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Linear study of the precessional fishbone instability

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The precessional fishbone instability is an $m = n = 1$ internal kink mode destabilized by a population of trapped energetic particles. The linear phase of this instability is studied here, analytically and numerically, with a simplified model. This model uses the reduced magneto-hydrodynamics (MHD) equations for the bulk plasma and the Vlasov equation for a population of energetic particles with a radially decreasing density. A threshold condition for the instability is found, as well as a linear growth rate and frequency. It is shown that the mode frequency is given by the precession frequency of the deeply trapped energetic particles at the position of strongest radial gradient. The growth rate is shown to scale with the energetic particle density and particles energy while it is decreased by continuum damping.

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I. INTRODUCTION

The resonant interaction between a distribution of particles and waves evolving in a plasma is one of the peculiarities of the kinetic dynamics of plasmas. It can excite or damp waves, depending on the sign of the gradient of the distribution of particles in momentum direction. Indeed, when in the neighborhood of a resonance, the distribution of the faster particles is less than that of slower particles, there is a net transfer of energy from the wave to the plasma, leading to the well-known phenomenon of Landau damping\textsuperscript{1}. In contrast, it is possible to excite a wave by taking energy from the resonating particles, in the case where there is a reversed gradient in the distribution function, such as when a beam of energetic particles is present in a cold plasma. This case is known as the bump-on-tail instability (referring to a bump of energetic particles on the tail of the distribution function of the thermal particles). This instability has been studied excluding\textsuperscript{2,3} or including\textsuperscript{4,5} in the model a finite dissipation for the wave. It was shown that without dissipation, the nonlinear dynamics would flatten the gradient around the position of energy deposition is that there are resonant instabilities that tend to take energy from energetic particles and, at the same time, to expel these particles from the plasma core, before they can transfer their energy to the plasma. One of these instabilities is the fishbone instability.

Fishbones were first observed in the Poloidal Divertor Experiment (PDX) tokamak\textsuperscript{8} in 1983. This instability was observed to have a kink-like mode structure, with dominant mode numbers $m = n = 1$, and to occur in repetitive bursts, where the mode frequency decreases by a factor of about 2 during each burst. It was seen when the Neutral Beam Injection (NBI) was active, in a direction nearly perpendicular to the magnetic field. The mode frequency was comparable to the precession frequency of the deeply trapped particles generated by the NBI. Furthermore, these bursts were associated with losses of fast particles, i.e. an outward radial displacement of these particles.

Two different theoretical interpretations were given for two different regimes. The first regime\textsuperscript{9}, of precessional fishbones, refers to the case where the mode frequency $\omega$ is much greater than the diamagnetic frequency $\omega_{*,i}$ associated with the radial pressure gradient of the thermal ions. In that regime, the trapped energetic particles destabilize a $m = n = 1$ mode with the frequency of their precessional motion $\omega_D$. This mode is purely kinetic and exists only when energetic particles are present, otherwise it is strongly damped by the Alfvén continuum\textsuperscript{10,11}. The second regime\textsuperscript{12} refers to a mode with frequency $\omega \approx \omega_{*,i} \approx \omega_D$. In that regime, the energetic particles excite a mode that already exists, a marginally stable internal kink mode that oscillates at the ion diamagnetic frequency.

It should be stressed that, in a torus, an $m = n = 1$ mode with an electric potential proportional to the minor radius within the $q = 1$ surface (kink-like mode) corresponds to a rigid displacement of the core region, without...
a bending of the field lines. Thus the mode, minimizing
the work that should be done against the magnetic ten-
tion, is the easiest one to excite. Here \( q \) is, as usual, the
safety factor.

One can make a few simplifications to analyze the
essence of the precessional branch of the instability. As
it was observed in the work by McGuire\(^8\), the mode fre-
cuency is close to the precessional frequency of deeply
trapped energetic ions. It is thus possible to simplify the
problem by assuming that all the particles are deeply
trapped. With this simplification, it is possible to reduce
the particle dynamics to a two-dimensional phase space
where the particle-wave interaction occurs only through
the electrostatic part of the wave field.

