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# Global stabilization of a Korteweg-de Vries equation with a distributed control saturated in $L^2$ -norm

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**Abstract:** This article deals with the design of saturated controls in the context of partial differential equations. It is focused on a Korteweg-de Vries equation, which is a nonlinear mathematical model of waves on shallow water surfaces. The aim of this article is to study the influence of a saturating in  $L^2$ -norm distributed control on the well-posedness and the stability of this equation. The well-posedness is proven applying a Banach fixed point theorem. The proof of the asymptotic stability of the closed-loop system is tackled with a Lyapunov function together with a sector condition describing the saturating input. Some numerical simulations illustrate the stability of the closed-loop nonlinear partial differential equation.

*Keywords:* Partial differential equation; saturation control; nonlinear control system; global stability; Lyapunov function.

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## 1. INTRODUCTION

The Korteweg-de Vries equation (KdV for short)

$$y_t + y_x + y_{xxx} + yy_x = 0, \quad (1)$$

is a mathematical model of waves on shallow water surfaces. Its stabilizability properties have been deeply studied with no constraints on the control, as reviewed in Cerpa (2014); Rosier and Zhang (2009). In this article, we focus on the following controlled KdV equation

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + f = 0, & (t, x) \in [0, +\infty) \times [0, L], \\ y(t, 0) = y(t, L) = y_x(t, L) = 0, & t \in [0, +\infty), \\ y(0, x) = y_0(x), & x \in [0, L], \end{cases} \quad (2)$$

where  $y$  stands for the state and  $f$  for the control. As studied in Rosier (1997), if  $f = 0$  and

$$L \in \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} / k, l \in \mathbb{N}^* \right\}, \quad (3)$$

then, there exist solutions of the linearized version of (2) written as follows

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, x) = y_0(x), \end{cases} \quad (4)$$

for which the  $L^2(0, L)$ -energy does not decay to zero. For instance, if  $L = 2\pi$  and  $y_0 = 1 - \cos(x)$  for all  $x \in [0, L]$ , then  $y(t, x) = 1 - \cos(x)$  is a stationary solution of (2) conserving the energy for any time  $t$ .

Note however that, if  $L = 2\pi$  and  $f = 0$ , the origin of (2) is locally asymptotically stable as stated in Chu et al. (2015). Then the nonlinear version of the Korteweg-de Vries equation has better results in terms of stability than the linearized one. A similar phenomenon also appears when studying the controllability of the equation with a Neumann control (see e.g. Coron and Crépeau (2004) or Cerpa (2007)).

In the literature there are some methods stabilizing the KdV equation (2) with boundary control, see Cerpa and Crépeau (2009); Cerpa and Coron (2013); Marx and Cerpa (2014) or distributed controls as studied in Pazoto (2005); Rosier and Zhang (2006).

In this paper, we deal with the case where the control is saturated. Indeed, in most applications, actuators are limited due to some physical constraints and the control input has to be bounded. Neglecting the amplitude actuator limitation can be source of undesirable and catastrophic behaviors for the closed-loop system. Nowadays, numerous techniques are available (see e.g. Tarbouriech et al. (2011); Teel (1992); Sussmann et al. (1994)) and such systems can be analyzed with an appropriate Lyapunov function and

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a sector condition of the saturation map, as introduced in Tarbouriech et al. (2011) or Zaccarian and Teel (2011).

To the best of our knowledge, there are few papers studying this topic in the infinite dimensional case. Among them, there are Lasiecka and Seidman (2003), Prieur et al. (2016), where a wave equation equipped with a saturated distributed actuator is considered and Daafouz et al. (2014), where a coupled PDE/ODE system modeling a switched power converter with a transmission line is considered and, due to some restrictions on the system, a saturated feedback has to be designed. There exist also some papers using the nonlinear semigroup theory and focusing on abstract systems (Logemann and Ryan (1998), Seidman and Li (2001), Slemrod (1989)).

