Long Brownian bridges in hyperbolic spaces converge to Brownian trees

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Abstract

We show that the range of a long Brownian bridge in the hyperbolic space converges after suitable renormalisation to the Brownian continuum random tree. This result is a relatively elementary consequence of

- A theorem by Bougerol and Jeulin, stating that the rescaled radial process converges to the normalized Brownian excursion,
- A property of invariance under re-rooting,
- The hyperbolicity of the ambient space in the sense of Gromov.

A similar result is obtained for the rescaled infinite Brownian loop in hyperbolic space.

1 Introduction

1.1 Brownian bridges in hyperbolic space

This work deals with geometric properties of the range of long Brownian bridges in hyperbolic space. For $d \geq 2$, let $H = H_d$ be the d-dimensional hyperbolic space, and let o be a distinguished point taken as origin. For every $T > 0$ we let b_T be the Brownian bridge on H from the origin o and with duration T. Heuristically, this can be seen as Brownian motion $(B(t), t \geq 0)$ (the diffusion on H whose generator is half the Laplace-Beltrami operator) in restriction to the interval $[0, T]$ and conditioned on the event $\{B(0) = B(T) = o\}$.

There are several natural (and equivalent) ways to make sense of this singular conditioning. Let $p_t(x, y)$ be the transition densities for standard Brownian motion on H, with

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respect to the standard volume measure m on H. Then the finite-dimensional distributions of b_T are given, for $0 < t_1 < \ldots < t_k < T$ and $x_1, \ldots, x_k \in H$, by the formula

$$\frac{\mathbb{P}(b_T(t_1) \in \mathrm{d}x_1, \ldots, b_T(t_k) \in \mathrm{d}x_k)}{m(\mathrm{d}x_1) \ldots m(\mathrm{d}x_k)} = \frac{p_{t_1}(o, x_1) p_{t_2 - t_1}(x_1, x_2) \ldots p_{t_k - t_{k-1}}(x_{k-1}, x_k) p_{T - t_k}(x_k, o)}{p_T(o, o)} .$$

Let also $\rho_T(t) = d_H(o, b_T(t)), 0 \leq t \leq T$ be the "radial part" of b_T.

We let $\mathbf{e}$ be a normalized Brownian excursion [27, Chapter XII.4]. Using analytical expressions for the heat kernel in H and stochastic differential equations techniques, Bougerol and Jeulin [8] proved the following limit theorem.

Theorem 1 ([8]). *One has the convergence in distribution*

$$\left(\frac{\rho_T(Tt)}{\sqrt{T}}, 0 \leq t \leq 1 \right) \xrightarrow[T \to \infty]{(d)} (\mathbf{e}_t, 0 \leq t \leq 1) ,$$

for the uniform topology on the space $\mathcal{C}([0,1], \mathbb{R})$ of continuous functions on $[0,1]$.

1.2 Main results

Our main result, Theorem 2 below, gives a geometric interpretation of Theorem 1. Recall that the Brownian continuum random tree [2, 4] is a random $\mathbb{R}$-tree coded by the function $\mathbf{e}$. More precisely, setting

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \inf_{s \wedge t \leq u \leq s \vee t} \mathbf{e}_u , \qquad s, t \in [0,1]$$

defines a pseudo-distance on $[0,1]$, and we let $(\mathcal{T}_{\mathbf{e}} = [0,1]/\{d_{\mathbf{e}} = 0\}, d_{\mathbf{e}})$ be the quotient metric space naturally associated with it. This space is called the Brownian continuum random tree. We naturally distinguish the point $o_{\mathbf{e}} = p_{\mathbf{e}}(0)$, where $p_{\mathbf{e}}$ is the canonical projection, and will usually write $\mathcal{T}_{\mathbf{e}}$ instead of $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}, o_{\mathbf{e}})$. This random metric space (or more precisely its isometry class) appears as the universal scaling limit of many tree-like random objects that naturally appear in combinatorics and probability, see for instance [20] for a survey, and [11, 12, 19, 24, 25, 26, 28, 30] for some recent developments on the topic. Here we show that the CRT also appears naturally in this more geometric context.

An important property of $\mathcal{T}_{\mathbf{e}}$ is the following strong re-rooting invariance property, first noted in [23, Proposition 4.9]. For every $s, t \in [0,1]$, if we let $s \oplus t = s + t$ if $s + t < 1$ and $s \oplus t = s + t - 1$ if $s + t \geq 1$, then for every $t \in [0,1]$

$$(d_{\mathbf{e}}(s \oplus t, s' \oplus t))_{s,s' \in [0,1]} \stackrel{(d)}{=} d_{\mathbf{e}} . \tag{1}$$

This rougly says that $\mathcal{T}_{\mathbf{e}}$ pointed at $p_{\mathbf{e}}(t)$ rather than $o_{\mathbf{e}}$ has the same distribution as $\mathcal{T}_{\mathbf{e}}$.

Let $\mathcal{R}_T = \{b_T(t) : 0 \leq t \leq T\} \subset H$ be the range of the Brownian loop b_T. We view it as a pointed metric space by endowing it with the restriction of the hyperbolic metric

$d_H/\sqrt{T}$ renormalized by $\sqrt{T}$, and by pointing it at o. As such, like $\mathcal{T}_{\mathbf{e}}$, it can be seen as a random element of the space $\mathbb{M}$ of isometry classes of pointed compact metric spaces (where two pointed metric spaces $(X, d_X, x), (Y, d_Y, y)$ are called isometric if there exists an isometry $\phi : X \to Y$ from X onto Y such that $\phi(x) = y$). This space is equipped with the pointed Gromov-Hausdorff distance [9, Chapter I.5].

Theorem 2. *One has the following convergence in distribution in* $\mathbb{M}$*:*

$$\left(\mathcal{R}_T, \frac{d_H}{\sqrt{T}}, o\right) \xrightarrow[T\to\infty]{(d)} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}, o_{\mathbf{e}})\,.$$

In fact, we will show that this convergence holds jointly with that of Theorem 1, meaning that

$$\left(\left(\frac{\rho_T(Tt)}{\sqrt{T}}, 0 \le t \le 1\right), \left(\mathcal{R}_T, \frac{d_H}{\sqrt{T}}, o\right)\right) \xrightarrow[T\to\infty]{(d)} (\mathbf{e}, \mathcal{T}_{\mathbf{e}}) \tag{2}$$

in distribution in the product topology of $\mathcal{C}([0,1],\mathbb{R}) \times \mathbb{M}$.

A couple of comments on Theorem 2 are in order. First, it is relatively natural to see a tree structure arise in this context, due to the fact that hyperbolic spaces can be seen as "fattened" trees. On the other hand, one should not think that the limiting tree naturally lives in the hyperbolic space H itself. Indeed, due to the renormalization by $\sqrt{T}$ of the distance d_H, one should rather imagine that the limiting CRT is a random subset in some asymptotic cone of H. It is well-known that H does not admit an asymptotic cone in a conventional (pointed Gromov-Hausdorff) sense, but that a substitute for this notion can be made sense of using ultralimits. A related striking property, already present in Theorem 1, is that the renormalization does not involve scaling constant depending on the dimension d of H, so indeed everything happens as if large hyperbolic Brownian bridges were living in the asymptotic cone (in the generalized sense), which does not depend on the dimension.