Again, with this simplification, and for a sufficiently
large magnetic field, the toroidal angular momentum \( P_\varphi \),
which is a constant of motion in the unperturbed field,
is approximately the poloidal magnetic flux (to within
a constant factor) and hence, the minor radius \( r \) is an
approximate constant of motion. This is an important
feature, as then a decrease in fast particle density along
radius can be seen as a positive gradient of the distribu-
tion function in the toroidal momentum direction\(^{13}\). A
decreasing density of deeply trapped energetic particles
play exactly the same role of a bump on the tail, in the
context of electrostatic waves. Many different works have
studied this problem, retaining more effects than we shall
use. The more general effects can include: gyrokinetic
ions\(^{14}\), vertical plasma elongation\(^{15}\), or a different kind
of energetic particles, such as barely trapped/circulating
electrons\(^{16,17}\).

In this paper, we will study the linear phase of the
precessional fishbone instability in the simplified model
discussed above. This model, due to its simplicity and
reduced dimensionality, has the great advantage of be-
ing analytically integrable in linear theory. It can thus
give direct indication of the roles played by the different
mechanisms that drive or suppress the instability. Fur-
thermore it can serve as a benchmark for nonlinear codes,
at least when the hypotheses are fulfilled. In section II,
we will introduce the model for the bulk plasma as well
as for the fast particles. This model is in the spirit of
the one developed by Ódblom et al.\(^{10}\), with the notable
difference that we have here a continuous radial distribu-
tion of fast particles but a single value for the magnetic
moment, where they use a Heavyside function for the ra-
dial distribution and a Maxwellian for the perpendicular
velocity distribution. In section III, the linear analysis
of the model is presented, and an approximate analytic
value is obtained for the instability threshold, the linear
growth rate and the frequency. Section IV describes the
numerical implementation of the model, and section V
discusses the linear numerical benchmarking of the code.
The summary and conclusions are found in section VI.

II. MODEL

The model will treat differently the dynamics of the
thermal plasma and the dynamics of the energetic par-
ticles. In this model, we consider a small inverse aspect
ratio tokamak where \( \varepsilon = r/R \ll 1 \) (\( R \) is the major radius
and \( r \) the minor radius). The thermal plasma will be de-
scribed by the reduced MHD equations in a cylindrical
geometry, meaning that in accord with the small inverse
aspect ratio approximation, the toroidal curvature effects
can be neglected, except for the interaction with energetic
particles. The fast particles will be described kinetically
and the toroidal curvature, which is crucial for the dy-
namics of the energetic particles, will be included. The
curvature will be used in the description of the phase
space evolution and, together with the energetic particle
pressure, in the vorticity equation.

A. MHD description of the thermal plasma

Let us define the magnetic field and the fluid velocity
field as

\[
B = B_T + \nabla \phi \times e_z
\]

(1)

\[
v = \nabla \phi \times e_z = \frac{1}{B_T} \nabla \phi_{el} \times e_z
\]

(2)

where \( \phi_{el} \) is the electric potential, \( \phi \) the stream func-
tion, \( B_T \) is the axial guide field in cylindrical geometry
\( B_T = B_T e_z \) and \( \psi = A_{ij} \) where \( A_{ij} \) is the axial com-
ponent of the vector potential. The plasma density vari-
ations \( (\rho(r) = \rho_0) \) and the thermal pressure effects are
neglected.

Keeping in the effects of energetic particle pressure,
and noting that, as only deeply trapped particles are
considered, the perpendicular component of the pressure
is much larger than the parallel one \( P_{\perp,h} \gg P_{\parallel,h} \), the
following two equations of the ideal reduced MHD are
obtained

\[
\frac{\partial \psi}{\partial t} = (B \cdot \nabla) \phi
\]

(3)

\[
\frac{\partial}{\partial t} \Omega + (v \cdot \nabla) \Omega = \frac{1}{\mu_0 \rho_0} (B \cdot \nabla) \Delta_{\perp} \psi - (b \times \kappa) \cdot \nabla \frac{P_{\perp,h}}{B_T}
\]

(4)

where \( b \) is the unit vector along the equilibrium magnetic
field, \( \kappa \) mimics the toroidal curvature, \( \Omega \) is the vorticity
\( \Omega = \Delta_{\perp} \phi \) and \( \Delta_{\perp} \) is the cylindrical laplacian in the
direction perpendicular to the magnetic guide field along
\( e_z \).
The fields are expanded in an equilibrium part and a perturbation as \( \psi = \psi_{eq} + \hat{\psi} \), \( \phi = \phi_{eq} + \hat{\phi} \). The perturbations are expressed in terms of the following Fourier series
\[
(\hat{\phi}, \hat{\psi}) = \sum_{m,n} (\hat{\phi}_{m,n}(r), \hat{\psi}_{m,n}(r)) e^{-i\omega t + im\varphi - in\theta} + c.c. \tag{5}
\]
where \( \theta \) is the poloidal angle and \( \varphi = \frac{2\pi z}{L_z} = \frac{z}{R} \) is equivalent to the toroidal angle, implying a periodic axial direction \( z \). For the linear study considered here, only the mode numbers \( m = n = 1 \) are relevant. As such, the perturbed fields become
\[
(\hat{\phi}, \hat{\psi}) = (\hat{\phi}_{1,1}(r), \hat{\psi}_{1,1}(r)) e^{-i\omega t + i\varphi - i\theta} + c.c. \tag{6}
\]
In the following, the "hat" notation, the indices, and the explicit radial dependence are dropped for simplicity.