In Marx et al. (2015), in which it is considered a linear Korteweg-de Vries equation with a saturated distributed control, nonlinear semigroup theory is applied. In the case of the present paper, since the term  $yy_x$  is not globally Lipschitz, such a theory is harder to use. Thus, we aim at studying a particular nonlinear partial differential equation without seeing it as an abstract control system and without using the nonlinear semigroup theory. Moreover, in the case of the present paper, the saturation is borrowed from Slemrod (1989) and allows us to give explicitly the decay rate for bounded initial conditions, although the saturation used in Marx et al. (2015) is the classical one and no decay rate is given.

This article is organized as follows. In Section 2, we present our main results about the well posedness and the stability of (2) in presence of saturating control. Sections 3.1 and 3.2 are devoted to prove these results by using respectively a Banach fixed-point theorem and some Lyapunov techniques together with a sector condition describing the saturating input. In Section 4, we give some simulations of the equation looped by a saturated feedback law. Section 5 collects some concluding remarks and possible further research lines.

**Notation:**  $y_t$  (resp.  $y_x$ ) stands for the partial derivative of the function  $y$  with respect to  $t$  (resp.  $x$ ). Given  $L > 0$ ,  $\|\cdot\|_{L^2(0,L)}$  denotes the norm in  $L^2(0,L)$  and  $H^1(0,L)$  (resp.  $H^3(0,L)$ ) is the set of all functions  $u \in L^2(0,L)$  such that  $u_x \in L^2(0,L)$  (resp.  $u_x, u_{xx}, u_{xxx} \in L^2(0,L)$ ). A function  $\alpha$  is said to be a class  $\mathcal{K}$  function if it is a nonnegative, an increasing function and if  $\alpha(0) = 0$ .

## 2. MAIN RESULTS

Setting a control  $f(t,x) = ay(t,x)$  with  $a > 0$ , we obtain that (2) is stabilized in the  $L^2(0,L)$ -norm topology. Indeed, performing some integrations by parts, we obtain, at least formally

$$\frac{1}{2} \frac{d}{dt} \int_0^L |y(t,x)|^2 dx = -|y_x(t,0)|^2 - a \int_0^L |y(t,x)|^2 dx \quad (5)$$

and thus

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq e^{-at} \|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \quad (6)$$

Let us consider the impact of the constraint on the control. For infinite-dimensional systems, a way to take into account this constraint is to use the following saturation function, that is for all  $s \in L^2(0,L)$  and for all  $x \in [0,L]$ ,

$$\text{sat}(s)(x) = \begin{cases} s(x) & \text{if } \|s\|_{L^2(0,L)} \leq u_s, \\ \frac{s(x)u_s}{\|s\|_{L^2(0,L)}} & \text{if } \|s\|_{L^2(0,L)} \geq u_s. \end{cases} \quad (7)$$

Let us consider the KdV equation controlled by a saturated distributed control as follows

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \text{sat}(ay) = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = 0, \\ y(0,x) = y_0(x). \end{cases} \quad (8)$$

Let us state the main results of this paper.

**Theorem 1.** [Well posedness] For any initial conditions  $y_0 \in L^2(0,L)$ , there exists a unique mild solution  $y \in C(0,T; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$  to (8).

**Theorem 2.** [Global asymptotic stability] Given positive values  $a$  and  $u_s$ , there exists a class  $\mathcal{K}$  function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that for a given  $y_0 \in L^2(0,L)$ , the mild solution  $y$  to (8) satisfies,

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \alpha(\|y_0\|_{L^2(0,L)}) e^{-at}, \quad \forall t \geq 0. \quad (9)$$

Moreover, for bounded initial conditions, we could estimate the decay rate of the solution. In other words, given a positive value  $r$ , for any initial condition  $y_0 \in L^2(0,L)$  such that  $\|y_0\|_{L^2(0,L)} \leq r$ , any mild solution  $y$  to (8) satisfies,

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)} e^{-\mu t}, \quad \forall t \geq 0, \quad (10)$$

where  $\mu$  is defined as follows

$$\mu := \min \left\{ a, \frac{u_s}{r} \right\}. \quad (11)$$

By a scaling, we may assume, without loss of generality, that either  $a$  or  $u_s$  is 1. However, Theorem 2 shows that saturating a controller insuring a rapid stabilization makes the origin globally asymptotically stable. We cannot select anymore the decay rate of the convergence.