Note that such a generalized asymptotic cone is a very ramified $\mathbb{R}$-tree (every point disconnects the tree into uncountably many connected components) which is in a sense much too large to consider random subsets on a mathematically sound basis, nevertheless, it is consistent with the idea that the (minuscule) sub-region of this cone that is explored by a very large loop should be a random $\mathbb{R}$-tree. Finally, given Theorem 1, it is very natural to guess that this random tree should be the Brownian continuum random tree.

In section 5 below we will also prove a result related to Theorem 2 dealing with the infinite Brownian loop in hyperbolic space, which is the "local limit" (with no rescaling involved) of b_T as $T \to \infty$. This is a random path taking values in H, and we will show that its range, equipped with the rescaled hyperbolic distance $a\,d_H$ for $a > 0$, converges as $a \to 0$ to a non-compact version of the continuum random tree, the so called self-similar CRT [3]. We refer the reader to section 5 for precise statements and continue our discussion of Theorem 2.

1.3 Motivation, methods and open questions

We will show that Theorem 2 is a relatively elementary consequence of

- Theorem 1,
- the hyperbolicity of H in the sense of Gromov
- a natural "re-rooting" invariance of Brownian loops under cyclic shifts.

The use of functional limit theorems as Theorem 1 and of re-rooting invariance properties are powerful tools in the study of random metric spaces, as exemplified by their use in the context of random maps. Our proofs borrow ideas of [21, 22] in particular.

To illustrate the robustness of the method, we will avoid as much as possible the use of specific properties of the hyperbolic spaces H, besides the fact that they satisfy the above three properties. In the rest of the paper, we will denote by δ a constant such that H is δ-hyperbolic [9]. For instance, Bougerol and Jeulin [8] proved Theorem 1 in the more general setting of non-compact rank 1 symmetric spaces instead of the hyperbolic space, and our proof applies almost *verbatim* to this situation, replacing hyperbolic isometries used in the re-rooting Lemma 3 below by a consistent choice of isometries of the symmetric space.

In a slightly different direction, Bougerol and Jeulin [8] also proved (and also using explicit representations of the probability densities) that the simple random walk $(S_n, n \geq 0)$ on a $k \geq 3$-regular tree $\mathbb{T}_k$ and conditioned to return to the origin o converges after rescaling to the Brownian excursion: if $d_{\mathbb{T}_k}$ denotes distance in the tree, then

$$\left(\frac{d_{\mathbb{T}_k}(o, S_{\lfloor 2nt \rfloor})}{\sqrt{2n}}\right) \qquad \text{given} \qquad \{S_0 = S_{2n} = o\} \xrightarrow[n\to\infty]{(d)} \mathbf{e} \tag{3}$$

in distribution in the Skorokhod space $\mathcal{D}([0,1], \mathbb{R})$ of càdlàg functions (the use of the Skorokhod space could be avoided by taking a continuous interpolation of the distance process above between integer times). Our methods allow to obtain that the range

$$\left(\{S_i, 0 \leq i \leq 2n\}, \frac{d_{\mathbb{T}_k}}{\sqrt{2n}}, o\right) \qquad \text{given} \qquad \{S_0 = S_{2n} = o\} \xrightarrow[n\to\infty]{(d)} \mathcal{T}_{\mathbf{e}}$$

in distribution in $\mathbb{M}$. This situation is in fact simpler since in this case the hyperbolic constant is $\delta = 0$ (so that we are already dealing with a tree metric). One should only adapt the re-rooting invariance Lemma 3 below by replacing the isometries of H with those of $\mathbb{T}_k$, making use of the fact that it is a transitive graph. We leave details to the reader.

As we were finishing this work, we became aware of the very recent PhD thesis of Andrew Stewart [29], who provides another proof of the result we just mentioned on the range of random walks on $\mathbb{T}_k$. Stewart's methods, based on the self-similar structure of the continuum random tree, are independent of ours and do not rely on Bougerol and Jeulin's result. It is indeed stressed in Appendix B of [29] that Bougerol and Jeulin's result can be used to obtain the convergence of the range, but the sketch of proof presented there seems quite different from our approach. There is also some overlap between conjectures made in [29] and some of the comments below.

Note that the recent work by Aïdékon and de Raphélis [1], proving convergence of the range of a null-recurrent biased walk on a infinite supercritical Galton-Watson tree to a Brownian forest, is in a similar spirit to the above discussion, but where the underlying (random) space is only supposed to be "statistically homogeneous". It would be interesting to see if the methods of [1] can be used to extend (3) with $\mathbb{T}_k$ replaced by a supercritical Galton-Watson tree. In a slightly different context, but in a very similar spirit, we also mention the work of Duquesne [15] on the range of barely transient random walks on regular trees.

In fact, we expect Theorem 2 to hold in a much wider context, and that the emergence of the Brownian continuum random tree as a limit of large Brownian loops is a signature of non-compact, negatively curved spaces that are "close to homogeneous". The intuition behind this result comes from the recent advances [17, 16] on local limit theorems for transition probabilities in hyperbolic groups. Namely, Gouëzel's results in [16] imply in particular that if G is a nonelementary Gromov-hyperbolic group, and if S is a finite symmetric subset of generators of G, then the number C_n of closed paths of length n in the Cayley graph of G associated with S is asymptotically

$$C_n \sim \alpha\, \beta^n\, n^{-3/2}$$

(modulo the usual periodicity caveat) for some $\alpha = \alpha(G,S) \in (0,\infty)$ and $\beta = \beta(G,S) \in (1,\infty)$. Note that, contrary to α, β which depend on G, S the exponent $-3/2$ is universal. In enumerative combinatorics, this kind of asymptotics is a distinctive signature of tree structures [14]. This is a first hint that a walk in G conditioned to come back at its starting point after n steps might approximate a tree in some sense. In fact, this general idea is present in the approach of [16].

However, besides these rough ideas, it is a challenge to prove a result such as Theorem 1 (or the weaker Theorem 2) in contexts where strong analytical or combinatorial tools, such as those used in [8], are not available.

The proof of Theorem 2 will be shown in Section 4 below, after two preliminary sections respectively on the re-rooting invariance and on a key tightness estimate. Finally, Section 5 is dedicated to study of the renormalized infinite Brownian loop.

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2 Invariance under re-rooting

If $\phi : H \to H$ is an isometry, then $p_t(\phi(x), \phi(y)) = p_t(x, y)$, as follows from invariance properties of the heat kernel on hyperbolic spaces ([6, 18]). From there, a natural property of invariance under cyclic shifts holds. For $x \in H, x \neq o$, we let $\phi_x : H \to H$ be the unique hyperbolic isometry sending x to o, and let ϕ_o be the identity map.

Lemma 3. *Fix $T > 0$ and $t \in [0, T]$. Then the processes*

$$b_T \,, \qquad \phi_{b_T(t)}(b_T(\cdot + t \mod T))\,,$$

have same distribution. Here, by convention, we let $s + t \mod T$ be the unique representative in $[0, T)$.

