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STochastic OPTimization library in C++

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Part I

Introduction

The STochastic OPTimization library (StOpt)
<https://gitlab.com/stochastic-control/StOpt>

aims at providing tools for solving some stochastic optimization problems encountered in finance or in the industry.

In a continuous setting, the controlled state is given by a stochastic differential equation

$$\begin{cases} dX_s^{x,t} &= b_a(t, X_s^{x,t})ds + \sigma_a(s, X_s^{x,t})dW_s \\ X_t^{x,t} &= x \end{cases}$$

where

- W_t is a d -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the natural (completed and right-continuous) filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by W up to some fixed time horizon $T > 0$,
- σ_a is a Lipschitz continuous function of (t, x, a) defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^n$ and taking values in the set of d -dimensional square matrices,
- b_a is a Lipschitz continuous function of (t, x, a) defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^n$ and taking values in \mathbb{R}^d ,
- a a control adapted to the filtration taking values in \mathbb{R}^n .

Suppose we want to minimize a cost function $J(t, x, a) = \mathbb{E}[\int_t^T f_a(s, X_s^{x,t})e^{\int_t^s c_a(u, X_u^{x,t})du}ds + e^{\int_t^T c_a(u, X_u^{x,t})}g(X_T^{x,t})]$ with respect to the control a . It is well known [1] that the optimal value $\hat{J}(t, x) = \inf_a J(T - t, x, a)$ is a viscosity solution of the equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) - \inf_{a \in A} \left(\frac{1}{2} \text{tr}(\sigma_a(t, x)\sigma_a(t, x)^T D^2 v(t, x)) + b_a(t, x)Dv(t, x) \right. \\ \left. + c_a(t, x)v(t, x) + f_a(t, x) \right) &= 0 \text{ in } \mathbb{R}^d \\ v(0, x) &= g(x) \text{ in } \mathbb{R}^d \end{aligned} \tag{1}$$

Under some classical assumptions on the coefficients [], the previous equation known as the Hamilton Jacobi Bellman equation admits an unique viscosity solution ([2]).

The resolution of the previous equation is quite hard especially in dimension greater than 3 or 4.

The library provides tools to solve this equation and simplified versions of it.

- a first method supposes that $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$ where $X_{1,s}^{x,t}$ is not controlled

$$\begin{cases} dX_{1,s}^{x,t} &= b(t, X_{1,s}^{x,t})ds + \sigma(s, X_{1,s}^{x,t})dW_s \\ X_{1,t}^{x,t} &= x \end{cases} \tag{2}$$

and $X_{2,s}^{x,t}$ has no diffusion term

$$\begin{cases} dX_{2,s}^{x,t} &= b_a(t, X_{2,s}^{x,t})ds \\ X_{2,t}^{x,t} &= x \end{cases}$$

In this case we can use Monte Carlo methods based on regression to solve the problem. The method is based on the Dynamic Programming principle and can be used even if the non controlled SDE is driven by a general Levy process. This method can be used even if the controlled state takes only some discrete values.

- The second case is a special case of the previous one when the problem to solve is linear and when the controlled state takes some values in some continuous intervals. The value function has to be convex or concave with respect to the controlled variables. This method, the SDDP method, is used when the dimension of the controlled state is high, preventing the use of the Dynamic Programming method.

Remark 1 *The use of this method requires other assumptions that will be described the devoted chapter.*

- A third method permits to solve the problem with Monte Carlo when a process is controlled but by the mean of an uncontrolled process. This typically the case of the optimization of a portfolio :
 - The portfolio value is controlled and deterministically discretized on a grid,
 - The portfolio evolution is driven by an exogenous process not controlled : the market prices.
- In the last method, we will suppose that the state takes continuous values, we will solve equation (1) using Semi Lagrangian methods discretizing the Brownian motion with two values and using some interpolations on grids.

In the sequel, we suppose that a time discretization is given for the resolution of the optimization problem. We suppose the step discretization is constant and equal to h such that $t_i = ih$. First, we describe some useful tools developed in the library for stochastic control. Then, we explain how to solve some optimization problems using these developed tools.

Remark 2 *In the library, we heavily relies on the Eigen library: “ArrayXd” stands for a vector of double, “ArrayXXd” for a matrix of double and “ArrayXi” a vector of integer.*

Part II

Useful tools for stochastic control

Chapter 1

The grids and their interpolators

In this chapter we develop the tools used to interpolate a function discretized on a given grid. A grid is a set of point in \mathbb{R}^d defining some meshes that can be used to interpolate a function on an open set in \mathbb{R}^d . These tools are used to interpolate a function given for example at some stock points, when dealing with storages. There are also useful for Semi Lagrangian methods, which need effective interpolation methods. In StOpt currently four kinds of grids are available :

- the first and second one are grids used to interpolate a function linearly on a grid,
- the third kind of grid, starting from a regular grid, permits to interpolate on a grid at the Gauss Lobatto points on each mesh.
- the last grid permits to interpolate a function in high dimension using the sparse grid method. The approximation is either linear, quadratic or cubic in each direction.

To each kind of grids are associated some iterators. An iterator on a grid permits to iterate on all points of the grids. All iterators derive from the abstract class “GridIterator”

```
1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef GRIDITERATOR_H
5 #define GRIDITERATOR_H
6 #include <Eigen/Dense>
7
8 /** \file GridIterator.h
9  * \brief Defines an iterator on the points of a grid
10 * \author Xavier Warin
11 */
12 namespace StOpt
13 {
14
15 /// \class GridIterator GridIterator.h
16 /// Iterator on a given grid
17 class GridIterator
18 {
19
```

```

20 public :
21
22     ///  

23     GridIterator() {}
24
25     ///  

26     virtual ~GridIterator() {}
27
28     ///  

29     virtual Eigen::ArrayXd getCoordinate() const = 0 ;
30
31     ///  

32     virtual bool isValid(void) const = 0;
33
34     ///  

35     virtual void next() = 0;
36
37     ///  

38     ///  

39     virtual void nextInc(const int &p_incr) = 0;
40
41     ///  

42     virtual int getCount() const = 0;
43
44     ///  

45     ///  

46     ///  

47     ///  

48     virtual void jumpToAndInc(const int &p_rank, const int &p_nbProc , const  

49         int &p_jump) = 0;
50
51     ///  

52     virtual int getRelativePosition() const = 0 ;
53
54     ///  

55     virtual int getNbPointRelative() const = 0 ;
56
57     ///  

58     virtual void reset() = 0 ;
59 };
60 }
61 #endif /* GRIDITERATOR_H */

```

All the iterators share some common features :

- the “getCount” method permits to get the number associated to the current grid point,
- the “next” method permits to go to the next point, while the “nextInc” method permits to jump forward to the “p_incr”the point,
- the “isValid” method permits to check that we are still on a grid point,

- the “getNbPointRelative” method permits to get the number of points that a given iterator can iterate on,
- the “getRelativePosition” get the number of points already iterated by the iterator.

Besides, we can directly jump to a given point : this feature is useful for “mpi” when a treatment on the grid is split between some processor and threads. This possibility is given by the “jumpToAndInc” method.

Using a grid “regGrid” the following source code permits to iterate on the points of the grids and get coordinates. For each coordinate, a function f is used to fill in an array of values. As pointed out before, each type of grid has its own grid iterator that can be obtained by the “getGridIterator” method.

```

1   ArrayXd data(regGrid.getNbPoints()); // create an array to store the
   values of the function f
2   shared_ptr<GridIterator> iterRegGrid = regGrid.getGridIterator();
3   while (iterRegGrid->isValid())
4   {
5       ArrayXd pointCoord = iterRegGrid->getCoordinate(); // store the
   coordinates of the point
6       data(iterRegGrid->getCount()) = f(pointCoord); // the value is stored
   in data at place iterRegGrid->getCount()
7       iterRegGrid->next(); // go to next point
8   }
```

It is also possible to “jump” some points and iterate to “p” points after. This possibility is useful for multithreaded tasks on points.

To each kind of grids, an interpolator is provided to interpolate a function given on a grid. Notice that the interpolator is created **for a given point** where we want to interpolate. All interpolators (not being spectral interpolators) derive from “Interpolator.h” whose source code is given below

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
   LGPL)
4 #ifndef INTERPOLATOR_H
5 #define INTERPOLATOR_H
6 #include <vector>
7 #include <Eigen/Dense>
8 /** \file Interpolator.h
9  * \brief Defines a interpolator on a full grid
10 * \author Xavier Warin
11 */
12 namespace StOpt
13 {
14
15 /// \class Interpolator Interpolator.h
16 /// Interpolation base class
17 class Interpolator
18 {
19 public :
20
```

```

21  /// \brief Default constructor
22  Interpolator() {}
23
24  /// \brief Default Destructor
25  virtual ~Interpolator() {}
26
27  /** \brief interpolate
28   * \param p_dataValues Values of the data on the grid
29   * \return interpolated value
30   */
31  virtual double apply(const Eigen::ArrayXd &p_dataValues) const = 0;
32
33  /** \brief interpolate and use vectorization
34   * \param p_dataValues Values of the data on the grid. Interpolation
35   * is achieved for all values in the first dimension
36   * \return interpolated value
37   */
38  virtual Eigen::ArrayXd applyVec(const Eigen::ArrayXXd &p_dataValues)
39  const = 0;
40 #endif

```

All interpolators provide a constructor specifying the point where the interpolation is achieved and the two functions “apply” and “applyVec” interpolating either a function (and sending back a value) or an array of functions sending back an array of interpolated values.

All the grid classes derive from an abstract class “SpaceGrids.h” below permitting to get back an iterator associated to the points of the grid (with possible jumps) and to create an interpolator associated to the grid.

```

1  // Copyright (C) 2016 EDF
2  // All Rights Reserved
3  // This code is published under the GNU Lesser General Public License (GNU
4  // LGPL)
5  #ifndef SPACEGRID_H
6  #define SPACEGRID_H
7  #include <array>
8  #include <memory>
9  #include <Eigen/Dense>
10 #include "StOpt/core/grids/GridIterator.h"
11 #include "StOpt/core/grids/Interpolator.h"
12 #include "StOpt/core/grids/InterpolatorSpectral.h"
13
14 /** \file SpaceGrid.h
15  * \brief Defines a base class for all the grids
16  * \author Xavier Warin
17  */
18 namespace StOpt
19 {
20 /// \class SpaceGrid SpaceGrid.h
21 /// Defines a base class for grids
22 class SpaceGrid

```

```

23 {
24 public :
25     /// \brief Default constructor
26     SpaceGrid() {}
27
28     /// \brief Default destructor
29     virtual ~SpaceGrid() {}
30
31     /// \brief Number of points of the grid
32     virtual size_t getNbPoints() const = 0;
33
34     /// \brief get back iterator associated to the grid
35     virtual std::shared_ptr< GridIterator> getGridIterator() const = 0;
36
37     /// \brief get back iterator associated to the grid (multi thread)
38     virtual std::shared_ptr< GridIterator> getGridIteratorInc(const int &
        p-iThread) const = 0;
39
40     /// \brief Get back interpolator at a point Interpolate on the grid
41     /// \param p_coord coordinate of the point for interpolation
42     /// \return interpolator at the point coordinates on the grid
43     virtual std::shared_ptr<Interpolator> createInterpolator(const Eigen::
        ArrayXd &p_coord) const = 0;
44
45     /// \brief Get back a spectral operator associated to a whole function
46     /// \param p_values Function value at the grids points
47     /// \return the whole interpolated value function
48     virtual std::shared_ptr<InterpolatorSpectral> createInterpolatorSpectral(
        const Eigen::ArrayXd &p_values) const = 0;
49
50     /// \brief Dimension of the grid
51     virtual int getDimension() const = 0 ;
52
53     /// \brief get back bounds associated to the grid
54     /// \return in each dimension give the extreme values (min, max) of the
        domain
55     virtual std::vector <std::array< double , 2> > getExtremeValues() const =
        0;
56
57     /// \brief test if the point is strictly inside the domain
58     /// \param p_point point to test
59     /// \return true if the point is strictly inside the open domain
60     virtual bool isStrictlyInside(const Eigen::ArrayXd &p_point) const = 0 ;
61
62     /// \brief test if a point is inside the grid (boundary include)
63     /// \param p_point point to test
64     /// \return true if the point is inside the open domain
65     virtual bool isInside(const Eigen::ArrayXd &p_point) const = 0 ;
66
67     /// \brief truncate a point that it stays inside the domain
68     /// \param p_point point to truncate
69     virtual void truncatePoint(Eigen::ArrayXd &p_point) const = 0 ;
70
71 };

```

```

72 }
73 #endif /* SPACEGRID.H */

```

All the grids objects, interpolators and iterators on grids point are in

StOpt/core/grids

The grids objects are mapped with python, giving the possibility to get back the iterators and the interpolators associated to a grid. Python examples can be found in

test/python/unit/grids

1.1 Linear grids

1.1.1 Definition and C++ API

Two kinds of grids are developed:

- the first one is the “GeneralSpaceGrid.h” with constructor

```

1 GeneralSpaceGrid( const std::vector<shared_ptr<Eigen::ArrayXd>> &
  p_meshPerDimension )

```

where *std::vector<shared_ptr<Eigen::ArrayXd>>* is a vector of (pointer of) arrays defining the grids points in each dimension. In this case the grid is not regular and the mesh size varies in space (see figure 1.1).

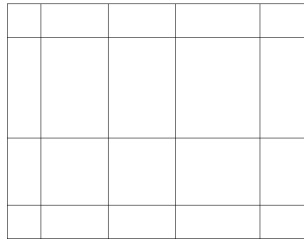


Figure 1.1: 2D general grid

- the second one is the “RegularSpaceGrid” with constructor

```

1 RegularSpaceGrid( const Eigen::ArrayXd &p_lowValues , const Eigen::ArrayXd
  &p_step , const Eigen::ArrayXi &p_nbStep )

```

The *p_lowValues* correspond to the bottom of the grid, *p_step* the size of each mesh, *p_nbStep* the number of steps in each direction (see figure 1.2)

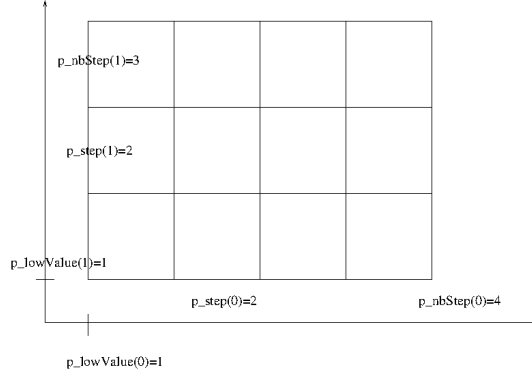


Figure 1.2: 2D regular grid

For each grid, a linear interpolator can be generated by call to the *createInterpolator* method or by creating directly the interpolator:

```

1  /** \brief Constructor
2  *  \param p_grid   is the grid used to interpolate
3  *  \param p_point  is the coordinates of the points used for interpolation
4  */
5  LinearInterpolator( const FullGrid *   p_grid , const Eigen::ArrayXd &
   p_point ):

```

Its construction from a grid (*regLin*) and an array *data* containing the values of the function at the grids points is given below (taking an example above to fill in the array *data*)

```

1  ArrayXd data(regGrid.getNbPoints()); // create an array to store the
   values of the function f
2  shared_ptr<GridIterator> iterRegGrid = regGrid.getGridIterator();
3  while (iterRegGrid->isValid())
4  {
5     ArrayXd pointCoord = iterRegGrid->getCoordinate(); // store the
   coordinate of the point
6     data(iterRegGrid->getCount()) = f(pointCoord); // the value is stored
   in data at place iterRegGrid->getCount()
7     iterRegGrid->next(); // go to next point
8  }
9  // point where to interpolate
10 ArrayXd point = ArrayXd::Constant(nDim, 1. / 3.);
11 // create the interpolator
12 LinearInterpolator regLin(&regGrid, point);
13 // get back the interpolated value
14 double interpReg = regLin.apply(data);

```

Let $I_{1,\Delta x}$ denote the linear interpolator where the mesh size is $\Delta x = (\Delta x^1, \dots, \Delta x^d)$. We get for a function f in $C^{k+1}(\mathbb{R}^d)$ with $k \leq 1$

$$\|f - I_{1,\Delta x}f\|_\infty \leq c \sum_{i=1}^d \Delta x_i^{k+1} \sup_{x \in [-1,1]^d} \left| \frac{\partial^{k+1} f}{\partial x_i^{k+1}} \right| \quad (1.1)$$

In particular if f is only Lipschitz

$$\|f - I_{1,\Delta x}f\|_\infty \leq K \sup_i \Delta x_i.$$

1.1.2 The python API

The python API makes it possible to use the grids with a similar syntax to the C++ API. We give here an example with a regular grid

```
1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import numpy as np
5 import unittest
6 import random
7 import math
8 import StOptGrids
9
10 # unit test for regular grids
11 #####
12
13 class testGrids(unittest.TestCase):
14
15     # 3 dimensional test for linear interpolation on regular grids
16     def testRegularGrids(self):
17         # low value for the meshes
18         lowValues = np.array([1., 2., 3.], dtype=np.float)
19         # size of the meshes
20         step = np.array([0.7, 2.3, 1.9], dtype=np.float)
21         # number of steps
22         nbStep = np.array([4, 5, 6], dtype=np.int32)
23         # create the regular grid
24         grid = StOptGrids.RegularSpaceGrid(lowValues, step, nbStep)
25         iterGrid = grid.getGridIterator()
26         # array to store
27         data = np.empty(grid.getNbPoints())
28         # iterates on points and store values
29         while( iterGrid.isValid()):
30             #get coordinates of the point
31             pointCoord = iterGrid.getCoordinate()
32             data[iterGrid.getCount()] = math.log(1. + pointCoord.sum())
33             iterGrid.next()
34         # get back an interpolator
35         ptInterp = np.array([2.3, 3.2, 5.9], dtype=np.float)
36         interpol = grid.createInterpolator(ptInterp)
37         # calculate interpolated value
38         interpValue = interpol.apply(data)
39         print(("Interpolated value" , interpValue))
40         # test grids function
41         iDim = grid.getDimension()
42         pt = grid.getExtremeValues()
43
44 if __name__ == '__main__':
45     unittest.main()
```

1.2 Legendre grids

With linear interpolation, in order to get an accurate solution, it is needed to refine the mesh so that Δx go to zero. Another approach consists in trying to fit on each mesh a polynomial by using a high degree interpolator.

1.2.1 Approximation of a function in 1 dimension.

From now, by re-scaling we suppose that we want to interpolate a function f on $[-1, 1]$. All the following results can be extended by tensorization in dimension greater than 1. P_N is the set of the polynomials of total degree below or equal to N . The minmax approximation of f of degree N is the polynomial $P_N^*(f)$ such that:

$$\|f - P_N^*(f)\|_\infty = \min_{p \in P_N} \|f - p\|_\infty$$

We call I_N^X interpolator from f on a grid of $N + 1$ points of $[-1, 1]$ $X = (x_0, \dots, x_N)$, the unique polynomial of degree N such that

$$I_N^X(f)(x_i) = f(x_i), 0 \leq i \leq N$$

This polynomial can be expressed in terms of the Lagrange polynomial $l_i^X, 0 \leq i \leq N$ associated to the grid (l_i^X is the unique polynomial of degree N taking value equal to 1 at point i and 0 at the other interpolation points).

$$I_N^X(f)(x) = \sum_{i=0}^N f(x_i) l_i^X(x)$$

The interpolation error can be expressed in terms of the interpolation points:

$$\|I_N^X(f)(x) - f\|_\infty \leq (1 + \lambda_N(X)) \|f - P_N^*(f)\|_\infty$$

where $\lambda_N(X)$ is the Lebesgue constant associated to Lagrange quadrature on the grid:

$$\lambda_N(X) = \max_{x \in [-1, 1]} \sum_{i=0}^N |l_i^X(x)|.$$

We have the following bound

$$\|I_N^X(f)(x)\|_\infty \leq \lambda_N(X) \sup_{x_i \in X} |f(x_i)| \leq \lambda_N(X) \|f\|_\infty$$

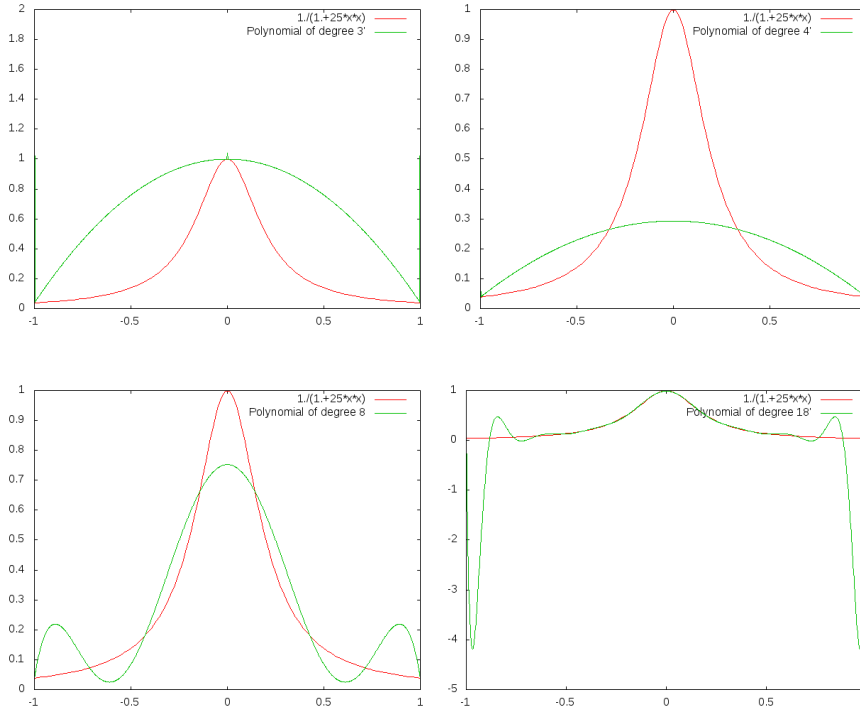
and the Erdős theorem states that

$$\lambda_N(X) > \frac{2}{\Pi} \log(N + 1) - C$$

It is well-known that the use of a uniform grid X_u is not optimal, because as $N \rightarrow \infty$, the Lebesgue constant satisfies

$$\lambda_N(X_u) \simeq \frac{2^{N+1}}{eN \ln N}$$

and the quadrature error in L_∞ increases a lot with N . Its use brings some oscillations giving the Runge effect. On figure ??, ??, ??, ??, we plot the Runge function $\frac{1}{1+25x^2}$ against its interpolation with polynomial with equidistant interpolation.



So we are interested in having quadrature with an “optimal” Lebesgue constant. For example Gauss-Chebyshev interpolation points (corresponding to the 0 of the polynomial $T_{N+1}(x) = \cos((N + 1)\arccos(x))$) give a Lebesgue constant $\lambda_N(X_{GC})$ equal to

$$\lambda_N(X_{GC}) \simeq \frac{2}{\pi} \ln(N + 1)$$

For our problem, we want to interpolate a function on meshes with high accuracy on the mesh while respecting the continuity of the function between the meshes. In order to ensure this continuity we want the extreme points on the re-scaled mesh $[-1, -1]$ (so $-1, 1$) to be on the interpolation grid. This leads to the Gauss Lobatto Chebyshev interpolation grid. In the library we choose to use the Gauss Lobatto Legendre interpolation grids which is as efficient as the Gauss Lobatto Chebyshev grids (in term of the Lebesgue constant) but computationally less costly due to absence of trigonometric function. We recall that the Legendre polynomial satisfies the recurrence

$$(N + 1)L_{N+1}(x) = (2N + 1)xL_N(x) - NL_{N-1}(x)$$

with $L_0 = 1$, $L_1(x) = x$.

These polynomials are orthogonal with the scalar product $(f, g) = \int_{-1}^1 f(x)g(x)dx$. We are interested in the derivatives of these polynomials L'_N that satisfy the recurrence

$$NL'_{N+1}(x) = (2N + 1)xL'_N(x) - (N + 1)L'_{N-1}(x)$$

these polynomials are orthogonal with the scalar product $(f, g) = \int_{-1}^1 f(x)g(x)(1-x^2)dx$. The Gauss Lobatto Legendre grids points for a grids with $N+1$ points are $\eta_1 = -1, \eta_{N+1} = 1$ and the η_i ($i = 2, \dots, N$) zeros of L'_N . The η_i ($i = 2, \dots, N$) are eigenvalues of the matrix P

$$P = \begin{pmatrix} 0 & \gamma_1 & \dots & 0 & 0 \\ \gamma_1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \gamma_{N-2} \\ 0 & 0 & \dots & \gamma_{N-2} & 0 \end{pmatrix},$$

$$\gamma_n = \frac{1}{2} \sqrt{\frac{n(n+2)}{(n+\frac{1}{2})(n+\frac{3}{2})}}, 1 \leq n \leq N-2,$$

The interpolation $I_N(f)$ is expressed in term of the Legendre polynomials by

$$I_N(f) = \sum_{k=0}^N \tilde{f}_k L_k(x),$$

$$\tilde{f}_k = \frac{1}{\gamma_k} \sum_{i=0}^N \rho_i f(\eta_i) L_k(\eta_i),$$

$$\gamma_k = \sum_{i=0}^N L_k(\eta_i)^2 \rho_i,$$

and the weights satisfies

$$\rho_i = \frac{2}{(M+1)ML_M^2(\eta_i)}, 1 \leq i \leq N+1.$$

More details can be found in [4]. In figure 1.4, we give the interpolation obtained with the Gauss Lobatto Legendre quadrature with two degrees of approximation.

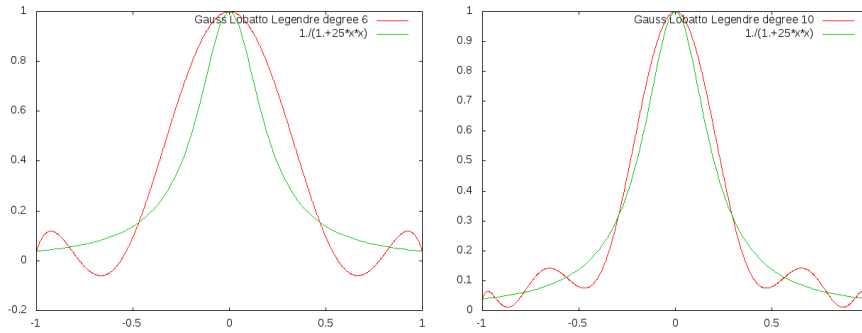


Figure 1.4: Interpolation with Gauss Legendre Lobatto grids

- When the function is not regular we introduce a notion weaker than the notion of derivatives. We note $w(f, \delta)$ the modulus of continuity on $[-1, 1]$ of a function f as

$$w(f, \delta) = \sup_{\substack{x_1, x_2 \in [-1, 1] \\ |x_1 - x_2| < \delta}} |f(x_1) - f(x_2)|$$

The modulus of continuity permits to express the best approximation of a function by a polynomial with the Jackson theorem:

Theorem 1 For a continuous function f on $[-1, 1]$

$$\|f - P_N^*(f)\|_\infty \leq Kw(f, \frac{1}{N})$$

and we deduce that for a grid of interpolation X

$$\begin{aligned} \|I_N^X(f)(x) - f\|_\infty &\leq M(N) \\ M(N) &\simeq Kw(f, \frac{1}{N})\lambda_N(X) \end{aligned}$$

a function is Dini–Lipschitz continuous if $w(f, \delta)\log(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. It is clear that Lipschitz functions are Dini–Lipschitz continuous because $w(f, \delta)\log(\delta) \leq K\log(\delta)\delta$.

- When the solution is more regular we can express the interpolation error as a function of its derivatives and we get the following Cauchy theorem for an interpolation grid X (see [3])

Theorem 2 If f is C^{N+1} , and X an interpolation grid with $N + 1$ points, then the interpolation error verifies

$$E(x) = f(x) - I_N^X(f)(x) = \frac{f^{N+1}(\eta)}{(N + 1)!} W_{N+1}^X(x) \quad (1.2)$$

where $\eta \in [-1, 1]$ and $W_{N+1}^X(x)$ is the nodal polynomial of degree $N + 1$ (the polynomial with the monomial of the highest degree with coefficient 1 being null at all the $N + 1$ points of X)

If we partition a domain $I = [a, b]$ in some meshes of size h and we use a Lagrange interpolator for the function $f \in C^{k+1}$, $k \leq N$ we obtain

$$\|f - I_{N,\Delta x}^X f\|_\infty \leq ch^{k+1} \|f^{(k+1)}\|_\infty$$

1.2.2 Extension in dimension d

In dimension d , we note P_N^* the best multivariate polynomial approximation of f of total degree lesser than N on $[-1, 1]^d$. On a d multidimensional grid $X = X_N^d$, we define the multivariate interpolator as the composition of one dimensional interpolator $I_N^X(f)(x) = I_N^{X_N,1} \times I_N^{X_N,2} \dots \times I_N^{X_N,d}(f)(x)$ where $I_N^{X_N,i}$ stands for the interpolator in dimension i . We get the following interpolation error

$$\|I_N^X(f) - f\|_\infty \leq (1 + \lambda_N(X_N))^d \|f - P_N^*(f)\|_\infty,$$

The error associated to the min max approximation is given by Feinerman and Newman [5], Soardi [6]

$$\|f - P_N^*(f)\|_\infty \leq (1 + \frac{\pi^2}{4}\sqrt{d})w(f, \frac{1}{N+2})$$

We deduce that if f is only Lipschitz

$$\|I_N^X(f)(x) - f\|_\infty \leq C\sqrt{d} \frac{(1 + \lambda_N(X))^d}{N + 2}$$

If the function is regular (in $C^{k+1}([-1, 1]^d)$, $k < N$) we get

$$\|f - P_N^*(f)\|_\infty \leq \frac{C_k}{N^k} \sum_{i=1}^d \sup_{x \in [-1, 1]^d} \left| \frac{\partial^{k+1} f}{\partial x_i^{k+1}} \right|$$

If we partition the domain $I = [a_1, b_1] \times \dots \times [a_d, b_d]$ in meshes of size $\Delta x = (\Delta x_1, \Delta x_2, \dots, \Delta x_d)$ such such we use a Lagrange interpolation on each mesh we obtain

$$\|f - I_{N, \Delta x}^X f\|_\infty \leq c \frac{(1 + \lambda_N(X))^d}{N^k} \sum_{i=1}^d \Delta x_i^{k+1} \sup_{x \in [-1, 1]^d} \left| \frac{\partial^{k+1} f}{\partial x_i^{k+1}} \right|$$

On figure 1.5 we give the Gauss Legendre Lobatto points in 2D for 2×2 meshes and a polynomial of degree 8 in each direction

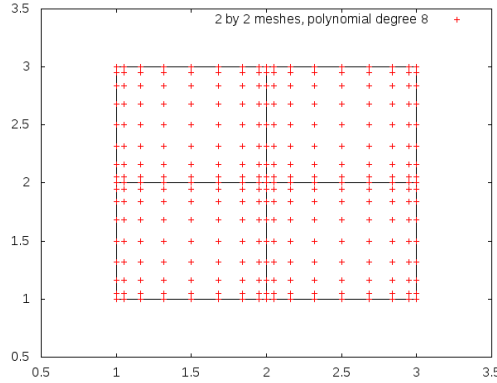


Figure 1.5: Gauss Legendre Lobatto points on 2×2 meshes.

1.2.3 Troncature

In order to avoid oscillations while interpolating, a troncature is used on each mesh such that the modified interpolator $\hat{I}_{N, \Delta x}^X$ satisfies :

$$\hat{I}_{N, \Delta x}^X f(x) = \min_{x_i \in M} f(x_i) \wedge I_{N, \Delta x}^X f(x) \vee \max_{x_i \in M} f(x_i) \quad (1.3)$$

where the x_i are the interpolation points on the mesh M containing the point x . For all characteristics of this modified operator, one can see [24].

1.2.4 The C++ API

The grid using Gauss Legendre Lobatto points can be created by the use of this constructor:

```
1 RegularLegendreGrid(const Eigen::ArrayXd &p_lowValues, const Eigen::
  ArrayXd &p_step, const Eigen::ArrayXi &p_nbStep, const Eigen::ArrayXi
  & p_poly);
```

The *p_lowValues* correspond to the bottom of the grid, *p_step* the size of each mesh, *p_nbStep* the number of steps in each direction (see figure 1.2). On each mesh the polynomial approximation in each dimension is specified by the *p_poly* array.

Remark 3 *If we take a polynomial of degree 1 in each direction this interpolator is equivalent to the linear interpolator. It is somehow slightly less efficient than the linear interpolator on a Regular grid described in the above section.*

We illustrate the use of the grid, its iterator and its interpolator used in order to draw the figures 1.4.