## 3. PROOF OF THE MAIN RESULTS

### 3.1 Well-posedness

*Linear system.* Before proving the well-posedness of (8), let us recall some useful results on the linear system (4). To do that, consider the operator defined by

$$\begin{aligned} D(A) &= \{w \in H^3(0,L), w(0) = w(L) = w'(L) = 0\}, \\ A : w &\in D(A) \subset L^2(0,L) \mapsto (-w' - w''') \in L^2(0,L). \end{aligned}$$

It is easy to prove that  $A$  generates a strongly continuous semigroup of contractions which we will denote by  $W(t)$ . We have the following theorem proven in Rosier (1997)

**Theorem 3.** [Well-posedness of (4), Rosier (1997)] For any initial conditions  $y_0 \in L^2(0,L)$ , there exists a unique mild solution  $y \in C(0,T; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$  to (4). Moreover, there exists  $C > 0$  such that the solution to (4) satisfies

$$\|y\|_{C(0,T; L^2(0,L))} + \|y\|_{L^2(0,T; H^1(0,L))} \leq C_1 \|y_0\|_{L^2(0,L)} \quad (12)$$

and the extra regularity

$$\|y_x(\cdot, 0)\|_{L^2(0,T)} \leq \|y_0\|_{L^2(0,L)}. \quad (13)$$

To ease the reading, let us denote the following Banach space, for all  $T > 0$ ,

$$\mathcal{B}(T) := C(0,T; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

endowed with the norm

$$\|y\|_{\mathcal{B}(T)} = \sup_{t \in [0, T]} \|y(t, \cdot)\|_{L^2(0, L)} + \left( \int_0^T \|y(t, \cdot)\|_{H^1(0, L)}^2 dt \right)^{\frac{1}{2}} \quad (14)$$

Before studying the well-posedness of (8), we need a well-posedness result with a right-hand side. Thus, given  $g \in L^1(0, T; L^2(0, L))$ , let us consider  $y$  the unique solution<sup>1</sup> to the following inhomogeneous problem:

$$\begin{cases} y_t + y_x + y_{xxx} = g, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, \cdot) = y_0. \end{cases} \quad (15)$$

Note that we need the following property on the saturation function, which will allow us to state that this type of nonlinearity belongs to the space  $L^1(0, T; L^2(0, L))$ .

**Lemma 1.** [Slemrod (1989), Theorem 5.1.] For all  $(s, \tilde{s}) \in L^2(0, L)^2$ , we have

$$\|\text{sat}(s) - \text{sat}(\tilde{s})\|_{L^2(0, L)} \leq 3\|s - \tilde{s}\|_{L^2(0, L)}. \quad (16)$$

Let us now state the following properties on the nonhomogeneous linearized KdV equation (15).

Firstly, we have this proposition borrowed from (Rosier, 1997, Proposition 4.1)

**Proposition 1.** [Rosier (1997)] If  $y \in L^2(0, T; H^1(0, L))$ , then  $yy_x \in L^1(0, T; L^2(0, L))$  and the map  $\psi_1 : y \in L^2(0, T; H^1(0, L)) \mapsto yy_x \in L^1(0, T; L^2(0, L))$  is continuous.

Secondly, we have the following proposition

**Proposition 2.** If  $y \in L^2(0, T; H^1(0, L))$ , then  $\text{sat}(ay) \in L^1(0, T; L^2(0, L))$  and the map  $\psi_2 : y \in L^2(0, T; H^1(0, L)) \mapsto \text{sat}(ay) \in L^1(0, T; L^2(0, L))$  is continuous.