Proof. For convenience, let $X_s := \phi_{b_T(t)}(b_T(s + t \mod T))$ for $s \in [0, T]$. Let $s, r \in (0, T)$ with $s < r$ and $F : H^2 \to \mathbb{R}_+$ be a measurable function. We will show that

$$\mathbb{E}\left(F(X_s, X_r)\right) = \mathbb{E}\left(F(b_T(s), b_T(r))\right)$$

We prove it in the case where $s < T - t < r < T$, the situations where $s < r < T - t$ and $T - t < s < r$ are easier and left to the reader. Observe that $0 < t + r - T < t < t + s < T$, so by the finite-dimensional distribution of b_T, we have

$$\begin{aligned}
&\mathbb{E}\left(F(X_s, X_r)\right) = \mathbb{E}\left[F\Big(\phi_{b_T(t)}(b_T(s+t)), \phi_{b_T(t)}(b_T(t + r - T))\Big)\right] \\
&= \int_H m(\mathrm{d}x_1) \int_H m(\mathrm{d}x_2) \int_H m(\mathrm{d}x_3) \\
&\qquad \times \frac{p_{t+r-T}(o, x_1) p_{T-r}(x_1, x_2) p_s(x_2, x_3) p_{T-t-s}(x_3, o)}{p_T(o, o)} F(\phi_{x_2}(x_3), \phi_{x_2}(x_1))
\end{aligned}$$

Since p_t is invariant by the isometry ϕ_{x_2}, we deduce that

$$\begin{aligned}
&\mathbb{E}\left(F(X_s, X_r)\right) = \int_H m(\mathrm{d}x_2) \int_{H^2} m(\mathrm{d}x_1) m(\mathrm{d}x_3) \\
&\times \frac{p_{t+r-T}(\phi_{x_2}(o), \phi_{x_2}(x_1)) p_{T-r}(\phi_{x_2}(x_1), o) p_s(o, \phi_{x_2}(x_3)) p_{T-t-s}(\phi_{x_2}(x_3), \phi_{x_2}(o))}{p_T(o, o)} F(\phi_{x_2}(x_3), \phi_{x_2}(x_1))
\end{aligned}$$

Let us write $y_1 = \phi_{x_2}(x_3)$,$y_2 = \phi_{x_2}(x_1)$, and $x_2' = \phi_{x_2}(o)$. Note that in the Poincaré ball model of H with origin $o = 0 \in \mathbb{R}^d$, x_2' is simply the point $-x_2$, so that clearly $\int_H f(x_2') m(\mathrm{d}x_2) = \int_H f(x_2) m(\mathrm{d}x_2)$ for every non-negative measurable f. It follows that

$$\begin{aligned}
&\mathbb{E}\left(F(X_s, X_r)\right) \\
&= \int_H m(\mathrm{d}x_2) \int_{H^2} m(\mathrm{d}y_1) m(\mathrm{d}y_2) \frac{p_{t+r-T}(x_2', y_2) p_{T-r}(y_2, o) p_s(o, y_1) p_{T-t-s}(y_1, x_2')}{p_T(o, o)} F(y_1, y_2) \\
&= \int_{H^2} m(\mathrm{d}y_1) m(\mathrm{d}y_2) \frac{p_s(o, y_1) p_{T-r}(y_2, o)}{p_T(o, o)} F(y_1, y_2) \int_H m(\mathrm{d}x_2) p_{T-t-s}(y_1, x_2') p_{t+r-T}(x_2', y_2) \\
&\int_{H^2} m(\mathrm{d}y_1) m(\mathrm{d}y_2) \frac{p_s(o, y_1) p_{T-r}(y_2, o)}{p_T(o, o)} F(y_1, y_2) \int_H m(\mathrm{d}x_2) p_{T-t-s}(y_1, x_2) p_{t+r-T}(x_2, y_2) \\
&= \int_{H^2} m(\mathrm{d}y_1) m(\mathrm{d}y_2) \frac{p_s(o, y_1) p_{T-r}(y_2, o)}{p_T(o, o)} F(y_1, y_2) p_{r-s}(y_1, y_2) \\
&= \mathbb{E}\left(F(b_T(s), b_T(r))\right)
\end{aligned}$$

The proof of the equality of all finite-dimensional marginals is similar to this case, with longer formulas, and we leave it as an exercise to the reader. This concludes Lemma 3 because of the continuity of b. ☐

Note that it does not really matter which isometry sending $b_T(t)$ to o we choose (we could also have chosen the unique parabolic isometry sending x to o) but of course one should perform this choice in a consistent way in such a way that the image of m under $x \mapsto \phi_x(o)$ is m (note that this image is clearly invariant under the action of isometries of H so is a constant multiple cm of m, and noting that $x \mapsto \phi_x(o)$ is a measurable involution from H onto H, this entails that $c = 1$.)

3 Tightness estimate

For $T, \eta > 0$, we let $\mathcal{N}(T, \eta)$ be the minimal number of balls of radius η (with respect to the metric $d_H/\sqrt{T}$) necessary to cover the range $\mathcal{R}_T$:

$$\mathcal{N}(T, \eta) = \inf \left\{ N \geq 1 : \exists x_1, \ldots, x_N \in H, \mathcal{R}_T \subset \bigcup_{k=1}^{N} B_{d_H}(x_k, \eta\sqrt{T}) \right\} .$$

Lemma 4. *It holds that for every $N \geq 2$ and $\eta > 0$,*

$$\limsup_{T\to\infty} \mathbb{P}\left(\mathcal{R}_T \not\subset \bigcup_{i=0}^{N-1} B_{d_H}(b_T(Ti/N), \eta\sqrt{T}) \right) \leq \frac{12}{\eta}\sqrt{\frac{N}{\pi}}\, e^{-\eta^2(N-1)/18} ,$$

and in particular one has $\lim_{N\to\infty} \limsup_{T\to\infty} \mathbb{P}(\mathcal{N}(T, \eta) > N) = 0$.

Proof. By the union bound and the re-rooting lemma 3,

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{R}_T \not\subset \bigcup_{i=0}^{N-1} B_{d_H}(b_T(Ti/N), \eta\sqrt{T}) \right) \\
&\leq \sum_{i=0}^{N-1} \mathbb{P}\left(\sup\left\{ d_H(b_T(iT/N), b_T((s + i/N)T)) : s \in [0, 1/N] \right\} \geq \eta\sqrt{T} \right) \\
&= N\, \mathbb{P}\left(\sup\left\{ d_H(o, b_T(Ts)) : s \in [0, 1/N] \right\} \geq \eta\sqrt{T} \right) .
\end{aligned}$$

Theorem 1 implies that

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup\left\{ d_H(o, b_T(Ts)) : s \in [0, 1/N] \right\} \geq \eta\sqrt{T} \right) \leq \mathbb{P}(\sup_{[0,1/N]} \mathbf{e} > \eta) .$$

To bound this probability, one can use for instance the fact (see Theorem XII.4.2 and Exercise XI.3.6 in [27]) that $((1 - s)X_{s/(1-s)}, 0 \leq s \leq 1)$ has same distribution as $\mathbf{e}$ if X is a 3-dimensional Bessel process. This shows that $\sup_{[0,1/N]} \mathbf{e}$ is stochastically dominated by $\sup_{[0,1/(N-1)]} X$. Then one can use the fact that X has same distribution as the Euclidean norm of a standard 3-dimensional Brownian motion. Using this, we easily get

$$\mathbb{P}\left(\sup_{[0,1/(N-1)]} X > \eta \right) \leq 6\, \mathbb{P}\left(\sup\{W_s : 0 \leq s \leq 1/(N - 1)\} > \eta/3 \right) ,$$

where $(W_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}$. Using the fact that $\sup\{W_s : 0 \leq s \leq t\}$ has same distribution as $|W_t|$ and the estimate $\mathbb{P}(|W_1| \geq x) \leq 2\exp(-x^2/2)/x\sqrt{2\pi}$, we get the wanted bound. We conclude since clearly $\mathcal{N}(T, \eta) > N$ implies that $\mathcal{R}_T \not\subset \bigcup_{i=0}^{N-1} B_{d_H}(b_T(Ti/N), \eta\sqrt{T})$. □