Equations (3) and (4) are now linearized assuming that there is no equilibrium flow \( (\phi_{eq} = 0) \)
\[
-i\omega \psi = i \frac{B_T}{R_0} \left( 1 - \frac{1}{q} \right) \phi \tag{7}
\]
\[
-i\omega \Omega = i \frac{B_T}{\mu_0 \rho_0 R_0} \left( 1 - \frac{1}{q} \right) \Delta_{\perp} \psi + \frac{B_T}{\mu_0 \rho_0 R_0} \left( 1 - \frac{1}{q} \right) \frac{\phi}{r} \partial_r J_{eq} - (b \times \kappa) \cdot \nabla \frac{P_{\parallel,h}}{B_T} \tag{8}
\]
where \( J_{eq} \) is the equilibrium toroidal current, which can be expressed as \( J_{eq} = \Delta_{\perp} \psi_{eq} = \frac{1}{q} \partial_r r \partial_t \psi_{eq} \).

Combining equations (7) and (8) yields
\[
\frac{1}{r^2} \partial_r \left( r^3 \left[ \omega_A^2(r) - \omega^2 \right] \partial_r \left( \frac{\phi}{r} \right) \right) = -i\omega \frac{B_T}{\rho_0} (b \times \kappa) \cdot \nabla \frac{P_{\parallel,h}}{B_T} \tag{9}
\]
which, once integrated with respect to radius, gives
\[
[\omega_A^2(r) - \omega^2] \partial_r \left( \frac{\phi}{r} \right) = -i\omega \hat{\rho} \tag{10}
\]
where \( \hat{\rho} = \frac{1}{q} \int_0^r r^2 \frac{B_T}{\rho_0} (b \times \kappa) \cdot \nabla \frac{P_{\parallel,h}}{B_T} dr \) is the fast particle contribution term and \( \omega_A(r) = \frac{v_A \gamma}{R} \left( 1 - \frac{1}{q} \right) \) is the Alfvén frequency associated with the mode helicity.

### B. Fast particle response

We now describe the kinetic evolution of the fast particles, which satisfies the Vlasov equation
\[
\partial_t f = \{ H, f \} = 0 \tag{11}
\]
where \( \{g,h\} = \partial_x g \partial_P h - \partial_x h \partial_P g \) are the Poisson brackets, which operate on the toroidal angle \( \varphi \) and its conjugate momentum variable \( P_\varphi \). The fast particle distribution \( f = F_{eq} + \delta f \) is decomposed into an equilibrium part \( F_{eq} \) and a perturbation \( \delta f \). The Hamiltonian \( H = H_{eq} + H_1 \) is also decomposed into an equilibrium part \( H_{eq} \) and a perturbation \( H_1 \).

Then, it should be noted that we are considering a population of deeply trapped fast particles, for which \( v_\parallel \) is negligible compared with \( v_{el} \) and \( \frac{e}{M} \psi_{eq} \), where parallel and perpendicular correspond to directions relative to the equilibrium magnetic field, \( e \) is the particle charge and \( M \) its mass.

These considerations give a few important results. The first one is that all fast particles lie close to the \( \theta = 0 \) plane (the horizontal mid-plane of the tokamak, outside of the magnetic axis). Therefore, all interactions of the energetic particles with modes described as \( g(r, \theta, \varphi) = \sum_{m,n=0} g_{m,n}(r)e^{-i\omega t + im\varphi - in\theta} + c.c \) can be taken as \( g(r, \theta = 0, \varphi) = \sum_{m,n=0} g_{m,n}(r)e^{-i\omega t + im\varphi} + c.c \).