**Proof.** Let  $y, z \in L^2(0, T; H^1(0, L))$ . We have, using Lemma 1 and Hölder inequality

$$\begin{aligned} \|\text{sat}(ay) - \text{sat}(az)\|_{L^1(0, T; L^2(0, L))}, \\ \leq 3 \int_0^T a\|y - z\|_{L^2(0, L)}, \\ \leq 3La\sqrt{T}\|y - z\|_{L^2(0, T; H^1(0, L))}. \end{aligned} \quad (17)$$

Plugging  $z = 0$  in (17) yields  $\text{sat}(ay) \in L^1(0, T; L^2(0, L))$  and using (17) gives the continuity of the map  $\psi_2$ . It concludes the proof of Proposition 2. •

Finally, we have this result borrowed from (Rosier, 1997, Proposition 4.1)

**Proposition 3.** [Rosier (1997)] For  $g \in L^1(0, T; L^2(0, L))$  and  $y_0 \in L^2(0, L)$ , the mild solution of (15) belongs to  $\mathcal{B}(T)$ . Moreover the map  $\psi_3 : g \in L^1(0, T; L^2(0, L)) \mapsto y \in \mathcal{B}(T)$  is continuous.

*Proof of Theorem 1* We are now in position to prove Theorem 1. Let us begin this section with a technical lemma.

<sup>1</sup> It follows from the semigroup theory the existence and the unicity of  $y$  when  $g \in L^1(0, T; L^2(0, L))$  (see (Pazy, 1983, Chapter 4)).

**Lemma 2.** [(Chapouly, 2009, Lemma 18)] For any  $T > 0$  and  $y, z \in \mathcal{B}(T)$ ,

$$\int_0^T \|(y(t, \cdot)z(t, \cdot))_x\|_{L^2(0, L)} dt \leq 2\sqrt{T}\|y\|_{\mathcal{B}(T)}\|z\|_{\mathcal{B}(T)} \quad (18)$$

Let us now state our local well-posedness result.

**Lemma 3.** [Local well-posedness of (8)] Let  $T > 0$ , be given. For any  $y_0 \in L^2(0, L)$ , there exists  $T' \in [0, T]$  depending on  $y_0$  such that (8) admits a unique solution  $y \in \mathcal{B}(T')$ .

**Proof.** We follow the strategy of Chapouly (2009) and Rosier and Zhang (2006). From Propositions 1, 2 and 3, we know that, for all  $z \in L^2(0, T; H^1(0, L))$ , there exists a unique solution  $y$  in  $\mathcal{B}(T)$  to the following equation

$$\begin{cases} y_t + y_x + y_{xxx} = -zz_x - \text{sat}(az), \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, x) = y_0(x). \end{cases} \quad (19)$$

Solution  $y$  to (19) can be written in its integral form

$$\begin{aligned} y(t) = & W(t)y_0 - \int_0^t W(t - \tau)(zz_x)(\tau) d\tau \\ & - \int_0^t W(t - \tau)\text{sat}(az(\tau, \cdot)) d\tau \end{aligned} \quad (20)$$

For given  $y_0 \in L^2(0, L)$ , let  $r$  and  $T'$  be positive values to be chosen. Let us consider the following set

$$S_{T', r} = \{z \in \mathcal{B}(T'), \|z\|_{\mathcal{B}(T')} \leq r\}, \quad (21)$$

which is a closed, convex and bounded subset of  $\mathcal{B}(T')$ . Consequently,  $S_{T', r}$  is a complete metric space in the topology induced from  $\mathcal{B}(T)$ . We define the map  $\Gamma$  on  $S_{T', r}$  by, for all  $z \in S_{T', r}$

$$\begin{aligned} \Gamma(z) := & W(t)y_0 - \int_0^t W(t - \tau)(zz_x)(\tau) d\tau \\ & - \int_0^t W(t - \tau)\text{sat}(az(\tau, \cdot)) d\tau, \quad \forall z \in S_{T', r}. \end{aligned} \quad (22)$$

We aim at proving the existence of a fixed point for this operator.