A crucial corollary of the tightness estimate (and hyperbolicity) is the fact that the range $\mathcal{R}_T$ cannot avoid large portions of geodesics between the points it visits. For $x, y \in H$, let $[x, y]$ be the (hyperbolic) geodesic segment between x and y. For $0 \leq s \leq t \leq T$ we let $\mathcal{R}_T(s, t) = \{b_T(u) : s \leq u \leq t\}$, and for $r > 0$ we define the event

$$\Lambda_T(r) = \left\{ \exists\, s \leq t \in [0, T] : \sup_{y \in [b_T(s), b_T(t)]} d_H(y, \mathcal{R}_T(s, t)) \geq r \right\}.$$

Lemma 5. *For every $\eta > 0$, one has $\mathbb{P}(\Lambda_T(\eta\sqrt{T})) \to 0$ as $T \to \infty$.*

Proof. A standard property of δ-Gromov-hyperbolic spaces (see Proposition III.1.6 in [9]) is that if c is a continuous path that avoids a ball $B(z, r)$ around some vertex z on a geodesic between the endpoints of c, then c must be of length at least $2^{(r-1)/\delta}$. This implies the property that if moreover we assume that c avoids the larger ball $B(z, 2r)$, then its image cannot be covered by less than $2^{(r-1)/\delta}/(2r)$ balls of radius r: otherwise, by possibly modifying the path c by a piecewise geodesic path inside each ball of a cover of the image of c by balls of radius r, we would find a path that avoids $B(z, r)$ but is of length at most $2r \times 2^{(r-1)/\delta}/(2r)$, a contradiction.

On the event $\Lambda_T(2\eta\sqrt{T})$, there exist $s < t$ in $[0, 1]$ and $y \in [b_T(Ts), b_T(Tt)]$ such that $d_H(y, \mathcal{R}_T(s, t)) \geq 2\eta\sqrt{T}$, meaning that the portion of the path of b_T between times s and t avoids $B_{d_H}(y, 2\eta\sqrt{T})$. By the above discussion, this implies that

$$\mathcal{N}(T, \eta) > \frac{2^{(\eta\sqrt{T}-1)/\delta}}{2\eta\sqrt{T}}.$$

Since the latter lower bound diverges for any $\eta > 0$ as $T \to \infty$, we conclude immediately from Lemma 4. □

We now define a continuous random function $d_{(T)}$ on $[0, 1]^2$ by the formula

$$d_{(T)}(s, t) = \frac{d_H(b_T(Ts), b_T(Tt))}{\sqrt{T}}, \qquad 0 \leq s, t \leq 1.$$

Lemma 6. *The family of laws of $d_{(T)}$, for $T \geq 1$, is relatively compact for the weak topology on probability measures on $\mathcal{C}([0, 1]^2, \mathbb{R})$.*

Proof. Note that for every $s, s', t, t' \in [0, 1]$, one has, by the triangle inequality,

$$|d_{(T)}(s, t) - d_{(T)}(s', t')| \leq d_{(T)}(s, s') + d_{(T)}(t, t').$$

This shows that the modulus of continuity of $d_{(T)}$ is bounded as follows: for $\alpha > 0$,

$$\sup_{\substack{|s-s'|\leq\alpha \\ |t-t'|\leq\alpha}} |d_{(T)}(s,t) - d_{(T)}(s',t')| \leq 2 \sup_{|s-s'|\leq\alpha} d_{(T)}(s,s') \,.$$

Now, for every $\eta > 0$, we obtain

$$\mathbb{P}\left(\sup_{\substack{|s-s'|\leq\alpha \\ |t-t'|\leq\alpha}} |d_{(T)}(s,t) - d_{(T)}(s',t')| \geq 8\eta \right) \leq \mathbb{P}\left(\sup_{|s-s'|\leq\alpha} d_{(T)}(s,s') \geq 4\eta \right) . \tag{4}$$

By δ-hyperbolicity, for every $a, b, c \in H$, it holds that

$$2d_H(a,[b,c]) + d_H(b,c) \leq d_H(a,b) + d_H(a,c) + 4\delta \,,$$

see (8.4) in [10]. We apply this to $a = o, b = b_T(Ts), c = b_T(Ts')$ for some $s \leq s'$, so that, if we let $y \in [b_T(Ts), b_T(Ts')]$ be such that $d_H(o,y) = d_H(o,[b_T(Ts), b_T(Ts')])$,

$$2\, d_H(o,y) + d_H(b_T(Ts), b_T(Ts')) \leq d_H(o, b_T(Ts)) + d_H(o, b_T(Ts')) + 4\delta \,.$$

Outside the event $\Lambda_T(\eta\sqrt{T})$, we can find $u \in [s,s']$ such that $d_H(b_T(Tu), y) \leq \eta\sqrt{T}$, so that

$$\begin{aligned} d_H(b_T(Ts), b_T(Ts')) &\leq \rho_T(Ts) + \rho_T(Ts') - 2\rho_T(Tu) + 4\delta + 2\eta\sqrt{T} \\ &\leq \rho_T(Ts) + \rho_T(Ts') - 2 \inf_{v\in[s,s']} \rho_T(Tv) + 4\delta + 2\eta\sqrt{T} \,, \end{aligned}$$

which, by letting $\rho_{(T)} = \rho_T(T\cdot)/\sqrt{T}$, can be rewritten as

$$d_{(T)}(s,s') \leq \rho_{(T)}(s) + \rho_{(T)}(s') - 2 \inf_{v\in[s,s']} \rho_{(T)}(v) + \frac{4\delta}{\sqrt{T}} + 2\eta \,. \tag{5}$$

Hence, we have proved that outside $\Lambda_T(\eta\sqrt{T})$, we have

$$\sup_{|s-s'|\leq\alpha} d_{(T)}(s,s') \leq 2\omega(\rho_{(T)},\alpha) + \frac{4\delta}{\sqrt{T}} + 2\eta \,,$$

where $\omega(f,\cdot)$ denotes the modulus of continuity of the function f. Therefore,

$$\mathbb{P}\left(\sup_{|s-s'|\leq\alpha} d_{(T)}(s,s') \geq 4\eta \right) \leq \mathbb{P}(\Lambda_T(\eta\sqrt{T})) + \mathbb{P}\left(\omega(\rho_{(T)},\alpha) \geq \eta - \frac{2\delta}{\sqrt{T}} \right)$$

By (4), Theorem 1 and Lemma 5, we conclude that

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{\substack{|s-s'|\leq\alpha \\ |t-t'|\leq\alpha}} |d_{(T)}(s,t) - d_{(T)}(s',t')| \geq 8\eta \right) \leq \mathbb{P}\left(\omega(\mathbf{e},\alpha) \geq \eta\right) \,,$$

and this converges to 0 as $\alpha \to 0$ for any fixed value of η. Together with the fact that $d_{(T)}(0,0) = 0$, this allows to conclude by standard results [7]. □

4 Convergence

In this section, we finish the proof of Theorem 2.