Furthermore the perturbed Hamiltonian simplifies, from the form \( H_1 = e(\phi - v_{el} \delta A_\parallel) \) to \( H_1 = e\phi \), where \( \delta A_\parallel = A_\parallel - A_{\parallel,eq} \). Then the Hamiltonian is of the form
\[
H = H_{eq}(P_{\varphi}, \mu) + e\phi_{el,n}(P_{\varphi}) e^{-i\omega t + in\varphi} \tag{12}
\]
Here \( P_{\varphi} \) is the toroidal momentum, and it is equivalent to the radial variable. As it is defined \( P_{\varphi} = eR \psi \), we find
\[
\frac{\partial P_{\varphi}}{\partial r} = -\frac{eR}{q} \tag{13}
\]
The equilibrium Hamiltonian is simply given by \( H_{eq} = \mu B \) where \( \mu \) is the magnetic moment. Thus the equilibrium precessional frequency of particles is simply
\[
\omega_D = \frac{\partial H_{eq}}{\partial P_{\varphi}} = \frac{\mu B_T q}{\omega_C MRr} \tag{14}
\]
where \( \omega_C \) is the cyclotron frequency of the particles considered. It can be seen here that, for a relatively flat \( q \)-profile, the precession frequency will decrease with radius.

### III. LINEAR THEORY

#### A. Linear particle response

Using this Hamiltonian in Eq. (11), one gets the linearized Vlasov equation
\[
\partial_t \delta f = \omega_D \partial_x \delta f - \partial_x e\phi_{el} \partial_x F_{eq} \tag{15}
\]
which, once solved, gives for the perturbation
\[
\delta f = \frac{e\phi_{el} \partial_x F_{eq}}{\omega_D - \omega} \tag{16}
\]

Now, the shape of the MHD mode must be taken into consideration. The dominant mode is kink-like, which means that the mode numbers are \( m = n = 1 \) and, well inside the \( q = 1 \) surface, the electric potential is, at the lowest order, proportional to the radius
\[
\phi_{el}(r, \theta, \varphi) = r \left( \frac{\phi_{el}}{r} \right)_{r=r_h} e^{-i\omega t + i\varphi - i\theta} \tag{17}
\]
Here $r_0$ is an arbitrary radius that is small enough compared to $r_*$, the radial position of the $q = 1$ surface, so that $\phi/r$ has already reached the asymptotic value it has inside the core. Furthermore if we suppose that the energetic particles remain inside the volume defined by $r_0$, the radial shape of the electric potential given by Eq.17, can be used for determining $\delta f$ via Eq.16.

The contribution of fast particles to the MHD dynamics is given by

$$\hat{\rho} = \frac{B_T}{r^3 \rho_0} \int_0^r 2d_3r (b \times \kappa) \cdot \nabla P_{\perp,h}$$

(18)

where

$$(b \times \kappa) \cdot \nabla \approx e \frac{\omega D}{\mu} \partial_\phi$$

(19)

which, it should be pointed out, is independent of $\mu$. The perturbed pressure is given by

$$P_{\perp,h} = \int f \mu_B T d^3v = \int (F_{eq}(\mu,v_\parallel,P_\phi) + \delta f(\mu,v_\parallel,P_\phi,\phi)) \mu_B T d^3v$$

(20)

where $d^3v = 2\pi v_\parallel dv_\perp dv_\parallel = 2\pi B_T^2 d\mu dv_\parallel$ so that

$$\hat{\rho} = \frac{2\pi\mu B_T^2}{M r^3 \rho_0} \int_0^r 2d_3r \frac{\omega D}{\mu} \partial_\phi \int_0^r \mu f d\mu dv_\parallel$$

$$= \frac{2\pi\mu B_T^2}{Mr^3 \rho_0} \int_0^r 2d_3r \frac{\omega D}{\mu} \int_0^r \mu f d\mu dv_\parallel$$

(21)

$$\hat{\rho} = \frac{2\pi\mu B_T^2}{M r^3 \rho_0} \int_0^r 2d_3r \frac{\omega D}{\mu} \partial_\phi F_{eq} \partial_\phi r_\perp dv_\parallel$$

(22)

Now let us define a general form for the equilibrium distribution function

$$F_{eq} = A\delta \left( \frac{\mu_\perp}{\mu_*} - 1 \right) \left( \frac{v_\parallel}{v_\perp} \right) F_\parallel(r)$$