It follows immediately from (17), Lemma 2 and the linear estimates written in Theorem 3 that, for every  $z \in S_{T', r}$ , there exist positive values  $C_2$  and  $C_3$  such that

$$\begin{aligned} \|\Gamma(z)\|_{\mathcal{B}(T')} \leq & C_1\|y_0\|_{L^2(0, L)} + C_2\sqrt{T'}\|z\|_{\mathcal{B}(T')}^2 \\ & + C_3\sqrt{T'}\|z\|_{\mathcal{B}(T')} \end{aligned} \quad (23)$$

We choose  $r > 0$  and  $T' > 0$  such that

$$\begin{cases} r = 2C_1\|y_0\|_{L^2(0, L)} \\ C_2\sqrt{T'}r + C_3\sqrt{T'} \leq \frac{1}{2} \end{cases} \quad (24)$$

in order to obtain

$$\|\Gamma(z)\|_{\mathcal{B}(T')} \leq r, \quad \forall z \in S_{T', r}. \quad (25)$$

Thus, with such  $r$  and  $T'$ ,  $\Gamma$  maps  $S_{T', r}$  to  $S_{T', r}$ . Moreover, one can prove with the same inequalities that

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{B}(T')} \leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{B}(T')}, \quad \forall (z_1, z_2) \in S_{T', r}^2 \quad (26)$$

The existence of the solutions to the Cauchy problem (8) follows by using the Banach fixed point theorem (Brezis, 2011, Theorem 5.7). •

We need the following Lemma inspired by Coron and Crépeau (2004) and Chapouly (2009) which implies that if there exists a solution for all  $T > 0$  then the solution is unique.

**Lemma 4.** *For any  $T > 0$  and  $a > 0$ , there exists  $C_4(T, L)$  such that for every  $y_0, z_0 \in L^2(0, L)$  for which there exist mild solutions  $y$  and  $z$  to*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \text{sat}(ay) = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, x) = y_0(x), \end{cases} \quad (27)$$

and

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + \text{sat}(az) = 0, \\ z(t, 0) = z(t, L) = 0, \\ z_x(t, L) = 0, \\ z(0, x) = z_0(x), \end{cases} \quad (28)$$

one has the following inequalities

$$\begin{aligned} & \int_0^T \int_0^L (z_x(t, x) - y_x(t, x))^2 dx dt \\ & \leq e^{C_4(1 + \|y\|_{L^2(0, T; H^1(0, L))} + \|z\|_{L^2(0, T; H^1(0, L))})} \int_0^L (z_0 - y_0)^2 dx, \end{aligned} \quad (29)$$

$$\begin{aligned} & \int_0^T \int_0^L (z(t, x) - y(t, x))^2 dx dt \\ & \leq e^{C_4(1 + \|y\|_{L^2(0, T; H^1(0, L))} + \|z\|_{L^2(0, T; H^1(0, L))})} \int_0^L (z_0 - y_0)^2 dx. \end{aligned} \quad (30)$$

**Proof.** Due to space limitation the proof has not been written. The interested reader can however refer to Coron and Crépeau (2004) and Chapouly (2009) where proofs of very similar results are provided. •

We aim at removing the smallness condition given by  $T'$  in Lemma 3. Since we have the local well-posedness, we only need to prove the following a priori estimate for any solution to (8).

**Lemma 5.** *For given  $T > 0$ , there exists  $K := K(T) > 0$  such that for any  $y_0 \in L^2(0, L)$ , for any  $0 < T' \leq T$  and for any mild solution  $y \in \mathcal{B}(T')$  to (8), it holds*

$$\|y\|_{\mathcal{B}(T')} \leq K \|y_0\|_{L^2(0, L)}. \quad (31)$$

**Proof.** Let us fix  $0 < T' \leq T$ . Let us multiply the first line of (8) by  $y$  and integrate on  $(0, L)$ . Using the boundary conditions in (8), we get the following estimates

$$\begin{aligned} \int_0^L yy_x dx &= 0, \quad \int_0^L yy_{xxx} dx = \frac{1}{2} |y_x(t, 0)|^2, \\ \int_0^L y^2 y_x dx &= 0. \end{aligned}$$