Lemma 7. *It holds that*

$$\left(\rho_{(T)}, d_{(T)}\right) \xrightarrow[T\to\infty]{(d)} (\mathbf{e}, d_{\mathbf{e}}) \tag{6}$$

in distribution in $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1]^2,\mathbb{R})$.

Proof. By Prokhorov's Theorem, based on Theorem 1 and Lemma 6, the laws of the random variables in the left-hand side of (6) form a relatively compact family of probability measures on $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1]^2,\mathbb{R})$. We deduce that for any sequence $T_n \to \infty$, we can extract a subsequence along which the pair of random variables in (6) converges in distribution towards a certain limiting random variable $(\mathbf{e}, d)$. The slight abuse of notation in denoting the first component by $\mathbf{e}$ is motivated by the fact that its marginal law is that of the normalized Brownian excursion, by Theorem 1. By using Skorokhod's theorem, we may and will assume that the convergence holds in the almost sure sense, which will simplify some of the arguments to come.

To conclude, it suffices to show that $d = d_{\mathbf{e}}$ a.s., since this will characterize uniquely the limiting distribution, hence allowing to obtain the convergence result without having to take extractions. Note that $d(0,s) = \mathbf{e}_s = d_{\mathbf{e}}(0,s)$ for every $s \in [0,1]$ almost surely, since $d_{(T)}(0,s) = \rho_{(T)}(s)$ and by passing to the limit. But the re-rooting Lemma 3 implies that $d_{(T)}(s,t)$ has same distribution as $d_{(T)}(0,t-s)$ for every $s \le t$ in $[0,1]$. By passing to the limit, we thus see that $d(s,t)$ has same distribution as $d(0,t-s) = d_{\mathbf{e}}(0,t-s)$. Using the re-rooting invariance of the Brownian continuum random tree (1), we obtain that in turn, this has same distribution as $d_{\mathbf{e}}(s,t)$. On the other hand, taking the limit in (5) (and using Lemma 5) shows that $d(s,t) \le d_{\mathbf{e}}(s,t)$ almost surely. Therefore, equality must hold almost surely, because the expectation of the (nonnegative) difference is 0. □

It is now straightforward to conclude the proof of (2), hence of Theorem 2. Still assuming that the convergence (6) holds almost surely, the set $\{(b_T(sT), p_{\mathbf{e}}(s)) : s \in [0,1]\}$ defines a correspondence [10, Section 7.3.3] between $\mathcal{R}_T$ and $\mathcal{T}_{\mathbf{e}}$ containing $(o, o_{\mathbf{e}})$, and of distortion bounded above by

$$\sup_{s,t\in[0,1]} |d_{(T)}(s,t) - d_{\mathbf{e}}(s,t)| \xrightarrow[T\to\infty]{} 0\,, \qquad \text{a.s.}$$

This shows that the pointed Gromov-Hausdorff distance between $(\mathcal{R}_T, d_H/\sqrt{T}, o)$ and $\mathcal{T}_{\mathbf{e}}$ converges to 0 almost surely, as wanted.

5 The infinite Brownian loop and the self-similar CRT

We now argue that our methods also allow to prove a result related to Theorem 2, which deals with the so-called *infinite Brownian loop*. The latter can be obtained as a local limit of large Brownian loops. Specifically, let us extend the bridge b_T by T-periodicity and view it a random function $(b_T(t), t \in \mathbb{R})$. We equip the space $\mathcal{C}(\mathbb{R}, H)$ with the compact-open topology, so that convergence in this space is equivalent to uniform convergence over compact intervals.

An important result by Anker, Bougerol and Jeulin [5, Theorem 1.2, Proposition 2.6 and Proposition 4.2] implies that in every non-compact symmetric space H, as $T \to \infty$, the Brownian bridge b_T converges in distribution in $\mathcal{C}(\mathbb{R}, H)$ towards a limit b_∞, called the infinite Brownian loop. As before in this paper, we will only focus on the case where H is the hyperbolic space, which corresponds to rank 1 symmetric spaces.

Anker, Bougerol and Jeulin further show the following result. Let $\rho_\infty(t) = d_H(o, b_\infty(t))$ for $t \in \mathbb{R}$. Theorems 1.4, 1.5 and 7.1 (iii) in [5], again in the very special case of rank 1 symmetric spaces, can be stated as follows.

Theorem 8. *Let R, R' be two independent Bessel processes of dimension 3 started from 0, and let $X_t = R_t$ if $t \geq 0$, $X_t = R'_{-t}$ if $t < 0$. Then it holds that*

$$\left(a\,\rho_\infty(t/a^2), t \in \mathbb{R}\right) \xrightarrow[a \to 0]{(d)} X\,, \tag{7}$$

in distribution for the compact-open topology on $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

From the process X, we can build a locally compact pointed random metric space called the self-similar Brownian continuum random tree [3], in a similar way to Section 1.2. Namely, we define a pseudo-distance d_X on $\mathbb{R}$ by the formula

$$d_X(s,t) = X_s + X_t - \check{X}(s,t)\,,$$

where $\check{X}(s,t) = \inf_{s\wedge t \leq u \leq s\vee t} X_u$ whenever $st \geq 0$, and $\check{X}(s,t) = \inf_{u \notin [s\wedge t, s\vee t]} X_u$ otherwise. We let $\mathcal{T}_X = (X/\{d_X = 0\}, d_X, o_X)$ be the quotient metric space, pointed at $o_X = p_X(0)$ where p_X is the canonical projection. This defines a locally compact, complete pointed $\mathbb{R}$-tree.

We let $\mathcal{R}_\infty = \{b_\infty(t), t \in \mathbb{R}\}$ be the range of b_∞, which we canonically view as the pointed metric space $(\{b_\infty(t), t \in \mathbb{R}\}, d_H, o)$. We use the notation $a\,M = (M, ad, x)$ whenever (M, d, x) is a pointed metric space and $a > 0$.

Theorem 9. *It holds that*

$$a\,\mathcal{R}_\infty \xrightarrow[a \to 0]{(d)} \mathcal{T}_X\,,$$

in distribution for the local Gromov-Hausdorff topology. This convergence holds jointly with (7).

This result can be obtained by adapting our arguments, but since we are now dealing with local Gromov-Hausdorff convergence [10, Chapter 8.1], which (very) roughly speaking amounts to the Gromov-Hausdorff convergence of balls centered at the distinguished point, some extra care should be taken.

5.1 Basic properties of $\mathcal{T}_X$

Let us gather some of the important properties of the self-similar CRT. First, it also satisfies a property of invariance under re-rooting that will be crucial to us. Here and below, the set $\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ will be endowed with the compact-open topology.

