Gauss Legendre Lobatto points can be created by the use of this constructor:

```
1    RegularLegendreGrid(const Eigen::ArrayXd &p_lowValues, const Eigen::
        ArrayXd &p_step, const  Eigen::ArrayXi &p_nbStep, const Eigen::ArrayXi
         & p_poly);
```

The *p_lowValues* correspond to the bottom of the grid, *p_step* the size of each mesh, *p_nbStep* the number of steps in each direction (see figure 1.2). On each mesh the polynomial approximation in each dimension is specified by the *p_poly* array.

Remark 3 *If we take a polynomial of degree 1 in each direction this interpolator is equivalent to the linear interpolator. It is somehow slightly less efficient than the linear interpolator on a Regular grid described in the above section.*

We illustrate the use of the grid, its iterator and its interpolator used in order to draw the figures 1.4.

```
1
2    ArrayXd lowValues = ArrayXd::Constant(1,-1.); // corner  point
3    ArrayXd step=  ArrayXd::Constant(1,2.); // size of the meshes
4    ArrayXi  nbStep = ArrayXi::Constant(1,1); // number of mesh in each
        direction
5    ArrayXi nPol = ArrayXi::Constant(1,p_nPol); // polynomial approximation
6    // regular Legendre
7    RegularLegendreGrid regGrid(lowValues, step, nbStep, nPol);
8
9    // Data array to store values on the grid points
10   ArrayXd data(regGrid.getNbPoints());
11   shared_ptr<GridIterator> iterRegGrid  =  regGrid.getGridIterator();
12   while (iterRegGrid->isValid())
13   {
14     ArrayXd pointCoord = iterRegGrid->getCoordinate();
15     data(iterRegGrid->getCount()) = 1./(1.+25*pointCoord(0)*pointCoord(0));
           // store runge function
16     iterRegGrid->next();
17   }
18   // point
19   ArrayXd point(1);
20   int nbp =  1000;
```

```
    double dx = 2./nbp;
    for (int ip =0; ip<= nbp; ++ip)
      {
        point(0)= -1+ ip* dx;
   // create interpolator
        shared_ptr<Interpolator> interp = regGrid.createInterpolator( point);
        double  interpReg = interp->apply(data); // interpolated value
     }
```

The previously defined operator is more effective when we interpolate many function at the same point. Its is the case for example for the valorization of a storage with regression where you want to interpolate all the simulations at the same stock level.

In some case it is more convenient to construct an interpolator acting on a global function. It is the case when you have a single function and you want to interpolate at many points for this function. In this specific case an interpolator deriving from the class InterpolatorSpectral can be constructed:

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU
    LGPL)
#ifndef INTERPOLATORSPECTRAL_H
#define INTERPOLATORSPECTRAL_H
#include <Eigen/Dense>
//#include "StOpt/core/grids/SpaceGrid.h"

/** \file InterpolatorSpectral.h
 *  \brief Defines an interpolator for a grid : here is a global interpolator
    , storing the representation of the  function
 *         to interpolate : this interpolation is effective when
    interpolating the same function many times at different points
 *         Here it is an abstract class
 * \author Xavier Warin
 */
namespace StOpt
{

/// forward declaration
class SpaceGrid ;

/// \class InterpolatorSpectral InterpolatorSpectral.h
/// Abstract class for spectral operator
class InterpolatorSpectral
{

public :
    virtual ~InterpolatorSpectral() {}

    /**  \brief  interpolate
     *  \param  p_point  coordinates of the point for interpolation
     *  \return interpolated value
     */
    virtual double apply(const Eigen::ArrayXd &p_point) const = 0;

```

```
35
36      /** \brief Affect the grid
37       * \param p_grid  the grid to affect
38       */
39      virtual void setGrid(const StOpt::SpaceGrid *p_grid)  = 0 ;
40  };
41  }
42  #endif
```

Its constructor is given by :

```
1       /** \brief Constructor taking in values on the grid
2        *  \param p_grid   is the grid used to interpolate
3        *  \param p_values   Function value at the grids points
4        */
5       LegendreInterpolatorSpectral(const shared_ptr< RegularLegendreGrid> &
           p_grid , const Eigen::ArrayXd &p_values) ;
```

This class has a member permitting to interpolate at a given point:

```
1       /**  \brief  interpolate
2        *  \param  p_point  coordinates of the point for interpolation
3        *  \return interpolated value
4        */
5       inline double apply(const Eigen::ArrayXd &p_point) const
```

We give an example of the use of this class, interpolating a function f in dimension 2.

```
1       ArrayXd lowValues = ArrayXd::Constant(2,1.); // bottom of the domain
2       ArrayXd step  = ArrayXd::Constant(2,1.); // size of the mesh
3       ArrayXi  nbStep = ArrayXi::Constant(2,5); // number of meshes in each
           direction
4       ArrayXi  nPol = ArrayXi::Constant(2,2) ; // polynomial of degree 2 in
           each direction
5       // regular
6       shared_ptr<RegularLegendreGrid> regGrid(new RegularLegendreGrid(lowValues
           , step, nbStep, nPol));
7       ArrayXd data(regGrid->getNbPoints()); // Data array
8       shared_ptr<GridIterator> iterRegGrid  =  regGrid->getGridIterator(); //
           iterator on the grid points
9       while (iterRegGrid->isValid())
10      {
11          ArrayXd pointCoord = iterRegGrid->getCoordinate();
12          data(iterRegGrid->getCount()) =  f(pointCoord);
13          iterRegGrid->next();
14      }
15
16      // spectral interpolator
17      LegendreInterpolatorSpectral interpolator(regGrid,data);
18      // interpolation point
19      ArrayXd pointCoord(2, 5.2);
20      // interpolated value
21      double vInterp = interpolator.apply(pointCoord);
```

1.2.5 The python API

Here is an example using Legendre grids:

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU LGPL)
import numpy as np
import unittest
import random
import math
import StOptGrids

# unit test for Legendre grids
################################

class testGrids(unittest.TestCase):


    # test Legendre grids
    def testLegendreGrids(self):
        # low value for the mesh
        lowValues =np.array([1.,2.,3.],dtype=np.float)
        # size of the mesh
        step = np.array([0.7,2.3,1.9],dtype=np.float)
        # number of step
        nbStep = np.array([4,5,6], dtype=np.int32)
        # degree of the polynomial in each direction
        degree =  np.array([2,1,3], dtype=np.int32)
        # create the Legendre grid
        grid = StOptGrids.RegularLegendreGrid(lowValues,step,nbStep,degree )
        iterGrid = grid.getGridIterator()
        # array to store
        data = np.empty(grid.getNbPoints())
        # iterates on point
        while( iterGrid.isValid()):
          #get coordinates of the point
          pointCoord = iterGrid.getCoordinate()
          data[iterGrid.getCount()] = math.log(1. + pointCoord.sum())
          iterGrid.next()
        # get back an interpolator
        ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
        interpol = grid.createInterpolator(ptInterp)
        # calculate interpolated value
        interpValue = interpol.apply(data)
        print(("Interpolated value Legendre" , interpValue))
         # test grids function
        iDim = grid.getDimension()
        pt = grid.getExtremeValues()


if __name__ == '__main__':
    unittest.main()
```

1.3 Sparse grids

A representation of a function in dimension d for d small (less than 4) is achieved by tensorization in the previous interpolation methods. When the function is smooth and when its cross derivatives are bounded, one can represent the function using the sparse grid methods. This methods permits to represent the function with far less points than classical without losing too much while interpolating. The sparse grid method was first used supposing that the function f to represent is null at the boundary Γ of the domain. This assumption is important because it permits to limit the explosion of the number of points with the dimension of the problem. In many application this assumption is not realistic or it is impossible to work on $f - f_{|\Gamma}$. In this library we will suppose that the function is not null at the boundary and provide grid object, iterators and interpolators to interpolate some functions represented on the sparse grid. Nevertheless, for the sake of clarity of the presentation, we will begin with the case of a function vanishing on the boundary.

1.3.1 The linear sparse grid method

We recall some classical results on sparse grids that can be found in [9]. We first assume that the function we interpolate is null at the boundary. By a change of coordinate an hyper-cube domain can be changed to a domain $\omega = [0,1]^d$. Introducing the hat function $\phi^{(L)}(x) = \max(1 - |x|, 0)$ (where (L) stands for linear), we obtain the following local one dimensional hat function by translation and dilatation

$$\phi^{(L)}_{l,i}(x) = \phi^{(L)}(2^l x - i)$$

depending on the level l and the index i, $0 < i < 2^l$. The grid points used for interpolation are noted $x_{l,i} = 2^{-l} i$. In dimension d, we introduce the basis functions

$$\phi^{(L)}_{\underline{l},\underline{i}}(x) = \prod_{j=1}^{d} \phi^{(L)}_{l_j,i_j}(x_j)$$

via a tensor approach for a point $\underline{x} = (x_1, ...x_d)$, a multi-level $\underline{l} := (l_1, .., l_d)$ and a multi-index $\underline{i} := (i_1, .., i_d)$. The grid points used for interpolation are noted $x_{\underline{l},\underline{i}} := (x_{l_1,i_1}, .., x_{l_d,i_d})$. We next introduce the index set

$$B_{\underline{l}} := \left\{ \underline{i} : 1 \leq i_j \leq 2^{l_j} - 1, i_j \text{ odd }, 1 \leq j \leq d \right\}$$

and the space of hierarchical basis

$$W^{(L)}_{\underline{l}} \quad := \quad span \left\{ \phi^{(L)}_{\underline{l},\underline{i}}(\underline{x}) : \underline{i} \in B_{\underline{l}} \right\}$$

A representation of the space $W^{(L)}_{\underline{l}}$ is given in dimension 1 on figure 1.6. The sparse grid space is defined as:

$$V_n \quad = \quad \bigoplus_{|\underline{l}|_1 \leq n+d-1} W^{(L)}_{\underline{l}} \tag{1.4}$$

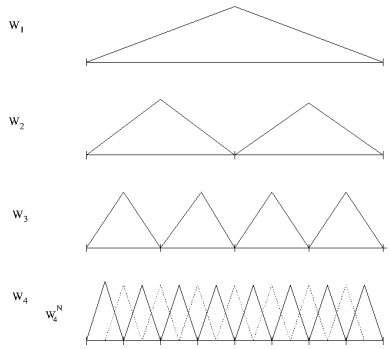


Figure 1.6: One dimensional $W^{(L)}$ spaces : $W_1^{(L)}$, $W_2^{(L)}$, $W_3^{(L)}$, $W_4^{(L)}$ and the nodal representation $W_4^{(L,N)}$

Remark 4 *The conventional full grid space is defined as* $V_n^F = \bigoplus_{|\underline{l}|_\infty \leq n} W_{\underline{l}}^{(L)}$

At a space of hierarchical increments $W_{\underline{l}}^{(L)}$ corresponds a space of nodal function $W_{\underline{l}}^{(L,N)}$ such that

$$W_{\underline{l}}^{(L,N)} \quad := \quad span\left\{\phi_{\underline{l},\underline{i}}^{(L)}(\underline{x}) : \underline{i} \in B_{\underline{l}}^N\right\}$$

with

$$B_{\underline{l}}^N := \left\{\underline{i} : 1 \leq i_j \leq 2^{l_j} - 1, 1 \leq j \leq d\right\}.$$

On figure 1.6 the one dimensional nodal base $W_4^{(L,N)}$ is spawned by $W_4^{(L)}$ and the dotted basis function. The space V_n can be represented as the space spawn by the $W_{\underline{l}}^{(L,N)}$ such that $|\underline{l}|_1 = n + d - 1$:

$$V_n \quad = \quad span\left\{\phi_{\underline{l},\underline{i}}^{(L)}(\underline{x}) : \underline{i} \in B_{\underline{l}}^N, |\underline{l}|_1 = n + d - 1\right\} \tag{1.5}$$

A function f is interpolated on the hierarchical basis as

$$I^{(L)}(f) \quad = \quad \sum_{|\underline{l}|_1 \leq n+d-1, \underline{i} \in B_{\underline{l}}} \alpha_{\underline{l},\underline{i}}^{(L)} \phi_{\underline{l},\underline{i}}^{(L)}$$

where $\alpha_{\underline{l},\underline{i}}^{(L)}$ are called the surplus (we give on figure 1.7 a representation of these coefficients). These surplus associated to a function f are calculated in the one dimension case for a node

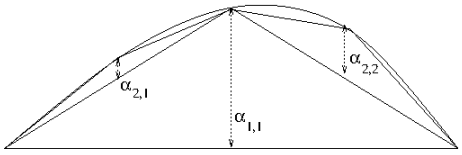


Figure 1.7: Example of hierarchical coefficients

$m = x_{l,i}$ as the difference of the value of the function at the node and the linear representation

of the function calculated with neighboring nodes. For example on figure 1.8, the hierarchical value is given by the relation:

$$\alpha^{(L)}(m) := \alpha_{l,i}^{(L)} = f(m) - 0.5(f(e(m)) + f(w(m)))$$

where $e(m)$ is the east neighbor of m and $w(m)$ the west one. The procedure is generalized in d dimension by successive hierarchization in all the directions. On figure 1.9, we give a

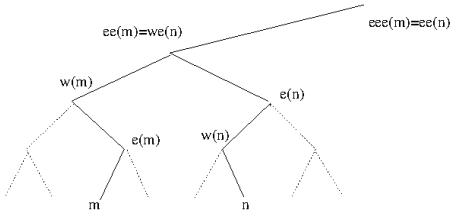


Figure 1.8: Node involved in linear, quadratic and cubic representation of a function at node m and n

representation of the W subspace for $\underline{l} \leq 3$ in dimension 2.

In order to deal with functions not null at the boundary, two more basis are added to the

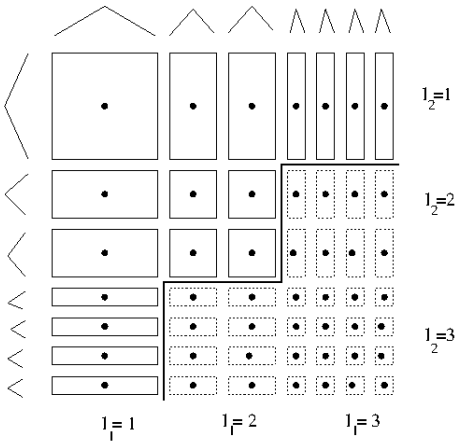


Figure 1.9: The two dimensional subspace $W_{\underline{l}}^{(L)}$ up to $l = 3$ in each dimension. The additional hierarchical functions corresponding to an approximation on the full grid are given in dashed lines.

first level as shown on figure 1.10. This approach results in many more points than the one without the boundary. As noted in [9] for n =5, in dimension 8 you have nearly 2.8 millions points in this approximation but only 6401 inside the domain. On figure 1.11 we give the grids points with boundary points in dimension 2 and 3 for a level 5 of the sparse grid.

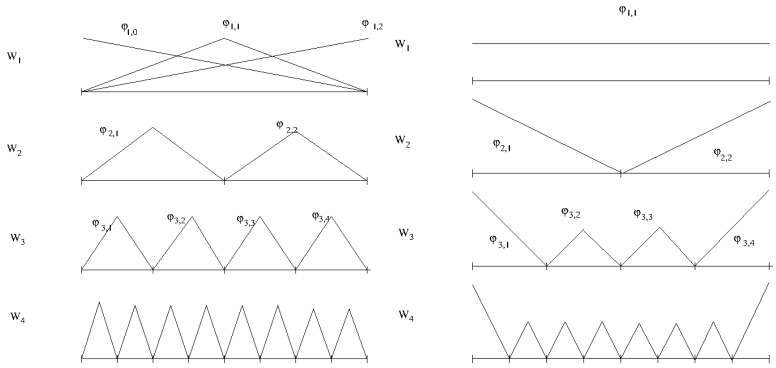


Figure 1.10: One dimensional $W^{(L)}$ spaces with linear functions with "exact " boundary (left) and "modified " boundary (right): $W_1^{(L)}$, $W_2^{(L)}$, $W_3^{(L)}$, $W_4^{(L)}$

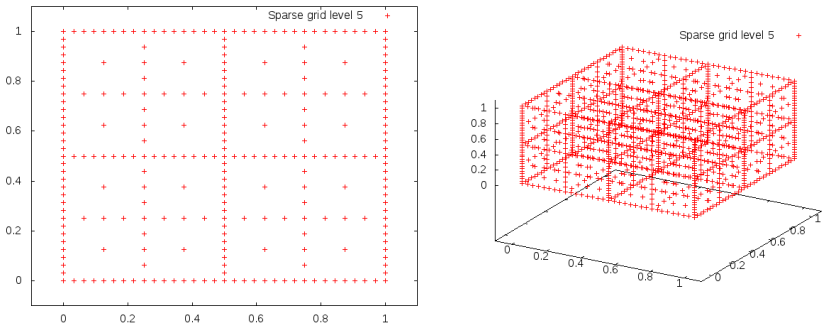


Figure 1.11: Sparse grid in dimension 2 and 3 with boundary points

If the boundary conditions are not important (infinite domain truncated in finance for example) the hat functions near the boundaries are modified by extrapolation (see figure 1.10) as explained in [9]. On level 1, we only have one degree of freedom assuming the function is constant on the domain. On all other levels, we extrapolate linearly towards the boundary the left and right basis functions, other functions remaining unchanged. So the new functions basis in 1D $\tilde{\phi}$ becomes

$$\tilde{\phi}_{l,i}^{(L)}(x) = \begin{cases} 1 & \text{if } l = 1 \text{ and } i = 1 \\ \left\{ \begin{array}{cc} 2 - 2^l x & \text{if } x \in [0, 2^{-l+1}] \\ 0 & \text{else} \end{array} \right\} & \text{if } l > 1 \text{ and } i = 1 \\ \left\{ \begin{array}{cc} 2^l(x-1) + 2 & \text{if } x \in [1 - 2^{-l+1}, 1] \\ 0 & \text{else} \end{array} \right\} & \text{if } l > 1 \text{ and } i = 2^l - 1 \\ \phi_{l,i}^{(L)}(x) & \text{otherwise} \end{cases}$$

On figure 1.12 we give the grids points eliminating boundary points in dimension 2 and 3 for a level 5 of the sparse grid.

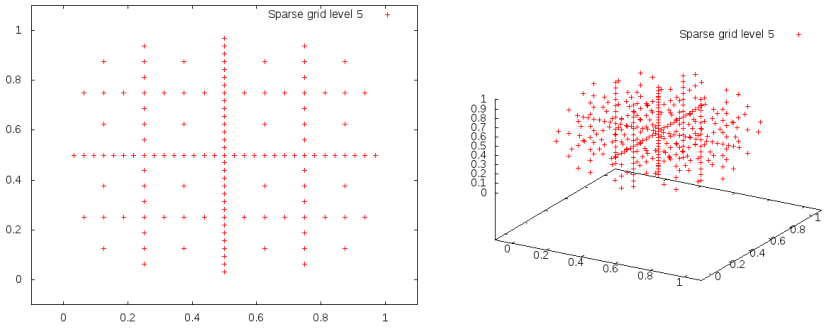


Figure 1.12: Sparse grid in dimension 2 and 3 without boundary points

The interpolation error associated to the linear operator $I^1 := I^{(L)}$ is linked to the regularity of the cross derivatives of the function [10, 17, 18]. If f is null at the boundary and admits derivatives such that $||\frac{\partial^{2d}u}{\partial x_1^2 ... \partial x_d^2}||_\infty < \infty$ then

$$||f - I^1(f)||_\infty = O(N^{-2} log(N)^{d-1}), \tag{1.6}$$

with N the number of points per dimension.

1.4 High order sparse grid methods

Changing the interpolator enables us to get a higher rate of convergence mainly in region where the solution is smooth. Following [17] and [18], it is possible to get higher order interpolators. Using a quadratic interpolator, the reconstruction on the nodal basis gives a quadratic function on the support of the previously defined hat function and a continuous function of the whole domain. The polynomial quadratic basis is defined on $[2^{-l}(i-1), 2^{-l}(i+1)]$ by

$$\phi_{l,i}^{(Q)}(x) = \phi^{(Q)}(2^l x - i)$$

with $\phi^{(Q)}(x) = 1 - x^2$.
The hierarchical surplus (coefficient on the basis) in one dimension is the difference between the value function at the node and the quadratic representation of the function using nodes available at the preceding level. With the notation of figure 1.8

$$\begin{aligned} \alpha(m)^{(Q)} &= f(m) - (\frac{3}{8}f(w(m)) + \frac{3}{4}f(e(m)) - \frac{1}{8}f(ee(m))) \\ &= \alpha(m)^{(L)}(m) - \frac{1}{4}\alpha(m)^{(L)}(e(m)) \\ &= \alpha(m)^{(L)}(m) - \frac{1}{4}\alpha(m)^{(L)}(df(m)) \end{aligned}$$

where $df(m)$ is the direct father of the node m in the tree.
Once again the quadratic surplus in dimension d is obtained by successive hierarchization in the different dimensions.
In order to take into account the boundary conditions, two linear functions $1 - x$ and x are added at the first level (see figure 1.13).

A version with modified boundary conditions can be derived for example by using linear interpolation at the boundary such that

$$\tilde{\phi}_{l,i}^{(Q)}(x) \;=\; \begin{cases} \tilde{\phi}_{l,i}^{(L)} & \text{if } i=1 \text{ or } i=2^l-1, \\ \phi_{l,i}^{(Q)}(x) & \text{otherwise} \end{cases}$$

In the case of the cubic representation, on figure 1.8 we need 4 points to define a function

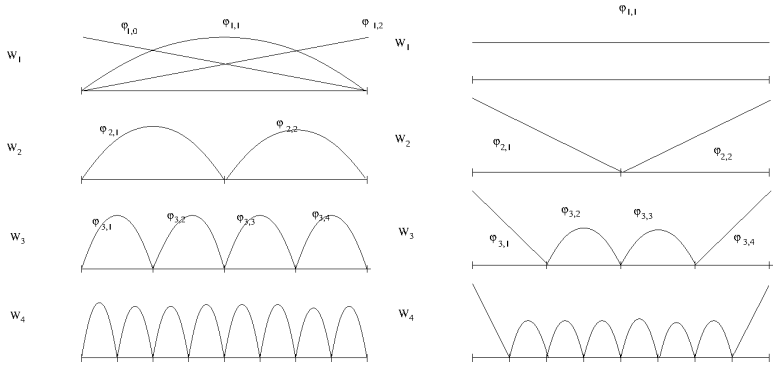


Figure 1.13: One dimensional $W^{(Q)}$ spaces with quadratic with "exact" boundary (left) and "modified" boundary (right): $W_1^{(Q)}$, $W_2^{(Q)}$, $W_3^{(Q)}$, $W_4^{(Q)}$

basis. In order to keep the same data structure, we use a cubic function basis at node m with value 1 at this node and 0 at the node $e(m)$, $w(m)$ and $ee(m)$ and we only keep the basis function between $w(m)$ and $e(m)$ [17].
Notice that there are two kinds of basis function depending of the position in the tree. The basis functions are given on $[2^{-l+1}i, 2^{-l+1}(i+1)]$ by

$$\begin{aligned} \phi_{l,2i+1}^{(C)}(x) &= \phi^{(C),1}(2^l x-(2i+1)), \text{ if } i \text{ even} \\ &= \phi^{(C),2}(2^l x-(2i+1)), \text{ if } i \text{ odd} \end{aligned}$$

with $\phi^{(C),1}(x)=\frac{(x^2-1)(x-3)}{3}$, $\phi^{(C),2}(x)=\frac{(1-x^2)(x+3)}{3}$.
The coefficient surplus can be defined as before as the difference between the value function at the node and the cubic representation of the function at the father node. Because of the two basis functions involved there are two kind of cubic coefficient.

- For a node $m=x_{l,8i+1}$ or $m=x_{l,8i+7}$, $\alpha^{(C)}(m)=\alpha^{(C,1)}(m)$, with

$$\alpha^{(C,1)}(m)=\alpha^{(Q)}(m)-\frac{1}{8}\alpha^{(Q)}(df(m))$$

- For a node $m=x_{l,8i+3}$ or $m=x_{l,8i+5}$, $\alpha^{(C)}(m)=\alpha^{(C,2)}(m)$, with

$$\alpha^{(C,2)}(m)=\alpha^{(Q)}(m)+\frac{1}{8}\alpha^{(Q)}(df(m))$$

Notice that a cubic representation is not available for $l=1$ so a quadratic approximation is used. As before boundary conditions are treated by adding two linear functions basis at

the first level and a modified version is available. We choose the following basis functions as defined on figure 1.14:

$$\tilde{\phi}_{l,i}^{(C)}(x) \quad = \quad \begin{cases} \tilde{\phi}_{l,i}^{(Q)} & \text{if } i \in \{1, 3, 2^l - 3, 2^l - 1\}, \\ \phi_{l,i}^{(C)}(x) & \text{otherwise} \end{cases}$$

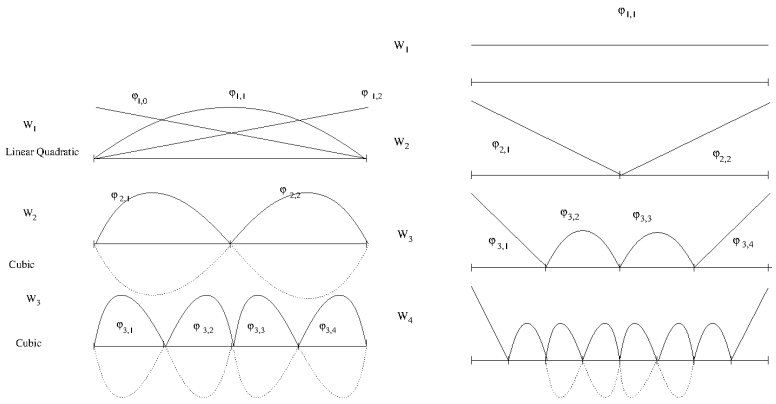


Figure 1.14: One dimensional $W^{(C)}$ spaces with cubic and "exact" boundary (left) and "modified" boundary (right): $W_1^{(C)}$, $W_2^{(C)}$, $W_3^{(C)}$, $W_4^{(C)}$

According to [10, 17, 18], if the function f is null at the boundary and admits derivatives such that $\sup_{\alpha_i \in \{2,\dots,p+1\}} \left\{ || \frac{\partial^{\alpha_1 + .. + \alpha_d} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} ||_\infty \right\} < \infty$ then the interpolation error can be generalized for $I^2 := I^{(Q)}$, $I^3 := I^{(C)}$ by:

$$||f - I^p(f)||_\infty = O(N^{-(p+1)} log(N)^{d-1}), \quad p = 2, 3$$

with N the number of points per dimension.

1.5 Anisotropy

In many situations, it is useless to refine as much in each direction. For example, when dealing with multidimensional storages we expect the mesh size to be of the same order in each direction. When the different storages have very different sizes, we want to refine more the storage with the highest capacity. In order to treat this anisotropy an extension of Sparse grids can be achieved by defining weight w in each direction. The definition 1.4 is replaced by:

$$V_n = \bigoplus_{\sum_{i=1}^d l_i w(i) \leq n + d - 1} W_{\underline{l}}^{(L)} \tag{1.7}$$

1.6 Adaptation

When the solution is not smooth, typically Lipschitz, there is no hope to get convergence results for classical Sparse Grids (see above the interpolation error linked to the cross derivatives of the function). So classical sparse grids have to be adapted such that the solution is

refined near singularities. In all adaptations methods hierarchical surplus $\alpha_{\underline{l},\underline{i}}$ are used to get an estimation of the local error. These coefficients give an estimation of the smoothness of the function value at the discrete points by representing the discrete mix second derivative of the function. There is mainly two kinds of adaptation used :

- the first one is performing local adaptation and only adds points locally [32, 8, 33, 12],
- the second one is performing adaptation at the level of the hierarchical space $W_{\underline{l}}$ (anisotropic sparse grid). This approach detects important dimensions that needs refinement and refines all the points in this dimension [11]. This refinement is also achieved in areas where the solution can be smooth. A more local version has been developed in [34].

In the current version of the library only dimension adaptation is available. Details on the algorithm can be bound in [11]. After a first initialization with a first initialization with a space

$$V_n = \bigoplus_{\sum_{i=1}^{d} l_i \leq n+d-1} W_{\underline{l}}^{(L)} \tag{1.8}$$

A set of active level $\mathcal{A}$ is created gathering all levels $\underline{l}$ such that $\sum_{i=1}^{d} l_i = n + d - 1$. All other levels are gathered in a set $\mathcal{O}$. At each level $\underline{l}$ in $\mathcal{A}$ an error is estimated $e_{\underline{l}}$ and with all local error $e_{\underline{l}}$ a global error E is calculated. Then the refinement algorithm 1 is used noting $\mathbf{e}_k$ the canonical basis in dimension k. Sometimes, using sparse grids during time iterations,

while $E > \eta$ **do**
  select $\underline{l}$ with the highest local error $e_{\underline{l}}$
  $\mathcal{A} = \mathcal{A} \backslash \{\underline{l}\}$
  $\mathcal{O} = \mathcal{O} \cup \{\underline{l}\}$
  for $k = 1$ to d **do**
    $\underline{m} = \underline{l} + \mathbf{e}_k$
    if $\underline{m} - \mathbf{e}_q \in \mathcal{O}$ for $q \in [1, d]$ **then**
      $\mathcal{A} = \mathcal{A} \cup \{\underline{m}\}$
      Hierarchize all points belonging to $\underline{m}$
      calculate $e_{\underline{m}}$
      update E
    end if
  end for
end while

Algorithm 1: Dimension refinement for a given tolerance η

it can be interesting to coarsen the meshes. A similar algorithm 2 can be used to eliminate levels with a very small local error.

$\mathcal{B}$ all elements of $\mathcal{A}$ with a local error below η
while $\mathcal{B}$ non nonempty **do**
  select $\underline{l} \in \mathcal{B}$ with the lowest local error $e_{\underline{l}}$
  for $k = 1$ to d **do**
    $\underline{m} = \underline{l} - \mathbf{e}_k$
    if $m_k > 0$ **then**
      if $\underline{m} + \mathbf{e}_q \in \mathcal{B}$ for $q \in [1, d]$ **then**
        $\mathcal{A} = \mathcal{A} \setminus \{\underline{m} + \mathbf{e}_q, q \in [1, d]\}$
        $\mathcal{B} = \mathcal{B} \setminus \{\underline{m} + \mathbf{e}_q, q \in [1, d]\}$
        $\mathcal{A} = \mathcal{A} \cup \{\underline{m}\}$
        Add $\underline{m}$ to $\mathcal{B}$ if local error below η
        $\mathcal{O} = \mathcal{O} \setminus \{\underline{m}\}$
        Break
      end if
    end if
  end for
  if $\underline{l} \in \mathcal{B}$ **then**
    $\mathcal{B} = \mathcal{B} \setminus \{\underline{l}\}$
  end if
end while

Algorithm 2: Dimension coarsening for a given tolerance η

1.7 C++ APi

The construction of the Sparse Grid including boundary point is done by the following constructor

```
1  SparseSpaceGridBound(const Eigen::ArrayXd  &p_lowValues, const Eigen::
       ArrayXd &p_sizeDomain, const int &p_levelMax,  const Eigen::ArrayXd &
       p_weight,
2              const size_t &p_degree)
```

with

- *p_lowValues* corresponds to the bottom of the grid,
- *p_sizeDomain* corresponds to the size of the resolution domain in each dimension,
- *p_levelMax* is the level of the sparse grids, the n in equation 1.7,
- *p_weight* the weight for anisotropic sparse grids, the w in equation 1.7,
- *p_degree* is equal to 1 (linear interpolator), or 2 (quadratic interpolator) or 3 (for cubic interpolator),

With the same notations the construction eliminating boundary points is done by the following constructor

```
SparseSpaceGridNoBound(const Eigen::ArrayXd &p_lowValues, const Eigen::ArrayXd &p_sizeDomain, const int &p_levelMax, const Eigen::ArrayXd &p_weight,
        const size_t &p_degree)
```

The data structure of type *SparseSet* to store the sparse grid is defined by a map with keys an array A storing a multi level and values a map with keys an array B storing the multi index associated to a point (A,B) and values the number of point (A,B) :

```
#define SparseSet std::map< Eigen::Array<char,Eigen::Dynamic,1 >, std::map< Eigen::Array<unsigned int,Eigen::Dynamic,1>, size_t, OrderTinyVector< unsigned int > >,OrderTinyVector< char> >
```

It is sometimes convenient to get back this data structure from the SparseGrid object : this is achieved by the following method :

```
std::shared_ptr<SparseSet> getDataSet() const ;
```

The previous two classes own two specific member functions to hierarchize (see section above) the value function known at the grids points for the whole grid.

- the first work on a single function:

```
/// \brief Hierarchize a function defined on the grid
/// \param p_toHierachize function to hierarchize
void toHierarchize( Eigen::ArrayXd & p_toHierachize );
```

- the second work on a matrix, permitting to hierarchize many functions in a single call (each row corresponds to a function representation)

```

/// \brief Hierarchize a set of functions defined on the grid
/// \param p_toHierachize function to hierarchize
void toHierarchizeVec( Eigen::ArrayXXd & p_toHierachize )
```

The two classes own two specific member functions to hierarchize point by point a value fonction at given points in the sparse grid :

- the first work on a single function:

```
/// \brief Hierarchize some points defined on the sparse grids
///        Hierarchization is performed point by point
/// \param p_nodalValues function to hierarchize
/// \param p_sparsePoints vector of sparse points to hierarchize (all points should belong to the dataset structure)
/// \param p_hierarchized array of all hierarchized values (it is updated)
virtual void toHierarchizePByP(const Eigen::ArrayXd &p_nodalValues, const std::vector<SparsePoint> &p_sparsePoints, Eigen::ArrayXd &p_hierarchized) const
```

- the second work on a matrix, permitting to hierarchize many functions in a single call (each row corresponds to a function representation)

```
    /// \brief Hierarchize some points defined on the sparse grids for a set of functions
    ///        Hierarchization is performed point by point
    /// \param p_nodalValues         functions to hierarchize (the row corresponds to the function number)
    /// \param p_sparsePoints        vector of sparse points to hierarchize (all points should belong to the dataset structure)
    /// \param p_hierarchized        array of all hierarchized values (it is updated)
    virtual void toHierarchizePByPVec(const Eigen::ArrayXXd &p_nodalValues, const  std::vector<SparsePoint> &p_sparsePoints, Eigen::ArrayXXd &p_hierarchized) const
```

The *SparsePoint* object is only a type def :

```
#define SparsePoint std::pair< Eigen::Array<char, Eigen::Dynamic, 1> , Eigen::Array<unsigned int, Eigen::Dynamic, 1> >
```

where the first array permits to store the multi level associated to the point and the second the multi index associated.
At last it is possible to hierarchize all points associated to a multi level. As before two methods are available :

- a first permits to hierarchize all the points associated to a given level. Hierarchized values are updated with these new values.

```
    /// \brief Hierarchize all points defined on a given level of the sparse grids
    ///        Hierarchization is performed point by point
    /// \param p_nodalValues         function to hierarchize
    /// \param p_iterLevel           iterator on the level of the point to hierarchize
    /// \param p_hierarchized        array of all hierarchized values (it is updated)
     virtual  void toHierarchizePByPLevel(const Eigen::ArrayXd &p_nodalValues, const  SparseSet::const_iterator &p_iterLevel, Eigen::ArrayXd &p_hierarchized) const
```

- the second permits to hierarchize differents functions together

```
    /// \brief Hierarchize all points defined on a given level of the sparse grids for a set of functions
    ///        Hierarchization is performed point by point
    /// \param p_nodalValues         function to hierarchize (the row corresponds to the function number)
    /// \param p_iterLevel           iterator on the level of the point to hierarchize
    /// \param p_hierarchized        array of all hierarchized values (it is updated)
    virtual void toHierarchizePByPLevelVec(const Eigen::ArrayXXd &p_nodalValues, const  SparseSet::const_iterator &p_iterLevel, Eigen::ArrayXXd &p_hierarchized) const
```

In the following example, the sparse grids with boundary points is constructed. The values of a function f at each coordinates are stored in an array $valuesFunction$, storing 2 functions to interpolate. The 2 global functions are hierarchized (see section above) in the array $hierarValues$, and then the interpolation can be achieved using these hierarchized values.

```
ArrayXd  lowValues = ArrayXd::Zero(5); // bottom of the grid
ArrayXd  sizeDomain = ArrayXd::Constant(5,1.); // size of the grid
ArrayXd  weight  = ArrayXd::Constant(5,1.); // weights
int degree =1 ; // linear interpolator
bool bPrepInterp = true; // precalculate neighbors of nodes
level = 4 ; // level of the sparse grid

// sparse grid generation
SparseSpaceGridBound sparseGrid(lowValues, sizeDomain, level, weight,
    degree, bPrepInterp);

//  grid iterators
shared_ptr<GridIterator > iterGrid = sparseGrid.getGridIterator();
ArrayXXd valuesFunction(1,sparseGrid.getNbPoints());
while (iterGrid->isValid())
{
    ArrayXd pointCoord = iterGrid->getCoordinate();
    valuesFunction(0,iterGrid->getCount()) = f(pointCoord) ;
    valuesFunction(1,iterGrid->getCount()) = f(pointCoord)+1 ;
    iterGrid->next();
}

// Hierarchize
ArrayXXd hieraValues =valuesFunction;
sparseGrid.toHierarchizeVec(hieraValues);

// interpolate
 ArrayXd pointCoord = ArrayXd::Constant(5,0.66);
 shared_ptr<Interpolator > interpolator = sparseGrid.createInterpolator(
     pointCoord);
 ArrayXd  interVal  = interpolator->applyVec(hieraValues);
```

Remark 5 *Point by point hierarchization on the global grid could have been calculated as below*

```
std::vector<SparsePoint> sparsePoints(sparseGrid.getNbPoints());
std::shared_ptr<SparseSet> dataSet =  sparseGrid.getDataSet();
// iterate on points
for (typename SparseSet::const_iterator iterLevel = dataSet->begin();
    iterLevel != dataSet->end(); ++iterLevel)
    for (typename SparseLevel::const_iterator iterPosition = iterLevel->
        second.begin(); iterPosition != iterLevel->second.end(); ++
        iterPosition)
    {
        sparsePoints[iterPosition->second] = make_pair(iterLevel->first,
            iterPosition->first);
    }
```

```
        ArrayXXd hieraValues = sparseGrid.toHierarchizePByPVec(
            valuesFunction, sparsePoints);
```

In some cases, it is more convenient to construct an interpolator acting on a global function. It is the case when you have a single function and you want to interpolate at many points for this function. In this specific case an interpolator deriving from the class InterpolatorSpectral (similarly to Legendre grid interpolators) can be constructed :

```
    /** \brief Constructor taking in values on the grid
     *  \param p_grid      is the sparse  grid used to interpolate
     *  \param p_values    Function values on the sparse grid
     */
  SparseInterpolatorSpectral(const shared_ptr< SparseSpaceGrid>  &p_grid ,
     const Eigen::ArrayXd &p_values)
```

This class has a member to interpolate at a given point:

```
    /**  \brief  interpolate
     *  \param  p_point  coordinates of the point for interpolation
     *  \return interpolated value
     */
    inline double apply(const Eigen::ArrayXd &p_point) const
```

See section 1.2 for an example (similar but with Legendre grids) to use this object.
Sometimes, one wish to iterate on points on a givel level. In the example below , for each level an iterator on all points belonging to a given level is got back and the values of a function f at each point are calculated and stored.

```
  // sparse grid generation
    SparseSpaceGridNoBound sparseGrid(lowValues, sizeDomain, p_level,
       p_weight, p_degree, bPrepInterp);

    //  test iterator on each level
    ArrayXd valuesFunctionTest(sparseGrid.getNbPoints());
    std::shared_ptr<SparseSet> dataSet =  sparseGrid.getDataSet();
     for (SparseSet::const_iterator iterLevel = dataSet->begin(); iterLevel
        != dataSet->end(); ++iterLevel)
      {
      // get back iterator on this level
   shared_ptr<SparseGridIterator> iterGridLevel = sparseGrid.
      getLevelGridIterator(iterLevel);
   while(iterGridLevel->isValid())
     {
       Eigen::ArrayXd pointCoord = iterGridLevel->getCoordinate();
       valuesFunctionTest(iterGridLevel->getCount()) = f(pointCoord);
       iterGridLevel->next();
     }
      }
```

At last adaptation can be realized with two member functions :

- A first one permits to refine adding points where the error is important. Notice that a function is provided to calculate from the hierarchical values the error at each level of

the sparse grid and that a second one is provided to get a global error from the error calculated at each level. This permits to specialize the refining depending for example if the calculation is achieved for integration or interpolation purpose.

```
1     /// \brief Dimension adaptation nest
2     /// \param p_precision       precision required for adaptation
3     /// \param p_fInterpol      function to interpolate
4     /// \param p_phi            function for the error on a given level
          in the m_dataSet structure
5     /// \param p_phiMult         from an error defined on different
          levels, send back a global error on the different levels
6     /// \param p_valuesFunction  an array storing the nodal values
7     /// \param  p_hierarValues  an array storing hierarchized values (
          updated)
8     void refine(const double &p_precision, const std::function<double(
          const Eigen::ArrayXd &p_x)> &p_fInterpol,
9                 const std::function< double(const SparseSet::
                      const_iterator &, const Eigen::ArrayXd &)> &p_phi,
10                const std::function< double(const std::vector< double> &)
                      > &p_phiMult,
11                Eigen::ArrayXd &p_valuesFunction,
12                Eigen::ArrayXd &p_hierarValues);
```

with

- $p\_precision$ the η tolerance in the algorithm,
- $p\_fInterpol$ the function permitting to calculate the nodal values,
- p_phi function permitting to calculate $e_{\underline{l}}$ the local error for a given $\underline{l}$,
- $p\_phiMult$ a function taking as argument all the $e_{\underline{l}}$ (local errors) and giving back the global error E,
- $p\_valuesFunction$ an array storing the nodal values (updated during refinement)
- $p\_hierarValues$ an array storing the hierarchized values (updated during refinement)

- A second one permits to coarsen the mesh, eliminating point where the error is too small

```
1    /// \brief Dimension adaptation coarsening : modify data structure by
         trying to remove all levels with local error
2     ///        below a local precision
3     /// \param p_precision      Precision under which coarsening  will be
          realized
4     /// \param p_phi            function for the error on a given level
          in the m_dataSet structure
5     /// \param p_valuesFunction  an array storing the nodal values (
          modified on the new structure)
6     /// \param p_hierarValues   Hierarchical values on a data structure (
          modified on the new structure)
7     void coarsen(const double &p_precision,  const std::function< double(
          const SparseSet::const_iterator &, const Eigen::ArrayXd &)> &
          p_phi,
```

```
                    Eigen::ArrayXd &p_valuesFunction,
                    Eigen::ArrayXd    &p_hierarValues);
```

with arguments similar to the previous function.