```
1 ArrayXd lowValues = ArrayXd::Constant(1, -1.); // corner point
2 ArrayXd step= ArrayXd::Constant(1, 2.); // size of the meshes
3 ArrayXi nbStep = ArrayXi::Constant(1, 1); // number of mesh in each
4 direction
5 ArrayXi nPol = ArrayXi::Constant(1, p_nPol); // polynomial approximation
6 // regular Legendre
7 RegularLegendreGrid regGrid(lowValues, step, nbStep, nPol);
8
9 // Data array to store values on the grid points
10 ArrayXd data(regGrid.getNbPoints());
11 shared_ptr<GridIterator> iterRegGrid = regGrid.getGridIterator();
12 while (iterRegGrid->isValid())
13 {
14   ArrayXd pointCoord = iterRegGrid->getCoordinate();
15   data(iterRegGrid->getCount()) = 1./(1.+25*pointCoord(0)*pointCoord(0));
16   // store runge function
17   iterRegGrid->next();
18 }
19 // point
20 ArrayXd point(1);
21 int nbp = 1000;
22 double dx = 2./nbp;
23 for (int ip =0; ip<= nbp; ++ip)
24 {
25   point(0)= -1+ ip* dx;
26 // create interpolator
27   shared_ptr<Interpolator> interp = regGrid.createInterpolator( point);
28   double interpReg = interp->apply(data); // interpolated value
29 }
```

The previously defined operator is more effective when we interpolate many function at the same point. Its is the case for example for the valorization of a storage with regression where you want to interpolate all the simulations at the same stock level.

In some case it is more convenient to construct an interpolator acting on a global function. It is the case when you have a single function and you want to interpolate at many points for this function. In this specific case an interpolator deriving from the class InterpolatorSpectral can be constructed:

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef INTERPOLATORSPECTRALH
5 #define INTERPOLATORSPECTRALH
6 #include <Eigen/Dense>
7 // #include "StOpt/core/grids/SpaceGrid.h"
8
9 /** \file InterpolatorSpectral.h
10 * \brief Defines an interpolator for a grid : here is a global interpolator
  , storing the representation of the function
11 * to interpolate : this interpolation is effective when
  interpolating the same function many times at different points
12 * Here it is an abstract class
13 * \author Xavier Warin
14 */
15 namespace StOpt
16 {
17
18 /// forward declaration
19 class SpaceGrid ;
20
21 /// \class InterpolatorSpectral InterpolatorSpectral.h
22 /// Abstract class for spectral operator
23 class InterpolatorSpectral
24 {
25
26 public :
27     virtual ~InterpolatorSpectral() {}
28
29     /** \brief interpolate
30     * \param p_point coordinates of the point for interpolation
31     * \return interpolated value
32     */
33     virtual double apply(const Eigen::ArrayXd &p_point) const = 0;
34
35
36     /** \brief Affect the grid
37     * \param p_grid the grid to affect
38     */
39     virtual void setGrid(const StOpt::SpaceGrid *p_grid) = 0 ;
40 };
41 }
42 #endif

```

Its constructor is given by :

```

1     /** \brief Constructor taking in values on the grid
2     * \param p_grid is the grid used to interpolate

```



```

3     * \param p-values   Function value at the grids points
4     */
5     LegendreInterpolatorSpectral(const shared_ptr< RegularLegendreGrid> &
        p_grid , const Eigen::ArrayXd &p-values) ;

```

This class has a member permitting to interpolate at a given point:

```

1     /** \brief interpolate
2     * \param p-point   coordinates of the point for interpolation
3     * \return interpolated value
4     */
5     inline double apply(const Eigen::ArrayXd &p-point) const

```

We give an example of the use of this class, interpolating a function f in dimension 2.

```

1     ArrayXd lowValues = ArrayXd::Constant(2,1.); // bottom of the domain
2     ArrayXd step     = ArrayXd::Constant(2,1.); // size of the mesh
3     ArrayXi nbStep  = ArrayXi::Constant(2,5); // number of meshes in each
        direction
4     ArrayXi nPol    = ArrayXi::Constant(2,2) ; // polynomial of degree 2 in
        each direction
5     // regular
6     shared_ptr<RegularLegendreGrid> regGrid(new RegularLegendreGrid(lowValues
        , step , nbStep , nPol));
7     ArrayXd data(regGrid->getNbPoints()); // Data array
8     shared_ptr<GridIterator> iterRegGrid = regGrid->getGridIterator(); //
        iterator on the grid points
9     while (iterRegGrid->isValid())
10    {
11        ArrayXd pointCoord = iterRegGrid->getCoordinate();
12        data(iterRegGrid->getCount()) = f(pointCoord);
13        iterRegGrid->next();
14    }
15
16    // spectral interpolator
17    LegendreInterpolatorSpectral interpolator(regGrid , data);
18    // interpolation point
19    ArrayXd pointCoord(2, 5.2);
20    // interpolated value
21    double vInterp = interpolator.apply(pointCoord);

```

1.2.5 The python API

Here is an example using Legendre grids:

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
        LGPL)
4 import numpy as np
5 import unittest
6 import random
7 import math
8 import StOptGrids

```

```

9
10 # unit test for Legendre grids
11 #####
12
13 class testGrids(unittest.TestCase):
14
15
16     # test Legendre grids
17     def testLegendreGrids(self):
18         # low value for the mesh
19         lowValues = np.array([1., 2., 3.], dtype=np.float)
20         # size of the mesh
21         step = np.array([0.7, 2.3, 1.9], dtype=np.float)
22         # number of step
23         nbStep = np.array([4, 5, 6], dtype=np.int32)
24         # degree of the polynomial in each direction
25         degree = np.array([2, 1, 3], dtype=np.int32)
26         # create the Legendre grid
27         grid = StOptGrids.RegularLegendreGrid(lowValues, step, nbStep, degree)
28         iterGrid = grid.getGridIterator()
29         # array to store
30         data = np.empty(grid.getNbPoints())
31         # iterates on point
32         while( iterGrid.isValid()):
33             #get coordinates of the point
34             pointCoord = iterGrid.getCoordinate()
35             data[iterGrid.getCount()] = math.log(1. + pointCoord.sum())
36             iterGrid.next()
37         # get back an interpolator
38         ptInterp = np.array([2.3, 3.2, 5.9], dtype=np.float)
39         interpol = grid.createInterpolator(ptInterp)
40         # calculate interpolated value
41         interpValue = interpol.apply(data)
42         print("Interpolated value Legendre", interpValue)
43         # test grids function
44         iDim = grid.getDimension()
45         pt = grid.getExtremeValues()
46
47
48 if __name__ == '__main__':
49     unittest.main()

```

1.3 Sparse grids

A representation of a function in dimension d for d small (less than 4) is achieved by tensorization in the previous interpolation methods. When the function is smooth and when its cross derivatives are bounded, one can represent the function using the sparse grid methods. This methods permits to represent the function with far less points than classical without losing too much while interpolating. The sparse grid method was first used supposing that the function f to represent is null at the boundary Γ of the domain. This assumption is important because it permits to limit the explosion of the number of points with the dimen-

sion of the problem. In many application this assumption is not realistic or it is impossible to work on $f - f|_{\Gamma}$. In this library we will suppose that the function is not null at the boundary and provide grid object, iterators and interpolators to interpolate some functions represented on the sparse grid. Nevertheless, for the sake of clarity of the presentation, we will begin with the case of a function vanishing on the boundary.

1.3.1 The linear sparse grid method

We recall some classical results on sparse grids that can be found in [9]. We first assume that the function we interpolate is null at the boundary. By a change of coordinate an hyper-cube domain can be changed to a domain $\omega = [0, 1]^d$. Introducing the hat function $\phi^{(L)}(x) = \max(1 - |x|, 0)$ (where (L) stands for linear), we obtain the following local one dimensional hat function by translation and dilatation

$$\phi_{l,i}^{(L)}(x) = \phi^{(L)}(2^l x - i)$$

depending on the level l and the index i , $0 < i < 2^l$. The grid points used for interpolation are noted $x_{l,i} = 2^{-l}i$. In dimension d , we introduce the basis functions

$$\phi_{\underline{l},\underline{i}}^{(L)}(x) = \prod_{j=1}^d \phi_{l_j,i_j}^{(L)}(x_j)$$

via a tensor approach for a point $\underline{x} = (x_1, \dots, x_d)$, a multi-level $\underline{l} := (l_1, \dots, l_d)$ and a multi-index $\underline{i} := (i_1, \dots, i_d)$. The grid points used for interpolation are noted $x_{\underline{l},\underline{i}} := (x_{l_1,i_1}, \dots, x_{l_d,i_d})$. We next introduce the index set

$$B_{\underline{l}} := \{ \underline{i} : 1 \leq i_j \leq 2^{l_j} - 1, i_j \text{ odd}, 1 \leq j \leq d \}$$

and the space of hierarchical basis

$$W_{\underline{l}}^{(L)} := \text{span} \left\{ \phi_{\underline{l},\underline{i}}^{(L)}(\underline{x}) : \underline{i} \in B_{\underline{l}} \right\}$$

A representation of the space $W_{\underline{l}}^{(L)}$ is given in dimension 1 on figure 1.6. The sparse grid space is defined as:

$$V_n = \bigoplus_{|\underline{l}|_1 \leq n+d-1} W_{\underline{l}}^{(L)} \tag{1.4}$$

Remark 4 *The conventional full grid space is defined as $V_n^F = \bigoplus_{|\underline{l}|_{\infty} \leq n} W_{\underline{l}}^{(L)}$*

At a space of hierarchical increments $W_{\underline{l}}^{(L)}$ corresponds a space of nodal function $W_{\underline{l}}^{(L,N)}$ such that

$$W_{\underline{l}}^{(L,N)} := \text{span} \left\{ \phi_{\underline{l},\underline{i}}^{(L)}(\underline{x}) : \underline{i} \in B_{\underline{l}}^N \right\}$$

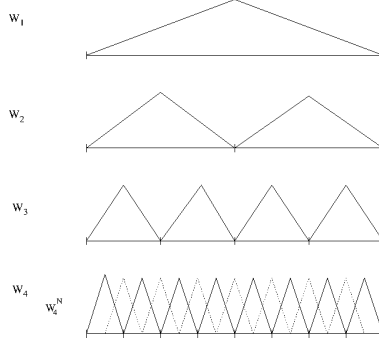


Figure 1.6: One dimensional $W^{(L)}$ spaces : $W_1^{(L)}$, $W_2^{(L)}$, $W_3^{(L)}$, $W_4^{(L)}$ and the nodal representation $W_4^{(L,N)}$

with

$$B_{\underline{l}}^N := \{ \underline{i} : 1 \leq i_j \leq 2^{l_j} - 1, 1 \leq j \leq d \}.$$

On figure 1.6 the one dimensional nodal base $W_4^{(L,N)}$ is spanned by $W_4^{(L)}$ and the dotted basis function. The space V_n can be represented as the space spanned by the $W_{\underline{l}}^{(L,N)}$ such that $|\underline{l}|_1 = n + d - 1$:

$$V_n = \text{span} \left\{ \phi_{\underline{l},\underline{i}}^{(L)}(\underline{x}) : \underline{i} \in B_{\underline{l}}^N, |\underline{l}|_1 = n + d - 1 \right\} \quad (1.5)$$

A function f is interpolated on the hierarchical basis as

$$I^{(L)}(f) = \sum_{|\underline{l}|_1 \leq n+d-1, \underline{i} \in B_{\underline{l}}} \alpha_{\underline{l},\underline{i}}^{(L)} \phi_{\underline{l},\underline{i}}^{(L)}$$

where $\alpha_{\underline{l},\underline{i}}^{(L)}$ are called the surplus (we give on figure 1.7 a representation of these coefficients). These surplus associated to a function f are calculated in the one dimension case for a node

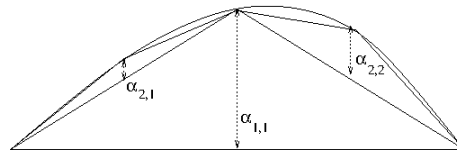


Figure 1.7: Example of hierarchical coefficients

$m = x_{\underline{l},\underline{i}}$ as the difference of the value of the function at the node and the linear representation of the function calculated with neighboring nodes. For example on figure 1.8, the hierarchical value is given by the relation:

$$\alpha^{(L)}(m) := \alpha_{\underline{l},\underline{i}}^{(L)} = f(m) - 0.5(f(e(m)) + f(w(m)))$$

where $e(m)$ is the east neighbor of m and $w(m)$ the west one. The procedure is generalized in d dimension by successive hierarchization in all the directions. On figure 1.9, we give a representation of the W subspace for $\underline{l} \leq 3$ in dimension 2.

In order to deal with functions not null at the boundary, two more basis are added to the

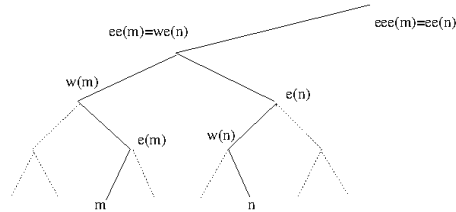


Figure 1.8: Node involved in linear, quadratic and cubic representation of a function at node m and n

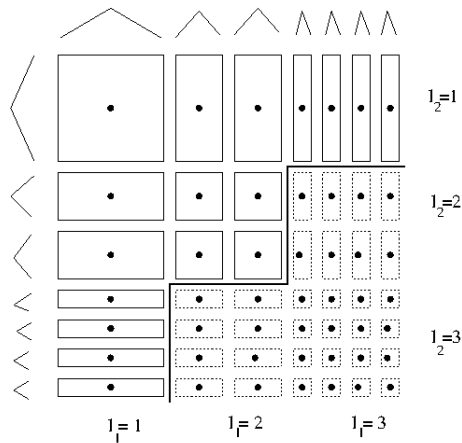


Figure 1.9: The two dimensional subspace $W_l^{(L)}$ up to $l = 3$ in each dimension. The additional hierarchical functions corresponding to an approximation on the full grid are given in dashed lines.

first level as shown on figure 1.10. This approach results in many more points than the one without the boundary. As noted in [9] for $n = 5$, in dimension 8 you have nearly 2.8 millions points in this approximation but only 6401 inside the domain. On figure 1.11 we give the grids points with boundary points in dimension 2 and 3 for a level 5 of the sparse grid.

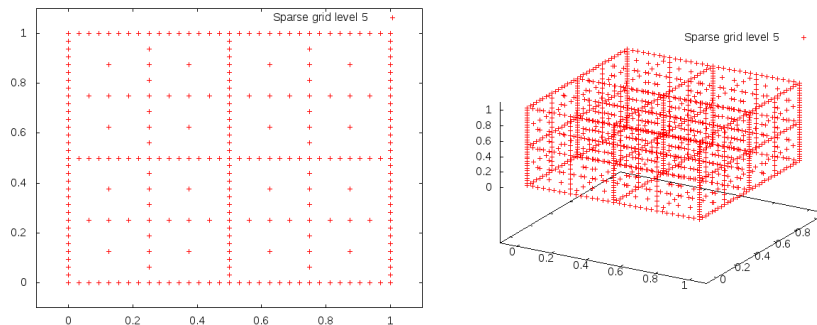


Figure 1.11: Sparse grid in dimension 2 and 3 with boundary points

If the boundary conditions are not important (infinite domain truncated in finance for example) the hat functions near the boundaries are modified by extrapolation (see figure

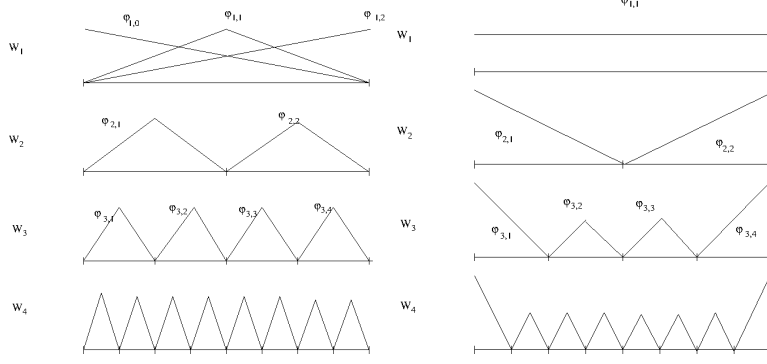


Figure 1.10: One dimensional $W^{(L)}$ spaces with linear functions with “exact ” boundary (left) and “modified ” boundary (right): $W_1^{(L)}$, $W_2^{(L)}$, $W_3^{(L)}$, $W_4^{(L)}$

1.10) as explained in [9]. On level 1, we only have one degree of freedom assuming the function is constant on the domain. On all other levels, we extrapolate linearly towards the boundary the left and right basis functions, other functions remaining unchanged. So the new functions basis in 1D $\tilde{\phi}$ becomes

$$\tilde{\phi}_{l,i}^{(L)}(x) = \begin{cases} 1 & \text{if } l = 1 \text{ and } i = 1 \\ \begin{cases} 2 - 2^l x & \text{if } x \in [0, 2^{-l+1}] \\ 0 & \text{else} \end{cases} & \text{if } l > 1 \text{ and } i = 1 \\ \begin{cases} 2^l(x - 1) + 2 & \text{if } x \in [1 - 2^{-l+1}, 1] \\ 0 & \text{else} \end{cases} & \text{if } l > 1 \text{ and } i = 2^l - 1 \\ \phi_{l,i}^{(L)}(x) & \text{otherwise} \end{cases}$$

On figure 1.12 we give the grids points eliminating boundary points in dimension 2 and 3 for a level 5 of the sparse grid.

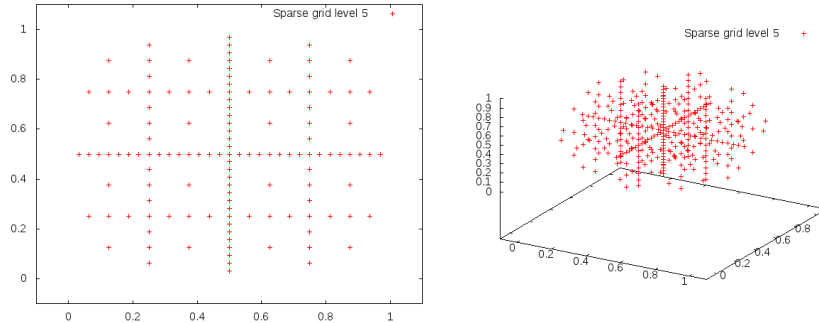


Figure 1.12: Sparse grid in dimension 2 and 3 without boundary points

The interpolation error associated to the linear operator $I^1 := I^{(L)}$ is linked to the regularity of the cross derivatives of the function [10, 17, 18]. If f is null at the boundary and admits derivatives such that $\|\frac{\partial^{2d} u}{\partial x_1^2 \dots \partial x_d^2}\|_\infty < \infty$ then

$$\|f - I^1(f)\|_\infty = O(N^{-2} \log(N)^{d-1}), \quad (1.6)$$

with N the number of points per dimension.

1.4 High order sparse grid methods

Changing the interpolator enables us to get a higher rate of convergence mainly in region where the solution is smooth. Following [17] and [18], it is possible to get higher order interpolators. Using a quadratic interpolator, the reconstruction on the nodal basis gives a quadratic function on the support of the previously defined hat function and a continuous function of the whole domain. The polynomial quadratic basis is defined on $[2^{-l}(i-1), 2^{-l}(i+1)]$ by

$$\phi_{l,i}^{(Q)}(x) = \phi^{(Q)}(2^l x - i)$$

with $\phi^{(Q)}(x) = 1 - x^2$.

The hierarchical surplus (coefficient on the basis) in one dimension is the difference between the value function at the node and the quadratic representation of the function using nodes available at the preceding level. With the notation of figure 1.8

$$\begin{aligned} \alpha(m)^{(Q)} &= f(m) - \left(\frac{3}{8}f(w(m)) + \frac{3}{4}f(e(m)) - \frac{1}{8}f(ee(m))\right) \\ &= \alpha(m)^{(L)}(m) - \frac{1}{4}\alpha(m)^{(L)}(e(m)) \\ &= \alpha(m)^{(L)}(m) - \frac{1}{4}\alpha(m)^{(L)}(df(m)) \end{aligned}$$

where $df(m)$ is the direct father of the node m in the tree.

Once again the quadratic surplus in dimension d is obtained by successive hierarchization in the different dimensions.

In order to take into account the boundary conditions, two linear functions $1 - x$ and x are added at the first level (see figure 1.13).

A version with modified boundary conditions can be derived for example by using linear interpolation at the boundary such that

$$\tilde{\phi}_{l,i}^{(Q)}(x) = \begin{cases} \tilde{\phi}_{l,i}^{(L)} & \text{if } i = 1 \text{ or } i = 2^l - 1, \\ \phi_{l,i}^{(Q)}(x) & \text{otherwise} \end{cases}$$

In the case of the cubic representation, on figure 1.8 we need 4 points to define a function basis. In order to keep the same data structure, we use a cubic function basis at node m with value 1 at this node and 0 at the node $e(m)$, $w(m)$ and $ee(m)$ and we only keep the basis function between $w(m)$ and $e(m)$ [17].

Notice that there are two kinds of basis function depending of the position in the tree. The basis functions are given on $[2^{-l+1}i, 2^{-l+1}(i+1)]$ by

$$\begin{aligned} \phi_{l,2i+1}^{(C)}(x) &= \phi^{(C),1}(2^l x - (2i+1)), \text{ if } i \text{ even} \\ &= \phi^{(C),2}(2^l x - (2i+1)), \text{ if } i \text{ odd} \end{aligned}$$

with $\phi^{(C),1}(x) = \frac{(x^2-1)(x-3)}{3}$, $\phi^{(C),2}(x) = \frac{(1-x^2)(x+3)}{3}$.

The coefficient surplus can be defined as before as the difference between the value function at the node and the cubic representation of the function at the father node. Because of the two basis functions involved there are two kind of cubic coefficient.

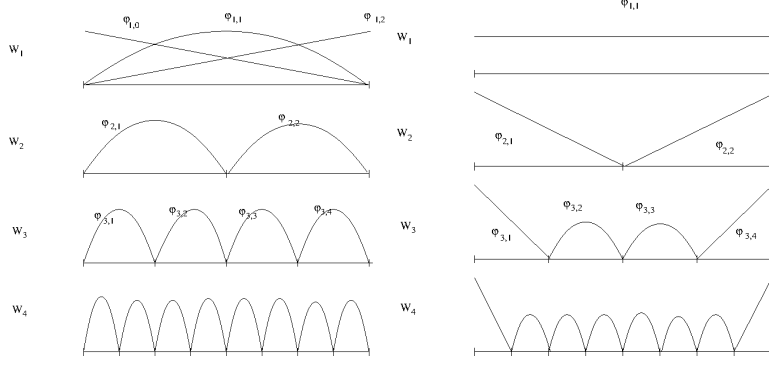


Figure 1.13: One dimensional $W^{(Q)}$ spaces with quadratic with “exact” boundary (left) and “modified” boundary (right): $W_1^{(Q)}$, $W_2^{(Q)}$, $W_3^{(Q)}$, $W_4^{(Q)}$

- For a node $m = x_{l,8i+1}$ or $m = x_{l,8i+7}$, $\alpha^{(C)}(m) = \alpha^{(C,1)}(m)$, with

$$\alpha^{(C,1)}(m) = \alpha^{(Q)}(m) - \frac{1}{8}\alpha^{(Q)}(df(m))$$

- For a node $m = x_{l,8i+3}$ or $m = x_{l,8i+5}$, $\alpha^{(C)}(m) = \alpha^{(C,2)}(m)$, with

$$\alpha^{(C,2)}(m) = \alpha^{(Q)}(m) + \frac{1}{8}\alpha^{(Q)}(df(m))$$

Notice that a cubic representation is not available for $l = 1$ so a quadratic approximation is used. As before boundary conditions are treated by adding two linear functions basis at the first level and a modified version is available. We choose the following basis functions as defined on figure 1.14:

$$\tilde{\phi}_{l,i}^{(C)}(x) = \begin{cases} \tilde{\phi}_{l,i}^{(Q)} & \text{if } i \in \{1, 3, 2^l - 3, 2^l - 1\}, \\ \phi_{l,i}^{(C)}(x) & \text{otherwise} \end{cases}$$

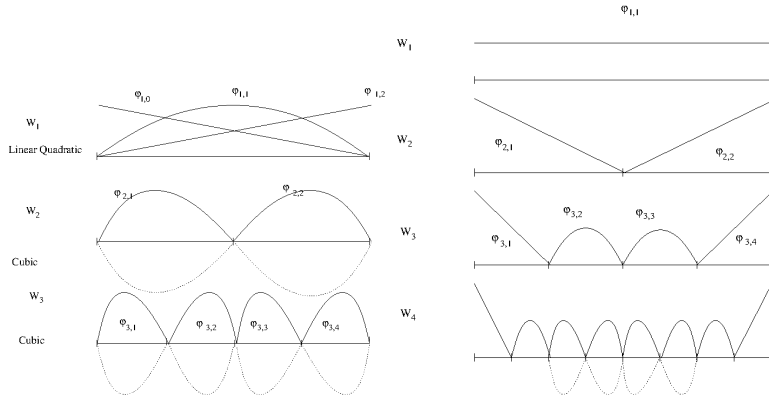


Figure 1.14: One dimensional $W^{(C)}$ spaces with cubic and “exact“ boundary (left) and “modified” boundary (right): $W_1^{(C)}$, $W_2^{(C)}$, $W_3^{(C)}$, $W_4^{(C)}$

According to [10, 17, 18], if the function f is null at the boundary and admits derivatives such that $\sup_{\alpha_i \in \{2, \dots, p+1\}} \left\{ \left\| \frac{\partial^{\alpha_1 + \dots + \alpha_d} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \right\|_{\infty} \right\} < \infty$ then the interpolation error can be generalized for $I^2 := I^{(Q)}$, $I^3 := I^{(C)}$ by:

$$\|f - I^p(f)\|_{\infty} = O(N^{-(p+1)} \log(N)^{d-1}), \quad p = 2, 3$$

with N the number of points per dimension.

1.5 Anisotropy

In many situations, it is useless to refine as much in each direction. For example, when dealing with multidimensional storages we expect the mesh size to be of the same order in each direction. When the different storages have very different sizes, we want to refine more the storage with the highest capacity. In order to treat this anisotropy an extension of Sparse grids can be achieved by defining weight w in each direction. The definition 1.4 is replaced by:

$$V_n = \bigoplus_{\sum_{i=1}^d l_i w(i) \leq n+d-1} W_{\underline{l}}^{(L)} \quad (1.7)$$

1.6 Adaptation

When the solution is not smooth, typically Lipschitz, there is no hope to get convergence results for classical Sparse Grids (see above the interpolation error linked to the cross derivatives of the function). So classical sparse grids have to be adapted such that the solution is refined near singularities. In all adaptations methods hierarchical surplus $\alpha_{\underline{l}, i}$ are used to get an estimation of the local error. These coefficients give an estimation of the smoothness of the function value at the discrete points by representing the discrete mix second derivative of the function. There is mainly two kinds of adaptation used :

- the first one is performing local adaptation and only adds points locally [32, 8, 33, 12],
- the second one is performing adaptation at the level of the hierarchical space $W_{\underline{l}}$ (anisotropic sparse grid). This approach detects important dimensions that needs refinement and refines all the points in this dimension [11]. This refinement is also achieved in areas where the solution can be smooth. A more local version has been developed in [34].

In the current version of the library only dimension adaptation is available. Details on the algorithm can be bound in [11]. After a first initialization with a first initialization with a space

$$V_n = \bigoplus_{\sum_{i=1}^d l_i \leq n+d-1} W_{\underline{l}}^{(L)} \quad (1.8)$$

A set of active level \mathcal{A} is created gathering all levels \underline{l} such that $\sum_{i=1}^d l_i = n + d - 1$. All other levels are gathered in a set \mathcal{O} . At each level \underline{l} in \mathcal{A} an error is estimated $e_{\underline{l}}$ and with all

```

while  $E > \eta$  do
  select  $\underline{l}$  with the highest local error  $e_{\underline{l}}$ 
   $\mathcal{A} = \mathcal{A} \setminus \{\underline{l}\}$ 
   $\mathcal{O} = \mathcal{O} \cup \{\underline{l}\}$ 
  for  $k = 1$  to  $d$  do
     $\underline{m} = \underline{l} + \mathbf{e}_k$ 
    if  $\underline{m} - \mathbf{e}_q \in \mathcal{O}$  for  $q \in [1, d]$  then
       $\mathcal{A} = \mathcal{A} \cup \{\underline{m}\}$ 
      Hierarchize all points belonging to  $\underline{m}$ 
      calculate  $e_{\underline{m}}$ 
      update  $E$ 
    end if
  end for
end while

```

Algorithm 1: Dimension refinement for a given tolerance η

local error $e_{\underline{l}}$ a global error E is calculated. Then the refinement algorithm 1 is used noting \mathbf{e}_k the canonical basis in dimension k . Sometimes, using sparse grids during time iterations, it can be interesting to coarsen the meshes. A similar algorithm 2 can be used to eliminate levels with a very small local error.

\mathcal{B} all elements of \mathcal{A} with a local error below η

```

while  $\mathcal{B}$  non nonempty do
  select  $\underline{l} \in \mathcal{B}$  with the lowest local error  $e_{\underline{l}}$ 
  for  $k = 1$  to  $d$  do
     $\underline{m} = \underline{l} - \mathbf{e}_k$ 
    if  $m_k > 0$  then
      if  $\underline{m} + \mathbf{e}_q \in \mathcal{B}$  for  $q \in [1, d]$  then
         $\mathcal{A} = \mathcal{A} \setminus \{\underline{m} + \mathbf{e}_q, q \in [1, d]\}$ 
         $\mathcal{B} = \mathcal{B} \setminus \{\underline{m} + \mathbf{e}_q, q \in [1, d]\}$ 
         $\mathcal{A} = \mathcal{A} \cup \{\underline{m}\}$ 
        Add  $\underline{m}$  to  $\mathcal{B}$  if local error below  $\eta$ 
         $\mathcal{O} = \mathcal{O} \setminus \{\underline{m}\}$ 
        Break
      end if
    end if
  end for
  if  $\underline{l} \in \mathcal{B}$  then
     $\mathcal{B} = \mathcal{B} \setminus \{\underline{l}\}$ 
  end if
end while

```

Algorithm 2: Dimension coarsening for a given tolerance η

1.7 C++ APi

The construction of the Sparse Grid including boundary point is done by the following constructor

```

1 SparseSpaceGridBound(const Eigen::ArrayXd &p_lowValues, const Eigen::
  ArrayXd &p_sizeDomain, const int &p_levelMax, const Eigen::ArrayXd &
  p_weight,
2 const size_t &p_degree)

```

with

- $p_lowValues$ corresponds to the bottom of the grid,
- $p_sizeDomain$ corresponds to the size of the resolution domain in each dimension,
- $p_levelMax$ is the level of the sparse grids, the n in equation 1.7,
- p_weight the weight for anisotropic sparse grids, the w in equation 1.7,
- p_degree is equal to 1 (linear interpolator), or 2 (quadratic interpolator) or 3 (for cubic interpolator),

With the same notations the construction eliminating boundary points is done by the following constructor

```

1 SparseSpaceGridNoBound(const Eigen::ArrayXd &p_lowValues, const Eigen::
  ArrayXd &p_sizeDomain, const int &p_levelMax, const Eigen::ArrayXd &
  p_weight,
2 const size_t &p_degree)

```

The data structure of type *SparseSet* to store the sparse grid is defined by a map with keys an array A storing a multi level and values a map with keys an array B storing the multi index associated to a point (A,B) and values the number of point (A,B) :

```

1 #define SparseSet std::map< Eigen::Array<char, Eigen::Dynamic, 1 >, std
  ::map< Eigen::Array<unsigned int, Eigen::Dynamic, 1>, size_t,
  OrderTinyVector< unsigned int > >, OrderTinyVector< char> >

```

It is sometimes convenient to get back this data structure from the SparseGrid object : this is achieved by the following method :

```

1 std::shared_ptr<SparseSet> getDataSet() const ;

```

The previous two classes own two specific member functions to hierarchize (see section above) the value function known at the grids points for the whole grid.

- the first work on a single function:

```

1 /// \brief Hierarchize a function defined on the grid
2 /// \param p_toHierachize function to hierarchize
3 void toHierarchize( Eigen::ArrayXd & p_toHierachize );

```

- the second work on a matrix, permitting to hierarchize many functions in a single call (each row corresponds to a function representation)

```

1
2 /// \brief Hierarchize a set of functions defined on the grid
3 /// \param p_toHierachize function to hierarchize
4 void toHierarchizeVec( Eigen::ArrayXXd & p_toHierachize )

```

The two classes own two specific member functions to hierarchize point by point a value function at given points in the sparse grid :

- the first work on a single function:

```

1 /// \brief Hierarchize some points defined on the sparse grids
2 /// Hierarchization is performed point by point
3 /// \param p_nodalValues function to hierarchize
4 /// \param p_sparsePoints vector of sparse points to
  hierarchize (all points should belong to the dataset structure)
5 /// \param p_hierarchized array of all hierarchized values (it
  is updated)
6 virtual void toHierarchizePByP(const Eigen::ArrayXd &p_nodalValues,
  const std::vector<SparsePoint> &p_sparsePoints, Eigen::ArrayXd
  &p_hierarchized) const

```

- the second work on a matrix, permitting to hierarchize many functions in a single call (each row corresponds to a function representation)

```

1  /// \brief Hierarchize some points defined on the sparse grids for a
    set of functions
2  ///      Hierarchization is performed point by point
3  /// \param p_nodalValues      functions to hierarchize (the row
    corresponds to the function number)
4  /// \param p_sparsePoints     vector of sparse points to
    hierarchize (all points should belong to the dataset structure)
5  /// \param p_hierarchized     array of all hierarchized values (it
    is updated)
6  virtual void toHierarchizePByPVec(const Eigen::ArrayXXd &
    p_nodalValues, const std::vector<SparsePoint> &p_sparsePoints,
    Eigen::ArrayXXd &p_hierarchized) const

```

The *SparsePoint* object is only a type def :

```

1 #define SparsePoint std::pair< Eigen::Array<char, Eigen::Dynamic, 1> , Eigen
    ::Array<unsigned int, Eigen::Dynamic, 1> >

```

where the first array permits to store the multi level associated to the point and the second the multi index associated.

At last it is possible to hierarchize all points associated to a multi level. As before two methods are available :

- a first permits to hierarchize all the points associated to a given level. Hierarchized values are updated with these new values.

```

1  /// \brief Hierarchize all points defined on a given level of the
    sparse grids
2  ///      Hierarchization is performed point by point
3  /// \param p_nodalValues      function to hierarchize
4  /// \param p_iterLevel        iterator on the level of the point
    to hierarchize
5  /// \param p_hierarchized     array of all hierarchized values (it
    is updated)
6  virtual void toHierarchizePByPLevel(const Eigen::ArrayXd &
    p_nodalValues, const SparseSet::const_iterator &p_iterLevel,
    Eigen::ArrayXd &p_hierarchized) const

```

- the second permits to hierarchize different functions together

```

1  /// \brief Hierarchize all points defined on a given level of the
    sparse grids for a set of functions
2  ///      Hierarchization is performed point by point
3  /// \param p_nodalValues      function to hierarchize (the row
    corresponds to the function number)
4  /// \param p_iterLevel        iterator on the level of the point
    to hierarchize
5  /// \param p_hierarchized     array of all hierarchized values (it
    is updated)
6  virtual void toHierarchizePByPLevelVec(const Eigen::ArrayXXd &
    p_nodalValues, const SparseSet::const_iterator &p_iterLevel,
    Eigen::ArrayXXd &p_hierarchized) const

```

In the following example, the sparse grids with boundary points is constructed. The values of a function f at each coordinates are stored in an array *valuesFunction*, storing 2 functions to interpolate. The 2 global functions are hierarchized (see section above) in the array *hierarValues*, and then the interpolation can be achieved using these hierarchized values.

```

1   ArrayXd  lowValues = ArrayXd::Zero(5); // bottom of the grid
2   ArrayXd  sizeDomain = ArrayXd::Constant(5,1.); // size of the grid
3   ArrayXd  weight = ArrayXd::Constant(5,1.); // weights
4   int degree =1 ; // linear interpolator
5   bool bPrepInterp = true; // precalculate neighbors of nodes
6   level = 4 ; // level of the sparse grid
7
8   // sparse grid generation
9   SparseSpaceGridBound sparseGrid(lowValues , sizeDomain , level , weight ,
   degree , bPrepInterp);
10
11  // grid iterators
12  shared_ptr<GridIterator > iterGrid = sparseGrid.getGridIterator();
13  ArrayXXd valuesFunction(1,sparseGrid.getNbPoints());
14  while (iterGrid->isValid())
15  {
16      ArrayXd pointCoord = iterGrid->getCoordinate();
17      valuesFunction(0,iterGrid->getCount()) = f(pointCoord) ;
18      valuesFunction(1,iterGrid->getCount()) = f(pointCoord)+1 ;
19      iterGrid->next();
20  }
21
22  // Hierarchize
23  ArrayXXd hieraValues =valuesFunction;
24  sparseGrid.toHierarchizeVec(hieraValues);
25
26  // interpolate
27  ArrayXd pointCoord = ArrayXd::Constant(5,0.66);
28  shared_ptr<Interpolator > interpolator = sparseGrid.createInterpolator(
   pointCoord);
29  ArrayXd interVal = interpolator->applyVec(hieraValues);

```

Remark 5 *Point by point hierarchization on the global grid could have been calculated as below*

```

1   std::vector<SparsePoint> sparsePoints(sparseGrid.getNbPoints());
2   std::shared_ptr<SparseSet> dataSet = sparseGrid.getDataSet();
3   // iterate on points
4   for (typename SparseSet::const_iterator iterLevel = dataSet->begin();
   iterLevel != dataSet->end(); ++iterLevel)
5       for (typename SparseLevel::const_iterator iterPosition = iterLevel->
   second.begin(); iterPosition != iterLevel->second.end(); ++
   iterPosition)
6           {
7               sparsePoints[iterPosition->second] = make_pair(iterLevel->first ,
   iterPosition->first);
8           }

```

```

9     ArrayXXd hieraValues = sparseGrid.toHierarchizePByPVec(
        valuesFunction , sparsePoints);

```

In some cases, it is more convenient to construct an interpolator acting on a global function. It is the case when you have a single function and you want to interpolate at many points for this function. In this specific case an interpolator deriving from the class `InterpolatorSpectral` (similarly to Legendre grid interpolators) can be constructed :

```

1     /** \brief Constructor taking in values on the grid
2     * \param p_grid      is the sparse grid used to interpolate
3     * \param p_values    Function values on the sparse grid
4     */
5     SparseInterpolatorSpectral(const shared_ptr< SparseSpaceGrid> &p_grid ,
        const Eigen::ArrayXd &p_values)

```

This class has a member to interpolate at a given point:

```

1     /** \brief interpolate
2     * \param p_point    coordinates of the point for interpolation
3     * \return interpolated value
4     */
5     inline double apply(const Eigen::ArrayXd &p_point) const

```

See section 1.2 for an example (similar but with Legendre grids) to use this object. Sometimes, one wish to iterate on points on a givel level. In the example below , for each level an iterator on all points belonging to a given level is got back and the values of a function f at each point are calculated and stored.

```

1     // sparse grid generation
2     SparseSpaceGridNoBound sparseGrid(lowValues , sizeDomain , p_level ,
        p_weight , p_degree , bPrepInterp);
3
4     // test iterator on each level
5     ArrayXd valuesFunctionTest(sparseGrid.getNbPoints());
6     std::shared_ptr<SparseSet> dataSet = sparseGrid.getDataSet();
7     for (SparseSet::const_iterator iterLevel = dataSet->begin(); iterLevel
        != dataSet->end(); ++iterLevel)
8     {
9         // get back iterator on this level
10    shared_ptr<SparseGridIterator> iterGridLevel = sparseGrid.
        getLevelGridIterator(iterLevel);
11    while(iterGridLevel->isValid())
12    {
13        Eigen::ArrayXd pointCoord = iterGridLevel->getCoordinate();
14        valuesFunctionTest(iterGridLevel->getCount()) = f(pointCoord);
15        iterGridLevel->next();
16    }
17    }

```

At last adaptation can be realized with two member functions :

- A first one permits to refine adding points where the error is important. Notice that a function is provided to calculate from the hierarchical values the error at each level of

the sparse grid and that a second one is provided to get a global error from the error calculated at each level. This permits to specialize the refining depending for example if the calculation is achieved for integration or interpolation purpose.