Proposition 10. *For every $t\in\mathbb{R}$, the random function $(d_X(s+t,s'+t))_{s,s'\in\mathbb{R}}$ in $\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ has same distribution as d_X.*

Proof. This can be shown from the re-rooting invariance of the CRT, by a limiting argument. However, some care has to be taken. Let $\mathbf{e}^\lambda_t=\sqrt{\lambda}\,\mathbf{e}(t/\lambda), 0\le t\le\lambda$ be the Brownian excursion with duration λ. We let $d^\lambda_{\mathbf{e}}(s,t)=\sqrt{\lambda}\,d_{\mathbf{e}}(s/\lambda,t/\lambda)$, defining a random pseudo-distance on $[0,\lambda]$. By [13, Proposition 3], for any $A\in(0,\lambda/2)$, the triplet

$$\left((\mathbf{e}^\lambda_t)_{0\le t\le A},(\mathbf{e}^\lambda_{\lambda-t})_{0\le t\le A},\min_{A\le t\le\lambda-A}\mathbf{e}^\lambda_t\right)$$

is absolutely continuous with respect to the law of $((X_t)_{0\le t\le A},(X_{-t})_{0\le t\le A},\check{X}(-A,A))$, with a density $\Delta_{\lambda,A}(\omega(A),\omega'(A),z)$ such that $\Delta_{\lambda,A}(x,y,z)$ converges to 1 as $\lambda\to\infty$ whenever $0<z<x\wedge y$. Therefore, for every $\varepsilon,A>0$, there exists $\lambda_0=\lambda_0(\varepsilon,A)>2A$ and a coupling of $\mathbf{e}^\lambda$ and X on some probability space such that for every $\lambda\ge\lambda_0$, outside an event $\mathcal{A}=\mathcal{A}(\varepsilon,A)$ of probability at most ε, we have

$$\mathbf{e}^\lambda_t=X_t\,,\quad \mathbf{e}^\lambda_{\lambda-t}=X_{-t}\quad\text{for } t\in[0,A]\,,\qquad\text{and}\qquad \min_{A\le t\le\lambda-A}\mathbf{e}^\lambda_t=\check{X}(-A,A)\,.$$

In particular, still on $\mathcal{A}^c$, it holds that for $s,s'\in[0,A]$,

$$d^\lambda_{\mathbf{e}}(s,s')=d_X(s,s')\,,\quad d^\lambda_{\mathbf{e}}(\lambda-s,\lambda-s')=d_X(-s,-s')\,,\quad d^\lambda_{\mathbf{e}}(s,T-s')=d_X(s,-s')\,.$$

Defining $\tilde{\mathbf{e}}^\lambda_t=\mathbf{e}^\lambda_t$ for $t\in[0,\lambda/2]$ and $\tilde{\mathbf{e}}^\lambda_t=\mathbf{e}^\lambda_{\lambda+t}$ for $t\in[-\lambda/2,0]$, we let

$$\tilde{d}^\lambda_{\mathbf{e}}(s,t)=\tilde{\mathbf{e}}^\lambda_s+\tilde{\mathbf{e}}^\lambda_t-2\check{\mathbf{e}}^\lambda(s,t)\,,\qquad s,t\in[-\lambda/2,\lambda/2]$$

where $\check{\mathbf{e}}^\lambda(s,t)=\min_{s\wedge t\le u\le s\vee t}\tilde{\mathbf{e}}^\lambda_u$ if $st\ge0$, and $\min_{u\in[-\lambda/2,s\wedge t]\cup[s\vee t,\lambda/2]}\tilde{\mathbf{e}}^\lambda_u$ otherwise. Then on the coupling event $\mathcal{A}^c$, one has $\tilde{d}_{\mathbf{e}}(s,s')=d_X(s,s')$ for every $s,s'\in[-A,A]$.

The re-rooting invariance for $d_{\mathbf{e}}$ together with the definition of $\mathbf{e}^\lambda$ shows that if $s\oplus_\lambda t$ denotes the representative of $s+t$ modulo λ in the interval $[-\lambda/2,\lambda/2)$, then $(\tilde{d}^\lambda_{\mathbf{e}}(s\oplus_\lambda t,s'\oplus_\lambda t),s,s'\in[-\lambda/2,\lambda/2])$ has same distribution as $\tilde{d}^\lambda_{\mathbf{e}}$. Fixing the value of t and fixing $A>2|t|$, for $\lambda\ge\lambda_0$, we see that $s\oplus_\lambda t=s+t\in[-A,A]$ for every $s\in[-A/2,A/2]$, so on the coupling event $\mathcal{A}^c$

$$(\tilde{d}^\lambda_{\mathbf{e}}(s\oplus_\lambda t,s'\oplus_\lambda t))_{s,s'\in[-A/2,A/2]}=(d_X(s+t,s'+t))_{s,s'\in[-A/2,A/2]}$$

while this has same law as the restriction of $\tilde{d}^\lambda_{\mathbf{e}}$ to $[-A/2,A/2]$. Since the left-hand side has same distribution as $\tilde{d}^\lambda_{\mathbf{e}}$ restricted to $[-A/2,A/2]^2$, and that the latter is equal on the coupling event $\mathcal{A}^c$ to the restriction of d_X to $[-A/2,A/2]^2$, we see for every $A,\varepsilon>0$, the total variation distance between the laws of $(d_X(t+s,t+s'))_{s,s'\in[-A/2,A/2]}$ and $(d_X(s,s'))_{s,s'\in[-A/2,A/2]}$ is at most $2\mathbb{P}(\mathcal{A}(\varepsilon,A))\le2\varepsilon$. Since ε is arbitrary, we see that these laws are equal, and since A is arbitrary as well, we can conclude. $\square$

The next property is a geometric property, which is often referred to as the fact that $\mathcal{T}_X$ has "a unique infinite spine", also called "baseline" in [3]. Recall that a geodesic ray in a length metric space is a subset that is isometric to $\mathbb{R}_+$ (that is identified with its natural parametrization by $\mathbb{R}_+$). This fact is essentially a consequence of the way it is introduced in [3], but it is also easy to prove it directly from the above definition, and we leave it as an exercise.

Proposition 11. *Almost surely, $\mathcal{T}_X$ has a unique geodesic ray starting from o_X.*

It is not difficult to see that if we let

$$\Gamma_+(r) = \sup\{t \geq 0 : X_t = r\}\,, \qquad \Gamma_-(r) = \inf\{t \leq 0 : X_t = r\}\,, \qquad r \geq 0\,, \tag{8}$$

then the unique geodesic ray of the last proposition is $p_X(\Gamma_+(r)) = p_X(\Gamma_-(r)), r \geq 0$, which means that

$$d_X(\Gamma_\pm(r), \Gamma_\pm(r')) = |r - r'| \quad \text{for every} \quad r, r' \geq 0\,. \tag{9}$$

5.2 Basic properties of $\mathcal{R}_\infty$

We now discuss the results of Sections 2 and 3 that are easily generalized to the infinite loop. The re-rooting Lemma 3 generalizes indeed, by a simple passage as $T \to \infty$ that we leave as an exercise to the reader.

Lemma 12. *For every $t \in \mathbb{R}$, the processes b_∞ and $\phi_{b_\infty(t)}(b_\infty(\cdot + t))$ have the same distribution.*

The tightness estimate of Lemma 4 does not generalize *verbatim*, but should be adapted in the following way. For $s \leq t$, we let $\mathcal{R}_\infty(s,t) = \{b_\infty(u) : s \leq u \leq t\}$. For simplicity, for $A > 0$ we let $\mathcal{R}_\infty(A) = \mathcal{R}_\infty(-A, A)$. As for $\mathcal{R}_\infty$, this set is canonically endowed with the restriction of d_H and pointed at o.