(23)

where $\delta$ is the Dirac delta function and we have chosen the single value of magnetic moment to be $\mu_*$. In this way Eq.22 reads

$$\hat{\rho} = -\frac{2\sqrt{2\pi} \mu B_T^2}{M^{3/2} r_0^3} \frac{\partial_\phi}{\mu_*} \frac{\mu_*^{3/2}}{\mu} \int_0^r q_\perp^2 dr \frac{\omega D_{\parallel} \partial_\phi F_\parallel(r)}{\omega_{D_*} - \omega}$$

(24)

Changing the integration variable to $y = \frac{r}{r_0}$ where $r_*$ is the radius of the $q = 1$ surface, and expressing $\omega_{D_*}$ as $\omega_{D_*} = \frac{q_\perp}{r_0}$ where $T$ is a characteristic period $T = \frac{1}{\omega_{D_*}(r_*)}$, one gets

$$\hat{\rho} = -\frac{2\sqrt{2\pi} \mu A}{M^{3/2} r_0^3 r_*} \frac{\mu_*^{3/2} B_T^2}{\mu} \int_0^y y^2 q^2 \partial_\phi F_\parallel(y) \frac{dy}{q - \omega T y}$$

(25)

B. MHD response and dispersion relation

Starting back from Eq. (10), a stretched variable can be defined as $x = (r - r_*)/\epsilon$ around the position where all variations of $\phi/r$ are located, i.e. around the $q = 1$ surface. These localized variations allow us to take the value of $r$ in Eq. (25) to be $r = r_*$ i.e. $y = 1$. As a result, one can integrate in radius around the $q = 1$ surface, so that

$$\frac{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}} = -\frac{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}$$

Here $\Delta_r$ is then defined as $\Delta_r = r_* - r_b$ and $\phi$ goes to zero at $r = r_b + 2\Delta_r$.

Linearizing $\omega_A$ so that $\omega_A = \epsilon \omega_A' x$ one finds

$$\frac{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}} = -\frac{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}{\int_{r=r_0}^{r=r_{\perp}+2\Delta_r} 2\pi r^2 \rho_0 \frac{\partial_\phi}{\phi}}$$

(26)

which translates to the following dispersion relation

$$i = -K \int_0^1 y^2 q^2 \partial_\phi F_\parallel(y) \frac{dy}{q - \omega T y}$$

(28)

where

$$K = \frac{2\sqrt{2\pi} \mu A_\parallel B_T^2}{M^{3/2} r_0^3 \rho_0 \omega A'}$$

(29)

is a factor taking into account the energetic content of the fast particles as well as the thermal plasma, geometric and magnetic parameters.

C. Threshold, frequency and growth rate

In order to perform an analytic treatment, some additional assumptions need to be made about the distribution function. As it was mentioned, all fast particles are assumed to be well contained within the $q = 1$ surface, and in addition it is assumed that the distribution function’s radial gradient goes to zero at the magnetic axis. Furthermore, in order to be able to get an analytic solution, it is assumed that $q(y) \simeq 1$ in the entire integration domain.

General distribution

Taking a $F_y$ in a form such that

- it is monotonic
- its gradient is centered around $y = y_0$
- its gradient is peaked enough around $y = y_0$
The last integral term can be redefined as

\[ K_0 = -\frac{1}{\pi y_0^3 F_y'(y_0)} ; \quad \omega_0 = \omega_{D*}(y_0) \] (30)

This solution shows that the threshold value \( K_0 \) is inversely proportional to the maximum (negative) value of the distribution function’s gradient, meaning that the mode will be more easily unstable if the gradient of the distribution function is more peaked. Furthermore, as expected, the mode frequency is consistent with the precessional frequency of particles at the location of the maximum gradient value.

**Specific distribution**

Let us now give an analytic profile for the radial distribution

\[ F_y(y) = \frac{1}{2} (1 - \text{erf}(\beta(y - y_0))) \] (31)

with a \( \beta \) and a \( y_0 \) chosen so that \( F_y \) goes to a constant at \( y = 0 \) and \( y = 1 \) (giving the “narrow gradient condition” \( \beta y_0 > 1 \); \( 1 - y_0 \beta > 1 \)). This distribution has a radial gradient

\[ \partial_y F_y = -\frac{\beta}{\sqrt{\pi}} e^{-\beta^2(y-y_0)^2} \] (32)

which allows for an exact solution of the dispersion relation eq. (28) to be expressed as

\[ i = -\frac{K y_0}{\omega T} - \frac{K}{(\omega T)^4} \int_0^1 F_y(y) dy \]

where

\[ \int_0^1 F_y(y) dy \approx \int_0^1 F_y(y) dy \]

which can be simplified, noticing that \( [y F_y]_0^1 = 0 \) and that \( F_y(y_0) \approx y_0 \), to

\[ i = -\frac{K y_0}{\omega T y_0} \frac{K}{(\omega T)^4} \int_0^1 F_y(y) dy \]