```

1  /// \brief Dimension adaptation nest
2  /// \param p_precision      precision required for adaptation
3  /// \param p_fInterpol     function to interpolate
4  /// \param p_phi           function for the error on a given level
5  /// \param p_phiMult       from an error defined on different
6  /// \param p_valuesFunction an array storing the nodal values
7  /// \param p_hierarValues  an array storing hierarchized values (
8  void refine(const double &p_precision , const std::function<double(
9  const Eigen::ArrayXd &p_x)> &p_fInterpol ,
10             const std::function< double(const SparseSet::
11             const_iterator &, const Eigen::ArrayXd &)> &p_phi ,
12             const std::function< double(const std::vector< double> &
13             > &p_phiMult ,
14             Eigen::ArrayXd &p_valuesFunction ,
15             Eigen::ArrayXd &p_hierarValues );

```

with

- *p_precision* the η tolerance in the algorithm,
 - *p_fInterpol* the function permitting to calculate the nodal values,
 - *p_phi* function permitting to calculate e_l the local error for a given l ,
 - *p_phiMult* a function taking as argument all the e_l (local errors) and giving back the global error E ,
 - *p_valuesFunction* an array storing the nodal values (updated during refinement)
 - *p_hierarValues* an array storing the hierarchized values (updated during refinement)
- A second one permits to coarsen the mesh, eliminating point where the error is too small

```

1  /// \brief Dimension adaptation coarsening : modify data structure by
2  ///      trying to remove all levels with local error
3  ///      below a local precision
4  /// \param p_precision      Precision under which coarsening will be
5  /// \param p_phi           function for the error on a given level
6  /// \param p_valuesFunction an array storing the nodal values (
7  /// \param p_hierarValues  Hierarchical values on a data structure (
8  void coarsen(const double &p_precision , const std::function< double(
9  const SparseSet::const_iterator &, const Eigen::ArrayXd &)> &
10             p_phi ,

```



```

8 Eigen::ArrayXd &p_valuesFunction ,
9 Eigen::ArrayXd &p_hierarValues);

```

with arguments similar to the previous function.

1.8 Python API

Here is an example of the python API used for interpolation with Sparse grids with boundary points and without boundary points. The adaptation and coarsening is available with an error calculated for interpolation only.

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import numpy as np
5 import unittest
6 import random
7 import math
8 import StOptGrids
9
10 # function used
11 def funcToInterpolate( x):
12     return math.log(1. + x.sum())
13
14 # unit test for sparse grids
15 #####
16
17 class testGrids(unittest.TestCase):
18
19
20     # test sparse grids with boundaries
21     def testSparseGridsBounds(self):
22         # low values
23         lowValues =np.array([1.,2.,3.])
24         # size of the domain
25         sizeDomValues = np.array([3.,4.,3.])
26         # anisotropic weights
27         weights = np.array([1.,1.,1.])
28         # level of the sparse grid
29         level =3
30         # create the sparse grid with linear interpolator
31         sparseGridLin = StOptGrids.SparseSpaceGridBound(lowValues ,
32             sizeDomValues , level , weights,1)
33         iterGrid = sparseGridLin.getGridIterator()
34         # array to store
35         data = np.empty(sparseGridLin.getNbPoints())
36         # iterates on point
37         while( iterGrid.isValid()):
38             data[iterGrid.getCount()] = funcToInterpolate(iterGrid.
39                 getCoordinate())
39             iterGrid.next()
39         # Hierarchize the data

```

```

40     hierarData = sparseGridLin.toHierarchize(data)
41     # get back an interpolator
42     ptInterp = np.array([2.3,3.2,5.9], dtype=np.float)
43     interpol = sparseGridLin.createInterpolator(ptInterp)
44     # calculate interpolated value
45     interpValue = interpol.apply(hierarData)
46     print(("Interpolated value sparse linear" , interpValue))
47     # create the sparse grid with quadratic interpolator
48     sparseGridQuad = StOptGrids.SparseSpaceGridBound(lowValues ,
49         sizeDomValues , level , weights ,2)
49     # Hierarchize the data
50     hierarData = sparseGridQuad.toHierarchize(data)
51     # get back an interpolator
52     ptInterp = np.array([2.3,3.2,5.9], dtype=np.float)
53     interpol = sparseGridQuad.createInterpolator(ptInterp)
54     # calculate interpolated value
55     interpValue = interpol.apply(hierarData)
56     print(("Interpolated value sparse quadratic " , interpValue))
57     # now refine
58     precision = 1e-6
59     print(("Size of hierarchical array " , len(hierarData)))
60     valueAndHierar = sparseGridQuad.refine(precision , funcToInterpolate ,
61         data , hierarData)
61     print(("Size of hierarchical array after refinement " , len(
62         valueAndHierar[0])))
62     # calculate interpolated value
63     interpol1 = sparseGridQuad.createInterpolator(ptInterp)
64     interpValue = interpol1.apply(valueAndHierar[1])
65     print(("Interpolated value sparse quadratic after refinement " ,
66         interpValue))
66     # coarsen the grid
67     precision = 1e-4
68     valueAndHierarCoarsen = sparseGridQuad.coarsen(precision ,
69         valueAndHierar[0] , valueAndHierar[1])
69     print(("Size of hierarchical array after coarsening " , len(
70         valueAndHierarCoarsen[0])))
70     # calculate interpolated value
71     interpol2 = sparseGridQuad.createInterpolator(ptInterp)
72     interpValue = interpol2.apply(valueAndHierarCoarsen[1])
73     print(("Interpolated value sparse quadratic after refinement " ,
74         interpValue))
74
75
76     # test sparse grids eliminating boundaries
77     def testSparseGridsNoBounds(self):
78         # low values
79         lowValues =np.array([1.,2.,3.] , dtype=np.float)
80         # size of the domain
81         sizeDomValues = np.array([3.,4.,3.] , dtype=np.float)
82         # anisotropic weights
83         weights = np.array([1.,1.,1.])
84         # level of the sparse grid
85         level =3
86         # create the sparse grid with linear interpolator

```

```

87 sparseGridLin = StOptGrids.SparseSpaceGridNoBound(lowValues ,
    sizeDomValues , level , weights ,1)
88 iterGrid = sparseGridLin.getGridIterator()
89 # array to store
90 data = np.empty(sparseGridLin.getNbPoints())
91 # iterates on point
92 while( iterGrid.isValid()):
93     data[iterGrid.getCount()] = funcToInterpolate(iterGrid.
        getCoordinate())
94     iterGrid.next()
95 # Hierarchize the data
96 hierarData = sparseGridLin.toHierarchize(data)
97 # get back an interpolator
98 ptInterp = np.array([2.3,3.2,5.9], dtype=np.float)
99 interpol = sparseGridLin.createInterpolator(ptInterp)
100 # calculate interpolated value
101 interpValue = interpol.apply(hierarData)
102 print("Interpolated value sparse linear" , interpValue)
103 # create the sparse grid with quadratic interpolator
104 sparseGridQuad = StOptGrids.SparseSpaceGridNoBound(lowValues ,
    sizeDomValues , level , weights ,2)
105 # Hierarchize the data
106 hierarData = sparseGridQuad.toHierarchize(data)
107 # get back an interpolator
108 ptInterp = np.array([2.3,3.2,5.9], dtype=np.float)
109 interpol = sparseGridQuad.createInterpolator(ptInterp)
110 # calculate interpolated value
111 interpValue = interpol.apply(hierarData)
112 print("Interpolated value sparse quadratic " , interpValue)
113 # test grids function
114 iDim = sparseGridQuad.getDimension()
115 pt = sparseGridQuad.getExtremeValues()
116 # now refine
117 precision = 1e-6
118 print("Size of hierarchical array " , len(hierarData))
119 valueAndHierar = sparseGridQuad.refine(precision , funcToInterpolate ,
    data , hierarData)
120 print("Size of hierarchical array after refinement " , len(
    valueAndHierar[0]))
121 # calculate interpolated value
122 interpol1 = sparseGridQuad.createInterpolator(ptInterp)
123 interpValue = interpol1.apply(valueAndHierar[1])
124 print("Interpolated value sparse quadratic after coarsening " ,
    interpValue)
125 # coarsen the grid
126 precision = 1e-4
127 valueAndHierarCoarsen = sparseGridQuad.coarsen(precision ,
    valueAndHierar[0] , valueAndHierar[1])
128 print("Size of hierarchical array after coarsening " , len(
    valueAndHierarCoarsen[0]))
129 # calculate interpolated value
130 interpol2 = sparseGridQuad.createInterpolator(ptInterp)
131 interpValue = interpol2.apply(valueAndHierarCoarsen[1])
132 print("Interpolated value sparse quadratic after coarsening " ,

```

```
        interpValue))
133
134 if __name__ == '__main__':
135     unittest.main()
```

Chapter 2

Introducing the regression resolution

Suppose the the stochastic differential equation in the optimization problem is not controlled:

$$dX^{x,t} = b(t, X_s^{x,t})ds + \sigma(s, X_s^{x,t})dW_s$$

This case is for example encountered while valuing American options in finance, when an arbitrage is realized between the pay off and the expected future gain if not exercising at the current time. In order to estimate this conditional expectation (depending of the Markov state), first suppose that a set of N Monte Carlo Simulation are available at dates t_i for a process $X_t := X_t^{0,x}$ where x is the initial state at date $t = 0$ and that we want to estimate $f(x) := \mathbb{E}[g(t+h, X_{t+h}) | X_t = x]$ for a given x and a given function g . This function f lies the infinite dimensional space of the L_2 functions. In order to approximate it, we try to find it in a finite dimensional space. Choosing a set of basis functions ψ_k for $k = 1$ to M , the conditional expectation can be approximated by

$$f(x) \simeq \sum_{k=1}^M \alpha_k \psi_k(X_t) \quad (2.1)$$

where $(\hat{\alpha}_k^{t_i, N})_{k \leq M}$ minimizes

$$\sum_{\ell=1}^N \left| g(X_{t+h}^\ell) - \sum_{k=1}^M \alpha_k \psi_k(X_t^\ell) \right|^2 \quad (2.2)$$

over $(\alpha_k)_{k \leq M} \in \mathbb{R}^M$. We have to solve a quadratic optimization problem of the form

$$\min_{\alpha \in \mathbb{R}^M} \|A\alpha - B\|^2 \quad (2.3)$$

Classically the previous equation is reduced to the normal equation

$$A'A\alpha = A'B, \quad (2.4)$$

which is solved by a Cholesky like approach when the matrix $A'A$ is definite otherwise the solution with the minimum L_2 norm can be computed.

2.1 C++ global API

All the regression classes derive from the BaseRegression abstract class, which stores a pointer to the “particles” (a matrix storing the simulations of $X^{x,t}$: the first dimension of the matrix corresponds to the dimension of $X^{x,t}$, and the second dimension corresponds to the particle number), and stores if the current date t is 0 (then the conditional expectation is only an expectation).

```
1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef BASEREGRESSION_H
5 #define BASEREGRESSION_H
6 #include <memory>
7 #include <vector>
8 #include <Eigen/Dense>
9 #include "StOpt/core/grids/InterpolatorSpectral.h"
10
11 /** \file BaseRegression.h
12  * \brief Base class to define regressor for stochastic optimization by
13  * Monte Carlo
14  * \author Xavier Warin
15  */
16 namespace StOpt
17 {
18   /// \class BaseRegression BaseRegression.h
19   /// Base class for regression
20   class BaseRegression
21   {
22   protected :
23
24     bool m_bZeroDate ;          ///< Is the regression date zero ?
25     std::shared_ptr<Eigen::ArrayXXd> m_particles;  ///< Particles used to
26     regress: first dimension : dimension of the problem , second
27     dimension : the number of particles
28
29 public :
30
31     /// \brief Default constructor
32     BaseRegression() {}
33
34     /// \brief Default destructor
35     virtual ~BaseRegression() {}
36
37     /// \brief Constructor storing the particles
38     /// \param p_bZeroDate first date is 0?
39     /// \param p_particles particles used for the meshes.
40     /// First dimension : dimension of the problem,
41     /// second dimension : the number of particles
42     BaseRegression(const bool &p_bZeroDate, const std::shared_ptr< Eigen::
43     ArrayXXd> &p_particles) : m_bZeroDate(p_bZeroDate), m_particles(
44     p_particles) {}
45 }
```

```

41
42  /// \brief Last constructor used in simulation
43  /// \param p_bZeroDate    first date is 0?
44  BaseRegression(const bool &p_bZeroDate) : m_bZeroDate(p_bZeroDate) {}
45
46  /// \brief Copy constructor
47  /// \param p_object    object to copy
48  BaseRegression(const BaseRegression &p_object): m_bZeroDate(p_object .
    getBZeroDate()), m_particles(p_object.getParticles()) {}
49
50  /// \brief update the particles used in regression  and construct the
    matrices
51  /// \param p_bZeroDate    first date is 0?
52  /// \param p_particles    particles used for the meshes.
53  ///                      First dimension  : dimension of the problem ,
54  ///                      second dimension : the number of particles
55  void updateSimulationsBase(const bool &p_bZeroDate, const std::shared_ptr
    < Eigen::ArrayXXd> &p_particles)
56  {
57      m_bZeroDate = p_bZeroDate;
58      m_particles = p_particles;
59
60  }
61
62  /// \brief Get some local accessors
63  ///@{
64  inline std::shared_ptr< Eigen::ArrayXXd > getParticles() const
65  {
66      return m_particles ;
67  }
68
69  /// \brief Get the object by reference
70  inline const Eigen::ArrayXXd &getParticlesRef() const
71  {
72      return *m_particles;
73  }
74
75  /// \brief Get dimension of the problem
76  inline int getDimension() const
77  {
78      return m_particles->rows();
79  }
80
81  /// \brief Get the number of simulations
82  inline int getNbSimul() const
83  {
84      return m_particles->cols() ;
85  }
86
87  /// \brief get the number of basis functions
88  virtual int getNumberOfFunction() const = 0 ;
89
90  ///@}
91  /// \brief Constructor storing the particles

```

```

92  /// \brief update the particles used in regression and construct the
    matrices
93  /// \param p_bZeroDate first date is 0?
94  /// \param p_particles particles used for the meshes.
95  /// First dimension : dimension of the problem,
96  /// second dimension : the number of particles
97  virtual void updateSimulations(const bool &p_bZeroDate, const std::
    shared_ptr< Eigen::ArrayXXd> &p_particles) = 0 ;
98
99  /// \brief conditional expectation basis function coefficient calculation
100  /// \param p_fToRegress function to regress associated to each
    simulation used in optimization
101  /// \return regression coordinates on the basis (size : number of meshes
    multiplied by the dimension plus one)
102  /// @{
103  virtual Eigen::ArrayXd getCoordBasisFunction(const Eigen::ArrayXd &
    p_fToRegress) const = 0;
104  virtual Eigen::ArrayXXd getCoordBasisFunctionMultiple(const Eigen::
    ArrayXXd &p_fToRegress) const = 0 ;
105  ///@}
106
107  /// \brief conditional expectation calculation
108  /// \param p_fToRegress simulations to regress used in optimization
109  /// \return regressed value function
110  /// @{
111  virtual Eigen::ArrayXd getAllSimulations(const Eigen::ArrayXd &
    p_fToRegress) const = 0;
112  virtual Eigen::ArrayXXd getAllSimulationsMultiple(const Eigen::ArrayXXd &
    p_fToRegress) const = 0;
113  ///@}
114
115  /// \brief Use basis functions to reconstruct the solution
116  /// \param p_basisCoefficients basis coefficients
117  ///@{
118  virtual Eigen::ArrayXd reconstruction(const Eigen::ArrayXd &
    p_basisCoefficients) const = 0 ;
119  virtual Eigen::ArrayXXd reconstructionMultiple(const Eigen::ArrayXXd &
    p_basisCoefficients) const = 0;
120  ///@}
121
122  /// \brief use basis function to reconstruct a given simulation
123  /// \param p_isim simulation number
124  /// \param p_basisCoefficients basis coefficients to reconstruct a given
    conditional expectation
125  virtual double reconstructionASim(const int &p_isim , const Eigen::
    ArrayXd &p_basisCoefficients) const = 0 ;
126
127  /// \brief conditional expectation reconstruction
128  /// \param p_coordinates coordinates to interpolate (uncertainty
    sample)
129  /// \param p_coordBasisFunction regression coordinates on the basis (
    size: number of meshes multiplied by the dimension plus one)
130  /// \return regressed value function reconstructed for each simulation
131  virtual double getValue(const Eigen::ArrayXd &p_coordinates ,

```



```

132         const Eigen::ArrayXd &p_coordBasisFunction)
133             const = 0;
134     /// \brief permits to reconstruct a function with basis functions
135     /// coefficients values given on a grid
136     /// \param p_coordinates coordinates (uncertainty sample)
137     /// \param p_ptOfStock grid point
138     /// \param p_interpFuncBasis spectral interpolator to interpolate
139     /// the basis functions coefficients used in regression on the grid (
140     /// given for each basis function)
141     virtual double getAValue(const Eigen::ArrayXd &p_coordinates, const
142     Eigen::ArrayXd &p_ptOfStock,
143     const std::vector< std::shared_ptr<
144     InterpolatorSpectral>> &p_interpFuncBasis)
145     const = 0;
146
147     /// \brief is the regression date zero
148     inline bool getBZeroDate() const
149     {
150         return m_bZeroDate;
151     }
152
153     /// \brief Clone the regressor
154     virtual std::shared_ptr<BaseRegression> clone() const = 0 ;
155 };
156 }
157 #endif

```

All regression classes share the same constructors:

- a first constructor stores the members of the class and computes the matrices for the regression: it is used for example to build a regression object at each time step of a resolution method,
- the second constructor is used to prepare some data which will be shared by all future regressions. It has to be used with the 'updateSimulation' method to update the effective matrix construction. In a resolution method with many time steps, the object will be constructed only once and at each time step the Markov state will be updated by the 'updateSimulation' method.

All regression classes share the common methods:

- "updateSimulationBase" (see above),
- "getCoordBasisFunction" takes the values $g(t+h, X_{t+h})$ for all simulations and returns the coefficients α_k of the basis functions,
- "getCoordBasisFunctionMultiple" is used if we want to do the previous calculation on multiple g functions in one call. In the matrix given as argument, the first dimension has a size equal to the number of Monte Carlo simulations, while the second dimension

has a size equal to the number of functions to regress. As output, the first dimension has a size equal to the number of function to regress and the second equal to the number of basis functions.

- “getAllSimulations” takes the values $g(t+h, X_{t+h})$ for all simulations and returns the regressed values for all simulations $f(X_t)$
- “getAllSimulationMultiple” is used if we want to do the previous calculation on multiple g functions in one call. In the matrix given as argument, the first dimension has a size equal to the number of Monte Carlo simulations, while the second dimension has a size equal to the number of functions to regress. The regressed values are given back in the same format.
- “reconstruction” takes the α_k coefficient of the basis functions as input and returns all the $f(X_t)$ for the simulations stored by applying equation (2.1).
- “reconstructionMultiple” is used if we want to do the previous calculation on multiple g functions in one call. As input the α_k coefficients of the basis functions are given (number of function to regress for first dimension, number of basis functions for second dimension). As a result the $f(X_t)$ for all simulations and all f functions are sent back (number of Monte Carlo simulations in first dimension, number of function to regress en second dimension).
- “reconstructionASim” takes a simulation number $isim$ (optimization part) and α_k coefficient of the basis functions as input and returns $f(X_t^{isim})$ by applying equation (2.1),
- “getValue” takes as first argument a sample of X_t , the basis function α_k and reconstruct the regressed solution of equation (2.1).

2.2 Adapted local polynomial basis

The description of the method and its properties can be found in [16]. We just recall the methodology

2.2.1 Description of the method

The method essentially consists in applying a non-conform finite element approach rather than a spectral like method as presented above.

The idea is to use, at each time step t_i , a set of functions $\psi_q, q \in [0, M_M]$ having local hyper cube support D_{i_1, i_2, \dots, i_d} where $i_j = 1$ to I_j , $M_M = \prod_{k=1, d} I_k$, and $\{D_{i_1, \dots, i_d}\}_{(i_1, \dots, i_d) \in [1, I_1] \times \dots \times [1, I_d]}$ is a partition of $[\min_{k=1, N} X_{t_i}^{1, (k)}, \max_{k=1, N} X_{t_i}^{1, (k)}] \times \dots \times [\min_{k=1, N} X_{t_i}^{d, (k)}, \max_{k=1, N} X_{t_i}^{d, (k)}]$. On each $D_l, l = (i_1, \dots, i_d)$, depending on the selected method, ψ_l is

- either a constant function, so the global number of degrees of freedom is equal to M_M ,
- or a linear function with $1 + d$ degrees of freedom, so the global number of degrees of freedom is equal to $M_M * (1 + d)$.

This approximation is “non-conform” in the sense that we do not assure the continuity of the approximation. However, it has the advantage to be able to fit any, even discontinuous, function. In order to avoid oscillations and to allow classical regression by the Choleski method, the supports are chosen so that they contain roughly the same number of particles.

On Figure 2.1, we have plotted an example of supports in the case of $6 = 4 \times 4$ local basis cells, in dimension 2.

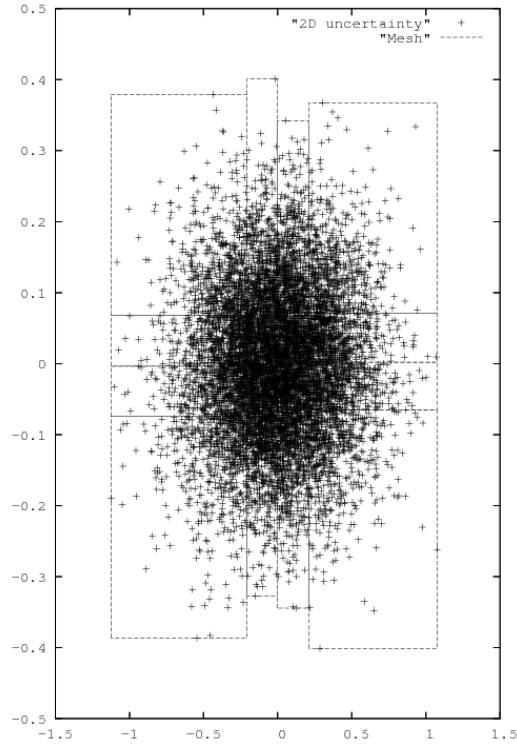


Figure 2.1: Support of 2D function basis

2.3 C++ api

2.3.1 The constant per cell approximation

The constructor of the local constant regression object is achieved by

```
1 LocalConstRegression(const Eigen::ArrayXi &p_nbMesh);
```

where *p_nbMesh* is an array giving the number of meshes used in each direction ((4,4) for the figure 2.1 for example).

The second constructor permits the construct the regression matrix,

```
1 LocalConstRegression(const bool &p_bZeroDate ,
2     const shared_ptr< ArrayXXd> &p_particles ,
3     const Eigen::ArrayXi &p_nbMesh)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_nbMesh* is an array giving the number of meshes used in each directions (4,4) for the figure 2.1.

2.3.2 The linear per cell approximation

The constructor of the local linear regression object is achieved by

```
1 LocalLinearRegression(const Eigen::ArrayXi &p_nbMesh);
```

where *p_nbMesh* is an array giving the number of meshes used in each direction ((4,4) for the figure 2.1 for example).

The second constructor permits the construct the regression matrix,

```
1 LocalLinearRegression(const bool &p_bZeroDate ,
2     const shared_ptr< ArrayXXd> &p_particles ,
3     const Eigen::ArrayXi &p_nbMesh)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_nbMesh* is an array giving the number of meshes used in each directions (4,4) for the figure 2.1.

2.3.3 An example in the linear case

Below we give a small example where “toRegress” corresponds to $g(t + h, X_{t+h})$ for all simulations and x store X_t for all simulations.

```
1 // create the mesh for a 2 dim problem , 4 meshes per direction
2 ArrayXi nbMesh = ArrayXi::Constant(2, 4);
3 // t is not zero
4 bool bZeroDate = 0;
5 // constructor
6 LocalLinearRegression localRegressor(nbMesh);
7 // update particles values
8 localRegressor.updateSimulations(bZeroDate, x);
9 // regressed values
10 ArrayXd regressedValues = localRegressor.getAllSimulations(toRegress);
```

2.4 Python API

Here is a similar example using the second constructor of the linear case

```
1 import StOptReg
2 nbSimul = 5000000;
3 np.random.seed(000)
4 x = np.random.uniform(-.,1.,size=(1,nbSimul));
5 # real function
6 toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
7 # function to regress
8 toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
9 # mesh
10 nbMesh = np.array([6],dtype=np.int32)
11 # Regressor
12 regressor = StOptReg.LocalLinearRegression(False,x,nbMesh)
13 y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Of course the constant per cell case in python is similar.

2.5 Local polynomial basis with meshes of same size

In some cases, instead of using adapted meshes, one can prefer to fix the mesh with a constant step in each direction with I_k meshes in each direction so that the total number of cells is $M_M = \prod_{k=1,d} I_k$. On each cell as in section 2.2, one can have two approximations :

- either a constant function, so the global number of degrees of freedom is equal to M_M ,
- or a linear function with $1 + d$ degrees of freedom, so the global number of degrees of freedom is equal to $M_M * (1 + d)$.

2.6 C++ api

2.6.1 The constant per cell approximation

The constructor of the local constant regression object is achieved by

```
1 LocalSameSizeConstRegression(const Eigen::ArrayXd &p_lowValues, const
Eigen::ArrayXd &p_step, const Eigen::ArrayXi &p_nbStep);
```

- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

The second constructor permits the construct the regression matrix,

```

1   LocalSameSizeConstRegression( const bool &p_bZeroDate ,
2                                 const std::shared_ptr< Eigen::ArrayXXd > &
3                                 p_particles ,
4                                 const Eigen::ArrayXd &p_lowValues ,
5                                 const Eigen::ArrayXd &p_step ,
6                                 const Eigen::ArrayXi &p_nbStep );

```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

2.6.2 The linear per cell approximation

The constructor of the local linear regression object is achieved by

```

1   LocalSameSizeLinearRegression( const Eigen::ArrayXd &p_lowValues , const
2                                 Eigen::ArrayXd &p_step , const Eigen::ArrayXi &p_nbStep );

```

where

- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

The second constructor permits the construct the regression matrix,

```

1   LocalSameSizeLinearRegression( const bool &p_bZeroDate ,
2                                 const std::shared_ptr< Eigen::ArrayXXd > &
3                                 p_particles ,
4                                 const Eigen::ArrayXd &p_lowValues ,
5                                 const Eigen::ArrayXd &p_step ,
6                                 const Eigen::ArrayXi &p_nbStep );

```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_lowValues* is an array giving the first point of the grid in each direction,
- *p_step* is an array giving the size of the meshes in each direction,
- *p_nbStep* is an array giving the number of meshes used in each direction.

2.6.3 An example in the linear case

Below we give a small example where “toRegress” is the array to regress with respect to an array “x” in dimension p_nDim :

```
1 // create a random ‘x’ array
2 shared_ptr<ArrayXXd> x(new ArrayXXd(ArrayXXd::Random(p_nDim, p_nbSimul))
3 );
4 // create the mesh by getting min and max value on the samples
5 double xMin = x->minCoeff() - tiny;
6 double xMax = x->maxCoeff() + tiny;
7 ArrayXd lowValues = ArrayXd::Constant(p_nDim, xMin);
8 ArrayXd step = ArrayXd::Constant(p_nDim, (xMax - xMin) / p_nMesh);
9 ArrayXi nbStep = ArrayXi::Constant(p_nDim, p_nMesh);
10 // constructor
11 LocalLinearRegression localRegressor(lowValues, step, nbStep);
12 // update particles values
13 localRegressor.updateSimulations(bZeroDate, x);
14 // regressed values
15 ArrayXd regressedValues = localRegressor.getAllSimulations(toRegress);
```

2.7 Python API

Here is a similar example using the second constructor of the linear case

```
1 import StOptReg
2 nbSimul = 5000000;
3 np.random.seed(000)
4 x = np.random.uniform(-., 1., size=(1, nbSimul));
5 # real function
6 toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
7 # function to regress
8 toRegress = toReal + 4*np.random.normal(0., , nbSimul)
9 # mesh
10 nStep = 20
11 lowValue = np.array([-1.0001], dtype=np.float)
12 step = np.array([2.0002/nStep], dtype=np.float)
13 nbMesh = np.array([nStep], dtype=np.int32)
14 # Regressor
15 regressor = StOptReg.LocalSameSizeLinearRegression(False, x, lowValue,
16 step, nbMesh)
17 y = regressor.getAllSimulations(toRegress).transpose()[0]
```

Of course the constant per cell case in python is similar.

2.8 Sparse grid regressor

In the case of a sparse regressor, the grid is an object “SparseSpaceGridNoBound” (extrapolation for the boundary conditions). The basis functions are given by the section 1.3 for linear, quadratic or cubic function basis.

2.8.1 C++ API

Two specific constructor are available:

- The first one to be used with the “updateSimulations” methods

```
1 SparseRegression(const int &p_levelMax, const Eigen::ArrayXd &  
p_weight, const int &p_degree, bool p_bNoRescale = false);
```

where

- *p_levelMax* corresponds to n in the equation (1.4),
 - *p_weight* the weight for anisotropic sparse grids (see equation (1.7),
 - *p_degree* is equal to (linear basis function), or 2 (quadratic basis) or 3 (for cubic basis functions),
 - *p_bNoRescale* if true no re scaling of the particles is used. Otherwise a re scaling of the mesh size is achieved (as for local basis functions, see section 2.2)
- The second one take the same arguments as the first constructor but adds a Boolean to check if the regression date is 0 and the particles X_t (here the re scaling is always achieved):

```
1 SparseRegression(const bool &p_bZeroDate,  
2                 const shared_ptr< Eigen::ArrayXXd > &p_particles,  
3                 const int &p_levelMax, const Eigen::ArrayXd &  
4                 p_weight,  
3                 const int &p_degree);
```

A simple example to express the regression of “toRegress”

```
1 // second member to regress  
2 ArrayXd toRegress(p_nbSimul);  
3 // for testing  
4 toRegress.setConstant(.);  
5 shared_ptr<ArrayXXd> x(new ArrayXXd(ArrayXXd::Random(p_nDim,  
6 p_nbSimul)));  
7 // constructor : the current date is not zero  
8 bool bZeroDate = 0;  
9 // constructor  
10 SparseRegression sparseRegressor(p_level, weight, p_degree);  
11 sparseRegressor.updateSimulations(bZeroDate, x); // update the state  
12 // then just calculate function basis coefficient  
13 ArrayXd regressedFunctionCoeff = sparseRegressor.getCoordBasisFunction  
14 (toRegress);  
15 // use the getValue method to get back the regressed values  
16 for (int is = 0; is < p_nbSimul; ++is)  
17 {  
18     Map<ArrayXd> xloc(x->col(is).data(), p_nDim);  
19     double reg = sparseRegressor.getValue(xloc, regressedFunctionCoeff  
20 );  
21 }
```


2.8.2 Python API

Here is a simple example of the python API:

```
1 import StOptReg
2 nbSimul = 2000000;
3 np.random.seed(000)
4 x = np.random.uniform(-.,1.,size=(1,nbSimul));
5 # real function
6 toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
7 # function to regress
8 toRegress = toReal + 4*np.random.normal(0.,,nbSimul)
9 # level for sparse grid
10 iLevel = 5;
11 # weight for anisotropic sparse grids
12 weight= np.array([], dtype=np.int32)
13 # Regressor degree
14 regressor = StOptReg.SparseRegression(False,x,iLevel,weight, )
15 y = regressor.getAllSimulations(toRegress)
```

2.9 Global polynomial basis

2.9.1 Description of the method

In this section, the $\psi_k(X_t)$ involved in equation 2.1 are some given polynomials. Available polynomials are the canonical one, the Hermite and the Chebyshev ones.

- Hermite polynomials $H_m(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$ are orthogonal with respect to the weight $w(x) = e^{-\frac{x^2}{2}}$ and we get

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) dx = \delta_{mn} \sqrt{2\pi n!}$$

they satisfy the recurrence :

$$H_{n+1}(x) = xH_n(x) - H'_n(x)$$

assuming $H_n(x) = \sum_{k=0}^n a_{n,k} x^k$, we get the recurrence

$$a_{n+1,k} = a_{n,k-1} - n a_{n-1,k}, k > 0 \quad (2.5)$$

$$a_{n+1,0} = -n a_{n-1,0} \quad (2.6)$$

- Chebyshev polynomials are $T_{N+1}(x) = \cos((N+1)\arcs(x))$. They are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ and

$$\int_{-1}^1 T_N(x) T_M(x) w(x) dx = \begin{cases} 0, & \text{if } M \neq N \\ \pi, & \text{if } M = N = 0 \\ \frac{\pi}{2}, & \text{if } M = N \neq 0 \end{cases}$$

They satisfy the following recurrence :

$$T_{N+2}(x) = 2xT_{N+1}(x) - T_N(x)$$

2.9.2 C++ API

The “GlobalRegression” class is template by the type of the polynomial (“Canonical”, “Tchebychev” or “Hermite”) The first constructor :

```
1 GlobalRegression(const int & p_degree, const int & p_dim);
```

where *p_degree* is the total degree of the polynomial approximation, *p_dim* is the dimension of the problem.

A second constructor is provided:

```
1 GlobalRegression(const bool &p_bZeroDate,
2                 const std::shared_ptr< Eigen::ArrayXXd > &p_particles,
3                 const int & p_degree)
```

where

- *p_bZeroDate* is true if the regression date is 0,
- *p_particles* the particles X_t for all simulations (dimension of X_t for first dimension, number of Monte Carlo simulations in second dimension),
- *p_degree* is the total degree of the polynomial approximation.

Below we give a small example where “toRegress” corresponds to $g(t + h, X_{t+h})$ for all simulations and *x* store X_t for all simulations.