1.8 Python APi

Here is an example of the python API used for interpolation with Sparse grids with boundary points and without boundary points. The adaptation and coarsening is available with an error calculated for interpolation only.

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU LGPL)
import numpy as np
import unittest
import random
import math
import StOptGrids

# function used
def funcToInterpolate( x):
    return math.log(1. + x.sum())

# unit test for sparse grids
################################

class testGrids(unittest.TestCase):


    # test sparse grids  with boundaries
    def testSparseGridsBounds(self):
        # low values
        lowValues =np.array([1.,2.,3.])
        # size of the domain
        sizeDomValues = np.array([3.,4.,3.])
        # anisotropic weights
        weights =  np.array([1.,1.,1.])
        # level of the sparse grid
        level =3
        # create the sparse grid with linear interpolator
        sparseGridLin = StOptGrids.SparseSpaceGridBound(lowValues, sizeDomValues, level, weights,1)
        iterGrid = sparseGridLin.getGridIterator()
         # array to store
        data = np.empty(sparseGridLin.getNbPoints())
        # iterates on point
        while( iterGrid.isValid()):
          data[iterGrid.getCount()] = funcToInterpolate(iterGrid.getCoordinate())
          iterGrid.next()
        # Hierarchize the data
```

```
40          hierarData = sparseGridLin.toHierarchize(data)
41          # get back an interpolator
42          ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
43          interpol = sparseGridLin.createInterpolator(ptInterp)
44          # calculate interpolated value
45          interpValue = interpol.apply(hierarData)
46          print(("Interpolated value sparse linear" , interpValue))
47           # create the sparse grid with quadratic interpolator
48          sparseGridQuad = StOptGrids.SparseSpaceGridBound(lowValues,
                sizeDomValues, level, weights,2)
49          # Hierarchize the data
50          hierarData = sparseGridQuad.toHierarchize(data)
51          # get back an interpolator
52          ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
53          interpol = sparseGridQuad.createInterpolator(ptInterp)
54          # calculate interpolated value
55          interpValue = interpol.apply(hierarData)
56          print(("Interpolated value sparse quadratic " , interpValue))
57          # now refine
58          precision = 1e-6
59          print(("Size of hierarchical array " , len(hierarData)))
60          valueAndHierar = sparseGridQuad.refine(precision,funcToInterpolate,
                data,hierarData)
61          print(("Size of hierarchical array after refinement " , len(
                valueAndHierar[0])))
62          # calculate interpolated value
63          interpol1 = sparseGridQuad.createInterpolator(ptInterp)
64          interpValue = interpol1.apply(valueAndHierar[1])
65          print(("Interpolated value sparse quadratic after refinement " ,
                interpValue))
66          # coarsen the grid
67          precision = 1e-4
68          valueAndHierarCoarsen = sparseGridQuad.coarsen(precision,
                valueAndHierar[0],valueAndHierar[1])
69          print(("Size of hierarchical array after coarsening " , len(
                valueAndHierarCoarsen[0])))
70          # calculate interpolated value
71          interpol2 = sparseGridQuad.createInterpolator(ptInterp)
72          interpValue = interpol2.apply(valueAndHierarCoarsen[1])
73          print(("Interpolated value sparse quadratic after refinement " ,
                interpValue))
74
75
76      # test sparse grids eliminating boundaries
77      def testSparseGridsNoBounds(self):
78          # low values
79          lowValues =np.array([1.,2.,3.],dtype=np.float)
80          # size of the domain
81          sizeDomValues = np.array([3.,4.,3.],dtype=np.float)
82          # anisotropic weights
83          weights =  np.array([1.,1.,1.])
84          # level of the sparse grid
85          level =3
86          # create the sparse grid with linear interpolator
```

```
87          sparseGridLin = StOptGrids.SparseSpaceGridNoBound(lowValues,
                sizeDomValues, level, weights,1)
88          iterGrid = sparseGridLin.getGridIterator()
89           # array to store
90          data = np.empty(sparseGridLin.getNbPoints())
91          # iterates on point
92          while( iterGrid.isValid()):
93            data[iterGrid.getCount()] = funcToInterpolate(iterGrid.
                  getCoordinate())
94            iterGrid.next()
95          # Hierarchize the data
96          hierarData = sparseGridLin.toHierarchize(data)
97          # get back an interpolator
98          ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
99          interpol = sparseGridLin.createInterpolator(ptInterp)
100         # calculate interpolated value
101         interpValue = interpol.apply(hierarData)
102         print(("Interpolated value sparse linear", interpValue))
103          # create the sparse grid with quadratic interpolator
104         sparseGridQuad = StOptGrids.SparseSpaceGridNoBound(lowValues,
                sizeDomValues, level, weights,2)
105         # Hierarchize the data
106         hierarData = sparseGridQuad.toHierarchize(data)
107         # get back an interpolator
108         ptInterp = np.array([2.3,3.2,5.9],dtype=np.float)
109         interpol = sparseGridQuad.createInterpolator(ptInterp)
110         # calculate interpolated value
111         interpValue = interpol.apply(hierarData)
112         print(("Interpolated value sparse quadratic ", interpValue))
113         # test grids function
114         iDim = sparseGridQuad.getDimension()
115         pt = sparseGridQuad.getExtremeValues()
116         # now refine
117         precision = 1e-6
118         print(("Size of hierarchical array ", len(hierarData)))
119         valueAndHierar = sparseGridQuad.refine(precision,funcToInterpolate,
                data,hierarData)
120         print(("Size of hierarchical array after refinement ", len(
                valueAndHierar[0])))
121         # calculate interpolated value
122         interpol1 = sparseGridQuad.createInterpolator(ptInterp)
123         interpValue = interpol1.apply(valueAndHierar[1])
124         print(("Interpolated value sparse quadratic after coarsening ",
                interpValue))
125         # coarsen the grid
126         precision = 1e-4
127         valueAndHierarCoarsen = sparseGridQuad.coarsen(precision,
                valueAndHierar[0],valueAndHierar[1])
128         print(("Size of hierarchical array after coarsening ", len(
                valueAndHierarCoarsen[0])))
129         # calculate interpolated value
130         interpol2 = sparseGridQuad.createInterpolator(ptInterp)
131         interpValue = interpol2.apply(valueAndHierarCoarsen[1])
132         print(("Interpolated value sparse quadratic after coarsening ",
```

```
            interpValue))

if __name__ == '__main__':
    unittest.main()
```

Chapter 2

Introducing the regression resolution

Suppose the the stochastic differential equation in the optimization problem is not controlled:

$$dX^{x,t} = b(t, X_s^{x,t})ds + \sigma_(s, X_s^{x,t})dW_s$$

This case is for example encountered while valuing American options in finance, when an arbitrage is realized between the pay off and the expected future gain if not exercising at the current time. In order to estimate this conditional expectation (depending of the Markov state), first suppose that a set of N Monte Carlo Simulation are available at dates t_i for a process $X_t := X_t^{0,x}$ where x is the initial state at date $t = 0$ and that we want to estimate $f(x) := \mathbb{E}[g(t+h, X_{t+h}) \mid X_t = x]$ for a given x and a given function g. This function f lies the infinite dimensional space of the L_2 functions. In order to approximate it, we try to find it in a finite dimensional space. Choosing a set of basis functions ψ_k for $k = 1$ to M, the conditional expectation can be approximated by

$$f(x) \simeq \sum_{k=1}^{M} \alpha_k \psi_k(X_t) \tag{2.1}$$

where $(\hat{\alpha}_k^{t_i,N})_{k \leq M}$ minimizes

$$\sum_{\ell=1}^{N} \left| g(X_{t+h}^l) - \sum_{k=1}^{M} \alpha_k \psi_k(X_t^l) \right|^2 \tag{2.2}$$

over $(\alpha_k)_{k \leq M} \in \mathbb{R}^M$. We have to solve a quadratic optimization problem of the form

$$\min_{\alpha \in \mathbb{R}^M} \|A\alpha - B\|^2 \tag{2.3}$$

Classically the previous equation is reduced to the normal equation

$$A'A\alpha = A'B \ , \tag{2.4}$$

which is solved by a Cholesky like approach when the matrix $A'A$ is definite otherwise the solution with the minimum L_2 norm can be computed using tghe pseudo inverse of $A'A$.
When the different component of $X^{x,t}$ are highly correlated i can be convinient to rotate

the data set onto its principal components using the PCA method. Rotating the dataset before doing regression has been advocated in [37] and [38] for example. The right-hand side of Figure 2.1 illustrates the new evaluation grid obtained on the same dataset. One can observe the better coverage and the fewer empty areas when using local regession that we will detail in this section.

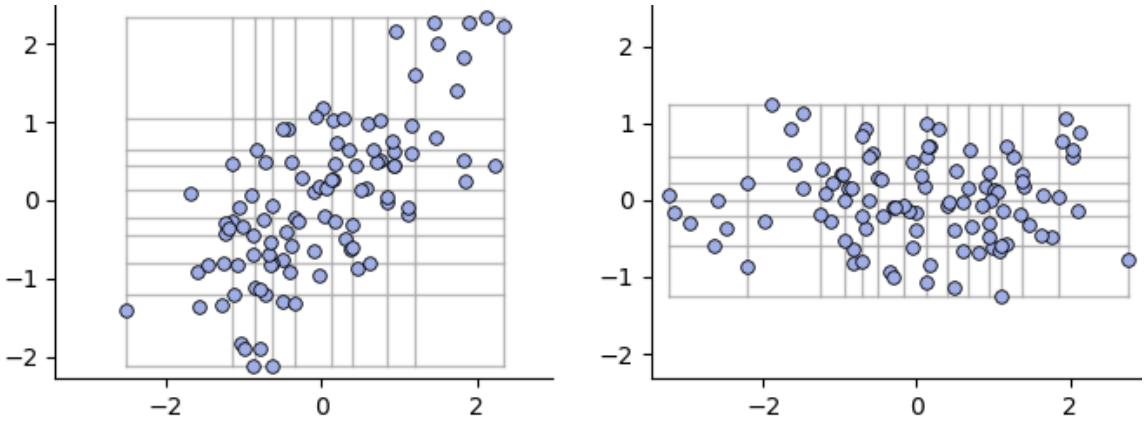


Figure 2.1: Evaluation grid: rotation

2.1 C++ global API

All the regression classes derive from the BaseRegression abstract class, which stores a pointer to the "particles" (a matrix storing the simulations of $X^{x,t}$: the first dimension of the matrix corresponds to the dimension of $X^{x,t}$, and the second dimension corresponds to the particle number), and stores if the current date t is 0 (then the conditional expectation is only an expectation).

```
1  // Copyright (C) 2016, 2017 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #ifndef BASEREGRESSION_H
5  #define BASEREGRESSION_H
6  #include <memory>
7  #include <vector>
8  #include <iostream>
9  #include <Eigen/Dense>
10 #include <Eigen/SVD>
11 #include "StOpt/core/grids/InterpolatorSpectral.h"
12
13 /** \file BaseRegression.h
14  *  \brief Base class to define regressor for stochastic optimization by
        Monte Carlo
15  *  \author Xavier Warin
16  */
17 namespace StOpt
18 {
```

```
19 /// \class BaseRegression BaseRegression.h
20 /// Base class for regression
21 class BaseRegression
22 {
23 protected :
24
25     bool m_bZeroDate ;                    ///< Is the regression date zero ?
26     bool m_bRotationAndRescale ; ///< do we rescale particles and do a
       rotation with SVD on data
27     Eigen::ArrayXd m_meanX ; ///< store  scaled factor in each direction (
       average of particles values in each direction)
28     Eigen::ArrayXd m_etypX ; ///< store  scaled factor in each direction (
       standard deviation of particles in each direction)
29     Eigen::MatrixXd m_svdMatrix ; ///< svd matrix transposed  used to
       transform particles
30     Eigen::ArrayXd  m_sing      ; ///< singular values associated to SVD
31     Eigen::ArrayXXd m_particles;  ///< Particles used to regress: first
       dimension  : dimension of the problem , second dimension : the  number
        of particles. These particles are rescaled and a rotation with SVD is
        achieved to avoid degeneracy in case of high correlations
32
33     // rotation for data and rescaling
34     void preProcessData();
35
36 public :
37
38     /// \brief Default constructor
39     BaseRegression();
40
41     /// \brief Default destructor
42     virtual ~BaseRegression() {}
43
44     /// \brief Default constructor
45     BaseRegression(const bool &p_bRotationAndRescale);
46
47     /// \brief Constructor storing the particles
48     /// \param  p_bZeroDate        first date is 0?
49     /// \param  p_particles        particles used for the meshes.
50     ///                            First dimension  : dimension of the
       problem,
51     ///                            second dimension : the  number of
       particles
52     /// \param  p_bRotationAndRescale do we rescale particle
53     /// Data are rescaled and a
54     BaseRegression(const bool &p_bZeroDate, const std::shared_ptr< Eigen::
       ArrayXXd> &p_particles, const bool &p_bRotationAndRescale);
55
56     /// \brief Constructor used in simulation, no rotation
57     /// \param  p_bZeroDate    first date is 0?
58     /// \param  p_bRotationAndRescale do we rescale particle
59     BaseRegression(const bool &p_bZeroDate, const bool &p_bRotationAndRescale
       );
60
61
```

```
62      /// \brief Last constructor used in simulation
63      /// \param  p_bZeroDate    first date is 0?
64      /// \param  p_meanX           scaled factor in each direction (average
            of particles values in each direction)
65      /// \param  p_etypX           scaled factor in each direction (standard
            deviation of particles in each direction)
66      /// \param  p_svdMatrix       svd matrix transposed  used to transform
            particles
67      /// \param  p_bRotationAndRescale do we rescale particle
68
69      BaseRegression(const bool &p_bZeroDate, const   Eigen::ArrayXd &p_meanX,
            const   Eigen::ArrayXd   &p_etypX, const   Eigen::MatrixXd   &
            p_svdMatrix, const bool &p_bRotationAndRescale);
70
71      /// \brief Copy constructor
72      /// \param p_object  object to copy
73      BaseRegression(const BaseRegression &p_object);
74
75      /// \brief update the particles used in regression  and construct the
            matrices
76      /// \param  p_bZeroDate    first date is 0?
77      /// \param  p_particles    particles used for the meshes.
78      ///                        Firs dimension  : dimension of the problem,
79      ///                        second dimension : the  number of particles
80      void updateSimulationsBase(const bool &p_bZeroDate, const std::shared_ptr
            < Eigen::ArrayXXd> &p_particles);
81
82      /// \brief Get some local accessors
83      ///@{
84      virtual inline Eigen::ArrayXXd  getParticles() const
85      {
86          return m_particles ;
87      }
88
89      /// \brief Get bRotationAndRescale
90      virtual inline bool getBRotationAndRescale() const
91      {
92          return m_bRotationAndRescale ;
93      }
94
95      /// \brief Get average of simulation per dimension
96      virtual inline Eigen::ArrayXd getMeanX() const
97      {
98          return m_meanX;
99      }
100
101     /// \brief get standard deviation per dimension
102     virtual inline Eigen::ArrayXd getEtypX() const
103     {
104         return m_etypX;
105     }
106
107     /// \brief get back the SVD matrix used for rescaling particles
108     virtual inline Eigen::MatrixXd getSvdMatrix() const
```

```
109     {
110         return m_svdMatrix;
111     }
112 
113     /// \brief get back singular values
114     virtual inline Eigen::ArrayXd getSing() const
115     {
116         return m_sing;
117     }
118 
119     /// \brief Get dimension of the problem
120     virtual inline int getDimension() const
121     {
122         return m_particles.rows();
123     }
124 
125     /// \brief Get the number of simulations
126     virtual inline int  getNbSimul()const
127     {
128         return m_particles.cols() ;
129     }
130 
131     /// \brief get back particle by its number
132     /// \param p_iPart   particle number
133     /// \return the particle (if no particle, send back an empty array)
134     virtual Eigen::ArrayXd getParticle(const int &p_iPart) const;
135 
136     /// \brief get the number of basis functions
137     virtual  int getNumberOfFunction() const = 0 ;
138 
139     ///@}
140     /// \brief Constructor storing the particles
141     /// \brief update the particles used in regression  and construct the
        matrices
142     /// \param  p_bZeroDate    first date is 0?
143     /// \param  p_particles    particles used for the meshes.
144     ///                        First dimension  : dimension of the problem,
145     ///                        second dimension : the  number of particles
146     virtual void updateSimulations(const bool &p_bZeroDate, const std::
        shared_ptr< Eigen::ArrayXXd> &p_particles) = 0 ;
147 
148     /// \brief conditional expectation basis function coefficient calculation
149     /// \param  p_fToRegress  function to regress associated to each
        simulation used in optimization
150     /// \return regression coordinates on the basis  (size : number of meshes
         multiplied by the dimension plus one)
151     /// @{
152     virtual Eigen::ArrayXd getCoordBasisFunction(const Eigen::ArrayXd &
        p_fToRegress) const = 0;
153     ///@}
154     /// \brief conditional expectation basis function coefficient calculation
         for multiple functions to regress
155     /// \param  p_fToRegress  function to regress associated to each
        simulation used in optimization (size : number of functions to regress
```

```
        \times the number of Monte Carlo simulations)
156     /// \return regression coordinates on the basis  (size :  number of
           function to regress  \times number of meshes multiplied by the
           dimension plus one)
157     /// @{
158     virtual Eigen::ArrayXXd getCoordBasisFunctionMultiple(const Eigen::
           ArrayXXd &p_fToRegress) const = 0 ;
159     ///@}
160
161     /// \brief conditional expectation calculation
162     /// \param  p_fToRegress  simulations  to regress used in optimization
163     /// \return regressed value function
164     /// @{
165     virtual Eigen::ArrayXd getAllSimulations(const Eigen::ArrayXd &
           p_fToRegress) const = 0;
166     virtual Eigen::ArrayXXd getAllSimulationsMultiple(const Eigen::ArrayXXd &
           p_fToRegress) const = 0;
167     ///@}
168
169     /// \brief Use basis functions to reconstruct the solution
170     /// \param p_basisCoefficients basis coefficients
171     ///@{
172     virtual Eigen::ArrayXd reconstruction(const Eigen::ArrayXd   &
           p_basisCoefficients) const = 0 ;
173     virtual Eigen::ArrayXXd reconstructionMultiple(const Eigen::ArrayXXd   &
           p_basisCoefficients) const = 0;
174     /// @}
175
176     /// \brief use basis function to reconstruct a given simulation
177     /// \param p_isim               simulation number
178     /// \param p_basisCoefficients  basis coefficients to reconstruct a given
           conditional expectation
179     virtual double reconstructionASim(const int &p_isim, const Eigen::ArrayXd
            &p_basisCoefficients) const = 0 ;
180
181     /// \brief conditional expectation reconstruction
182     /// \param  p_coordinates        coordinates to interpolate (uncertainty
           sample)
183     /// \param  p_coordBasisFunction regression coordinates on the basis  (
           size: number of meshes multiplied by the dimension plus one)
184     /// \return regressed value function reconstructed for each simulation
185     virtual double getValue(const Eigen::ArrayXd   &p_coordinates,
186                             const Eigen::ArrayXd   &p_coordBasisFunction)
                                const = 0;
187
188     /// \brief conditional expectation reconstruction for a lot of
           simulations
189     /// \param  p_coordinates        coordinates to interpolate (uncertainty
           sample) size uncertainty dimension by number of samples
190     /// \param  p_coordBasisFunction regression coordinates on the basis  (
           size: number of meshes multiplied by the dimension plus one)
191     /// \return regressed value function reconstructed for each simulation
192     Eigen::ArrayXd  getValues(const Eigen::ArrayXXd   &p_coordinates,
193               const Eigen::ArrayXd   &p_coordBasisFunction) const
```

```
194       {
195    Eigen::ArrayXd valRet(p_coordinates.cols());
196    for (int is  =0; is < p_coordinates.cols(); ++is)
197      valRet(is)= getValue(p_coordinates.col(is),p_coordBasisFunction);
198    return valRet;
199       }
200
201     /// \brief permits to reconstruct a function with basis functions
          coefficients values given on a grid
202     /// \param  p_coordinates          coordinates  (uncertainty sample)
203     /// \param  p_ptOfStock            grid point
204     /// \param  p_interpFuncBasis      spectral interpolator to interpolate
          the basis functions  coefficients used in regression on the grid (
          given for each basis function)
205     virtual double getAValue(const Eigen::ArrayXd &p_coordinates,  const
          Eigen::ArrayXd &p_ptOfStock,
206                              const std::vector< std::shared_ptr<
                                 InterpolatorSpectral> > &p_interpFuncBasis)
                                 const = 0;
207
208     /// \brief is the regression date zero
209     inline bool getBZeroDate() const
210     {
211         return m_bZeroDate;
212     }
213
214     /// \brief Clone the regressor
215     virtual std::shared_ptr<BaseRegression> clone() const = 0 ;
216
217
218 };
219
220 }
221
222 #endif
```

All regression classes share the same constructors:

- a first constructor stores the members of the class and computes the matrices for the regression: it is used for example to build a regression object at each time step of a resolution method,
- the second constructor is used to prepare some data which will be shared by all future regressions. It has to be used with the 'updateSimulation' method to update the effective matrix construction. In a resolution method with many time steps, the object will be constructed only once and at each time step the Markov state will be updated by the 'updateSimulation' method.

All regression classes share the common methods:

- "updateSimulationBase' (see above),
- "getCoordBasisFunction" takes the values $g(t+h, X_{t+h})$ for all simulations and returns the coefficients α_k of the basis functions,

- "getCoordBasisFunctionMultiple" is used if we want to do the previous calculation on multiple g functions in one call. In the matrix given as argument, the first dimension has a size equal to the number of Monte Carlo simulations, while the second dimension has a size equal to the number of functions to regress. As output, the first dimension has a size equal to the number of function to regress and the second equal to the number of basis functions.

- "getAllSimulations" takes the values $g(t+h, X_{t+h})$ for all simulations and returns the regressed values for all simulations $f(X_t)$

- "getAllSimulationMultiple" is used if we want to do the previous calculation on multiple g functions in one call. In the matrix given as argument, the first dimension has a size equal to the number of Monte Carlo simulations, while the second dimension has a size equal to the number of functions to regress. The regressed values are given back in the same format.

- "reconstruction" takes the α_k coefficient of the basis functions as input and returns all the $f(X_t)$ for the simulations stored by applying equation (2.1).

- "reconstructionMultiple" is used if we want to do the previous calculation on multiple g functions in one call. As input the α_k coefficients of the basis functions are given (number of function to regress for first dimension, number of basis functions for second dimension). As a result the $f(X_t)$ for all simulations and all$f$ functions are sent back (number of Monte Carlo simulations in first dimension, number of function to regress en second dimension).

- "reconstructionASim" takes a simulation number $isim$ (optimization part) and α_k coefficient of the basis functions as input and returns $f(X_t^{isim})$ by applying equation (2.1),

- "getValue" takes as first argument a sample of X_t, the basis function α_k and reconstruct the regressed solution of equation (2.1).

- "getValues" takes as first argument some samples of X_t (array size dimension of uncertainty by number of samples), the basis function α_k and reconstruct the regressed solution of equation (2.1) (an array).

2.2 Adapted local polynomial basis

The description of the method and its properties can be found in [16]. We just recall the methodology. These local adapted methods can benefit from a rotation in the its principal axis using the PCA method. The rotation is activated by a flag in the constructor of the objects?

2.2.1 Description of the method

The method essentially consists in applying a non-conform finite element approach rather than a spectral like method as presented above.

The idea is to use, at each time step t_i, a set of functions $\psi_q, q \in [0, M_M]$ having local hyper cube support $D_{i_1,i_2,..,i_d}$ where $i_j = 1$ to I_j, $M_M = \prod_{k=1,d} I_k$, and $\{D_{i_1,..i_d}\}_{(i_1,..,i_d)\in[1,I_1]\times\cdots\times[1,I_d]}$ is a partition of $[\min_{k=1,N} X_{t_i}^{1,(k)}, \max_{k=1,N} X_{t_i}^{1,(k)}]\times \cdots \times[\min_{k=1,N} X_{t_i}^{d,(k)}, \max_{k=1,N} X_{t_i}^{d,(k)}]$. On each D_l, $l = (i_1, .., i_d)$, depending on the selected method, ψ_l is

- either a constant function, so the global number of degrees of freedom is equal to M_M,
- or a linear function with $1 + d$ degrees of freedom, so the global number of degrees of freedom is equal to $M_M * (1 + d)$.

This approximation is "non-conform" in the sense that we do not assure the continuity of the approximation. However, it has the advantage to be able to fit any, even discontinuous, function. In order to avoid oscillations and to allow classical regression by the Choleski method, the supports are chosen so that they contain roughly the same number of particles.

On Figure 2.2, we have plotted an example of supports in the case of $6 = 4 \times 4$ local basis cells, in dimension 2.

Sometimes we can do further exploiting knowledge on the continuation value. In the case

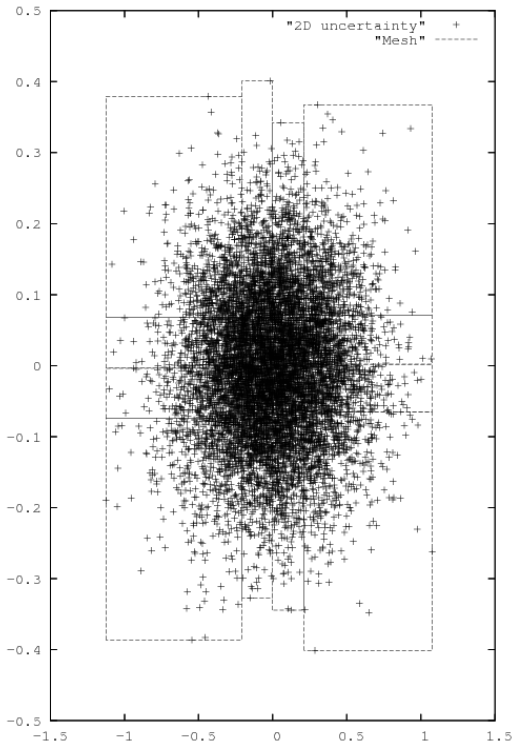


Figure 2.2: Support of 2D function basis

of an american basket option for example we have convexity of this continuation value with respect to the underlying prices. It is possible to modify the previous algorithm to try to

impose that the numerical method repects this convexity. The algorithm in [36] has been implemented as an option. This algorithm may not converge when used in multidimension but it permits to improve the convexity of the solution while iterating a few times.

2.3 C++ api

2.3.1 The constant per cell approximation

The constructor of the local constant regression object is achieved by

```
1    LocalConstRegression(const Eigen::ArrayXi &p_nbMesh, bool
        p_bRotationAndRecale = false);
```

where :

- *p_nbMesh* is an array giving the number of meshes used in each direction ($(4, 4)$ for the figure 2.2 for example).
- *p_bRotationAndRecale* is an optimal argument by default set to False meaning that no rotation of the data in its principal components axis is achieved. In the case of rotation, the direction are sorted with their singular values decreasing and the number of meshes in *p_nbMesh* are defined for these sorted directions : $p\_nbMesh(0)$ is associated with first direction with the highest singular value, $p\_nbMesh(1)$ with the direction associated to the second highest singular value etc..

The second constructor permits the construct the regression matrix,

```
1    LocalConstRegression(const bool &p_bZeroDate,
2        const shared_ptr< ArrayXXd> &p_particles,
3        const Eigen::ArrayXi &p_nbMesh,
4        bool p_bRotationAndRecale = false)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_nbMesh* is an array giving the number of meshes used in each directions $(4, 4)$ for the figure 2.2,
- *p_bRotationAndRecale* is an optimal argument by default set to False meaning that no rotation of the data in its principal components axis is achieved. In the case of rotation, the direction are sorted with their singular values decreasing and the number of meshes in *p_nbMesh* are defined for these sorted directions : $p\_nbMesh(0)$ is associated with first direction with the highest singular value, $p\_nbMesh(1)$ with the direction associated to the second highest singular value etc...

2.3.2 The linear per cell approximation

The constructor of the local linear regression object is achieved by

```
LocalLinearRegression(const Eigen::ArrayXi &p_nbMesh, bool
    p_bRotationAndRecale = false);
```

where

- *p_nbMesh* is an array giving the number of meshes used in each direction ($(4, 4)$ for the figure 2.2 for example),
- *p_bRotationAndRecale* is an optimal argument by default set to False meaning that no rotation of the data in its principal components axis is achieved. In the case of rotation, the direction are sorted with their singular values decreasing and the number of meshes in *p_nbMesh* are defined for these sorted directions : $p\_nbMesh(0)$ is associated with first direction with the highest singular value, $p\_nbMesh(1)$ with the direction associated to the second highest singular value etc...

The second constructor permits the construct the regression matrix,

```
LocalLinearRegression(const bool &p_bZeroDate,
    const shared_ptr< ArrayXXd> &p_particles,
    const Eigen::ArrayXi &p_nbMesh,
    bool p_bRotationAndRecale = false)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_nbMesh* is an array giving the number of meshes used in each directions $(4, 4)$ for the figure 2.2
- *p_bRotationAndRecale* is an optimal argument by default set to False meaning that no rotation of the data in its principal components axis is achieved. In the case of rotation, the direction are sorted with their singular values decreasing and the number of meshes in *p_nbMesh* are defined for these sorted directions : $p\_nbMesh(0)$ is associated with first direction with the highest singular value, $p\_nbMesh(1)$ with the direction associated to the second highest singular value etc...

This class can benefit of the methodology in [36] implementing a generalization of the member function "getAllSimulations' :

```
Eigen::ArrayXd getAllSimulationsConvex(const Eigen::ArrayXd &p_fToRegress,
    const int &p_nbIterMax)
```

where

- *p_fToRegress* is the set of points we want to regress preserving convexity of the regressed function value
- *p_nbIterMax* is the maximal number of iteration of the method.

It returns the regressed values for all simulations of the uncertainties.

2.3.3 An example in the linear case

Below we give a small example where "toRegress" corresponds to $g(t+h, X_{t+h})$ for all simulations and x store X_t for all simulations.

```
// create the mesh for a 2 dim problem, 4 meshes per direction
 ArrayXi  nbMesh = ArrayXi::Constant(2, 4);
 // t is not zero
 bool bZeroDate = 0;
 // constructor, no rotation of the data
 LocalLinearRegression localRegressor(nbMesh);
 // update particles values
 localRegressor.updateSimulations(bZeroDate, x);
 // regressed values
  ArrayXd  regressedValues = localRegressor.getAllSimulations(toRegress);
```

2.4 Python API

Here is a similar example using the second constructor of the linear case

```
        import StOptReg
        nbSimul = 5000000;
        np.random.seed(000)
        x = np.random.uniform(-.,1.,size=(1,nbSimul));
        # real function
        toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
        # function to regress
        toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
        # mesh
        nbMesh = np.array([6],dtype=np.int32)
        # Regressor without rotation of data
        regressor = StOptReg.LocalLinearRegression(False,x,nbMesh)
        y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Of course the constant per cell case in python is similar. As in C++ the linear case permits to try to regress preserving convexity by using the *getAllSimulationsConvex* method.

2.5 Local polynomial basis with meshes of same size

In some cases, instead of using adapted meshes, on can prefer to fix the mesh with a constant step in each direction with I_k meshes in each direction so that the total number of cells is $M_M = \prod_{k=1,d} I_k$. On each cell as in section 2.2, one can have two approximations :

- either a constant function, so the global number of degrees of freedom is equal to M_M,
- or a linear function with $1+d$ degrees of freedom, so the global number of degrees of freedom is equal to $M_M * (1+d)$.

Because we define in each direction, the domain for the local basis, we don't use any rotation of the data.

2.6 C++ api

2.6.1 The constant per cell approximation

The constructor of the local constant regression object is achieved by

```
LocalSameSizeConstRegression(const Eigen::ArrayXd &p_lowValues, const
    Eigen::ArrayXd &p_step, const Eigen::ArrayXi &p_nbStep);
```

- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

The second constructor permits the construct the regression matrix,

```
LocalSameSizeConstRegression(const bool &p_bZeroDate,
                             const std::shared_ptr< Eigen::ArrayXXd > &
                                 p_particles,
                             const Eigen::ArrayXd &p_lowValues,
                             const Eigen::ArrayXd &p_step,
                             const Eigen::ArrayXi &p_nbStep);
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

2.6.2 The linear per cell approximation

The constructor of the local linear regression object is achieved by

```
LocalSameSizeLinearRegression(const Eigen::ArrayXd &p_lowValues, const
    Eigen::ArrayXd &p_step, const Eigen::ArrayXi &p_nbStep);
```

where

- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

The second constructor permits the construct the regression matrix,

```
        LocalSameSizeLinearRegression(const bool &p_bZeroDate,
                                      const std::shared_ptr< Eigen::ArrayXXd > &
                                          p_particles,
                                      const Eigen::ArrayXd &p_lowValues,
                                      const Eigen::ArrayXd &p_step,
                                      const  Eigen::ArrayXi &p_nbStep)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

2.6.3 An example in the linear case

Below we give a small example where "toRegress" is the array to regress with respect to an array "x" in dimension *p_nDim* :

```
    // create a random  ''x'' array
    shared_ptr<ArrayXXd> x(new ArrayXXd(ArrayXXd::Random(p_nDim, p_nbSimul))
        );
   // create the mesh by getting min and max value on the samples
   double xMin = x->minCoeff() - tiny;
   double xMax = x->maxCoeff() + tiny;
   ArrayXd  lowValues = ArrayXd::Constant(p_nDim, xMin);
   ArrayXd  step = ArrayXd::Constant(p_nDim, (xMax - xMin) / p_nMesh);
   ArrayXi  nbStep = ArrayXi::Constant(p_nDim, p_nMesh);
   // constructor
   LocalLinearRegression localRegressor(lowValues,step,  nbStep);
   // update particles values
   localRegressor.updateSimulations(bZeroDate, x);
   // regressed values
    ArrayXd  regressedValues = localRegressor.getAllSimulations(toRegress);
```

2.7 Python API

Here is a similar example using the second constructor of the linear case

```
        import StOptReg
        nbSimul = 5000000;
        np.random.seed(000)
        x = np.random.uniform(-.,1.,size=(1,nbSimul));
        # real function
        toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
```

```
        # function to regress
        toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
        # mesh
        nStep = 20
        lowValue =  np.array([-1.0001],dtype=np.float)
        step =  np.array([2.0002/nStep],dtype=np.float)
        nbMesh = np.array([nStep],dtype=np.int32)
        # Regressor
        regressor = StOptReg.LocalSameSizeLinearRegression(False,x,lowValue,
            step,nbMesh)
        y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Of course the constant per cell case in python is similar.

2.8 Sparse grid regressor

In the case of a sparse regressor, the grid is an object "SparseSpaceGridNoBound" (extrapolation for the boundary conditions). The basis functions are given by the section 1.3 for linear, quadratic or cubic function basis. No rotation of the data is available.

2.8.1 C++ API

Two specific constructor are available:

- The first one to be used with the "updateSimulations" methods

```
SparseRegression(const int &p_levelMax, const Eigen::ArrayXd &
    p_weight, const int &p_degree, bool p_bNoRescale = false);
```

  where

  - *p_levelMax* corresponds to n in the equation (1.4),
  - *p_weight* the weight for anisotropic sparse grids (see equation (1.7),
  - *p_degree* is equal to (linear basis function), or 2 (quadratic basis) or 3 (for cubic basis functions),
  - *p_bNoRescale* if true no re scaling of the particles is used. Otherwise a re scaling of the mesh size is achieved (as for local basis functions, see section 2.2)

- The second one take the same arguments as the first constructor but adds a Boolean to check if the regression date is 0 and the particles X_t (here the re scaling is always achieved):

```
SparseRegression(const bool &p_bZeroDate,
                 const shared_ptr< Eigen::ArrayXXd > &p_particles,
                 const int &p_levelMax, const Eigen::ArrayXd &
                     p_weight,
                 const int &p_degree);
```

A simple example to express the regression of "toRegress"

```
        // second member to regress
         ArrayXd toRegress(p_nbSimul);
        // for testing
        toRegress.setConstant(.);
        shared_ptr<ArrayXXd> x(new ArrayXXd(ArrayXXd::Random(p_nDim,
            p_nbSimul)));
        // constructor : the current date is not  zero
        bool bZeroDate = 0;
        // constructor
        SparseRegression sparseRegressor(p_level , weight, p_degree);
        sparseRegressor.updateSimulations(bZeroDate, x); // update the state
         // then just calculate function basis coefficient
        ArrayXd regressedFuntionCoeff = sparseRegressor.getCoordBasisFunction
            (toRegress);
        // use the getValue method to get back the regressed values
        for (int is = 0; is < p_nbSimul; ++is)
        {
            Map<ArrayXd> xloc(x->col(is).data(), p_nDim);
            double reg = sparseRegressor.getValue(xloc, regressedFuntionCoeff
                );
        }
        // get back all values once for all
        ArrayXd regressedAllValues  = localRegressor.getValues(*x,
            regressedFuntionCoeff) ;
```

2.8.2 Python API

Here is a simple example of the python API:

```
        import StOptReg
        nbSimul = 2000000;
        np.random.seed(000)
        x = np.random.uniform(-.,1.,size=(1,nbSimul));
        # real function
        toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
        # function to regress
        toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
        # level for sparse grid
        iLevel = 5;
        # weight for anisotropic sparse grids
        weight= np.array([],dtype=np.int32)
        # Regressor degree
        regressor = StOptReg.SparseRegression(False,x,iLevel, weight, )
        y = regressor.getAllSimulations(toRegress)
         # get back basis function
        regressedFuntionCoeff= regressor.getCoordBasisFunction(toRegress)
        # get back all values
        ySecond= regressor.getValues(x,regressedFuntionCoeff)
```

2.9 Global polynomial basis

2.9.1 Description of the method

In this section, the $\psi_k(X_t)$ involved in equation 2.1 are some given polynomials. Available polynomials are the canonical one, the Hermite and the Chebyshev ones.

- Hermite polynomials $H_m(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$ are orthogonal with respect to the weight $w(x) = e^{-\frac{x^2}{2}}$ and we get

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) dx = \delta_{mn} \sqrt{2\pi} n!$$

  they satisfy the recurrence :

$$H_{n+1}(x) = x H_n(x) - H_n^{'}(x)$$

  assuming $H_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$, we get the recurrence

$$a_{n+1,k} = a_{n,k-1} - n a_{n-1,k}, k > 0 \tag{2.5}$$

$$a_{n+1,0} = -n a_{n-1,0} \tag{2.6}$$

- Chebyshev polynomials are $T_{N+1}(x) = cos((N+1) arcs(x))$. They are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ and

$$\int_{-1}^{1} T_N(x) T_M(x) w(x) dx = \begin{cases} 0, \text{ if } M \neq N \\ \pi, \text{ if } M = N = 0 \\ \frac{\pi}{2}, \text{ if } M = N \neq 0 \end{cases}$$

  They satisfy the following recurrence :

$$T_{N+2}(x) = 2x T_{N+1}(x) - T_N(x)$$

As an option rotation of the data is possible even if the avantage of the rotation seem to be limited for global polynomials.

2.9.2 C++ API

The "GlobalRegression" class is template by the type of the polynomial ("Canonical","Tchebychev" or "Hermite") The first constructor :

```
1    GlobalRegression(const int & p_degree, const int & p_dim, bool
       p_bRotationAndRecale = false);
```

where *p_degree* is the total degree of the polynomial approximation, *p_dim* is the dimension of the problem, *p_bRotationAndRecale* is an optional flag set to true if rotation of the data should be achieved (default is no rotation). A second constructor is provided:

```
GlobalRegression(const bool &p_bZeroDate,
        const std::shared_ptr< Eigen::ArrayXXd > &p_particles,
        const int & p_degree, bool  p_bRotationAndRecale = false)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_degree* is the total degree of the polynomial approximation,
- *p_bRotationAndRecale* is an optional flag set to true if rotation of the data should be achieved (default is no rotation)

Below we give a small example where "toRegress" corresponds to $g(t+h, X_{t+h})$ for all simulations and x store X_t for all simulations.

```
// total degree equal to 2
 int degree=2;
 // t is not zero
 bool bZeroDate = 0;
 // constructor with Hermite polynomials, no rotation
 GlobalRegression<Hermite> localRegressor(degree,x.rows());
 // update particles values
 localRegressor.updateSimulations(bZeroDate, x);
 // regressed values
  ArrayXd  regressedValues = localRegressor.getAllSimulations(toRegress);
```

In the above example the Hermite regression can be replaced by the canonical one :

```
GlobalRegression<Canonical> localRegressor(degree,x.rows());
```

or by a Chebyshev one :

```
GlobalRegression<Tchebychev> localRegressor(degree,x.rows());
```