The last integral term can be redefined as

\[ \int_0^1 \frac{F_y(y) dy}{1 - \omega T y} = -\frac{\beta}{\sqrt{\pi}} \int_0^1 \frac{e^{-\beta^2(y-y_0)^2} dy}{1 - \omega T y} \]

where \( z = \beta(y - y_0) \) and \( X = \frac{\beta}{\omega T} (1 - \omega T y_0) \). Using again the narrow gradient condition, the integration domain can be stretched to infinity. Here the causal prescription is chosen, meaning that \( \Im(\omega) \) is taken positive (positive growth rate), giving a negative \( \Im(X) \). In this case, the integral reads

\[ \int_{-\infty}^{+\infty} e^{-z^2} dz = i\pi(2e^{-X^2} - w(X)) \] (36)

where \( w(X) \) is the Faddeeva function. The exponential term is due to the pole coming from the causal prescription. The dispersion relation is then given as:

\[ i = -\frac{K y_0}{\omega T} - \frac{K}{(\omega T)^2} \frac{iK\beta\sqrt{\pi}}{(\omega T)^3} (2e^{-X^2} - w(X)) \] (37)

We can show that the frequency at the threshold is given by \( \frac{1}{\omega_0 T} = \omega_{D*}(y_0) \) plus a small correction. Let us define \( \delta = 1 - \omega_0 T y_0 \ll 1 \), so that \( X = \frac{\beta}{\omega T} \delta \), and expand the dispersion relation while only keeping the first order terms in \( X \)

\[ i = -\frac{K_0 y_0^2}{1 - \delta} \frac{K_0 y_0^2}{(1 - \delta)^2} + \frac{iK_0 \beta y_0^3 \sqrt{\pi}}{(1 - \delta)^3} (1 - \frac{2iX}{\sqrt{\pi}}) + O(X^2) \] (38)

To the lowest order, this leads to

\[ i = -2K_0 y_0^2 (1 - \beta^2 y_0^2 \delta) + iK_0 \beta y_0^3 \sqrt{\pi} \] (39)

from where we get

\[ K_0 = \frac{1}{\sqrt{\pi} \beta y_0^3} ; \quad \omega_0 = \frac{1}{y_0 T} (1 - \delta); \quad \delta = \frac{1 - \beta^2 y_0^2}{\beta y_0^2} \] (40)

These relations are the threshold values given for the fishbone mode in our model. Note that the correction to the frequency at threshold \( \delta \) is indeed small. We can see that these relations are the same as the ones for a general distribution in Eq. 30, with a correction of order \( O(\delta) \) related to the finite width of the gradient.

It is now possible to find values for the growth rate and the real frequency close to the threshold values. Let us expand the dispersion relation around the threshold values and write \( K \) and \( \omega \) as \( K = K_0 + \delta K \) and \( \omega = \omega_0 + \delta \omega \). The dispersion relation Eq. (37), written as

\[ i = \mathcal{D}(K, \omega) \] reads

\[ i = \mathcal{D}(K_0, \omega_0) + \delta K \frac{\partial \mathcal{D}}{\partial K} \bigg|_{K_0, \omega_0} + \delta \omega \frac{\partial \mathcal{D}}{\partial \omega} \bigg|_{K_0, \omega_0} + O(\delta K^2, \delta \omega^2) \] (41)

It is straightforward to show that \( \frac{\partial \mathcal{D}}{\partial K} = \frac{D}{K} = \frac{K}{K_0} \) at the same time, to zeroth order in \( \delta \),

\[ \frac{\partial \mathcal{D}}{\partial \omega} = -i\sqrt{\pi} K_0 T \beta y_0^4 - 2K_0 T \beta^2 y_0^5 \gamma \] (42)

which leads to the relation

\[ \gamma = \delta K \frac{\sqrt{\pi}}{2\beta y_0} \omega_0 + O(\delta K^2) ; \quad \delta \omega = \frac{1}{2} \beta y_0 \gamma + O(\delta K^2) \] (44)

that represent the growth rate and the real frequency correction close to the threshold.
IV. NUMERICAL IMPLEMENTATION OF THE MODEL

The numerical model is based on the same assumptions as the linear theory. In this case a domain decomposition is done, separating the area of interest in two regions. The first region is the core region, centered on the magnetic axis, and extending up to \( r_s \), a radius close to \( r_\ast \), that of the \( q = 1 \) surface. It is considered that all fast particles are contained in this region. The second domain is an annular region centered around the \( q = 1 \) surface, with \( r_0 < r < r_b + 2\Delta_r \), in which there are no fast particles. Here, a 2 dimensional code will be used instead of simply integrating numerically the linear equations, as it will serve for future nonlinear studies.