```
1 // total degree equal to 2
2 int degree=2;
3 // t is not zero
4 bool bZeroDate = 0;
5 // constructor with Hermite polynomials
6 GlobalRegression<Hermite> localRegressor (degree, x.rows());
7 // update particles values
8 localRegressor.updateSimulations(bZeroDate, x);
9 // regressed values
10 ArrayXd regressedValues = localRegressor.getAllSimulations(toRegress);
```

In the above example the Hermite regression can be replaced by the canonical one :

```
1 GlobalRegression<Canonical> localRegressor (degree, x.rows());
```

or by a Chebyshev one :

```
1 GlobalRegression<Tchebychev> localRegressor (degree, x.rows());
```

2.9.3 Python API

Here is a similar example using the second constructor

```
1 import StOptReg
2 nbSimul = 5000000;
3 np.random.seed(000)
4 x = np.random.uniform(-., 1., size=(1, nbSimul));
5 # real function
```

```

6     toReal = (2+x[0,:]+(+x[0,:])*(1+x[0,:]))
7     # function to regress
8     toRegress = toReal + 4*np.random.normal(0., , nbSimul)
9     # degree
10    degree =2
11    # Regressor
12    regressor = StOptReg.GlobalHermiteRegression(False ,x, degree)
13    y = regressor.getAllSimulations(toRegress).transpose()[0]

```

Available regressors are “GlobalHermiteRegression” as in the example above , “Global-CanonicalRegression” and “GlobalTchebychevRegression” with an obvious correspondence.

Chapter 3

Continuation values objects and similar ones

In a first part we describe a way to store and use continuation values calculated during the use of regression methods to estimate conditional expectations. In a second part, we introduce an object used to interpolate a function both discretized on grids for its deterministic part and estimated by regressor for its stochastic part. The second object is similar to the first in spirit but being dedicated to interpolation is more effective to use in simulations realized after the optimization part of a problem.

3.1 Continuation values object

A special case is the case where the state $X^{x,t}$ in equation (1) can be separated into two parts $X^{x,t} = (X_1^{x,t}, X_2^{x,t})$ where

1. the first part is given by the following equation

$$dX_1^{x,t} = b(t, X_s^{x,t})ds + \sigma(s, X_s^{x,t})dW_s$$

and is not controlled: the stochastic process is exogenous,

2. the second part is given by the following equation

$$dX_2^{x,t} = b_a(t)ds$$

such that the $X_2^{x,t}$ is a degenerated version of 1 without diffusion, a representing the control.

This first case is for example encountered while valuing American options in finance. In this case, $X_1^{x,t}$ holds the values of the stocks involved in the option and $X_2^{x,t}$ is for example an integer valued process equal to one if the option is not exercised and 0 if it has already been exercised.

Another classical case happening while dealing with stocks for example is a Gas Storage valuation. In this simple case, the process $X_1^{x,t}$ is the value of the gas on the market and $X_2^{x,t}$ is the position (in volume) in the gas storage. The library offers to store the conditional expectation for all the states $X_2^{x,t}$.

- $X_2^{x,t}$ will be stored on a grid of points (see section 1)
- for each point i of the grid the conditional expectation of a function $g_i(X_2^{x,t})$ associated to the point i using a regressor (see section 1) can be calculated and stored such that the continuation value C is a function of $(X_1^{x,t}, X_2^{x,t})$.

3.1.1 C++ API

As for regressions two constructors are provided

- The first one is the default construction: it is used in simulation algorithm with the “loadForSimulation” method to store the basis coefficients α_k^i for the grid point i (see equation (2.1)),
- The second one

```

1 ContinuationValue(const shared_ptr< SpaceGrid > & p_grid ,
2                  const shared_ptr< BaseRegression > & p_condExp ,
3                  const Eigen::ArrayXXd &p_cash )

```

with

- p_grid the grids associated to the control deterministic space,
- $p_condExp$ the conditional expectation operator
- p_cash the function to regress depending on the grid position (first dimension the number of simulations, second dimension the grid size)

This constructor constructs for all point i all the α_k^i (see equation (2.1)).

The main methods provided are:

- a first method used in simulation permitting to load for grid point i the coefficient α_k^i associated to the function g_i ,

```

1 void loadForSimulation(const shared_ptr< SpaceGrid > & p_grid ,
2                       const shared_ptr< BaseRegression > &
3                       p_condExp ,
4                       const Eigen::ArrayXXd &p_values )

```

with

- p_grid the grid associated to the controlled deterministic space,
 - $p_condExp$ the conditional expectation operator,
 - p_values the α_k^i for all grid points i (size the number of function basis, the number of grid points)
- a second method taking as input a point to be interpolated in the grid and returning the conditional expectation at the interpolated point for all simulations:

```
1 Eigen::ArrayXd getAllSimulations(const Eigen::ArrayXd &p_ptOfStock)
```

- a method taking as input an interpolator in the grid and returning the conditional expectation for all simulations at the interpolated point used to construct the interpolator :

```
1 Eigen::ArrayXd getAllSimulations(const Interpolator &p_interpol)
```

- a method taking as input a simulation number used in optimization and a point used to interpolate in the grid and returning the conditional expectation at the interpolated point for the given simulation used in optimization.

```
1 double getASimulation(const int &p_isim, const Eigen::ArrayXd &
    p_ptOfStock)
```

- a method taking as input a simulation number used in optimization and an interpolator in the grid and returning the conditional expectation at the interpolated point used to construct the interpolator for the given simulation used in optimization :

```
1 double getASimulation(const int &p_isim, const Interpolator &
    p_interpol)
```

- a method that permits to calculate the conditional expectation for a sample of $X_1^{x,t}$:

```
1 double getValue(const Eigen::ArrayXd &p_ptOfStock, const Eigen::
    ArrayXd &p_coordinates) const
```

where:

- *p_ptOfStock* the point where we interpolate the conditional expectation (a realization of $X_2^{x,t}$)
- *p_coordinates* the sample of $X_1^{x,t}$ used to estimate the conditional expectation
- and the function returns $C(X_1^{x,t}, X_2^{x,t})$.

Below we regress an identical function for all grid points (here a grid of 4 points in dimension 1):

```
1 int sizeForStock = 4;
2 // second member to regress with one stock
3 ArrayXXd toRegress = ArrayXXd::Constant(p_nbSimul, sizeForStock, 1.);
4 // grid for stock
5 Eigen::ArrayXd lowValues(1), step(1);
6 lowValues(0) = 0. ;
7 step(0) = 1;
8 Eigen::ArrayXi nbStep(1);
9 nbStep(0) = sizeForStock - 1;
10 // grid
11 shared_ptr< RegularSpaceGrid > regular = MyMakeShared<
    RegularSpaceGrid>(lowValues, step, nbStep);
12 // conditional expectation (local basis functions)
```

```

13 ArrayXi nbMesh = ArrayXi::Constant(p_nDim, p_nbMesh);
14 shared_ptr<LocalLinearRegression> localRegressor = MyMakeShared<
    LocalLinearRegression>(false, x, nbMesh);
15
16 // creation continuation value object
17 ContinuationValue continuation(regular, localRegressor, toRegress);
18
19 // regress with continuation value object
20 ArrayXd ptStock(1);
21 ptStock(0) = sizeForStock / 2; // point where we regress
22 // calculation the regression values for the current point for all
    the simulations
23 ArrayXd regressedByContinuation = continuation.getAllSimulations(
    ptStock);

```

3.1.2 Python API

Here is an example of the use of the mapping

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
    LGPL)
4 import numpy as np
5 import unittest
6 import random
7 import math
8 import StOptGrids
9 import StOptReg
10
11
12 # unit test for continuation values
13 #####
14
15 class testContValues(unittest.TestCase):
16
17     # test a regular grid for stocks and a local function basis for
    regression
18     def testSimpleGridsAndRegressor(self):
19         # low value for the meshes
20         lowValues = np.array([1., 2., 3.], dtype=np.float)
21         # size of the meshes
22         step = np.array([0.7, 2.3, 1.9], dtype=np.float)
23         # number of steps
24         nbStep = np.array([3, 2, 4], dtype=np.int32)
25         # create the regular grid
26         #####
27         grid = StOptGrids.RegularSpaceGrid(lowValues, step, nbStep)
28         # simulation
29         nbSimul = 10000
30         np.random.seed(1000)
31         x = np.random.uniform(-1., 1., size=(1, nbSimul));
32         # mesh
33         nbMesh = np.array([16], dtype=np.int32)

```

```

34     # Create the regressor
35     #####
36     regressor = StOptReg.LocalLinearRegression(False, x, nbMesh)
37     # regressed values
38     toReal = (2+x[0,:]+(1+x[0,:])*(1+x[0,:]))
39     # function to regress
40     toRegress = toReal + 4*np.random.normal(0., 1, nbSimul)
41     # create a matrix (number of stock points by number of simulations)
42     toRegressMult = np.zeros(shape=(len(toRegress), grid.getNbPoints()))
43     for i in range(toRegressMult.shape[1]):
44         toRegressMult[:, i] = toRegress
45     # Now create the continuation object
46     #####
47     contOb = StOptReg.ContinuationValue(grid, regressor, toRegressMult)
48     # get back the regressed values at the point stock
49     ptStock= np.array([1.2, 3.1, 5.9], dtype=np.float)
50     regressValues = contOb.getAllSimulations(ptStock)
51
52
53
54 if __name__ == '__main__':
55     unittest.main()

```

3.2 The GridAndRegressedValue object

As explained above, when we want to interpolate a function discretized partly on a grid and by regression a specific object can be used. As for the continuation it has a “getValue” to estimate the function at a state with both a deterministic, and a stochastic part.

3.2.1 C++ API

The object has five constructors and we only described the two more commonly used :

- The first one

```

1   GridAndRegressedValue(const std::shared_ptr< SpaceGrid > &p_grid ,
2                         const std::shared_ptr< BaseRegression > &
3                         p-reg ,
4                         const Eigen::ArrayXXd &p_values)

```

with

- *p_grid* the grid associated to the control deterministic space,
- *p_reg* the regressor object
- *p_values* the functions at some points on the deterministic and stochastic grid.

- A second constructor only stores the grid and regressor :

```

1   GridAndRegressedValue(const std::shared_ptr< SpaceGrid > &p_grid ,
2                         const std::shared_ptr< BaseRegression > &
3                         p-reg)

```


The main methods are the following ones :

- the main method that permits to calculate the function $C(X_{1,s}^{x,t}, X_{2,s}^{x,t})$ value for a point $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$ where $X_{2,s}^{x,t}$ is on the grid and $X_{1,s}^{x,t}$ is the part treated by regression.

```
1 double getValue(const Eigen::ArrayXd &p_ptOfStock, const Eigen::
    ArrayXd &p_coordinates) const
```

where:

- *p_ptOfStock* $X_{2,s}^{x,t}$ part of $X_s^{x,t}$
- *p_coordinates* $X_{1,s}^{x,t}$ part of $X_s^{x,t}$.

- the method ‘getRegressedValues’ that permits to get all regression coefficients for all points of the grid. The array returned has a size (number of function basis, number of points on the grid)

```
1 Eigen::ArrayXXd getRegressedValues() const
```

- the method ‘setRegressedValues’ permits to store all the values regressed coefficients on a grid of a function of $X_s^{x,t} = (X_{1,s}^{x,t}, X_{2,s}^{x,t})$.

```
1 void setRegressedValues(const Eigen::ArrayXXd &p_regValues)
```

where *p_regValues* has a size (number of function basis, number of points on the grid).

3.2.2 Python API

The python API is similar to the one of the ContinuationValue object (voir section 3.1.2).

Part III

Solving optimization problems with dynamic programming methods

In the sequel, we suppose that we have developed a Simulator generating some Monte Carlo simulations at the different optimization dates. In order to use the different frameworks developed in the sequel we suppose that the Simulator is derived from the abstract class “ SimulatorDPBase.h”

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef SIMULATORDPBASE_H
5 #define SIMULATORDPBASE_H
6 #include <Eigen/Dense>
7
8 /* \file SimulatorDPBase.h
9  * \brief Abstract class for simulators for Dynamic Programming Programms
10 * \author Xavier Warin
11 */
12
13 namespace StOpt
14 {
15   /// \class SimulatorDPBase SimulatorDPBase.h
16   /// Abstract class for simulator used in dynamic programming
17   class SimulatorDPBase
18   {
19
20
21   public :
22     virtual ~SimulatorDPBase() {}
23     /// \brief get current markovian state : dimension of the problem for
24     the first dimension , second dimension the number of Monte Carlo
25     simulations
26     virtual Eigen::MatrixXd getParticles() const = 0;
27     /// \brief a step forward for simulations
28     virtual void stepForward() = 0;
29     /// \brief a step backward for simulations
30     virtual void stepBackward() = 0;
31     /// \brief a step forward for simulations
32     /// \return current particles (markovian state as assets for example) (
33     dimension of the problem times simulation number)
34     virtual Eigen::MatrixXd stepForwardAndGetParticles() = 0;
35     /// \brief a step backward for simulations
36     /// \return current particles (markovian state as assets for example) (
37     dimension of the problem times simulation number)
38     virtual Eigen::MatrixXd stepBackwardAndGetParticles() = 0;
39     /// \brief get back dimension of the regression
40     virtual int getDimension() const = 0;
41     /// \brief get the number of steps
42     virtual int getNbStep() const = 0;
43     /// \brief Get the current step size
44     virtual double getStep() const = 0;
45     /// \brief Get current time
46     virtual double getCurrentStep() const = 0 ;
47     /// \brief Number of Monte Carlo simulations
48     virtual int getNbSimul() const = 0;

```

```

45  /// \brief Permit to actualize for one time step (interest rate)
46  virtual double getActuStep() const = 0;
47  /// \brief Permits to actualize at the initial date (interest rate)
48  virtual double getActu() const = 0 ;
49
50 };
51 }
52 #endif /* SIMULATORDPBASE.H */

```

Supposing that the Simulator is a Black Scholes simulator for P assets, simulating M Monte Carlo simulations, at $N + 1$ dates t_0, \dots, t_N , the Markov state for particle j , date t_i , Monte Carlo simulation k and asset p is $X_{p,i}^k$ and we give below the meaning of the different methods of “ SimulatorDPBase.h”:

- the “getParticle” method gives at the current optimization/simulation date t_i the Markov states $X_{p,i}^k$ in a matrix A such that $A(p, k) = X_{p,i}^k$,
- the “stepForward” method is used while simulating the assets evolution in forward: a step forward is realized from t_i to t_{i+1} and Brownian motions used for the assets are updated at the new time step,
- the “stepBackward” method is used for simulation of the asset from the last date to time 0. This method is used during an asset optimization by Dynamic Programming,
- the “stepForwardAndGetParticles” method: second and first method in one call,
- the “stepBackwardAndGetParticles” method: third and first method in one call,
- the “getDimension” method returns the number of assets,
- the “getNbStep” method returns the number of step (N),
- the “getStep” method returns the time step $t_{i+1} - t_i$ at the current time t_i ,
- the “getNbSimul” method returns M .
- the “getActuStep” method return the actualization factor on one time step
- the “getActu” method returns an actualization factor at the “0” date.

Chapter 4

Using conditional expectation estimated by regressions to solve simple problems

In this chapter we give some examples to value an American option. This use of the conditional expectation operators can be extended to many stochastic problem using this previously developed objects.

4.1 The American option valuing by Longstaff Schwartz

Suppose in this example that the payoff of the American option is given by g and that the interest rate is 0. The value of the option is given by

$$P_t = \text{esssup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}(g(\tau, X_\tau) \mid \mathcal{F}_t) \quad \text{for } t \leq T \quad \mathbb{P} - \text{a.s.}, \quad (4.1)$$

where $\mathcal{T}_{[t,T]}$ denotes the set of stopping times with values in $[t, T]$.

We recall the classical Longstaff Schwartz algorithm 3 estimating the empirical conditional expectation using the regression estimation previously seen.

Initialization:

Set $\hat{\tau}_\kappa^{1,\pi,(j)} := T, j \leq N$

Backward induction:

for $i = \kappa - 1$ to 0 **do**

 set $\hat{\tau}_i^{1,\pi} := t_i \mathbf{1}_{A_i^1} + \hat{\tau}_{i+1}^{1,\pi} \mathbf{1}_{(A_i^1)^c}$ where $A_i^1 := \{g(t_i, X_{t_i}) \geq \hat{\mathbb{E}}[g(\hat{\tau}_{i+1}^{1,\pi}, X_{\hat{\tau}_{i+1}^{1,\pi}}) \mid \mathcal{F}_{t_i}]\}$.

end for

Price estimator at 0: $\hat{P}_0^{1,\pi} := \hat{\mathbb{E}}[g(\hat{\tau}_0^{1,\pi}, X_{\hat{\tau}_0^{1,\pi}})]$.

Algorithm 3: Algorithm with regression [optimal exercise time estimation]

4.1.1 American option with the C++ API

We value in the algorithm below an American option using a simulator *p_sim*, a regressor *p_regressor*, a payoff function *p_payoff*:

```
1  double step = p_sim.getStep(); // time step increment
2  // asset simulated under the neutral risk probability: get the trend of
   the first asset to get the interest rate
3  double expRate = exp(-step * p_sim.getMu()(0));
4  // Terminal pay off
5  VectorXd Cash(p_payOff(p_sim.getParticles()));
6  for (int iStep = 0; iStep < p_sim.getNbStep(); ++iStep)
7  {
8      shared_ptr<ArrayXXd> asset(new ArrayXXd(p_sim.
           stepBackwardAndGetParticles())); // asset = Markov state
9      VectorXd payOffLoc = p_payOff(*asset); // pay off
10     // update conditional expectation operator for current Markov state
11     p_regressor.updateSimulations(((iStep == (p_sim.getNbStep() - 1)) ?
           true : false), asset);
12     // conditional expectation
13     VectorXd condEspec = p_regressor.getAllSimulations(Cash) * expRate;
14     // arbitrage between pay off and cash delivered after
15     Cash = (condEspec.array() < payOffLoc.array()).select(payOffLoc, Cash
           * expRate);
16 }
17 return Cash.mean();
```

4.2 American option with the Python API

Using the python API the American resolution is given below :

```
1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
   LGPL)
4 import numpy as np
5 import math as maths
6
7 # american option by Longstaff-Schwartz
8 # p_sim Monte Carlo simulator
9 # p_payOff Option pay off
10 # p_regressor regressor object
11 def resolution(p_simulator, p_payOff, p_regressor) :
12
13     step = p_simulator.getStep()
14     # asset simulated under the neutral risk probability : get the trend of
       first asset to get interest rate
15     expRate = np.exp(-step * p_simulator.getMu()[0])
16     # Terminal
17     particle = p_simulator.getParticles()
18     Cash = p_payOff.operator(particle)
19
20     for iStep in range(0, p_simulator.getNbStep()):
```

```
21     asset = p_simulator.stepBackwardAndGetParticles()
22     payOffLoc = p_payOff.operator(asset)
23     isLastStep = False
24     if iStep == p_simulator.getNbStep() - 1 :
25         isLastStep = True
26
27     p_regressor.updateSimulations(isLastStep, asset)
28     # conditional expectation
29     condEspece = p_regressor.getAllSimulations(Cash).squeeze() * expRate
30     # arbitrage
31     Cash = np.where(condEspece < payOffLoc, payOffLoc, Cash * expRate)
32
33     return maths.fsum(Cash) / len(Cash)
```

Chapter 5

Using the general framework to manage stock problems

In this chapter the state is separated into three parts $X^{x,t} = (X_1^{x,t}, X_2^{x,t}, I_t)$. $(X_1^{x,t}, X_2^{x,t})$, which corresponds to the special case of chapter 3 where $X_1^{x,t}$ is not controlled and $X_2^{x,t}$ is controlled. Two cases can be tackled :

- the first case corresponds to the case where $X_2^{x,t}$ is deterministic (think of storage management),
- the second case corresponds to the case where $X_2^{x,t}$ is stochastic (think of portfolio optimization).

I_t takes some integers values and is here to describe some finite discrete regimes (to treat some switching problems). A general framework is available to solve this kind of problem. First, the second part $X_2^{x,t}$ is discretized on a grid as explained in chapter 3.

- Either a full grid is used for $X_2^{x,t}$ and two types of resolutions either sequential or parallel be can considered :
 - a resolution can be achieved sequentially or a parallelization with MPI on the calculations can be achieved (speed up but no size up). This approach can be used for problems in small dimension.
 - a resolution can be achieved with a parallelization by the MPI framework by spreading the work to be achieved on the grid points, and spread the data between processors (speed up and size up). We will denote this parallelization technique a “distribution” technique. This approach is necessary to tackle very big optimization problems where the global solution cannot be stored in the memory of a single processor.
- or the grid for $X_2^{x,t}$ is not full (so sparse) and only a parallelization by thread and MPI can be achieved on the calculations (speed up and no size up). With sparse grids, only the case $X_2^{x,t}$ deterministic is treated.

In the case of the MPI parallelization technique distributing task and data (full grids only), [19] and [20] are used. Suppose that the grid is the same at each time step (only here to ease the case), and that we have 4 processors (figure 5.1) then:

- at the last time step, the final values at each point for each simulation are computed (each processor computes the values for its own grid points),
- at the previous time step, from a grid point own by a processor, we are able to localize the grids points attained at the next time step by all the commands,
- on figure 5.1, we give the points owned by other processors that can be reached from points owned by processor 3,
- some MPI communications are achieved bringing back the data (values calculated at the previous treated time step) needed by processor 3 to be able to update the value calculated by dynamic programming at the current time for all the points owned by processor 3,
- all the communications between all processors are achieved together.

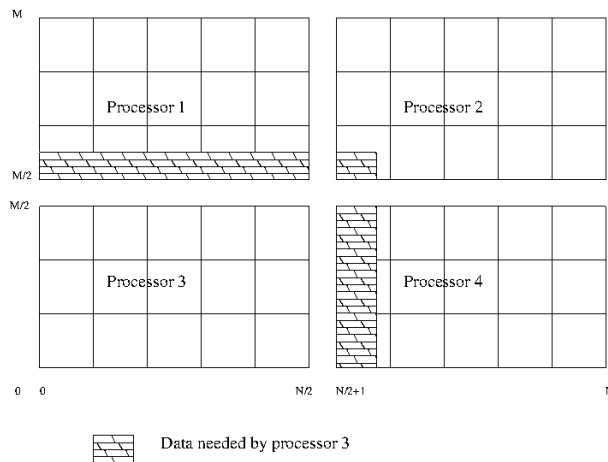


Figure 5.1: Data to send to processor 3

The global state of the the problem is store in the StateWithStocks object.

5.1 General requirement about business object

In order to use the framework, the developer has to describe the problem he wants to solve on one time step staring from a state $X^{x,t}$. This business object has to offer some common methods and it is derived from “OptimizerBase.h”

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef OPTIMIZERBASE_H
5 #define OPTIMIZERBASE_H
6 #include <Eigen/Dense>
7 #include "StOpt/core/utils/StateWithStocks.h"

```

```

8 #include "StOpt/core/grids/SpaceGrid.h"
9 #include "StOpt/regression/BaseRegression.h"
10 #include "StOpt/regression/ContinuationValue.h"
11 #include "StOpt/regression/GridAndRegressedValue.h"
12 #include "StOpt/dp/SimulatorDPBase.h"
13
14 /** \file OptimizerBase.h
15 * \brief Define an abstract class for Dynamic Programming problems solved
16 *       by Monte Carlo methods
17 *       \author Xavier Warin
18 */
19 namespace StOpt
20 {
21
22 /// \class OptimizerBase OptimizerBase.h
23 /// Base class for optimizer for Dynamic Programming with and without
24   regression methods
25 class OptimizerBase
26 {
27
28 public :
29
30     OptimizerBase() {}
31
32     virtual ~OptimizerBase() {}
33
34     /// \brief defines the dimension to split for MPI parallelism
35     /// For each dimension return true is the direction can be split
36     virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
37         const = 0 ;
38
39     /// \brief defines the diffusion cone for parallelism
40     /// \param p_regionByProcessor region (min max) treated by the
41     processor for the different regimes treated
42     /// \return returns in each dimension the min max values in the stock
43     that can be reached from the grid p_gridByProcessor for each regime
44     virtual std::vector< std::array< double, 2> > getCone(const std::vector<
45         std::array< double, 2> > &p_regionByProcessor) const = 0;
46
47     /// \brief Defines a step in simulation using interpolation in controls
48     /// \param p_grid grid at arrival step after command
49     /// \param p_control defines the controls
50     /// \param p_state defines the state value (modified)
51     /// \param p_phiInOut defines the value function (modified): size
52     number of functions to follow
53     virtual void stepSimulateControl(const std::shared_ptr< StOpt::SpaceGrid>
54         &p_grid, const std::vector< StOpt::GridAndRegressedValue > &
55         p_control,
56
57         StOpt::StateWithStocks &p_state,
58         Eigen::Ref<Eigen::ArrayXd> p_phiInOut)
59         const = 0 ;
60
61

```

```

52
53
54  /// \brief Get the number of regimes allowed for the asset to be reached
      at the current time step
55  virtual int getNbRegime() const = 0 ;
56
57  /// \brief get the simulator back
58  virtual std::shared_ptr< StOpt:: SimulatorDPBase > getSimulator() const =
      0;
59
60  /// \brief get back the dimension of the control
61  virtual int getNbControl() const = 0 ;
62
63  /// \brief get size of the function to follow in simulation
64  virtual int getSimuFuncSize() const = 0;
65
66 };
67 }
68 #endif /* OPTIMIZERBASE_H */

```

We detail all the methods that have to be implemented for all resolution methods (with or without regressions).

- the “getNbRegime” permits to get the number of regimes of the problem: for example, in switching problems, when there is a cost of switching, the working regime has to be incorporated in the state. Another example is the case of conditional delta to calculate for an asset: two regimes can be used: one to calculate the asset value and the second one to calculate the Δ . This number of regimes can be time dependent : in this case for a current resolution date t the “getNbRegime” method send the number of regimes at the very beginning of the time step (in t^-) such that a switch to a new regime can occurred in t^+ .
- the “getSimulator” method is used to get back the simulator giving the Monte Carlo simulations,
- the “getSimuFuncSize” method is used in simulation to define the number of functions to follow in the simulation part. For example in a stochastic target problem where the target is a given wealth with a given probability, one may want to follow the evolution of the probability at each time step and the wealth obtained while trading. In this case the “getSimuFuncSize” returns 2.
- the “getCone” method is only relevant if the MPI framework with distribution is used. As argument it take a vector of size the dimension of the grid. Each component of the vector is an array containing the minimal and maximal coordinates values of points in the current grid defining an hyper cube $H1$. It returns for each dimension, the coordinates min and max of the hyper cube $H2$ containing the points that can be reached by applying a command from a grid point in $H1$.
- the “getDimensionToSplit” method permits to define in the MPI framework with distribution which directions to split for solution distribution on processors. For each

dimension it returns a Boolean where “true” means that the direction is a candidate for splitting.

- the “stepSimulateControl” method is used after optimization using the optimal controls calculated in the optimization part. From a state p_state (storing the $X^{x,t}$), the optimal control calculated in optimization $p_control$, the optimal functions values along the current trajectory are stored in $p_phiInOut$. The state p_state is updated during at the end of the call function.

In a first part we present the framework for problems where conditional expectation is calculated by regression (case where $X_2^{t,x}$ is not controlled). Then we develop the framework not using regression for conditional expectation calculations. All conditional expectation are calculated using exogenous particles and interpolation. This will be typically the case for portfolio optimization.

5.2 Solving the problem using conditional expectation calculated by regressions

In this part we suppose that $X_2^{x,t}$ is controlled and deterministic so regression methods can be used.

5.2.1 Requirement to use the framework

In order to use the framework with regression for conditional expectation, a business object describing the business on one time step from one state is derived from “OptimizerDPBase.h” itself derived from “OptimizerBase.h” .

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef OPTIMIZERDPBASE_H
5 #define OPTIMIZERDPBASE_H
6 #include <Eigen/Dense>
7 #include "StOpt/core/Utils/StateWithStocks.h"
8 #include "StOpt/core/grids/SpaceGrid.h"
9 #include "StOpt/regression/BaseRegression.h"
10 #include "StOpt/regression/ContinuationValue.h"
11 #include "StOpt/regression/GridAndRegressedValue.h"
12 #include "StOpt/dp/SimulatorDPBase.h"
13 #include "StOpt/dp/OptimizerBase.h"
14
15 /** \file OptimizerDPBase.h
16 * \brief Define an abstract class for Dynamic Programming problems solved
  by regression methods
17 * \author Xavier Warin
18 */
19
20 namespace StOpt

```

```

21 {
22
23 /// \class OptimizerDPBase OptimizerDPBase.h
24 /// Base class for optimizer for Dynamic Programming with regression
    methods
25 class OptimizerDPBase : public OptimizerBase
26 {
27
28
29 public :
30
31     OptimizerDPBase() {}
32
33     virtual ~OptimizerDPBase() {}
34
35     /// \brief defines the diffusion cone for parallelism
36     /// \param p_regionByProcessor region (min max) treated by the
        processor for the different regimes treated
37     /// \return returns in each dimension the min max values in the stock
        that can be reached from the grid p_gridByProcessor for each regime
38     virtual std::vector< std::array< double, 2> > getCone(const std::vector<
        std::array< double, 2> > &p_regionByProcessor) const = 0;
39
40     /// \brief defines the dimension to split for MPI parallelism
41     /// For each dimension return true is the direction can be split
42     virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
        const = 0 ;
43
44     /// \brief defines a step in optimization
45     /// \param p_grid grid at arrival step after command
46     /// \param p_stock coordinates of the stock point to treat
47     /// \param p_condEsp continuation values for each regime
48     /// \param p_phiIn for each regime gives the solution calculated at
        the previous step ( next time step by Dynamic Programming resolution)
        : structure of the 2D array ( nb simulation ,nb stocks )
49     /// \return a pair :
50     /// - for each regimes (column) gives the solution for each
        particle (row)
51     /// - for each control (column) gives the optimal control
        for each particle (rows)
52     /// .
53     virtual std::pair< Eigen::ArrayXXd, Eigen::ArrayXXd> stepOptimize(const
        std::shared_ptr< StOpt::SpaceGrid> &p_grid, const Eigen::ArrayXd
        &p_stock,
54         const std::vector< StOpt::ContinuationValue > &p_condEsp,
55         const std::vector< std::shared_ptr< Eigen::ArrayXXd > > &p_phiIn
        ) const = 0;
56
57
58     /// \brief defines a step in simulation
59     /// Notice that this implementation is not optimal but is convenient if
        the control is discrete.
60     /// By avoiding interpolation in control we avoid non admissible control
61     /// Control are recalculated during simulation.

```

```

62  /// \param p_grid      grid at arrival step after command
63  /// \param p_continuation defines the continuation operator for each
    regime
64  /// \param p_state     defines the state value (modified)
65  /// \param p_phiInOut  defines the value functions (modified) : size
    number of functions to follow
66  virtual void stepSimulate(const std::shared_ptr< StOpt:: SpaceGrid> &
    p_grid , const std::vector< StOpt:: GridAndRegressedValue > &
    p_continuation ,
67                                StOpt:: StateWithStocks &p_state ,
68                                Eigen:: Ref<Eigen:: ArrayXd> p_phiInOut) const =
    0 ;
69
70
71  /// \brief Defines a step in simulation using interpolation in controls
72  /// \param p_grid      grid at arrival step after command
73  /// \param p_control   defines the controls
74  /// \param p_state     defines the state value (modified)
75  /// \param p_phiInOut  defines the value function (modified): size
    number of functions to follow
76  virtual void stepSimulateControl(const std::shared_ptr< StOpt:: SpaceGrid>
    &p_grid , const std::vector< StOpt:: GridAndRegressedValue > &
    p_control ,
77                                StOpt:: StateWithStocks &p_state ,
78                                Eigen:: Ref<Eigen:: ArrayXd> p_phiInOut)
    const = 0 ;
79
80
81
82  /// \brief Get the number of regimes allowed for the asset to be reached
    at the current time step
83  /// If \f$ t \f$ is the current time, and \f$ dt \f$ the resolution
    step, this is the number of regime allowed on \f$[ t- dt, t[\f$
84  virtual int getNbRegime() const = 0 ;
85
86  /// \brief get the simulator back
87  virtual std::shared_ptr< StOpt:: SimulatorDPBase > getSimulator() const =
    0;
88
89  /// \brief get back the dimension of the control
90  virtual int getNbControl() const = 0 ;
91
92  /// \brief get size of the function to follow in simulation
93  virtual int getSimuFuncSize() const = 0;
94
95 };
96 }
97 #endif /* OPTIMIZERDPBASE_H */

```

We detail the different methods to implement in addition to the methods of “Optimizer-Base.h”:

- the “stepOptimize” methods is used in optimization. We want to calculate the optimal value at current t_i at a grid point p_stock using a grid p_grid at the next date t_{i+1} ,

the continuation values for all regimes $p_condEsp$ permitting to calculate conditional expectation of the optimal value function calculated at the previously treated time step t_{i+1} . From a grid point p_stock it calculates the function values and the optimal controls. It returns a pair where the

- first element is a matrix (first dimension is the number of simulations, second dimension the number of regimes) giving the function value,
 - second element is a matrix (first dimension is the number of simulations, second dimension the number of controls) giving the optimal control.
- the “stepSimulate” method is used after optimization using the continuation values calculated in the optimization part. From a state p_state (storing the $X^{x,t}$), the continuation values calculated in optimization $p_continuation$, the optimal functions values along the current trajectory are stored in $p_phiInOut$.

In the case of a gas storage [21], the holder of the storage can inject gas from the network in the storage (paying the market price) or withdraw gas from the storage on the network (receiving the market price). In this case the Optimize object is given in this file. You can have a look at the implementation of the “getCone” method.

5.2.2 The framework in optimization

Once an Optimizer is derived for the project, and supposing that a full grid is used for the stock discretization, the framework provides a “TransitionStepRegressionDPDist” object in MPI that permits to solve the optimization problem with distribution of the data on one time step with the following constructor:

```

1  TransitionStepRegressionDPDist( const  shared_ptr<FullGrid> &
                                  p_pGridCurrent ,
2                                  const  shared_ptr<FullGrid> &
                                  p_pGridPrevious ,
3                                  const  shared_ptr<OptimizerDPBase > &
                                  p_pOptimize) :
```

with

- $p_pGridCurrent$ is the grid at the current time step (t_i),
- $p_pGridPrevious$ is the grid at the previously treated time step (t_{i+1}),
- $p_pOptimize$ the optimizer object

Remark 6 A similar object is available without the MPI distribution framework “TransitionStepRegressionDP” with still enabling parallelization with threads and MPI on the calculations on the full grid points.

Remark 7 In the case of sparse grids with only parallelization on the calculations (threads and MPI) “TransitionStepRegressionDPSparse” object can be used

The main method is

```

1     std::vector< shared_ptr< Eigen::ArrayXXd > > OneStep(const std::vector<
      shared_ptr< Eigen::ArrayXXd > > &p_phiIn ,
2         const shared_ptr< BaseRegression> &p_condExp)

```

with

- p_phiIn the vector (its size corresponds to the number of regimes) of matrix of optimal values calculated at the previous time iteration for each regime . Each matrix is a number of simulations by number of stock points matrix.
- $p_condExp$ the conditional expectation operator,

returning a pair :

- first element is a vector of matrix with new optimal values at the current time step (each element of the vector corresponds to a regime and each matrix is a number of simulations by number of stock points matrix).
- second element is a vector of matrix with new optimal controls at the current time step (each element of the vector corresponds to a control and each matrix is a number of simulations by number of stock points matrix).

Remark 8 All “*TransitionStepRegressionDP*” derive from a “*TransitionStepRegressionBase*” object having a pure virtual “*OneStep*” method.

A second method is provided permitting to dump the continuation values of the problem and the optimal control at each time step :

```

1     void dumpContinuationValues(std::shared_ptr<gs::BinaryFileArchive> p_ar ,
      const std::string &p_name, const int &p_iStep ,
2         const std::vector< std::shared_ptr< Eigen::
      ArrayXXd > > &p_phiInPrev ,
3         const std::vector< std::shared_ptr< Eigen::
      ArrayXXd > > &p_control ,
4         const std::shared_ptr<BaseRegression> &
      p_condExp ,
5         const bool &p_bOneFile) const

```

with :

- p_ar is the archive where controls and solutions are dumped,
- p_name is a base name used in the archive to store the solution and the control,
- $p_phiInPrev$ is the solution at the previous time step used to calculate the continuation values that are stored,
- $p_control$ stores the optimal controls calculated at the current time step,
- $p_condExp$ is the conditional expectation object permitting to calculate conditional expectation of functions defined at the previous time step treated $p_phiInPrev$ and permitting to store a representation of the optimal control.

- *p_bOneFile* is set to one if the continuation and optimal controls calculated by each processor are dumped on a single file. Otherwise the continuation and optimal controls calculated by each processor are dumped on different files (one by processor). If the problem gives continuation and optimal control values on the global grid that can be stored in the memory of the computation node, it can be more interesting to dump the continuation/control values in one file for the simulation of the optimal policy.

Remark 9 *As for the “TransitionStepRegressionDP” and the “TransitionStepRegressionDPSparse” object, their “dumpContinuationValues” doesn’t need a p_bOneFile argument: obviously optimal controls and solutions are stored in a single file.*

We give here a simple example of a time resolution using this method when the MPI distribution of data is used

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef USE_MPI
5 #include <fstream>
6 #include <memory>
7 #include <functional>
8 #include <boost/lexical_cast.hpp>
9 #include <boost/mpi.hpp>
10 #include <Eigen/Dense>
11 #include "geners/BinaryFileArchive.hh"
12 #include "StOpt/core/grids/FullGrid.h"
13 #include "StOpt/regression/BaseRegression.h"
14 #include "StOpt/dp/FinalStepRegressionDPDist.h"
15 #include "StOpt/dp/TransitionStepRegressionDPDist.h"
16 #include "StOpt/core/parallelism/reconstructProc0Mpi.h"
17 #include "StOpt/dp/OptimizerDPBase.h"
18 #include "StOpt/dp/SimulatorDPBase.h"
19
20
21 using namespace std;
22
23 double DynamicProgrammingByRegressionDist(const shared_ptr<StOpt::FullGrid>
  &p_grid ,
24     const shared_ptr<StOpt::OptimizerDPBase > &p_optimize ,
25     shared_ptr<StOpt::BaseRegression> &p_regressor ,
26     const function<double(const int &, const Eigen::ArrayXd &, const
  Eigen::ArrayXd &)> &p_funcFinalValue ,
27     const Eigen::ArrayXd &p_pointStock ,
28     const int &p_initialRegime ,
29     const string &p_fileToDump ,
30     const bool &p_bOneFile)
31 {
32     // from the optimizer get back the simulator
33     shared_ptr< StOpt::SimulatorDPBase> simulator = p_optimize->getSimulator
  ();
34     // final values

```

Table 5.1: Which “TransitionStepRegression” object to use depending on the grid used and the type of parallelization used.