Using the fact that  $\text{sat}$  is odd, we get that

$$\frac{1}{2} \frac{d}{dt} \|y(t, \cdot)\|_{L^2(0, L)}^2 \leq -\frac{1}{2} |y_x(t, 0)|^2 - \int_0^L y \text{sat}(ay) dx \leq 0 \quad (32)$$

and consequently

$$\|y\|_{L^\infty(0, T'; L^2(0, L))} \leq \|y_0\|_{L^2(0, L)}. \quad (33)$$

It remains to prove a similar inequality for

$$\|y_x\|_{L^2(0, T'; L^2(0, L))},$$

to achieve the proof. To do that, we multiply by  $xy$  (8), integrate on  $(0, L)$  and use the following

$$\begin{aligned} \int_0^L xyy_x dx &= -\frac{1}{2} \|y\|_{L^2(0, L)}^2, \quad \int_0^L yy_{xxx} dx = \frac{3}{2} \|y_x\|_{L^2(0, L)}^2, \\ -\int_0^L xy^2 y_x dx &= \left| \frac{1}{3} \int_0^L y^3(t, x) dx \right|, \\ &\leq \frac{1}{3} \sup_{x \in [0, L]} |y(t, x)| \|y\|_{L^\infty(0, T'; L^2(0, L))}^2, \\ &\leq \frac{\sqrt{L}}{3} \|y_x\|_{L^2(0, L)} \|y\|_{L^\infty(0, T'; L^2(0, L))}^2, \\ &\leq \frac{\sqrt{L}\delta}{6} \|y_x\|_{L^2(0, L)} + \frac{\sqrt{L}}{6\delta} \|y\|_{L^\infty(0, T'; L^2(0, L))}^4, \end{aligned} \quad (34)$$

where  $\delta$  is chosen such that  $\delta := \frac{3}{\sqrt{L}}$ . In this way, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L |x^{1/2} y(t, \cdot)|^2 dx &- \frac{1}{2} \int_0^L y^2 dx + \frac{3}{2} \int_0^L |y_x|^2 dx \\ &- \int_0^L xy^2 y_x dx = - \int_0^L x \text{sat}(ay) y dx. \end{aligned} \quad (35)$$

We get using (34) and the fact that  $\text{sat}$  is odd

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L |x^{1/2} y(t, \cdot)|^2 dx &+ \int_0^L |y_x|^2 dx \leq \\ &\frac{1}{2} \|y\|_{L^\infty(0, T'; L^2(0, L))}^2 + \frac{L}{18} \|y\|_{L^\infty(0, T'; L^2(0, L))}^4. \end{aligned} \quad (36)$$

Using (33) and a Grönwall inequality, we get the existence of a positive value  $C_5 = C_5(L) > 0$  such that

$$\|y_x\|_{L^2(0, T'; L^2(0, L))} \leq C_5 \|y_0\|_{L^2(0, L)}, \quad (37)$$

which concludes the proof of Lemma 5. •

Using Lemmas 3, 4 and 5, for any  $T > 0$ , we can conclude that there exists a unique mild solution in  $\mathcal{B}(T)$  to (8). Indeed, with Lemma 3, we know that there exists  $T' \in (0, T)$  such that there exists a unique solution to (8) in  $\mathcal{B}(T')$ . Lemma 4 states that if there exists a solution to  $\mathcal{B}(T)$ , then it is unique. Finally, Lemma 5 allows us to state the well-posedness for every  $T > 0$ : since the solution  $y$  to (8) is bounded by its initial condition for every  $T' > 0$  belonging to  $[0, T]$  as stated in (32), we know that there exists a unique solution to (8) in  $\mathcal{B}(T)$ . This concludes the proof of Theorem 1. □

### 3.2 Stability

This section is devoted to the proof of Theorem 2. We need then to prove this lemma.

**Lemma 6.** *System (8) is semi-globally exponentially stable in  $L^2(0, L)$ . In other words, for any  $r > 0$  there exists a positive constant  $\mu$  given in Theorem 2 such that for any  $y_0 \in L^2(0, L)$  satisfying  $\|y_0\|_{L^2(0, L)} \leq r$ , the mild solution  $y = y(t, x)$  to (8) satisfies*

$$\|y(t, \cdot)\|_{L^2(0, L)} \leq \|y_0\|_{L^2(0, L)} e^{-\mu t} \quad \forall t \geq 0. \quad (38)$$

From this result, we will be able to prove the global asymptotic stability of (8), as done at the end of this section. Note moreover that this result is indeed the second part of Theorem 2.