A. Core region

This region is the part of the \( \theta = 0 \) plane that is located between the magnetic axis and the \( q = 1 \) surface. As it was said previously, the radius is parametrized by the toroidal momentum, and therefore this region can be described kinetically in angle-action variables, the action being the toroidal momentum \( P_\varphi \), and the angle the toroidal angle \( \varphi \). As before the Hamiltonian is given by an equilibrium part \( H_0 = \mu_s B_T \) and a time varying part, i.e. the electric potential of the mode:

\[
H_1(\varphi, P_\varphi) = e r(P_\varphi) \left( \frac{\phi_\text{el}(t)}{r(P_\varphi)} \right)_{r(P_\varphi) = r_b} e^{i \varphi} \tag{45}
\]

The evolution of the distribution function \( f \) is given by Eq.11.

The Vlasov equation in this region is treated numerically using a semi-lagrangian scheme with the code VLAP, using a CIP-CSL method\(^{19,20}.\) It can be seen that this is a nonlinear equation, but for the study of interest here, we will look only at the linear phase of the evolution.

For consistency with the analytic prediction, the \( q \)-profile will be taken as \( q(r) = q_0 \simeq 1 \) inside the core region. This allows us to get an explicit relation between \( P_\varphi \) and \( r \):

\[
P_\varphi(r) - P_\varphi,0 = -e(r^2 - r_0^2) \frac{B_T}{2} \tag{46}
\]

where \( r_0 \) is the minimal radius of the core region, and \( P_\varphi,0 \) is the minimal value for the toroidal momentum.

B. Annular region

This region is a narrow layer centered around \( r_\ast \). By considering a single helicity response \((m = n = 1)\), and the fact that its width is small compared with its length \((\Delta_r / r_\ast \ll 1)\), it is possible to simplify the geometry to a two dimensional slab. As there are no fast particles in this region, the dynamics is described by the usual reduced MHD model, without considering the fast particle contribution

\[
\frac{\partial \psi}{\partial t} = [\psi, \phi] + \eta \Delta_{\perp} \psi \tag{47}
\]

\[
\frac{\partial \Omega}{\partial t} = [\Omega, \phi] + [\psi, \Delta_{\perp} \psi] + \nu \Delta_{\perp} \Omega \tag{48}
\]

where \([f, g] = \frac{1}{r^2} \partial_r (r f \partial_r g) - \frac{1}{r} \partial_\varphi f \partial_\varphi g\). A small but finite resistivity \( \eta \) and viscosity \( \nu \) have been added for numerical stability.

These equations have been cast in a dimensionless form by normalizing all length scales to the \( q = 1 \) radius \( r_\ast \) (so that \( r = 1 \) corresponds to the \( q = 1 \) surface), all time scales to the Alfvén time, and the magnetic fields to the equilibrium magnetic field. Only the linearized equations for the mode \( m = 1 \) are retained in our simulations. Once linearized, they read

\[
\frac{\partial \psi}{\partial \tau} = i \phi \partial_r \psi_{eq} + \eta \Delta_{\perp} \psi \tag{49}
\]

\[
\frac{\partial \Omega}{\partial \tau} = \partial_\varphi \psi_{eq} \Delta_{\perp} \psi - i \psi \partial_{rr} \psi_{eq} + \nu \Delta_{\perp} \Omega \tag{50}
\]

where \( \frac{1}{r^2} \partial_\varphi = i k_\varphi = i \). The MHD equations in this region are treated numerically using a semi-spectral finite differences scheme with the code AMON\(^{21}.\) It should be noted here that, in accordance with the analytic theory, the equilibrium poloidal magnetic field profile is taken linear. It is of the form \( \partial_r \psi_{eq} = B_\theta_{eq} = \frac{B_T}{r_s} \frac{r - r_\ast}{r_\ast L_s} \), where \( L_s \) is a characteristic shear length of the magnetic field.