	Full grid	Sparse grid
Sequential	“TransitionStepRegressionDP”	“TransitionStepRegressionDPSparse”
Parallelization on calculations threads and MPI	“TransitionStepRegressionDP”	“TransitionStepRegressionDPSparse”
Distribution of calculations and data	“TransitionStepRegressionDPDist”	Not available

```

35     vector< shared_ptr< Eigen::ArrayXXd > > valuesNext = StOpt::
        FinalStepRegressionDPDist(p_grid, p_optimize->getNbRegime(),
        p_optimize->getDimensionToSplit())(p_funcFinalValue, simulator->
        getParticles().array());
36     // dump
37     boost::mpi::communicator world;
38     string toDump = p_fileToDump;
39     // test if one file generated
40     if (!p_bOneFile)
41         toDump += "_" + boost::lexical_cast<string>(world.rank());
42     shared_ptr<gs::BinaryFileArchive> ar;
43     if ((!p_bOneFile) || (world.rank() == 0))
44         ar = make_shared<gs::BinaryFileArchive>(toDump.c_str(), "w");
45     // name for object in archive
46     string nameAr = "Continuation";
47     for (int iStep = 0; iStep < simulator->getNbStep(); ++iStep)
48     {
49         shared_ptr<Eigen::ArrayXXd> asset(new Eigen::ArrayXXd(simulator->
        stepBackwardAndGetParticles()));
50         // conditional expectation operator
51         p_regressor->updateSimulations(((iStep == (simulator->getNbStep() -
        1)) ? true : false), asset);
52         // transition object
53         StOpt::TransitionStepRegressionDPDist transStep(p_grid, p_grid,
        p_optimize);
54         pair< vector< shared_ptr< Eigen::ArrayXXd > >, vector< shared_ptr<
        Eigen::ArrayXXd > > > valuesAndControl = transStep.oneStep(
        valuesNext, p_regressor);
55         transStep.dumpContinuationValues(ar, nameAr, iStep, valuesNext,
        valuesAndControl.second, p_regressor, p_bOneFile);
56         valuesNext = valuesAndControl.first;
57     }
58     // reconstruct a small grid for interpolation
59     return StOpt::reconstructProc0Mpi(p_pointStock, p_grid, valuesNext[
        p_initialRegime], p_optimize->getDimensionToSplit());
60 }
61 }
62 #endif

```

An example without distribution of the data can be found in this file. We give at last a table with the different “TransitionStepRegression” objects to use depending on the type of parallelization used.

5.2.3 The framework in simulation

Once the optimization has been achieved, continuation values are dumped in one file (or some files) at each time step. In order to simulate the optimal policy, we can use the continuation values previously calculated (see chapter 3) or we can use the optimal controls calculated in optimization. In continuous optimization, using the control is more effective in term of computational cost. When the control is discrete, interpolation of the controls can lead to non admissible controls and simulation with the value function is more accurate.

While simulating the optimal control, two cases can occur :

- For most of the case (small dimensional case), the optimal control or the optimal function value can be stored in the memory of the computing node and function values and controls are stored in a single file. In this case a simulation of the optimal control can easily be achieved by distributing the Monte Carlo simulations on the available calculations nodes : this can be achieved by using the “SimulateStepRegression” or “SimulateStepRegressionControl” objects at each time step of the simulation.
- When dealing with very large problems, optimization is achieved by distributing the data on the processors and it is impossible to store the optimal command on one node. In this case, optimal controls and optimal solutions are stored in the memory of the node that has been used to calculate them in optimization. Simulations are reorganized at each time step and gathered so that they occupy the same part of the global grid. Each processor will then get from other processors a localized version of the optimal control or solution that it needs. This methodology is used in the “SimulateStepRegressionDist” and “SimulateStepRegressionControlDist” objects.

We detail the simulations objects using the optimal function value calculated in optimization and the optimal control for the case of very big problems.

- Simulation step using the value function calculated in optimization :

In order to simulate one step of the optimal policy, an object “SimulateStepRegressionDist” is provided with constructor

```

1 SimulateStepRegressionDist(gs:: BinaryFileArchive &p_ar ,  const int &
   p_iStep ,  const std::string &p_nameCont ,
2                                     const shared_ptr<FullGrid> &
   p_pGridFollowing ,  const shared_ptr<
3                                     OptimizerDPBase > &p_pOptimize ,
   const bool &p_bOneFile)

```

where

- *p_ar* is the binary archive where the continuation values are stored,
- *p_iStep* is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
- *p_nameCont* is the base name for continuation values,
- *p_GridFollowing* is the grid at the next time step (*p_iStep* + 1),

- *p_Optimize* the Optimizer describing the transition from one time step to the following one,
- *p_OneFile* equal to true if a single archive is used to store continuation values.

Remark 10 *A version without distribution of data but with multithreaded and with MPI possible on calculations is available with the object “SimulateStepRegression”. The *p_OneFile* argument is omitted during construction.*

This object implements the method “oneStep”

```
1 void oneStep(std::vector<StateWithStocks > &p_statevector , Eigen::
   ArrayXXd &p_phiInOut)
```

where:

- *p_statevector* store the states for the all the simulations: this state is updated by application of the optimal command,
- *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where *nb* is given by the “getSimuFuncSize” method of the optimizer and *nbSimul* the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
1 // Copyright (C) 2016 EDF
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3 // This code is published under the GNU Lesser General Public License (
   GNU LGPL)
4 #ifndef SIMULATEREGRESSIONDIST_H
5 #define SIMULATEREGRESSIONDIST_H
6 #include <functional>
7 #include <memory>
8 #include <Eigen/Dense>
9 #include <boost/mpi.hpp>
10 #include "geners/BinaryFileArchive.hh"
11 #include "StOpt/core/grids/FullGrid.h"
12 #include "StOpt/core/Utils/StateWithStocks.h"
13 #include "StOpt/dp/SimulateStepRegressionDist.h"
14 #include "StOpt/dp/OptimizerDPBase.h"
15 #include "StOpt/dp/SimulatorDPBase.h"
16
17
18 /** \file SimulateRegressionDist.h
19  * \brief Defines a simple program showing how to use simulation
20  *       A simple grid is used
21  * \author Xavier Warin
22  */
23
24
25 /// \brief Simulate the optimal strategy , mpi version
```

```

26  /// \param p_grid          grid used for deterministic state (
    stocks for example)
27  /// \param p_optimize     optimizer defining the optimization
    between two time steps
28  /// \param p_funcFinalValue function defining the final value
29  /// \param p_pointStock   initial point stock
30  /// \param p_initialRegime regime at initial date
31  /// \param p_fileToDump   name associated to dumped bellman
    values
32  /// \param p_bOneFile     do we store continuation values in
    only one file
33 double SimulateRegressionDist(const std::shared_ptr<StOpt::FullGrid> &
    p_grid ,
34                               const std::shared_ptr<StOpt::
    OptimizerDPBase > &p_optimize ,
35                               const std::function<double(const int &,
    const Eigen::ArrayXd &, const Eigen::
    ArrayXd &>> &p_funcFinalValue ,
36                               const Eigen::ArrayXd &p_pointStock ,
37                               const int &p_initialRegime ,
38                               const std::string &p_fileToDump ,
39                               const bool &p_bOneFile)
40 {
41     boost::mpi::communicator world;
42     // from the optimizer get back the simulator
43     std::shared_ptr< StOpt:: SimulatorDPBase> simulator = p_optimize->
    getSimulator();
44     int nbStep = simulator->getNbStep();
45     std::vector< StOpt:: StateWithStocks> states;
46     states.reserve(simulator->getNbSimul());
47     for (int is = 0; is < simulator->getNbSimul(); ++is)
48         states.push_back(StOpt:: StateWithStocks(p_initialRegime ,
    p_pointStock , Eigen::ArrayXd::Zero(simulator->getDimension())
    ));
49     std::string toDump = p_fileToDump ;
50     // test if one file generated
51     if (!p_bOneFile)
52         toDump += "_ " + boost::lexical_cast<std::string>(world.rank());
53     gs::BinaryFileArchive ar(toDump.c_str(), "r");
54     // name for continuation object in archive
55     std::string nameAr = "Continuation";
56     // cost function
57     Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
    getSimuFuncSize(), simulator->getNbSimul());
58     for (int istep = 0; istep < nbStep; ++istep)
59     {
60         StOpt:: SimulateStepRegressionDist(ar, nbStep - 1 - istep, nameAr,
    p_grid, p_optimize, p_bOneFile).oneStep(states, costFunction
    );
61
62         // new stochastic state
63         Eigen::ArrayXXd particles = simulator->
    stepForwardAndGetParticles();
64         for (int is = 0; is < simulator->getNbSimul(); ++is)

```

```

65         states[is].setStochasticRealization(particles.col(is));
66
67     }
68     // final : accept to exercise if not already done entirely (here
        suppose one function to follow)
69     for (int is = 0; is < simulator->getNbSimul(); ++is)
70         costFunction(0, is) += p_funcFinalValue(states[is].getRegime(),
            states[is].getPtStock(), states[is].getStochasticRealization
            ()) * simulator->getActu();
71
72     return costFunction.mean();
73 }
74
75 #endif /* SIMULATEREGRESSIONDIST_H */

```

The version of the previous example using a single archive storing the control/solution is given in this file.

- Simulation step using the optimal controls calculated in optimization :

```

1     SimulateStepRegressionControlDist(gs::BinaryFileArchive &p_ar, const
        int &p_iStep, const std::string &p_nameCont,
2                                     const std::shared_ptr<FullGrid> &
        p_pGridCurrent,
3                                     const std::shared_ptr<FullGrid> &
        p_pGridFollowing,
4                                     const std::shared_ptr<
        OptimizerDPBase > &p_pOptimize,
5                                     const bool &p_bOneFile);

```

where

- p_ar is the binary archive where the continuation values are stored,
- p_iStep is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
- $p_nameCont$ is the base name for control values,
- $p_GridCurrent$ is the grid at the current time step (p_iStep),
- $p_GridFollowing$ is the grid at the next time step ($p_iStep + 1$),
- $p_Optimize$ is the Optimizer describing the transition from one time step to the following one,
- $p_OneFile$ equals to true if a single archive is used to store continuation values.

Remark 11 *A version where a single archive storing the control/solution is used is available with the object “SimulateStepRegressionControl”*

This object implements the method “oneStep”

```

1 void oneStep(std::vector<StateWithStocks > &p_statevector , Eigen::
  ArrayXd &p_phiInOut)

```

where:

- *p_statevector* stores for all the simulations the state : this state is updated by application of the optimal commands,
- *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where *nb* is given by the “getSimuFuncSize” method of the optimizer and *nbSimul* the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (
  GNU LGPL)
4 #ifndef USE_MPI
5 #ifndef SIMULATEREGREGRESSIONCONTROLDIST_H
6 #define SIMULATEREGREGRESSIONCONTROLDIST_H
7 #include <functional>
8 #include <memory>
9 #include <Eigen/Dense>
10 #include <boost/mpi.hpp>
11 #include "generators/BinaryFileArchive.hh"
12 #include "StOpt/core/grids/FullGrid.h"
13 #include "StOpt/core/Utils/StateWithStocks.h"
14 #include "StOpt/dp/SimulateStepRegressionControlDist.h"
15 #include "StOpt/dp/OptimizerDPBase.h"
16 #include "StOpt/dp/SimulatorDPBase.h"
17
18
19 /** \file SimulateRegressionControlDist.h
20 * \brief Defines a simple program showing how to use simulation
21 * A simple grid is used
22 * \author Xavier Warin
23 */
24
25
26 /// \brief Simulate the optimal strategy using optimal controls
  calculated in optimization , mpi version
27 /// \param p_grid grid used for deterministic state (
  stocks for example)
28 /// \param p_optimize optimizer defining the optimization
  between two time steps
29 /// \param p_funcFinalValue function defining the final value
30 /// \param p_pointStock initial point stock
31 /// \param p_initialRegime regime at initial date
32 /// \param p_fileToDump name associated to dumped bellman
  values

```

```

33  /// \param p_bOneFile          do we store continuation values in
    only one file
34  double SimulateRegressionControlDist(const std::shared_ptr<StOpt::
    FullGrid> &p_grid ,
35                                          const std::shared_ptr<StOpt::
    OptimizerDPBase > &p_optimize ,
36                                          const std::function<double(const int
    &, const Eigen::ArrayXd &,
    const Eigen::ArrayXd &)> &
    p_funcFinalValue ,
37                                          const Eigen::ArrayXd &p_pointStock ,
38                                          const int &p_initialRegime ,
39                                          const std::string &p_fileToDump ,
40                                          const bool &p_bOneFile)
41  {
42      boost::mpi::communicator world;
43      // from the optimizer get back the simulator
44      std::shared_ptr< StOpt:: SimulatorDPBase> simulator = p_optimize->
    getSimulator();
45      int nbStep = simulator->getNbStep();
46      std::vector< StOpt:: StateWithStocks> states;
47      states.reserve(simulator->getNbSimul());
48      for (int is = 0; is < simulator->getNbSimul(); ++is)
49          states.push_back(StOpt:: StateWithStocks(p_initialRegime ,
    p_pointStock , Eigen::ArrayXd::Zero(simulator->getDimension())
    ));
50      std::string toDump = p_fileToDump ;
51      // test if one file generated
52      if (!p_bOneFile)
53          toDump += "_" + boost::lexical_cast<std::string>(world.rank());
54      gs::BinaryFileArchive ar(toDump.c_str(), "r");
55      // name for continuation object in archive
56      std::string nameAr = "Continuation";
57      // cost function
58      Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
    getSimuFuncSize(), simulator->getNbSimul());
59      for (int istep = 0; istep < nbStep; ++istep)
60      {
61          StOpt:: SimulateStepRegressionControlDist(ar, nbStep - 1 - istep ,
    nameAr, p_grid , p_grid , p_optimize , p_bOneFile).oneStep(
    states , costFunction);
62
63          // new stochastic state
64          Eigen::ArrayXXd particules = simulator->
    stepForwardAndGetParticles();
65          for (int is = 0; is < simulator->getNbSimul(); ++is)
66              states[is].setStochasticRealization(particules.col(is));
67      }
68      // final : accept to exercise if not already done entirely (here
    suppose one function to follow)
69      for (int is = 0; is < simulator->getNbSimul(); ++is)
70          costFunction(0, is) += p_funcFinalValue(states[is].getRegime() ,
    states[is].getPtStock(), states[is].getStochasticRealization
    ()) * simulator->getActu();

```



```

71
72     return costFunction.mean();
73 }
74
75 #endif /* SIMULATEREGRESSIONCONTROLDIST_H */
76 #endif

```

The version of the previous example using a single archive storing the control/solution is given in this file.

In the table below we indicate which simulation object should be used at each time step depending on the “TransitionStepRegressionD” object used in optimization.

5.3 Solving the problem for $X_2^{x,t}$ stochastic

In this part we suppose that $X_2^{x,t}$ is controlled but is stochastic.

5.3.1 Requirement to use the framework

In order to use the framework, a business object describing the business on one time step from one state is derived from “OptimizerNoRegressionDPBase.h” itself derived from “OptimizerBase.h” .

```

1 // Copyright (C) 2016 EDF
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3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef OPTIMIZERNOREGRESSIONDPBASE_H
5 #define OPTIMIZERNOREGRESSIONDPBASE_H
6 #include <Eigen/Dense>
7 #include "StOpt/core/Utils/StateWithStocks.h"
8 #include "StOpt/core/grids/SpaceGrid.h"
9 #include "StOpt/regression/BaseRegression.h"
10 #include "StOpt/regression/GridAndRegressedValue.h"
11 #include "StOpt/dp/SimulatorDPBase.h"
12 #include "StOpt/dp/OptimizerBase.h"
13
14 /** \file OptimizerNoRegressionDPBase.h
15 * \brief Define an abstract class for Dynamic Programming problems solve by
  Monte Carlo but without regression method
16 * to compute conditional expectation.
17 * \author Xavier Warin
18 */
19
20 namespace StOpt
21 {
22
23 /// \class OptimizerNoRegressionDPBase OptimizerNoRegressionDPBase.h
24 /// Base class for optimizer for Dynamic Programming solved without
  regression method to compute conditional expectation.
25 class OptimizerNoRegressionDPBase : public OptimizerBase

```

Table 5.2: Which simulation object to use depending on the TransitionStepRegression object used.

	"TransitionStepRegressionDP"	"TransitionStepRegressionDPDist"	"TransitionStepRegressionDPDist"	"TransitionStepRegressionDFSparsed"
"SimulateStepRegression"	Yes	"bOneFile" = True	No	Yes
"SimulateStepRegressionControl"	Yes	Yes	No	Yes
"SimulateStepRegressionDist"	No	Yes	Yes	No
"SimulateStepRegressionControlDist"	No	Yes	Yes	No

```

26 {
27
28
29 public :
30
31     OptimizerNoRegressionDPBase() {}
32
33     virtual ~OptimizerNoRegressionDPBase() {}
34
35     /// \brief defines the diffusion cone for parallelism
36     /// \param p_regionByProcessor region (min max) treated by the
37     /// processor for the different regimes treated
38     /// \return returns in each dimension the min max values in the stock
39     /// that can be reached from the grid p_gridByProcessor for each regime
40     virtual std::vector< std::array< double, 2> > getCone(const std::vector<
41     std::array< double, 2> > &p_regionByProcessor) const = 0;
42
43     /// \brief defines the dimension to split for MPI parallelism
44     /// For each dimension return true is the direction can be split
45     virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
46     const = 0 ;
47
48     /// \brief defines a step in optimization
49     /// \param p_stock coordinates of the stock point to treat
50     /// \param p_valNext Optimized values at next time step for each
51     /// regime
52     /// \param p_regressorCur Regressor at the current date
53     /// \return a pair :
54     /// - for each regimes (column) gives the solution for each
55     /// particle (row)
56     /// - for each control (column) gives the optimal control
57     /// for each particle (rows)
58     /// .
59     virtual std::pair< Eigen::ArrayXXd, Eigen::ArrayXXd> stepOptimize(const
60     Eigen::ArrayXd &p_stock ,
61     const std::vector< GridAndRegressedValue > &p_valNext ,
62     std::shared_ptr< BaseRegression > p_regressorCur) const = 0;
63
64     /// \brief Defines a step in simulation using interpolation in controls
65     /// \param p_grid grid at arrival step after command
66     /// \param p_control defines the controls
67     /// \param p_state defines the state value (modified)
68     /// \param p_phiInOut defines the value function (modified): size
69     /// number of functions to follow
70     virtual void stepSimulateControl(const std::shared_ptr< StOpt::SpaceGrid>
71     &p_grid , const std::vector< StOpt::GridAndRegressedValue > &
72     p_control ,
73     StOpt::StateWithStocks &p_state ,
74     Eigen::Ref<Eigen::ArrayXd> p_phiInOut)
75     const = 0 ;
76
77
78
79
80
81
82
83
84
85
86
87

```

```

68
69  /// \brief Get the number of regimes allowed for the asset to be reached
      at the current time step
70  ///   If \f$ t \f$ is the current time, and \f$ dt \f$ the resolution
      step, this is the number of regime allowed on \f$[ t- dt, t[\f$
71  virtual int getNbRegime() const = 0 ;
72
73  /// \brief get the simulator back
74  virtual std::shared_ptr< StOpt:: SimulatorDPBase > getSimulator() const =
      0;
75
76  /// \brief get back the dimension of the control
77  virtual int getNbControl() const = 0 ;
78
79  /// \brief get size of the function to follow in simulation
80  virtual int getSimuFuncSize() const = 0;
81
82 };
83 }
84 #endif /* OPTIMIZERDPBASE_H */

```

In addition to the methods of “OptimizerBase.h” the following method is needed :

- the “stepOptimize” methods is used in optimization. We want to calculate the optimal value regressed at current t_i at a grid point p_stock using a grid p_grid at the next date t_{i+1} ,

From a grid point p_stock it calculates the function values regressed and the optimal controls regressed. It returns a pair where the

- first element is a matrix (first dimension is the number of functions in the regression, second dimension the number of regimes) giving the function value regressed,
- second element is a matrix (first dimension is the number of functions in the regression, second dimension the number of controls) giving the optimal control regressed.

In this case of the optimization of an actualized portfolio with dynamic:

$$dX_2^{x,t} = X_2^{x,t} \frac{dX_1^{x,t}}{X_1^{x,t}}$$

where $X_1^{x,t}$ is the risky asset value, the Optimize object is given in this file.

5.3.2 The framework in optimization

Once an Optimizer is derived for the project, and supposing that a full grid is used for the stock discretization, the framework provides a “TransitionStepDPDist” object in MPI that permits to solve the optimization problem with distribution of the data on one time step with the following constructor:

```

1 TransitionStepDPDist( const shared_ptr<FullGrid> &p_pGridCurrent ,
2 const shared_ptr<FullGrid> &p_pGridPrevious ,
3 const std::shared_ptr<BaseRegression> &p_regressorCurrent ,
4 const std::shared_ptr<BaseRegression> &p_regressorPrevious ,
5 const shared_ptr<OptimizerNoRegressionDPBase > &p_pOptimize):

```

with

- *p_pGridCurrent* is the grid at the current time step (t_i),
- *p_pGridPrevious* is the grid at the previously treated time step (t_{i+1}),
- *p_regressorCurrent* is a regressor at the current date (to evaluate the function at the current date)
- *p_regressorPrevious* is a regressor at the previously treated time step (t_{i+1}) permitting to evaluate a function at date t_{i+1} ,
- *p_pOptimize* the optimizer object

Remark 12 *A similar object is available without the MPI distribution framework “TransitionStepDP” with still enabling parallelization with threads and MPI on the calculations on the full grid points.*

Remark 13 *The case of sparse grids is currently not treated in the framework.*

The main method is

```

1 std::pair< std::shared_ptr< std::vector< Eigen::ArrayXXd > > , std::
  shared_ptr< std::vector< Eigen::ArrayXXd > > > oneStep( const std::
  vector< Eigen::ArrayXXd > &p_phiIn )

```

with

- *p_phiIn* the vector (its size corresponds to the number of regimes) of matrix of optimal values calculated regressed at the previous time iteration for each regime . Each matrix is a number of function regressor at the previous date by number of stock points matrix.

returning a pair :

- first element is a vector of matrix with new optimal values regressed at the current time step (each element of the vector corresponds to a regime and each matrix is a number of regressed functions at the current date by the number of stock points matrix).
- second element is a vector of matrix with new optimal regressed controls at the current time step (each element of the vector corresponds to a control and each matrix is a number of regressed controls by the number of stock points matrix).

Remark 14 *All “TransitionStepDP” derive from a “TransitionStepBase” object having a pure virtual “OneStep” method.*

A second method is provided permitting to dump the the optimal control at each time step:

```

1 void dumpValues(std::shared_ptr<gs::BinaryFileArchive> p_ar ,
2               const std::string &p_name, const int &p_iStep ,
3               const std::vector< Eigen::ArrayXXd > &p_control , const
               bool &p_bOneFile) const

```

with :

- *p_ar* is the archive where controls and solutions are dumped,
- *p_name* is a base name used in the archive to store the solution and the control,
- *p_control* stores the optimal controls calculated at the current time step,
- *p_bOneFile* is set to one if the optimal controls calculated by each processor are dumped on a single file. Otherwise the optimal controls calculated by each processor are dumped on different files (one by processor). If the problem gives optimal control values on the global grid that can be stored in the memory of the computation node, it can be more interesting to dump the control values in one file for the simulation of the optimal policy.

Remark 15 *As for the “TransitionStepDP”, its “dumpValues” doesn’t need a p_bOneFile argument: obviously optimal controls are stored in a single file.*

We give here a simple example of a time resolution using this method when the MPI distribution of data is used

```

1 // Copyright (C) 2016 EDF
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  LGPL)
4 #ifndef USE_MPI
5 #include <fstream>
6 #include <boost/mpi.hpp>
7 #include <memory>
8 #include <functional>
9 #include <boost/lexical_cast.hpp>
10 #include <Eigen/Dense>
11 #include "gensers/BinaryFileArchive.hh"
12 #include "StOpt/core/grids/FullGrid.h"
13 #include "StOpt/regression/LocalConstRegression.h"
14 #include "StOpt/regression/GridAndRegressedValue.h"
15 #include "StOpt/dp/FinalStepRegressionDPDist.h"
16 #include "StOpt/dp/TransitionStepDPDist.h"
17 #include "StOpt/core/parallelism/reconstructProc0Mpi.h"
18 #include "test/c++/tools/dp/OptimizePortfolioDP.h"
19
20 using namespace std;
21 using namespace Eigen;
22
23 double DynamicProgrammingPortfolioDist(const shared_ptr<StOpt::FullGrid> &
  p_grid ,

```

```

24         const shared_ptr<OptimizePortfolioDP>
25             &p_optimize ,
26         const ArrayXi &p_nbMesh ,
27         const function<double(const int &,
28             const ArrayXd &, const ArrayXd &)>
29             &p_funcFinalValue ,
30         const ArrayXd &p_initialPortfolio ,
31         const string &p_fileToDump ,
32         const bool &p_bOneFile
33     )
34 {
35     // initialize simulation
36     p_optimize->initializeSimulation();
37     // store regressor
38     shared_ptr<StOpt::LocalConstRegression> regressorPrevious;
39
40     // store final regressed values in object valuesStored
41     shared_ptr<vector<ArrayXXd>> valuesStored = make_shared<vector<
42     ArrayXXd>>(p_optimize->getNbRegime());
43     {
44         vector<shared_ptr<ArrayXXd>> valuesPrevious = StOpt::
45             FinalStepRegressionDPDist(p_grid , p_optimize->getNbRegime() ,
46             p_optimize->getDimensionToSplit())(p_funcFinalValue , *p_optimize->
47             getCurrentSim());
48         // regressor operator
49         regressorPrevious = make_shared<StOpt::LocalConstRegression>(false ,
50             p_optimize->getCurrentSim() , p_nbMesh);
51         for (int iReg = 0; iReg < p_optimize->getNbRegime(); ++iReg)
52             (*valuesStored)[iReg] = regressorPrevious->
53             getCoordBasisFunctionMultiple(valuesPrevious[iReg]->transpose
54             ()).transpose();
55     }
56     boost::mpi::communicator world;
57     string toDump = p_fileToDump ;
58     // test if one file generated
59     if (!p_bOneFile)
60         toDump += "_" + boost::lexical_cast<string>(world.rank());
61     shared_ptr<gs::BinaryFileArchive> ar;
62     if ((!p_bOneFile) || (world.rank() == 0))
63         ar = make_shared<gs::BinaryFileArchive>(toDump.c_str() , "w");
64     // name for object in archive
65     string nameAr = "OptimizePort";
66     // iterate on time steps
67     for (int iStep = 0; iStep < p_optimize->getNbStep(); ++iStep)
68     {
69         // step backward for simulations
70         p_optimize->oneStepBackward();
71         // create regressor at the given date
72         bool bZeroDate = (iStep == p_optimize->getNbStep() - 1);
73         shared_ptr<StOpt::LocalConstRegression> regressorCur = make_shared<
74             StOpt::LocalConstRegression>(bZeroDate , p_optimize->getCurrentSim
75             (), p_nbMesh);
76         // transition object
77         StOpt::TransitionStepDPDist transStep(p_grid , p_grid , regressorCur ,

```

```

    regressorPrevious , p_optimize);
66 pair< shared_ptr< vector< ArrayXXd> >, shared_ptr< vector< ArrayXXd >
    >> valuesAndControl = transStep.oneStep(*valuesStored);
67 // dump control values
68 transStep.dumpValues(ar , nameAr , iStep , *valuesAndControl.second ,
    p_bOneFile);
69 valuesStored = valuesAndControl.first ;
70 // shift regressor
71 regressorPrevious = regressorCur;
72 }
73 // interpolate at the initial stock point and initial regime( 0 here) (
    take first particle)
74 shared_ptr<ArrayXXd> topRows = make_shared<ArrayXXd>((*valuesStored)[0].
    topRows(1));
75 return StOpt::reconstructProc0Mpi(p_initialPortfolio , p_grid , topRows ,
    p_optimize->getDimensionToSplit());
76 }
77 #endif

```

An example without distribution of the data can be found in this file.

5.3.3 The framework in simulation

Not special framework is available in simulation. Use the function “SimulateStepRegressionControl” or “SimulateStepRegressionControlDist” described in the section 5.2.3.

Chapter 6

The Python API

In order to use the Python API, it is possible to use only the mapping of the grids, continuation values, and regression object and to program an equivalent of “TransitionStepRegressionDP” and of “SimulateStepRegression”, “SimulateStepRegressionControl” in python. No mapping is currently available for “TransitionStepDP”. An example using python is given by

```
1 # Copyright (C) 2016 EDF
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3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import numpy as np
5 import StOptReg as reg
6 import StOptGrids
7 import StOptGlobal
8
9 class TransitionStepRegressionDP:
10
11     def __init__(self, p_pGridCurrent, p_pGridPrevious, p_pOptimize):
12
13         self.m_pGridCurrent = p_pGridCurrent
14         self.m_pGridPrevious = p_pGridPrevious
15         self.m_pOptimize = p_pOptimize
16
17     def oneStep(self, p_phiIn, p_condExp):
18
19         nbRegimes = self.m_pOptimize.getNbRegime()
20         phiOut = list(range(nbRegimes))
21         nbControl = self.m_pOptimize.getNbControl()
22         controlOut = list(range(nbControl))
23
24         # only if the processor is working
25         if self.m_pGridCurrent.getNbPoints() > 0:
26
27             # allocate for solution
28             for iReg in range(nbRegimes):
29                 phiOut[iReg] = np.zeros((p_condExp.getNbSimul(), self.
                    m_pGridCurrent.getNbPoints()))
30
31             for iCont in range(nbControl):
```

```

32         controlOut[iCont] = np.zeros((p_condExp.getNbSimul(), self.
33             m_pGridCurrent.getNbPoints()))
34
35         # number of threads
36         nbThreads = 1
37
38         contVal = []
39
40         for iReg in range(len(p_phiIn)):
41             contVal.append(reg.ContinuationValue(self.m_pGridPrevious,
42                 p_condExp, p_phiIn[iReg]))
43
44         # create iterator on current grid treated for processor
45         iterGridPoint = self.m_pGridCurrent.getGridIteratorInc(0)
46
47         # iterates on points of the grid
48         for iIter in range(self.m_pGridCurrent.getNbPoints()):
49
50             if iterGridPoint.isValid():
51                 pointCoord = iterGridPoint.getCoordinate()
52                 # optimize the current point and the set of regimes
53                 solutionAndControl = self.m_pOptimize.stepOptimize(self.
54                     m_pGridPrevious, pointCoord, contVal, p_phiIn)
55
56                 # copy solution
57                 for iReg in range(self.m_pOptimize.getNbRegime()):
58                     phiOut[iReg][:, iterGridPoint.getCount()] =
59                         solutionAndControl[0][:, iReg]
60
61                 for iCont in range(nbControl):
62                     controlOut[iCont][:, iterGridPoint.getCount()] =
63                         solutionAndControl[1][:, iCont]
64
65                 iterGridPoint.nextInc(nbThreads)
66
67         res = []
68         res.append(phiOut)
69         res.append(controlOut)
70         return res

```

This object can be used as in a time step optimization as follows

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
4 # LGPL)
5 import StOptReg
6 import TransitionStepRegressionDP as trans
7 import FinalStepRegressionDP as final
8
9 def DynamicProgrammingByRegression(p_grid, p_optimize, p_regressor,
10     p_funcFinalValue, p_pointStock, p_initialRegime, p_fileToDump, key="
    Continuation"):

```

```

11 # from the optimizer get back the simulation
12 simulator = p_optimize.getSimulator()
13 # final values
14 valuesNext = final.FinalStepRegressionDP(p_grid, p_optimize.getNbRegime()
    ).operator(p_funcFinalValue, simulator.getParticles())
15
16 archiveToWrite = StOptReg.BinaryFileArchiveStOpt(p_fileToDump, "w")
17 nsteps = simulator.getNbStep()
18 # iterate on time steps
19 for iStep in range(nsteps):
20     asset = simulator.stepBackwardAndGetParticles()
21
22     # conditional expectation operator
23     if iStep == (simulator.getNbStep() - 1):
24         p_regressor.updateSimulations(True, asset)
25     else:
26         p_regressor.updateSimulations(False, asset)
27
28     # transition object
29     transStep = trans.TransitionStepRegressionDP(p_grid, p_grid,
    p_optimize)
30     valuesAndControl = transStep.oneStep(valuesNext, p_regressor)
31     valuesNext = valuesAndControl[0]
32
33     # Dump the continuation values in the archive:
34     archiveToWrite.dumpGridAndRegressedValue(key, nsteps - 1 - iStep,
    valuesNext, p_regressor, p_grid)
35
36 # interpolate at the initial stock point and initial regime
37 return (p_grid.createInterpolator(p_pointStock).applyVec(valuesNext [
    p_initialRegime])).mean()

```

Some examples are available in the test directory (for example for swing options).

Another approach more effective in term of computational cost consists in mapping the simulator object derived from the SimulatorDPBase object and optimizer object derived from the OptimizerDPBase object and to use the high level python mapping of and “Simulat-eStepRegression”. In the test part of the library some Black-Scholes simulator and some Mean reverting simulator for a future curve deformation are developed and some examples of the mapping are achieved in the BoostPythonSimulators.cpp file. Similarly the optimizer class for swings options, optimizer for a fictitious swing in dimension 2, optimizer for a gas storage, optimizer for a gas storage with switching cost are mapped to python in the BoostPythonOptimizers.cpp file.

In the example below we describe the use of this high level interface for the swing options with a Black Scholes simulator : we give in this example the mapping of the mostly used objects:

```

1 # Copyright (C) 2016 EDF
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3 # This code is published under the GNU Lesser General Public License (GNU
    LGPL)
4 import math
5 import numpy as np
6 import unittest

```

```

7 import StOptGrids
8 import StOptReg
9 import StOptGlobal
10 import Utils
11 import Simulators as sim
12 import Optimizers as opt
13
14 # unit test for global shape
15 #####
16
17 class OptimizerConstruction(unittest.TestCase):
18
19     def test(self):
20         try:
21             imp.find_module('mpi4py')
22             found = True
23         except:
24             print("Not parallel module found ")
25             found = False
26
27         if found :
28             from mpi4py import MPI
29             comm = MPI.COMM_WORLD
30             initialValues = np.zeros(1, dtype=np.float) + 1.
31             sigma = np.zeros(1.) + 0.2
32             mu = np.zeros(1.) + 0.05
33             corr = np.ones((1.,1.), dtype=np.float)
34             # number of step
35             nStep = 30
36             # exercise dates
37             dates = np.linspace(0., 1., nStep + 1)
38             T= dates[len(dates) - 1]
39             nbSimul = 10 # simulation number (optimization and simulation)
40             # simulator
41             #####
42             bsSim = sim.BlackScholesSimulator(initialValues , sigma , mu , corr ,
43                 T, len(dates) - 1, nbSimul, False)
44             strike = 1.
45             # Pay off
46             payOff= Utils.BasketCall(strike)
47             # optimizer
48             #####
49             N = 3 # number of exercise dates
50             swiOpt = opt.OptimizerSwingBlackScholes(payOff,N)
51             # link simulator to optimizer
52             swiOpt.setSimulator(bsSim)
53             # archive
54             #####
55             ar = StOptGlobal.BinaryFileArchive("Archive", "w")
56             # regressor
57             #####
58             nMesh = 1
59             regressor = StOptReg.LocalLinearRegression(nMesh)
60             # Grid

```