2.9.3 Python API

Here is a similar example using the second constructor

```
        import StOptReg
        nbSimul = 5000000;
        np.random.seed(1000)
        x = np.random.uniform(-.,1.,size=(1,nbSimul));
        # real function
        toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
        # function to regress
        toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
        # degree
        degree =2
        # Regressor, no rotation
        regressor = StOptReg.GlobalHermiteRegression(False,x,degree)
        y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Available regressors are "GlobalHermiteRegression" as in the example above , "GlobalCanonicalRegression" and "GlobalTchebychevRegression" with an obvious correspondence.

2.10 Kernel regression

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ be a sample of N input points x_i and output points y_i drawn from a joint distribution (X, Y). The kernel density estimator (aka Parzen-Rosenblatt estimator) of the density of X at the evaluation point z is given by:

$$\hat{f}_{\mathrm{KDE}}(z) := \frac{1}{N} \sum_{i=1}^{N} K_h(x_i - z) \tag{2.7}$$

where $K_h(u) := \frac{1}{h} K\left(\frac{u}{h}\right)$ with kernel K and bandwidth h. The Nadaraya-Watson kernel regression estimator of $\mathbb{E}[Y \,|X = z]$ is given by:

$$\hat{f}_{\mathrm{NW}}(z) := \frac{\sum_{i=1}^{N} K_h(x_i - z) y_i}{\sum_{i=1}^{N} K_h(x_i - z)} \tag{2.8}$$

The estimator $\hat{f}_{\mathrm{NW}}(z)$ performs a kernel-weighted local average of the response points y_i that are such that their corresponding inputs x_i are close to the evaluation point z. It can be described as a locally constant regression. More generally, locally linear regressions can be performed:

$$\hat{f}_{\mathrm{L}}(z) := \min_{\alpha(z), \beta(z)} \sum_{i=1}^{N} K_h(x_i - z) \left[y_i - \alpha(z) - \beta(z) x_i\right]^2 \tag{2.9}$$

The well known computational problem with the implementation of the kernel smoothers (2.7)-(2.8)-(2.9) is that their direct evaluation on a set of M evaluation points would require $\mathcal{O}(M \times N)$ operations. In particular, when the evaluation points coincide with the input points $x_1, x_2, \ldots, x_N$, a direct evaluation requires a quadratic $\mathcal{O}(N^2)$ number of operations. In StOpt we develop the methodology described in [39] permitting to get a $N \log N$ cost function.

2.10.1 The univariate case

In one dimension, StOpt uses the one dimensional Epanechnikov kernel

$$K(u) = \frac{3}{4}(1 - u^2) \mathbb{1}\{|u| \leq 1\}$$

and the fast suming algorithm is used : Let $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ be a sample of N input (source) points x_i and output points y_i, and let $z_1, z_2, \ldots, z_M$ be a set of M evaluation (target) points. Without loss of generality, we assume that the input points and evaluation points are sorted: $x_1 \leq x_2 \leq \ldots \leq x_N$ and $z_1 \leq z_2 \leq \ldots \leq z_M$. In order to compute the kernel density estimator (2.7), the kernel regression (2.8) and the locally linear regression (2.9) for every evaluation point z_j, one needs to compute sums of the type

$$\mathbf{S}_j = \mathbf{S}_j^{p,q} := \frac{1}{N} \sum_{i=1}^{N} K_h(x_i - z_j) x_i^p y_i^q = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{x_i - z_j}{h}\right) x_i^p y_i^q \,, p = 0, 1, \, q = 0, 1 \tag{2.10}$$

for every $j \in \{1, 2, \ldots, M\}$. The direct, independent evaluation of these sums would require $\mathcal{O}(N \times M)$ operations (a sum of N terms for each $j \in \{1, 2, \ldots, M\}$). The idea of fast sum updating is to use the information from the sum $\mathbf{S}_j$ to compute the next sum $\mathbf{S}_{j+1}$ without going through all the N input points again. Using the Epanechnikov (parabolic) kernel $K(u) = \frac{3}{4}(1-u^2)\mathbb{1}\{|u| \leq 1\}$ we get:

$$\begin{aligned}
\mathbf{S}_j^{p,q} &= \frac{1}{Nh}\sum_{i=1}^{N}\frac{3}{4}\left(1-\left(\frac{x_i-z_j}{h}\right)^2\right)x_i^p y_i^q \mathbb{1}\{z_j-h \leq x_i \leq z_j+h\} \\
&= \frac{1}{Nh}\frac{3}{4}\sum_{i=1}^{N}\left(1-\frac{z_j^2}{h^2}+2\frac{z_j}{h^2}x_i-\frac{1}{h^2}x_i^2\right)x_i^p y_i^q \mathbb{1}\{z_j-h \leq x_i \leq z_j+h\} \\
&= \frac{3}{4Nh}\left\{\left(1-\frac{z_j^2}{h^2}\right)\mathcal{S}^{p,q}([z_j-h, z_j+h]) + 2\frac{z_j}{h^2}\mathcal{S}^{p+1,q}([z_j-h, z_j+h]) - \frac{1}{h^2}\mathcal{S}^{p+2,q}([z_j-h, z_j+h])\right\}
\end{aligned} \tag{2.11}$$

where

$$\mathcal{S}^{p,q}([L, R]) := \sum_{i=1}^{N} x_i^p y_i^q \mathbb{1}\{L \leq x_i \leq R\} \tag{2.12}$$

These sums $\mathcal{S}^{p,q}([z_j - h, z_j + h])$ can be evaluated quickly from $j = 1$ to $j = M$ as long as the input points x_i and the evaluation points z_j are sorted in increasing order. Indeed,

$$\begin{aligned}
\mathcal{S}^{p,q}([z_{j+1}-h, z_{j+1}+h]) &= \sum_{i=1}^{N} x_i^p y_i^q \mathbb{1}\{z_{j+1}-h \leq x_i \leq z_{j+1}+h\} \\
&= \sum_{i=1}^{N} x_i^p y_i^q \mathbb{1}\{z_j-h \leq x_i \leq z_j+h\} \\
&- \sum_{i=1}^{N} x_i^p y_i^q \mathbb{1}\{z_j-h \leq x_i < z_{j+1}-h\} + \sum_{i=1}^{N} x_i^p y_i^q \mathbb{1}\{z_j+h < x_i \leq z_{j+1}+h\} \\
&= \mathcal{S}^{p,q}([z_j-h, z_j+h]) - \mathcal{S}^{p,q}([z_j-h, z_{j+1}-h[) + \mathcal{S}^{p,q}(]z_j+h, z_{j+1}+h])
\end{aligned} \tag{2.13}$$

Therefore one can simply update the sum $\mathcal{S}^{p,q}([z_j - h, z_{j+1} + h])$ for the evaluation point z_j to obtain the next sum $\mathcal{S}^{p,q}([z_{j+1} - h, z_{j+1} + h])$ for the next evaluation point z_{j+1} by subtracting the terms $x_i^p y_i^q$ for which x_i lie between $z_j - h$ and $z_{j+1} - h$, and adding the terms $x_i^p y_i^q$ for which x_i lie between $z_j + h$ and $z_{j+1} + h$. This can be achieved in a fast $\mathcal{O}(M + N)$ operations by going through the input points x_i, stored in increasing order at a cost of $\mathcal{O}(N \log N)$ operations, and through the evaluation points z_j, stored in increasing order at a cost of $\mathcal{O}(M \log M)$ operations.

2.10.2 The multivariate case

We now turn to the multivariate case. Let d be the dimension of the inputs. We consider again a sample $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ of N input points x_i and output points y_i,

where the input points are now multivariate:

$$x_i = (x_{1,i}, x_{2,i}, \ldots, x_{d,i}) \ , \ i \in \{1, 2, \ldots, N\}$$

StOpt library uses the additive Epanechnikov kernel in the muti-dimensional case.

$$K_d(u_1, \ldots, u_d) = \frac{1}{d2^{d-1}} \sum_{k=1}^{d} K(u_k) \prod_{k_0=1}^{d} \mathbb{1}\{|u_{k_0}| < 1\} = \frac{3}{d2^{d+1}} \sum_{k=1}^{d} \left(1 - u_k^2\right) \prod_{k_0=1}^{d} \mathbb{1}\{|u_{k_0}| < 1\} \tag{2.14}$$

One can show ([39]) that the computation of the multivariate version of the kernels smoothers (2.7), (2.8) and (2.9) boils down to the computation of the following sums:

$$\begin{aligned} \mathbf{S}_j &= \mathbf{S}^{p_1,p_2,q}_{k_1,k_2,j} := \frac{1}{N} \sum_{i=1}^{N} K_{d,h}(x_i - z_j) x^{p_1}_{k_1,i} x^{p_2}_{k_2,i} y^q_i \\ &= \frac{1}{N \Pi_{k=1}^d h_k} \sum_{i=1}^{N} K_d \left(\frac{x_{1,i} - z_{1,j}}{h_1}, \frac{x_{2,i} - z_{2,j}}{h_2}, \ldots, \frac{x_{d,i} - z_{d,j}}{h_d} \right) x^{p_1}_{k_1,i} x^{p_2}_{k_2,i} y^q_i \end{aligned} \tag{2.15}$$

for each evaluation point $z_j = (z_{1,j}, z_{2,j}, \ldots, z_{d,j}) \in \mathbb{R}^d$, $j \in \{1, 2, \ldots, M\}$, for powers $p_1, p_2, q = 0, 1$ and for dimension index $k_1, k_2 = 1, 2, \ldots, d$.

Kernel development

Using the multivariate kernel (2.14), one can develop the sum (2.15) as follows:

$$\begin{aligned} &\mathbf{S}^{p_1,p_2,q}_{k_1,k_2,j} = \frac{1}{N \prod_{k=1}^d h_k} \sum_{i=1}^{N} K_d \left(\frac{x_{1,i} - z_{1,j}}{h_1}, \frac{x_{2,i} - z_{2,j}}{h_2}, \ldots, \frac{x_{d,i} - z_{d,j}}{h_d} \right) x^{p_1}_{k_1,i} x^{p_2}_{k_2,i} y^q_i \\ &= \frac{3}{d2^{d+1} N \prod_{k=1}^d h_k} \sum_{i=1}^{N} \sum_{k=1}^{d} \left(1 - \frac{(x_{k,i} - z_{k,j})^2}{h_k^2} \right) x^{p_1}_{k_1,i} x^{p_2}_{k_2,i} y^q_i \prod_{k_0=1}^{d} \mathbb{1}\{|x_{k_0,i} - z_{k_0,j}| \leq 1\} \\ &= \frac{3}{d2^{d+1} N \prod_{k=1}^d h_k} \sum_{k=1}^{d} \sum_{i=1}^{N} \left(1 - \frac{z^2_{k,j}}{h_k^2} + 2\frac{z_{k,j}}{h_k^2} x_{k,i} - \frac{1}{h_k^2} x^2_{k,i} \right) x^{p_1}_{k_1,i} x^{p_2}_{k_2,i} y^q_i \prod_{k_0=1}^{d} \mathbb{1}\{|x_{k_0,i} - z_{k_0,j}| \leq 1\} \\ &= \frac{3}{d2^{d+1} N \prod_{k=1}^d h_k} \sum_{k=1}^{d} \left\{ \left(1 - \frac{z^2_{k,j}}{h_k^2} \right) \mathcal{S}^{[0,p_1,p_2],q}_{[k,k_1,k_2]}([z_j - h_j, z_j + h_j]) + \right. \\ &= \left. 2\frac{z_{k,j}}{h_k^2} \mathcal{S}^{[1,p_1,p_2],q}_{[k,k_1,k_2]}([z_j - h_j, z_j + h_j]) - \frac{1}{h_k^2} \mathcal{S}^{[2,p_1,p_2],q}_{[k,k_1,k_2]}([z_j - h_j, z_j + h_j]) \right\} \end{aligned} \tag{2.16}$$

where

$$\mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}([\mathbf{L}, \mathbf{R}]) := \sum_{i=1}^{N} \left(\prod_{l=1}^{3} (x_{k_l,i})^{p_l} \right) y^q_i \prod_{k_0=1}^{d} \mathbb{1}\{L_{k_0} \leq x_{k_0,i} \leq R_{k_0}\} \tag{2.17}$$

for any hypercube $[\mathbf{L}, \mathbf{R}] := [L_1, R_1] \times [L_2, R_2] \times \ldots \times [L_d, R_d] \subseteq \mathbb{R}^d$, powers $\mathbf{p} := (p_1, p_2, p_3) \in \mathbb{N}^3$, $q \in \mathbb{N}$ and indices $\mathbf{k} := (k_1, k_2, k_3) \in \{1, 2, \ldots, d\}^3$, and where $[z_j - h_j, z_j + h_j] := [z_{1,j} - h_{1,j}, z_{1,j} + h_{1,j}] \times [z_{2,j} - h_{2,j}, z_{2,j} + h_{2,j}] \times \ldots \times [z_{d,j} - h_{d,j}, z_{d,j} + h_{d,j}]$

To sum up what has been obtained so far, computing multivariate kernel smoothers (kernel density estimation, kernel regression, locally linear regression) boils down to computing sums of the type (2.17) on hypercubes of the type $[z_j - h_j, z_j + h_j]$ for every evaluation point $j \in \{1, 2, \ldots, M\}$. In the univariate case, these sums could be computed efficiently by sorting the input points x_i, $i \in \{1, 2, \ldots, N\}$ and updating the sums from one evaluation point to the next (equation (2.13)). Our goal is now to set up a similar efficient fast sum updating algorithm for the multivariate sums (2.17). To do so, we are first going to partition the input data into a multivariate rectilinear grid (subsection 2.10.2), by taking advantage of the fact that the evaluation grid is rectilinear and that the supports of the kernels have a hypercube shape. Then, we are going to set up a fast sweeping algorithm using the sums on each hypercube of the partition as the unit blocks to be added and removed (subsection 2.10.2), unlike the univariate case where the input points themselves were being added and removed iteratively.

Data partition

The first stage of the multivariate fast sum updating algorithm is to partition the sample of input points into a rectilinear grid. To do so, we partition each dimension independently as follows: for each dimension $k \in \{1, 2, \ldots, d\}$, the set of threshold points $\tilde{\mathcal{G}}_k := \{z_{k,j_k} - h_{k,j_k}\}_{j_k \in \{1,2,\ldots,M_k\}} \cup \{z_{k,j_k} + h_{k,j_k}\}_{j_k \in \{1,2,\ldots,M_k\}}$ is used to partition the k-th axis. The second row of Figure 2.3 illustrates this partition on a set of 4 points, where for simplicity the evaluation points are the same as the input points. Denote the sorted points of the partition $\tilde{\mathcal{G}}_k$ as $\tilde{g}_{k,1} \leq \tilde{g}_{k,2} \leq \ldots \leq \tilde{g}_{k,2M_k}$

$$\tilde{\mathcal{G}}_k = \{\tilde{g}_{k,1}, \tilde{g}_{k,2}, \ldots, \tilde{g}_{k,2M_k}\}$$

and define the partition intervals $\tilde{I}_{k,l} := [\tilde{g}_{k,l}, \tilde{g}_{k,l+1}]$ for $l \in \{1, 2, \ldots, 2M_k - 1\}$.

Because for each dimension $k \in \{1, 2, \ldots, d\}$, all the bandwidths edges $z_{k,j_k} - h_{k,j_k}$ and $z_{k,j_k} + h_{k,j_k}$, $j_k \in \{1, 2, \ldots, M_k\}$, belong to $\tilde{\mathcal{G}}_k$, there exists, for any evaluation point $z_j = (z_{1,j_1}, z_{2,j_2}, \ldots, z_{d,j_d}) \in \mathbb{R}^d$, some indices $\left(\tilde{L}_{1,j_1}, \tilde{L}_{2,j_2}, \ldots, \tilde{L}_{d,j_d}\right)$ and $\left(\tilde{R}_{1,j_1}, \tilde{R}_{2,j_2}, \ldots, \tilde{R}_{d,j_d}\right)$ such that

$$\begin{aligned}
[z_j - h_j, z_j + h_j] &= [z_{1,j_1} - h_{1,j_1}, z_{1,j_1} + h_{1,j_1}] \times \ldots \times [z_{d,j_d} - h_{d,j_d}, z_{d,j_d} + h_{d,j_d}] \\
&= \left[\tilde{g}_{1,\tilde{L}_{1,j_1}}, \tilde{g}_{1,\tilde{R}_{1,j_1}+1}\right] \times \ldots \times \left[\tilde{g}_{d,\tilde{L}_{d,j_d}}, \tilde{g}_{d,\tilde{R}_{d,j_d}+1}\right] \\
&= \bigcup_{(l_1,\ldots,l_d) \in \{\tilde{L}_{1,j_1},\ldots,\tilde{R}_{1,j_1}\} \times \ldots \times \{\tilde{L}_{d,j_d},\ldots,\tilde{R}_{d,j_d}\}} \tilde{I}_{1,l_1} \times \ldots \times \tilde{I}_{d,l_d}
\end{aligned} \tag{2.18}$$

and, consequently, such that the sum (2.17) on the hypercube $[z_j - h_j, z_j + h_j]$ is equal to the sum of sums (2.17) on all the hypercubes of the partition included in $[z_j - h_j, z_j + h_j]$ (namely all the hypercubes $\tilde{I}_{1,l_1} \times \tilde{I}_{2,l_2} \times \ldots \times \tilde{I}_{d,l_d}$ such that $l_k \in \{\tilde{L}_{k,j_k}, \tilde{L}_{k,j_k} + 1, \ldots, \tilde{R}_{k,j_k}\}$ in each dimension $k \in \{1, 2, \ldots, d\}$):

$$\mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}([z_j - h_j, z_j + h_j]) = \bigcup_{(l_1,\ldots,l_d) \in \{\tilde{L}_{1,j_1},\ldots,\tilde{R}_{1,j_1}\} \times \ldots \times \{\tilde{L}_{d,j_d},\ldots,\tilde{R}_{d,j_d}\}} \mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}\left(\tilde{I}_{1,l_1} \times \ldots \times \tilde{I}_{d,l_d}\right) \tag{2.19}$$

where we assume without loss of generality that the bandwidth grid $h_j = (h_{1,j_1}, h_{2,j_2}, \ldots, h_{d,j_d})$, $j_k \in \{1, 2, \ldots, M_k\}$, $k \in \{1, 2, \ldots d\}$ is such that $\tilde{\mathcal{G}}_k$ does not contain any input $x_{k,i}$, $i \in \{1, 2, \ldots, N\}$, to ensure there is no input point on the boundaries of the inner hypercubes.

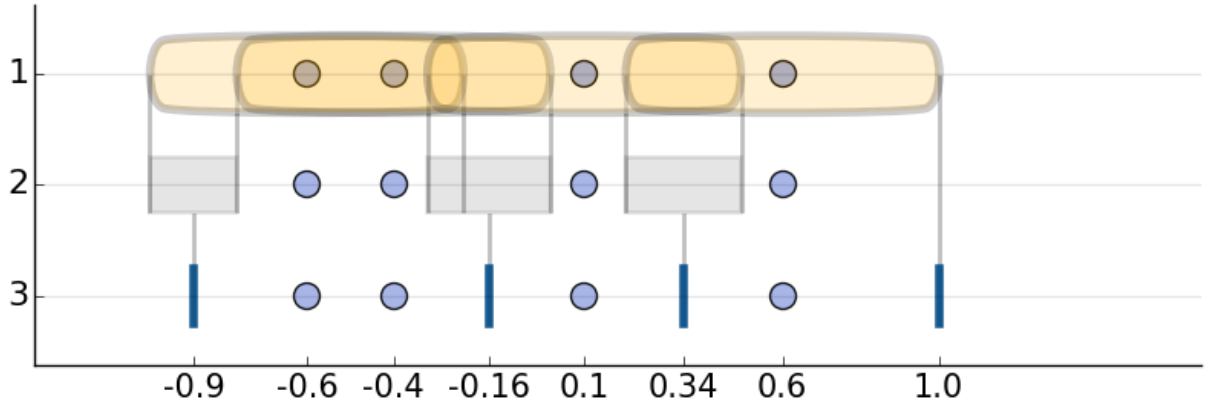


Figure 2.3: From bandwidths to partition (1D)

The sum decomposition (2.19) is the cornerstone of the fast multivariate sum updating algorithm, but before going further, one can simplify the partitions $\tilde{\mathcal{G}}_k$, $k \in \{1, 2, \ldots, d\}$ while maintaining a sum decomposition of the type (2.19). Indeed, the partitions $\tilde{\mathcal{G}}_k = \{z_{k,j_k} - h_{k,j_k}, z_{k,j_k} + h_{k,j_k};\ j_k = 1, \ldots, M_k\}$ can in general produce empty intervals (intervals which do not contain any input points, cf. the grey intervals on the second row of Figure 2.3). To avoid keeping track of sums $\mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}$ on the corresponding hypercubes known to be empty, one can trim the partitions $\tilde{\mathcal{G}}_k$ by shrinking each succession of empty intervals into one new partition threshold (cf. the final partition on the third row of Figure 2.3). Denote as $\mathcal{G}_k$ the resulting simplified partitions, containing the points $g_{k,1} < g_{k,2} < \ldots < g_{k,m_k}$:

$$\mathcal{G}_k = \{g_{k,1}, g_{k,2}, \ldots, g_{k,m_k}\}$$

where $2 \leq m_k \leq 2M_k$, $k \in \{1, 2, \ldots, d\}$, and $m := \prod_{k=1}^{d} m_k \leq 2^d M$. Define the partition intervals $I_{k,l} := [g_{k,l}, g_{k,l+1}]$, $l \in \{1, 2, \ldots, m_k - 1\}$. Because the only intervals to have been modified from $\tilde{\mathcal{G}}_k$ to $\mathcal{G}_k$ were empty, the following still holds:
For any evaluation point $z_j = (z_{1,j_1}, z_{2,j_2}, \ldots, z_{d,j_d}) \in \mathbb{R}^d$, $j_k \in \{1, 2, \ldots, M_k\}$, $k \in \{1, 2, \ldots d\}$, there exists indices $(L_{1,j_1}, L_{2,j_2}, \ldots, L_{d,j_d})$ and $(R_{1,j_1}, R_{2,j_2}, \ldots, R_{d,j_d})$, where $L_{k,j_k} \in \{1, 2, \ldots, m_k - 1\}$ and $R_{k,j_k} \in \{1, 2, \ldots, m_k - 1\}$ with $L_{k,j_k} \leq R_{k,j_k}$, $k \in \{1, 2, \ldots d\}$, such that

$$\mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}([z_j - h_j, z_j + h_j]) = \bigcup_{(l_1,\ldots,l_d) \in \{L_{1,j_1},\ldots,R_{1,j_1}\} \times \ldots \times \{L_{d,j_d},\ldots,R_{d,j_d}\}} \mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}(I_{1,l_1} \times \ldots \times I_{d,l_d}) \qquad (2.20)$$

To complement the illustration of univariate partition given by Figure 2.3, Figure 2.4 provides a bivariate partition example. There are four points, each at the center of their respective rectangular kernel (in orange). On the left-hand side, the bandwidths boundaries are used to produce the partitions $\tilde{\mathcal{G}}_k$ in each dimension. One can see that most of the resulting hypercubes (rectangles) are empty. On the right-hand side, the empty hypercubes are removed/merged, resulting in the trimmed partitions $\mathcal{G}_k$ in each dimension. Remark that this is a simple example for which every final hypercube only contains one point.

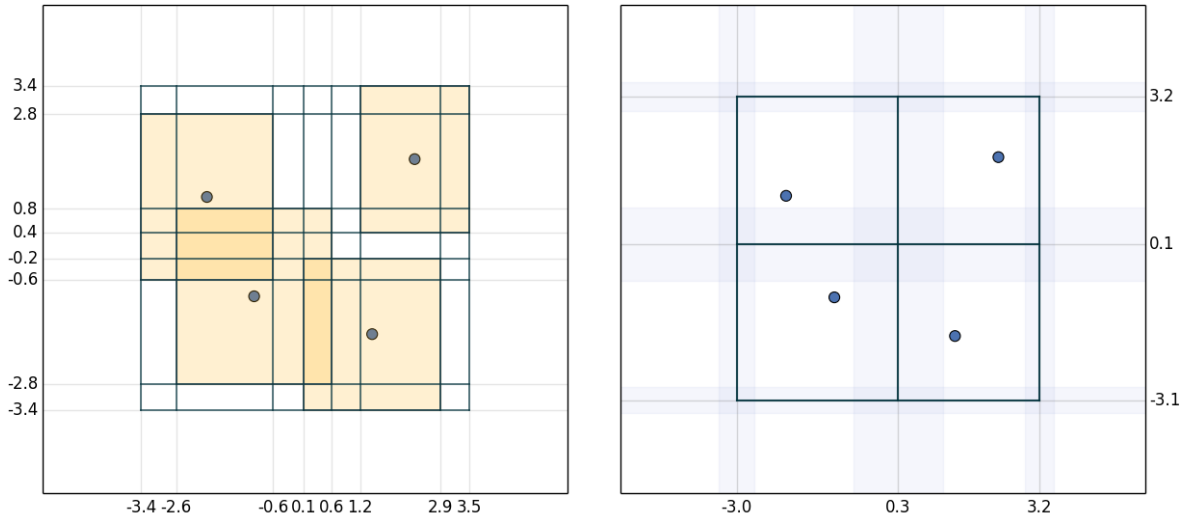


Figure 2.4: From bandwidths to partition (2D)

Fast multivariate sweeping algorithm

So far, we have shown that computing multivariate kernel smoothers is based on the computation of the kernel sums (2.15), which can be decomposed into sums of the type (2.17), which themselves can be decomposed into the smaller sums (2.20) by decomposing every kernel support of every evaluation point onto the rectilinear partition described in the previous subsection 2.10.2. The final task is to define an efficient algorithm to traverse all the hypercube unions $\bigcup_{(l_1,\ldots,l_d)\in\{L_{1,j_1},\ldots,R_{1,j_1}\}\times\ldots\times\{L_{d,j_d},\ldots,R_{d,j_d}\}} I_{1,l_1}\times\ldots\times I_{d,l_d}$, so as to compute the right-hand side sums (2.20) in an efficient fast sum updating fashion that extends the univariate updating (2.13).

First, to simplify notations, we introduce the multi-index $\text{idx} := (\mathbf{p}, q, \mathbf{k}) \in \{0,1,2\}\times\{0,1\}^3\times\{1,2,\ldots,d\}^3$ to summarize the polynomial $\left(\prod_{l=1}^3 (x_{k_l,i})^{p_l}\right) y_i^q$ in the sum $\mathcal{S}_{\mathbf{k}}^{\mathbf{p},q}([\mathbf{L},\mathbf{R}])$ (equation (2.17)), and introduce the compact notation

$$\mathcal{S}^{\text{idx}}_{l_1,l_2,\ldots,l_d} := \mathcal{S}_{\mathbf{k}}^{\mathbf{p},q}\left(I_{1,l_1}\times\ldots\times I_{d,l_d}\right) \tag{2.21}$$

to simplify the notation on the right-hand side of equation (2.20). In summary, $\mathcal{S}^{\text{idx}}_{l_1,l_2,\ldots,l_d}$ corresponds to the sum of the polynomials $\left(\prod_{l=1}^3 (x_{k_l,i})^{p_l}\right) y_i^q$ over all the data points within the hypercube $I_{1,l_1}\times\ldots\times I_{d,l_d}$. We precompute all the sums $\mathcal{S}^{\text{idx}}_{l_1,l_2,\ldots,l_d}$, and use them as the input material for the fast multivariate sum updating.

In the bivariate case, we first provide an algorithm to compute the sums $\mathcal{T}^{\text{idx}}_{1,l_2} := \sum_{l_1=L_{1,j_1}}^{R_{1,j_1}} \mathcal{S}^{\text{idx}}_{l_1,l_2}$, for every $l_2 \in \{1,2,\ldots,m_2-1\}$ and every indices interval $[L_{1,j_1}, R_{1,j_1}]$, $j_1 \in \{1,2,\ldots,M_1\}$. Starting with $j_1 = 1$, we first compute $\mathcal{T}^{\text{idx}}_{1,l_2} = \sum_{l_1=L_{1,1}}^{R_{1,1}} \mathcal{S}^{\text{idx}}_{l_1,l_2}$ for every $l_2 \in \{1,2,\ldots,m_2-1\}$. Then we iteratively increment j_1 from $j_1 = 1$ to $j_1 = M_1$. After each incrementation of j_1, we update $\mathcal{T}^{\text{idx}}_{1,l_2}$ by fast sum updating

$$\sum_{l_1=L_{1,j_1}}^{R_{1,j_1}} \mathcal{S}^{\text{idx}}_{l_1,l_2} = \sum_{l_1=L_{1,j_1-1}}^{R_{1,j_1-1}} \mathcal{S}^{\text{idx}}_{l_1,l_2} + \sum_{l_1=R_{1,j_1-1}+1}^{R_{1,j_1}} \mathcal{S}^{\text{idx}}_{l_1,l_2} - \sum_{l_1=L_{1,j_1-1}}^{L_{1,j_1}-1} \mathcal{S}^{\text{idx}}_{l_1,l_2} \tag{2.22}$$

The second stage is to perform a fast sum updating in the second dimension, with the sums $\mathcal{T}^{\text{idx}}_{1,l_2} = \sum_{l_1=L_{1,j_1}}^{R_{1,j_1}} \mathcal{S}^{\text{idx}}_{l_1,l_2}$ as input material. Our goal is to compute the sums $\mathcal{T}^{\text{idx}}_{2} := \sum_{l_2=L_{2,j_2}}^{R_{2,j_2}} \mathcal{T}^{\text{idx}}_{1,l_2}$ for every indices interval $[L_{2,j_2}, R_{2,j_2}]$, $j_2 \in \{1, 2, \ldots, M_2\}$. In a similar manner, we start with $j_2 = 1$ and the initial sum $\mathcal{T}^{\text{idx}}_{2} = \sum_{l_2=L_{2,1}}^{R_{2,1}} \mathcal{T}^{\text{idx}}_{1,l_2}$. We then increment j_2 from $j_2 = 1$ to $j_2 = M_2$ iteratively. After each incrementation of j_2, we update $\mathcal{T}^{\text{idx}}_{2}$ by fast sum updating:

$$\sum_{l_2=L_{2,j_2}}^{R_{2,j_2}} \mathcal{T}^{\text{idx}}_{1,l_2} = \sum_{l_2=L_{2,j_2-1}}^{R_{2,j_2-1}} \mathcal{T}^{\text{idx}}_{1,l_2} + \sum_{l_2=R_{2,j_2-1}+1}^{R_{2,j_2}} \mathcal{T}^{\text{idx}}_{1,l_2} - \sum_{l_2=L_{2,j_2-1}}^{L_{2,j_2-1}} \mathcal{T}^{\text{idx}}_{1,l_2} \tag{2.23}$$

Using the notation change (2.21) and equation (2.20), the resulting sum $\sum_{l_2=L_{2,j_2}}^{R_{2,j_2}} \mathcal{T}^{\text{idx}}_{1,l_2} = \sum_{l_1=L_{1,j_1}}^{R_{1,j_1}} \sum_{l_2=L_{2,j_2}}^{R_{2,j_2}} \mathcal{S}^{\text{idx}}_{l_1,l_2}$ is equal to $\mathcal{S}^{\mathbf{p},q}_{\mathbf{k}}([z_j - h_j, z_j + h_j])$, which can be used to compute the kernel sums $\mathbf{S}_j = \mathbf{S}^{p_1,p_2,q}_{k_1,k_2,j}$ using equation (2.16), from which the bivariate kernel smoothers (kernel density estimator, kernel regression, locally linear regression) can be computed.

This ends the description of the fast sum updating algorithm in the bivariate case. Finally, the general multivariate case is a straightforward extension of the bivariate case.

2.10.3 C++ APi

The constructor permits to defines the kernel regressor :

```
1 LocalGridKernelRegression(const bool &p_bZeroDate,
2                           const std::shared_ptr< Eigen::ArrayXXd > &
                                p_particles,
3                           const double &p_coeffBandWidth,
4                           const double   &p_coefNbGridPoint,
5                           const bool &p_bLinear);
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_coeffBandWidth* between 0 and 1 defines the percentage of points to be used to define the bandwidth for each point.
- *p_coefNbGridPoint* is a multiplicative factor defining the number of points z used for the multigrid approximation : a PCA is used to define a rotation of the data. The kernel regression is achieved according the base defined by the eigenvectors associated to the PCA. The number of points along the axes defined by the eigenvectors is given according the singular value associated to the eigenvector. The total number of evaluation points along the axes of the new base is roughly the number of simulations (p_particles.cols()) by *p_coefNbGridPoint*.
- *p_bLinear* when set to false states that the simple kernel density estimation (2.8) is used. When *p_bLinear* is true, the linear kernel regression (2.9) is used.

Below we give a small example where "toRegress" corresponds to $g(t+h, X_{t+h})$ for all simulations and x store X_t for all simulations.

```
1
2       // t is not zero
3       bool bZeroDate = 0;
4       // proportion of points used to define bandwidth
5       double prop =0.1;
6       //  multiplicative factor equal to one : number of evaluation points
           equals to the number of particles
7       double q  =1.
8       // choose a linear regression
9       bool bLin= true;
10      // constructor
11      LocalGridKernelRegression kernelReg(bZeroDate, x, prop, q, bLin);
12      // update particles values
13      localRegressor.updateSimulations(bZeroDate, x);
14     // regressed values
15      ArrayXd  regressedValues = localRegressor.getAllSimulations(toRegress);
```

2.10.4 Python API

As usual the python constructors are similar to the c++ constructors. here is a small example the use of the kernel regression method.

```
1       import StOptReg
2       nbSimul = 5000000;
3       np.random.seed(1000)
4       x = np.random.uniform(-.,1.,size=(1,nbSimul));
5       # real function
6       toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
7       # function to regress
8       toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
9       # degree
10      degree =2
11      # Regressor, no rotation
12      regressor = StOptReg.GlobalHermiteRegression(False,x,degree)
13      y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Chapter 3

Continuation values objects and similar ones

In a first part we describe a way to store and use continuation values calculated during the use of regression methods to estimate conditional expectations. In a second part, we introduce an object used to interpolate a function both discretized on grids for its deterministic part and estimated by regressor for its stochastic part. The second object is similar to the first in spirit but being dedicated to interpolation is more effective to use in simulations realized after the optimization part of a problem.

3.1 Continuation values object

A special case is the case where the state $X^{x,t}$ in equation (1) can be separated into two parts $X^{x,t} = (X_1^{x,t}, X_2^{x,t})$ where

1. the first part is given by the following equation

$$dX_1^{x,t} = b(t, X_s^{x,t})ds + \sigma_(s, X_s^{x,t})dW_s$$

   and is not controlled: the stochastic process is exogenous,

2. the second part is given by the following equation

$$dX_2^{x,t} = b_a(t)ds$$

   such that the $X_2^{x,t}$ is a degenerated version of 1 without diffusion, a representing the control.

This first case is for example encountered while valuing American options in finance. In this case, $X_1^{x,t}$ holds the values of the stocks involved in the option and $X_2^{x,t}$ is for example an integer valued process equal to one if the option is not exercised and 0 if it has already been exercised.
Another classical case happening while dealing with stocks for example is a Gas Storage valuation. In this simple case, the process $X_1^{x,t}$ is the value of the gas on the market and $X_2^{x,t}$ is the position (in volume) in the gas storage. The library offers to store the conditional expectation for all the states $X_2^{x,t}$.

- $X_2^{x,t}$ will be stored on a grid of points (see section 1)
- for each point i of the grid the conditional expectation of a function $g_i(X_2^{x,t})$ associated to the point i using a regressor (see section 1) can be calculated and stored such that the continuation value C is a function of $(X_1^{x,t}, X_2^{x,t})$.

3.1.1 C++ API

As for regressions two constructors are provided

- The first one is the default construction: it is used in simulation algorithm with the "loadForSimulation" method to store the basis coefficients α_k^i for the grid point i (see equation (2.1)),
- The second one

```
1    ContinuationValue(const  shared_ptr< SpaceGrid > &  p_grid ,
2                      const shared_ptr< BaseRegression >  & p_condExp,
3                      const Eigen::ArrayXXd &p_cash)
```

  with

  - *p_grid* the grids associated to the control deterministic space,
  - *p_condExp* the conditional expectation operator
  - *p_cash* the function to regress depending on the grid position (first dimension the number of simulations, second dimension the grid size)

  This constructor constructs for all point i all the α_k^i (see equation (2.1)).

The main methods provided are:

- a first method used in simulation permitting to load for grid point i the coefficient α_k^i associated to the function g_i,

```
1    void loadForSimulation(const  shared_ptr< SpaceGrid > & p_grid ,
2                           const shared_ptr< BaseRegression >  &
                                p_condExp,
3                           const Eigen::ArrayXXd &p_values)
```

  with

  - *p_grid* the grid associated to the controlled deterministic space,
  - *p_condExp* the conditional expectation operator,
  - *p_values* the α_k^i for all grid points i (size the number of function basis, the number of grid points)

- a second method taking as input a point to be interpolated in the grid and returning the conditional expectation at the interpolated point for all simulations:

```
Eigen::ArrayXd getAllSimulations(const Eigen::ArrayXd &p_ptOfStock)
```

- a method taking as input an interpolator in the grid and returning the conditional expectation for all simulations at the interpolated point used to construct the interpolator :

```
Eigen::ArrayXd getAllSimulations(const Interpolator &p_interpol)
```

- a method taking as input a simulation number used in optimization and a point used to interpolate in the grid and returning the conditional expectation at the interpolated point for the given simulation used in optimization.

```
double getASimulation(const int &p_isim, const Eigen::ArrayXd &
    p_ptOfStock)
```

- a method taking as input a simulation number used in optimization and an interpolator in the grid and returning the conditional expectation at the interpolated point used to construct the interpolator for the given simulation used in optimization :

```
double getASimulation(const int &p_isim, const Interpolator &
    p_interpol)
```

- a method that permits to calculate the conditional expectation for a sample of $X_1^{x,t}$:

```
double getValue(const Eigen::ArrayXd &p_ptOfStock, const Eigen::
    ArrayXd &p_coordinates) const
```

  where:

  - *p_ptOfStock* the point where we interpolate the conditional expectation (a realization of $X_2^{x,t}$)
  - *p_coordinates* the sample of $X_1^{x,t}$ used to estimate the conditional expectation
  - and the function returns $C(X_1^{x,t}, X_2^{x,t})$.

Below we regress an identical function for all grid points (here a grid of 4 points in dimension 1):

```
        int sizeForStock = 4;
        // second member to regress with one stock
        ArrayXXd toRegress = ArrayXXd::Constant(p_nbSimul,sizeForStock, 1.);
        // grid for stock
        Eigen::ArrayXd lowValues(1), step(1);
        lowValues(0) = 0. ;
        step(0) = 1;
        Eigen::ArrayXi nbStep(1);
        nbStep(0) = sizeForStock - 1;
        // grid
        shared_ptr< RegularSpaceGrid > regular = MyMakeShared<
            RegularSpaceGrid>(lowValues, step, nbStep);
        // conditional espectation (local basis functions)
```

```
        ArrayXi  nbMesh = ArrayXi::Constant(p_nDim, p_nbMesh);
         shared_ptr<LocalLinearRegression> localRegressor = MyMakeShared<
            LocalLinearRegression>(false, x, nbMesh);

        // creation continuation value object
        ContinuationValue  continuation(regular, localRegressor,  toRegress);

        // regress with continuation value object
        ArrayXd ptStock(1) ;
        ptStock(0) = sizeForStock / 2; // point where we regress
        // calculation the regression values for the current point for all
           the simulations
        ArrayXd regressedByContinuation = continuation.getAllSimulations(
           ptStock);
```

3.1.2 Python API

Here is an example of the use of the mapping

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU
   LGPL)
import numpy as np
import unittest
import random
import math
import StOptGrids
import StOptReg


# unit test for continuation values
#####################################

class testContValues(unittest.TestCase):

    # test a regular grid  for stocks and a local function basis for
       regression
    def testSimpleGridsAndRegressor(self):
        # low value for the meshes
        lowValues =np.array([1.,2.,3.],dtype=np.float)
        # size of the meshes
        step = np.array([0.7,2.3,1.9],dtype=np.float)
        # number of steps
        nbStep = np.array([3,2,4], dtype=np.int32)
        # create the regular grid
        ############################
        grid = StOptGrids.RegularSpaceGrid(lowValues,step,nbStep)
        # simulation
        nbSimul =10000
        np.random.seed(1000)
        x = np.random.uniform(-1.,1.,size=(1,nbSimul));
        # mesh
        nbMesh = np.array([16],dtype=np.int32)
```

```
        # Create the regressor
        ######################
        regressor = StOptReg.LocalLinearRegression(False,x,nbMesh)
        # regressed values
        toReal = (2+x[0,:]+(1+x[0,:])*(1+x[0,:]))
        # function to regress
        toRegress = toReal + 4*np.random.normal(0.,1,nbSimul)
        # create a matrix (number of stock points by number of simulations)
        toRegressMult = np.zeros(shape=(len(toRegress),grid.getNbPoints()))
        for i in range(toRegressMult.shape[1]):
            toRegressMult[:,i] = toRegress
        # Now create the continuation object
        ####################################
        contOb = StOptReg.ContinuationValue(grid,regressor,toRegressMult)
        # get back the regressed values at the point stock
        ptStock=  np.array([1.2,3.1,5.9],dtype=np.float)
        regressValues = contOb.getAllSimulations(ptStock)


if __name__ == '__main__':
    unittest.main()
```

3.2 The GridAndRegressedValue object

As explained above, when we want to interpolate a function discretized partly on a grid and by regression a specific object can we used. As for the continuation it has a "getValue" to estimate the function at a state with both a deterministic ,and a stochastic part.

3.2.1 C++ API

The object has five constructors and we only described the two more commonly used :

- The first one

```
GridAndRegressedValue(const  std::shared_ptr< SpaceGrid >   &p_grid ,
                      const std::shared_ptr< BaseRegression >   &
                          p_reg ,
                      const Eigen::ArrayXXd &p_values)
```

  with

  - *p_grid* the grid associated to the control deterministic space,
  - *p_reg* the regressor object
  - *p_values* the functions at some points on the deterministic and stochastic grid.

- A second constructor only stores the grid and regressor :

```
GridAndRegressedValue(const  std::shared_ptr< SpaceGrid >   &p_grid ,
                      const std::shared_ptr< BaseRegression >   &
                          p_reg)
```

The main methods are the following ones :

- the main method that permits to calculate the function $C(X_{1,s}^{x,t}, X_{2,s}^{x,t})$ value for a point $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$ where $X_{2,s}^{x,t}$ is on the grid and $X_{1,s}^{x,t}$ is the part treated by regression.

```
1    double getValue(const Eigen::ArrayXd &p_ptOfStock, const Eigen::
        ArrayXd &p_coordinates) const
```

  where:

  - $p\_ptOfStock$ $X_{2,s}^{x,t}$ part of $X_s^{x,t}$
  - $p\_coordinates$ $X_{1,s}^{x,t}$ part of $X_s^{x,t}$.

- the method 'getRegressedValues' that permits to get all regression coefficients for all points of the grid. The array returned has a size (number of function basis, number of points on the grid)

```
1 Eigen::ArrayXXd getRegressedValues() const
```

- the method 'setRegressedValues' permits to store all the values regressed coefficients on a grid of a function of $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$.

```
1 void setRegressedValues(const Eigen::ArrayXXd &p_regValues)
```

  where $p\_regValues$ has a size (number of function basis, number of points on the grid).

3.2.2 Python API

The python API is similar to the one of the ContinuationValue object (voir section 3.1.2).