Lemma 13. *For every integers $A, N \geq 2$ and every $\eta > 0$, it holds that*

$$\limsup_{a \to 0} \mathbb{P}\left(\mathcal{R}_\infty(A/a^2) \not\subset \bigcup_{i=-AN}^{AN} B_{d_H}(b_\infty(i/Na^2), \eta/a)\right) \leq \frac{25\,A}{\eta}\sqrt{\frac{N}{\pi}}e^{-\eta^2(N-1)/18}\,.$$

The proof is the same as Lemma 4, using the union bound, and then the re-rooting Lemma 12 and the convergence (7). The following analog of Lemma 5 is deduced in exactly the same way, letting

$$\Lambda_\infty(r, A) = \left\{\exists\, s \leq t \in [-A, A] : \sup_{y \in [b_\infty(s), b_\infty(t)]} d_H(y, \mathcal{R}_\infty(s,t)) \geq r\right\}.$$

We also define a distance function and a renormalized radial process by the formula

$$d_{(a)}(s,t) = a\, d_H(b_\infty(s/a^2), b_\infty(t/a^2))\,, \qquad \rho_{(a)}(t) = a\,\rho_\infty(t/a^2) = d_{(a)}(0,t)\,,$$

for every $s, t \in \mathbb{R}$. These should not be mistaken for $d_{(T)}, \rho_{(T)}$ used in earlier sections. We state a consequence of Lemma 13, proved in the same way as Lemma 5 and the beginning of the proof of Lemma 6.

Lemma 14. *For every $A, \eta > 0$, one has $\mathbb{P}(\Lambda_\infty(\eta/a, A/a^2)) \to 0$ as $a \to 0$. Moreover, outside the event $\Lambda_\infty(\eta/a, A/a^2)$, one has, for every $s \le s'$ in the interval $[-A, A]$,*

$$d_{(a)}(s, s') \le \rho_{(a)}(s) + \rho_{(a)}(s') - 2 \inf_{v \in [s,s']} \rho_{(a)}(v) + 4a\delta + 2\eta \tag{10}$$

5.3 Convergence

We can now state the following key lemma.

Lemma 15. *We have the following convergence in distribution in $\mathcal{C}(\mathbb{R}, \mathbb{R}) \times \mathcal{C}(\mathbb{R}^2, \mathbb{R})$:*

$$(\rho_{(a)}, d_{(a)}) \xrightarrow[a \to 0]{(d)} (X, d_X)\,.$$

Proof. Using Lemma 14 instead of Lemma 5, we deduce exactly as in Lemma 6 that the family of laws of $d_{(a)}$ for $a \le 1$ is a tight family of random variables (due to the fact that we are considering the compact-open topology, it suffices to control the modulus of continuity of $d_{(a)}$ restricted to compact subsets of $\mathbb{R}^2$ of the form $[-A, A]^2$). As in the beginning of the proof of Lemma 7 it holds that for any sequence $a_n \to 0$, we can extract a subsequence along which $(\rho_{(a)}, d_{(a)})$ converges in distribution in $\mathcal{C}(\mathbb{R}, \mathbb{R}) \times \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ to some limit (X, d). Without loss of generality, we may assume that the convergence holds almost surely, and it remains to check that $d = d_X$ almost surely.

Note that by using Lemma 12 and passing to the limit, the function d satisfies the same re-rooting invariance property as d_X: namely, for every $t \in \mathbb{R}$, the function $(d(s + t, s' + t), s, s' \in \mathbb{R})$ has same distribution as d. Moreover, passing to the limit in (10) and using Lemma 14 shows that $d(s, t) \le X_s + X_t - 2 \inf_{[s,t]} X$ for every $s \le t$ in $\mathbb{R}$. However, at this point one should note that this upper-bound is equal to $d_X(s, t)$ only if $st \ge 0$. For such s, t, the rest of the argument applies without change: assuming for instance $0 \le s \le t$, $d(s, t)$ has same distribution as $d(0, t - s)$ by re-rooting invariance. Then $d(0, t - s) = X_{t-s} = d_X(0, t - s)$, which has same distribution as $d_X(s, t)$ by the re-rooting Proposition 10. Since $d(s, t) \le d_X(s, t)$ almost surely, we deduce that they are in fact equal almost surely. We obtain that the restrictions of d to $(\mathbb{R}_+)^2$ and $(\mathbb{R}_-)^2$ are respectively equal the same restrictions of d_X.

By this last fact and re-rooting invariance, we obtain that for every $A > 0$,

$$(d(s, t))_{s,t \ge -A} \overset{(d)}{=} (d(s + A, t + A))_{s,t \ge -A} = (d_X(s + A, t + A))_{s,t \ge -A} \overset{(d)}{=} (d_X(s, t))_{s,t \ge -A}\,,$$

and by letting $A \to \infty$ we obtain that d has same distribution as d_X. In particular, d is a pseudo-distance on $\mathbb{R}$ such that the quotient space $\mathcal{T} = (\mathbb{R}/\{d = 0\}, d, o_{\mathcal{T}})$ is a real tree with the same distribution as $\mathcal{T}_X$ (with $o_{\mathcal{T}} = p(0)$ where p is canonical projection). Therefore, the uniqueness of the geodesic ray stated in Proposition 11 must also be true for $\mathcal{T}$. On the other hand, since the restrictions of d and d_X to $\mathbb{R}_+^2$ and $\mathbb{R}_-^2$ are equal, the images by p of the two functions Γ_+ and Γ_- of (8) are two geodesic rays γ_+, γ_- from $o_{\mathcal{T}}$. This is due to the fact that these functions take values in $\mathbb{R}_+$ and $\mathbb{R}_-$, on which $d = d_X$, and to (9).

By uniqueness of the geodesic ray starting from the root, these path must be one and only, so $\gamma_+(r) = \gamma_-(r)$ for every $r \geq 0$. Let $s < 0 < t$, and let $h_t = \inf_{[t,\infty)} X$ and $h_s = \inf_{(-\infty,s]} X$. We define

$$\Gamma_t(r) = \inf\{u \geq t : X_u = X_t - r\}, \qquad \Gamma_s(r) = \sup\{u \leq s : X_u = X_s - r\},$$

which take finite values respectively if $0 \leq r \leq X_t - h_t$ and $0 \leq r \leq X_s - h_s$. The images $p(\Gamma_t)$ and $p(\Gamma_s)$ are geodesic paths in $\mathcal{T}$ respectively from $p(t), p(s)$ to the points $\gamma_+(h_t)$ and $\gamma_-(h_s)$. This is due to the fact that Γ_t, Γ_s take their values respectively in $\mathbb{R}_+$ and $\mathbb{R}_-$, on which the restrictions of d and d_X coincide, and to the fact that,

$$d_X(\Gamma_t(r), \Gamma_t(r')) = |r - r'|, \qquad r, r' \in [0, X_t - h_t],$$

as is easily checked, together with the similar identity for Γ_s. By connecting $\gamma_+(h_t)$ and $\gamma_-(h_s)$ along $\gamma_+ = \gamma_-$, we can construct a path from $p(s)$ to $p(t)$ with length

$$(X_s - h_s) + (X_t - h_t) + |h_t - h_s| = X_s + X_t - 2h_t \wedge h_s = d_X(s,t).$$

Therefore, we have obtained that $d(s,t) \leq d_X(s,t)$ also for $st < 0$. So we can again apply a re-rooting argument in this situation, and conclude that $d = d_X$ everywhere, almost surely. □

The proof of Theorem 9 does not follow directly from Lemma 15, due to the fact that there could be, in principle, points of the infinite Brownian loop that are visited at large times, but are close to the origin, a phenomenon that is not detected by the compact-open topology used so far. Therefore, the discussion from this point will be longer than in Section 4. In the sequel, we again assume without loss of generality that the convergence in Lemma 15 holds almost surely.