```

60 #####
61 # low value for the meshes
62 lowValues =np.array ([0.], dtype=np.float)
63 # size of the meshes
64 step = np.array ([1.], dtype=np.float)
65 # number of steps
66 nbStep = np.array ([N-1], dtype=np.int32)
67 grid = StOptGrids.RegularSpaceGrid(lowValues , step , nbStep)
68 # pay off function for swing
69 #####
70 payOffBasket = Utils.BasketCall(strike);
71 payoff = Utils.PayOffSwing(payOffBasket ,N)
72 dir(payoff)
73 print("payoff" , payoff.set(0,np.array ([0.5], dtype=np.float) , np.
      array ([1.], dtype=np.float)))
74 # final step
75 #####
76 asset =bsSim.getParticles()
77 fin = StOptGlobal.FinalStepRegressionDP(grid ,1)
78 values = fin.set( payoff , asset)
79 # transition time step
80 #####
81 # on step backward and get asset
82 asset = bsSim.stepBackwardAndGetParticles()
83 # update regressor
84 regressor.updateSimulations(0, asset)
85 transStep = StOptGlobal.TransitionStepRegressionDP(grid , grid ,
      swiOpt)
86 valuesNextAndControl=transStep.oneStep(values , regressor)
87 transStep.dumpContinuationValues(ar , "Continuation" ,1,
      valuesNextAndControl [0] , valuesNextAndControl [1] , regressor)
88 # simulate time step
89 #####
90 nbSimul= 10
91 vecOfStates =[] # state of each simulation
92 for i in np.arange(nbSimul):
93     # one regime, all with same stock level (dimension 2), same
      realization of simulation (dimension 3)
94     vecOfStates.append(StOptGlobal.StateWithStocks(1, np.array
      ([1.]) , np.zeros(1)))
95 arRead = StOptGlobal.BinaryFileArchive("Archive" ,"r")
96 simStep = StOptGlobal.SimulateStepRegression(arRead ,1 , "
      Continuation" , grid , swiOpt)
97 phi = np.zeros((1 , nbSimul))
98 NewState = VecOfStateNext = simStep.oneStep(vecOfStates , phi)
99 # print new state (different of C++)
100 print("New vector of state" , NewState [0])
101 for i in np.arange(len(NewState [0])):
102     print(" i" , i , " Stock " , NewState [0][i].getPtStock() , "
      Regime " , NewState [0][i].getRegime() , " Stochastic
      realization" , NewState [0][i].getStochasticRealization())
103 print("New cost function" , NewState [1])
104
105 if __name__ == '__main__':

```

Its declination in term of a time nest for optimization is given below (please notice that the “TransitionStepRegressionDP” object is the result of the mapping between python and c++ and given in the “StOptGlobal” module)

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved
3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import StOptGrids
5 import StOptReg
6 import StOptGlobal
7
8
9 def DynamicProgrammingByRegressionHighLevel(p_grid , p_optimize , p_regressor ,
  p_funcFinalValue , p_pointStock , p_initialRegime , p_fileToDump) :
10
11     # from the optimizer get back the simulation
12     simulator = p_optimize.getSimulator()
13     # final values
14     fin = StOptGlobal.FinalStepRegressionDP(p_grid , p_optimize.getNbRegime())
15     valuesNext = fin.set(p_funcFinalValue , simulator.getParticles())
16     ar = StOptGlobal.BinaryFileArchive(p_fileToDump , "w")
17     nameAr = "Continuation"
18     # iterate on time steps
19     for iStep in range(simulator.getNbStep()) :
20         asset = simulator.stepBackwardAndGetParticles()
21         # conditional expectation operator
22         if iStep == (simulator.getNbStep() - 1):
23             p_regressor.updateSimulations(True , asset)
24         else:
25             p_regressor.updateSimulations(False , asset)
26
27     # transition object
28     transStep = StOptGlobal.TransitionStepRegressionDP(p_grid , p_grid ,
  p_optimize)
29     valuesAndControl = transStep.oneStep(valuesNext , p_regressor)
30     transStep.dumpContinuationValues(ar , nameAr , iStep , valuesNext ,
  valuesAndControl[1] , p_regressor)
31     valuesNext = valuesAndControl[0]
32
33     # interpolate at the initial stock point and initial regime
34     return (p_grid.createInterpolator(p_pointStock).applyVec(valuesNext [
  p_initialRegime])).mean()

```

Similarly a python time nest in simulation using the control previously calculated in optimization can be given as an example by :

```

1 # Copyright (C) 2016 EDF
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3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import numpy as np
5 import StOptReg as reg

```

```

6 import StOptGrids
7 import StOptGlobal
8
9
10 # Simulate the optimal strategy , threaded version
11 # p_grid                grid used for deterministic state (stocks for
    example)
12 # p_optimize            optimizer defining the optimization between two
    time steps
13 # p_funcFinalValue     function defining the final value
14 # p_pointStock         initial point stock
15 # p_initialRegime     regime at initial date
16 # p_fileToDump         name of the file used to dump continuation values
    in optimization
17 def SimulateRegressionControl(p_grid , p_optimize , p_funcFinalValue ,
    p_pointStock , p_initialRegime , p_fileToDump) :
18
19     simulator = p_optimize.getSimulator()
20     nbStep = simulator.getNbStep()
21     states = []
22
23     for i in range(simulator.getNbSimul()) :
24         states.append(StOptGlobal.StateWithStocks(p_initialRegime ,
            p_pointStock , np.zeros(simulator.getDimension())))
25
26     ar = StOptGlobal.BinaryFileArchive(p_fileToDump , "r")
27     # name for continuation object in archive
28     nameAr = "Continuation"
29     # cost function
30     costFunction = np.zeros((p_optimize.getSimuFuncSize() , simulator.
        getNbSimul()))
31
32     # iterate on time steps
33     for istep in range(nbStep) :
34         NewState = StOptGlobal.SimulateStepRegressionControl(ar , nbStep - 1 -
            istep , nameAr , p_grid , p_optimize).oneStep(states , costFunction)
35         # different from C++
36         states = NewState[0]
37         costFunction = NewState[1]
38         # new stochastic state
39         particles = simulator.stepForwardAndGetParticles()
40
41         for i in range(simulator.getNbSimul()) :
42             states[i].setStochasticRealization(particles[:,i])
43
44         # final : accept to exercise if not already done entirely
45         for i in range(simulator.getNbSimul()) :
46             costFunction[0,i] += p_funcFinalValue.set(states[i].getRegime() ,
                states[i].getPtStock() , states[i].getStochasticRealization()) *
                simulator.getActu()
47
48     # average gain/cost
49     return costFunction.mean()

```

Equivalent using MPI and the distribution of calculations and data can be used using the

“mpi4py” package. An example of its use can be found in the MPI version of a swing optimization and valorization.

Part IV

Semi Lagrangian methods

For the Semi Lagrangian methods the C++ API is the only one available (no python API is currently developed).

Chapter 7

Theoretical background

In this part, we are back to the resolution of equation (1).

7.1 Notation and regularity results

We denote by \wedge the minimum and \vee the maximum. We denote by $|\cdot|$ the Euclidean norm of a vector, $Q := (0, T] \times \mathbb{R}^d$. For a bounded function w , we set

$$|w|_0 = \sup_{(t,x) \in Q} |w(t,x)|, \quad [w]_1 = \sup_{(s,x) \neq (t,y)} \frac{|w(s,x) - w(t,y)|}{|x-y| + |t-s|^{\frac{1}{2}}}$$

and $|w|_1 = |w|_0 + [w]_1$. $C_1(Q)$ will stand for the space of functions with a finite $|\cdot|_1$ norm. For t given, we denote

$$\|w(t, \cdot)\|_\infty = \sup_{x \in \mathbb{R}^d} |w(t,x)|$$

We use the classical assumption on the data of (1) for a given \hat{K} :

$$\sup_a |g|_1 + |\sigma_a|_1 + |b_a|_1 + |f_a|_1 + |c_a|_1 \leq \hat{K} \quad (7.1)$$

A classical result [2] gives us the existence and uniqueness of the solution in the space of bounded Lipschitz functions:

Proposition 1 *If the coefficients of the equation (1) satisfy (7.1), there exists a unique viscosity solution of the equation (1) belonging to $C_1(Q)$. If u_1 and u_2 are respectively sub and super solution of equation (1) satisfying $u_1(0, \cdot) \leq u_2(0, \cdot)$ then $u_1 \leq u_2$.*

A spatial discretization length of the problem Δx being given, thereafter $(i_1 \Delta x, \dots, i_d \Delta x)$ with $\bar{i} = (i_1, \dots, i_d) \in \mathbf{Z}^d$ will correspond to the coordinates of a mesh $M_{\bar{i}}$ defining a hyper-cube in dimension d . For an interpolation grid $(\xi_i)_{i=0, \dots, N} \in [-1, 1]^N$, and for a mesh \bar{i} , the point $y_{\bar{i}, \tilde{j}}$ with $\tilde{j} = (j_1, \dots, j_d) \in [0, N]^d$ will have the coordinate $(\Delta x(i_1 + 0.5(1 + \xi_{j_1})), \dots, \Delta x(i_d + 0.5(1 + \xi_{j_d})))$. We denote $(y_{\bar{i}, \tilde{j}})_{\bar{i}, \tilde{j}}$ the set of all the grids points on the whole domain.

We notice that for regular mesh with constant volume Δx^d , we have the following relation for all $x \in \mathbb{R}^d$:

$$\min_{\bar{i}, \tilde{j}} |x - y_{\bar{i}, \tilde{j}}| \leq \Delta x. \quad (7.2)$$

7.2 Time discretization for HJB equation

The equation (1) is discretized in time by the scheme proposed by Camilli Falcone [22] for a time discretization h .

$$\begin{aligned} v_h(t+h, x) &= \inf_{a \in A} \left[\sum_{i=1}^q \frac{1}{2q} (v_h(t, \phi_{a,h,i}^+(t, x)) + v_h(t, \phi_{a,h,i}^-(t, x))) \right. \\ &\quad \left. + f_a(t, x)h + c_a(t, x)hv_h(t, x) \right] \\ &:= v_h(t, x) + \inf_{a \in A} L_{a,h}(v_h)(t, x) \end{aligned} \quad (7.3)$$

with

$$\begin{aligned} L_{a,h}(v_h)(t, x) &= \sum_{i=1}^q \frac{1}{2q} (v_h(t, \phi_{a,h,i}^+(t, x)) + v_h(t, \phi_{a,h,i}^-(t, x)) - 2v_h(t, x)) \\ &\quad + hc_a(t, x)v_h(t, x) + hf_a(t, x) \\ \phi_{a,h,i}^+(t, x) &= x + b_a(t, x)h + (\sigma_a)_i(t, x)\sqrt{hq} \\ \phi_{a,h,i}^-(t, x) &= x + b_a(t, x)h - (\sigma_a)_i(t, x)\sqrt{hq} \end{aligned}$$

where $(\sigma_a)_i$ is the i -th column of σ_a . We note that it is also possible to choose other types of discretization in the same style as those defined in [23].

In order to define the solution at each date, a condition on the value chosen for v_h between 0 and h is required. We choose a time linear interpolation once the solution has been calculated at date h :

$$v_h(t, x) = (1 - \frac{t}{h})g(x) + \frac{t}{h}v_h(h, x), \forall t \in [0, h]. \quad (7.4)$$

We first recall the following result :

Proposition 2 *Under the condition on the coefficients given by equation (7.1), the solution v_h of equations (7.3) and (7.4) is uniquely defined and belongs to $C_1(Q)$. We check that if $h \leq (16 \sup_a \{|\sigma_a|_1^2 + |b_a|_1^2 + 1\} \wedge 2 \sup_a |c_a|_0)^{-1}$, there exists C such that*

$$|v - v_h|_0 \leq Ch^{\frac{1}{4}}. \quad (7.5)$$

Moreover, there exists C independent of h such that

$$|v_h|_0 \leq C, \quad (7.6)$$

$$|v_h(t, x) - v_h(t, y)| \leq C|x - y|, \forall (x, y) \in Q^2. \quad (7.7)$$

7.3 Space interpolation

The space resolution of equation (7.3) is achieved on a grid. The ϕ^+ and ϕ^- have to be computed by the use of an interpolator I such that:

$$\begin{aligned} v_h(t, \phi_{a,h,i}^+(t, x)) &\simeq I(v_h(t, \cdot))(\phi_{a,h,i}^+(t, x)), \\ v_h(t, \phi_{a,h,i}^-(t, x)) &\simeq I(v_h(t, \cdot))(\phi_{a,h,i}^-(t, x)). \end{aligned}$$

In order to easily prove the convergence of the scheme to the viscosity solution of the problem, the monotony of the scheme is generally required leading to some linear interpolator slowly converging. An adaptation to high order interpolator where the function is smooth can be achieved using Legendre grids and Sparse grids with some truncation (see [24], [25]).

Chapter 8

C++ API

In order to achieve the interpolation and calculate the semi Lagrangian value

$$\sum_{i=1}^q \frac{1}{2q} (v_h(t, \phi_{a,h,i}^+(t, x)) + v_h(t, \phi_{a,h,i}^-(t, x)))$$

a first object “SemiLagrangEspCond” is available:

```
1 // Copyright (C) 2016 EDF
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3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef SEMILAGRANGESPCOND_H
5 #define SEMILAGRANGESPCOND_H
6 #include <Eigen/Dense>
7 #include <map>
8 #include <array>
9 #include <vector>
10 #include "StOpt/core/utils/constant.h"
11 #include "StOpt/core/grids/InterpolatorSpectral.h"
12
13 /** \file SemiLagrangEspCond.h
14 * \brief Semi Lagrangian method for process \f$ d x_t = b dt + \sigma dW_t
  \f$
15 * where \f$ X_t, b \f$ with values in \f$ \{\mathbb{R}\}^n \f$, \f$ \sigma
  \f$ a \f$ \mathbf{R}^n
16 * \times \mathbf{R}^m \f$ matrix and \f$ W_t \f$ with values in \f$ \
  \mathbf{R}^m \f$
17 */
18
19 namespace StOpt
20 {
21
22 /// \class SemiLagrangEspCond SemiLagrangEspCond.h
23 /// calculate semi Lagrangian operator for previously defined process.
24 class SemiLagrangEspCond
25 {
26     ///\brief interpolator
27     std::shared_ptr<InterpolatorSpectral> m_interpolator;
28
```

```

29  /// \brief store extremal values for the grid (min, max coordinates in
    each dimension)
30  std::vector <std::array< double , 2> > m_extremalValues;
31
32  /// \brief Do we use modification of volatility to stay in the domain
33  bool m_bModifVol ;
34
35 public :
36
37  /// \brief Constructor
38  /// \param p_interpolator Interpolator storing the grid
39  /// \param p_extremalValues Extremal values of the grid
40  /// \param p_bModifVol do we modify volatility to stay in the
    domain
41  SemiLagrangEspCond(const std::shared_ptr<InterpolatorSpectral> &
    p_interpolator , const std::vector <std::array< double , 2> > &
    p_extremalValues , const bool &p_bModifVol);
42
43  /// \brief Calculate  $\frac{1}{2d} \sum_{i=1}^d \phi(x + b dt + \sigma_i \sqrt{dt}) + \phi(x + b dt - \sigma_i \sqrt{dt})$ 
    where  $\sigma_i$  is column  $i$  of  $\sigma$ 
44  /// \param p_x beginning point
45  /// \param p_b trend
46  /// \param p_sig volatility matrix
47  /// \param p_dt Time step size
48  /// \return (the value calculated, true) if point inside the domain,
    otherwise (0., false)
49
50  std::pair<double, bool> oneStep(const Eigen::ArrayXd &p_x, const Eigen
    ::ArrayXd &p_b, const Eigen::ArrayXXd &p_sig, const double &p_dt)
    const;
51
52
53 };
54 }
55 #endif

```

Its constructor uses the following arguments :

- a first one “p_interpolator” defines a “spectral” interpolator on a grid : this “spectral” interpolator is constructed from a grid and a function to interpolate (see section 1). In our case, it will be used to interpolate the solution from the previous time step,
- a second one “p_extremalValues” defines for each dimension the minimal and maximal coordinates of points belonging to the grid,
- a third one “p_bModifVol” if set to “true” permits to achieve a special treatment when points to interpolate are outside the grid : the volatility of the underlying process is modified (keeping the same mean and variance) trying to keep points inside the domain (see [24]).

This object has the method “oneStep” taking

- p_x the foot of the characterize (for each dimension),

- p_b the trend of the process (for each dimension),
- p_sig the matrix volatility of the process,

such that the interpolation is achieved for a time step h at points $p_x + p_bh \pm p_sig\sqrt{h}$. It returns a pair (a, b) where a contains the calculated value if the b value is true. When the interpolation is impossible to achieve, the b value is set to false.

In order to use the API, an object deriving from the “OptimizerSLBase.h” object has to be constructed. This object permits to define the PDE to solve (with its optimization problem if any).

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef OPTIMIZERSLBASE_H
5 #define OPTIMIZERSLBASE_H
6 #include <vector>
7 #include <Eigen/Dense>
8 #include "StOpt/core/grids/SpaceGrid.h"
9 #include "StOpt/core/grids/FullGrid.h"
10 #include "StOpt/core/grids/InterpolatorSpectral.h"
11 #include "StOpt/semilagrangien/SemiLagrangEspCond.h"
12
13 /** \file OptimizerSLBase.h
14 * \brief Define an abstract class for Dynamic Programming problems
15 * \author Xavier Warin
16 */
17
18 namespace StOpt
19 {
20
21 /// \class OptimizerSLBase OptimizerSLBase.h
22 /// Base class for optimizer for resolution by semi Lagrangian methods of
  HJB equations
23 class OptimizerSLBase
24 {
25
26
27 public :
28
29     OptimizerSLBase() {}
30
31     virtual ~OptimizerSLBase() {}
32
33
34     /// \brief define the diffusion cone for parallelism
35     /// \param p_regionByProcessor region (min max) treated by the
  processor for the different regimes treated
36     /// \return returns in each dimension the min max values in the stock
  that can be reached from the grid p_gridByProcessor for each regime
37     virtual std::vector< std::array< double, 2> > getCone(const std::vector<
  std::array< double, 2> > &p_regionByProcessor) const = 0;

```



```

38
39  /// \brief defines the dimension to split for MPI parallelism
40  ///      For each dimension return true is the direction can be split
41  virtual Eigen::Array< bool, Eigen::Dynamic, 1> getDimensionToSplit()
      const = 0 ;
42
43  /// \brief defines a step in optimization
44  /// \param p_point      coordinates of the point to treat
45  /// \param p_semiLag    semi Lagrangian operator for each regime for
      solution at the previous step
46  /// \param p_time      current date
47  /// \param p_phiInPt   value of the function at the previous time step at
      p_point for each regime
48  /// \return a pair :
49  ///      - first an array of the solution (for each regime)
50  ///      - second an array of the optimal controls ( for each control
      )
51  virtual std::pair< Eigen::ArrayXd, Eigen::ArrayXd> stepOptimize(const
      Eigen::ArrayXd &p_point ,
52      const std::vector< std::shared_ptr<SemiLagrangEspCond> > &
      p_semiLag ,
53      const double &p_time ,
54      const Eigen::ArrayXd &p_phiInPt) const = 0;
55
56
57  /// \brief defines a step in simulation
58  /// \param p_gridNext   grid at the next step
59  /// \param p_semiLag    semi Lagrangian operator at the current step
      in each regime
60  /// \param p_state      state array (can be modified)
61  /// \param p_iReg       regime number
62  /// \param p_gaussian   unitary Gaussian realization
63  /// \param p_phiInPt   value of the function at the next time step at
      p_point for each regime
64  /// \param p_phiInOut   defines the value functions (modified) to
      follow
65  virtual void stepSimulate(const SpaceGrid &p_gridNext ,
66      const std::vector< std::shared_ptr< StOpt::
      SemiLagrangEspCond> > &p_semiLag ,
67      Eigen::Ref<Eigen::ArrayXd> p_state ,   int &
      p_iReg ,
68      const Eigen::ArrayXd &p_gaussian ,
69      const Eigen::ArrayXd &p_phiInPt ,
70      Eigen::Ref<Eigen::ArrayXd> p_phiInOut) const =
      0 ;
71
72
73  /// \brief defines a step in simulation using the control calculated in
      optimization
74  /// \param p_gridNext   grid at the next step
75  /// \param p_controlInterp the optimal controls interpolator
76  /// \param p_state      state array (can be modified)
77  /// \param p_iReg       regime number
78  /// \param p_gaussian   unitary Gaussian realization

```

```

79  /// \param p_phiInOut      defines the value functions (modified) to
    follow
80  virtual void stepSimulateControl(const SpaceGrid    &p_gridNext ,
81                                const std::vector< std::shared_ptr<
    InterpolatorSpectral> > &
    p_controlInterp ,
82                                Eigen::Ref<Eigen::ArrayXd> p_state ,
    int &p_iReg ,
83                                const Eigen::ArrayXd &p_gaussian ,
84                                Eigen::Ref<Eigen::ArrayXd> p_phiInOut)
    const = 0 ;
85
86  /// \brief get number of regimes
87  virtual int getNbRegime() const = 0 ;
88
89  /// \brief get back the dimension of the control
90  virtual int getNbControl() const = 0 ;
91
92  /// \brief do we modify the volatility to stay in the domain
93  virtual bool getBModifVol() const = 0 ;
94
95  /// \brief get the number of Brownians involved in semi Lagrangian for
    simulation
96  virtual int getBrownianNumber() const = 0 ;
97
98  /// \brief get size of the function to follow in simulation
99  virtual int getSimuFuncSize() const = 0;
100
101  /// \brief Permit to deal with some boundary points that do not need
    boundary conditions
102  ///      Return false if all points on the boundary need some boundary
    conditions
103  /// \param p_point potentially on the boundary
104  virtual bool isNotNeedingBC(const Eigen::ArrayXd &p_point) const = 0;
105 };
106 }
107 #endif /* OPTIMIZERSLBASE_H */

```

The main methods associated to this object are :

- “stepOptimize” is use to calculate the solution of the PDE at one point.
 - It takes a point of the grid used *p_point*,
 - and apply the semi Lagrangian scheme *p_semiLag* at this point,
 - at a date given by *p_time*.

It returns a pair containing:

- the function value calculated at *p_point* for each regime,
 - the optimal control calculated at *p_point* for each control.
- “stepSimulate” is used when the PDE is associated to an optimization problem and we want to simulate an optimal policy using the function values calculated in the optimization part. The arguments are:

- *p_gridNext* defining the grid used at the following time step,
 - *p_semiLag* the semi Lagrangian operator constructed with an interpolator using the following time solution,
 - *p_state* the vector defining the current state for the current regime,
 - *p_iReg* the current regime number,
 - *p_gaussian* is the vector of gaussian random variables used to calculate the Brownian involved in the underlying process for the current simulation,
 - *p_phiInP* at the value of the function calculated in optimization at next time step for the given point,
 - *p_phiInOut* storing the cost functions : the size of the array is the number of functions to follow in simulation.
- “stepSimulateControl” is used when the PDE is associated to an optimization problem and we want to simulate an optimal policy using the optimal controls calculated in the optimization part. The arguments are:
 - *p_gridNext* defining the grid used at the following time step,
 - *p_controlInterp* a vector (for each control) of interpolators in controls
 - *p_state* the vector defining the current state for the current regime,
 - *p_iReg* the current regime number,
 - *p_gaussian* is the vector of gaussian random variables used to calculate the Brownian involved in the underlying process for the current simulation.
 - *p_phiInOut* storing the cost functions : the size of the array is the number of functions to follow in simulation.

On return the *p_state* vector is modified, the *p_iReg* is modified and the cost function *p_phiInOut* is modified for the current trajectory.

- the “getCone” method is only relevant if the distribution for data (so MPI) is used. As argument it take a vector of size the dimension of the grid. Each component of the vector is an array containing the minimal and maximal coordinates values of points of the current grid defining an hyper cube *H1* . It returns for each dimension, the coordinates min and max of the hyper cube *H2* containing the points that can be reached by applying a command from a grid point in *H1*. If no optimization is achieved, it returns the hyper cube *H2* containing the points reached by the semi Lagrangian scheme. For explanation of the parallel formalism see chapter 5.
- the “getDimensionToSplit” method is only relevant if the distribution for data (so MPI) is used. The method permits to define which directions to split for solution distribution on processors. For each dimension it returns a Boolean where “true” means that the direction is a candidate for splitting,
- the “isNotNeedingBC” permits to define for a point on the boundary of the grid if a boundary condition is needed (“True” is returned) or if no boundary is needed (return “false”).

And example of the derivation of such an optimizer for a simple stochastic target problem (described in paragraph 5.3.4 in [24]) is given below :

```

1 #include <iostream>
2 #include "StOpt/core/utils/constant.h"
3 #include "test/c++/tools/semilagrangien/OptimizeSLCase3.h"
4
5 using namespace StOpt;
6 using namespace Eigen ;
7 using namespace std ;
8
9 OptimizerSLCase3::OptimizerSLCase3(const double &p_mu, const double &p_sig ,
10     const double &p_dt, const double &p_alphaMax, const double &p_stepAlpha):
11     m_dt(p_dt), m_mu(p_mu), m_sig(p_sig), m_alphaMax(p_alphaMax), m_stepAlpha
12     (p_stepAlpha) {}
13
14 vector< array< double, 2> > OptimizerSLCase3::getCone(const vector< array<
15     double, 2> > &p_xInit) const
16 {
17     vector< array< double, 2> > xReached(1);
18     xReached[0][0] = p_xInit[0][0] - m_alphaMax * m_mu / m_sig * m_dt -
19         m_alphaMax * sqrt(m_dt);
20     xReached[0][1] = p_xInit[0][1] + m_alphaMax * sqrt(m_dt) ;
21     return xReached;
22 }
23
24 pair< ArrayXd, ArrayXd> OptimizerSLCase3::stepOptimize(const ArrayXd &
25     p_point,
26     const vector< shared_ptr<SemiLagrangEspCond> > &p_semiLag, const
27     double &, const Eigen::ArrayXd &) const
28 {
29     pair< ArrayXd, ArrayXd> solutionAndControl;
30     solutionAndControl.first.resize(1);
31     solutionAndControl.second.resize(1);
32     ArrayXd b(1);
33     ArrayXXd sig(1, 1) ;
34     double vMin = StOpt::infy;
35     for (int iA1 = 0; iA1 < m_alphaMax / m_stepAlpha; ++iA1)
36     {
37         double alpha = iA1 * m_stepAlpha;
38         b(0) = -alpha * m_mu / m_sig; // trend
39         sig(0) = alpha; // volatility with one Brownian
40         pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_point, b, sig,
41             m_dt); // test the control
42         if (lagrang.second)
43         {
44             if (lagrang.first < vMin)
45             {
46                 vMin = lagrang.first;
47                 solutionAndControl.second(0) = alpha;
48             }
49         }
50     }
51     solutionAndControl.first(0) = vMin;

```

```

46     return solutionAndControl;
47 }
48
49 void OptimizerSLCase3::stepSimulate(const StOpt::SpaceGrid &p_gridNext ,
50                                     const std::vector< shared_ptr< StOpt::
51                                         SemiLagrangEspCond >> &p_semiLag ,
52                                     Eigen::Ref<Eigen::ArrayXd> p_state , int
53                                         &,
54                                     const Eigen::ArrayXd &p_gaussian , const
55                                         Eigen::ArrayXd &,
56                                     Eigen::Ref<Eigen::ArrayXd>) const
57 {
58     double vMin = StOpt::infy;
59     double alphaOpt = -1;
60     ArrayXd b(1);
61     ArrayXXd sig(1, 1) ;
62     ArrayXd proba = p_state ;
63     // recalculate the optimal alpha
64     for (int iAl = 0; iAl < m_alphaMax / m_stepAlpha; ++iAl)
65     {
66         double alpha = iAl * m_stepAlpha;
67         b(0) = -alpha * m_mu / m_sig; // trend
68         sig(0) = alpha; // volatility with one Brownian
69         pair<double, bool> lagrang = p_semiLag[0]->oneStep(proba, b, sig ,
70             m_dt); // test the control
71         if (lagrang.second)
72         {
73             if (lagrang.first < vMin)
74             {
75                 vMin = lagrang.first;
76                 alphaOpt = alpha;
77             }
78         }
79     }
80     proba(0) += alphaOpt * p_gaussian(0) * sqrt(m_dt);
81     // truncate if necessary
82     p_gridNext.truncatePoint(proba);
83     p_state = proba ;
84 }
85
86 void OptimizerSLCase3::stepSimulateControl(const SpaceGrid &p_gridNext ,
87     const vector< shared_ptr< InterpolatorSpectral>> &p_controlInterp
88     ,
89     Eigen::Ref<Eigen::ArrayXd> p_state , int &,
90     const ArrayXd &p_gaussian ,
91     Eigen::Ref<Eigen::ArrayXd>) const
92 {
93     ArrayXd proba = p_state ;
94     double alphaOpt = p_controlInterp[0]->apply(p_state);
95     proba(0) += alphaOpt * p_gaussian(0) * sqrt(m_dt);
96     // truncate if necessary
97     p_gridNext.truncatePoint(proba);

```

```

95     p_state = proba ;
96 }

```

8.1 PDE resolution

Once the problem is described, a time recursion can be achieved using the “Transition-StepSemilagrang” object in a sequential resolution of the problem. This object permits to solve the problem on one time step.

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef TRANSITIONSTEPSEMLAGRANG_H
5 #define TRANSITIONSTEPSEMLAGRANG_H
6 #ifdef OMP
7 #include <omp.h>
8 #endif
9 #include <functional>
10 #include <memory>
11 #include <Eigen/Dense>
12 #include "geners/BinaryFileArchive.hh"
13 #include "StOpt/semilagrangien/TransitionStepSemilagrangBase.h"
14 #include "StOpt/core/grids/SpaceGrid.h"
15 #include "StOpt/core/grids/InterpolatorSpectral.h"
16 #include "StOpt/semilagrangien/OptimizerSLBase.h"
17
18 /** \file TransitionStepSemilagrang.h
19  * \brief Solve one step of explicit semi Lagrangian scheme
20  * \author Xavier Warin
21  */
22
23
24 namespace StOpt
25 {
26
27   /// \class TransitionStepSemilagrang TransitionStepSemilagrang.h
28   /// One step of semi Lagrangian scheme
29   class TransitionStepSemilagrang : public TransitionStepSemilagrangBase
30   {
31   private :
32
33     std::shared_ptr<SpaceGrid> m_gridCurrent ; ///< global grid at current
34     std::shared_ptr<SpaceGrid> m_gridPrevious ; ///< global grid at previous
35     std::shared_ptr<OptimizerSLBase > m_optimize ; ///< optimizer solving
36     the problem for one point and one step
37
38   public :
39     /// \brief Constructor

```

```

40 TransitionStepSemilagrang(const std::shared_ptr<SpaceGrid> &
    p_gridCurrent ,
41                          const std::shared_ptr<SpaceGrid> &
    p_gridPrevious ,
42                          const std::shared_ptr<OptimizerSLBase > &
    p_optimize);
43
44 /// \brief One time step for resolution
45 /// \param p_phiIn      for each regime the function value ( on the
    grid)
46 /// \param p_time      current date
47 /// \param p_boundaryFunc Function at the boundary to impose Dirichlet
    conditions (depending on regime and position)
48 /// \return           solution obtained after one step of dynamic programming
    and the optimal control
49 std::pair< std::vector< std::shared_ptr< Eigen::ArrayXd > >, std::vector<
    std::shared_ptr< Eigen::ArrayXd > > > oneStep(const std::vector<
    std::shared_ptr< Eigen::ArrayXd > > &p_phiIn , const double &p_time ,
    const std::function<double(const int &, const Eigen::ArrayXd &)> &
    p_boundaryFunc) const;
50
51 /// \brief Permits to dump continuation values on archive
52 /// \param p_ar         archive to dump in
53 /// \param p_name      name used for object
54 /// \param p_iStep     Step number or identifier for time step
55 /// \param p_phiIn     for each regime the function value
56 /// \param p_control   for each control, the optimal value
57 void dumpValues(std::shared_ptr<gs::BinaryFileArchive> p_ar , const std::
    string &p_name, const int &p_iStep , const std::vector< std::shared_ptr
    < Eigen::ArrayXd > > &p_phiIn ,
58               const std::vector< std::shared_ptr< Eigen::ArrayXd > > &
    p_control) const;
59 };
60 }
61 #endif /* TRANSITIONSTEPSEMLAGRANG_H */

```

It constructor takes the following arguments:

- *p_gridCurrent* a grid describing the meshes at the current date,
- *p_gridPrevious* a grid describing the meshes at the previously treated date,
- *p_optimize* an object derived from the “OptimizerSLBase” and describing the problem to solve at a given date and a given point of the current grid.

A first method “oneStep” take the following arguments :

- *p_phiIn* describes for each regime the solution previously calculated on the grid at the previous time,
- *p_time* is the current time step,
- *p_boundaryFunc* is a function giving the Dirichlet solution of the problem depending on the number of regimes and the position on the boundary.

Table 8.1: Which “TransitionStepSemilagrang” object to use depending on the grid used and the type of parallelization used.

	Full grid	Sparse grid
Sequential	“TransitionStepSemilagrang”	“TransitionStepSemilagrang”
Parallelization on calculations threads and MPI	“TransitionStepSemilagrang”	“TransitionStepSemilagrang”
Distribution of calculations and data (MPI)	“TransitionStepSemilagrangDist”	Not available

It gives back an estimation of the solution at the current date on the current grid for all the regimes and an estimation of the optimal control calculated for all the controls.

A last method “dumpValues” method permits to dump the solution calculated p_phiIn at the step $p_istep + 1$ and the optimal control at step p_istep in an archive p_ar .

A version using the distribution of the data and calculations can be found in the TransitionStepSemilagrangDist object. An example of a time recursion in sequential can be found in the semiLagrangianTime function and an example with distribution can be found in the semiLagrangianTimeDist function. In both functions developed in the test chapter the analytic solution of the problem is known and compared to the numerical estimation obtained with the semi Lagrangian method.

8.2 Simulation framework

Once the optimal controls and the value functions are calculated, one can simulate the optimal policy by using the function values (recalculating the optimal control for each simulation) or using directly the optimal controls calculated in optimization

- Calculate the optimal strategy in simulation

by using the function values calculated in optimization :

In order to simulate one step of the optimal policy, an object “SimulateStepSemilagrangDist” is provided with constructor

```

1   SimulateStepSemilagrangDist(gs:: BinaryFileArchive &p_ar ,   const int &
      p_iStep ,   const std::string &p_name ,
2                                     const std::shared_ptr<FullGrid> &
      p_gridNext ,   const std::shared_ptr<
      OptimizerSLBase > &p_pOptimize ,
3                                     const bool &p_bOneFile);
```

where

- p_ar is the binary archive where the continuation values are stored,
- p_iStep is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
- p_name is the base name to search in the archive,
- $p_GridNext$ is the grid at the next time step ($p_iStep + 1$),

- *p_Optimize* is the Optimizer describing the transition from one time step to the following one,
- *p_OneFile* equals to true if a single archive is used to store continuation values.

Remark 16 *A version without distribution of data but only multithreaded and parallelized with MPI on data is available with the object “SimulateStepSemilagrang”*

This object implements the method “oneStep”

```
1 void oneStep(const Eigen::ArrayXXd & p_gaussian, Eigen::ArrayXXd &
  p_statevector, Eigen::ArrayXi & p_iReg, Eigen::ArrayXd &
  p_phiInOuts)
```

where:

- *p_gaussian* is a two dimensional array (number of Brownian in the modelization by the number of Monte Carlo simulations).
- *p_statevector* store the continuous state (continuous state size by number of simulations)
- *p_iReg* for each simulation give the current regime number,
- *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where *nb* is given by the “getSimuFuncSize” method of the optimizer and *nbSimul* the number of Monte Carlo simulations.