Part III

Solving optimization problems with dynamic programming methods

In the sequel, we suppose that we have developed a Simulator generating some Monte Carlo simulations at the different optimization dates. In order to use the different frameworks developed in the sequel we suppose that the Simulator is derived from the abstract class " SimulatorDPBase.h"

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifndef SIMULATORDPBASE_H
#define SIMULATORDPBASE_H
#include <Eigen/Dense>

/* \file SimulatorDPBase.h
 * \brief Abstract class for simulators for Dynamic Programming Programms
 * \author Xavier Warin
 */

namespace StOpt
{
/// \class SimulatorDPBase SimulatorDPBase.h
/// Abstract class for simulator used in dynamic programming
class SimulatorDPBase
{


public :

    /// \brief Constructor
    SimulatorDPBase() {}
    /// \brief Destructor
    virtual ~SimulatorDPBase() {}
    /// \brief get current  markovian state : dimension of the problem for the  first dimension , second dimension the number of Monte Carlo simulations
    virtual Eigen::MatrixXd getParticles() const = 0;
    /// \brief a step forward for simulations
    virtual void  stepForward() = 0;
    /// \brief a step backward for simulations
    virtual void  stepBackward() = 0;
    /// \brief a step forward for simulations
    /// \return  current particles (markovian state as assets for example) ( dimension of the problem times simulation number)
    virtual Eigen::MatrixXd  stepForwardAndGetParticles() = 0;
    /// \brief a step backward for simulations
    /// \return  current particles (markovian state as assets for example) ( dimension of the problem times simulation number)
    virtual Eigen::MatrixXd stepBackwardAndGetParticles() = 0;
    /// \brief get back dimension of the regression
    virtual int getDimension() const = 0;
    /// \brief get the number of steps
    virtual  int getNbStep() const = 0;
    /// \brief Get the current step size
    virtual double getStep() const = 0;
```

```
45      /// \brief Get current time
46      virtual double getCurrentStep() const = 0 ;
47      /// \brief Number of Monte Carlo simulations
48      virtual int getNbSimul() const = 0;
49      /// \brief Permit to actualize for one time step (interest rate)
50      virtual double getActuStep() const = 0;
51      /// \brief Permits to actualize at the initial date (interest rate)
52      virtual double getActu() const = 0 ;
53
54  };
55  }
56  #endif /* SIMULATORDPBASE_H */
```

Supposing that the Simulator is a Black Scholes simulator for P assets, simulating M Monte Carlo simulations, at $N+1$ dates $t_0, ..., t_N$, the Markov state for particle j, date t_i, Monte Carlo simulation k and asset p is $X_{p,i}^k$ and we give below the meaning of the different methods of " SimulatorDPBase.h":

- the "getParticle" method gives at the current optimization/simulation date t_i the Markov states $X_{p,i}^k$ in a matrix A such that $A(p,k) = X_{p,i}^k$,
- the "stepForward" method is used while simulating the assets evolution in forward: a step forward is realized from t_i to t_{i+1} and Brownian motions used for the assets are updated at the new time step,
- the "stepBackward" method is used for simulation of the asset from the last date to time 0. This method is used during an asset optimization by Dynamic Programming,
- the "stepForwardAndGetParticles" method: second and first method in one call,
- the "stepBackwardAndGetParticles" method: third and first method in one call,
- the "getDimension" method returns the number of assets,
- the "getNbStep" method returns the number of step (N),
- the "getStep" method returns the time step $t_{i+1} - t_i$ at the current time t_i,
- the "getNbSimul" method returns M.
- the "getActuStep" method return the actualization factor on one time step
- the "getActu" method returns an actualization factor at the "0" date.

Chapter 4

Using conditional expectation estimated by regressions to solve simple problems

In this chapter we give some examples to value an American option. This use of the conditional expectation operators can be extended to many stochastic problem using this previously developed objects.

4.1 The American option valuing by Longstaff Schwartz

Suppose in this example that the payoff of the American option is given by g and that the interest rate is 0. The value of the option is given by

$$P_t = \text{esssup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}(g(\tau, X_\tau) \mid \mathcal{F}_t) \quad \text{for} \quad t \leq T \quad \mathbb{P} - \text{a.s.}\,, \tag{4.1}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of stopping times with values in $[t, T]$.

We recall the classical Longstaff Schwartz algorithm 3 estimating the empirical conditional expectation using the regression estimation previously seen.

Initialization:
Set $\hat{\tau}_\kappa^{1,\pi,(j)} := T$, $j \leq N$
Backward induction:
for $i = \kappa - 1$ to 0 **do**
  set $\hat{\tau}_i^{1,\pi} := t_i \mathbf{1}_{A_i^1} + \hat{\tau}_{i+1}^{1,\pi} \mathbf{1}_{(A_i^1)^c}$ where $A_i^1 := \{g(t_i, X_{t_i}) \geq \hat{\mathbb{E}}[g(\hat{\tau}_{i+1}^{1,\pi}, X_{\hat{\tau}_{i+1}^{1,\pi}}) \mid \mathcal{F}_{t_i}]\}$.
end for
Price estimator at 0: $\hat{P}_0^{1,\pi} := \hat{\mathbb{E}}[g(\hat{\tau}_0^{1,\pi}, X_{\hat{\tau}_0^{1,\pi}})]$.

Algorithm 3: Algorithm with regression [optimal exercise time estimation]

4.1.1 American option with the C++ API

We value in the algorithm below an American option using a simulator p_sim, a regressor $p\_regressor$, a payoff function p_payoff:

```
    double step = p_sim.getStep(); // time step increment
    // asset simulated under the neutral risk probability: get the trend of
       the first asset to get the interest rate
    double expRate = exp(-step * p_sim.getMu()(0));
    // Terminal pay off
    VectorXd Cash(p_payOff(p_sim.getParticles()));
    for (int iStep = 0; iStep < p_sim.getNbStep(); ++iStep)
    {
        shared_ptr<ArrayXXd> asset(new ArrayXXd(p_sim.
           stepBackwardAndGetParticles())); // asset = Markov state
        VectorXd payOffLoc = p_payOff(*asset); // pay off
        // update conditional expectation operator for current Markov state
        p_regressor.updateSimulations(((iStep == (p_sim.getNbStep() - 1)) ?
           true : false), asset);
        // conditional  expectation
        VectorXd condEspec = p_regressor.getAllSimulations(Cash) * expRate;
        // arbitrage between pay off and cash delivered after
        Cash = (condEspec.array() < payOffLoc.array()).select(payOffLoc, Cash
            * expRate);
    }
    return Cash.mean();
```

4.2 American option with the Python API

Using the python API the American resolution is given below :

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU
   LGPL)
import numpy as np
import math as maths

# american option by Longstaff-Schwartz
# p_sim         Monte Carlo simulator
# p_payOff      Option pay off
# p_regressor   regressor object
def resolution(p_simulator, p_payOff, p_regressor) :

    step = p_simulator.getStep()
    # asset simulated under the neutral risk probability : get the trend of
       first asset to get interest rate
    expRate = np.exp(-step * p_simulator.getMu()[0])
    # Terminal
    particle = p_simulator.getParticles()
    Cash = p_payOff.operator(particle)

    for iStep in range(0, p_simulator.getNbStep()):
```

```
        asset = p_simulator.stepBackwardAndGetParticles()
        payOffLoc = p_payOff.operator(asset)
        isLastStep = False
        if iStep == p_simulator.getNbStep() - 1 :
            isLastStep = True

        p_regressor.updateSimulations(isLastStep, asset)
        # conditional expectation
        condEspec = p_regressor.getAllSimulations(Cash).squeeze() * expRate
        # arbitrage
        Cash = np.where(condEspec < payOffLoc, payOffLoc, Cash * expRate)

    return maths.fsum(Cash) / len(Cash)
```

Chapter 5

Using the general framework to manage stock problems

In this chapter the state is separated into three parts $X^{x,t} = (X_1^{x,t}, X_2^{x,t}, I_t)$. $(X_1^{x,t}, X_2^{x,t})$, which corresponds to the special case of chapter 3 where $X_1^{x,t}$ is not controlled and $X_2^{x,t}$ is controlled. Two cases can be tackled :

- the first case corresponds to the case where $X_2^{x,t}$ is deterministic (think of storage management),
- the second case corresponds to the case where $X_2^{x,t}$ is stochastic (think of portfolio optimization).

I_t takes some integers values and is here to describe some finite discrete regimes (to treat some switching problems). A general framework is available to solve this kind of problem. First, the second part $X_2^{x,t}$ is discretized on a grid as explained in chapter 3.

- Either a full grid is used for $X_2^{x,t}$ and two types of resolutions either sequential or parallel be can considered :
  - a resolution can be achieved sequentially or a parallelization with MPI on the calculations can be achieved (speed up but no size up). This approach can be used for problems in small dimension.
  - a resolution can be achieved with a parallelization by the MPI framework by spreading the work to be achieved on the grid points, and spread the data between processors (speed up and size up). We will denote this parallelization technique a "distribution" technique. This approach is necessary to tackle very big optimization problems where the global solution cannot be stored in the memory of a single processor.
- or the grid for $X_2^{x,t}$ is not full (so sparse) and only a parallelization by thread and MPI can be achieved on the calculations (speed up and no size up). With sparse grids, only the case $X_2^{x,t}$ deterministic is treated.

In the case of the MPI parallelization technique distributing task and data (full grids only), [19] and [20] are used. Suppose that the grid is the same at each time step (only here to ease the case), and that we have 4 processors (figure 5.1) then:

- at the last time step, the final values at each point for each simulation are computed (each processor computes the values for its own grid points),
- at the previous time step, from a grid point own by a processor, we are able to localize the grids points attained at the next time step by all the commands,
- on figure 5.1, we give the points owned by other processors that can be reached from points owned by processor 3,
- some MPI communications are achieved bringing back the data (values calculated at the previous treated time step) needed by processor 3 to be able to update the value calculated by dynamic programming at the current time for all the points owned by processor 3,
- all the communications between all processors are achieved together.

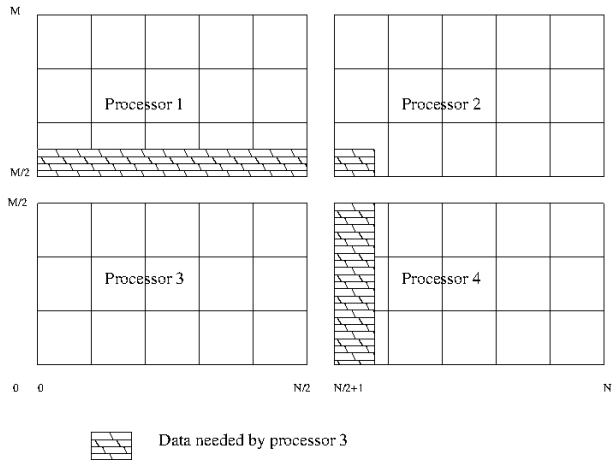


Figure 5.1: Data to send to processor 3

The global state of the the problem is store in the StateWithStocks object.

5.1 General requirement about business object

In order to use the framework, the developer has to describe the problem he wants to solve on one time step staring from a state $X^{x,t}$. This business object has to offer some common methods and it is derived from "OptimizerBase.h"

```
1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
    LGPL)
4 #ifndef OPTIMIZERBASE_H
5 #define OPTIMIZERBASE_H
6 #include <Eigen/Dense>
7 #include "StOpt/core/utils/StateWithStocks.h"
```

```cpp
#include "StOpt/core/grids/SpaceGrid.h"
#include "StOpt/regression/BaseRegression.h"
#include "StOpt/regression/ContinuationValue.h"
#include "StOpt/regression/GridAndRegressedValue.h"
#include "StOpt/dp/SimulatorDPBase.h"

/** \file OptimizerBase.h
 *  \brief Define an abstract class for Dynamic Programming problems solved by Monte Carlo methods
 *     \author Xavier Warin
 */

namespace StOpt
{

/// \class OptimizerBase OptimizerBase.h
///  Base class for optimizer for Dynamic Programming with and without regression methods
class OptimizerBase
{


public :

    OptimizerBase() {}

    virtual ~OptimizerBase() {}

    /// \brief defines the dimension to split for MPI parallelism
    ///        For each dimension return true is the direction can be split
    virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit() const = 0 ;

    /// \brief defines the diffusion cone for parallelism
    /// \param  p_regionByProcessor         region (min max) treated by the processor for the different regimes treated
    /// \return returns in each dimension the min max values in the stock that can be reached from the grid p_gridByProcessor for each regime
    virtual std::vector< std::array< double, 2> > getCone(const  std::vector< std::array< double, 2>  > &p_regionByProcessor) const = 0;

    /// \brief Defines a step in simulation using interpolation in controls
    /// \param p_grid           grid at arrival step after command
    /// \param p_control        defines the controls
    /// \param p_state          defines the state value (modified)
    /// \param p_phiInOut       defines the value function (modified): size number of functions to follow
    virtual void stepSimulateControl(const std::shared_ptr< StOpt::SpaceGrid> &p_grid, const std::vector< StOpt::GridAndRegressedValue  > & p_control,
                                     StOpt::StateWithStocks &p_state,
                                     Eigen::Ref<Eigen::ArrayXd> p_phiInOut) const = 0 ;

```

```
52
53
54      /// \brief Get the number of regimes allowed for the asset to be reached
            at the current time step
55      virtual    int getNbRegime() const = 0 ;
56
57      /// \brief get the simulator back
58      virtual std::shared_ptr< StOpt::SimulatorDPBase > getSimulator() const =
            0;
59
60      /// \brief get back the dimension of the control
61      virtual int getNbControl() const = 0 ;
62
63      /// \brief get size of the  function to follow in simulation
64      virtual int getSimuFuncSize() const = 0;
65
66  };
67  }
68  #endif /* OPTIMIZERBASE_H */
```

We detail all the methods that have to be implemented for all resolution methods (with or without regressions).

- the "getNbRegime" permits to get the number of regimes of the problem: for example, in switching problems, when there is a cost of switching, the working regime has to be incorporated in the state. Another example is the case of conditional delta to calculate for an asset: two regimes can be used: one to calculate the asset value and the second one to calculate the Δ. This number of regimes can be time dependent : in this case for a current resolution date t the "getNbRegime" method send the number of regimes at the very beginning of the time step (in t^-) such that a switch to a new regime can occurred in t^+.

- the "getSimulator" method is used to get back the simulator giving the Monte Carlo simulations,

- the " getSimuFuncSize" method is used in simulation to define the number of functions to follow in the simulation part. For example in a stochastic target problem where the target is a given wealth with a given probability, one may want to follow the evolution of the probability at each time step and the wealth obtained while trading. In this case the " getSimuFuncSize" returns 2.

- the "getCone" method is only relevant if the MPI framework with distribution is used. As argument it take a vector of size the dimension of the grid. Each component of the vector is an array containing the minimal and maximal coordinates values of points in the current grid defining an hyper cube $H1$. It returns for each dimension, the coordinates min and max of the hyper cube $H2$ containing the points that can be reached by applying a command from a grid point in $H1$.

- the "getDimensionToSplit" method permits to define in the MPI framework with distribution which directions to split for solution distribution on processors. For each

dimension it returns a Boolean where "true" means that the direction is a candidate for splitting.

- the "stepSimulateControl" method is used after optimization using the optimal controls calculated in the optimization part. From a state *p_state* (storing the $X^{x,t}$), the optimal control calculated in optimization *p_control*, the optimal functions values along the current trajectory are stored in *p_phiInOut*. The state *p_state* is updated during at the end of the call function.

In a first part we present the framework for problems where conditional expectation is calculated by regression (case where $X_2^{t,x}$ is not controlled). Then we develop the framework not using regression for conditional expectation calculations. All conditional expectation are calculated using exogenous particles and interpolation. This will be typically the case for portfolio optimization.

5.2 Solving the problem using conditional expectation calculated by regressions

In this part we suppose that $X_2^{x,t}$ is controlled and deterministic so regression methods can be used.

5.2.1 Requirement to use the framework

In order to use the framework with regression for conditional expectation, a business object describing the business on one time step from one state is derived from "OptimizerDPBase.h" itself derived from "OptimizerBase.h" .

```
1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #ifndef OPTIMIZERDPBASE_H
5  #define OPTIMIZERDPBASE_H
6  #include <Eigen/Dense>
7  #include "StOpt/core/utils/StateWithStocks.h"
8  #include "StOpt/core/grids/SpaceGrid.h"
9  #include "StOpt/regression/BaseRegression.h"
10 #include "StOpt/regression/ContinuationValue.h"
11 #include "StOpt/regression/GridAndRegressedValue.h"
12 #include "StOpt/dp/SimulatorDPBase.h"
13 #include "StOpt/dp/OptimizerBase.h"
14
15 /** \file OptimizerDPBase.h
16  *  \brief Define an abstract class for Dynamic Programming problems solved
       by regression  methods
17  *     \author Xavier Warin
18  */
19
20 namespace StOpt
```

```
21 {
22
23 /// \class OptimizerDPBase OptimizerDPBase.h
24 ///  Base class for optimizer for Dynamic Programming  with regression
       methods
25 class OptimizerDPBase : public OptimizerBase
26 {
27
28
29 public :
30
31     OptimizerDPBase() {}
32
33     virtual ~OptimizerDPBase() {}
34
35     /// \brief defines the diffusion cone for parallelism
36     /// \param  p_regionByProcessor         region (min max) treated by the
          processor for the different regimes treated
37     /// \return returns in each dimension the min max values in the stock
          that can be reached from the grid p_gridByProcessor for each regime
38     virtual std::vector< std::array< double, 2> > getCone(const  std::vector<
           std::array< double, 2>  > &p_regionByProcessor) const = 0;
39
40     /// \brief defines the dimension to split for MPI parallelism
41     ///        For each dimension return true is the direction can be split
42     virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
          const = 0 ;
43
44     /// \brief defines a step in optimization
45     /// \param p_grid      grid at arrival step after command
46     /// \param p_stock     coordinates of the stock point to treat
47     /// \param p_condEsp   continuation values for each regime
48     /// \param p_phiIn     for each regime  gives the solution calculated at
          the previous step ( next time step by Dynamic Programming resolution)
          : structure of the 2D array ( nb simulation ,nb stocks )
49     /// \return   a pair  :
50     ///             - for each regimes (column) gives the solution for each
          particle (row)
51     ///             - for each control (column) gives the optimal control
          for each particle (rows)
52     ///              .
53     virtual std::pair< Eigen::ArrayXXd, Eigen::ArrayXXd>   stepOptimize(const
           std::shared_ptr< StOpt::SpaceGrid> &p_grid, const Eigen::ArrayXd
          &p_stock,
54             const std::vector< StOpt::ContinuationValue  > &p_condEsp,
55             const std::vector < std::shared_ptr< Eigen::ArrayXXd > > &p_phiIn
                  ) const = 0;
56
57
58     /// \brief defines a step in simulation
59     /// Notice that this implementation is not optimal but is convenient if
          the control is discrete.
60     /// By avoiding interpolation in control we avoid non admissible control
61     /// Control are recalculated during simulation.
```

```
62      /// \param p_grid           grid at arrival step after command
63      /// \param p_continuation   defines the continuation operator for each
            regime
64      /// \param p_state          defines the state value (modified)
65      /// \param p_phiInOut       defines the value functions (modified) : size
            number of functions  to follow
66      virtual void stepSimulate(const std::shared_ptr< StOpt::SpaceGrid>   &
            p_grid, const std::vector< StOpt::GridAndRegressedValue  > &
            p_continuation,
67                                StOpt::StateWithStocks &p_state,
68                                Eigen::Ref<Eigen::ArrayXd> p_phiInOut) const =
                                    0 ;
69
70
71      /// \brief Defines a step in simulation using interpolation in controls
72      /// \param p_grid           grid at arrival step after command
73      /// \param p_control        defines the controls
74      /// \param p_state          defines the state value (modified)
75      /// \param p_phiInOut       defines the value function (modified): size
            number of functions to follow
76      virtual void stepSimulateControl(const std::shared_ptr< StOpt::SpaceGrid>
              &p_grid, const std::vector< StOpt::GridAndRegressedValue  > &
            p_control,
77                                       StOpt::StateWithStocks &p_state,
78                                       Eigen::Ref<Eigen::ArrayXd> p_phiInOut)
                                           const = 0 ;
79
80
81
82      /// \brief Get the number of regimes allowed for the asset to be reached
             at the current time step
83      ///     If \f$ t \f$ is the current time, and $\f$ dt \f$  the resolution
            step,  this is the number of regime allowed on \f$[ t- dt, t[\f$
84      virtual   int getNbRegime() const = 0 ;
85
86      /// \brief get the simulator back
87      virtual std::shared_ptr< StOpt::SimulatorDPBase > getSimulator() const =
            0;
88
89      /// \brief get back the dimension of the control
90      virtual int getNbControl() const = 0 ;
91
92      /// \brief get size of the  function to follow in simulation
93      virtual int getSimuFuncSize() const = 0;
94
95  };
96  }
97  #endif /* OPTIMIZERDPBASE_H */
```

We detail the different methods to implement in addition to the methods of "Optimizer-Base.h":

- the "stepOptimize" methods is used in optimization. We want to calculate the optimal value at current t_i at a grid point *p_stock* using a grid *p_grid* at the next date t_{i+1},

the continuation values for all regimes *p_condEsp* permitting to calculate conditional expectation of the optimal value function calculated at the previously treated time step t_{i+1}. From a grid point *p_stock* it calculates the function values and the optimal controls. It returns a pair where the

  - first element is a matrix (first dimension is the number of simulations, second dimension the number of regimes) giving the function value,
  - second element is a matrix (first dimension is the number of simulations, second dimension the number of controls) giving the optimal control.

- the "stepSimulate" method is used after optimization using the continuation values calculated in the optimization part. From a state *p_state* (storing the $X^{x,t}$), the continuation values calculated in optimization *p_continuation*, the optimal functions values along the current trajectory are stored in *p_phiInOut*.

In the case of a gas storage [21], the holder of the storage can inject gas from the network in the storage (paying the market price) or withdraw gas from the storage on the network (receiving the market price). In this case the Optimize object is given in this file. You can have a look at the implementation of the "getCone" method.

5.2.2 The framework in optimization

Once an Optimizer is derived for the project, and supposing that a full grid is used for the stock discretization, the framework provides a "TransitionStepRegressionDPDist" object in MPI that permits to solve the optimization problem with distribution of the data on one time step with the following constructor:

```
1  TransitionStepRegressionDPDist(const shared_ptr<FullGrid> &
       p_pGridCurrent,
2                                 const shared_ptr<FullGrid> &
                                      p_pGridPrevious,
3                                 const shared_ptr<OptimizerDPBase > &
                                      p_pOptimize):
```

with

- *p_pGridCurrent* is the grid at the current time step (t_i),
- *p_pGridPrevious* is the grid at the previously treated time step (t_{i+1}),
- *p_pOptimize* the optimizer object

Remark 6 *A similar object is available without the MPI distribution framework "TransitionStepRegressionDP" with still enabling parallelization with threads and MPI on the calculations on the full grid points.*

Remark 7 *In the case of sparse grids with only parallelization on the calculations (threads and MPI) "TransitionStepRegressionDPSparse" object can be used*

The main method is

```
1    std::vector< shared_ptr< Eigen::ArrayXXd > > OneStep(const std::vector<
         shared_ptr< Eigen::ArrayXXd > > &p_phiIn,
2            const shared_ptr< BaseRegression> &p_condExp)
```

with

- *p_phiIn* the vector (its size corresponds to the number of regimes) of matrix of optimal values calculated at the previous time iteration for each regime . Each matrix is a number of simulations by number of stock points matrix.
- *p_condExp* the conditional expectation operator,

returning a pair :

- first element is a vector of matrix with new optimal values at the current time step (each element of the vector corresponds to a regime and each matrix is a number of simulations by number of stock points matrix).
- second element is a vector of matrix with new optimal controls at the current time step (each element of the vector corresponds to a control and each matrix is a number of simulations by number of stock points matrix).

Remark 8 *All "TransitionStepRegressionDP" derive from a "TransitionStepRegressionBase" object having a pure virtual "OneStep" method.*

A second method is provided permitting to dump the continuation values of the problem and the optimal control at each time step :

```
1    void dumpContinuationValues(std::shared_ptr<gs::BinaryFileArchive> p_ar,
         const std::string &p_name, const int &p_iStep,
2                                const std::vector< std::shared_ptr< Eigen::
                                     ArrayXXd > > &p_phiInPrev,
3                                const std::vector< std::shared_ptr< Eigen::
                                     ArrayXXd > > &p_control,
4                                const std::shared_ptr<BaseRegression> &
                                     p_condExp,
5                                const bool &p_bOneFile) const
```

with :

- *p_ar* is the archive where controls and solutions are dumped,
- *p_name* is a base name used in the archive to store the solution and the control,
- *p_phiInPrev* is the solution at the previous time step used to calculate the continuation values that are stored,
- *p_control* stores the optimal controls calculated at the current time step,
- *p_condExp* is the conditional expectation object permitting to calculate conditional expectation of functions defined at the previous time step treated *p_phiInPrev* and permitting to store a representation of the optimal control.

- *p_bOneFile* is set to one if the continuation and optimal controls calculated by each processor are dumped on a single file. Otherwise the continuation and optimal controls calculated by each processor are dumped on different files (one by processor). If the problem gives continuation and optimal control values on the global grid that can be stored in the memory of the computation node, it can be more interesting to dump the continuation/control values in one file for the simulation of the optimal policy.

Remark 9 *As for the "TransitionStepRegressionDP" and the "TransitionStepRegressionDPSparse" object, their "dumpContinuationValues" doesn't need a p_bOneFile argument: obviously optimal controls and solutions are stored in a single file.*

We give here a simple example of a time resolution using this method when the MPI distribution of data is used

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifdef USE_MPI
#include <fstream>
#include <memory>
#include <functional>
#include <boost/lexical_cast.hpp>
#include <boost/mpi.hpp>
#include <Eigen/Dense>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/core/grids/FullGrid.h"
#include "StOpt/regression/BaseRegression.h"
#include "StOpt/dp/FinalStepRegressionDPDist.h"
#include "StOpt/dp/TransitionStepRegressionDPDist.h"
#include "StOpt/core/parallelism/reconstructProc0Mpi.h"
#include "StOpt/dp/OptimizerDPBase.h"
#include "StOpt/dp/SimulatorDPBase.h"


using namespace std;

double DynamicProgrammingByRegressionDist(const shared_ptr<StOpt::FullGrid> &p_grid,
        const shared_ptr<StOpt::OptimizerDPBase > &p_optimize,
        shared_ptr<StOpt::BaseRegression> &p_regressor,
        const function<double(const int &, const Eigen::ArrayXd &, const Eigen::ArrayXd &)> &p_funcFinalValue,
        const Eigen::ArrayXd &p_pointStock,
        const int &p_initialRegime,
        const string &p_fileToDump,
        const bool &p_bOneFile)
{
    // from the optimizer get back the simulator
    shared_ptr< StOpt::SimulatorDPBase> simulator = p_optimize->getSimulator();
    // final values
```

Table 5.1: Which "TransitionStepRegression" object to use depending on the grid used and the type of parallelization used.

	Full grid	Sparse grid
Sequential	"TransitionStepRegressionDP"	"TransitionStepRegressionDPSparse"
Parallelization on calculations threads and MPI	"TransitionStepRegressionDP"	"TransitionStepRegressionDPSparse"
Distribution of calculations and data	"TransitionStepRegressionDPDist"	Not available

```
35      vector< shared_ptr< Eigen::ArrayXXd > >  valuesNext = StOpt::
            FinalStepRegressionDPDist(p_grid, p_optimize->getNbRegime(),
            p_optimize->getDimensionToSplit())(p_funcFinalValue, simulator->
            getParticles().array());
36      // dump
37      boost::mpi::communicator world;
38      string toDump = p_fileToDump ;
39      // test if one file generated
40      if (!p_bOneFile)
41          toDump += "_" + boost::lexical_cast<string>(world.rank());
42      shared_ptr<gs::BinaryFileArchive> ar;
43      if ((!p_bOneFile) || (world.rank() == 0))
44          ar = make_shared<gs::BinaryFileArchive>(toDump.c_str(), "w");
45      // name for object in archive
46      string nameAr = "Continuation";
47      for (int iStep = 0; iStep < simulator->getNbStep(); ++iStep)
48      {
49          shared_ptr<Eigen::ArrayXXd> asset = make_shared<Eigen::ArrayXXd>(
                simulator->stepBackwardAndGetParticles());
50          // conditional expectation operator
51          p_regressor->updateSimulations(((iStep == (simulator->getNbStep() -
                1)) ? true : false), asset);
52          // transition object
53          StOpt::TransitionStepRegressionDPDist transStep(p_grid, p_grid,
                p_optimize);
54          pair< vector< shared_ptr< Eigen::ArrayXXd > >, vector< shared_ptr<
                Eigen::ArrayXXd > > > valuesAndControl  = transStep.oneStep(
                valuesNext, p_regressor);
55          transStep.dumpContinuationValues(ar, nameAr, iStep, valuesNext,
                valuesAndControl.second, p_regressor, p_bOneFile);
56          valuesNext = valuesAndControl.first;
57      }
58      // reconstruct a small grid for interpolation
59      return StOpt::reconstructProc0Mpi(p_pointStock, p_grid, valuesNext[
            p_initialRegime], p_optimize->getDimensionToSplit());
60
61  }
62  #endif
```

An example without distribution of the data can be found in this file. We give at last a table with the different "TransitionStepRegression" objects to use depending on the type of parallelization used.

5.2.3 The framework in simulation

Once the optimization has been achieved, continuation values are dumped in one file (or some files) at each time step. In order to simulate the optimal policy, we can use the continuation values previously calculated (see chapter 3) or we can use the optimal controls calculated in optimization. In continuous optimization, using the control is more effective in term of computational cost. When the control is discrete, interpolation of the controls can lead to non admissible controls and simulation with the value function is more accurate.
While simulating the optimal control, two cases can occur :

- For most of the case (small dimensional case), the optimal control or the optimal function value can be stored in the memory of the computing node and function values and controls are stored in a single file. In this case a simulation of the optimal control can easily be achieved by distributing the Monte Carlo simulations on the available calculations nodes : this can be achieved by using the "SimulateStepRegression"or "SimulateStepRegressionControl" objects at each time step of the simulation.

- When dealing with very large problems, optimization is achieved by distributing the data on the processors and it is impossible to store the optimal command on one node. In this case, optimal controls and optimal solutions are stored in the memory of the node that has been used to calculate them in optimization. Simulations are reorganized at each time step and gathered so that they occupy the same part of the global grid. Each processor will then get from other processors a localized version of the optimal control or solution that it needs. This methodology is used in the "SimulateStepRegressionDist" and "SimulateStepRegressionControlDist" objects.

We detail the simulations objects using the optimal function value calculated in optimization and the optimal control for the case of very big problems.

- Simulation step using the value function calculated in optimization :

  In order to simulate one step of the optimal policy, an object "SimulateStepRegressionDist" is provided with constructor

```
1  SimulateStepRegressionDist(gs::BinaryFileArchive &p_ar, const int &
       p_iStep, const std::string &p_nameCont,
2                             const shared_ptr<FullGrid> &
                                  p_pGridFollowing, const shared_ptr<
                                  OptimizerDPBase > &p_pOptimize,
3                             const bool &p_bOneFile)
```

  where

  - *p_ar* is the binary archive where the continuation values are stored,
  - *p_iStep* is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
  - *p_nameCont* is the base name for continuation values,
  - *p_GridFollowing* is the grid at the next time step ($p\_iStep + 1$),

– *p_Optimize* the Optimizer describing the transition from one time step to the following one,

– *p_OneFile* equal to true if a single archive is used to store continuation values.

Remark 10 *A version without distribution of data but with multithreaded and with MPI possible on calculations is available with the object "SimulateStepRegression". The p_OneFile argument is omitted during construction.*

This object implements the method "oneStep"

```
void oneStep(std::vector<StateWithStocks > &p_statevector , Eigen::
    ArrayXXd  &p_phiInOut)
```

where:

– *p_statevector* store the states for the all the simulations: this state is updated by application of the optimal command,

– *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where nb is given by the "getSimuFuncSize" method of the optimizer and $nbSimul$ the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (
    GNU LGPL)
#ifndef SIMULATEREGREGRESSIONDIST_H
#define SIMULATEREGREGRESSIONDIST_H
#include <functional>
#include <memory>
#include <Eigen/Dense>
#include <boost/mpi.hpp>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/core/grids/FullGrid.h"
#include "StOpt/core/utils/StateWithStocks.h"
#include "StOpt/dp/SimulateStepRegressionDist.h"
#include "StOpt/dp/OptimizerDPBase.h"
#include "StOpt/dp/SimulatorDPBase.h"


/** \file SimulateRegressionDist.h
 *  \brief Defines a simple program showing how to use simulation
 *         A simple grid  is used
 *  \author Xavier Warin
 */


/// \brief Simulate the optimal strategy , mpi version
```

```
26 /// \param p_grid                    grid used for  deterministic state (
       stocks for example)
27 /// \param p_optimize                optimizer defining the optimization
       between two time steps
28 /// \param p_funcFinalValue          function defining the final value
29 /// \param p_pointStock              initial point stock
30 /// \param p_initialRegime           regime at initial date
31 /// \param p_fileToDump              name associated to dumped bellman
       values
32 /// \param p_bOneFile                do we store continuation values  in
       only one file
33 double SimulateRegressionDist(const std::shared_ptr<StOpt::FullGrid> &
       p_grid,
34                               const std::shared_ptr<StOpt::
                                      OptimizerDPBase > &p_optimize,
35                               const std::function<double(const int &,
                                      const Eigen::ArrayXd &, const Eigen::
                                      ArrayXd &)>  &p_funcFinalValue,
36                               const Eigen::ArrayXd &p_pointStock,
37                               const int &p_initialRegime,
38                               const std::string   &p_fileToDump,
39                               const bool &p_bOneFile)
40 {
41     boost::mpi::communicator world;
42     // from the optimizer get back the simulator
43     std::shared_ptr< StOpt::SimulatorDPBase> simulator = p_optimize->
           getSimulator();
44     int nbStep = simulator->getNbStep();
45     std::vector< StOpt::StateWithStocks> states;
46     states.reserve(simulator->getNbSimul());
47     for (int is = 0; is < simulator->getNbSimul(); ++is)
48         states.push_back(StOpt::StateWithStocks(p_initialRegime,
               p_pointStock, Eigen::ArrayXd::Zero(simulator->getDimension())
               ));
49     std::string toDump = p_fileToDump ;
50     // test if one file generated
51     if (!p_bOneFile)
52         toDump += "_" + boost::lexical_cast<std::string>(world.rank());
53     gs::BinaryFileArchive ar(toDump.c_str(), "r");
54     // name for continuation object in archive
55     std::string nameAr = "Continuation";
56     // cost function
57     Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
           getSimuFuncSize(), simulator->getNbSimul());
58     for (int istep = 0; istep < nbStep; ++istep)
59     {
60         StOpt::SimulateStepRegressionDist(ar, nbStep - 1 - istep, nameAr,
               p_grid, p_optimize, p_bOneFile).oneStep(states, costFunction
               );
61
62         // new stochastic state
63         Eigen::ArrayXXd particles =  simulator->
               stepForwardAndGetParticles();
64         for (int is = 0; is < simulator->getNbSimul(); ++is)
```

```
65              states[is].setStochasticRealization(particles.col(is));
66
67      }
68      // final : accept to exercise if not already done entirely (here
            suppose one function to follow)
69      for (int is = 0; is < simulator->getNbSimul(); ++is)
70          costFunction(0, is) += p_funcFinalValue(states[is].getRegime(),
                states[is].getPtStock(), states[is].getStochasticRealization
                ()) * simulator->getActu();
71
72      return costFunction.mean();
73  }
74
75  #endif /* SIMULATEREGRESSIONDIST_H */
```

The version of the previous example using a single archive storing the control/solution is given in this file.

- Simulation step using the optimal controls calculated in optimization :

```
1       SimulateStepRegressionControlDist(gs::BinaryFileArchive &p_ar,  const
            int &p_iStep,  const std::string &p_nameCont,
2                                         const   std::shared_ptr<FullGrid> &
                                              p_pGridCurrent,
3                                         const   std::shared_ptr<FullGrid> &
                                              p_pGridFollowing,
4                                         const  std::shared_ptr<
                                              OptimizerDPBase > &p_pOptimize,
5                                         const bool &p_bOneFile);
```

where

- *p_ar* is the binary archive where the continuation values are stored,
- *p_iStep* is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
- *p_nameCont* is the base name for control values,
- *p_GridCurrent* is the grid at the current time step (*p_iStep*),
- *p_GridFollowing* is the grid at the next time step ($p\_iStep + 1$),
- *p_Optimize* is the Optimizer describing the transition from one time step to the following one,
- *p_OneFile* equals to true if a single archive is used to store continuation values.

Remark 11 *A version where a single archive storing the control/solution is used is available with the object "SimulateStepRegressionControl"*

This object implements the method "oneStep"

```
void oneStep(std::vector<StateWithStocks > &p_statevector , Eigen::
    ArrayXd  &p_phiInOut)
```

where:

- *p_statevector* stores for all the simulations the state : this state is updated by application of the optimal commands,
- *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where nb is given by the "getSimuFuncSize" method of the optimizer and $nbSimul$ the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (
    GNU LGPL)
#ifdef USE_MPI
#ifndef SIMULATEREGREGRESSIONCONTROLDIST_H
#define SIMULATEREGREGRESSIONCONTROLDIST_H
#include <functional>
#include <memory>
#include <Eigen/Dense>
#include <boost/mpi.hpp>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/core/grids/FullGrid.h"
#include "StOpt/core/utils/StateWithStocks.h"
#include "StOpt/dp/SimulateStepRegressionControlDist.h"
#include "StOpt/dp/OptimizerDPBase.h"
#include "StOpt/dp/SimulatorDPBase.h"


/** \file SimulateRegressionControlDist.h
 *  \brief Defines a simple program showing how to use simulation
 *         A simple grid  is used
 *  \author Xavier Warin
 */


/// \brief Simulate the optimal strategy using optimal controls
    calculated in optimization , mpi version
/// \param p_grid                   grid used for  deterministic state (
    stocks for example)
/// \param p_optimize               optimizer defining the optimization
    between two time steps
/// \param p_funcFinalValue         function defining the final value
/// \param p_pointStock             initial point stock
/// \param p_initialRegime          regime at initial date
/// \param p_fileToDump             name associated to dumped bellman
    values
```

```
33 /// \param p_bOneFile                do we store continuation values  in
       only one file
34 double SimulateRegressionControlDist(const std::shared_ptr<StOpt::
       FullGrid> &p_grid,
35                                      const std::shared_ptr<StOpt::
                                           OptimizerDPBase > &p_optimize,
36                                      const std::function<double(const int
                                            &, const Eigen::ArrayXd &,
                                           const Eigen::ArrayXd &)>  &
                                           p_funcFinalValue,
37                                      const Eigen::ArrayXd &p_pointStock,
38                                      const int &p_initialRegime,
39                                      const std::string   &p_fileToDump,
40                                      const bool &p_bOneFile)
41 {
42     boost::mpi::communicator world;
43     // from the optimizer get back the simulator
44     std::shared_ptr< StOpt::SimulatorDPBase> simulator = p_optimize->
           getSimulator();
45     int nbStep = simulator->getNbStep();
46     std::vector< StOpt::StateWithStocks> states;
47     states.reserve(simulator->getNbSimul());
48     for (int is = 0; is < simulator->getNbSimul(); ++is)
49         states.push_back(StOpt::StateWithStocks(p_initialRegime,
               p_pointStock, Eigen::ArrayXd::Zero(simulator->getDimension())
               ));
50     std::string toDump = p_fileToDump ;
51     // test if one file generated
52     if (!p_bOneFile)
53         toDump += "_" + boost::lexical_cast<std::string>(world.rank());
54     gs::BinaryFileArchive ar(toDump.c_str(), "r");
55     // name for continuation object in archive
56     std::string nameAr = "Continuation";
57     // cost function
58     Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
           getSimuFuncSize(), simulator->getNbSimul());
59     for (int istep = 0; istep < nbStep; ++istep)
60     {
61         StOpt::SimulateStepRegressionControlDist(ar, nbStep - 1 - istep,
               nameAr, p_grid, p_grid, p_optimize, p_bOneFile).oneStep(
               states, costFunction);
62
63         // new stochastic state
64         Eigen::ArrayXXd particules =  simulator->
               stepForwardAndGetParticles();
65         for (int is = 0; is < simulator->getNbSimul(); ++is)
66             states[is].setStochasticRealization(particules.col(is));
67     }
68     // final : accept to exercise if not already done entirely (here
           suppose one function to follow)
69     for (int is = 0; is < simulator->getNbSimul(); ++is)
70         costFunction(0, is) += p_funcFinalValue(states[is].getRegime(),
               states[is].getPtStock(), states[is].getStochasticRealization
               ()) * simulator->getActu();
```

```
71
72      return costFunction.mean();
73  }
74
75  #endif /* SIMULATEREGRESSIONCONTROLDIST_H */
76  #endif
```

The version of the previous example using a single archive storing the control/solution is given in this file.

In the table below we indicate which simulation object should be used at each time step depending on the "TransitionStepRegressionD" object used in optimization.

5.3 Solving the problem for $X_2^{x,t}$ stochastic

In this part we suppose that $X_2^{x,t}$ is controlled but is stochastic.

5.3.1 Requirement to use the framework

In order to use the framework, a business object describing the business on one time step from one state is derived from "OptimizerNoRegressionDPBase.h" itself derived from "OptimizerBase.h" .