*Technical lemma.* Before starting the proof of the Lemma 6, let us state and prove the following lemma

**Lemma 7.** [Sector condition] Let  $r$  be a positive value. Given a positive value  $a$  and  $s \in L^2(0, L)$  such that  $\|s\|_{L^2(0, L)} \leq r$ , we have

$$\beta(x) := \left( \text{sat}(as)(x) - k(r)as(x) \right) s(x) \geq 0, \quad \forall x \in [0, L], \quad (39)$$

with

$$k(r) = \min \left\{ \frac{u_s}{ar}, 1 \right\}. \quad (40)$$

**Proof.** Consider the two following cases

1.  $\|as\|_{L^2(0, L)} \geq u_s$ ;
2.  $\|as\|_{L^2(0, L)} \leq u_s$ .

The first case implies that, for all  $x \in [0, L]$

$$\text{sat}(as)(x) = \frac{as(x)}{\|as\|_{L^2(0, L)}} u_s.$$

Thus, for all  $x \in [0, L]$

$$\beta(x) = as(x)^2 \left( \frac{u_s}{\|as\|_{L^2(0, L)}} - k(r) \right). \quad (41)$$

Since we have

$$\frac{u_s}{\|as\|_{L^2(0, L)}} \geq \frac{u_s}{ar} \geq k(r),$$

then, we obtain

$$\beta(x) \geq 0.$$

The second case implies that, for all  $x \in [0, L]$

$$\text{sat}(as)(x) = as(x)$$

We obtain that

$$\beta(x) := (1 - k(r))as(x)^2 \geq 0.$$

It concludes the proof of Lemma 7. •

*Semi-global exponential stability.* Now we are able to prove Lemma 6. Let  $r$  be a positive value such that  $\|y_0\|_{L^2(0, L)} \leq r$ .

Note that from (32), we get

$$\|y\|_{L^2(0, L)} \leq \|y_0\|_{L^2(0, L)} \leq r \quad (42)$$

With the Lemma 7 and (32), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^L |y(t, x)|^2 dx \leq - \int_0^L k(r)a|y(t, x)|^2 dx \quad (43)$$

Thus, with (40), applying the Grönwall lemma leads to

$$\|y(t, \cdot)\|_{L^2(0, L)} \leq e^{-\mu t} \|y_0\|_{L^2(0, L)} \quad (44)$$

where  $\mu$  is defined in the statement of Theorem 2. Therefore, it concludes the proof of Lemma 6. □

*Proof of Theorem 2* We are now in position to prove Theorem 2, inspired by Rosier and Zhang (2006).

**Proof.** By Lemma 6 and (32), if

$$\|\tilde{y}_0\|_{L^2(0, L)} \leq \frac{u_s}{a}, \quad (45)$$

then, by dissipativity, we have  $\|\tilde{y}(t, \cdot)\|_{L^2(0, L)} \leq \frac{u_s}{a}$  for all  $t \geq 0$ . Thus,  $\text{sat}(a\tilde{y}) = a\tilde{y}$  and we know, from (6), that the corresponding solution  $\tilde{y}$  to (8) satisfies

$$\|\tilde{y}(t, \cdot)\|_{L^2(0, L)} \leq \|\tilde{y}_0\|_{L^2(0, L)} e^{-at} \quad \forall t \geq 0. \quad (46)$$

In addition, for a given  $r > 0$ , there exists a positive constant  $\mu$ , which is given in (11), such that if  $\|y_0\|_{L^2(0, L)} \leq r$ , then any mild solution  $y$  to (8) satisfies

$$\|y(t, \cdot)\|_{L^2(0, L)} \leq \|y_0\|_{L^2(0, L)} e^{-\mu t} \quad \forall t \geq 0 \quad (47)$$