![FIG. 1. Growth rate of the mode as a function of \( K/K_a \). Solid line is the theoretical analytic result, stars linked with the dashed line are the numerical points.](image-url)
The outer boundary condition is simply given by $r = r_s$, the equation ends up being
\[
[\omega_A^2(r) - \omega^2] \partial_r \phi|_{r_s} = -ir_s \omega \rho \quad (51)
\]
Noting here that the normalizations are such that $\mu_0 = \rho_0 = 1$, we can see that the Alfvén frequency is $\omega_A(r) = k_0 v_A \phi = k_0 B_{\phi,eq}$. Using that definition for $\omega_A$ for the annular region one can express $K$ as $K_{\text{sland}} = 2\sqrt{2\pi^2 L_S \mu_0^{3/2} B_{\phi,eq}^{3/2}} A$.

The retroaction of the MHD dynamics on the particles’ one is thus simply given by the amplitude and phase of $\phi/r$ at the inner boundary. Indeed the MHD evolution provides $\frac{\partial \phi(r)}{\partial r(P_s)} = \frac{\partial \phi(r)}{\partial r(P_o)} = r_b$, which is used as the input for the perturbed Hamiltonian. These interactions between the two different domains are done at each time step.

V. LINEAR NUMERICAL BENCHMARKS

The numerical results for the linear analysis are presented in Fig. 1-3. In the results presented below $L_S$ has been taken to be 2.5. The half-width of the annular layer is $\Delta r = 0.15$. That way, the Alfvén frequency continuum varies linearly from $\omega_A(-\Delta r) = -0.06$ to $\omega_A(\Delta r) = 0.06$. The distribution function parameters are set as $\beta = 20$ and $y_0 = 0.35$ (verifying the narrow gradient condition). For Fig. 1 and Fig. 3, $\mu_\ast = 2 \times 10^{-5}$, which leads to a theoretical linear frequency $\omega = 5.7 \times 10^{-3}$. The small normalized values taken for the dissipation parameters are $\nu = \eta = 10^{-9}$.

It can be seen from Fig. 1 that the numerical results fit well the theory in the region linear in $\delta K$. The growth rate is seen to depart from the linear analytic value when $K/K_0$ grows far from 1. This is expected as the growth rate is calculated analytically for $\delta K \ll K_0$. It can be seen in Fig. 2 that the numerical mode frequency matches the analytic predictions as well for three cases near threshold, where the resonance frequency is changed via the parameter $\mu_\ast$. If, while keeping $\mu_\ast$ constant, $y_0$ is changed, the mode frequency also follows the expected scaling.

It can be seen on the profile shown in Fig. 3 that, as expected, the strong variations of the mode $\phi(r)$ are only present around $r = 1$ ($q = 1$). The nearly constant profile on the left of the $q = 1$ surface describes, in a slab geometry, a rigid displacement of the plasma core, as expected. Around the $q = 1$ surface, one can observe the expected two MHD resonances located at $\omega_A(r) = \pm \omega$, whose width depends on the growth rate and on the resistivity and viscosity.

VI. CONCLUSION

A simplified model for the precessional fishbone instability, based on a separation between the thermal plasma,
treated with a fluid approach, and an energetic particle population, treated kinetically, has been developed. A linear analytic treatment has been done, allowing us to find a threshold condition, a linear frequency and a linear growth rate for the instability within the scope of our model. It is shown that the threshold value depends on geometric and MHD parameters, as well as energetic particle parameters. A linear numerical study of this model has been performed and is in good agreement with the analytic theory, within the bounds of its validity.

It is shown that the internal kink mode can be destabilized by a negative radial gradient in a deeply trapped energetic particle population. This mode is easier to excite as this gradient gets steeper. The frequency of the mode is determined by the precessional frequency of the particles at the position of steepest gradient. It is seen that the kink mode radial structure, when it has a finite frequency, changes from a single step function centered at the \( q = 1 \) surface to a two-step structure, where the two steps are centered around the Alfvén resonances \( \omega_A = \pm \omega \).

We calculated an instability threshold condition, showing that it is more easily exceeded with increase in the following:

- the radial density gradient of energetic particles
- the perpendicular velocity
- the radial position of the gradient.

The latter is likely due to the increase of the electric potential with the increase in minor radius. On the other hand, the mode will be stabilized by an increase in magnetic shear that causes a greater continuum damping, and by a higher bulk plasma density.

The longer term objectives of this work are to study the nonlinear evolution of the fishbone mode in a simplified model. The results obtained here have allowed us to benchmark the linear evolution of the mode in the numerical simulations for our model.

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