```
1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #ifndef OPTIMIZERNOREGRESSIONDPBASE_H
5  #define OPTIMIZERNOREGRESSIONDPBASE_H
6  #include <Eigen/Dense>
7  #include "StOpt/core/utils/StateWithStocks.h"
8  #include "StOpt/core/grids/SpaceGrid.h"
9  #include "StOpt/regression/BaseRegression.h"
10 #include "StOpt/regression/GridAndRegressedValue.h"
11 #include "StOpt/dp/SimulatorDPBase.h"
12 #include "StOpt/dp/OptimizerBase.h"
13
14 /** \file OptimizerNoRegressionDPBase.h
15  *  \brief Define an abstract class for Dynamic Programming problems solve by
        Monte Carlo but without regression method
16  *         to compute conditional expectation.
17  *     \author Xavier Warin
18  */
19
20 namespace StOpt
21 {
22
23 /// \class OptimizerNoRegressionDPBase OptimizerNoRegressionDPBase.h
24 ///  Base class for optimizer for Dynamic Programming solved without
       regression method  to compute conditional expectation.
25 class OptimizerNoRegressionDPBase : public OptimizerBase
```

Table 5.2: Which simulation object to use depending on the TransitionStepRegression object used.

	" TransitionStepRegressionDP"	"TransitionStepRegressionDPDist" "bOneFile"=True	"TransitionStepRegressionDPDist" "bOneFile"= False	"TransitionStepRegressionDPSparse"
"SimulateStepRegression"	Yes	Yes	No	Yes
"SimulateStepRegressionControl"	Yes	Yes	No	Yes
"SimulateStepRegressionDist"	No	Yes	Yes	No
"SimulateStepRegressionControlDist"	No	Yes	Yes	No

```
{


public :

    OptimizerNoRegressionDPBase() {}

    virtual ~OptimizerNoRegressionDPBase() {}

    /// \brief defines the diffusion cone for parallelism
    /// \param  p_regionByProcessor         region (min max) treated by the processor for the different regimes treated
    /// \return returns in each dimension the min max values in the stock that can be reached from the grid p_gridByProcessor for each regime
    virtual std::vector< std::array< double, 2> > getCone(const  std::vector< std::array< double, 2>  > &p_regionByProcessor) const = 0;

    /// \brief defines the dimension to split for MPI parallelism
    ///        For each dimension return true is the direction can be split
    virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit() const = 0 ;

    /// \brief defines a step in optimization
    /// \param p_stock          coordinates of the stock point to treat
    /// \param p_valNext        Optimized values at next time step for each regime
    /// \param p_regressorCur    Regressor at the current  date
    /// \return   a pair  :
    ///             - for each regimes (column) gives the solution for each particle (row)
    ///             - for each control (column) gives the optimal control for each particle (rows)
    ///              .
    virtual std::pair< Eigen::ArrayXXd, Eigen::ArrayXXd>   stepOptimize(const Eigen::ArrayXd   &p_stock,
            const std::vector< GridAndRegressedValue  > &p_valNext,
            std::shared_ptr< BaseRegression  >     p_regressorCur) const = 0;



    /// \brief Defines a step in simulation using interpolation in controls
    /// \param p_grid           grid at arrival step after command
    /// \param p_control        defines the controls
    /// \param p_state          defines the state value (modified)
    /// \param p_phiInOut       defines the value function (modified): size number of functions to follow
    virtual void stepSimulateControl(const std::shared_ptr< StOpt::SpaceGrid> &p_grid, const std::vector< StOpt::GridAndRegressedValue  > &p_control,
                                     StOpt::StateWithStocks &p_state,
                                     Eigen::Ref<Eigen::ArrayXd> p_phiInOut) const = 0 ;


```

```
68
69      /// \brief Get the number of regimes allowed for the asset to be reached
            at the current time step
70      ///     If \f$ t \f$ is the current time, and $\f$ dt \f$  the resolution
            step,  this is the number of regime allowed on \f$[ t- dt, t[\f$
71      virtual    int getNbRegime() const = 0 ;
72
73      /// \brief get the simulator back
74      virtual std::shared_ptr< StOpt::SimulatorDPBase > getSimulator() const =
            0;
75
76      /// \brief get back the dimension of the control
77      virtual int getNbControl() const = 0 ;
78
79      /// \brief get size of the  function to follow in simulation
80      virtual int getSimuFuncSize() const = 0;
81
82  };
83  }
84  #endif /* OPTIMIZERDPBASE_H */
```

In addition to the methods of "OptimizerBase.h" the following method is needed :

- the "stepOptimize" methods is used in optimization. We want to calculate the optimal value regressed at current t_i at a grid point *p_stock* using a grid *p_grid* at the next date t_{i+1},

  From a grid point *p_stock* it calculates the function values regressed and the optimal controls regressed. It returns a pair where the

  - first element is a matrix (first dimension is the number of functions in the regression, second dimension the number of regimes) giving the function value regressed,
  - second element is a matrix (first dimension is the number of functions in the regression, second dimension the number of controls) giving the optimal control regressed.

In this case of the optimization of an actualized portfolio with dynamic:

$$dX_2^{x,t} = X_2^{x,t} \frac{dX_1^{x,t}}{X_1^{x,t}}$$

where $X_1^{x,t}$ is the risky asset value, the Optimize object is given in this file.

5.3.2 The framework in optimization

Once an Optimizer is derived for the project, and supposing that a full grid is used for the stock discretization, the framework provides a "TransitionStepDPDist" object in MPI that permits to solve the optimization problem with distribution of the data on one time step with the following constructor:

```
TransitionStepDPDist(const  shared_ptr<FullGrid> &p_pGridCurrent,
const  shared_ptr<FullGrid> &p_pGridPrevious,
const  std::shared_ptr<BaseRegression> &p_regressorCurrent,
const  std::shared_ptr<BaseRegression> &p_regressorPrevious,
const  shared_ptr<OptimizerNoRegressionDPBase > &p_pOptimize):
```

with

- *p_pGridCurrent* is the grid at the current time step (t_i),
- *p_pGridPrevious* is the grid at the previously treated time step (t_{i+1}),
- *p_regressorCurrent* is a regressor at the current date (to evaluate the function at the current date)
- *p_regressorPrevious* is a regressor at the previously treated time step (t_{i+1}) permitting to evaluate a function at date t_{i+1},
- *p_pOptimize* the optimizer object

Remark 12 *A similar object is available without the MPI distribution framework "TransitionStepDP" with still enabling parallelization with threads and MPI on the calculations on the full grid points.*

Remark 13 *The case of sparse grids in currently not treated in the framework.*

The main method is

```
std::pair< std::shared_ptr< std::vector< Eigen::ArrayXXd > > , std::
    shared_ptr< std::vector< Eigen::ArrayXXd  > > >  oneStep(const std::
    vector< Eigen::ArrayXXd  > &p_phiIn)
```

with

- *p_phiIn* the vector (its size corresponds to the number of regimes) of matrix of optimal values calculated regressed at the previous time iteration for each regime . Each matrix is a number of function regressor at the previous date by number of stock points matrix.

returning a pair :

- first element is a vector of matrix with new optimal values regressed at the current time step (each element of the vector corresponds to a regime and each matrix is a number of regressed functions at the current date by the number of stock points matrix).
- second element is a vector of matrix with new optimal regressed controls at the current time step (each element of the vector corresponds to a control and each matrix is a number of regressed controls by the number of stock points matrix).

Remark 14 *All "TransitionStepDP" derive from a "TransitionStepBase" object having a pure virtual "OneStep" method.*

A second method is provided permitting to dump the the optimal control at each time step:

```
    void dumpValues(std::shared_ptr<gs::BinaryFileArchive> p_ar ,
                    const std::string &p_name, const int &p_iStep,
                    const std::vector< Eigen::ArrayXXd > &p_control, const
                        bool &p_bOneFile) const
```

with :

- *p_ar* is the archive where controls and solutions are dumped,
- *p_name* is a base name used in the archive to store the solution and the control,
- *p_control* stores the optimal controls calculated at the current time step,
- *p_bOneFile* is set to one if the optimal controls calculated by each processor are dumped on a single file. Otherwise the optimal controls calculated by each processor are dumped on different files (one by processor). If the problem gives optimal control values on the global grid that can be stored in the memory of the computation node, it can be more interesting to dump the control values in one file for the simulation of the optimal policy.

Remark 15 *As for the "TransitionStepDP", its "dumpValues" doesn't need a p_bOneFile argument: obviously optimal controls are stored in a single file.*

We give here a simple example of a time resolution using this method when the MPI distribution of data is used

```
// Copyright (C) 2016 EDF
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// This code is published under the GNU Lesser General Public License (GNU
    LGPL)
#ifdef USE_MPI
#include <fstream>
#include <boost/mpi.hpp>
#include <memory>
#include <functional>
#include <boost/lexical_cast.hpp>
#include <Eigen/Dense>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/core/grids/FullGrid.h"
#include "StOpt/regression/LocalConstRegression.h"
#include "StOpt/regression/GridAndRegressedValue.h"
#include "StOpt/dp/FinalStepRegressionDPDist.h"
#include "StOpt/dp/TransitionStepDPDist.h"
#include "StOpt/core/parallelism/reconstructProc0Mpi.h"
#include "test/c++/tools/dp/OptimizePortfolioDP.h"

using namespace std;
using namespace Eigen;

double DynamicProgrammingPortfolioDist(const shared_ptr<StOpt::FullGrid> &
    p_grid,
```

```
24                                          const shared_ptr<OptimizePortfolioDP>
                                                &p_optimize,
25                                          const ArrayXi &p_nbMesh,
26                                          const function<double(const int &,
                                                const ArrayXd &, const ArrayXd &)>
                                                 &p_funcFinalValue,
27                                          const ArrayXd &p_initialPortfolio,
28                                          const string   &p_fileToDump,
29                                          const bool &p_bOneFile
30                                         )
31 {
32     // initialize simulation
33     p_optimize->initializeSimulation();
34     // store regressor
35     shared_ptr<StOpt::LocalConstRegression> regressorPrevious;
36 
37     // store final regressed values in  object valuesStored
38     shared_ptr< vector< ArrayXXd > > valuesStored = make_shared< vector<
           ArrayXXd> >(p_optimize->getNbRegime());
39     {
40         vector< shared_ptr< ArrayXXd > >  valuesPrevious = StOpt::
               FinalStepRegressionDPDist(p_grid, p_optimize->getNbRegime(),
               p_optimize->getDimensionToSplit())(p_funcFinalValue, *p_optimize->
               getCurrentSim());
41         // regressor operator
42         regressorPrevious = make_shared<StOpt::LocalConstRegression>(false,
               p_optimize->getCurrentSim(), p_nbMesh);
43         for (int iReg = 0; iReg < p_optimize->getNbRegime(); ++iReg)
44             (*valuesStored)[iReg] = regressorPrevious->
                   getCoordBasisFunctionMultiple(valuesPrevious[iReg]->transpose
                   ()).transpose();
45     }
46     boost::mpi::communicator world;
47     string toDump = p_fileToDump ;
48     // test if one file generated
49     if (!p_bOneFile)
50         toDump +=  "_" + boost::lexical_cast<string>(world.rank());
51     shared_ptr<gs::BinaryFileArchive> ar;
52     if ((!p_bOneFile) || (world.rank() == 0))
53         ar = make_shared<gs::BinaryFileArchive>(toDump.c_str(), "w");
54     // name for object in archive
55     string nameAr = "OptimizePort";
56     // iterate on time steps
57     for (int iStep = 0; iStep < p_optimize->getNbStep(); ++iStep)
58     {
59         // step backward for simulations
60         p_optimize->oneStepBackward();
61         // create regressor at the given date
62         bool bZeroDate = (iStep == p_optimize->getNbStep() - 1);
63         shared_ptr<StOpt::LocalConstRegression> regressorCur = make_shared<
               StOpt::LocalConstRegression>(bZeroDate, p_optimize->getCurrentSim
               (), p_nbMesh);
64         // transition object
65         StOpt::TransitionStepDPDist transStep(p_grid, p_grid, regressorCur,
```

```
            regressorPrevious, p_optimize);
66          pair< shared_ptr< vector< ArrayXXd> >, shared_ptr< vector< ArrayXXd >
                > > valuesAndControl = transStep.oneStep(*valuesStored);
67          // dump control values
68          transStep.dumpValues(ar, nameAr, iStep,  *valuesAndControl.second,
                p_bOneFile);
69          valuesStored = valuesAndControl.first;
70          // shift regressor
71          regressorPrevious = regressorCur;
72      }
73      // interpolate at the initial stock point and initial regime( 0 here)  (
            take first particle)
74      shared_ptr<ArrayXXd> topRows = make_shared<ArrayXXd>((*valuesStored)[0].
            topRows(1));
75      return StOpt::reconstructProc0Mpi(p_initialPortfolio, p_grid, topRows,
            p_optimize->getDimensionToSplit());
76  }
77  #endif
```

An example without distribution of the data can be found in this file.

5.3.3 The framework in simulation

Not special framework is available in simulation. Use the function "SimulateStepRegressionControl" or "SimulateStepRegressionControlDist" described in the section 5.2.3.

Chapter 6

The Python API

6.1 Mapping to the framework

In order to use the Python API, it is possible to use only the mapping of the grids, continuation values, and regression object and to program an equivalent of “TransitionStepRegressionDP” and of “ SimulateStepRegression”, “ SimulateStepRegressionControl” in python. No mapping is currently available for “TransitionStepDP”. An example using python is given by

```
1  # Copyright (C) 2016 EDF
2  # All Rights Reserved
3  # This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  import numpy as np
5  import StOptReg as reg
6
7  class TransitionStepRegressionDP:
8
9      def __init__(self, p_pGridCurrent, p_pGridPrevious, p_pOptimize):
10
11         self.m_pGridCurrent = p_pGridCurrent
12         self.m_pGridPrevious = p_pGridPrevious
13         self.m_pOptimize = p_pOptimize
14
15     def oneStep(self, p_phiIn, p_condExp):
16
17         nbRegimes = self.m_pOptimize.getNbRegime()
18         phiOut = list(range(nbRegimes))
19         nbControl = self.m_pOptimize.getNbControl()
20         controlOut = list(range(nbControl))
21
22         # only if the processor is working
23         if self.m_pGridCurrent.getNbPoints() > 0:
24
25             # allocate for solution
26             for iReg in range(nbRegimes):
27                 phiOut[iReg] = np.zeros((p_condExp.getNbSimul(), self.
                       m_pGridCurrent.getNbPoints()))
28
```

```
            for iCont in range(nbControl):
                controlOut[iCont] = np.zeros((p_condExp.getNbSimul(), self.
                    m_pGridCurrent.getNbPoints()))

            # number of threads
            nbThreads = 1

            contVal = []

            for iReg in range(len(p_phiIn)):
                contVal.append(reg.ContinuationValue(self.m_pGridPrevious,
                    p_condExp, p_phiIn[iReg]))

            # create iterator on current grid treated for processor
            iterGridPoint = self.m_pGridCurrent.getGridIteratorInc(0)

            # iterates on points of the grid
            for iIter in range(self.m_pGridCurrent.getNbPoints()):

                if iterGridPoint.isValid():
                    pointCoord = iterGridPoint.getCoordinate()
                    # optimize the current point and the set of regimes
                    solutionAndControl = self.m_pOptimize.stepOptimize(self.
                        m_pGridPrevious, pointCoord, contVal, p_phiIn)

                    # copy solution
                    for iReg in range(self.m_pOptimize.getNbRegime()):
                        phiOut[iReg][:,iterGridPoint.getCount()] =
                            solutionAndControl[0][:,iReg]

                    for iCont in range(nbControl):
                        controlOut[iCont][:,iterGridPoint.getCount()] =
                            solutionAndControl[1][:,iCont]

                    iterGridPoint.nextInc(nbThreads)

        res = []
        res.append(phiOut)
        res.append(controlOut)
        return res
```

This object can be used as in a time step optimization as follows

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU
    LGPL)
import StOptReg
import StOptGeners
import TransitionStepRegressionDP as trans
import FinalStepRegressionDP as final


def DynamicProgrammingByRegression(p_grid, p_optimize, p_regressor,
    p_funcFinalValue, p_pointStock, p_initialRegime, p_fileToDump, key="
```

```
    Continuation"):

    # from the optimizer get back the simulation
    simulator = p_optimize.getSimulator()
    # final values
    valuesNext = final.FinalStepRegressionDP(p_grid, p_optimize.getNbRegime()
        ).operator(p_funcFinalValue, simulator.getParticles())

    archiveToWrite = StOptGeners.BinaryFileArchive(p_fileToDump, "w")
    nsteps = simulator.getNbStep()
    # iterate on time steps
    for iStep in range(nsteps):
        asset = simulator.stepBackwardAndGetParticles()

        # conditional expectation operator
        if iStep == (simulator.getNbStep() - 1):
            p_regressor.updateSimulations(True, asset)
        else:
            p_regressor.updateSimulations(False, asset)

        # transition object
        transStep = trans.TransitionStepRegressionDP(p_grid, p_grid,
            p_optimize)
        valuesAndControl = transStep.oneStep(valuesNext, p_regressor)
        valuesNext = valuesAndControl[0]

        # Dump the continuation values in the archive:
        archiveToWrite.dumpGridAndRegressedValue(key, nsteps - 1 - iStep,
            valuesNext, p_regressor, p_grid)

    # interpolate at the initial stock point and initial regime
    return (p_grid.createInterpolator(p_pointStock).applyVec(valuesNext[
        p_initialRegime])).mean()
```

Some examples are available in the test directory (for example for swing options).
Another approach more effective in term of computational cost consists in mapping the simulator object derived from the SimulatorDPBase object and optimizer object derived from the OptimizerDPBase object and to use the high level python mapping of and "SimulateStepRegression". In the test part of the library some Black-Scholes simulator and some Mean reverting simulator for a future curve deformation are developed and some examples of the mapping are achieved in the BoostPythonSimulators.cpp file. Similarly the optimizer class for swings options, optimizer for a fictitious swing in dimension 2, optimizer for a gas storage, optimizer for a gas storage with switching cost are mapped to python in the BoostPythonOptimizers.cpp file.
In the example below we describe the use of this high level interface for the swing options with a Black Scholes simulator : we give in this example the mapping of the mostly used objects:

```
# Copyright (C) 2016 EDF
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU
    LGPL)
import math
```

```
import numpy as np
import unittest
import StOptGrids
import StOptReg
import StOptGlobal
import Utils
import Simulators as sim
import Optimizers as opt

# unit test for global shape
################################

class OptimizerConstruction(unittest.TestCase):

    def test(self):
        try:
            imp.find_module('mpi4py')
            found =True
        except:
            print("Not parallel module found ")
            found = False

        if found :
            from mpi4py import MPI
            comm = MPI.COMM_WORLD
            initialValues = np.zeros(1,dtype=np.float) + 1.
            sigma = np.zeros(1.) + 0.2
            mu = np.zeros(1.) + 0.05
            corr = np.ones((1.,1.),dtype=np.float)
            # number of step
            nStep = 30
            # exercise dates
            dates = np.linspace(0., 1., nStep + 1)
            T= dates[len(dates) - 1]
            nbSimul = 10 # simulation number (optimization and simulation)
            # simulator
            ############
            bsSim = sim.BlackScholesSimulator(initialValues, sigma, mu, corr,
                T, len(dates) - 1, nbSimul, False)
            strike = 1.
            # Pay off
            payOff= Utils.BasketCall(strike)
            # optimizer
            ############
            N = 3   # number of exercise dates
            swiOpt = opt.OptimizerSwingBlackScholes(payOff,N)
            # link simulator to optimizer
            swiOpt.setSimulator(bsSim)
            # archive
            #########
            ar = StOptGlobal.BinaryFileArchive("Archive","w")
            # regressor
            ############
            nMesh = 1
```

```
58              regressor = StOptReg.LocalLinearRegression(nMesh)
59              # Grid
60              ######
61              # low value for the meshes
62              lowValues =np.array([0.],dtype=np.float)
63              # size of the meshes
64              step = np.array([1.],dtype=np.float)
65              # number of steps
66              nbStep = np.array([N-1], dtype=np.int32)
67              gridArrival =  StOptGrids.RegularSpaceGrid(lowValues,step,nbStep)
68              gridStart   =  StOptGrids.RegularSpaceGrid(lowValues,step,nbStep
                    -1)
69              # pay off function for swing
70              ##############################
71              payOffBasket = Utils.BasketCall(strike);
72              payoff = Utils.PayOffSwing(payOffBasket,N)
73              dir(payoff)
74              print("payoff" , payoff.set(0,np.array([0.5],dtype=np.float), np.
                    array([1.],dtype=np.float)))
75              # final step
76              #############
77              asset =bsSim.getParticles()
78              fin = StOptGlobal.FinalStepRegressionDP(gridArrival,1)
79              values = fin.set( payoff,asset)
80              # transition time step
81              ######################
82              # on step backward and get  asset
83              asset = bsSim.stepBackwardAndGetParticles()
84              # update regressor
85              regressor.updateSimulations(0,asset)
86              transStep = StOptGlobal.TransitionStepRegressionDP(gridStart,
                    gridArrival,swiOpt)
87              valuesNextAndControl=transStep.oneStep(values,regressor)
88              transStep.dumpContinuationValues(ar,"Continuation",1,
                    valuesNextAndControl[0],valuesNextAndControl[1],regressor)
89              # simulate time step
90              ####################
91              nbSimul= 10
92              vecOfStates =[] # state of each simulation
93              for i in np.arange(nbSimul):
94                  # one regime, all with same stock level (dimension 2), same
                        realization of simulation (dimension 3)
95                  vecOfStates.append(StOptGlobal.StateWithStocks(1, np.array
                        ([0.]) , np.zeros(1)))
96              arRead = StOptGlobal.BinaryFileArchive("Archive","r")
97              simStep = StOptGlobal.SimulateStepRegression(arRead,1,"
                    Continuation",grid,swiOpt)
98              phi = np.zeros((1,nbSimul))
99              NewState = VecOfStateNext = simStep.oneStep(vecOfStates, phi)
100             # print new state (different of C++)
101             print("New vector of state", NewState[0])
102             for i in np.arange(len(NewState[0])):
103                 print(" i" , i , " Stock " , NewState[0][i].getPtStock(), "
                        Regime " ,  NewState[0][i].getRegime(), " Stochastic
```

```
                    realization" , NewState[0][i].getStochasticRealization())
            print("New cost function", NewState[1])

if __name__ == '__main__':
    unittest.main()
```

Its declination in term of a time nest for optimization is given below (please notice that the "TransitionStepRegressionDP" object is the result of the mapping between python and c++ and given in the "StOptGlobal" module)

```
# Copyright (C) 2016 EDF
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# This code is published under the GNU Lesser General Public License (GNU LGPL)
import StOptGrids
import StOptReg
import StOptGlobal
import StOptGeners


def DynamicProgrammingByRegressionHighLevel(p_grid, p_optimize, p_regressor, p_funcFinalValue, p_pointStock, p_initialRegime, p_fileToDump) :

    # from the optimizer get back the simulation
    simulator = p_optimize.getSimulator()
    # final values
    fin = StOptGlobal.FinalStepRegressionDP(p_grid, p_optimize.getNbRegime())
    valuesNext = fin.set(p_funcFinalValue, simulator.getParticles())
    ar = StOptGeners.BinaryFileArchive(p_fileToDump, "w")
    nameAr = "Continuation"
    # iterate on time steps
    for iStep in range(simulator.getNbStep()) :
        asset = simulator.stepBackwardAndGetParticles()
        # conditional expectation operator
        if iStep == (simulator.getNbStep() - 1):
            p_regressor.updateSimulations(True, asset)
        else:
            p_regressor.updateSimulations(False, asset)

        # transition object
        transStep = StOptGlobal.TransitionStepRegressionDP(p_grid, p_grid, p_optimize)
        valuesAndControl = transStep.oneStep(valuesNext, p_regressor)
        transStep.dumpContinuationValues(ar, nameAr, iStep, valuesNext, valuesAndControl[1], p_regressor)
        valuesNext = valuesAndControl[0]

    # interpolate at the initial stock point and initial regime
    return (p_grid.createInterpolator(p_pointStock).applyVec(valuesNext[p_initialRegime])).mean()
```

Similarly a python time nest in simulation using the control previously calculated in optimization can be given as an example by :

```
# Copyright (C) 2016 EDF
```

```
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
    LGPL)
4 import numpy as np
5 import StOptReg as reg
6 import StOptGrids
7 import StOptGeners
8 import StOptGlobal
9
10
11 # Simulate the optimal strategy , threaded version
12 # p_grid                    grid used for  deterministic state (stocks for
    example)
13 # p_optimize                optimizer defining the optimization between two
    time steps
14 # p_funcFinalValue          function defining the final value
15 # p_pointStock              initial point stock
16 # p_initialRegime           regime at initial date
17 # p_fileToDump              name of the file used to dump continuation values
    in optimization
18 def SimulateRegressionControl(p_grid, p_optimize, p_funcFinalValue,
    p_pointStock, p_initialRegime, p_fileToDump) :
19
20     simulator = p_optimize.getSimulator()
21     nbStep = simulator.getNbStep()
22     states = []
23
24     for i in range(simulator.getNbSimul()) :
25         states.append(StOptGlobal.StateWithStocks(p_initialRegime,
               p_pointStock, np.zeros(simulator.getDimension())))
26
27     ar = StOptGeners.BinaryFileArchive(p_fileToDump, "r")
28     # name for continuation object in archive
29     nameAr = "Continuation"
30     # cost function
31     costFunction = np.zeros((p_optimize.getSimuFuncSize(), simulator.
           getNbSimul()))
32
33     # iterate on time steps
34     for istep in range(nbStep) :
35         NewState = StOptGlobal.SimulateStepRegressionControl(ar, nbStep - 1 -
               istep, nameAr, p_grid, p_optimize).oneStep(states, costFunction)
36         # different from C++
37         states = NewState[0]
38         costFunction = NewState[1]
39         # new stochastic state
40         particles = simulator.stepForwardAndGetParticles()
41
42         for i in range(simulator.getNbSimul()) :
43             states[i].setStochasticRealization(particles[:,i])
44
45     # final : accept to exercise if not already done entirely
46     for i in range(simulator.getNbSimul()) :
47         costFunction[0,i] += p_funcFinalValue.set(states[i].getRegime(),
```

```
            states[i].getPtStock(), states[i].getStochasticRealization()) *
            simulator.getActu()
48
49      # average gain/cost
50      return costFunction.mean()
```

Equivalent using MPI and the distribution of calculations and data can be used using the "mpi4py" package. An example of its use can be found in the MPI version of a swing optimization and valorization.

6.2 Special python binding

Some specific features have been added to the python interface to increase the flexibility of the library. A special mapping of the geners library has been achieved for some specific needs.

6.2.1 A first binding to use the framework

The "BinaryFileArchive" in the python module "StOptGeners" permits for:

- a grid on point,
- a list of numpy array (dimension 2) of size the number of simulations used by the number of points on the grid (the size of the list corresponds to the number of regimes used in case of a regime switching problem : if one regime, this list contains only one item which is a two dimensional array)
- a regressor

to create a set of regressed values of the numpy arrays values and store them in the archive. This functionality permits to store the continuation values associated to a problem.
The dump method "dumpGridAndRegressedValue" in the "BinaryFileArchive" class permits this dump.
It is also possible to get back the continuation values obtained using the "readGridAndRegressedValue" method.

```
1  # Copyright (C) 2016 EDF
2  # All Rights Reserved
3  # This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  import numpy as np
5  import unittest
6  import random
7  import StOptGrids
8  import StOptReg
9  import StOptGeners
10
11
12 # unit test for dumping binary archive of regressed value and Read then
13 ##########################################################################
```

```
14
15 class testBinaryArchiveStOpt(unittest.TestCase):
16
17       def testSimpleStorageAndLecture(self):
18
19         # low value for the mesh
20         lowValues =np.array([1.,2.,3.],dtype=np.float)
21         # size of the mesh
22         step = np.array([0.7,2.3,1.9],dtype=np.float)
23         # number of step
24         nbStep = np.array([4,5,6], dtype=np.int32)
25         # degree of the polynomial in each direction
26         degree =  np.array([2,1,3], dtype=np.int32)
27         # create the Legendre grid
28         grid = StOptGrids.RegularLegendreGrid(lowValues,step,nbStep,degree )
29
30
31         # simulate the perburbed values
32         ###################################
33         nbSimul =40000
34         np.random.seed(1000)
35         x = np.random.uniform(-1.,1.,size=(1,nbSimul));
36         # mesh
37         nbMesh = np.array([16],dtype=np.int32)
38         # Create the regressor
39         ########################
40         regressor = StOptReg.LocalLinearRegression(False,x,nbMesh)
41
42         # regressed values same values for each point of the grid
43         ##########################################################################
44         toReal = (2+x[0,:]+(1+x[0,:])*(1+x[0,:]))
45         # function to regress
46         toRegress = toReal + 4*np.random.normal(0.,1,nbSimul)
47         # create a matrix (number of stock points by number of simulations)
48         toRegressMult = np.zeros(shape=(len(toRegress),grid.getNbPoints()))
49         for i in range(toRegressMult.shape[1]):
50            toRegressMult[:,i] = toRegress
51         # into a list  : two times to test 2 regimes
52         listToReg = []
53         listToReg.append(toRegressMult)
54         listToReg.append(toRegressMult)
55
56
57         # Create the binary archive to dump
58         ###########################################
59         archiveToWrite = StOptGeners.BinaryFileArchive("MyArchive","w")
60         # step 1
61         archiveToWrite.dumpGridAndRegressedValue("toStore", 1,listToReg,
               regressor,grid)
62         # step 3
63         archiveToWrite.dumpGridAndRegressedValue("toStore", 3,listToReg,
               regressor,grid)
64
65
```

```
        # Read the regressed values
        ############################
        archiveToRead =  StOptGeners.BinaryFileArchive("MyArchive","r")
        contValues = archiveToRead.readGridAndRegressedValue(3,"toStore")


        # list of 2 continuation values
        ################################
        stockPoint = np.array([1.5,3.,5.])
        uncertainty = np.array([0.])
        value =contValues[0].getValue(stockPoint,uncertainty)
        print "Value found", value



if __name__ == '__main__':
    unittest.main()
```

6.2.2 Binding to store/read a regressor and some two dimensional array

Sometimes, users prefer to avoid the use of the framework provided and prefer to only use the python binding associated to the regression methods. When some regressions are achieved for different set of particules (meaning that one or more functions are regressed), it it possible to dump the regressor used and some values associated to these regressions :

- the "dumpSome2DArray", "readSome2DArray" permits to dump and read 2 dimensional numpy arrays,
- the "dumpSomeRegressor" , "readSomeRegressor" permits to dump and read a regressor.

```
# Copyright (C) 2017 EDF
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import numpy as np
import StOptReg
import StOptGeners

# unit test to show how to store some regression object and basis function coefficients associated
#############################################################################################


def  createData():

    X1=np.arange(0.0  ,  2.2  ,  0.01  )
    X2=np.arange(0.0  ,  1.1  ,  0.005  )
    Y=np.zeros((len(X1),len(X2)))
```

```
    for i in range(len(X1)):
        for j in range(len(X2)):
            if i < len(X1)//2:
                if j < len(X2)//2:
                    Y[i,j]=X1[i]+X2[j]
                else:
                   Y[i,j]=4*X1[i]+4*X2[j]
            else:
                if j < len(X2)//2:
                    Y[i,j]=2*X1[i]+X2[j]
                else:
                    Y[i,j]=2*X1[i]+3*X2[j]

    XX1, XX2 = np.meshgrid(X1,X2)
    Y=Y.T

    r,c = XX1.shape

    X =  np.reshape(XX1,(r*c,1))[:,0]
    I =  np.reshape(XX2,(r*c,1))[:,0]
    Y =  np.reshape(Y,(r*c,1))[:,0]

    xMatrix = np.zeros((2,len(X)))
    xMatrix[0,:] = X
    xMatrix[1,:] = I

    return xMatrix, Y




# main

xMatrix, y = createData()

# 2 dimensional regression 2 by 2 meshes
nbMesh = np.array([2,2],dtype=np.int32)
regressor = StOptReg.LocalLinearRegression(False,xMatrix,nbMesh)

# coefficients
coeff = regressor.getCoordBasisFunction(y)

print("Regressed coeff", coeff)


# store them in a matrix
coeffList = np.zeros(shape=(1,3*2*2))
coeffList[0,:]=coeff.transpose()


# archive write for  regressors
archiveWriteForRegressor =StOptGeners.BinaryFileArchive("archive","w")

# store
```

```
step =1
archiveWriteForRegressor.dumpSome2DArray("RegCoeff",step,coeff)
archiveWriteForRegressor.dumpSomeRegressor("Regressor",step,regressor)

# archive Read for  regressors
archiveReadForRegressor =StOptGeners.BinaryFileArchive("archive","r")

# get back
values = archiveReadForRegressor.readSome2DArray("RegCoeff",step)
reg = archiveReadForRegressor.readSomeRegressor("Regressor",step)
print("Regressed coeff ", values)
print("Reg",reg)

print "newValue", reg.getValue([0.4,0.6],values)
```

Part IV

Semi Lagrangian methods

For the Semi Lagrangian methods the C++ API is the only one available (no python API is currently developed).

Chapter 7

Theoretical background

In this part, we are back to the resolution of equation (1).

7.1 Notation and regularity results

We denote by $\wedge$ the minimum and $\vee$ the maximum. We denote by $|\quad|$ the Euclidean norm of a vector, $Q := (0,T] \times \mathbb{R}^d$. For a bounded function w, we set

$$|w|_0 = \sup_{(t,x)\in Q} |w(t,x)|, \qquad [w]_1 = \sup_{(s,x)\neq(t,y)} \frac{|w(s,x) - w(t,y)|}{|x-y| + |t-s|^{\frac{1}{2}}}$$

and $|w|_1 = |w|_0 + [w]_1$. $C_1(Q)$ will stand for the space of functions with a finite $|\quad|_1$ norm. For t given, we denote

$$||w(t,.)||_\infty = \sup_{x\in\mathbb{R}^d} |w(t,x)|$$

We use the classical assumption on the data of (1) for a given $\hat{K}$:

$$\sup_a |g|_1 + |\sigma_a|_1 + |b_a|_1 + |f_a|_1 + |c_a|_1 \leq \hat{K} \tag{7.1}$$

A classical result [2] gives us the existence and uniqueness of the solution in the space of bounded Lipschitz functions:

Proposition 1 *If the coefficients of the equation* (1) *satisfy* (7.1)*, there exists a unique viscosity solution of the equation* (1) *belonging to* $C_1(Q)$*. If* u_1 *and* u_2 *are respectively sub and super solution of equation* (1) *satisfying* $u_1(0,.) \leq u_2(0,.)$ *then* $u_1 \leq u_2$*.*

A spatial discretization length of the problem Δx being given, thereafter $(i_1\Delta x, .., i_d\Delta x)$ with $\bar{i} = (i_1, ..., i_d) \in \mathbf{Z}^d$ will correspond to the coordinates of a mesh $M_{\bar{i}}$ defining a hyper-cube in dimension d. For an interpolation grid $(\xi_i)_{i=0,..N} \in [-1,1]^N$, and for a mesh $\bar{i}$, the point $y_{\bar{i},\tilde{j}}$ with $\tilde{j} = (j_1, .., j_d) \in [0,N]^d$ will have the coordinate $(\Delta x(i_1 + 0.5(1+\xi_{j_1})), .., \Delta x(i_d + 0.5(1+\xi_{j_d}))$. We denote $(y_{\bar{i},\tilde{j}})_{\bar{i},\tilde{j}}$ the set of all the grids points on the whole domain.
We notice that for regular mesh with constant volume Δx^d, we have the following relation for all $x \in \mathbb{R}^d$:

$$\min_{\bar{i},\tilde{j}} |x - y_{\bar{i},\tilde{j}}| \leq \Delta x. \tag{7.2}$$

7.2 Time discretization for HJB equation

The equation (1) is discretized in time by the scheme proposed by Camilli Falcone [22] for a time discretization h.

$$\begin{aligned} v_h(t+h,x) &= \inf_{a\in A}\left[\sum_{i=1}^{q}\frac{1}{2q}(v_h(t,\phi^+_{a,h,i}(t,x))+v_h(t,\phi^-_{a,h,i}(t,x))) \right. \\ &\quad \left. +f_a(t,x)h+c_a(t,x)hv_h(t,x)\right] \\ &:= v_h(t,x)+\inf_{a\in A}L_{a,h}(v_h)(t,x) \end{aligned} \tag{7.3}$$

with

$$\begin{aligned} L_{a,h}(v_h)(t,x) &= \sum_{i=1}^{q}\frac{1}{2q}(v_h(t,\phi^+_{a,h,i}(t,x))+v_h(t,\phi^-_{a,h,i}(t,x))-2v_h(t,x)) \\ &\quad +hc_a(t,x)v_h(t,x)+hf_a(t,x) \\ \phi^+_{a,h,i}(t,x) &= x+b_a(t,x)h+(\sigma_a)_i(t,x)\sqrt{hq} \\ \phi^-_{a,h,i}(t,x) &= x+b_a(t,x)h-(\sigma_a)_i(t,x)\sqrt{hq} \end{aligned}$$

where $(\sigma_a)_i$ is the i-th column of σ_a. We note that it is also possible to choose other types of discretization in the same style as those defined in [23].
In order to define the solution at each date, a condition on the value chosen for v_h between 0 and h is required. We choose a time linear interpolation once the solution has been calculated at date h:

$$v_h(t,x)=(1-\frac{t}{h})g(x)+\frac{t}{h}v_h(h,x),\forall t\in[0,h]. \tag{7.4}$$

We first recall the following result :

Proposition 2 *Under the condition on the coefficients given by equation* (7.1)*, the solution* v_h *of equations* (7.3) *and* (7.4) *is uniquely defined and belongs to* $C_1(Q)$*. We check that if* $h\leq(16\sup_a\{|\sigma_a|_1^2+|b_a|_1^2+1\}\wedge 2\sup_a|c_a|_0)^{-1}$*, there exists* C *such that*

$$|v-v_h|_0\leq Ch^{\frac{1}{4}}. \tag{7.5}$$

Moreover, there exists C *independent of* h *such that*

$$|v_h|_0 \leq C, \tag{7.6}$$

$$|v_h(t,x)-v_h(t,y)| \leq C|x-y|,\forall(x,y)\in Q^2. \tag{7.7}$$

7.3 Space interpolation

The space resolution of equation (7.3) is a achieved on a grid. The ϕ^+ and ϕ^- have to be computed by the use of an interpolator I such that:

$$\begin{aligned} v_h(t,\phi^+_{a,h,i}(t,x)) &\simeq I(v_h(t,.))(\phi^+_{a,h,i}(t,x)), \\ v_h(t,\phi^-_{a,h,i}(t,x)) &\simeq I(v_h(t,.))(\phi^-_{a,h,i}(t,x)). \end{aligned}$$

In order to easily prove the convergence of the scheme to the viscosity solution of the problem, the monotony of the scheme is generally required leading to some linear interpolator slowly converging. An adaptation to high order interpolator where the function is smooth can be achieved using Legendre grids and Sparse grids with some truncation (see [24], [25]).

Chapter 8

C++ API

In order to achieve the interpolation and calculate the semi Lagrangian value

$$\sum_{i=1}^{q} \frac{1}{2q}(v_h(t,\phi^+_{a,h,i}(t,x)) + v_h(t,\phi^-_{a,h,i}(t,x))$$

a first object "SemiLagrangEspCond" is available:

```
1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #ifndef SEMILAGRANGESPCOND_H
5  #define SEMILAGRANGESPCOND_H
6  #include <Eigen/Dense>
7  #include <map>
8  #include <array>
9  #include <vector>
10 #include "StOpt/core/utils/constant.h"
11 #include "StOpt/core/grids/InterpolatorSpectral.h"
12
13 /** \file SemiLagrangEspCond.h
14  *  \brief Semi Lagrangian method for process \f$ d x_t = b dt + \sigma dW_t
        \f$
15  *  where \f$ X_t, b \f$ with values in \f$ {\mathbb R}^n \f$ , \f$ \sigma
        \f$ a \f$ \mathbf{R}^n
16  *  \times \mathbf{R}^m \f$ matrix and \f$ W_t \f$ with values in \f$ \
        mathbf{R}^m \f$
17  */
18
19 namespace StOpt
20 {
21
22 /// \class SemiLagrangEspCond SemiLagrangEspCond.h
23 /// calculate semi Lagrangian operator for previously defined process.
24 class SemiLagrangEspCond
25 {
26     ///\brief interpolator
27     std::shared_ptr<InterpolatorSpectral> m_interpolator;
28
```

```
29      /// \brief store extremal values for the grid (min, max coordinates in
            each dimension)
30      std::vector <std::array< double, 2> > m_extremalValues;
31
32      /// \brief Do we use modification of volatility  to stay in the domain
33      bool m_bModifVol ;
34
35  public :
36
37      /// \brief Constructor
38      /// \param  p_interpolator    Interpolator storing the grid
39      /// \param  p_extremalValues  Extremal values of the grid
40      /// \param  p_bModifVol        do we modify volatility to stay in the
            domain
41      SemiLagrangEspCond(const std::shared_ptr<InterpolatorSpectral> &
            p_interpolator, const std::vector <std::array< double, 2> > &
            p_extremalValues, const bool &p_bModifVol);
42
43      /// \brief Calculate \f$ \frac{1}{2d} \sum_{i=1}^d \phi(x+ b dt + \
            sigma_i \sqrt{dt})+  \phi(x+ b dt - \sigma_i \sqrt{dt} \f$
44      ///        where \f$ \sigma_i \f$ is column \f$ i\f$ of \f$ \sigma \f$
45      /// \param p_x                 beginning point
46      /// \param p_b                 trend
47      /// \param p_sig               volatility matrix
48      /// \param  p_dt               Time step size
49      /// \return  (the value calculated,true) if point inside the domain,
            otherwise (0., false)
50      std::pair<double, bool>  oneStep(const Eigen::ArrayXd  &p_x, const Eigen
            ::ArrayXd &p_b, const Eigen::ArrayXXd &p_sig, const double &p_dt)
            const;
51
52
53  };
54  }
55  #endif
```

Its constructor uses the following arguments :

- a first one "p_interpolator" defines a "spectral" interpolator on a grid : this "spectral" interpolator is constructed from a grid and a function to interpolate (see section 1). In our case, it will be used to interpolate the solution from the previous time step,
- a second one "p_extremalValues" defines for each dimension the minimal and maximal coordinates of points belonging to the grid,
- a third one "p_bModifVol" if set to "true" permits to achieve a special treatment when points to interpolate are outside the grid : the volatility of the underlying process is modified (keeping the same mean and variance) trying to keep points inside the domain (see [24]).

This object has the method "oneStep" taking

- p_x the foot of the characterize (for each dimension),

- p_b the trend of the process (for each dimension),
- p_sig the matrix volatility of the process,

such that the interpolation is achieved for a time step h at points $p\_x + p\_bh \pm p\_sig\sqrt{h}$. It returns a pair (a, b) where a contains the calculated value if the b value is true. When the interpolation is impossible to achieve, the b value is set to false.

In order to use the API, an object deriving from the "OptimizerSLBase.h" object has to be constructed. This object permits to define the PDE to solve (with it optimization problem if any).