Remark 17 *The previous object “SimulateStepSemilagrangDist” is used with MPI for problems of quite high dimension. In the case of small dimension (below or equal to three), the parallelization with MPI or the sequential calculations can be achieved by the “SimulateStepSemilagrang” object.*

An example of the use of this method to simulate an optimal policy with distribution is given below:

```
1 // Copyright (C) 2016 EDF
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3 // This code is published under the GNU Lesser General Public License (
  GNU LGPL)
4 #ifdef USE_MPI
5 #include <boost/random.hpp>
6 #include <memory>
7 #include <Eigen/Dense>
8 #include "geners/BinaryFileArchive.hh"
9 #include "StOpt/semilagrangien/OptimizerSLBase.h"
10 #include "StOpt/semilagrangien/SimulateStepSemilagrangDist.h"
11
12 using namespace std;
13
14 double semiLagrangianSimuDist(const shared_ptr<StOpt::FullGrid> &p_grid,
```

```

15         const shared_ptr<StOpt::OptimizerSLBase > &
16             p_optimize ,
17         const function<double(const int &, const
18             Eigen::ArrayXd &)> &p_funcFinalValue ,
19         const int &p_nbStep ,
20         const Eigen::ArrayXd &p_stateInit ,
21         const int &p_initialRegime ,
22         const int &p_nbSimul ,
23         const string &p_fileToDump ,
24         const bool &p_bOneFile)
25 {
26     boost::mpi::communicator world;
27     // store states in a regime
28     Eigen::ArrayXXd states(p_stateInit.size(), p_nbSimul);
29     for (int is = 0; is < p_nbSimul; ++is)
30         states.col(is) = p_stateInit;
31     // store the regime number
32     Eigen::ArrayXi regime = Eigen::ArrayXi::Constant(p_nbSimul ,
33         p_initialRegime);
34     // test if one file generated
35     string toDump = p_fileToDump ;
36     if (!p_bOneFile)
37         toDump += " " + boost::lexical_cast<string>(world.rank());
38     gs::BinaryFileArchive ar(toDump.c_str(), "r");
39     // name for continuation object in archive
40     string nameAr = "Continuation";
41     // cost function
42     Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
43         getSimuFuncSize(), p_nbSimul);
44     // random generator and Gaussian variables
45     boost::mt19937 generator;
46     boost::normal_distribution<double> normalDistrib;
47     boost::variate_generator<boost::mt19937 &, boost::normal_distribution
48         <double>> normalRand(generator, normalDistrib);
49     Eigen::ArrayXXd gaussian(p_optimize->getBrownianNumber(), p_nbSimul);
50     // iterate on time steps
51     for (int istep = 0; istep < p_nbStep; ++istep)
52     {
53         for (int is = 0; is < gaussian.cols(); ++is)
54             for (int id = 0; id < gaussian.rows(); ++id)
55                 gaussian(id, is) = normalRand();
56
57         StOpt::SimulateStepSemilagrangDist(ar, p_nbStep - 1 - istep ,
58             nameAr, p_grid, p_optimize, p_bOneFile).oneStep(gaussian ,
59             states, regime, costFunction);
60     }
61     // final cost to add
62     for (int is = 0; is < p_nbSimul; ++is)
63         costFunction(0, is) += p_funcFinalValue(regime(is), states.col(is)
64             ));
65     // average gain/cost
66     return costFunction.mean();
67 }
68 #endif

```

A sequential or parallelized on calculations version of the previous example is given in this file.

- Calculate the optimal strategy in simulation

by interpolation of the optimal control calculated in optimization :

In order to simulate one step of the optimal policy, an object “SimulateStepSemilagrangControlDist” is provided with constructor

```

1   SimulateStepSemilagrangControlDist ( gs :: BinaryFileArchive &p_ar ,
2       const int &p_iStep ,   const std::string &p_name ,
3                               const std::shared_ptr<FullGrid> &
4                               p_gridCur ,
5                               const std::shared_ptr<FullGrid> &
                               p_gridNext ,
                               const std::shared_ptr<
                                   OptimizerSLBase > &p_pOptimize
                               ,
                               const bool &p_bOneFile )

```

where

- *p_ar* is the binary archive where the continuation values are stored,
- *p_iStep* is the number associated to the current time step (0 at the beginning date of simulation, the number is increased by one at each time step simulated),
- *p_name* is the base name to search in the archive,
- *p_GridCur* is the grid at the current time step (*p_iStep*),
- *p_GridNext* is the grid at the next time step (*p_iStep* + 1),
- *p_Optimize* is the Optimizer describing the transition from one time step to the following one,
- *p_OneFile* equals to true if a single archive is used to store continuation values.

Remark 18 *The previous object “SimulateStepSemilagrangControlDist” is used with MPI distribution of data for problems of quite high dimension. In the case of small dimension (below or equal to three), the parallelization with MPI or the sequential calculations can be achieved by the “SimulateStepSemilagrangControl” object.*

This object implements the method “oneStep”

```

1 void oneStep((const Eigen::ArrayXXd & p_gaussian , Eigen::ArrayXXd &
   p_statevector , Eigen::ArrayXi &p_iReg , Eigen::ArrayXd &
   p_phiInOuts)

```

where:

- *p_gaussian* is a two dimensional array (number of Brownian in the modelization by the number of Monte Carlo simulations).

- *p_statevector* stores the continuous state (continuous state size by number of simulations)
- *p_iReg* for each simulation gives the current regime number,
- *p_phiInOut* stores the gain/cost functions for all the simulations: it is updated by the function call. The size of the array is $(nb, nbSimul)$ where *nb* is given by the “getSimuFuncSize” method of the optimizer and *nbSimul* the number of Monte Carlo simulations.

An example of the use of this method to simulate an optimal policy with distribution is given below:

```

1 // Copyright (C) 2016 EDF
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3 // This code is published under the GNU Lesser General Public License (
  GNU LGPL)
4 #ifdef USE_MPI
5 #include <memory>
6 #include <boost/random.hpp>
7 #include <Eigen/Dense>
8 #include "generators/BinaryFileArchive.hh"
9 #include "StOpt/semilagrangien/OptimizerSLBase.h"
10 #include "StOpt/semilagrangien/SimulateStepSemilagrangControlDist.h"
11
12 using namespace std;
13
14 double semiLagrangianSimuControlDist(const shared_ptr<StOpt::FullGrid> &
  p_grid ,
15                                     const shared_ptr<StOpt::
  OptimizerSLBase > &p_optimize ,
16                                     const function<double(const int &,
  const Eigen::ArrayXd &> &
  p_funcFinalValue ,
17                                     const int &p_nbStep ,
18                                     const Eigen::ArrayXd &p_stateInit ,
19                                     const int &p_initialRegime ,
20                                     const int &p_nbSimul ,
21                                     const string &p_fileToDump ,
22                                     const bool &p_bOneFile)
23 {
24     boost::mpi::communicator world;
25     // store states in a regime
26     Eigen::ArrayXXd states(p_stateInit.size(), p_nbSimul);
27     for (int is = 0; is < p_nbSimul; ++is)
28         states.col(is) = p_stateInit;
29     // store the regime number
30     Eigen::ArrayXi regime = Eigen::ArrayXi::Constant(p_nbSimul ,
  p_initialRegime);
31     // test if one file generated
32     string toDump = p_fileToDump ;
33     if (!p_bOneFile)
34         toDump += " " + boost::lexical_cast<string>(world.rank());
35     gs::BinaryFileArchive ar(toDump.c_str(), "r");
36     // name for continuation object in archive

```

```

37 string nameAr = "Continuation";
38 // cost function
39 Eigen::ArrayXXd costFunction = Eigen::ArrayXXd::Zero(p_optimize->
    getSimuFuncSize(), p_nbSimul);
40 // random generator and Gaussian variables
41 boost::mt19937 generator;
42 boost::normal_distribution<double> normalDistrib;
43 boost::variate_generator<boost::mt19937 &, boost::normal_distribution
    <double>> normalRand(generator, normalDistrib);
44 Eigen::ArrayXXd gaussian(p_optimize->getBrownianNumber(), p_nbSimul);
45 // iterate on time steps
46 for (int istep = 0; istep < p_nbStep; ++istep)
47 {
48     for (int is = 0; is < gaussian.cols(); ++is)
49         for (int id = 0; id < gaussian.rows(); ++id)
50             gaussian(id, is) = normalRand();
51
52     StOpt::SimulateStepSemilagrangControlDist(ar, p_nbStep - 1 -
        istep, nameAr, p_grid, p_grid, p_optimize, p_bOneFile).
        oneStep(gaussian, states, regime, costFunction);
53 }
54 // final cost to add
55 for (int is = 0; is < p_nbSimul; ++is)
56     costFunction(0, is) += p_funcFinalValue(regime(is), states.col(is)
        ));
57 // average gain/cost
58 return costFunction.mean();
59 }
60 #endif

```

The sequential (or parallelized on calculations) version of the previous example is given in this file

Remark 19 *In the previous example, we suppose that only one function is followed in simulation, and that we send back an average for this value function as a result.*

Table 8.2: Which simulation object to use depending on the TransitionStepSemilagrang object used.

	“TransitionStepSemilagrang”	“TransitionStepSemilagrangDist” “bOneFile” = True	“TransitionStepSemilagrangDist” “bOneFile” = False
“SimulateStepSemilagrang”	Yes	Yes	No
“SimulateStepSemilagrangControl”	Yes	Yes	No
“SimulateStepSemilagrangDist”	No	Yes	Yes
“SimulateStepSemilagrangControlDist”	No	Yes	Yes

Part V

An example with both dynamic programming with regression and PDE

In this chapter we give an example where both dynamic programming with regressions and PDE can be used. It permits to compare the resolution and the solution obtained by both methods. All information about the modelization can be obtained by [26].

In this example we take the following notations :

- D_t is a demand process (in electricity) with an Ornstein Uhlenbeck dynamic :

$$dD_t = \alpha(m - D_t)dt + \sigma dW_t,$$

- Q_t is the cumulative carbon emission due to electricity production to satisfy the demand,

$$dQ_t = (D_t - L_t)^+ dt,$$

- L_t the total investment capacity in non emissive technology to produce electricity

$$L_t = \int_0^t l_s ds$$

where l_s is an intensity of investment in non emissive technology at date s ,

- Y_t is the carbon price where

$$Y_t = \mathbb{E}_t(\lambda 1_{Q_T \geq H}),$$

with λ and H given.

We introduce the following functions :

- the electricity price function which is a function of demand and the global investment of non emissive technology.

$$p_t = (1 + D_t)^2 - L_t,$$

- the profit function by selling electricity is given by

$$\Pi(D_t, L_t) = p_t D_t - (D_t - L_t)^+,$$

- $\tilde{c}(l_t, L_t)$ is the investment cost for new capacities of non emissive technology.

$$\tilde{c}(l, L) = \bar{\beta}(c_\infty + (c_0 - c_\infty)e^{\beta L})(1 + l)l$$

The value of the firm selling electricity is given by $V(t, D_t, Q_t, L_t)$. It satisfies the coupling equations :

$$\begin{cases} \partial_t v + \alpha(m - D)\partial_D v + \frac{1}{2}\sigma^2\partial_{DD}^2 v + (D - L)^+\partial_Q v + \Pi(D, L) \\ + sL^{1-\alpha} - y(D - L)^+ + \sup_l \{l\partial_L v - \tilde{c}(l, L)\} = 0 \\ v_T = 0 \end{cases} \quad (8.1)$$

and the carbon price $y(t, D_t, Q_t, L_t)$ is given by :

$$\begin{cases} \partial_t y + \alpha(m - D)\partial_D y + \frac{1}{2}\sigma^2\partial_{DD}^2 y + (D - L)^+\partial_Q y + l^*\partial_L y = 0 \\ y_T = \lambda 1_{Q_T \geq K} \end{cases} \quad (8.2)$$

and l^* is the optimal control in equation (8.1). The previous equation can be solved with the Semi Lagrangian method.

After a time discretization with a step δt a dynamic programming equation can be given by

$$\begin{aligned} v(T - \delta t, D, Q, L) &= \sup_l (\Pi(D, L) + sL^{1-\alpha} - y_{T-\delta t}(D - L)^+ - \tilde{c}(l, L))\delta t + \\ &\quad \mathbb{E}_{T-\delta t}(V(T, D_T^{T-\delta t, D}, Q + (D - L)^+\delta t, L + l\delta t)) \end{aligned} \quad (8.3)$$

$$Y(T - \delta t, D, Q, L) = \mathbb{E}_{T-\delta t}(Y(T, D_T^{T-\delta t, D}, Q + (D - L)^+\delta t, L + l^*\delta t)) \quad (8.4)$$

The previous equations (8.3) and (8.4) can be solved with the regression methods.

In order to use the previously developed frameworks in parallel, we have to define for both method some common variables.

- The number of regimes to use (obtained by the “getNbRegime” method) is 2 : one to store the v value, one for the y value,
- In the example we want to follow during simulations the functions values v and y so we set the number of function obtained by the “getSimuFuncSize” method to 2.
- In order to test the controls in optimization and simulation we define a maximal intensity of investment “lMax” and a discretization step to test the controls “lStep”.

In the sequel we store the optimal functions in optimization and recalculate the optimal control in simulation.

8.3 The dynamic programming with regression approach

All we have to do is to specify an optimizer defining the methods used to optimize and simulate, and the “getCone” method for parallelization :

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #include "StOpt/core/utils/constant.h"
5 #include "OptimizeDPEmissive.h"
6
7 using namespace std ;
8 using namespace StOpt;
9 using namespace Eigen;
10
11
12 // constructor
13 OptimizeDPEmissive::OptimizeDPEmissive(const double &p_alpha ,
14                                         const std::function<double(double ,
  double)> &p_PI ,

```

```

15         const std::function< double(double,
16             double) > &p_cBar, const
17             double &p_s, const double &
18             p_lambda,
19             const double &p_dt,
20             const double &p_maturity,
21             const double &p_lMax, const double &
22             p_lStep, const std::vector<std::
23             array< double, 2> > &p_extrem):
24     m_alpha(p_alpha), m_PI(p_PI),
25     m_cBar(p_cBar), m_s(p_s), m_lambda(p_lambda), m_dt(p_dt), m_maturity(
26         p_maturity), m_lMax(p_lMax), m_lStep(p_lStep),
27     m_extrem(p_extrem)
28 {}
29
30 Array< bool, Dynamic, 1> OptimizeDPEmissive::getDimensionToSplit() const
31 {
32     Array< bool, Dynamic, 1> bDim = Array< bool, Dynamic, 1>::Constant(2,
33         true);
34     return bDim ;
35 }
36
37 // for parallelism
38 std::vector< std::array< double, 2> > OptimizeDPEmissive::getCone(const
39     vector< std::array< double, 2> > &p_xInit) const
40 {
41     vector< array< double, 2> > xReached(2);
42     xReached[0][0] = p_xInit[0][0] ; // Q only increases
43     xReached[0][1] = m_extrem[0][1] ; // whole domain due to demand which is
44         unbounded
45     xReached[1][0] = p_xInit[1][0] ; // L only increases
46     xReached[1][1] = p_xInit[1][1] + m_lMax * m_dt ; // maximal increase
47         given by the control
48     return xReached;
49 }
50
51 // one step in optimization from stock point for all simulations
52 std::pair< ArrayXXd, ArrayXXd> OptimizeDPEmissive::stepOptimize(const std::
53     shared_ptr< StOpt::SpaceGrid> &p_grid, const ArrayXd &p_stock,
54     const std::vector< ContinuationValue> &p_condEsp,
55     const std::vector< std::shared_ptr< ArrayXXd > > &) const
56 {
57     std::pair< ArrayXXd, ArrayXXd> solutionAndControl;
58     // to store final solution (here two regimes)
59     solutionAndControl.first = ArrayXXd::Constant(m_simulator->getNbSimul(),
60         2, -StOpt::infy);
61     solutionAndControl.second = ArrayXXd::Constant(m_simulator->getNbSimul()
62         , 1, -StOpt::infy);
63     // demand
64     ArrayXd demand = m_simulator->getParticles().array().row(0).transpose();
65     // Gain (size number of simulations)
66     ArrayXd gain(m_simulator->getNbSimul());
67     double gainSubvention = m_s * pow(p_stock(1), 1. - m_alpha); //
68         subvention for non emissive energy

```

```

55 for (int is = 0 ; is < m_simulator->getNbSimul(); ++is)
56     gain(is) = m_PI(demand(is), p_stock(1)) + gainSubvention ; // gain by
        production and subvention
57 ArrayXd ptStockNext(2);
58 // time to maturity
59 double timeToMat = m_maturity - m_simulator->getCurrentStep();
60 // interpolator at the new step
61 for (int is = 0 ; is < m_simulator->getNbSimul(); ++is)
62 {
63     for (int iA1 = 0; iA1 < m_lMax / m_lStep ; ++iA1) // test all
        command for investment between 0 and lMax
64     {
65         double l = iA1 * m_lStep;
66         // interpolator at the new step
67         ptStockNext(0) = p_stock(0) + std::max(demand(is) - p_stock(1),
            0.) * m_dt;
68         ptStockNext(1) = p_stock(1) + l * m_dt ;
69         // first test we are inside the domain
70         if (p_grid->isInside(ptStockNext))
71         {
72             // create an interpolator at the arrival point
73             std::shared_ptr<StOpt::Interpolator> interpolator = p_grid->
                createInterpolator(ptStockNext);
74             // calculate Y for this simulation with the optimal control
75             double yLoc = p_condEsp[1].getASimulation(is, *interpolator);
76             // local gain
77             double gainLoc = (gain(is) - yLoc * std::max(demand(is) -
                p_stock(1), 0.) - m_cBar(l, p_stock(1))) * m_dt;
78             // gain + conditional expectation of future gains
79             double condExp = gainLoc + p_condEsp[0].getASimulation(is, *
                interpolator);
80             if (condExp > solutionAndControl.first(is, 0)) // test
                optimality of the control
81             {
82                 solutionAndControl.first(is, 0) = condExp;
83                 solutionAndControl.first(is, 1) = yLoc;
84                 solutionAndControl.second(is, 0) = 1;
85             }
86         }
87     }
88     // test if solution acceptable
89     if (StOpt::almostEqual(solutionAndControl.first(is, 0), - StOpt::
        infty, 10))
90     {
91         // fix boundary condition
92         solutionAndControl.first(is, 0) = timeToMat * (m_PI(demand(is),
            p_stock(1)) + m_s * pow(p_stock(1), 1. - m_alpha) - m_lambda *
            std::max(demand(is) - p_stock(1), 0.));
93         solutionAndControl.first(is, 1) = m_lambda ; // Q est maximal !!
94         solutionAndControl.second(is, 0) = 0. ; // fix control to zero
95     }
96 }
97 return solutionAndControl;
98 }

```

```

99
100 // one step in simulation for current simulation
101 void OptimizedDPEmissive::stepSimulate(const std::shared_ptr< StOpt::SpaceGrid
    > &p_grid, const std::vector< StOpt::GridAndRegressedValue > &
    p_continuation,
102                                     StOpt::StateWithStocks &p_state,
103                                     Ref<ArrayXd> p_phiInOut) const
104 {
105     ArrayXd ptStock = p_state.getPtStock();
106     ArrayXd ptStockNext(ptStock.size());
107     double vOpt = - StOpt::infty;
108     double gainOpt = 0.;
109     double lOpt = 0. ;
110     double demand = p_state.getStochasticRealization()(0); // demand for this
        simulation
111     ptStockNext(0) = ptStock(0) + std::max(demand - ptStock(1), 0.) * m_dt;
112     double gain = m_PI(demand, ptStock(1)) + m_s * pow(ptStock(1), 1. -
        m_alpha) ; // gain from production and subvention
113     double yOpt = 0. ;
114     for (int iA1 = 0; iA1 < m_lMax / m_lStep ; ++iA1) // test all command
        for investment between 0 and lMax
115     {
116         double l = iA1 * m_lStep;
117         // interpolator at the new step
118         ptStockNext(1) = ptStock(1) + l * m_dt ;
119         // first test we are inside the domain
120         if (p_grid->isInside(ptStockNext))
121         {
122             // calculate Y for this simulation with the control
123             double yLoc = p_continuation[1].getValue(ptStockNext, p_state.
                getStochasticRealization());
124             // local gain
125             double gainLoc = (gain - yLoc * std::max(demand - ptStock(1), 0.)
                - m_cBar(1, ptStock(1))) * m_dt;
126             // gain + conditional expectation of future gains
127             double condExp = gainLoc + p_continuation[0].getValue(
                ptStockNext, p_state.getStochasticRealization());
128
129             if (condExp > vOpt) // test optimality of the control
130             {
131                 vOpt = condExp;
132                 gainOpt = gainLoc;
133                 lOpt = l;
134                 yOpt = yLoc;
135             }
136         }
137     }
138     p_phiInOut(0) += gainOpt; // follow v value
139     p_phiInOut(1) = yOpt ; // follow y value
140     ptStockNext(1) = ptStock(1) + lOpt * m_dt ; // update state due to
        control
141     p_state.setPtStock(ptStockNext);
142 }

```

This case in dimension 2 for the stocks can be treated with interpolation on the full 2

dimensional grid and on a 2 dimensional sparse grid. Both versions of the resolution are given in a test case.

8.4 The PDE approach

We can do the same with the PDE approach using a simulator for the OU demand. We then define an optimizer and the methods used to optimize and simulate, and the “getCone” method for parallelization :

```

1 #include <iostream>
2 #include "StOpt/core/utils/constant.h"
3 #include "OptimizeSLEmissive.h"
4
5 using namespace StOpt;
6 using namespace Eigen ;
7 using namespace std ;
8
9 // constructor
10 OptimizeSLEmissive::OptimizeSLEmissive(const double &p_alpha, const double &
    p_m, const double &p_sig, const std::function<double(double, double)> &
    p_PI,
11
12
13
14
15
16
17
18
19
20
21
22
23 // for parallelism
24 vector< array< double, 2>> OptimizeSLEmissive::getCone(const vector<
    array< double, 2>> &p_xInit) const
25 {
26     vector< array< double, 2>> xReached(3);
27     xReached[0][0] = p_xInit[0][0] + m_alpha * (m_m - m_extrem[0][1]) *
        m_dt - m_sig * sqrt(m_dt); // demand "cone" driven by maximal value
        allowed for demand
28     xReached[0][1] = p_xInit[0][1] + m_alpha * m_m * m_dt + m_sig * sqrt(
        m_dt) ; // low value for demand is taken equal to 0
29     xReached[1][0] = p_xInit[1][0] ; // Q only increases
30     xReached[1][1] = p_xInit[1][1] + m_extrem[0][1] * m_dt ; // Q increase
        bounded by maximal demand
31     xReached[2][0] = p_xInit[2][0] ; // L only increases

```

```

32     xReached[2][1] = p_xInit[2][1] + m_lMax * m_dt ;// maximal increase
           given by the control
33     return xReached;
34 }
35
36
37 // one step in optimization from current point
38 std::pair< ArrayXd, ArrayXd> OptimizeSLEmissive::stepOptimize(const ArrayXd
           &p_point ,
39         const vector< shared_ptr<SemiLagrangEspCond> > &p_semiLag ,
40         const double &, const ArrayXd &) const
41 {
42     pair< ArrayXd, ArrayXd> solutionAndControl;
43     solutionAndControl.first.resize(2);
44     solutionAndControl.second.resize(1);
45     ArrayXXd sig = ArrayXXd::Zero(3, 1) ;
46     sig(0, 0) = m_sig;
47     double vOpt = - StOpt::infy;
48     double yOpt = 0. ;
49     double lOpt = 0 ;
50     ArrayXd b(3);
51     b(0) = m_alpha * (mm - p_point(0)) ; // trend
52     b(1) = max(p_point(0) - p_point(2), 0.);
53     // gain already possible to calculate (production and subvention)
54     double gainFirst = m_PI(p_point(0), p_point(2)) + m_s * pow(p_point(2),
           1. - m_alpha) ;
55     for (int iA1 = 0; iA1 < m_lMax / m_lStep ; ++iA1) // test all command for
           investment between 0 and lMax
56     {
57         double l = iA1 * m_lStep;
58         b(2) = l ;
59         pair<double, bool> lagrangY = p_semiLag[1]->oneStep(p_point , b, sig ,
           m_dt); // for the control calculate y
60         if (lagrangY.second) // is the control admissible
61         {
62             pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_point , b,
           sig , m_dt); // one step for v
63             // gain function
64             double gain = m_dt * (gainFirst - lagrangY.first * b(1) - m_cBar
           (1, p_point(2)));
65             double arbitrage = gain + lagrang.first;
66             if (arbitrage > vOpt) // optimality of the control
67             {
68                 vOpt = arbitrage; // upgrade solution v
69                 yOpt = lagrangY.first; // store y
70                 lOpt = l; // upgrade optimal control
71             }
72         }
73     }
74
75     if (StOpt::almostEqual(vOpt, - StOpt::infy , 10))
76     {
77         std::cout << " Reduce time step " << std::endl ;
78         abort();

```

```

79     }
80     solutionAndControl.first(0) = vOpt; // send back v function
81     solutionAndControl.first(1) = yOpt; // send back y function
82     solutionAndControl.second(0) = lOpt; // send back optimal control
83     return solutionAndControl;
84 }
85
86 // one step in simulation for current simulation
87 void OptimizeSLEmissive::stepSimulate(const SpaceGrid &p_gridNext,
88                                       const std::vector< std::shared_ptr<
89                                           StOpt::SemiLagrangEspCond> > &
90                                           p_semiLag,
91                                       Ref<ArrayXd> p_state, int &,
92                                       const ArrayXd &p_gaussian,
93                                       const ArrayXd &,
94                                       Ref<ArrayXd> p_phiInOut) const
95 {
96     ArrayXd state = p_state;
97     ArrayXXd sig = ArrayXXd::Zero(3, 1); // diffusion matrix for semi
98     Lagrangian
99     sig(0, 0) = m_sig;
100    double vOpt = - StOpt::infty;
101    double lOpt = 0;
102    double yOpt = 0;
103    ArrayXd b(3);
104    b(0) = m_alpha * (mm - p_state(0)); // trend for D (independent of
105    control)
106    b(1) = max(p_state(0) - p_state(2), 0.); // trend for Q (independent of
107    control)
108    double gainFirst = m_PI(p_state(0), p_state(2)) + m_s * pow(p_state(2),
109    1. - m_alpha); // gain for production and subvention
110    for (int iA1 = 0; iA1 < m_lMax / m_lStep; ++iA1) // recalculate the
111    optimal control
112    {
113        double l = iA1 * m_lStep;
114        b(2) = l;
115        pair<double, bool> lagrangY = p_semiLag[1]->oneStep(p_state, b, sig,
116        m_dt); // calculate y for this control
117        if (lagrangY.second)
118        {
119            pair<double, bool> lagrang = p_semiLag[0]->oneStep(p_state, b,
120            sig, m_dt); // calculate the function value v
121            // gain function
122            double gain = m_dt * (gainFirst - lagrangY.first * b(1) - m_cBar
123            (1, p_state(2)));
124            double arbitrage = gain + lagrang.first;
125            if (arbitrage > vOpt) // arbitrage
126            {
127                vOpt = arbitrage; // upgrade solution
128                yOpt = lagrangY.first; // upgrade y value
129                lOpt = l; // upgrade optimal control
130            }
131        }
132    }
133 }

```