Consequently, setting  $T_r := \mu^{-1} \ln \left( \frac{ar}{u_s} \right)$ , we have

$$\|y_0\|_{L^2(0, L)} \leq r \Rightarrow \|y(T_r, \cdot)\|_{L^2(0, L)} \leq e^{-\ln \left( \frac{ar}{u_s} \right)} r = \frac{u_s}{a}$$

Therefore, using (46), we obtain

$$\begin{aligned} \|y(t, \cdot)\|_{L^2(0, L)} &\leq \|y(T_r, \cdot)\|_{L^2(0, L)} e^{-a(t-T_r)}, \quad \forall t \geq T_r \\ &\leq \|y_0\|_{L^2(0, L)} e^{aT_r} e^{-at}, \quad \forall t \geq 0. \end{aligned}$$

Thus it concludes the proof of Theorem 2. •

## 4. SIMULATION

Let us discretize the PDE (8) by means of finite difference method (see e.g. Pazoto et al. (2010) for an introduction on the numerical scheme of a generalized Korteweg-de Vries equation). The time and the space steps are chosen such that the stability condition of the numerical scheme is satisfied (see Pazoto et al. (2010) where this stability condition is clearly established).

Given  $y_0(x) = 1 - \cos(x)$ ,  $L = 2\pi$ ,  $a = 1$  and  $T_{final} = 6$ , Figure 1 shows the solution to (8) and with the unsaturated control  $f = ay$ . Figure 2 illustrates the simulated solution with the same initial condition and a saturated control  $f = \text{sat}(ay)$  where  $u_s = 0.5$ . The evolution of the  $L^2$ -energy of the solution in these two cases is given by Figure 3.

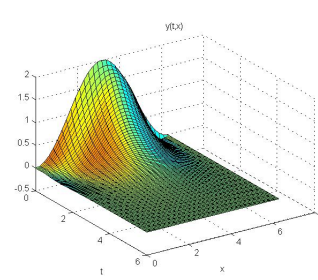


Fig. 1. Solution  $y(t, x)$  with a feedback law  $f = ay$ .

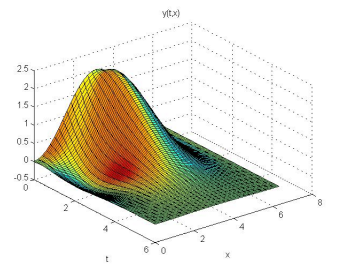


Fig. 2. Solution  $y(t, x)$  with a feedback  $f = \text{sat}(ay)$  where  $u_s = 0.5$ .

## 5. CONCLUSION

In this paper, we have studied the well-posedness and the asymptotic stability of a Korteweg-de Vries equation with a saturated distributed control. The well-posedness issue has been tackled by using the Banach fixed-point theorem and we proved the stability by using a sector condition and Lyapunov theory for infinite dimensional

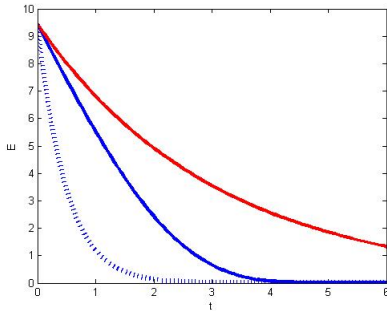


Fig. 3. Blue: Time evolution of the energy function  $\|y\|_{L^2(0,L)}^2$  with a saturation  $u_s = 0.5$  and  $a = 1$ . Red: Time evolution of the theoretical energy  $\|y_0\|_{L^2(0,L)}^2 e^{-2\mu t}$ . Dotted line: Time evolution of the solution without saturation and  $a = 1$ .

systems. Two questions may arise in this context. In Marx et al. (2015), an other saturation function is used. Is the equation with a control saturated with such a function still globally asymptotically stable? Some boundary controls have been already designed in Cerpa and Coron (2013), Coron and Lü (2014), Tang and Krstic (2013) or Cerpa and Crépeau (2009). By saturating this controllers, are the corresponding equations still stable?

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