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifndef OPTIMIZERSLBASE_H
#define OPTIMIZERSLBASE_H
#include <vector>
#include <Eigen/Dense>
#include "StOpt/core/grids/SpaceGrid.h"
#include "StOpt/core/grids/FullGrid.h"
#include "StOpt/core/grids/InterpolatorSpectral.h"
#include "StOpt/semilagrangien/SemiLagrangEspCond.h"

/** \file OptimizerSLBase.h
 *  \brief Define an abstract class for Dynamic Programming problems
 *     \author Xavier Warin
 */

namespace StOpt
{

/// \class OptimizerSLBase OptimizerSLBase.h
///  Base class for optimizer for resolution by semi Lagrangian methods of HJB equations
class OptimizerSLBase
{


public :

    OptimizerSLBase() {}

    virtual ~OptimizerSLBase() {}


    /// \brief define the diffusion cone for parallelism
    /// \param  p_regionByProcessor         region (min max) treated by the processor for the different regimes treated
    /// \return returns in each dimension the min max values in the stock that can be reached from the grid p_gridByProcessor for each regime
    virtual std::vector< std::array< double, 2> > getCone(const  std::vector< std::array< double, 2>  > &p_regionByProcessor) const = 0;
```

```
38
39      /// \brief defines the dimension to split for MPI parallelism
40      ///        For each dimension return true is the direction can be split
41      virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
            const = 0 ;
42
43      /// \brief defines a step in optimization
44      /// \param p_point     coordinates of the point to treat
45      /// \param p_semiLag   semi Lagrangian operator for each regime for
            solution at the previous step
46      /// \param p_time      current date
47      /// \param p_phiInPt   value of the function at the previous time step at
             p_point for each regime
48      /// \return a pair :
49      ///          - first an array of the solution (for each regime)
50      ///          - second an array of the optimal controls ( for each control
            )
51      virtual std::pair< Eigen::ArrayXd, Eigen::ArrayXd>  stepOptimize(const
            Eigen::ArrayXd   &p_point,
52              const std::vector< std::shared_ptr<SemiLagrangEspCond> > &
                    p_semiLag,
53              const double &p_time,
54              const Eigen::ArrayXd &p_phiInPt) const = 0;
55
56
57      /// \brief defines a step in simulation
58      /// \param p_gridNext      grid at the next step
59      /// \param p_semiLag       semi Lagrangian operator at the current step
            in each regime
60      /// \param p_state         state array (can be modified)
61      /// \param p_iReg          regime number
62      /// \param p_gaussian      unitary Gaussian realization
63      /// \param p_phiInPt       value of the function at the next time step at
             p_point for each regime
64      /// \param p_phiInOut      defines the value functions (modified) to
            follow
65      virtual void stepSimulate(const SpaceGrid   &p_gridNext,
66                                const  std::vector< std::shared_ptr< StOpt::
                                       SemiLagrangEspCond> > &p_semiLag,
67                                Eigen::Ref<Eigen::ArrayXd>  p_state,   int &
                                       p_iReg,
68                                const Eigen::ArrayXd &p_gaussian,
69                                const Eigen::ArrayXd &p_phiInPt,
70                                Eigen::Ref<Eigen::ArrayXd> p_phiInOut) const  =
                                       0 ;
71
72
73      /// \brief defines a step in simulation using the control calculated in
            optimization
74      /// \param p_gridNext      grid at the next step
75      /// \param p_controlInterp the optimal controls interpolator
76      /// \param p_state         state array (can be modified)
77      /// \param p_iReg          regime number
78      /// \param p_gaussian      unitary Gaussian realization
```

```
79      /// \param p_phiInOut       defines the value functions (modified) to
            follow
80      virtual void stepSimulateControl(const SpaceGrid    &p_gridNext,
81                                       const  std::vector< std::shared_ptr<
                                             InterpolatorSpectral>>   &
                                             p_controlInterp,
82                                       Eigen::Ref<Eigen::ArrayXd>  p_state,
                                             int &p_iReg,
83                                       const Eigen::ArrayXd &p_gaussian,
84                                       Eigen::Ref<Eigen::ArrayXd> p_phiInOut)
                                             const  = 0 ;
85
86      /// \brief get number of regimes
87      virtual   int getNbRegime() const = 0 ;
88
89      /// \brief get back the dimension of the control
90      virtual int getNbControl() const = 0 ;
91
92      /// \brief do we modify the volatility to stay in the domain
93      virtual  bool getBModifVol() const = 0 ;
94
95      /// \brief get the number of Brownians involved in semi Lagrangian for
            simulation
96      virtual int getBrownianNumber() const = 0 ;
97
98      /// \brief get size of the  function to follow in simulation
99      virtual int getSimuFuncSize() const = 0;
100
101     /// \brief Permit to deal with some boundary points that do not need
            boundary conditions
102     ///        Return false if all points on the boundary need some boundary
            conditions
103     /// \param  p_point  potentially on the  boundary
104     virtual bool isNotNeedingBC(const Eigen::ArrayXd &p_point)  const = 0;
105 };
106 }
107 #endif /* OPTIMIZERSLBASE_H */
```

The main methods associated to this object are :

- "stepOptimize" is use to calculate the solution of the PDE at one point.
  - It takes a point of the grid used *p_point*,
  - and apply the semi Lagrangian scheme *p_semiLag* at this point,
  - at a date given by *p_time*.

  It returns a pair containing:
  - the function value calculated at *p_point* for each regime,
  - the optimal control calculated at *p_point* for each control.
- "stepSimulate" is used when the PDE is associated to an optimization problem and we want to simulate an optimal policy using the function values calculated in the optimization part. The arguments are:

- $p\_gridNext$ defining the grid used at the following time step,
- $p\_semiLag$ the semi Lagrangian operator constructed with an interpolator using the following time solution,
- p_state the vector defining the current state for the current regime,
- p_iReg the current regime number,
- $p\_gaussian$ is the vector of gaussian random variables used to calculate the Brownian involved in the underlying process for the current simulation,
- p_phiInP at the value of the function calculated in optimization at next time step for the given point,
- $p\_phiInOut$ storing the cost functions : the size of the array is the number of functions to follow in simulation.

- "stepSimulateControl" is used when the PDE is associated to an optimization problem and we want to simulate an optimal policy using the optimal controls calculated in the optimization part. The arguments are:
  - $p\_gridNext$ defining the grid used at the following time step,
  - $p\_controlInterp$ a vector (for each control) of interpolators in controls
  - p_state the vector defining the current state for the current regime,
  - p_iReg the current regime number,
  - $p\_gaussian$ is the vector of gaussian random variables used to calculate the Brownian involved in the underlying process for the current simulation.
  - $p\_phiInOut$ storing the cost functions : the size of the array is the number of functions to follow in simulation.

  On return the p_state vector is modified, the p_iReg is modified and the cost function $p\_phiInOut$ is modified for the current trajectory.

- the "getCone" method is only relevant if the distribution for data (so MPI) is used. As argument it take a vector of size the dimension of the grid. Each component of the vector is an array containing the minimal and maximal coordinates values of points of the current grid defining an hyper cube $H1$. It returns for each dimension, the coordinates min and max of the hyper cube $H2$ containing the points that can be reached by applying a command from a grid point in $H1$. If no optimization is achieved, it returns the hyper cube $H2$ containing the points reached by the semi Lagrangian scheme. For explanation of the parallel formalism see chapter 5.

- the "getDimensionToSplit" method is only relevant if the distribution for data (so MPI) is used. The method permits to define which directions to split for solution distribution on processors. For each dimension it returns a Boolean where "true" means that the direction is a candidate for splitting,

- the "isNotNeedingBC" permits to define for a point on the boundary of the grid if a boundary condition is needed ("True" is returned) or if no boundary is needed (return "false").

And example of the derivation of such an optimizer for a simple stochastic target problem (described in paragraph 5.3.4 in [24]) is given below :

```
#include <iostream>
#include "StOpt/core/utils/constant.h"
#include "test/c++/tools/semilagrangien/OptimizeSLCase3.h"

using namespace StOpt;
using namespace Eigen ;
using namespace std ;

OptimizerSLCase3::OptimizerSLCase3(const double &p_mu, const double &p_sig, const double &p_dt, const double &p_alphaMax, const double &p_stepAlpha):
    m_dt(p_dt), m_mu(p_mu), m_sig(p_sig), m_alphaMax(p_alphaMax), m_stepAlpha(p_stepAlpha) {}

vector< array< double, 2> > OptimizerSLCase3::getCone(const vector< array< double, 2> > &p_xInit) const
{
    vector< array< double, 2> > xReached(1);
    xReached[0][0] = p_xInit[0][0] - m_alphaMax * m_mu / m_sig * m_dt - m_alphaMax * sqrt(m_dt);
    xReached[0][1] = p_xInit[0][1] + m_alphaMax * sqrt(m_dt) ;
    return xReached;
}

pair< ArrayXd, ArrayXd> OptimizerSLCase3::stepOptimize(const ArrayXd &p_point,
        const vector< shared_ptr<SemiLagrangEspCond> > &p_semiLag, const double &, const Eigen::ArrayXd &) const
{
    pair< ArrayXd, ArrayXd> solutionAndControl;
    solutionAndControl.first.resize(1);
    solutionAndControl.second.resize(1);
    ArrayXd b(1);
    ArrayXXd sig(1, 1) ;
    double vMin = StOpt::infty;
    for (int iAl = 0; iAl < m_alphaMax / m_stepAlpha; ++iAl)
    {
        double alpha = iAl * m_stepAlpha;
        b(0) = -alpha * m_mu / m_sig; // trend
        sig(0) = alpha; // volatility with one Brownian
        pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_point, b, sig, m_dt); // test the control
        if (lagrang.second)
        {
            if (lagrang.first < vMin)
            {
                vMin = lagrang.first;
                solutionAndControl.second(0) = alpha;
            }
        }
    }

    solutionAndControl.first(0) = vMin;
```

```
46      return solutionAndControl;
47  }
48
49  void OptimizerSLCase3::stepSimulate(const StOpt::SpaceGrid    &p_gridNext,
50                                      const std::vector< shared_ptr< StOpt::
                                            SemiLagrangEspCond > > &p_semiLag,
51                                      Eigen::Ref<Eigen::ArrayXd>  p_state,  int
                                             &,
52                                      const Eigen::ArrayXd &p_gaussian, const
                                            Eigen::ArrayXd &,
53                                      Eigen::Ref<Eigen::ArrayXd>) const
54  {
55      double vMin = StOpt::infty;
56      double alphaOpt = -1;
57      ArrayXd b(1);
58      ArrayXXd sig(1, 1) ;
59      ArrayXd proba = p_state ;
60      // recalculate the optimal alpha
61      for (int iAl = 0; iAl < m_alphaMax / m_stepAlpha; ++iAl)
62      {
63          double alpha = iAl * m_stepAlpha;
64          b(0) = -alpha * m_mu / m_sig;// trend
65          sig(0) = alpha;// volatility with one Brownian
66          pair<double, bool> lagrang = p_semiLag[0]->oneStep(proba, b, sig,
                m_dt);// test the control
67          if (lagrang.second)
68          {
69              if (lagrang.first < vMin)
70              {
71                  vMin = lagrang.first;
72                  alphaOpt = alpha;
73              }
74          }
75      }
76      proba(0) += alphaOpt * p_gaussian(0) * sqrt(m_dt);
77      // truncate if necessary
78      p_gridNext.truncatePoint(proba);
79      p_state = proba ;
80
81  }
82
83
84  void OptimizerSLCase3:: stepSimulateControl(const SpaceGrid    &p_gridNext,
85          const  vector< shared_ptr< InterpolatorSpectral> >  &p_controlInterp
                ,
86          Eigen::Ref<Eigen::ArrayXd>  p_state,   int &,
87          const ArrayXd &p_gaussian,
88          Eigen::Ref<Eigen::ArrayXd>) const
89  {
90      ArrayXd proba = p_state ;
91      double alphaOpt = p_controlInterp[0]->apply(p_state);
92      proba(0) += alphaOpt * p_gaussian(0) * sqrt(m_dt);
93      // truncate if necessary
94      p_gridNext.truncatePoint(proba);
```

```
95      p_state = proba ;
96  }
```

8.1 PDE resolution

Once the problem is described, a time recursion can be achieved using the "TransitionStepSemilagrang" object in a sequential resolution of the problem. This object permits to solve the problem on one time step.

```
1   // Copyright (C) 2016 EDF
2   // All Rights Reserved
3   // This code is published under the GNU Lesser General Public License (GNU
        LGPL)
4   #ifndef TRANSITIONSTEPSEMILAGRANG_H
5   #define TRANSITIONSTEPSEMILAGRANG_H
6   #ifdef OMP
7   #include <omp.h>
8   #endif
9   #include <functional>
10  #include <memory>
11  #include <Eigen/Dense>
12  #include "geners/BinaryFileArchive.hh"
13  #include "StOpt/semilagrangien/TransitionStepSemilagrangBase.h"
14  #include "StOpt/core/grids/SpaceGrid.h"
15  #include "StOpt/core/grids/InterpolatorSpectral.h"
16  #include "StOpt/semilagrangien/OptimizerSLBase.h"
17
18  /** \file TransitionStepSemilagrang.h
19   * \brief Solve one step  of explicit semi Lagrangian scheme
20   * \author Xavier Warin
21   */
22
23
24  namespace StOpt
25  {
26
27  /// \class TransitionStepSemilagrang TransitionStepSemilagrang.h
28  ///        One step  of semi Lagrangian scheme
29  class TransitionStepSemilagrang : public TransitionStepSemilagrangBase
30  {
31  private :
32
33      std::shared_ptr<SpaceGrid>  m_gridCurrent ; ///< global grid at current
            time step
34      std::shared_ptr<SpaceGrid>  m_gridPrevious ; ///< global grid at previous
             time step
35      std::shared_ptr<OptimizerSLBase  >  m_optimize ; ///< optimizer solving
            the problem for one point and one step
36
37  public :
38
39      /// \brief Constructor
```

```
40      TransitionStepSemilagrang(const  std::shared_ptr<SpaceGrid> &
            p_gridCurrent,
41                                const  std::shared_ptr<SpaceGrid> &
                                       p_gridPrevious,
42                                const  std::shared_ptr<OptimizerSLBase > &
                                       p_optimize);
43
44      /// \brief One time step for resolution
45      /// \param p_phiIn          for each regime the function value ( on the
            grid)
46      /// \param p_time           current date
47      /// \param p_boundaryFunc  Function at the boundary to impose Dirichlet
            conditions  (depending on regime and position)
48      /// \return     solution obtained after one step of dynamic programming
            and the optimal control
49      std::pair< std::vector< std::shared_ptr< Eigen::ArrayXd > >, std::vector<
             std::shared_ptr< Eigen::ArrayXd > > > oneStep(const std::vector<
            std::shared_ptr< Eigen::ArrayXd > > &p_phiIn, const double &p_time,
            const std::function<double(const int &, const Eigen::ArrayXd &)> &
            p_boundaryFunc) const;
50
51      /// \brief Permits to dump continuation values on archive
52      /// \param p_ar                    archive to dump in
53      /// \param p_name                  name used for object
54      /// \param  p_iStep                Step number or identifier for time step
55      /// \param p_phiIn                 for each regime the function value
56      /// \param p_control               for each control, the optimal value
57      void dumpValues(std::shared_ptr<gs::BinaryFileArchive> p_ar, const std::
            string &p_name, const int &p_iStep, const std::vector< std::shared_ptr
            < Eigen::ArrayXd > > &p_phiIn,
58                      const std::vector< std::shared_ptr< Eigen::ArrayXd > > &
                            p_control) const;
59  };
60  }
61  #endif /* TRANSITIONSTEPSEMILAGRANG_H */
```

It constructor takes the following arguments:

- *p_gridCurrent* a grid describing the meshes at the current date,
- *p_gridPrevious* a grid describing the meshes at the previously treated date,
- *p_optimize* an object derived from the "OptimizerSLBase" and describing the problem to solve at a given date and a given point of the current grid.

A first method "oneStep" take the following arguments :

- *p_phiIn* describes for each regime the solution previously calculated on the grid at the previous time,
- *p_time* is the current time step,
- *p_boundaryFunc* is a function giving the Dirichlet solution of the problem depending on the number of regimes and the position on the boundary.

Table 8.1: Which "TransitionStepSemilagrang" object to use depending on the grid used and the type of parallelization used.

	Full grid	Sparse grid
Sequential	"TransitionStepSemilagrang"	"TransitionStepSemilagrang"
Parallelization on calculations threads and MPI	"TransitionStepSemilagrang"	"TransitionStepSemilagrang"
Distribution of calculations and data (MPI)	"TransitionStepSemilagrangDist"	Not available

It gives back an estimation of the solution at the current date on the current grid for all the regimes and an estimation of the optimal control calculated for all the controls.
A last method "dumpValues" method permits to dump the solution calculated p_phiIn at the step $p\_istep + 1$ and the optimal control at step p_istep in an archive p_ar.
A version using the distribution of the data and calculations can be found in the TransitionStepSemilagrangDist object. An example of a time recursion in sequential can be found in the semiLagrangianTime function and an example with distribution can be found in the semiLagrangianTimeDist function. In both functions developed in the test chapter the analytic solution of the problem is known and compared to the numerical estimation obtained with the semi Lagrangian method.

8.2 Simulation framework

Once the optimal controls and the value functions are calculated, one can simulate the optimal policy by using the function values (recalculating the optimal control for each simulation) or using directly the optimal controls calculated in optimization

- Calculate the optimal strategy in simulation
  by using the function values calculated in optimization :
  In order to simulate one step of the optimal policy, an object "SimulateStepSemilagrangDist" is provided with constructor

```
1    SimulateStepSemilagrangDist(gs::BinaryFileArchive &p_ar, const int &
       p_iStep, const std::string &p_name ,
2                                const std::shared_ptr<FullGrid> &
                                   p_gridNext, const std::shared_ptr<
                                   OptimizerSLBase > &p_pOptimize ,
3                                const bool &p_bOneFile);
```

  where

  - p_ar is the binary archive where the continuation values are stored,
  - p_iStep is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
  - p_name is the base name to search in the archive,
  - $p\_GridNext$ is the grid at the next time step ($p\_iStep + 1$),

– *p_Optimize* is the Optimizer describing the transition from one time step to the following one,

– *p_OneFile* equals to true if a single archive is used to store continuation values.

Remark 16 *A version without distribution of data but only multithreaded and parallelized with MPI on data is available with the object "SimulateStepSemilagrang"*

This object implements the method "oneStep"

```
void oneStep(const Eigen::ArrayXXd & p_gaussian,Eigen::ArrayXXd &
    p_statevector , Eigen::ArrayXi  &p_iReg, Eigen::ArrayXd &
    p_phiInOuts)
```

where:

– *p_gaussian* is a two dimensional array (number of Brownian in the modelization by the number of Monte Carlo simulations).

– *p_statevector* store the continuous state (continuous state size by number of simulations)

– *p_iReg* for each simulation give the current regime number,

– *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where nb is given by the "getSimuFuncSize" method of the optimizer and $nbSimul$ the number of Monte Carlo simulations.

Remark 17 *The previous object "SimulateStepSemilagrangDist" is used with MPI for problems of quite high dimension. In the case of small dimension (below or equal to three), the parallelization with MPI or the sequential calculations can be achieved by the "SimulateStepSemilagrang" object.*

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (
    GNU LGPL)
#ifdef USE_MPI
#include <boost/random.hpp>
#include <memory>
#include <Eigen/Dense>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/semilagrangien/OptimizerSLBase.h"
#include "StOpt/semilagrangien/SimulateStepSemilagrangDist.h"

using namespace std;

double semiLagrangianSimuDist(const shared_ptr<StOpt::FullGrid> &p_grid,
```

```
15                                  const shared_ptr<StOpt::OptimizerSLBase > &
                                        p_optimize,
16                                  const function<double(const int &, const
                                        Eigen::ArrayXd &)>   &p_funcFinalValue,
17                                  const int &p_nbStep,
18                                  const Eigen::ArrayXd &p_stateInit,
19                                  const int &p_initialRegime,
20                                  const int &p_nbSimul,
21                                  const string   &p_fileToDump,
22                                  const bool &p_bOneFile)
23 {
24     boost::mpi::communicator world;
25     // store states in a regime
26     Eigen::ArrayXXd states(p_stateInit.size(), p_nbSimul);
27     for (int is = 0; is < p_nbSimul; ++is)
28         states.col(is) = p_stateInit;
29     // sore the regime number
30     Eigen::ArrayXi regime = Eigen::ArrayXi::Constant(p_nbSimul,
           p_initialRegime);
31     // test if one file generated
32     string toDump = p_fileToDump ;
33     if (!p_bOneFile)
34         toDump += "_" + boost::lexical_cast<string>(world.rank());
35     gs::BinaryFileArchive ar(toDump.c_str(), "r");
36     // name for continuation object in archive
37     string nameAr = "Continuation";
38     // cost function
39     Eigen::ArrayXXd costFunction =  Eigen::ArrayXXd::Zero(p_optimize->
           getSimuFuncSize(), p_nbSimul);
40     // random generator and Gaussian variables
41     boost::mt19937 generator;
42     boost::normal_distribution<double> normalDistrib;
43     boost::variate_generator<boost::mt19937 &, boost::normal_distribution
           <double> > normalRand(generator, normalDistrib);
44     Eigen::ArrayXXd gaussian(p_optimize->getBrownianNumber(), p_nbSimul);
45     // iterate on time steps
46     for (int istep = 0; istep < p_nbStep; ++istep)
47     {
48         for (int is = 0; is < gaussian.cols(); ++is)
49             for (int id  = 0; id < gaussian.rows(); ++id)
50                 gaussian(id, is) = normalRand();
51
52         StOpt::SimulateStepSemilagrangDist(ar, p_nbStep - 1 - istep,
               nameAr, p_grid, p_optimize, p_bOneFile).oneStep(gaussian,
               states, regime, costFunction);
53     }
54     // final cost to add
55     for (int is = 0; is < p_nbSimul; ++is)
56         costFunction(0, is) += p_funcFinalValue(regime(is), states.col(is
               ));
57     // average gain/cost
58     return costFunction.mean();
59 }
60 #endif
```

A sequential or parallelized on calculations version of the previous example is given in this file.

- Calculate the optimal strategy in simulation
 by interpolation of the optimal control calculated in optimization :
In order to simulate one step of the optimal policy, an object "SimulateStepSemilagrangControlDist" is provided with constructor

```
1    SimulateStepSemilagrangControlDist(gs::BinaryFileArchive &p_ar,
        const int &p_iStep,  const std::string &p_name ,
2                                       const std::shared_ptr<FullGrid>   &
                                            p_gridCur,
3                                       const std::shared_ptr<FullGrid> &
                                            p_gridNext,
4                                       const std::shared_ptr<
                                            OptimizerSLBase > &p_pOptimize
                                            ,
5                                       const bool &p_bOneFile)
```

where

  - *p_ar* is the binary archive where the continuation values are stored,
  - *p_iStep* is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
  - *p_name* is the base name to search in the archive,
  - *p_GridCur* is the grid at the current time step (*p_iStep*),
  - *p_GridNext* is the grid at the next time step (*p_iStep* + 1),
  - *p_Optimize* is the Optimizer describing the transition from one time step to the following one,
  - *p_OneFile* equals to true if a single archive is used to store continuation values.

Remark 18 *The previous object "SimulateStepSemilagrangControlDist" is used with MPI distribution of data for problems of quite high dimension. In the case of small dimension (below or equal to three), the parallelization with MPI or the sequential calculations can be achieved by the "SimulateStepSemilagrangControl" object.*

This object implements the method "oneStep"

```
1 void oneStep((const Eigen::ArrayXXd & p_gaussian, Eigen::ArrayXXd &
    p_statevector , Eigen::ArrayXi   &p_iReg, Eigen::ArrayXd  &
    p_phiInOuts)
```

where:

  - *p_gaussian* is a two dimensional array (number of Brownian in the modelization by the number of Monte Carlo simulations).

– *p_statevector* stores the continuous state (continuous state size by number of simulations)

– *p_iReg* for each simulation gives the current regime number,

– *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where nb is given by the "getSimuFuncSize" method of the optimizer and $nbSimul$ the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifdef USE_MPI
#include <memory>
#include <boost/random.hpp>
#include <Eigen/Dense>
#include "geners/BinaryFileArchive.hh"
#include "StOpt/semilagrangien/OptimizerSLBase.h"
#include "StOpt/semilagrangien/SimulateStepSemilagrangControlDist.h"

using namespace std;

double semiLagrangianSimuControlDist(const shared_ptr<StOpt::FullGrid> &p_grid,
                                     const shared_ptr<StOpt::OptimizerSLBase> &p_optimize,
                                     const function<double(const int &, const Eigen::ArrayXd &)> &p_funcFinalValue,
                                     const int &p_nbStep,
                                     const Eigen::ArrayXd &p_stateInit,
                                     const int &p_initialRegime,
                                     const int &p_nbSimul,
                                     const string &p_fileToDump,
                                     const bool &p_bOneFile)
{
    boost::mpi::communicator world;
    // store states in a regime
    Eigen::ArrayXXd states(p_stateInit.size(), p_nbSimul);
    for (int is = 0; is < p_nbSimul; ++is)
        states.col(is) = p_stateInit;
    // sore the regime number
    Eigen::ArrayXi regime = Eigen::ArrayXi::Constant(p_nbSimul, p_initialRegime);
    // test if one file generated
    string toDump = p_fileToDump ;
    if (!p_bOneFile)
        toDump += "_" + boost::lexical_cast<string>(world.rank());
    gs::BinaryFileArchive ar(toDump.c_str(), "r");
    // name for continuation object in archive
```

```
37      string nameAr = "Continuation";
38      // cost function
39      Eigen::ArrayXXd costFunction =  Eigen::ArrayXXd::Zero(p_optimize->
            getSimuFuncSize(), p_nbSimul);
40      // random generator and Gaussian variables
41      boost::mt19937 generator;
42      boost::normal_distribution<double> normalDistrib;
43      boost::variate_generator<boost::mt19937 &, boost::normal_distribution
            <double> > normalRand(generator, normalDistrib);
44      Eigen::ArrayXXd gaussian(p_optimize->getBrownianNumber(), p_nbSimul);
45      // iterate on time steps
46      for (int istep = 0; istep < p_nbStep; ++istep)
47      {
48          for (int is = 0; is < gaussian.cols(); ++is)
49              for (int id  = 0; id < gaussian.rows(); ++id)
50                  gaussian(id, is) = normalRand();
51
52          StOpt::SimulateStepSemilagrangControlDist(ar, p_nbStep - 1 -
                istep, nameAr, p_grid, p_grid, p_optimize, p_bOneFile).
                oneStep(gaussian, states, regime, costFunction);
53      }
54      // final cost to add
55      for (int is = 0; is < p_nbSimul; ++is)
56          costFunction(0, is) += p_funcFinalValue(regime(is), states.col(is
                ));
57      // average gain/cost
58      return costFunction.mean();
59  }
60  #endif
```

The sequential (or parallelized on calculations) version of the previous example is given in this file

Remark 19 *In the previous example, we suppose that only one function is followed in simulation, and that we send back an average for this value function as a result.*

Table 8.2: Which simulation object to use depending on the TransitionStepSemilagrang object used.

	“TransitionStepSemilagrang”	“TransitionStepSemilagrangDist” “bOneFile”=True	“TransitionStepSemilagrangDist“ “bOneFile”= False
“SimulateStepSemilagrang”	Yes	Yes	No
“SimulateStepSemilagrangControl”	Yes	Yes	No
“SimulateStepSemilagrangDist”	No	Yes	Yes
‘SimulateStepSemilagrangControlDist”‘	No	Yes	Yes

Part V

An example with both dynamic programming with regression and PDE

In this chapter we give an example where both dynamic programming with regressions and PDE can be used. It permits to compare the resolution and the solution obtained by both methods. All information about the modelization can be obtained by [26].
In this example we take the following notations :

- D_t is a demand process (in electricity) with an Ornstein Uhlenbeck dynamic :

$$dD_t = \alpha(m - D_t)dt + \sigma dW_t,$$

- Q_t is the cumulative carbon emission due to electricity production to satisfy the demand,

$$dQ_t = (D_t - L_t)^+ dt,$$

- L_t the total investment capacity in non emissive technology to produce electricity

$$L_t = \int_0^t l_s ds$$

  where l_s is an intensity of investment in non emissive technology at date s,

- Y_t is the carbon price where

$$Y_t = \mathbb{E}_t(\lambda 1_{Q_T \geq H}),$$

  with λ and H given.

We introduce the following functions :

- the electricity price function which is a function of demand and the global investment of non emissive technology.

$$p_t = (1 + D_t)^2 - L_t,$$

- the profit function by selling electricity is given by

$$\Pi(D_t, L_t) = p_t D_t - (D_t - L_t)^+,$$

- $\tilde{c}(l_t, L_t)$ is the investment cost for new capacities of non emissive technology.

$$\tilde{c}(l, L) = \bar{\beta}(c_\infty + (c_0 - c_\infty)e^{\beta L})(1 + l)l$$

The value of the firm selling electricity is given by $V(t, D_t, Q_t, L_t)$. It satisfies the coupling equations :

$$\begin{cases} \partial_t v + \alpha(m - D)\partial_D v + \frac{1}{2}\sigma^2 \partial^2_{DD} v + (D - L)^+ \partial_Q v + \Pi(D, L) \\ +sL^{1-\alpha} - y(D - L)^+ + \sup_l \{l\partial_L v - \tilde{c}(l, L)\} = 0 \\ v_T = 0 \end{cases} \tag{8.1}$$

and the carbon price $y(t, D_t, Q_t, L_t)$ is given by :

$$\begin{cases} \partial_t y + \alpha(m - D)\partial_D y + \frac{1}{2}\sigma^2 \partial^2_{DD} y + (D - L)^+ \partial_Q y + l^* \partial_L y = 0 \\ y_T = \lambda 1_{Q_T \geq K} \end{cases} \tag{8.2}$$

and l^* is the optimal control in equation (8.1). The previous equation can be solved with the Semi Lagrangian method.
After a time discretization with a step δt a dynamic programming equation can be given by

$$\begin{aligned} v(T - \delta t, D, Q, L) &= \sup_l (\Pi(D, L) + sL^{1-\alpha} - y_{T-\delta t}(D - L)^+ - \tilde{c}(l, L))\delta t + \\ &\quad \mathbb{E}_{T-\delta t}(V(T, D_T^{T-\delta t, D}, Q + (D - L)^+ \delta t, L + l\delta t)) \qquad (8.3) \\ Y(T - \delta t, D, Q, L) &= \mathbb{E}_{T-\delta t}(Y(T, D_T^{T-\delta t, D}, Q + (D - L)^+ \delta t, L + l^* \delta t)) \qquad (8.4) \end{aligned}$$

The previous equations (8.3) and (8.4) can be solved with the regression methods.
In order to use the previously developed frameworks in parallel, we have to define for both method some common variables.

- The number of regimes to use (obtained by the "getNbRegime' method) is 2 : one to store the v value, one for the y value,
- In the example we want to follow during simulations the functions values v and y so we set the number of function obtained by the "getSimuFuncSize" method to 2.
- In order to test the controls in optimization and simulation we define a maximal intensity of investment "lMax" and a discretization step to test the controls "lStep".

In the sequel we store the optimal functions in optimization and recalculate the optimal control in simulation.

8.3 The dynamic programming with regression approach

All we have to do is to specify an optimizer defining the methods used to optimize and simulate, and the "getCone" method for parallelization :

```
1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
       LGPL)
4  #include "StOpt/core/utils/constant.h"
5  #include "OptimizeDPEmissive.h"
6
7  using namespace std ;
8  using namespace StOpt;
9  using namespace Eigen;
10
11
12 // constructor
13 OptimizeDPEmissive::OptimizeDPEmissive(const double &p_alpha,
14                                        const std::function<double(double,
                                             double)> &p_PI,
```

```
15                                          const std::function< double(double,
                                                double) >      &p_cBar,  const
                                                double   &p_s, const double &
                                                p_lambda,
16                                          const double &p_dt,
17                                          const double   &p_maturity,
18                                          const  double &p_lMax, const double &
                                                p_lStep, const  std::vector <std::
                                                array< double, 2>  >   &p_extrem):
19      m_alpha(p_alpha), m_PI(p_PI),
20      m_cBar(p_cBar), m_s(p_s), m_lambda(p_lambda), m_dt(p_dt), m_maturity(
            p_maturity), m_lMax(p_lMax), m_lStep(p_lStep),
21      m_extrem(p_extrem)
22  {}
23
24  Array< bool, Dynamic, 1> OptimizeDPEmissive::getDimensionToSplit() const
25  {
26      Array< bool, Dynamic, 1> bDim = Array< bool, Dynamic, 1>::Constant(2,
            true);
27      return  bDim ;
28  }
29
30  // for parallelism
31  std::vector< std::array< double, 2> > OptimizeDPEmissive::getCone(const
        vector<  std::array< double, 2>  > &p_xInit) const
32  {
33      vector< array< double, 2> > xReached(2);
34      xReached[0][0] = p_xInit[0][0]   ; // Q only increases
35      xReached[0][1] = m_extrem[0][1]   ; // whole domain due to demand which is
            unbounded
36      xReached[1][0] = p_xInit[1][0]   ;  // L only increases
37      xReached[1][1] = p_xInit[1][1] + m_lMax * m_dt   ; // maximal increase
            given by the control
38      return xReached;
39  }
40
41  // one step in optimization from stock point for all simulations
42  std::pair< ArrayXXd, ArrayXXd> OptimizeDPEmissive::stepOptimize(const   std::
        shared_ptr< StOpt::SpaceGrid> &p_grid, const ArrayXd   &p_stock,
43          const  std::vector< ContinuationValue> &p_condEsp,
44          const std::vector < std::shared_ptr< ArrayXXd > > &) const
45  {
46      std::pair< ArrayXXd, ArrayXXd> solutionAndControl;
47      // to store final solution (here two regimes)
48      solutionAndControl.first = ArrayXXd::Constant(m_simulator->getNbSimul(),
            2, -StOpt::infty);
49      solutionAndControl.second =  ArrayXXd::Constant(m_simulator->getNbSimul()
            , 1, -StOpt::infty);
50      // demand
51      ArrayXd demand = m_simulator->getParticles().array().row(0).transpose();
52      // Gain (size number of simulations)
53      ArrayXd gain(m_simulator->getNbSimul());
54      double gainSubvention =  m_s * pow(p_stock(1), 1. - m_alpha); //
            subvention for non emissive energy
```

```
55      for (int is = 0 ; is < m_simulator->getNbSimul(); ++is)
56          gain(is) = m_PI(demand(is), p_stock(1)) + gainSubvention ; // gain by
                production and subvention
57      ArrayXd ptStockNext(2);
58      // time to maturity
59      double timeToMat = m_maturity - m_simulator->getCurrentStep();
60      // interpolator at the new step
61      for (int is = 0 ; is < m_simulator->getNbSimul(); ++is)
62      {
63          for (int  iAl = 0; iAl < m_lMax / m_lStep ; ++iAl) // test all
                command for investment between 0 and lMax
64          {
65              double  l = iAl * m_lStep;
66              // interpolator at the new step
67              ptStockNext(0) =  p_stock(0) + std::max(demand(is) - p_stock(1),
                    0.) * m_dt;
68              ptStockNext(1) =  p_stock(1) +  l * m_dt ;
69              // first test we are inside the domain
70              if (p_grid->isInside(ptStockNext))
71              {
72                  // create an interpolator at the arrival point
73                  std::shared_ptr<StOpt::Interpolator>  interpolator = p_grid->
                        createInterpolator(ptStockNext);
74                  // calculate Y for this simulation with the optimal control
75                  double yLoc = p_condEsp[1].getASimulation(is, *interpolator);
76                  // local gain
77                  double gainLoc = (gain(is) - yLoc * std::max(demand(is) -
                        p_stock(1), 0.) - m_cBar(l, p_stock(1))) * m_dt;
78                  //  gain + conditional expectation of future gains
79                  double condExp = gainLoc + p_condEsp[0].getASimulation(is, *
                        interpolator);
80                  if (condExp >  solutionAndControl.first(is, 0)) // test
                        optimality of the control
81                  {
82                      solutionAndControl.first(is, 0) = condExp;
83                      solutionAndControl.first(is, 1) = yLoc;
84                      solutionAndControl.second(is, 0) = l;
85                  }
86              }
87          }
88          // test if solution acceptable
89          if (StOpt::almostEqual(solutionAndControl.first(is, 0), - StOpt::
                infty, 10))
90          {
91              // fix boundary condition
92              solutionAndControl.first(is, 0) =  timeToMat * (m_PI(demand(is),
                    p_stock(1)) + m_s * pow(p_stock(1), 1. - m_alpha) - m_lambda *
                     std::max(demand(is) - p_stock(1), 0.));
93              solutionAndControl.first(is, 1) = m_lambda ; // Q est maximal !!
94              solutionAndControl.second(is, 0) = 0. ; // fix control to zero
95          }
96      }
97      return solutionAndControl;
98  }
```

```
99
100 // one step in simulation for current simulation
101 void OptimizeDPEmissive::stepSimulate(const std::shared_ptr< StOpt::SpaceGrid
    >  &p_grid, const std::vector< StOpt::GridAndRegressedValue > &
    p_continuation,
102                                       StOpt::StateWithStocks &p_state,
103                                       Ref<ArrayXd> p_phiInOut) const
104 {
105     ArrayXd ptStock = p_state.getPtStock();
106     ArrayXd ptStockNext(ptStock.size());
107     double vOpt = - StOpt::infty;
108     double gainOpt = 0.;
109     double lOpt = 0. ;
110     double demand = p_state.getStochasticRealization()(0); // demand for this
        simulation
111     ptStockNext(0) =  ptStock(0) + std::max(demand - ptStock(1), 0.) * m_dt;
112     double   gain  = m_PI(demand, ptStock(1)) +  m_s * pow(ptStock(1), 1. -
        m_alpha) ; // gain from production and subvention
113     double yOpt = 0. ;
114     for (int  iAl = 0; iAl < m_lMax / m_lStep ; ++iAl) // test all command
        for investment between 0 and lMax
115     {
116         double  l = iAl * m_lStep;
117         // interpolator at the new step
118         ptStockNext(1) =  ptStock(1) +  l * m_dt ;
119         // first test we are inside the domain
120         if (p_grid->isInside(ptStockNext))
121         {
122             // calculate Y for this simulation with the control
123             double yLoc = p_continuation[1].getValue(ptStockNext, p_state.
                getStochasticRealization());
124             // local gain
125             double gainLoc = (gain - yLoc * std::max(demand - ptStock(1), 0.)
                 - m_cBar(l, ptStock(1))) * m_dt;
126             //  gain + conditional expectation of future gains
127             double condExp = gainLoc +  p_continuation[0].getValue(
                ptStockNext, p_state.getStochasticRealization());
128
129             if (condExp >  vOpt) // test optimality of the control
130             {
131                 vOpt = condExp;
132                 gainOpt = gainLoc;
133                 lOpt = l;
134                 yOpt = yLoc;
135             }
136         }
137     }
138     p_phiInOut(0) += gainOpt; // follow v value
139     p_phiInOut(1) = yOpt ; // follow y value
140     ptStockNext(1) =  ptStock(1) +  lOpt * m_dt ; // update state due to
        control
141     p_state.setPtStock(ptStockNext);
142 }
```

This case in dimension 2 for the stocks can be treated with interpolation on the full 2

dimensional grid and on a 2 dimensional sparse grid. Both versions of the resolution are given in a test case.

8.4 The PDE approach

We can do the same with the PDE approach using a simulator for the OU demand. We then define an optimizer and the methods used to optimize and simulate, and the "getCone" method for parallelization :

```
#include <iostream>
#include "StOpt/core/utils/constant.h"
#include "OptimizeSLEmissive.h"

using namespace StOpt;
using namespace Eigen ;
using namespace std ;

// constructor
OptimizeSLEmissive::OptimizeSLEmissive(const double &p_alpha, const double &p_m, const double &p_sig, const std::function<double(double, double)> &p_PI,
                                       const std::function< double(double, double) > &p_cBar, const double &p_s, const double &p_dt,
                                       const double &p_lMax, const double &p_lStep, const std::vector <std::array< double, 2> > &p_extrem):
    m_alpha(p_alpha), m_m(p_m), m_sig(p_sig), m_PI(p_PI), m_cBar(p_cBar), m_s(p_s), m_dt(p_dt),
    m_lMax(p_lMax), m_lStep(p_lStep), m_extrem(p_extrem) {}

Array< bool, Dynamic, 1> OptimizeSLEmissive::getDimensionToSplit() const
{
    Array< bool, Dynamic, 1> bDim = Array< bool, Dynamic, 1>::Constant(3, true);
    return bDim ;
}


// for parallelism
vector< array< double, 2> > OptimizeSLEmissive::getCone(const vector< array< double, 2> > &p_xInit) const
{
    vector< array< double, 2> > xReached(3);
    xReached[0][0] = p_xInit[0][0] + m_alpha * (m_m - m_extrem[0][1]) * m_dt - m_sig * sqrt(m_dt); // demand "cone" driven by maximal value allowed for demand
    xReached[0][1] = p_xInit[0][1] + m_alpha * m_m * m_dt + m_sig * sqrt(m_dt) ; // low value for demand is taken equal to 0
    xReached[1][0] = p_xInit[1][0] ;// Q only increases
    xReached[1][1] = p_xInit[1][1] + m_extrem[0][1] * m_dt ; // Q increase bounded by maximal demand
    xReached[2][0] = p_xInit[2][0] ; // L only increases
```

```
32      xReached[2][1] = p_xInit[2][1] + m_lMax * m_dt  ;// maximal increase
            given by the control
33      return xReached;
34  }
35
36
37  // one step in optimization from current point
38  std::pair< ArrayXd, ArrayXd> OptimizeSLEmissive::stepOptimize(const ArrayXd
        &p_point,
39          const vector< shared_ptr<SemiLagrangEspCond> > &p_semiLag,
40          const double &, const ArrayXd &) const
41  {
42      pair< ArrayXd, ArrayXd> solutionAndControl;
43      solutionAndControl.first.resize(2);
44      solutionAndControl.second.resize(1);
45      ArrayXXd sig = ArrayXXd::Zero(3, 1) ;
46      sig(0, 0) = m_sig;
47      double vOpt = - StOpt::infty;
48      double yOpt = 0. ;
49      double lOpt = 0 ;
50      ArrayXd b(3);
51      b(0) =  m_alpha * (m_m - p_point(0)) ; // trend
52      b(1) =  max(p_point(0) - p_point(2), 0.);
53      // gain already possible to calculate (production and subvention)
54      double gainFirst = m_PI(p_point(0), p_point(2)) + m_s * pow(p_point(2),
            1. - m_alpha)  ;
55      for (int iAl = 0; iAl < m_lMax / m_lStep ; ++iAl) // test all command for
            investment between 0 and lMax
56      {
57          double  l = iAl * m_lStep;
58          b(2) =  l ;
59          pair<double, bool> lagrangY = p_semiLag[1]->oneStep(p_point, b, sig,
                m_dt); // for the control calculate y
60          if (lagrangY.second) // is the control admissible
61          {
62              pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_point, b,
                    sig, m_dt);  // one step for v
63              // gain function
64              double gain = m_dt * (gainFirst - lagrangY.first * b(1)  - m_cBar
                    (l, p_point(2)));
65              double arbitrage = gain + lagrang.first;
66              if (arbitrage > vOpt) // optimality  of the control
67              {
68                  vOpt = arbitrage; // upgrade solution v
69                  yOpt =  lagrangY.first; // store y
70                  lOpt = l; // upgrade optimal control
71              }
72          }
73      }
74
75      if (StOpt::almostEqual(vOpt, - StOpt::infty, 10))
76      {
77          std::cout << " Reduce time step " << std::endl ;
78          abort();
```