For every $A > 0$, we denote by $\mathcal{T}_X(A) = p_X([-A, A])$, which is a compact subset of $\mathcal{T}_X$ (in fact, it is an $\mathbb{R}$-tree in its own right). Then the proof of (2) given in Section 4 generalizes immediately to the following: almost surely, for every $A > 0$,

$$\big(\rho_{(a)}, a\,\mathcal{R}_\infty(A/a^2)\big) \underset{a\to 0}{\longrightarrow} (X, \mathcal{T}_X(A)).$$

In turn, because of the fact that $\mathcal{T}_X(A)$ is a length space (it is indeed an $\mathbb{R}$-tree), this implies that the balls of radius r in these spaces converge in the pointed Gromov-Hausdorff topology, as a simple variation of Exercise 8.1.3 in [10]:

$$B(a\,\mathcal{R}_\infty(A/a^2), r) \underset{a\to 0}{\longrightarrow} B(\mathcal{T}_X(A), r), \tag{11}$$

with $B(M, r) = \{y \in M : d(x,y) \leq r\}$ denoting the closed ball centered at x with radius r in the pointed metric space (M, d, x), and $B(M, r)$ is seen as a metric space pointed at o and endowed with the restriction of d.

Note that from the transience of the Bessel processes of dimension greater than 2, for any $r, \varepsilon > 0$, the set $\mathcal{T}_X(A)$ contains the ball $B_{d_X}(o_X, r)$ with probability at least $1-\varepsilon$ if we choose A large enough, namely if

$$\mathbb{P}\left(\inf_{|u|>A} X_u \leq r\right) \leq \varepsilon\,.$$

On this likely event, one has $B(\mathcal{T}_X(A), r) = B(\mathcal{T}_X, r)$. We will be able to conclude the proof of Theorem 9 if we can show that for any $r > 0$, the set $\mathcal{R}_\infty(A/a^2)$ contains the ball $B(a\,\mathcal{R}_\infty, r)$ with high probability, uniformly in a small enough:

$$\lim_{A\to\infty} \limsup_{a\to 0} \mathbb{P}\left(B(a\,\mathcal{R}_\infty, r) \not\subset \mathcal{R}_\infty(A/a^2)\right) = 0\,. \tag{12}$$

Indeed, in this case, this shows that $B(a\,\mathcal{R}_\infty(A/a^2), r) = B(a\,\mathcal{R}_\infty, r)$ with high probability uniformly in a small enough, so that (11) implies that for every $r > 0$,

$$B(a\,\mathcal{R}_\infty, r) \underset{a\to 0}{\longrightarrow} B(\mathcal{T}_X, r)\,,$$

in probability in the pointed Gromov-Hausdorff topology, and this implies that $a\,\mathcal{R}_\infty$ converges in probability to $\mathcal{T}_X$ in the local Gromov-Hausdorff topology, as wanted.

It remains to prove (12). For this, we will use Proposition 5.3 in [5] and its proof, where it is shown that the process $(d_H(o, b_\infty(t))^2, t \geq 0)$ is a diffusion $(Y_t, t \geq 0)$ in $\mathbb{R}_+$ satisfying the stochastic differential equation

$$Y_t = Y_0 + 2\int_0^t \sqrt{Y_s}\,\mathrm{d}\beta_s + t + 2\int_0^t g(Y_s)\,\mathrm{d}s\,, \tag{13}$$

where β is a standard Brownian motion and g is a nonnegative continuous function that converges to 1 at infinity by Proposition 8.2 in [5] (one has $g(x) = \chi(\sqrt{x})$ with the notation therein). Recall that the (strong) solution of the stochastic differential equation

$$\mathcal{Z}^{(n)}_t = \mathcal{Z}^{(n)}_0 + 2\int_0^t \sqrt{\mathcal{Z}^{(n)}_s}\,\mathrm{d}\beta_s + nt$$

is a squared Bessel process of dimension n (for any real number $n > 0$). From this, and using standard comparison principles [27, Theorem IX.3.7], one concludes that we can couple the process Y with $\mathcal{Z}^{(1)}$ (the square of a reflected Brownian motion) in such a way that $Y \geq \mathcal{Z}^{(1)}$ almost surely. In particular, Y is a.s. unbounded, so that for every $M > 0$, the time $\tau_M = \inf\{t \geq 0 : Y_t = M\}$ is a.s. finite. Let C be large enough so that $g(x) > 3/4$ for every $x \geq C$, and for a fixed $\varepsilon > 0$, let $M > C$ be such that

$$\mathbb{P}\left(\inf_{t\geq 0} \mathcal{Z}^{(5/2)}_t \geq C \,\middle|\, \mathcal{Z}^{(5/2)}_0 = M\right) \geq 1 - \varepsilon\,.$$

By applying the Markov property at time τ_M, the process $(Y_{\tau_M + t}, t \geq 0)$ satisfies (13) starting from the value M, and by the choice of C it can be coupled with the squared

Bessel process $\mathcal{Z}^{(5/2)}$ starting from M in such a way that $\mathcal{Z}_t^{(5/2)} \le Y_{\tau_M+t}$ on the event that $\inf_{t\ge 0} \mathcal{Z}_t^{(5/2)} \ge C$ (so that the drift coefficient in (13) remains bounded from below by $5/2$). Finally, let T be large enough so that $\mathbb{P}(\tau_M > T) \le \varepsilon$. Upon a further application of the Markov property at time T, we have shown that outside the event

$$\mathcal{A} = \{\tau_M > T\} \cup \left(\left\{ \inf_{t\ge 0} \mathcal{Z}_t^{(5/2)} < C \right\} \cap \{\tau_M \le T\} \right)$$

of probability at most 2ε, we can couple b_∞ with a Bessel process of dimension $5/2$, say Z, in such a way that $d_H(o, b_\infty(t+T)) \ge Z_t$ for every $t \ge 0$. Without loss of generality, we may assume that Z starts from 0, again by standard monotone coupling.

Finally, one has

$$\begin{aligned} \mathbb{P}(B(a\,\mathcal{R}_\infty, r) \not\subset \mathcal{R}_\infty(A/a^2)) &= \mathbb{P}(\exists\, t \notin [-A/a^2, A/a^2] : a\,\rho_\infty(t) \le r) \\ &\le 2\,\mathbb{P}(\exists\, t > A : a\,\rho_\infty(t/a^2) \le r) \\ &\le 2\mathbb{P}(\mathcal{A}) + 2\mathbb{P}(\exists\, t > A : a\,Z_{t/a^2 - T} \le r) \\ &\le 4\varepsilon + 2\mathbb{P}(\exists\, t > A : Z_{t-a^2T} \le r)\,, \end{aligned}$$

where we used symmetry at the first step, the coupling with Z at the penultimate step, and the scaling property of Bessel processes at the last step. Choosing a_0 small enough so that $a_0^2 T \le 1$ say, we can find $A > 1$ large enough so that $\mathbb{P}(\exists\, t > A : Z_{t-a^2T} \le r) \le \varepsilon$ for every $a \in (0, a_0)$ by the transience of Z. This concludes the proof of (12), and thus of Theorem 8.

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