```

123 // gain function
124 p_phiInOut(0) += m_dt * (gainFirst - yOpt * b(1) - m_cBar(lOpt, state(2))
    ); // store v value
125 p_phiInOut(1) = yOpt; // store y value
126 // update state
127 state(0) += m_alpha * (mm - p_state(0)) * m_dt + m_sig * p_gaussian(0) *
    sqrt(m_dt); // demand (no control)
128 state(1) += b(1) * m_dt; //Q
129 state(2) += lOpt * m_dt; //L
130 // truncate if necessary to stay inside domain.
131 p_gridNext.truncatePoint(state);
132 p_state = state ;
133 }

```

The three dimensional grids used can be some full grids or some sparse grids. Both versions of the resolution can be found in a test case.

Part VI

**Stochastic Dual Dynamic
Programming**

Chapter 9

SDDP algorithm

9.1 Some general points about SDDP

SDDP is an approximate dynamic programming algorithm developed by Pereira and Pinto in 1991 [27].

To describe how SDDP works, we will consider a class of linear programs that have T stages denoted $\{0, 1, \dots, t, \dots, T\}$. We restrict our class of problems to linear programs with relatively complete recourse : the feasible region of the linear program in each stage is nonempty and bounded.

Let us formalize now the variables and constraints used in the SDDP problem.

Notations used

The notations described here are used in the general case.

- x_t the state variable at time t ,
- $\omega_t \in \Omega_t$ the random data process at time t , where Ω_t is the set of random data.
- c_t is the cost vector at time t ,
- A_t and E_t denote constraints matrices.
- $Q_t(x_{t-1}, \omega_t)$ is the expected value of the problem at time t , knowing the state x_{t-1} and the random data ω_t .
- $\mathcal{Q}_t(x_{t-1}) = \mathbb{E}[Q_t(x_{t-1}, \omega_t)]$

Decision process

The random data process ω_t is discovered gradually. Thus from an initial state x_0 , the state variables $(x_t)_{t \in \{0, 1, \dots, T\}}$ are determined in a non-anticipative way. The scheme is the following :

$x_0 \rightarrow$ observation of $\omega_1 \rightarrow$ decision of x_1
 \rightarrow decision of $x_{T-1} \rightarrow$ observation of $\omega_T \rightarrow$ decision of x_T

A rigorous formulation of the multistage stochastic linear program to solve is the following:

$$V^* = \min_{\substack{A_0 x_0 = \omega_0 \\ x_1 \geq 0}} c_0^\top x_0 + \mathbb{E} \left[\min_{\substack{E_1 x_0 + A_1 x_1 = \omega_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E} \left[\dots + \mathbb{E} \left[\min_{\substack{E_T x_{T-1} + A_T x_T = \omega_T \\ x_T \geq 0}} c_T^\top x_T \right] \right] \right] \quad (9.1)$$

The deterministic equivalent of this problem (9.1) is achieved by discretizing ω_t (or by using directly ω_t if discrete). The number of variables of this problem increases exponentially with the number of stages. It cannot be solved directly even if T or $(\Omega_t)_{t \in \{0,1,\dots,T\}}$ are of reasonable size.

Dynamic programming principle

Dynamic programming involves splitting up the problem (9.1) in a series of sub-problem bounded together by a state variable. The aim is to compute backwards the functions Q_t and \mathcal{Q}_t . They fulfill the following equations :

$$[LP_t] \begin{cases} s.c. & Q_t(x_{t-1}, \omega_t) = \min c_t^\top x_t + \mathcal{Q}_{t+1}(x_t) \\ & A_t x_t = \omega_t - E_t x_{t-1}, \quad [\pi_t(\omega_t)] \\ & x_t \geq 0 \end{cases} \quad (9.2)$$

$$\mathcal{Q}_t(x_{t-1}) = \mathbb{E}[Q_t(x_{t-1}, \omega_t)] \quad (9.3)$$

The function $Q(x_{t-1}, \omega_t)$ stands for the expected value of a future cost knowing the state x_{t-1} the random data ω_t . $\mathcal{Q}_t(x_{t-1})$ is the expected value of the future cost knowing the state. The dynamic programming principle insures that $V^* = \mathcal{Q}_1(x_0)$.

Given $\mathcal{Q}_T(\cdot)$, the successive computations are achieved backwards switching between the resolution of the linear sub-problem (9.2) and the computation of (9.3).

The implementation of dynamic programming involves approximating successively the two value functions with equations (9.2 - 9.3) by discretizing the state space and solving the linear sub-problems. The number of discretization points increases exponentially with the dimension of the state vector and becomes huge for our applications ("curse of dimensionality").

SDDP algorithm

SDDP is a method used to solve stochastic multi-stage problem described in [27]. SDDP is based on Benders decomposition described in [28]. Please note that SDDP was developed in order to solve hydro thermal scheduling problem.

SDDP gets round the curse of dimensionality by avoiding *a priori* complete discretization of the state space. Each SDDP iteration is a two-stage process. The first step involves generating a sequence of realistic states x_t^* from which in the second step the value functions are estimated in their neighborhood. By repeating successively these two steps the approximation of the value function becomes more and more accurate. SDDP is also made of two passes computed alternatively :

- a backward pass : the aim is to improve the number of Benders cut in the neighborhood of well-chosen candidate states. It provides also a lower bound of the optimal cost.
- a forward pass : the aim is to provide a set of new candidate states. An estimation of the upper bound of the optimal cost is also computed.

On the other hand SDDP method stands on the shape of the future value function $\mathcal{Q}_t(x_{t-1})$. Indeed in the frame of a linear problem with complete recurse the value function is convex and piecewise linear. It can therefore be approximated by taking the supremum of a family of minoring affine functions. These affine functions are called *optimality cuts* or *Benders cuts*.

9.2 A method, different algorithms

The method implemented in this library is based on the different situations shown in a technical report of PSR program [29] where three different cases of the basic problem are solved by SDDP. The three cases are implemented in the library. Other cases could be added to those existing in the future.

Notations

These notations will be used to present the different algorithm of SDDP.

- \bar{z} denotes the optimal cost obtained in forward pass.
- \underline{z} denotes the optimal cost obtained in backward pass.
- β_t^j denotes the slope of the j^{th} Benders cut.
- α_t^j denotes the intercept of the j^{th} Benders cut.

9.2.1 The basic case

To describe this case the notations shown above are used. We focus on stochastic multi-stage problems with the following properties.

- Random quantities in different stages are independent.
- The random quantities at time t is summarized in ω_t .
- At each stage, the linear sub-problem solution space is non-empty and bounded.

In this case the functions $\mathcal{Q}_t(\cdot)$ are convex. The primal and dual solutions of the linear problem exist and define optimal cuts. We can now describe precisely how the implemented algorithm is working.

Initialization

The following values are fixed :

- $\{0, 1, \dots, T\}$, the time horizon.
- $n = 0$, is the counter of the number of iterations (*backward-forward*). n is incremented at the end of each iteration.
- $p \in \mathbb{R}$, the precision to reach for the convergence test.
- $n_{step} \in \mathbb{N}$, the number of iterations achieved between 2 convergence tests.
- $n_{iterMax} \in \mathbb{N}$, the maximal number of iterations.
- $x_0 \in \mathbb{R}_+^n$, the initial vector state.
- $L \in \mathbb{N}$, the number of scenarios used in the backward pass.
- $G \in \mathbb{N}$, the number of scenarios used in the forward pass. It gives also the number of new cuts computed at every iteration (*backward-forward*) and the number of states near which the Benders cuts are computed.

Forward pass

The aim of this pass is to explore new feasible vector state and to get an estimation of the upper bound of the optimal cost. To this end the current strategy is simulated for a set of G scenarios. The set of scenarios could be historical chronicles or random draws.

Algorithm 4: Run of forward pass (n^{th} iteration)

Simulate sets $\{(\omega_t^g), t \in \{1, \dots, T\}\}$ of equally distributed scenarios : for $g \in \Omega^g = \{1, \dots, G\}$;

for $g \in \Omega^g$ **do**

Solve the following linear sub-problem. ;

$$[AP_0^n] \begin{cases} Q_0 = \min_{x_0, \theta_1} c_0^\top x_0 + \theta_1 \\ \text{u.c.} & A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ & x_0 \geq 0 \\ & \theta_1 + (\beta_1^j)^\top x_0 \geq \alpha_1^j \quad j \in \{1, \dots, G, \dots, nG\} \end{cases} \quad (9.4)$$

Store the primal solution (x_0^g) of the problem $[AP_{0,g}^n]$.

for $t \in \{1, \dots, T\}$ **do**

Solve the following linear sub-problem. ;

$$[AP_{t,g}^n] \begin{cases} Q_t^g(x_{t-1}^g, \omega_t^g) = \min_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ \text{s.c.} & A_t x_t = \omega_t^g - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^g)] \\ & x_t \geq 0 \\ & \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geq \alpha_{t+1}^j, \quad j \in \{1, \dots, G, \dots, nG\} \end{cases} \quad (9.5)$$

Store the primal solution (x_t^g) of the problem $[AP_{t,g}^n]$

end

Compute the cost for scenario g , at iteration n : $\bar{z}_n^g = \sum_{t=0}^T c_t x_t^g$;

end

Compute the total cost in forward pass at iteration n : $\bar{z}_n = \frac{1}{G} \sum_{g=1}^G \bar{z}_n^g$

Backward pass

The aim of the backward pass is to add at each stage a set of new Benders cut and to provide a new estimation of the lower bound of the optimal operational cost. To this end we have scenarios set of the random quantities (dimension of the set is L) recorded during the initialization.

Algorithm 5: Run of backward pass

for $t = T, T - 1, \dots, 1$ **do**

for $x_{t-1}^g, g \in \{1, \dots, G\}$ **do**

for $\omega_t^l, l \in \{1, \dots, L\}$ **do**

 Solve the following linear sub-problem. ;

$$[AP_{t,l}^{n,g}] \begin{cases} Q_t^l(x_{t-1}^g, \omega_t^l) = \min_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ \text{s.c.} \quad A_t x_t = \omega_t^l - E_t x_{t-1}^g, \quad [\pi_t(\omega_t^l)] \\ x_t \geq 0 \\ \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geq \alpha_{t+1}^j, \quad j \in \{1, \dots, G, \dots, (n+1)G\} \end{cases} \quad (9.6)$$

 Store the dual solution $\pi_t(\omega_t^l)$ and the primal one $Q_t^l(x_{t-1}^g, \omega_t^l)$ of the linear sub-problem $[AP_{t,l}^{n,g}]$

 Compute the cut that goes with the l^{th} hazard draw :

$$\begin{cases} \alpha_{t,l}^g = Q_t^l(x_{t-1}^g, \omega_t^l) + \pi_t(\omega_t^l)^\top E_t x_{t-1}^g \\ \beta_{t,l}^g = E_t^\top \pi_t(\omega_t^l) \end{cases} \quad (9.7)$$

end

 Compute the g^{th} new Benders cut at time t at iteration n : is defined as the mean value of the cuts obtained before:

$$\begin{cases} \alpha_t^k = \frac{1}{L} \sum_{l=1}^L \alpha_{t,l}^g \\ \beta_t^k = \frac{1}{L} \sum_{l=1}^L \beta_{t,l}^g \\ k = nG + g \end{cases} \quad (9.8)$$

end

end

Solve the following linear sub-problem. ;

$$[AP_0^n] \begin{cases} Q_0 = \min_{x_0, \theta_1} c_0^\top x_0 + \theta_1 \\ \text{s.c.} \quad A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ x_0 \geq 0 \\ \theta_1 + (\beta_1^j)^\top x_0 \geq \alpha_1^j \quad j \in \{1, \dots, G, \dots, (n+1)G\} \end{cases} \quad (9.9)$$

Save the cost *backward* $z_n = Q_0$

Stopping test

In the literature about SDDP lots of stopping criterion were used and their efficiency has been proved. However a criterion is suitable for each particular problem. Thus it is tough to bring out one which is generic. Due to genericity requirements, two classical criterion are implemented in the library. These can be customized by the user. The first one defines a maximal number of iterations $n_{iterMax}$ (an iteration is made of the succession of *backward-forward* passes) which shall not be exceeded. The second one is a test of convergence

towards each other between the forward and the backward cost. The convergence test uses the following indicator :

$$\psi_{n_{step}i} = \left| \frac{\bar{z}_{n_{step}i} - z_{n_{step}i}}{\bar{z}_{n_{step}i}} \right|, \quad \text{with } i \in \mathbb{N} \quad (9.10)$$

This one is computed every n_{step} iterations. If it is lesser than a threshold p the process stops, otherwise it goes on. The threshold is fixed by the user.

9.2.2 Dependence of the random quantities

In the previous case we restrict our problem to independent random quantities in the different stages. The resolution of the SDDP was achieved on the state vector x_t in the basic case.

But sometimes in real life the random quantities can be temporarily correlated. In a hydraulic problem for example there exists time-related dependency of the outcomes. Time-related dependencies can also exist in the demand. Yet with time-related random quantities the Bellman recurrence formula (9.2 - 9.3) does not hold and the classical SDDP can not be applied.

However if the Bellman functions are convex with respect to the time-related random quantities one has only to increase the dimension of the state vector by the dimension of the time-related random quantities to be back in the configuration of the basic case. In this case solving a linear program of reasonable size for each hazard draw is enough to compute new Benders cuts computation in the neighborhood of a candidate state.

There exists a few options to represent the time-related dependency of the random quantities. However in order to not increase too much the dimension of the problem, a ARMA process of order 1 is often chosen. In the random data vector ω_t two different parts has to be distinguished from now on:

- ω_t^{ind} is the random data vector corresponding to the independent random quantities.
- ω_t^{dep} is the random data vector corresponding to the time-related random quantities.

And ω_t^{dep} fulfills the following recurrence equation :

$$\frac{\omega_t^{dep} - \mu_{\omega,t}}{\sigma_{\omega,t}} = \psi_1 \frac{\omega_{t-1}^{dep} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t \quad (9.11)$$

To apply the Bellman recurrence formula the vector state should be made of the decision variable x_t and the time-related random quantities ω_t^{dep} . Dimension of the vector state is then increased. $x_t^{dep} = (x_t, \omega_t^{dep})^\top$ denotes the new state vector. The Bellman function satisfies from now on the following two-stages linear problem at time t :

$$[LP_t'] \begin{cases} Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t) = \min c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \omega_t^{dep}) \\ u.c. \quad A_t x_t = P \omega_t^{dep} - E_t x_{t-1}, \quad [\pi_t(\omega_t)] \\ x_t \geq 0 \end{cases} \quad (9.12)$$

with P the matrix such that $\omega_t = P \omega_t^{dep}$.

The variable ω_t^{dep} is a random process. Thus the above problem is solved using specific values ω_t^l of this variable. To get them we apply a Markov process that is we simulate different values of the white noise ϵ_t^l .

The new form of the state vector implies changes in the sensitivity of the Bellman function. Thus it is a function depending on the decision variable x_t but also on the the time-related random quantity vector ω_t^{dep} . The computation of Benders cuts is then a bit different :

$$\begin{aligned} \frac{\partial Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t)}{\partial \omega_{t-1}^{dep}} &= \frac{\partial Q_t(x_{t-1}, \omega_{t-1}^{dep}, \omega_t)}{\partial \omega_t^{dep}} \frac{\partial \omega_t^{dep}}{\partial \omega_{t-1}^{dep}} \\ &= \pi_t(\omega_t)^\top P \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}}, \end{aligned} \tag{9.13}$$

Backward pass has to be modified in the following manner. Some new computation steps have to be taken into account.

Algorithm 6: Run of backward pass with time-related random quantities (AR1 process)

Pick up the set of the following pairs: $\{x_t^g, \omega_t^{g,dep}\}$ for $g \in \{1, \dots, G\}$, $t \in \{1, \dots, T\}$

for $t = T, T-1, \dots, 1$ **do**

for $(x_{t-1}^g, \omega_{t-1}^{dep,g})$, $g \in \{1, \dots, G\}$ **do**

for $l \in \{1, \dots, L\}$ **do**

Produce a value for the white noise ϵ_t^l ;

Compute the element $\hat{\omega}_t^l$ knowing the previous random quantity $\omega_{t-1}^{dep,g}$:

$$\hat{\omega}_t^l = \sigma_{\omega,t} \left(\psi_1 \frac{\omega_{t-1}^{dep,g} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t^l \right) + \mu_{\omega,t} \quad (9.14)$$

Solve the following linear sub-problem ;

$$[AP'_{t,l}] \begin{cases} Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l) = \min_{x_t, \theta_{t+1}} c_t^\top x_t + \theta_{t+1} \\ u.c. \quad A_t x_t = P \hat{\omega}_t^l - E_t x_{t-1}^g, \quad [\pi_t(\hat{\omega}_t^l)] \\ x_t \geq 0 \\ \theta_{t+1} + (\beta_{t+1}^j)^\top x_t + (\gamma_{t+1}^j)^\top \hat{\omega}_t^l \geq \alpha_{t+1}^j, \\ j \in \{1, \dots, G, \dots, (n+1)G\} \end{cases} \quad (9.15)$$

Store the dual solution $\pi_t(\omega_t^l)$ and the primal one $Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l)$ of the primal problem $[AP'_{t,l}]$

Compute the cut that goes with the l^{th} hazard draw :

$$\begin{cases} \alpha_{t,l}^g = Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l) + \pi_t(\hat{\omega}_t^l)^\top \left(E_t x_{t-1}^g - \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P \omega_{t-1}^{dep,g} \right) \\ \beta_{t,l}^g = E_t^\top \pi_t(\hat{\omega}_t^l) \\ \gamma_{t,l}^g = \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P^\top \pi_t(\hat{\omega}_t^l) \end{cases} \quad (9.16)$$

end

Compute the g^{th} new Benders cut at time t at iteration n defined as the mean value of the cuts obtained before:

$$\begin{cases} \alpha_t^k = \frac{1}{L} \sum_{l=1}^L \alpha_{t,l}^g \\ \beta_t^k = \frac{1}{L} \sum_{l=1}^L \beta_{t,l}^g \\ \gamma_t^k = \frac{1}{L} \sum_{l=1}^L \gamma_{t,l}^g \\ k = nG + g \end{cases} \quad (9.17)$$

end

end

Solve the following linear sub-problem. ;

$$[AP_0^n] \begin{cases} Q_0 = \min_{x_0, \theta_1} c_0^\top x_0 + \theta_1 \\ u.c. \quad A_0 x_0 = \omega_0, \quad [\pi_0(\omega_0)] \\ x_0 \geq 0 \\ \theta_1 + (\beta_1^j)^\top x_0 + (\gamma_1^j)^\top \omega_0^{dep} \geq \alpha_1^j, \\ j \in \{1, \dots, G, \dots, (n+1)G\} \end{cases} \quad (9.18)$$

Save the backward cost $z_n = Q_0$

9.2.3 Non-convexity and conditionnal cuts

Some random quantities may introduce non-convexity preventing us to apply the classical algorithm of SDDP. Indeed when the random quantities appear on the right-hand side of the linear constraints or in the cost function (typically A_t and/or c_t become random) the property of non-convexity of the Bellman functions with respect to the random quantities is not anymore observed.

In the frame of a management production problem the situation happened often. For example sometimes the unit operation cost of plants are random. It is also observed when we deal with spot price uncertainty for use in stochastic mid-term scheduling.

In a technical report [29] Pereira and Pinto suggested a new algorithm in order to efficiently approximate the Bellman functions using explicitly the dependence of the Bellman functions with respect to these random quantities. This new algorithm is based on a combination of SDDP and ordinary stochastic dynamic programming. The SDP part deals with the non-convex random quantities, whereas the other random quantities are treated in the SDDP part. It is an extension of the classical SDDP algorithm. It is described in detail in [29] and in [30].

In that case the modelization used in the library is somewhat different from the one described in both articles. Indeed in the articles it is based on a finite number of non-convex random quantities. A discretization of the non-convex random quantity space is the necessary.

In [30] spot price p_t is regarded as a state. The set of feasible spot price is discretized into in a set of M points ζ_1, \dots, ζ_M . The following Markov model is then used :

$$\mathbb{P}(p_t = \zeta_j | p_{t-1} = \zeta_i) = \rho_{ij}(t)$$

This model makes easier the implementation of the SDP. But it implies discretization mistakes that are hard to quantify. It is also tough to discretize with efficiency a random process.

In order to overcome these points in the library the evolution of the non-convex random quantities is decided by Monte Carlo simulations. At each stage a fixed number of Monte-Carlo simulations is provided. Anyway in spite of this difference the global view of this brand new algorithm is similar to that one described in both articles :

- The non-convex random quantities depend on the realization of the previous one according to a mathematical model (Markov chain).
- At each stage Bellman functions are approximated through the conditional realization of these random quantities.
- We used conditional cuts to give an estimation of the Bellman functions.

In our algorithm the features of the conditional cuts are revealed thanks to a conditional expectation computation.

Yet conditional expectation computations are not easy when the exact distribution of the random variable is not known. A few techniques exist but in the library a specific one is used and described above in chapter 2 : it is based on local linear regression.

Regression, stochastic dynamic programming and SDDP

The run of the backward pass in the new algorithm combining SDDP and SDP using local linear regression is described below.

Before describing in detail this algorithm, let us introduce a few notations :

- S is the space of the non-convex random quantities.
- d is the dimension of the space of the non-convex random quantities S
- At each stage U Monte Carlo simulations in S are provided. Thus we get U scenarios denoted s_t^u at each stage t
- \tilde{I} is a partition of the space of the non-convex random quantities S .

$$\tilde{I} = \{\underline{I} = (i_1, \dots, i_d), i_1 \in \{1, \dots, I_1\}, \dots, i_d \in \{1, \dots, I_d\}\}$$

- $\{D_{\underline{I}}\}_{\underline{I} \in \tilde{I}}$ is the set of meshes of the set of scenarios.
- $M_M = \prod_{k=1}^d I_k$ denotes the number of meshes at each stage.

Algorithm 7: Run of the backward pass with time-related (AR1) and non-convex random quantities

Pick up the set of the following pairs : $\{x_t^g, \omega_t^{g,dep}\}$ pour $g \in \{1, \dots, G\}$, $t \in \{1, \dots, T\}$

for $t = T, T-1, \dots, 1$ **do**

Generate values for the non-convex random quantities at time t knowing the scenarios at time $t-1$: s_t^u , $u \in \{1, \dots, U\}$

for $(x_{t-1}^g, \omega_{t-1}^{dep,g})$, $g \in \{1, \dots, G\}$ **do**

for $u \in \{1, \dots, U\}$ **do**

Consider a scenario s_t^u in the mesh D_I ;

for $l \in \{1, \dots, L\}$ **do**

Produce a value for the white noise ϵ_t^l ;

Compute the element $\hat{\omega}_t^l$ knowing the previous random quantity $\omega_{t-1}^{dep,g}$:

$$\hat{\omega}_t^l = \sigma_{\omega,t} \left(\psi_1 \frac{\omega_{t-1}^{dep,g} - \mu_{\omega,t-1}}{\sigma_{\omega,t-1}} + \psi_2 \epsilon_t^l \right) + \mu_{\omega,t} \quad (9.19)$$

Pick up the cuts corresponding the mesh D_I : $\{\alpha_{t+1}^{I,j}(s), \beta_{t+1}^{I,j}(s), \gamma_{t+1}^{I,j}(s)\}$, $j \in \{1; \dots; (n+1)G\}$

Solve the following linear sub-problem ;

$$[AP'_{t,l}{}^{n,g}] \left\{ \begin{array}{l} Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l, s_t^u) = \min_{x_t, \theta_{t+1}} c_t(s_t^u)^\top x_t + \theta_{t+1}(s_t^u) \\ \text{s.c.} \quad A_t(s_t^u)x_t = P\hat{\omega}_t^l - E_t x_{t-1}^g, \quad [\pi_t(\hat{\omega}_t^l, s_t^u)] \\ x_t \geq 0 \\ \theta_{t+1}(s_t^u) + (\beta_{t+1}^{I,j}(s_t^u))^\top x_t + (\gamma_{t+1}^{I,j}(s_t^u))^\top \hat{\omega}_t^l \geq \alpha_{t+1}^{I,j}(s_t^u), \quad j \in \{1, \dots, G, \dots, (n+1)G\} \end{array} \right. \quad (9.20)$$

Store the dual solution $\pi_t(\hat{\omega}_t^l)$ and the primal solution $Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l, s_t^u)$ of the problem $[AP'_{t,l}{}^{n,g}]$

Calculate the corresponding cut at the l^{th} draw of uncertainties :

$$\left\{ \begin{array}{l} \hat{\alpha}_{t,l}^{g,I}(s_t^u) = Q_t^l(x_{t-1}^g, \omega_{t-1}^{dep,g}, \hat{\omega}_t^l) + \pi_t(\hat{\omega}_t^l)^\top \left(E_t x_{t-1}^g - \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P \omega_{t-1}^{dep,g} \right) \\ \hat{\beta}_{t,l}^{g,I}(s_t^u) = E_t^\top \pi_t(\hat{\omega}_t^l) \\ \hat{\gamma}_{t,l}^{g,I}(s_t^u) = \psi_1 \frac{\sigma_{\omega,t}}{\sigma_{\omega,t-1}} P^\top \pi_t(\hat{\omega}_t^l) \end{array} \right. \quad (9.21)$$

end

Compute the cut for a non-convex random quantity s_t^u at time t at iteration n : it is defined as the weighted average on the L Benders cut obtained before :

$$\left\{ \begin{array}{l} \hat{\alpha}_t^{g,I}(s_t^u) = \frac{1}{L} \sum_{l=1}^L \hat{\alpha}_{t,l}^{g,I}(s_t^u) \\ \hat{\beta}_t^{g,I}(s_t^u) = \frac{1}{L} \sum_{l=1}^L \hat{\beta}_{t,l}^{g,I}(s_t^u), \quad j = nG + g \\ \hat{\gamma}_t^{g,I}(s_t^u) = \frac{1}{L} \sum_{l=1}^L \hat{\gamma}_{t,l}^{g,I}(s_t^u) \end{array} \right. \quad (9.22)$$

end

for $\underline{I}_i, i \in \{1, \dots, M_M\}$ **do**

Compute the g^{th} new cut of the mesh $D_{\underline{I}_i}$ at time t at iteration n defined as the conditional expectation with respect to the scenario u at time t

$$\left\{ \begin{array}{l} \alpha_t^{j,I}(s_{t-1}^u) = \mathbb{E} \left[\hat{\alpha}_t^{g,I}(s_t^u) | s_{t-1}^u \right], \\ \beta_t^{j,I}(s_{t-1}^u) = \mathbb{E} \left[\hat{\beta}_t^{g,I}(s_t^u) | s_{t-1}^u \right], \\ \gamma_t^{j,I}(s_{t-1}^u) = \mathbb{E} \left[\hat{\gamma}_t^{g,I}(s_t^u) | s_{t-1}^u \right] \end{array} \right., \quad j = nG + g \quad (9.23)$$

end

end

end

Solve the initial linear sub problem $[AP_0^n]$;

Save the backward cost $z_n = Q_0$

9.3 C++ API

The SDDP part of the stochastic library is in C++ code. This unit is a classical black box : specific inputs have to be provided in order to get the expected results. In the SDDP unit backward and forward pass are achieved successively until the stopping criterion is reached. In this unit the succession of passes is realized by `backwardForwardSDDP` class. This class takes as input three non-defined classes.

9.3.1 Inputs

The user has to implement three classes.

- One class where the transition problem is described which is denoted in the example `TransitionOptimizer`. This class is at the core of the problem resolution. Therefore much flexibility is let to the user to implement this class. In some ways this class is the place where the technical aspects of the problem are adjusted. This class describes backward and forward pass. Four methods should be implemented :
 - `oneStepForward` : solves the different transition linear problem during the forward pass for a particle, a random vector and an initial state.
 - `oneStepBackward` : solves the different transition linear problem during the backward pass for a particle, a random vector and an initial state.
 - `updateDates` : update the dates at time t .
 - `oneAdmissibleState` : returns an admissible state at time t (respect only the constraints)

`TransitionOptimizer` should derive from the `OptimizerSDDPBase` class defined below.

```
1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
4 // LGPL)
5 #ifndef OPTIMIZERSDDPBASE_H
6 #define OPTIMIZERSDDPBASE_H
7 #include <Eigen/Dense>
8 #include "StOpt/sddp/SDDPCutBase.h"
9 #include "StOpt/core/grids/OneDimRegularSpaceGrid.h"
10 #include "StOpt/core/grids/OneDimData.h"
11 #include "StOpt/sddp/SimulatorSDDPBase.h"
12
13 /** \file OptimizerSDDPBase.h
14 * \brief Define an abstract class for Stochastic Dual Dynamic Programming
15 * \author Xavier Warin
16 */
```

```

17
18 namespace StOpt
19 {
20
21 /// \class OptimizerSDDPBase OptimizerSDDPBase.h
22 /// Base class for optimizer for Dynamic Programming
23 class OptimizerSDDPBase
24 {
25
26
27 public :
28
29     OptimizerSDDPBase() {}
30
31     virtual ~OptimizerSDDPBase() {}
32
33
34     /// \brief Optimize the LP during backward resolution
35     /// \param p_linCut cuts used for the PL (Benders for the Bellman value
36     ///         at the end of the time step)
37     /// \param p_aState store the state, and 0.0 values
38     /// \param p_particle the particle n dimensional value associated to the
39     ///         regression
40     /// \param p_isample sample number for independant uncertainties
41     /// \return a vector with the optimal value and the derivatives if the
42     ///         function value with respect to each state
43     virtual Eigen::ArrayXd oneStepBackward(const std::unique_ptr< StOpt::
44     SDDPCutBase > &p_linCut, const std::tuple< std::shared_ptr<Eigen::
45     ArrayXd>, int, int > &p_aState, const Eigen::ArrayXd &p_particle,
46     const int &p_isample) const = 0;
47
48     /// \brief Optimize the LP during forward resolution
49     /// \param p_aParticle a particule in simulation part to get back cuts
50     /// \param p_linCut cuts used for the PL (Benders for the Bellman value
51     ///         at the end of the time step)
52     /// \param p_state store the state, the particle number used in
53     ///         optimization and mesh number associated to the particle. As an input
54     ///         it constains the current state
55     /// \param p_stateToStore for backward resolution we need to store \f$ (
56     S_t,A_{t-1},D_{t-1}) \f$ where p_state in output is \f$ (S_t,A_{t},D_{
57     t}) \f$
58     /// \param p_isample sample number for independant uncertainties
59     virtual double oneStepForward(const Eigen::ArrayXd &p_aParticle, Eigen::
60     ArrayXd &p_state, Eigen::ArrayXd &p_stateToStore, const std::
61     unique_ptr< StOpt::SDDPCutBase > &p_linCut, const int &p_isample)
62     const = 0 ;
63
64
65     /// brief update the optimizer for new date
66     virtual void updateDates(const double &p_date, const double &p_dateNext)
67     = 0 ;
68
69
70     /// \brief Get an admissible state for a given date
71     /// \param p_date current date

```

```

56     /// \return an admissible state
57     virtual std::shared_ptr<Eigen::ArrayXd> oneAdmissibleState(double p_date)
        = 0 ;
58
59     /// \brief get back state size
60     virtual int getStateSize() const = 0;
61
62     /// \brief get the backward simulator back
63     virtual std::shared_ptr< StOpt:: SimulatorSDDPBase > getSimulatorBackward
        () const = 0;
64
65     /// \brief get the forward simulator back
66     virtual std::shared_ptr< StOpt:: SimulatorSDDPBase > getSimulatorForward()
        const = 0;
67
68 };
69 }
70 #endif /* OPTIMIZERSDDPBASE_H */

```

- A simulator for forward pass : SimulatorSim
- A simulator for backward pass : SimulatorOpt. This simulator can use an underlying process to generate scenarios, a set of historical chronicles or a discrete set of scenarios. Often in the realized test case a Boolean is enough to distinguish the forward and the backward simulator.

An abstract class for simulators is defined below.

```

1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #ifndef SIMULATORSDDPBASE_H
5 #define SIMULATORSDDPBASE_H
6 #include <Eigen/Dense>
7
8 /* \file SimulatorBase.h
9 * \brief Abstract class for simulators for SDDP method
10 * \author Xavier Warin
11 */
12 namespace StOpt
13 {
14     /// \class SimulatorSDDPBase SimulatorSDDPBase.h
15     /// Abstract class for simulators used for SDDP
16     class SimulatorSDDPBase
17     {
18     public :
19
20         /// \brief Get back the number of particles (used in regression part)
21         virtual int getNbSimul() const = 0;
22         /// \brief Get back the number of sample used (simulation at each time
          step , these simulations are independent of the state)
23         virtual int getNbSample() const = 0;

```



```

24  /// \brief Update the simulator for the date
25  virtual void updateDates(const double &p_date) = 0;
26  /// \brief get one simulation
27  /// \param p_isim simulation number
28  /// \return the particle associated to p_isim
29  /// \brief get current Markov state
30  virtual Eigen::VectorXd getOneParticle(const int &p_isim) const = 0;
31  /// \brief get current Markov state
32  virtual Eigen::MatrixXd getParticles() const = 0;
33  /// \brief Reset the simulator (to use it again for another SDDP sweep)
34  virtual void resetTime() = 0;
35  /// \brief in simulation part of SDDP reset time and reinitialize
    uncertainties
36  /// \param p_nbSimul Number of simulations to update
37  /// \param p_nbSample Number of sample to update
38  virtual void updateSimulationNumberAndResetTime(const int &p_nbSimul ,
    const int &p_nbSample) = 0;
39 };
40 }
41 #endif /* SIMULATOR_SDDP_BASE_H */

```

9.3.2 Set of parameters

We have seen above that providing a set of parameters in the initialization phase was necessary (convergence criterion, initial state vector, ...). They depend on each problem and are based on technical skills. Suggestion to the intention of the user : provide a *main* where all these parameters are defined and where the sequence of computations to achieve is described. Basic class `backwardForwardSDDP` should be called.

9.3.3 The black box

The algorithms described above are applied. As said before the user controls the implementation of the business side of the problem (transition problem). But in the library a few things are managed automatically and the user has to be aware of :

- The **Parallelization** during the problem resolution is managed automatically. During compilation, if the compiler detects an MPI (Message Passing Interface) library problem resolution will be achieved in a parallelized manner.
- The **cut management**. All the cuts added at each iteration are currently serialized and stored in an archive initialized by the user. No cuts are pruned. In the future one can consider to work on cuts management [31].
- A **double stopping criterion** is barely used by the library : a convergence test and a maximal number of iterations. If one of the two criteria goes over the thresholds defined by the user resolution stops automatically. Once again further work could be considered on that topic.

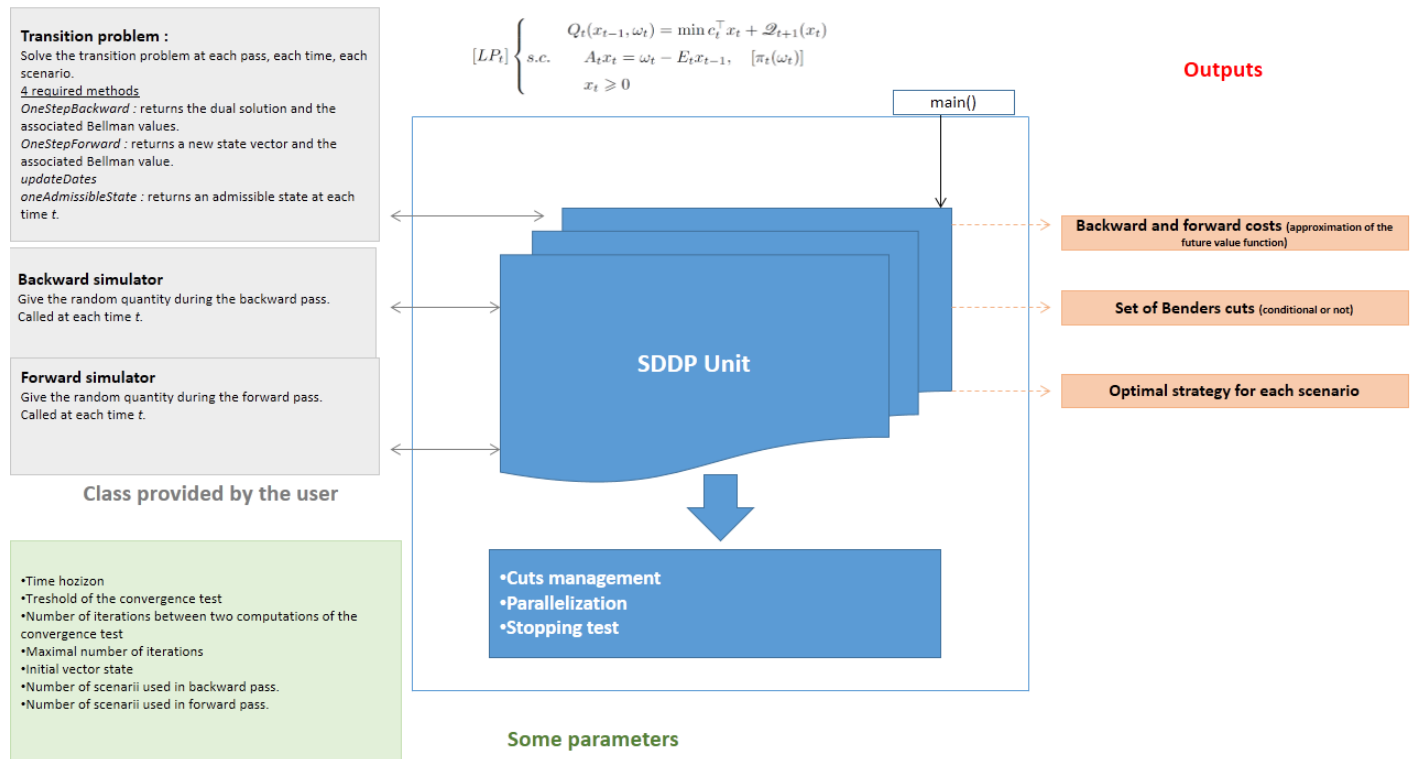


Figure 9.1: Current architecture of the generic SDDP unit

9.3.4 Outputs

The outputs of the SDDP library are not currently clearly defined. Thus during the resolution of a SDDP problem only the number of iterations, the evolution of the backward and forward costs and of the convergence criterion are logged.

Yet while iterating backward and forward pass the value of the Bellman functions and the related Benders cuts, the different states visited during the forward pass and the costs evolution are stored at each time of the time horizon. These information are helpful for the users and easy to catch.

But the outputs should be suitable for the operational requirements. That is a point that should be adjusted after a first testing phase.

9.4 Python API

A high level Python mapping is also available in the SDDP part. The backward-forward C++ function is exposed in Python by the SDDP module “StOptSDDP”.

```
1 import StOptSDDP
2 dir(StOptSDDP)
```

that should give

```
['OptimizerSDDPBase', 'SDDPFinalCut', 'SimulatorSDDPBase', __doc__, __file__, __name__, __package__, 'backwardForwardSDDP']
```

The “backwardForwardSDDP” realize the forward backard SDDP sweep giving a SDDP optimizer and a SDDP uncertainty simulator. The initial final cuts for the last time steps are provided by the “SDDPFinalCut” object.

To realize the mapping of SDDP optimizers and simulators written in C++ it is necessary to create a Boost Python wrapper. In order to expose the C++ optimizer class “OptimizeDemandSDDP” used in the test case “testDemandSDDP.cpp”, the following wrapper can be found in

“StOpt/test/c++/python/BoostPythonSDDPOptimizers.cpp”

```
1 // Copyright (C) 2016 EDF
2 // All Rights Reserved
3 // This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 #include <boost/python.hpp>
5 #include "StOpt/python/BoostToStdSharedPtr.h"
6 #include "StOpt/core/grids/OneDimRegularSpaceGrid.h"
7 #include "StOpt/core/grids/OneDimData.h"
8 #include "StOpt/sddp/OptimizerSDDPBase.h"
9 #include "test/c++/tools/sddp/OptimizeDemandSDDP.h"
10 #include "test/c++/python/FutureCurveWrap.h"
11 #include "test/c++/tools/simulators/SimulatorGaussianSDDP.h"
12
13
14 using namespace boost::python;
15
16 /** \file BoostPythonSDDPOptimizers.cpp
17 * \brief permits to map Optimizers for SDDP
18 * \author Xavier Warin
19 */
20
21 #ifdef _DEBUG
22 #undef _DEBUG
23 #include <Python.h>
24 #define _DEBUG
25 #else
26 #include <Python.h>
27 #endif
28
29
30 /// \wrapper for Optimizer for demand test case in SDDP
31 class OptimizeDemandSDDPWrap : public OptimizeDemandSDDP<
  SimulatorGaussianSDDP>
32 {
33 public :
34
35     /// \brief Constructor
36     /// \param p_sigD volatility for demand
37     /// \param p_kappaD AR coefficient for demand
38     /// \param p_timeDAverage average demand
```

```

39  /// \param p_spot          Spot price
40  /// \param p_simulatorBackward backward simulator
41  /// \param p_simulatorForward Forward simulator
42  OptimizeDemandSDDPWrap(const double &p_sigD, const double &p_kappaD,
43                          const FutureCurve &p_timeDAverage,
44                          const double &p_spot,
45                          const boost::shared_ptr<SimulatorGaussianSDDP> &
46                          p_simulatorBackward,
47                          const boost::shared_ptr<SimulatorGaussianSDDP> &
48                          p_simulatorForward):
49  OptimizeDemandSDDP(p_sigD, p_kappaD,
50                      std::make_shared< StOpt::OneDimData< StOpt::
51                      OneDimRegularSpaceGrid, double> >(static_cast<
52                      StOpt::OneDimData< StOpt::
53                      OneDimRegularSpaceGrid, double> >(
54                      p_timeDAverage)),
55                      p_spot, make_shared_ptr<SimulatorGaussianSDDP>(
56                      p_simulatorBackward), make_shared_ptr<
57                      SimulatorGaussianSDDP>(p_simulatorForward)) { }
58 };
59
60 // MSVC 2015 BUG
61 #if (_MSC_VER == 1900)
62 namespace boost
63 {
64 template <
65 OptimizeDemandSDDPWrap const volatile *get_pointer< class
66 OptimizeDemandSDDPWrap const volatile >(
67     class OptimizeDemandSDDPWrap const volatile *c)
68 {
69     return c;
70 }
71 }
72 #endif
73
74 BOOST_PYTHON_MODULE(SDDPOptimizers)
75 {
76
77     // map optimizer for demand test case
78     class_ <OptimizeDemandSDDPWrap, std::shared_ptr<OptimizeDemandSDDPWrap>,
79           bases<StOpt::OptimizerSDDPBase> >("OptimizeDemandSDDP",
80           init< const double &, const double &, const FutureCurve &,
81           const double &,
82           const boost::shared_ptr<SimulatorGaussianSDDP> &,
83           const boost::shared_ptr<SimulatorGaussianSDDP> &>());
84 }

```

The wrapper used to expose the SDDP simulator is given in
“StOpt/test/c++/python/BoostPythonSimulators.cpp”

Then it is possible to use the mapping to write a Python version of “testDemandSDDP.cpp”

```

1 # Copyright (C) 2016 EDF
2 # All Rights Reserved

```

```

3 # This code is published under the GNU Lesser General Public License (GNU
  LGPL)
4 import StOptGrids
5 import StOptSDDP
6 import StOptGlobal
7 import Simulators as sim
8 import SDDPOptimizers as opt
9 import numpy as NP
10 import unittest
11 import math
12 import imp
13
14 # unittest equivalent of testDemandSDDP : here MPI version
15 #####
16 def testDemandSDDPFunc(p_sigD , p_sampleOptim , p_sampleCheckSimul):
17
18     maturity = 40
19     nstep = 40;
20
21     # optimizer parameters
22     kappaD = 0.2; # mean reverting coef of demand
23     spot = 3 ; # spot price
24
25     # define a a time grid
26     timeGrid = StOptGrids.OneDimRegularSpaceGrid(0. , maturity / nstep ,
27         nstep)
28
29     # periodicity factor
30     iPeriod = 52;
31     # average demande values
32     demValues = []
33
34     for i in list(range(nstep + 1)) :
35         demValues.append(2. + 0.4 * math.cos((math.pi * i * iPeriod) /
36             nstep))
37
38     # define average demand
39     demGrid =sim.FutureCurve(timeGrid , demValues)
40
41     initialState = demGrid.get(0.)*NP.ones(1)
42
43     finCut = StOptSDDP.SDDPFinalCut(NP.zeros((2,1)))
44
45     # here cuts are not conditional to an uncertainty
46     nbMesh = NP.array([],NP.int32)
47     sampleSimul = 1
48     nbUncertainties = 1;
49
50     # backward simulator
51     backwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties ,
52         p_sampleOptim,0)
53     # forward simulator
54     forwardSimulator = sim.SimulatorGaussianSDDP(nbUncertainties ,

```

```

    sampleSimul,1)
53
54     # Create the optimizer
55     optimizer = opt.OptimizeDemandSDDP(p_sigD, kappaD, demGrid, spot,
        backwardSimulator, forwardSimulator)
56
57     # optimisation dates
58     dates = NP.linspace( 0., maturity, nstep + 1);
59
60     # names for archive
61     nameRegressor = "RegressorDemand";
62     nameCut = "CutDemand";
63     nameVisitedStates = "VisitedStateDemand";
64
65     # precision parameter
66     nIterMax = 40
67     accuracyClose = 1.
68     accuracy = accuracyClose / 100.
69     nstepIterations = 4; # check for convergence between nstepIterations
        step
70
71     values = StOptSDDP.backwardForwardSDDP(optimizer, p_sampleCheckSimul
        , initialState, finCut, dates, nbMesh, nameRegressor, nameCut,
        nameVisitedStates, nIterMax,
72
        accuracy, nstepIterations
        );
73
74     return values
75
76 class testDemandSDDP(unittest.TestCase):
77
78     def testDemandSDDP1D(self):
79         try:
80             imp.find_module('mpi4py')
81             found = True
82         except:
83             print("Not parallel module found ")
84             found = False
85
86         if found :
87             from mpi4py import MPI
88             world = MPI.COMM_WORLD
89
90             sigD = 0.6 ;
91             sampleOptim = 500;
92             sampleCheckSimul = 500;
93
94             values = testDemandSDDPFunc(sigD, sampleOptim, sampleCheckSimul)
95
96             if (world.rank==0):
97                 print values
98
99 if __name__ == '__main__':
100     unittest.main()

```

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