```
79      }
80      solutionAndControl.first(0) =  vOpt; // send back v function
81      solutionAndControl.first(1) =  yOpt; // send back y function
82      solutionAndControl.second(0) =  lOpt; // send back optimal control
83      return solutionAndControl;
84  }
85
86  // one step in simulation for current simulation
87  void OptimizeSLEmissive::stepSimulate(const SpaceGrid   &p_gridNext,
88                                        const  std::vector< std::shared_ptr<
                                              StOpt::SemiLagrangEspCond> > &
                                              p_semiLag,
89                                        Ref<ArrayXd>  p_state,   int &,
90                                        const ArrayXd &p_gaussian,
91                                        const ArrayXd &,
92                                        Ref<ArrayXd> p_phiInOut) const
93  {
94      ArrayXd state = p_state;
95      ArrayXXd sig = ArrayXXd::Zero(3, 1) ; // diffusion matrix for semi
            Lagrangian
96      sig(0, 0) = m_sig;
97      double vOpt = - StOpt::infty;
98      double lOpt = 0 ;
99      double yOpt = 0;
100     ArrayXd b(3);
101     b(0) =  m_alpha * (m_m - p_state(0)) ; // trend for D (independent of
            control)
102     b(1) =  max(p_state(0) - p_state(2), 0.); // trend for Q (independent of
            control)
103     double gainFirst = m_PI(p_state(0), p_state(2)) + m_s * pow(p_state(2),
            1. - m_alpha)  ; // gain for production and subvention
104     for (int iAl = 0; iAl < m_lMax / m_lStep ; ++iAl) // recalculate the
            optimal control
105     {
106         double  l = iAl * m_lStep;
107         b(2) =  l ;
108         pair<double, bool> lagrangY = p_semiLag[1]->oneStep(p_state, b, sig,
                m_dt); // calculate y for this control
109         if (lagrangY.second)
110         {
111             pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_state, b,
                    sig, m_dt); // calculate the function value v
112             // gain function
113             double gain = m_dt * (gainFirst - lagrangY.first * b(1)  - m_cBar
                    (l, p_state(2)));
114             double arbitrage = gain + lagrang.first;
115             if (arbitrage > vOpt) // arbitrage
116             {
117                 vOpt = arbitrage; // upgrade solution
118                 yOpt =  lagrangY.first; // upgrade y value
119                 lOpt = l; // upgrade optimal control
120             }
121         }
122     }
```

```
123     // gain function
124     p_phiInOut(0) += m_dt * (gainFirst - yOpt * b(1) - m_cBar(lOpt, state(2))
            ); // store v value
125     p_phiInOut(1) = yOpt; // store y value
126     // update state
127     state(0) += m_alpha * (m_m - p_state(0)) * m_dt + m_sig * p_gaussian(0) *
            sqrt(m_dt); // demand (no control)
128     state(1) += b(1) * m_dt; //Q
129     state(2) += lOpt * m_dt; //L
130     // truncate if necessary to stay inside domain.
131     p_gridNext.truncatePoint(state);
132     p_state = state ;
133 }
```

The three dimensional grids used can be some full grids or some sparse grids. Both versions of the resolution can be found in a test case.

Part VI

Stochastic Dual Dynamic Programming

Chapter 9

SDDP algorithm

9.1 Some general points about SDDP

SDDP is an approximate dynamic programming algorithm developed by Pereira and Pinto in 1991 [27].

To describe how SDDP works, we will consider a class of linear programs that have T stages denoted $\{0, 1, ..., t, ..., T\}$. We restrict our class of problems to linear programs with relatively complete recourse : the feasible region of the linear program in each stage is nonempty and bounded.

Let us formalize now the variables and constraints used in the SDDP problem.

Notations used

The notations described here are used in the general case.

- x_t the state variable at time t,
- $\omega_t \in \Omega_t$ the random data process at time t, where Ω_t is the set of random data.
- c_t is the cost vector at time t,
- A_t and E_t denote constraints matrices.
- $Q_t(x_{t-1}, \omega_t)$ is the expected value of the problem at time t, knowing the state x_{t-1} and the random data ω_t.
- $\mathscr{Q}_t(x_{t-1}) = \mathbb{E}[Q_t(x_{t-1}, \omega_t)]$

Decision process

The random data process ω_t is discovered gradually. Thus from an initial state x_0, the state variables $(x_t)_{t\in\{0,1,...,T\}}$ are determined in a non-anticipative way. The scheme is the following :

$x_0 \rightarrow$ observation of $\omega_1 \rightarrow$ decision of x_1 .....
$\rightarrow$ decision of $x_{T-1} \rightarrow$ observation of $\omega_T \rightarrow$ decision of x_T

A rigorous formulation of the multistage stochastic linear program to solve is the following:

$$V^* = \min_{\substack{A_0 x_0 = \omega_0 \\ x_1 \geq 0}} c_0^\top x_0 + \mathbb{E}\left[\min_{\substack{E_1 x_0 + A_1 x_1 = \omega_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[.... + \mathbb{E}\left[\min_{\substack{E_T x_{T-1} + A_T x_T = \omega_T \\ x_T \geq 0}} c_T^\top x_T\right]\right]\right] \quad (9.1)$$

The deterministic equivalent of this problem (9.1) is achieved by discretizing ω_t (or by using directly ω_t if discrete). The number of variables of this problem increases exponentially with the number of stages. It cannot be solved directly even if T or $(\Omega_t)_{t\in\{0,1,...,T\}}$ are of reasonable size.

Dynamic programming principle

Dynamic programming involves splitting up the problem (9.1) in a series of sub-problem bounded together by a state variable. The aim is to compute backwards the functions Q_t and $\mathscr{Q}_t$. They fulfill the following equations :

$$[LP_t] \begin{cases} & Q_t(x_{t-1}, \omega_t) = \min c_t^\top x_t + \mathscr{Q}_{t+1}(x_t) \\ s.c. & \quad A_t x_t = \omega_t - E_t x_{t-1}, \quad [\pi_t(\omega_t)] \\ & \quad x_t \geqslant 0 \end{cases} \quad (9.2)$$

$$\mathscr{Q}_t(x_{t-1}) = \mathbb{E}[Q_t(x_{t-1}, \omega_t)] \quad (9.3)$$

The function $Q(x_{t-1}, \omega_t)$ stands for the expected value of a future cost knowing the state x_{t-1} the random data ω_t. $\mathscr{Q}_t(x_{t-1})$ is the expected value of the future cost knowing the state. The dynamic programming principle insures that $V^* = \mathscr{Q}_1(x_0)$.

Given $\mathscr{Q}_T(\cdot)$, the successive computations are achieved backwards switching between the resolution of the linear sub-problem (9.2) and the computation of (9.3).

The implementation of dynamic programming involves approximating successively the two value functions with equations (9.2 - 9.3) by discretizing the state space and solving the linear sub-problems. The number of discretization points increases exponentially with the dimension of the state vector and becomes huge for our applications ("curse of dimensionality"). Besides a linear approximation of $\mathscr{Q}_{t+1}(x_t)$ must be available in order to cast the transition problem into a LP.

SDDP algorithm

SDDP is a method used to solve stochastic multi-stage problem described in [27]. SDDP is based on Benders decomposition described in [28]. Please note that SDDP was developed in order to solve hydro thermal scheduling problem.

SDDP limits the curse of dimensionality by avoiding *a priori* complete discretization of the state space. Each SDDP iteration is a two-stage process. The first step involves generating a sequence of realistic states x_t^* from which in the second step the value functions are estimated in their neighborhood. By repeating successively these two steps the approximation of the value function becomes more and more accurate. SDDP is also made of two passes computed alternatively :

- a backward pass : the aim is to improve the number of Benders cut in the neighborhood of well-chosen candidate states. It provides also a lower bound of the optimal cost.
- a forward pass : the aim is to provide a set of new candidate states. An estimation of the upper bound of the optimal cost is also computed.

On the other hand SDDP method stands on the shape of the future value function $\mathscr{Q}_t(x_{t-1})$. Indeed in the frame of a linear problem with complete recurse the value function is convex and piecewise linear. It can therefore be approximated by taking the supremum of a family of minoring affine functions. These affine functions are called *optimality cuts* or *Benders cuts*.

9.2 A method, different algorithms

The method implemented in this library is based on the different situations shown in a technical report of PSR program [29] where three different cases of the basic problem are solved by SDDP. The three cases are implemented in the library. Other cases could be added to those existing in the future.

Notations

These notations will be used to present the different algorithm of SDDP.

- $\bar{z}$ denotes the optimal cost obtained in forward pass.
- $\underline{z}$ denotes the optimal cost obtained in backward pass.
- β_t^j denotes the slope of the j^{th} Benders cut.
- α_t^j denotes the intercept of the j^{th} Benders cut.

9.2.1 The basic case

To describe this case the notations shown above are used. We focus on stochastic multi-stage problems with the following properties.

- Random quantities in different stages are independent.
- The random quantities at time t is summarized in ω_t .
- At each stage, the linear sub-problem solution space is non-empty and bounded.

In this case the functions $\mathscr{Q}_t(\cdot)$ are convex. The primal and dual solutions of the linear problem exist and define optimal cuts. We can now describe precisely how the implemented algorithm is working.

Initialization

The following values are fixed :

- $\{0, 1, ..., T\}$, the time horizon.
- $n = 0$, is the counter of the number of iterations (*backward-forward*). n is incremented at the end of each iteration.
- $p \in \mathbb{R}$, the precision to reach for the convergence test.
- $n_{step} \in \mathbb{N}$, the number of iterations achieved between 2 convergence tests.
- $n_{iterMax} \in \mathbb{N}$, the maximal number of iterations.
- $x_0 \in \mathbb{R}_+^n$, the initial vector state.
- $L \in \mathbb{N}$, the number of scenarios used in the backward pass.
- $G \in \mathbb{N}$, the number of scenarios used in the forward pass. It gives also the number of new cuts computed at every iteration (*backward-forward*) and the number of states near which the Benders cuts are computed.

Forward pass

The aim of this pass is to explore new feasible vector state and to get an estimation of the upper bound of the optimal cost. To this end the current strategy is simulated for a set of G scenarios. The set of scenarios could be historical chronicles or random draws.

Algorithm 4: Run of forward pass (n^{th} iteration)

Simulate sets $\{(\omega_t^g), t \in \{1,..,T\}\}$ of equally distributed scenarios : for $g \in \Omega^{\mathscr{G}} = \{1,...,G\}$;

for $g \in \Omega_{\mathscr{G}}$ **do**

Solve the following linear sub-problem. ;

$$[AP_0^n] \begin{cases} & Q_0 = \min\limits_{x_0,\theta_1} c_0^\top x_0 + \theta_1 \\ u.c. & A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ & x_0 \geqslant 0 \\ & \theta_1 + (\beta_1^j)^\top x_0 \geqslant \alpha_1^j \quad j \in \{1,...,G,...,nG\} \end{cases} \tag{9.4}$$

Store the primal solution (x_0^g) of the problem $[AP_{0,g}^n]$.

for $t \in \{1,...,T\}$ **do**

Solve the following linear sub-problem. ;

$$[AP_{t,g}^n] \begin{cases} & Q_t^g(x_{t-1}^g, \omega_t^g) = \min\limits_{x_t,\theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ s.c. & A_t x_t = \omega_t^g - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^g)] \\ & x_t \geqslant 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geqslant \alpha_{t+1}^j, \qquad j \in \{1,...,G,...,nG\} \end{cases} \tag{9.5}$$

Store the primal solution (x_t^g) of the problem $[AP_{t,g}^n]$

end

Compute the cost for scenario g, at iteration n : $\bar{z}_n^g = \sum_{t=0}^T c_t x_t^g$;

end

Compute the total cost in forward pass at iteration n : $\bar{z}_n = \frac{1}{G}\sum_{g=1}^G \bar{z}_n^g$

Backward pass

The aim of the backward pass is to add at each stage a set of new Benders cut and to provide a new estimation of the lower bound of the optimal operational cost. To this end we have scenarios set of the random quantities (dimension of the set is L) recorded during the initialization. At each time step G cuts are added using the G visited states $(x_t^g)_{g=1,..,G}$ obtained during the forwards pass.

Algorithm 5: Run of backward pass

for $t = T, T-1, ..., 1$ **do**
  for $x_{t-1}^g,\ g \in \{1, ..., G\}$ **do**
    for $\omega_t^l,\ l \in \{1, ..., L\}$ **do**
      Solve the following linear sub-problem. ;

$$[AP_{t,l}^{n,g}] \begin{cases} & Q_t^l(x_{t-1}^g, \omega_t^l) = \min\limits_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ s.c. & A_t x_t = \omega_t^l - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^l)] \\ & x_t \geqslant 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geqslant \alpha_{t+1}^j, \qquad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.6}$$

      Store the dual solution $\pi_t(\omega_t^l)$ and the primal one $Q_t^l(x_{t-1}^g, \omega_t^l)$ of the linear sub-problem $[AP_{t,l}^{n,g}]$
      Compute the cut that goes with the l^{th} hazard draw :

$$\begin{cases} \alpha_{t,l}^g = Q_t^l(x_{t-1}^g, \omega_t^l) + \pi_t(\omega_t^l)^\top E_t x_{t-1}^g \\ \beta_{t,l}^g = E_t^\top \pi_t(\omega_t^l) \end{cases} \tag{9.7}$$

    end
    Compute the g^{th} new Benders cut at time t at iteration n : is defined as the mean value of the cuts obtained before:

$$\begin{cases} \alpha_t^k = \frac{1}{L} \sum\limits_{l=1}^{L} \alpha_{t,l}^g \\ \beta_t^k = \frac{1}{L} \sum\limits_{l=1}^{L} \beta_{t,l}^g \\ \qquad\qquad k = nG + g \end{cases} \tag{9.8}$$

  end
end
Solve the following linear sub-problem. ;

$$[AP_0^n] \begin{cases} & Q_0 = \min\limits_{x_0, \theta_1} c_0^\top x_0 + \theta_1 \\ s.c. & A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ & x_0 \geqslant 0 \\ & \theta_1 + (\beta_1^j)^\top x_0 \geqslant \alpha_1^j \qquad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.9}$$

Save the cost *backward* $\underline{z}_n = Q_0$

Stopping test

In the literature about SDDP lots of stopping criterion were used and their efficiency has been proved. However a criterion is suitable for each particular problem. Thus it is tough to bring out one which is generic. Due to genericity requirements, two classical criterion are implemented in the library. These can be customized by the user. The first one defines a maximal number of iterations $n_{iterMax}$ (an iteration is made of the succession of *backward-forward* passes) which shall not be exceeded. The second one is a test of convergence

towards each other between the forward and the backward cost. The convergence test uses the following indicator :

$$\psi_{n_{step}i} = \left| \frac{\bar{z}_{n_{step}i} - \underline{z}_{n_{step}i}}{\bar{z}_{n_{step}i}} \right|, \quad \text{with } i \in \mathbb{N} \tag{9.10}$$

This one is computed every n_{step} iterations. If it is lesser than a threshold p the process stops, otherwise it goes on. The threshold is fixed by the user.

9.2.2 Dependence of the random quantities

In the previous case we restrict our problem to independent random quantities in the different stages. The resolution of the SDDP was achieved on the state vector x_t in the basic case.

But sometimes in real life the random quantities can be temporarily correlated. In a hydraulic problem for example there exists time-related dependency of the outcomes. Time-related dependencies can also exist in the demand. Yet with time-related random quantities the Bellman recurrence formula (9.2 - 9.3) does not hold and the classical SDDP can not be applied.

However if the Bellman functions are convex with respect to the time-related random quantities one has only to increase the dimension of the state vector by the dimension of the time-related random quantities to be back in the configuration of the basic case. In this case solving a linear program of reasonable size for each hazard draw is enough to compute new Benders cuts computation in the neighborhood of a candidate state.

There exists a few options to represent the time-related dependency of the random quantities. However in order to not increase too much the dimension of the problem, a ARMA process of order 1 is often chosen. In the random data vector ω_t two different parts has to be distinguished from now on:

- ω_t^{ind} is the random data vector corresponding to the independent random quantities.
- ω_t^{dep} is the random data vector corresponding to the time-related random quantities.

And ω_t^{dep} fulfills the following recurrence equation :

$$\frac{\omega_t^{dep} - \mu_{\omega,t}}{\sigma_{\omega,t}} = \psi_1 \frac{\omega_{t-1}^{dep} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t \tag{9.11}$$

To apply the Bellman recurrence formula the vector state should be made of the decision variable x_t and the time-related random quantities ω_t^{dep}. Dimension of the vector state is then increased. $x_t^{dep} = (x_t, \omega_t^{dep})^\top$ denotes the new state vector. The Bellman function satisfies from now on the following two-stages linear problem at time t :

$$[LP_t'] \begin{cases} & Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t) = \min c_t^\top x_t + \mathscr{Q}_{t+1}(x_t, \omega_t^{dep}) \\ u.c. & \quad A_t x_t = P\omega_t^{dep} - E_t x_{t-1}, \qquad [\pi_t(\omega_t)] \\ & \quad x_t \geqslant 0 \end{cases} \tag{9.12}$$

with P the matrix such that $\omega_t = P\omega_t^{dep}$.

The variable ω_t^{dep} is a random process. Thus the above problem is solved using specific values ω_t^l of this variable. To get them we apply a Markov process that is we simulate different values of the white noise ϵ_t^l.

The new form of the state vector implies changes in the sensitivity of the Bellman function. Thus it is a function depending on the decision variable x_t but also on the the time-related random quantity vector ω_t^{dep}. The computation of Benders cuts is then a bit different:

$$\begin{aligned}\frac{\partial Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t)}{\partial \omega_{t-1}^{dep}} &= \frac{\partial Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t)}{\partial \omega_t^{dep}} \frac{\partial \omega_t^{dep}}{\partial \omega_{t-1}^{dep}} \\ &= \pi_t(\omega_t)^\top P \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}},\end{aligned} \tag{9.13}$$

Backward pass has to be modified in the following manner. Some new computation steps have to be taken into account.

Algorithm 6: Run of backward pass with time-related random quantities (AR1 process)

Pick up the set of the following pairs: $\{x_t^g, \omega_t^{g,dep}\}$ for $g \in \{1, ..., G\}$, $t \in \{1, ..., T\}$
for $t = T, T-1, ..., 1$ **do**
  for $(x_{t-1}^g, \omega_{t-1}^{dep,g})$, $g \in \{1, ..., G\}$ **do**
    for $l \in \{1, ..., L\}$ **do**
      Produce a value for the white noise ϵ_t^l ;
      Compute the element $\hat{\omega_t^l}$ knowing the previous random quantity $\omega_{t-1}^{dep,g}$:

$$\hat{\omega_t^l} = \sigma_{\omega,t}\left(\psi_1 \frac{\omega_{t-1}^{dep,g} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t^l\right) + \mu_{\omega,t} \tag{9.14}$$

      Solve the following linear sub-problem ;

$$[AP_{t,l}^{'n,g}] \begin{cases} & Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l}) = \min\limits_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ u.c. & A_t x_t = P\hat{\omega_t^l} - E_t x_{t-1}^g, \quad [\pi_t(\hat{\omega_t^l})] \\ & x_t \geqslant 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t + (\gamma_{t+1}^j)^\top \hat{\omega_t^l} \geqslant \alpha_{t+1}^j, \\ & \quad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.15}$$

      Store the dual solution $\pi_t(\omega_t^l)$ and the primal one $Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l})$ of the primal problem $[AP_{t,l}^{'n,g}]$
      Compute the cut that goes with the l^{th} hazard draw :

$$\begin{cases} \alpha_{t,l}^g = Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l}) + \pi_t(\hat{\omega_t^l})^\top \left(E_t x_{t-1}^g - \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P \omega_{t-1}^{dep,g}\right) \\ \beta_{t,l}^g = E_t^\top \pi_t(\hat{\omega_t^l}) \\ \gamma_{t,l}^g = \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P^\top \pi_t(\hat{\omega_t^l}) \end{cases} \tag{9.16}$$

    end
    Compute the g^{th} new Benders cut at time t at iteration n defined as the mean value of the cuts obtained before:

$$\begin{cases} \alpha_t^k = \frac{1}{L}\sum\limits_{l=1}^{L} \alpha_{t,l}^g \\ \beta_t^k = \frac{1}{L}\sum\limits_{l=1}^{L} \beta_{t,l}^g \\ \gamma_t^k = \frac{1}{L}\sum\limits_{l=1}^{L} \gamma_{t,l}^g \\ \qquad k = nG + g \end{cases} \tag{9.17}$$

  end
end
Solve the following linear sub-problem. ;

$$[AP_0^{'n}] \begin{cases} & Q_0 = \min\limits_{x_0, \theta_1} c_0^\top x_0 + \theta_1 \\ u.c. & A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ & x_0 \geqslant 0 \\ & \theta_1 + (\beta_1^j)^\top x_0 + (\gamma_1^j)^\top \omega_0^{dep} \geqslant \alpha_1^j, \\ & \quad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.18}$$

Save the backward cost $\underline{z}_n = Q_0$

9.2.3 Non-convexity and conditionnal cuts

Some random quantities may introduce non-convexity preventing us to apply the classical algorithm of SDDP. Indeed when the random quantities appear on the left-hand side of the linear constraints or in the cost function (typically A_t and/or c_t become random) the property of non-convexity of the Bellman functions with respect to the random quantities is not anymore observed.

In the frame of a management production problem the situation happened often. For example sometimes the unit operation cost of plants are random. It is also observed when we deal with spot price uncertainty for use in stochastic mid-term scheduling.

In a technical report [29] Pereira and Pinto suggested a new algorithm in order to efficiently approximate the Bellman functions using explicitly the dependence of the Bellman functions with respect to these random quantities. This new algorithm is based on a combination of SDDP and ordinary stochastic dynamic programming. The SDP part deals with the non-convex random quantities, whereas the other random quantities are treated in the SDDP part. It is an extension of the classical SDDP algorithm. It is described in detail in [29] and in [30].

In that case the modelization used in the library is somewhat different from the one described in both articles. Indeed in the articles it is based on a finite number of non-convex random quantities. A discretization of the non-convex random quantity space is the necessary.

In [30] spot price p_t is regarded as a state. The set of feasible spot price is discretized into in a set of M points $\zeta_1, ..., \zeta_M$. The following Markov model is then used :

$$\mathbb{P}\left(p_t = \zeta_j | p_{t-1} = \zeta_i\right) = \rho_{ij}(t)$$

This model makes easier the implementation of the SDP. But it implies discretization mistakes that are hard to quantify. It is also tough to discretize with efficiency a random process.

In order to overcome these points in the library the evolution of the non-convex random quantities is decided by Monte Carlo simulations. At each stage a fixed number of Monte-Carlo simulations is provided. Anyway in spite of this difference the global view of this brand new algorithm is similar to that one described in both articles :

- The non-convex random quantities depend on the realization of the previous one according to a mathematical model (Markov chain).
- At each stage Bellman functions are approximated through the conditional realization of these random quantities.
- We used conditional cuts to give an estimation of the Bellman functions. These conditional cuts are computed using the methods in section 2 : two methods are available in the library. Both use adaptive support. The first uses a constant per cell approximation while the second uses a linear per cell approximation.

In our algorithm the features of the conditional cuts are revealed thanks to a conditional expectation computation.

Yet conditional expectation computations are not easy when the exact distribution of the random variable is not known. A few techniques exist but in the library a specific one is used and described above in chapter 2 : it is based on local linear regression.

Regression, stochastic dynamic programming and SDDP

The run of the backward pass in the new algorithm combining SDDP and SDP using local linear regression is described below.

Before describing in detail this algorithm, let us introduce a few notations :

- S is the space of the non-convex random quantities.
- d is the dimension of the space of the non-convex random quantities S
- At each stage U Monte Carlo simulations in S are provided. Thus we get U scenarios denoted s_t^u at each stage t
- $\tilde{I}$ is a partition of the space of the non-convex random quantities S.

$$\tilde{I} = \{\underline{I} = (i_1, ..., i_d), i_1 \in \{1, ..., I_1\}, ..., i_d \in \{1, ..., I_d\}\}$$

- $\{D_{\underline{I}}\}_{\underline{I} \in \tilde{I}}$ is the set of meshes of the set of scenarios.
- $M_M = \prod_{k=1}^{d} I_k$ denotes the number of meshes at each stage.

Algorithm 7: Run of the backward pass with time-related (AR1) and non-convex random quantities

Pick up the set of the following pairs : $\{x_t^g, \omega_t^{g,dep}\}$ pour $g \in \{1, ..., G\}$, $t \in \{1, ..., T\}$
for $\underline{t = T, T-1, ..., 1}$ **do**
Generate values for the non-convex random quantities at time t knowing the scenarios at time $t-1$: s_t^u, $u \in \{1, ..., U\}$
for $\underline{(x_{t-1}^g, \omega_{t-1}^{dep,g}),\ g \in \{1, ..., G\}}$ **do**
for $\underline{u \in \{1, ..., U\}}$ **do**
Consider a scenario s_t^u in the mesh $D_{\underline{I}}$;
for $\underline{l \in \{1, ..., L\}}$ **do**
Produce a value for the white noise ϵ_t^l ;
Compute the element $\hat{\omega_t^l}$ knowing the previous random quantity $\omega_{t-1}^{dep,g}$:

$$\hat{\omega_t^l} = \sigma_{\omega,t}\left(\psi_1 \frac{\omega_{t-1}^{dep,g} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t^l\right) + \mu_{\omega,t} \tag{9.19}$$

Pick up the cuts corresponding the mesh $D_{\underline{I}}$: $\left\{\alpha_{t+1}^{\underline{I},j}(s), \beta_{t+1}^{\underline{I},j}(s), \gamma_{t+1}^{\underline{I},j}(s)\right\}$, $j \in \{1; ...; (n+1)G\}$
Solve the following linear sub-problem ;

$$[AP_{t,l}^{'n,g}] \begin{cases} & Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l}, s_t^u) = \min\limits_{x_t, \theta_{t+1}} c_t(s_t^u)^\top x_t + \theta_{t+1}(s_t^u) \\ s.c. & A_t(s_t^u)x_t = P\hat{\omega_t^l} - E_t x_{t-1}^g, \quad [\pi_t(\hat{\omega_t^l}, s_t^u)] \\ & x_t \geqslant 0 \\ & \theta_{t+1}(s_t^u) + (\beta_{t+1}^{\underline{I},j}(s_t^u))^\top x_t + (\gamma_{t+1}^{\underline{I},j}(s_t^u))^\top \hat{\omega_t^l} \geqslant \alpha_{t+1}^{\underline{I},j}(s_t^u), \quad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.20}$$

Store the dual solution $\pi_t(\hat{\omega_t^l})$ and the primal solution $Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l}, s_t^u)$ of the problem $[AP_{t,l}^{'n,g}]$
Calculate the corresponding cut at the l^{th} draw of uncertainties :

$$\begin{cases} \hat{\alpha}_{t,l}^{g,\underline{I}}(s_t^u) = Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega_t^l}) + \pi_t(\hat{\omega_t^l})^\top \left(E_t x_{t-1}^g - \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P \omega_{t-1}^{dep,g}\right) \\ \hat{\beta}_{t,l}^{g,\underline{I}}(s_t^u) = E_t^\top \pi_t(\hat{\omega_t^l}) \\ \hat{\gamma}_{t,l}^{g,\underline{I}}(s_t^u) = \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P^\top \pi_t(\hat{\omega_t^l}) \end{cases} \tag{9.21}$$

end
Compute the cut for a non-convex random quantity s_t^u at time t at iteration n : it is defined as the weighted average on the L Benders cut obtained before :

$$\begin{cases} \hat{\alpha}_t^{g,\underline{I}}(s_t^u) = \frac{1}{L}\sum_{l=1}^{L} \hat{\alpha}_{t,l}^{g,\underline{I}}(s_t^u) \\ \hat{\beta}_t^{g,\underline{I}}(s_t^u) = \frac{1}{L}\sum_{l=1}^{L} \hat{\beta}_{t,l}^{g,\underline{I}}(s_t^u), \qquad j = nG + g \\ \hat{\gamma}_t^{g,\underline{I}}(s_t^u) = \frac{1}{L}\sum_{l=1}^{L} \hat{\gamma}_{t,l}^{g,\underline{I}}(s_t^u) \end{cases} \tag{9.22}$$

end
for $\underline{I_i, i \in \{1, ..., M_M\}}$ **do**
Compute the g^{th} new cut of the mesh $D_{\underline{I}_i}$ at time t at iteration n defined as the conditional expectation with respect to the scenario u at time t

$$\begin{cases} \alpha_t^{j,I}(s_{t-1}^u) = \mathbb{E}\left[\hat{\alpha}_t^{g,\underline{I}}(s_t^u)|s_{t-1}^u\right], \\ \beta_t^{j,I}(s_{t-1}^u) = \mathbb{E}\left[\hat{\beta}_t^{g,\underline{I}}(s_t^u)|s_{t-1}^u\right], \qquad j = nG + g \\ \gamma_t^{j,I}(s_{t-1}^u) = \mathbb{E}\left[\hat{\gamma}_t^{g,\underline{I}}(s_t^u)|s_{t-1}^u\right] \end{cases} \tag{9.23}$$

end
end
end
Solve the initial linear sub problem $[AP_0^{'n}]$;
Save the backward cost $\underline{z}_n = Q_0$

9.3 C++ API

The SDDP part of the stochastic library is in `C++` code. This unit is a classical black box : specific inputs have to be provided in order to get the expected results. In the SDDP unit backward and forward pass are achieved successively until the stopping criterion is reached. In this unit the succession of passes is realized by `backwardForwardSDDP` class. This class takes as input three non-defined classes.

9.3.1 Inputs

The user has to implement three classes.

- One class where the transition problem is described which is denoted in the example `TransitionOptimizer`. This class is at the core of the problem resolution. Therefore much flexibility is let to the user to implement this class. In some ways this class is the place where the technical aspects of the problem are adjusted. This class describes backward and forward pass. Four methods should be implemented :
  - updateDates : set new set of dates ("t", "t+dt")
  - oneStepForward : solves the different transition linear problem during the forward pass for a particle, a random vector and an initial state. :
    - the state (x_t, w_t^{dep}) is given as input of the function,
    - the s_t values are restored by the simulator,
    - the LP is solved between dates t and $t+dt$ for the given s_t and the constraints due to w_t^{dep} (demand, flow constraints) and permits to get the optimal x_{t+dt}.
    - Using iid sampling, w_{t+dt}^{dep} is estimated
    - return $(x_{t+dt}, w_{t+dt}^{dep})$ as the following state and (x_{t+dt}, w_t^{dep}) that will be used as the state to visit during next backward resolution.
  - oneStepBackward : solves the different transition linear problem during the backward pass for a particle, a random vector and an initial state.
    - The state (x_{t+dt}, w_t^{dep}) is given as input if $t \geq 0$ otherwise input is (x_0, w_0^{dep})
    - If $t \geq 0$, sample to calculate w_{t+dt}^{dep} in order to get the state $(x_{t+dt}, w_{t+dt}^{dep})$ at the beginning of the period of resolution of the LP.
    - Solve the LP from date $t+dt$ to next date $t+2dt$ (if equally spaced periods).
    - Return the function value and the dual that will be used for cuts estimations.
  - oneAdmissibleState : returns an admissible state at time t (respect only the constraints)

  `TransitionOptimizer` should derive from the *OptimizerSDDPBase* class defined below.

```cpp
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifndef OPTIMIZERSDDPBASE_H
#define OPTIMIZERSDDPBASE_H
#include <Eigen/Dense>
#include "StOpt/sddp/SDDPCutBase.h"
#include "StOpt/core/grids/OneDimRegularSpaceGrid.h"
#include "StOpt/core/grids/OneDimData.h"
#include "StOpt/sddp/SimulatorSDDPBase.h"


/** \file OptimizerSDDPBase.h
 *  \brief Define an abstract class for Stochastic Dual Dynamic Programming problems
 *     \author Xavier Warin
 */

namespace StOpt
{

/// \class OptimizerSDDPBase OptimizerSDDPBase.h
///  Base class for optimizer for Dynamic Programming
class OptimizerSDDPBase
{


public :

    OptimizerSDDPBase() {}

    virtual ~OptimizerSDDPBase() {}


    /// \brief Optimize the LP during backward resolution
    /// \param p_linCut cuts used for the PL (Benders for the Bellman value at the end of the time step)
    /// \param p_aState store the state, and 0.0 values
    /// \param p_particle  the particle n dimensional value associated to the regression
    /// \param p_isample   sample number for independant uncertainties
    /// \return  a vector with the optimal value and the derivatives if the function value with respect to each state
    virtual Eigen::ArrayXd oneStepBackward(const std::unique_ptr< StOpt::SDDPCutBase > &p_linCut, const std::tuple< std::shared_ptr<Eigen::ArrayXd>, int, int > &p_aState, const Eigen::ArrayXd &p_particle, const int &p_isample) const = 0;

    /// \brief Optimize the LP during forward resolution
    /// \param p_aParticle a particule in simulation part to get back cuts
    /// \param p_linCut cuts used for the PL  (Benders for the Bellman value at the end of the time step)
    /// \param p_state  store the state, the particle number used in
```

```
        optimization and mesh number associated to the particle. As an input
        it constains the current state
46      /// \param p_stateToStore for backward resolution we need to store \f$ (
        S_t,A_{t-1},D_{t-1}) \f$  where p_state in output is \f$ (S_t,A_{t},D_
        {t}) \f$
47      /// \param p_isimu           number of teh simulation used
48      virtual double oneStepForward(const Eigen::ArrayXd &p_aParticle, Eigen::
        ArrayXd &p_state,  Eigen::ArrayXd &p_stateToStore, const std::
        unique_ptr< StOpt::SDDPCutBase > &p_linCut,
49                                    const int &p_isimu) const = 0 ;
50
51
52      /// \brief update the optimizer for new date
53      ///         - In Backward mode, LP resolution achieved at date p_dateNext
        ,
54      ///           starting with uncertainties given at date p_date and
        evolving to give uncertainty at date p_dateNext,
55      ///         - In Forward mode,  LP resolution achieved  at date p_date,
56      ///           and uncertainties evolve till date p_dateNext
57      ///         .
58      virtual void updateDates(const double &p_date, const double &p_dateNext)
        = 0 ;
59
60      /// \brief Get an admissible state for a given date
61      /// \param  p_date   current date
62      /// \return an admissible state
63      virtual std::shared_ptr<Eigen::ArrayXd> oneAdmissibleState(const double &
        p_date) = 0 ;
64
65      /// \brief get back state size
66      virtual int getStateSize() const = 0;
67
68      /// \brief get the backward simulator back
69      virtual std::shared_ptr< StOpt::SimulatorSDDPBase > getSimulatorBackward
        () const = 0;
70
71      /// \brief get the forward simulator back
72      virtual std::shared_ptr< StOpt::SimulatorSDDPBase > getSimulatorForward()
         const = 0;
73
74  };
75  }
76  #endif /* OPTIMIZERSDDPBASE_H */
```

- A simulator for forward pass : `SimulatorSim`
- A simulator for backward pass : `SimulatorOpt`. This simulator can use an underlying process to generate scenarios, a set of historical chronicles or a discrete set of scenarios. Often in the realized test case a Boolean is enough to distinguish the forward and the backward simulator.

An abstract class for simulators is defined below.

```cpp
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#ifndef SIMULATORSDDPBASE_H
#define SIMULATORSDDPBASE_H
#include <Eigen/Dense>

/* \file SimulatorBase.h
 * \brief Abstract class for simulators for SDDP method
* \author Xavier Warin
*/
namespace StOpt
{
/// \class SimulatorSDDPBase SimulatorSDDPBase.h
/// Abstract class for simulators used for SDDP
class SimulatorSDDPBase
{
public :

    /// \brief Constructor
    SimulatorSDDPBase() {}

    /// \brief Destructor
    virtual ~SimulatorSDDPBase() {}

    /// \brief Get back the number of particles (used in regression part)
    virtual  int getNbSimul() const = 0;
    /// \brief Get back the number of sample used (simulation at each time step , these simulations are independent of the state)
    virtual  int getNbSample() const = 0;
    /// \brief Update the simulator for the date
    virtual  void updateDates(const double &p_date) = 0;
    /// \brief get one simulation
    /// \param p_isim  simulation number
    /// \return the particle associated to p_isim
    /// \brief get  current Markov state
    virtual  Eigen::VectorXd getOneParticle(const int &p_isim) const = 0;
    /// \brief get  current Markov state
    virtual  Eigen::MatrixXd getParticles() const = 0;
    /// \brief Reset the simulator (to use it again for another SDDP sweep)
    virtual  void resetTime() = 0;
    /// \brief in simulation  part of SDDP reset  time  and reinitialize uncertainties
    /// \param p_nbSimul  Number of simulations to update
    virtual  void updateSimulationNumberAndResetTime(const int &p_nbSimul) = 0;
};
}
#endif /* SIMULATORSDDPBASE_H */
```

9.3.2 Architecture

The SDDP handling part of the library is built following the scheme described below.

In the following pseudo-code you have to keep in mind that some small shortcuts have been used in view of making the reading reader-friendly (for example linear sub-problem in the initial case ($t = 0$) should be a bit different than the the one in other time-steps, `forwardSDDP()`,`backwardSDDP()`,`backwardforwardSDDP()` inputs have been omitted for simplification). A more rigorous theoretical explanation is available in the previous part.

Three colors have been used : blue parts correspond to the use of functions implemented in the **TransitionOptimizer** class, red parts correspond to the use of **Simulator**(Sim or Opt) functions while grey parts correspond to generic functions totally handled by the librabry. To be more accurate, what you have to implement as a StOpt user is only the **TransitionOptimizer** and the **Simulator** (blue and red part), other functions and described loops are already implemented and managed by the library.

Algorithm 8: Run of backwardforwardSDDP(),the main function)

Init: x_t^g=**TransitionOptimizer.oneadmissiblestate(t)**, for $g \in \{1, ..., G\}$ and $t \in \{1, ..., T-1\}$, $n = 0$

while $\psi > \epsilon$ and $n < n_{max}$ **do**

StOpt

V_b= backwardSDDP() *Using the previously computed set* $(x_t^g)_{t,g}$ *and creating a set of cuts*

V_f = forwardSDDP() *Simulation using the cuts created in all the backward passes and update the set* $(x_t^g)_{t,g}$

$$\psi = \frac{V_f - V_b}{V_f}$$

$$n = n + 1$$

end

Algorithm 9: Run of *forwardSDDP()* (n^{th} iteration)

for $g \in \Omega_{\mathscr{G}}$ **do**

for $t \in \{0, ..., T\}$ **do**

TransitionOptimizer.updatedates(t,t+1): update the required data following the current time step (iterator over current time step, average demand,...)

SimulatorSim.updatedates(t): give the random quantities (ω_t^g) for the scenario g at time t

StOpt Read the previously computed files to gather $\alpha_{t+1}^j, \beta_{t+1}^j$, for $j \in \{1, ..., G, ..., nG\}$

TransitionOptimizer.onestepforward():
Solve the following linear sub-problem. ;

$$[AP_{t,g}^n] \begin{cases} & Q_t^g(x_{t-1}^g, \omega_t^g) = \min\limits_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ s.c. \quad & A_t x_t = \omega_t^g - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^g)] \\ & x_t \geqslant 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geqslant \alpha_{t+1}^j, \qquad j \in \{1, ..., G, ..., nG\} \end{cases} \tag{9.24}$$

Compute the cost for current time step $c_t x_t^g$
Return: the primal solution (x_t^g) of the problem

StOpt Store the primal solution (x_t^g) of the problem $[AP_{t,g}^n]$

end

StOpt Compute the cost for scenario g, at iteration n : $\bar{z}_n^g = \sum_{t=0}^T c_t x_t^g$;

end

StOpt Compute the total cost in forward pass at iteration n : $\bar{z}_n = \frac{1}{G} \sum_{g=1}^G \bar{z}_n^g$

Algorithm 10: Run of *backwardSDDP()*

for $t = T, T-1, ..., 0$ **do**

StOpt Read the previously computed files to gather x_{t-1}^g, for $g \in \{1, ..., G, \}$

TransitionOptimizer.updatedates(t-1,t): update the required data following the current time step (iterator over current time step, average demand,...)

SimulatorOpt.updatedates(t): give the random quantities for the L scenarios at time t

StOpt Read the previously computed files to gather $\alpha_{t+1}^j, \beta_{t+1}^j$, for $j \in \{1, ..., G, ..., nG\}$

for x_{t-1}^g, $g \in \{1, ..., G\}$ **do**

for ω_t^l, $l \in \{1, ..., L\}$ **do**

TransitionOptimizer.onestepbackward()
Solve the following linear sub-problem. ;

$$[AP_{t,l}^{n,g}] \begin{cases} & Q_t^l(x_{t-1}^g, \omega_t^l) = \min\limits_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ s.c. & A_t x_t = \omega_t^l - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^l)] \\ & x_t \geqslant 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geqslant \alpha_{t+1}^j, \qquad j \in \{1, ..., G, ..., (n+1)G\} \end{cases} \tag{9.25}$$

Return: the dual solution $\pi_t(\omega_t^l)$ and the primal one $Q_t^l(x_{t-1}^g, \omega_t^l)$ of the linear sub-problem $[AP_{t,l}^{n,g}]$

end

StOpt Compute the g^{th} new Benders cut at time t at iteration n : α_t^j, β_t^j, for $j \in \{(n-1)G, (n-1)G+1, ..., nG\}$

end

end

StOpt Save the cost *backward* $\underline{z}_n = Q_0$

9.3.3 Implement your problem

In the following section, some tips and explanations will be given in view of helping you implementing your problem in the library. It is advised to have a look at the examples provided by the library. It will give you a better understanding of what is needed to compute the SDDP method through StOpt (folder test/c++/tools/sddp for the optimizer examples, test/c++/tools/simulators for the simulators one, and test/c++/functional for the main instances).

Implement your own TransitionOptimizer class

As described above, your **TransitionOptimizer** class should be specific to your problem (it's given as an argument of the `backwardforwardSDDP` function). Hence, you have to implement it by yourself following certain constraints in view of making it fitting the library requirements.
First, make it sure that your **TransitionOptimizer** class heritates from the class `OptimizerSDDPBase`. You will then have to implement the following functions.

- The `updateDates` function allows to update the data stored by the optimizer, fitting the times indicated as argument.

```
1     ///          .
```

If your transition problem depends on the time, you should for instance store those arguments value. Following your needs you could also update data such as the average demand at current and at next time step in a gas storage problem.
The $p_{dateNext}$ argument is used as the current time step in the backward pass. Hence, you should store the values for both the arguments current and next time step.

- The `oneAdmissibleState` function give an admissible state (that means a state respecting all the constraints) for the time step given as an argument.

```
1     /// \return an admissible state
```

- The `oneStepBackward` function allows to compute one step of the backward pass.

```
1     virtual Eigen::ArrayXd oneStepBackward(const std::unique_ptr< StOpt::
          SDDPCutBase > &p_linCut, const std::tuple< std::shared_ptr<Eigen
          ::ArrayXd>, int, int > &p_aState, const Eigen::ArrayXd &
          p_particle, const int &p_isample) const = 0;
```

The first argument is the cuts already selected for the current time step. It is easy to handle them, just use the `getCutsAssociatedToAParticle` function as described in the examples that you can find in the test folder (*OptimizeReservoirWithInflowsSDDP.h* without regression or *OptimizeGasStorageSDDP.h* with regression). You will then have the needed cuts as an array *cuts* that you can link to the values described in the theoretical part at the time step t by $cuts(0,j) = \alpha^j_{t+1}$, $cuts(i,j) = \beta^j_{i-1,t+1}$ $j \in \{1, ..., G, ..., (n+1)G\}$,$i \in \{1, ..., nb_{state}\}$.
You will have to add the cuts to your constraints by yourself, using this array and your solver functionnalities.
Moreover, as an argument you have the object containing the state at the beginning of the time step p_{astate} (**have in mind that this argument is given as an Eigen array**), $p_{particle}$ contains the random quantities in which the regression over the expectation of the value function will be based (the computational cost is high so have a look at the theoretical part to know when you really need to use this), finally the last argument is an integer giving in which scenario index the resolution will be done.
The function returns a 1-dimensional array of size $nb_{state}+1$ containing as a first argument the objective function, and then for $i \in \{1, ..., nb_{state}\}$ it contains the derivatives of the objective function compared to each of the i dimensions of the state (you have to find a way to have it by using the dual solution for instance).

- The `oneStepForward` function allows to compute one step of the foward pass.

```
1     virtual double oneStepForward(const Eigen::ArrayXd &p_aParticle,
          Eigen::ArrayXd &p_state,  Eigen::ArrayXd &p_stateToStore, const
          std::unique_ptr< StOpt::SDDPCutBase > &p_linCut,
```

As you can see, the `oneStepForward` is quite similar to the `oneStepBackward`. A tip, used in the examples and that you should use, is to build a function generating and solving the linear problem $[AP^n_{t,g}]$ (for a given scenario g and a given time step t) which appears for both the forward and the backward pass. This function creating and generating the linear problem will be called in both our functions `oneStepForward` and `oneStepBackward`. Take care that in the forward pass the current time step is given through the function `updateDates`(current date,next date) by the argument current date while in the backward pass the current time is given through the argument next date (this is a requirement needed to compute the regressions as exposed in the theoretical part). Finally note that the two previously described functions are `const` functions and you have to consider that during your implementation.

- The other functions that you have to implement are simple functions (accessors) easy to understand.

Implement your own Simulator class

This simulator should be the object that will allow you to build some random quantities following a desired law. It should be given as an argument of your optimizer. You can implement it by yourselft, however a set of simulators (gaussian, AR1, MeanReverting,...) are given in the test folder you could directly use it if it fits your problem requirements.
An implemented **Simulator** derivating from the `SimulatorSDDPBase` class needs to implement those functions:

- The `getNbSimul` function returns the number of simulations of random quantities used in regression part. It is the U hinted in the theoretical part.

```
1    virtual  int getNbSimul() const = 0;
```

- The `getNbSample` function returns the number of simulations of random quantities that are not used in the regression part. It is the G hinted in the theoretical part. For instance, in some instances we need a gaussian random quantity in view of computing the noise when we are in the "dependence of the random quantities" part.

```
1    virtual  int getNbSample() const = 0;
```

- The `updateDates` function is really similar to the optimizer one. However you just have one argument (the current time step) here. It is also here that you have to generate new random quantities for the resolution.

```
1    virtual  void updateDates(const double &p_date) = 0;
```

- The `getOneParticle` and the `getParticles` functions should return the quantities used in regression part.

```
1    virtual  Eigen::VectorXd getOneParticle(const int &p_isim) const = 0;
```

```
1    virtual  Eigen::MatrixXd getParticles() const = 0;
```

- The two last functions `resetTime` and `updateSimulationNumberAndResetTime` are quite explicit.

9.3.4 Set of parameters

The basic function `backwardForwardSDDP` should be called to use the SDDP part of the library. This function is templated by the regressor used :

- "LocalConstRegressionForSDDP" regressor permits to use a constant per mesh approximation of the SDDP cuts,
- " LocalLinearRegressionForSDDP" regressor permits to use a linear approximation per mesh of the SDDP cuts.

```
1  /// \param  p_accuracy            accuracy asked , on return estimation of
       accuracy achieved (expressed in %)
2  /// \param  p_nStepConv           every p_nStepConv convergence is checked
3  /// \param  p_stringStream        dump all print messages
4  /// \param  p_bPrintTime          if true print time at each backward and
       forward step
5  /// \return backward and forward valorization
6  template<  class LocalRegressionForSDDP>
7  std::pair<double, double> backwardForwardSDDP(std::shared_ptr<
       OptimizerSDDPBase>    &p_optimizer,
8          const int   &p_nbSimulCheckForSimu,
9          const Eigen::ArrayXd &p_initialState,
10         const SDDPFinalCut &p_finalCut,
11         const Eigen::ArrayXd &p_dates,
12         const Eigen::ArrayXi &p_meshForReg,
13         const std::string &p_nameRegressor,
14         const std::string &p_nameCut,
15         const std::string &p_nameVisitedStates,
```

Most of the arguments are pretty clear (You can see examples in test/c++/functional). The strings correspond to names that will be given by the files which will store cuts, visited states or regressor data. $p_{nbSimulCheckForSimu}$ corresponds to the number of simulations (number of foward pass called) when we have to check the convergence by comparing the outcome given by the forward pass and the one given by the backward pass. $p_{nStepConv}$ indicates when the convergence is checked (each $p_{nStepConv}$ iteration). $p_{finalCut}$ corresponds to the cut used at the last time step : when the final value function is zero, the last cut is given by an all zero array of size $nb_{state} + 1$. p_{dates} is an array made up with all the time steps of the study period given as doubles, p_{iter} correspond to the maximum number of iterations. Finally, $p_{stringStream}$ is an `ostringstream` in which the result of the optimization will be stored.

9.3.5 The black box

The algorithms described above are applied. As said before the user controls the implementation of the business side of the problem (transition problem). But in the library a few things are managed automatically and the user has to be aware of :

- The **Parallelization** during the problem resolution is managed automatically. During compilation, if the compiler detects an MPI (Message Passing Interface)library problem resolution will be achieved in a parallelized manner.
- The **cut management**. All the cuts added at each iteration are currently serialized and stored in an archive initialized by the user. No cuts are pruned. In the future one can consider to work on cuts management [31].
- A **double stopping criterion** is barely used by the library : a convergence test and a maximal number of iterations. If one of the two criteria goes over the thresholds defined by the user resolution stops automatically. Once again further work could be considered on that topic.

9.3.6 Outputs

The outputs of the SDDP library are not currently defined. Thus during the resolution of a SDDP problem only the number of iterations, the evolution of the backward and forward costs and of the convergence criterion are logged.

Yet while iterating backward and forward pass the value of the Bellman functions and the related Benders cuts , the different states visited during the forward pass and the costs evolution are stored at each time of the time horizon. These information are helpful for the users and easy to catch.
Once the convergence is achieved, the user should rerun some simulations adding some flag to store the results needed by the application (distribution cost etc...) : these results will be post-processed by the user.

9.4 Python API

A high level Python mapping is also available in the SDDP part. The backward-forward `C++` function is exposed in Python by the SDDP module "StOptSDDP". In this mapping only the linear per mesh regressor is used.

```
1 import StOptSDDP
2 dir(StOptSDDP)
```

that should give

```
['OptimizerSDDPBase','SDDPFinalCut','SimulatorSDDPBase','__doc__','__file__','__name__','__package__','backwardForwardSDDP']
```

The "backwardForwardSDDP" realize the forward backard SDDP sweep giving a SDDP optimizer and a SDDP uncertainty simulator. The initial final cuts for the last time steps

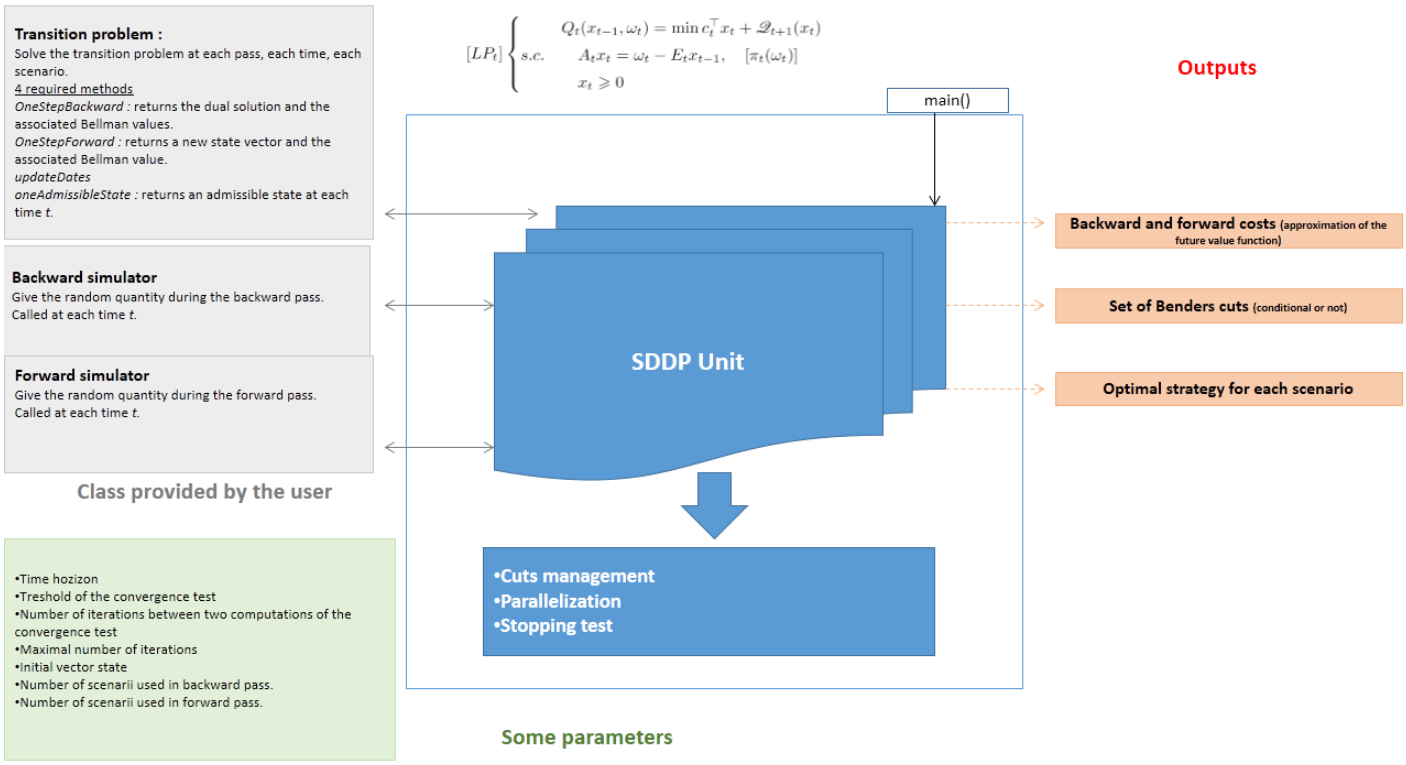

Figure 9.1: Current architecture of the generic SDDP unit

are provided by the "'SDDPFinalCut"' object.
To realize the mapping of SDDP optimizers and simulators written in C++ it is necessary to create a Boost Python wrapper. In order to expose the C++ optimizer class "OptimizeDemandSDDP"used in the test case "testDemandSDDP.cpp", the following wrapper can be found in
"StOpt/test/c++/python/BoostPythonSDDPOptimizers.cpp"

```
// Copyright (C) 2016 EDF
// All Rights Reserved
// This code is published under the GNU Lesser General Public License (GNU LGPL)
#include <boost/version.hpp>
#include "StOpt/core/grids/OneDimRegularSpaceGrid.h"
#include "StOpt/core/grids/OneDimData.h"
#include "StOpt/sddp/OptimizerSDDPBase.h"
#include "test/c++/tools/sddp/OptimizeDemandSDDP.h"
#include "test/c++/tools/simulators/SimulatorGaussianSDDP.h"
#ifdef _WIN32
#include <boost/shared_ptr.hpp>
#include "StOpt/python/BoostToStdSharedPtr.h"
#endif


#ifdef __linux__
#ifdef __clang__
#if BOOST_VERSION <  105600
// map std::shared ptr to boost python
namespace boost
{
template<class T> T *get_pointer(std::shared_ptr<T> p)
{
    return p.get();
}
}
#endif
#endif
#endif

#include <boost/python.hpp>
#include "test/c++/python/FutureCurveWrap.h"

/** \file BoostPythonSDDPOptimizers.cpp
*  \brief permits to map Optimizers for SDDP
* \author Xavier Warin
*/

#ifdef _DEBUG
#undef _DEBUG
#include <Python.h>
#define _DEBUG
#else
#include <Python.h>
#endif
```

```cpp
#include <numpy/arrayobject.h>
#include "StOpt/python/NumpyConverter.hpp"




using namespace boost::python;

/// \wrapper for Optimizer for demand test case in SDDP
class OptimizeDemandSDDPWrap : public OptimizeDemandSDDP<SimulatorGaussianSDDP>
{
public :

    /// \brief Constructor
    /// \param   p_sigD                volatility for demand
    /// \param   p_kappaD              AR coefficient for demand
    /// \param   p_timeDAverage        average demand
    /// \param   p_spot                Spot price
    /// \param   p_simulatorBackward   backward simulator
    /// \param   p_simulatorForward    Forward simulator
#ifdef _WIN32
    OptimizeDemandSDDPWrap(const double &p_sigD, const double &p_kappaD,
                           const FutureCurve &p_timeDAverage,
                           const double &p_spot,
                           const boost::shared_ptr<SimulatorGaussianSDDP> &p_simulatorBackward,
                           const boost::shared_ptr<SimulatorGaussianSDDP> &p_simulatorForward):
        OptimizeDemandSDDP(p_sigD, p_kappaD,
                           std::make_shared< StOpt::OneDimData< StOpt::OneDimRegularSpaceGrid, double> >(static_cast< StOpt::OneDimData< StOpt::OneDimRegularSpaceGrid, double> >(p_timeDAverage)),
                           p_spot, make_shared_ptr<SimulatorGaussianSDDP>(p_simulatorBackward), make_shared_ptr<SimulatorGaussianSDDP>(p_simulatorForward)) { }
#else
    OptimizeDemandSDDPWrap(const double &p_sigD, const double &p_kappaD,
                           const FutureCurve &p_timeDAverage,
                           const double &p_spot,
                           const std::shared_ptr<SimulatorGaussianSDDP> &p_simulatorBackward,
                           const std::shared_ptr<SimulatorGaussianSDDP> &p_simulatorForward):
        OptimizeDemandSDDP(p_sigD, p_kappaD,
                           std::make_shared< StOpt::OneDimData< StOpt::OneDimRegularSpaceGrid, double> >(static_cast< StOpt::OneDimData< StOpt::OneDimRegularSpaceGrid, double> >(p_timeDAverage)),
                           p_spot, p_simulatorBackward, p_simulatorForward) { }
```

```
#endif
};


// MSVC 2015 BUG
#if (_MSC_VER == 1900)
namespace boost
{
template <>
OptimizeDemandSDDPWrap  const volatile *get_pointer< class
    OptimizeDemandSDDPWrap const volatile >(
        class          OptimizeDemandSDDPWrap  const volatile *c)
{
    return c;
}
}
#endif

BOOST_PYTHON_MODULE(SDDPOptimizers)
{


    Register<Eigen::ArrayXd>();
    Register<Eigen::ArrayXXd>();

    // map optimizer for demand test case
#ifdef  _WIN32
    class_<OptimizeDemandSDDPWrap, std::shared_ptr<OptimizeDemandSDDPWrap>,
        bases<StOpt::OptimizerSDDPBase> >("OptimizeDemandSDDP",
            init< const  double &, const double &,  const  FutureCurve &,
            const double &,
            const boost::shared_ptr<SimulatorGaussianSDDP> &,
            const boost::shared_ptr<SimulatorGaussianSDDP> &>())
#else
    class_<OptimizeDemandSDDPWrap, std::shared_ptr<OptimizeDemandSDDPWrap>,
        bases<StOpt::OptimizerSDDPBase> >("OptimizeDemandSDDP",
            init< const  double &, const double &,  const  FutureCurve &,
            const double &,
            const std::shared_ptr<SimulatorGaussianSDDP> &,
            const std::shared_ptr<SimulatorGaussianSDDP> &>())
#endif
    .def("getSimulatorBackward", &OptimizeDemandSDDP<SimulatorGaussianSDDP>::
        getSimulatorBackward)
    .def("getSimulatorForward", &OptimizeDemandSDDP<SimulatorGaussianSDDP>::
        getSimulatorForward)
    .def("oneAdmissibleState", &OptimizeDemandSDDP<SimulatorGaussianSDDP>::
        oneAdmissibleState)
    ;
}
```

The wrapper used to expose the SDDP simulator is given in
"StOpt/test/c++/python/BoostPythonSimulators.cpp"
Then it is possible to use the mapping to write a Python version of "testDemandSDDP.cpp"

```
# Copyright (C) 2016 EDF
```

```
# All Rights Reserved
# This code is published under the GNU Lesser General Public License (GNU LGPL)
import StOptGrids
import StOptSDDP
import StOptGlobal
import sys
sys.path.append("./sddp")
import Utils
import SDDPSimulators as sim
import SDDPOptimizers as opt
import numpy as NP
import unittest
import math
import imp
import backwardForwardSDDP as bfSDDP # import of the function written in python

# unitest equivalent of testDemandSDDP : here MPI version
# High level python interface : at level of the backwardForwardSDDP c++ file
##########################################################################
def demandSDDPFunc(p_sigD, p_sampleOptim ,p_sampleCheckSimul):

        maturity = 40
        nstep = 40;

        # optimizer parameters
        kappaD = 0.2; # mean reverting coef of demand
        spot = 3 ; # spot price

         # define a a time grid
        timeGrid = StOptGrids.OneDimRegularSpaceGrid(0., maturity / nstep, nstep)

        # periodicity factor
        iPeriod = 52;
        # average demande values
        demValues = []

        for i in list(range(nstep + 1)) :
            demValues.append(2. + 0.4 * math.cos((math.pi * i * iPeriod) / nstep))

        # define average demand
        demGrid = Utils.FutureCurve(timeGrid, demValues)

        initialState = demGrid.get(0.)*NP.ones(1)

        finCut = StOptSDDP.SDDPFinalCut(NP.zeros((2,1)))

        # here cuts are not conditional to an uncertainty
        nbMesh = NP.array([],NP.int32)
        nbUncertainties = 1;

```

```
52          # backward simulator
53          backwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties,
                p_sampleOptim)
54          # forward simulator
55          forwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties)
56
57          # Create the optimizer
58          optimizer = opt.OptimizeDemandSDDP(p_sigD, kappaD,  demGrid, spot,
                backwardSimulator, forwardSimulator)
59
60          # optimisation dates
61          dates = NP.linspace( 0., maturity,nstep + 1);
62
63          # names for archive
64          nameRegressor = "RegressorDemand";
65          nameCut = "CutDemand";
66          nameVisitedStates = "VisitedStateDemand";
67
68          #  precision parameter
69          nIterMax = 40
70          accuracyClose =  1.
71          accuracy = accuracyClose / 100.
72          nstepIterations = 4; # check for convergence between nstepIterations
                step
73
74          values = StOptSDDP.backwardForwardSDDP(optimizer,  p_sampleCheckSimul
                , initialState, finCut, dates,  nbMesh, nameRegressor, nameCut,
                nameVisitedStates, nIterMax,
75                                                      accuracy, nstepIterations
                                                            );
76
77          print("Values " , values)
78          return values
79
80
81  # unitest equivalent of testDemandSDDP : here low interface python version
82  # Low level python interface : use backwardForwardSDDP.py
83  ##############################################################################
84  def demandSDDPFuncLowLevel(p_sigD, p_sampleOptim ,p_sampleCheckSimul):
85
86          maturity = 40
87          nstep = 40;
88
89          # optimizer parameters
90          kappaD = 0.2; # mean reverting coef of demand
91          spot = 3 ; # spot price
92
93           # define a a time grid
94          timeGrid = StOptGrids.OneDimRegularSpaceGrid(0., maturity / nstep,
                nstep)
95
96
97          # periodicity factor
98          iPeriod = 52;
```

```
99          # average demande values
100         demValues = []
101
102         for i in list(range(nstep + 1)) :
103             demValues.append(2. + 0.4 * math.cos((math.pi * i * iPeriod) /
                nstep))
104
105         # define average demand
106         demGrid =Utils.FutureCurve(timeGrid, demValues)
107
108         initialState = demGrid.get(0.)*NP.ones(1)
109
110         finCut = StOptSDDP.SDDPFinalCut(NP.zeros((2,1)))
111
112         # here cuts are not conditional to an uncertainty
113         nbMesh = NP.array([],NP.int32)
114         nbUncertainties = 1;
115
116         # backward simulator
117         backwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties,
                p_sampleOptim)
118         # forward simulator
119         forwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties)
120
121         # Create the optimizer
122         optimizer = opt.OptimizeDemandSDDP(p_sigD, kappaD,  demGrid, spot,
                backwardSimulator, forwardSimulator)
123
124         # optimisation dates
125         dates = NP.linspace( 0., maturity,nstep + 1);
126
127         # names for archive
128         nameRegressor = "RegressorDemand";
129         nameCut = "CutDemand";
130         nameVisitedStates = "VisitedStateDemand";
131
132         #  precision parameter
133         nIterMax = 40
134         accuracyClose =  1.
135         accuracy = accuracyClose / 100.
136         nstepIterations = 4; # check for convergence between nstepIterations
                step
137
138         values = bfSDDP.backwardForwardSDDP(optimizer,  p_sampleCheckSimul,
                initialState, finCut, dates,  nbMesh, nameRegressor,
139                                             nameCut, nameVisitedStates, nIterMax,
140                                             accuracy, nstepIterations);
141
142         return values
143
144
145 class testDemandSDDP(unittest.TestCase):
146     def testDemandSDDP1D(self):
147         try:
```

```
148             imp.find_module('mpi4py')
149             found = True
150         except:
151             print("Not parallel module found ")
152             found = False
153 
154         if found :
155             from mpi4py import MPI
156             world = MPI.COMM_WORLD
157 
158             sigD = 0.6 ;
159             sampleOptim = 500;
160             sampleCheckSimul = 500;
161 
162             values = demandSDDPFunc(sigD, sampleOptim ,sampleCheckSimul)
163 
164             if (world.rank==0):
165                     print("Values is ",values)
166 
167     def testDemandSDDP1DLowLevel(self):
168         sigD = 0.6 ;
169         sampleOptim = 500;
170         sampleCheckSimul = 500;
171         demandSDDPFuncLowLevel(sigD, sampleOptim ,sampleCheckSimul)
172 
173 
174 if __name__ == '__main__':
175     unittest.main()
```

Part VII

Some test cases description

In this part, we describe the functional test cases of the library. The c++ version of these test cases can be found in "test/c++/functional" while their python equivalent (when existing) can be found in "test/python/functional". We describe here in details the c++ test cases.

9.5 American option

The library gives some test cases for the Bermudean option problem ([16] for details on the bermudean option problem). All Bermudean test cases use a basket option payoff. The reference for the converged methods can be found in [16].

9.5.1 testAmerican

The test case in this file permits to test during the Dynamic Programming resolution different regressors :

- either using some local functions basis with support of same size :
  - Either using a constant per mesh representation of the function ("LocalSameSizeConstRegression" regressor)
  - Either using a linear per mesh representation of the function ("LocalSameSizeLinearRegression" regressor)
- either using some function basis with adaptive support ([16])
  - Either using a constant per mesh representation of the function ("LocalConstRegression" regressor)
  - Either using a linear per mesh representation of the function ("LocalLinearRegression" regressor)
- Either using global polynomial regressor :
  - Either using Hermite polynomials,
  - Either using Canonical polynomials (monomes),
  - Either using Tchebychev polynomials.
- Either using sparse regressor,
- Either using kernel regressors :
  - either using constant kernel regressor,
  - either using linear kernel regressor.

testAmericanLinearBasket1D

Test 1D problem with "LocalLinearRegression" regressor.

testAmericanConstBasket1D

Test 1D problem with "LocalConstRegression" regressor.

testAmericanSameSizeLinearBasket1D

Test 1D problem with "LocalSameSizeLinearRegression" regressor.

testAmericanSameSizeConstBasket1D

Test 1D problem with LocalSameSizeConstRegression regressor.

testAmericanGlobalBasket1D

Test 1D problem with global Hermite, Canonical and Tchebychev regressor.

testAmericanGridKernelConstBasket1D

Test 1D problem with classical kernel regression

testAmericanGridKernelLinearBasket1D

Test 1D problem with linear kernel regression

testAmericanLinearBasket2D

Test 2D problem with "LocalLinearRegression" regressor.

testAmericanConstBasket2D

Test 2D problem with "LocalConstRegression" regressor.

testAmericanSameSizeLinearBasket2D

Test 2D problem with "LocalSameSizeLinearRegression" regressor.

testAmericanSameSizeConstBasket2D

Test 2D problem with LocalSameSizeConstRegression regressor.

testAmericanGlobalBasket2D

Test 2D problem with global Hermite, Canonical and Tchebychev regressor.

testAmericanGridKernelConstBasket2D

Test 2D problem with classical kernel regression

testAmericanGridKernelLinearBasket1D

Test 2D problem with linear kernel regression

testAmericanBasket3D

Test 3D problem with "LocalLinearRegression" regressor.

testAmericanGlobalBasket3D

Test 3D problem with global Hermite, Canonical and Tchebychev regressor.

testAmericanGridKernelLinearBasket3D

Test 3D problem with linear kernel regression.

testAmericanBasket4D

Test 4D problem with "LocalLinearRegression" regressor.

9.5.2 testAmericanConvex

Three test cases with basket american options are implemented trying to keep convexity of the solution

testAmericanLinearConvexBasket1D

Linear adapted regression in 1D preserving the convexity at each time step.

testAmericanLinearConvexBasket2D

Linear adapted regression in 2D trying to preserve the convexity at each time step.

testAmericanLinearConvexBasket3D

Linear adapted regression in 3D trying to preserve the convexity at each time step.

9.5.3 testAmericanForSparse

This test case is here to test sparse grid regressors (see section 1.3). As described before we can use a linear, quadratic or cubic representation on each cell. The reference is the same as in the testAmerican subsection so linked to a Bermudean basket option.

testAmericanSparseBasket1D

Use sparse grids in 1D (so equivalent to full grid) for linear, quadratic or cubic representation.

testAmericanSparseBasket2D

Use sparse grids in 2D for linear, quadratic or cubic representation.

testAmericanSparseBasket3D

Use sparse grids in 3D for linear, quadratic or cubic representation.

testAmericanSparseBasket4D

Use sparse grids in 4D for linear, quadratic or cubic representation.

9.5.4 testAmericanOptionCorrel

Same case as before but with correlations betwwen assets. Permits to test that rotation due to the PCA analysis works correctly.

testAmericCorrel

Check in 2D that

- Local Constant per mesh regression with and without rotation give the same result,
- Local Linear per mesh regression with and without rotation give the same result,
- Global regression with and without rotation give the same result.

9.6 testSwingOption

The swing option problem is the generalization of the American option using a Black Scholes model for the underlying asset : out of a set of "nStep" dates (chosen equal to 20 here) we can choose N dates (N equal to three) to exercise the option. At each exercise date t , we get the pay-off $(S_t - K)^+$ where S_t is the value of the underlying asset at date t. See [35] for description of the swing problem and the backward resolution techniques. Due to classical results on the Snell envelop for European payoff, the analytical value of this problem is the sum of the N payoff at the N last dates where we can exercise (recall that the value of an American call is the value of the European one). The Markov state of the problem at a given date t is given by the value of the underlying (Markov) and the number of exercises already achieved at date t. This test case can be run in parallel with MPI. In all test cases, we use a "LocalLinearRegression" to evaluate the conditional expectations used during the Dynamic Programming approach.

testSwingOptionInOptimization

After having calculated the analytical solution for this problem,

- a first resolution is provided using the "resolutionSwing" function. For this simple problem, only a regressor is necessary to decide if we exercise at the current date of not.

- a second resolution is provided in the "resolutionSwingContinuation" function using the "Continuation" object (see chapter 3) permitting to store continuation values for a value of the underlying and for a stock level. This example is provide here to show how to use this object on a simple test case. This approach is here not optimal because getting the continuation value for an asset value and a stock level (only discrete here) means some unnecessary interpolation on the stock grids (here we choose a "RegularSpaceGrid" to describe the stock level and interpolate linearly between the stock grids). In the case of swing with varying quantities to exercise [35] or the gas storage problem, this object is very useful,

- A last resolution is provided using the general framework described and the "DynamicProgrammingByRegressionDist" function described in subsection 5.2.2. Once again the framework is necessary for this simple test case, but it shows that it can be used even for some very simple cases.

9.6.1 testSwingOption2D

Here we suppose that we have two similar swing options to price and we solve the problem ignoring that the stocks are independent : this means that we solve the problem on a two dimensional grid (for the stocks) instead of two times the same problem on a grid with one stock.

- we begin by an evaluation of the solution for a single swing with the "resolutionSwing" function giving a value A.

- then we solve the 2 dimensional (in stock) problem giving a value B with our framework with the "DynamicProgrammingByRegressionDist" function.

Then we check that $B = 2A$.

9.6.2 testSwingOption3

We do the same as previously but the management of three similar swing options is realized by solving as a three dimensional stock problem.

9.6.3 testSwingOptimSimu / testSwingOptimSimuMpi

This test case takes the problem described in section 9.6, solves it using the framework 5.2.2. Once the optimization using regression ("LocalLinearRegression" regressor) is achieved, a simulation part is used using the previously calculated Bellman values. We check the the values obtained in optimization and simulation are close. The two test case files (testSwingOptimSimu/testSwingOptimSimuMpi) use the two versions of MPI parallelization distributing or not the data on the processors.

9.6.4 testSwingOptimSimuWithHedge

The test case takes the problem described in section 9.6, solves it using regression ("LocalLinearRegression" regressor) while calculating the optimal hedge by the conditional tangent method as explained in [21]. After optimization, a simulation part implement the optimal control and the optimal hedge associated. We check :

- That values in optimization and simulation are close
- That the hedge simulated has an average nearly equal to zero,
- That the hedged swing simulations give a standard deviation reduced compared to the non hedged option value obtained by simulation without hedge.

This test case shows are that the multiple regimes introduced in the framework 5.2.2 can be used to calculate and store the optimal hedge. This is achieved by the creation of a dedicated optimizer "OptimizeSwingWithHedge".

9.6.5 testSwingOptimSimuND / testSwingOptimSimuNDMpi

The test case takes the problem described in section 9.6, suppose that we have two similar options to valuate and that we ignore that the options are independent giving a problem to solve with two stocks managed jointly as in subsection 9.6.1. After optimizing the problem using regression ("LocalLinearRegression" regressor) we simulate the optimal control for this two dimensional problem and check that values in optimization and simulation are close. In "testSwingOptimSimuND" Mpi parallelization, if activated, only parallelize the calculation, while in "testSwingOptimSimuNDMpi" the data are also distributed on processors. In the latter, two options are tested,

- in "testSwingOptionOptim2DSimuDistOneFile" the Bellman values are distributed on the different processors but before being dumped they are recombine to give a single file for simulation.
- in "testSwingOptionOptim2DSimuDistMultipleFile" the Bellman values are distributed on the different processors but each processor dumps its own Bellman Values. During the simulation, each processor rereads its own Bellman values.

In the same problem in high dimension may be only feasible with the second approach.

9.7 Gas Storage

9.7.1 testGasStorage / testGasStorageMpi

The model used is a mean reverting model similar to the one described in [21]. We keep only one factor in equation (8) in [21]. The problem consists in maximizing the gain from a gas storage by the methodology described in [21]. All test cases are composed of three parts :

- an optimization is realized by regression ("LocalLinearRegression" regressor),
- a first simulation of the optimal control using the continuation values stored during the optimization part,
- a second simulation directly using the optimal controls stored during the optimization part.

We check that the three previously calculated values are close.
Using dynamic programming method, we need to interpolate into the stock grid to get the Bellman values at one stock point. Generally a simple linear interpolator is used (giving a monotone scheme). As explicated in [24], it is possible to use higher order schemes still being monotone. We test different interpolators. In all test case we use a "LocalLinearRegression" to evaluate the conditional expectations. The MPI version permits to test the distribution of the data when using parallelization.

testSimpleStorage

We use a classical regular grid with equally spaces points to discretize the stock of gas and a linear interpolator to interpolate in the stock.

testSimpleStorageLegendreLinear

We use a Legendre grid with linear interpolation, so the result should be the same as above.

testSimpleStorageLegendreQuadratic

We use a quadratic interpolator for the stock level.

testSimpleStorageLegendreCubic

We use a cubic interpolator for the stock level.

testSimpleStorageSparse

We use a sparse grid interpolator (equivalent to a full grid interpolator because it is a one dimensional problem). We only test the sparse grid with a linear interpolator.

9.7.2 testGasStorageKernel

The model used is a mean reverting model similar to the one described in [21]. We keep only one factor in equation (8) in [21].The problem consists in maximizing the gain from a gas storage by the methodology described in [21]. The specificity here is that a kernel regression method is used.

testSimpleStorageKernel

Use the linear kernel regression method to solve the Gas Storage problem.

9.7.3 testGasStorageVaryingCavity

The stochastic model is the same as in section 9.7.1. As previously, all test cases are composed of three parts :

- an optimization is realized by regression ("LocalLinearRegression" regressor),
- a first simulation of the optimal control using the continuation values stored during the optimization part,
- a second simulation directly using the optimal controls stored during the optimization part.

We check that the three previously calculated values are close on this test case where the grid describing the gas storage constraint is time varying. This permits to check the splitting of the grids during parallelization.

9.7.4 testGasStorageSwitchingCostMpi

The test case is similar to the one in section 9.7.1 (so using regression methods) : we added some extra cost when switching from each regime to the other. The extra cost results in the fact that the Markov state is composed of the asset price, the stock level and the current regime we are (the latter is not present in other test case on gas storage). This test case shows that our framework permits to solve regime switching problems. As previously all test cases are composed of three parts :

- an optimization is realized by regression ("LocalLinearRegression" regressor),
- a first simulation of the optimal control using the continuation values stored during the optimization part,
- a second simulation directly using the optimal controls stored during the optimization part.

We check that the three previously calculated values are close.

9.7.5 testGasStorageSDDP

The modelization of the asset is similar to the other test case. We suppose that we have N similar independent storages. So solving the problem with N stocks should give N times the value of one stock.

- First the value of the storage is calculated by dynamic programming giving value A,
- then the SDDP method (chapter 9) is used to valuate the problem giving the B value. The Benders cuts have to be done conditionally to the price level.

We check that B is close to NA.

testSimpleStorageSDDP1D

Test the case $N = 1$.

testSimpleStorageSDDP2D

Test the case $N = 2$.

testSimpleStorageSDDP10D

Test the case $N = 10$.

9.8 testLake / testLakeMpi

This is the case of a reservoir with inflows following an AR1 model. We can withdraw water from the reservoir (maximal withdrawal rate given) to produce energy by selling it at a given price (taken equal to 1 by unit volume). We want to maximize the expected earnings obtained by an optimal management of the lake. The problem permits to show how some stochastic inflows can be taken into account with dynamic programming with regression ("LocalLinearRegression" regressor used).
The test case is compose of three parts :

- an optimization is realized by regression ("LocalLinearRegression" regressor),
- a first simulation of the optimal control using the continuation values stored during the optimization part,
- a second simulation directly using the optimal controls stored during the optimization part.

We check that the three previously calculated values are close.

9.9 testDemandSDDP

This test case is the most simple using the SDDP method. We suppose that we have a demand following an AR 1 model

$$D^{n+1} = k(D^n - D) + \sigma_d g + kD,$$

where D is the average demand, σ_d the standard deviation of the demand on one time step, k the mean reverting coefficient, $D^0 = D$, and g a unit centered Gaussian variable. We have to satisfy the demand by buying energy at a price P. We want to calculate the following expected value

$$\begin{aligned} V &= P\mathbb{E}(\sum_{i=0}^{N} D_i) \\ &= (N+1)D_0 P \end{aligned}$$

This can be done (artificially) using SDDP.

testDemandSDDP1DDeterministic

It takes $\sigma_d = 0$.

testDemandSDDP1D

It solves the stochastic problem.

9.10 Reservoir variations with SDDP

9.10.1 testReservoirWithInflowsSDDP

For this SDDP test case, we suppose that we dispose of N similar independent reservoirs with inflows given at each time time by independent centered Gaussian variables with standard deviation σ_i. We suppose that we have to satisfy at M dates a demand given by independent centered Gaussian variables with standard deviation σ_d. In order to satisfy the demand, we can buy some water with quantity q_t at a deterministic price S_t or withdraw water from the reservoir at a pace lower than a withdrawal rate. Under the demand constraint, we want to minimize :

$$\mathbb{E}(\sum_{i=0}^{M} q_t S_t)$$

Each time we check that forward and backward methods converge to the same value. Because of the independence of uncertainties the dimension of the Markov state is equal to N.

testSimpleStorageWithInflowsSDDP1DDeterminist

$\sigma_i = 0$ for inflows and $\sigma_d = 0.$ for demand. N taken equal to 1.

testSimpleStorageWithInflowsSDDP2DDeterminist

$\sigma_i = 0$ for inflows and $\sigma_d = 0.$ for demand. N taken equal to 2.

testSimpleStorageWithInflowsSDDP5DDeterminist

$\sigma_i = 0$ for inflows and $\sigma_d = 0.$ for demand. N taken equal to 5.

testSimpleStorageWithInflowsSDDP1D

$\sigma_i = 0.6$, $\sigma_d = 0.8$ for demand. $N = 1$

testSimpleStorageWithInflowsSDDP2D

$\sigma_i = 0.6$ for inflows, $\sigma_d = 0.8$ for demand. $N = 2$

testSimpleStorageWithInflowsSDDPD

$\sigma_i = 0.6$ for inflows, $\sigma_d = 0.8$ for demand. $N = 5$.

9.10.2 testStorageWithInflowsSDDP

For this SDDP test case, we suppose that we dispose of N similar independent reservoirs with inflows following an AR1 model :

$$X^{n+1} = k(X^n - X) + \sigma g + X,$$

with $X^0 = X$, σ the standard deviation associated, g some unit centered Gaussian variable. We suppose that we have to satisfy at M dates a demand following an AR1 process too. In order to satisfy the demand, we can buy some water with quantity q_t at a deterministic price S_t or withdraw water from the reservoir at a pace lower than a withdrawal rate. Under the demand constraint, we want to minimize :

$$\mathbb{E}(\sum_{i=0}^{M} q_t S_t)$$

Each time we check that forward and backward methods converge to the same value. Because of the structure of the uncertainties the dimension of the Markov state is equal to $2N + 1$ (N storage, N inflows, and demand).

testSimpleStorageWithInflowsSDDP1DDeterministic

All parameters σ are set to 0. $N = 1$.

testSimpleStorageWithInflowsSDDP2DDeterministic

All parameters σ are set to 0. $N = 2$.

testSimpleStorageWithInflowsSDDP5DDeterministic

All parameters σ are set to 0. $N = 5$.

testSimpleStorageWithInflowsSDDP10DDeterministic

All parameters σ are set to 0. $N = 10$.

testSimpleStorageWithInflowsSDDP1D

$\sigma = 0.3$ for inflows, $\sigma = 0.4$ for demand. $N = 1$.

testSimpleStorageWithInflowsSDDP5D

$\sigma = 0.3$ for inflows, $\sigma = 0.4$ for demand. $N = 5$.

9.10.3 testStorageWithInflowsAndMarketSDDP

This is the same problem as 9.10.2, but the price S_t follow an AR 1 model. We use a SDDP approach to solve this problem. Because of the price dependencies, the SDDP cut have to be done conditionally to the price level.

testSimpleStorageWithInflowsAndMarketSDDP1DDeterministic

All volatilities set to 0. $N = 1$.

testSimpleStorageWithInflowsAndMarketSDDP2DDeterministic

All volatilities set to 0. $N = 2$.

testSimpleStorageWithInflowsAndMarketSDDP5DDeterministic

All volatilities set to 0. $N = 5$.

testSimpleStorageWithInflowsAndMarketSDDP10DDeterministic

All volatilities set to 0. $N = 10$.

testSimpleStorageWithInflowsAndMarketSDDP1D

$\sigma = 0.3$ for inflows, $\sigma = 0.4$ for demand, $\sigma = 0.6$ for the spot price. $N = 1$.

testSimpleStorageWithInflowsAndMarketSDDP5D

$\sigma = 0.3$ for inflows, $\sigma = 0.4$ for demand, $\sigma = 0.6$ for the spot price. $N = 5$.

9.11 Semi-Lagrangian

9.11.1 testSemiLagrangCase1/testSemiLagrangCase1

Test Semi-Lagrangian deterministic methods for HJB equation. This corresponds to the second test case without control in [24] (2 dimensional test case).

TestSemiLagrang1Lin

Test the Semi-Lagrangian method with the linear interpolator.

TestSemiLagrang1Quad

Test the Semi-Lagrangian method with the quadratic interpolator.

TestSemiLagrang1Cubic

Test the Semi-Lagrangian method with the cubic interpolator.

TestSemiLagrang1SparseQuad

Test the sparse grid interpolator with a quadratic interpolation.

TestSemiLagrang1SparseQuadAdapt

Test the sparse grid interpolator with a quadratic interpolation and some adaptation in the meshing.

9.11.2 testSemiLagrangCase2/testSemiLagrangCase2

Test Semi-Lagrangian deterministic methods for HJB equation. This corresponds to the first case without control in [24] (2 dimensional test case).

TestSemiLagrang2Lin

Test the Semi-Lagrangian method with the linear interpolator.

TestSemiLagrang2Quad

Test the Semi-Lagrangian method with the quadratic interpolator.

TestSemiLagrang2Cubic

Test the Semi-Lagrangian method with the cubic interpolator.

TestSemiLagrang2SparseQuad

Test the sparse grid interpolator with a quadratic interpolation.

9.11.3 testSemiLagrangCase2/testSemiLagrangCase2

Test Semi-Lagrangian deterministic methods for HJB equation. This corresponds to the stochastic target test case 5.3.4 in [24].

TestSemiLagrang3Lin

Test the Semi-Lagrangian method with the linear interpolator.

TestSemiLagrang3Quad

Test the Semi-Lagrangian method with the quadratic interpolator.

TestSemiLagrang3Cubic

Test the Semi-Lagrangian method with the cubic interpolator.

9.12 Non emimissive test case

9.12.1 testDPNonEmissive

Solve the problem described in part V by dynamic programming and regression.

- first an optimization is realized,
- the an simulation part permit to test the controls obtained.

9.12.2 testSLNonEmissive

Solve the problem described in part V by the Semi-Lagrangian method.

- first an optimization is realized,
- the an simulation part permit to test the controls obtained.

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