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# Growing timescales and lengthscales characterizing vibrations of amorphous solids

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**Low-temperature properties of crystalline solids can be understood using harmonic perturbations around a perfect lattice, as in Debye's theory. Low-temperature properties of amorphous solids, however, strongly depart from such descriptions, displaying enhanced transport, activated slow dynamics across energy barriers, excess vibrational modes with respect to Debye's theory (i.e., a boson peak), and complex irreversible responses to small mechanical deformations. These experimental observations indirectly suggest that the dynamics of amorphous solids becomes anomalous at low temperatures. Here, we present direct numerical evidence that vibrations change nature at a well-defined location deep inside the glass phase of a simple glass former. We provide a real-space description of this transition and of the rapidly growing time- and lengthscales that accompany it. Our results provide the seed for a universal understanding of low-temperature glass anomalies within the theoretical framework of the recently discovered Gardner phase transition.**

glass transition | disordered solids | Gardner transition | computer simulations | hard spheres

Understanding the nature of the glass transition, which describes the gradual transformation of a viscous liquid into an amorphous solid, remains an open challenge in condensed matter physics (1, 2). As a result, the glass phase itself is not well understood either. The main challenge is to connect the localized, or “caged,” dynamics that characterizes the glass transition to the low-temperature anomalies that distinguish amorphous solids from their crystalline counterparts (3–7). Recent theoretical advances, building on the random first-order transition approach (8), have led to an exact mathematical description of both the glass transition and the amorphous phases of hard spheres in the mean-field limit of infinite-dimensional space (9). A surprising outcome has been the discovery of a novel phase transition inside the amorphous phase, separating the localized states produced at the glass transition from their inherent structures. This Gardner transition (10), which marks the emergence of a fractal hierarchy of marginally stable glass states, can be viewed as a glass transition deep within a glass, at which vibrational motion dramatically slows down and becomes spatially correlated (11). Although these theoretical findings promise to explain and unify the emergence of low-temperature anomalies in amorphous solids, the gap remains wide between mean-field calculations (9, 11) and experimental work. Here, we provide direct numerical evidence that vibrational motion in a simple 3D glass-former becomes anomalous at a well-defined location inside the glass phase. In particular, we report the rapid growth of a relaxation time related to cooperative vibrations, a nontrivial change in the probability distribution function of a global order parameter, and the rapid growth of a correlation length. We also relate these findings to observed anomalies in low-temperature laboratory glasses. These results provide key support for a universal understanding of the

anomalies of glassy materials, as resulting from the diverging length- and timescales associated with the criticality of the Gardner transition.

## Preparation of Glass States

Experimentally, glasses are obtained by a slow thermal or compression annealing, the rate of which determines the location of the glass transition (1, 2). We find that a detailed numerical analysis of the Gardner transition requires the preparation of extremely well-relaxed glasses (corresponding to structural relaxation timescales challenging to simulate) to study vibrational motion inside the glass without interference from particle diffusion. We thus combine a very simple glass-forming model—a polydisperse mixture of hard spheres—to an efficient Monte Carlo scheme to obtain equilibrium configurations at unprecedentedly high densities, i.e., deep in the supercooled regime. The optimized swap Monte Carlo algorithm (12), which combines standard local Monte Carlo moves with attempts at exchanging pairs of particle diameters, indeed enhances thermalization by several orders of magnitude. Configurations contain either  $N = 1,000$  or  $N = 8,000$  (results in Figs. 1–3 are for  $N = 1,000$ ; results in Fig. 4 are for  $N = 8,000$ ) hard spheres with equal unit mass  $m$  and diameters independently drawn from a probability distribution  $P_\sigma(\sigma) \sim \sigma^{-3}$ , for  $\sigma_{\min} \leq \sigma \leq \sigma_{\min}/0.45$ . We similarly study a 2D bidisperse model glass former and report the main results in *SI Appendix*.

We mimic slow annealing in two steps (Fig. 1). First, we produce equilibrated liquid configurations at various densities  $\varphi_g$  using our efficient simulation scheme, concurrently

## Significance

**Amorphous solids constitute most of solid matter but remain poorly understood. The recent solution of the mean-field hard-sphere glass former provides, however, deep insights into their material properties. In particular, this solution predicts a Gardner transition below which the energy landscape of glasses becomes fractal and the solid is marginally stable. Here we provide, to our knowledge, the first direct evidence for the relevance of a Gardner transition in physical systems. This result thus opens the way toward a unified understanding of the low-temperature anomalies of amorphous solids.**

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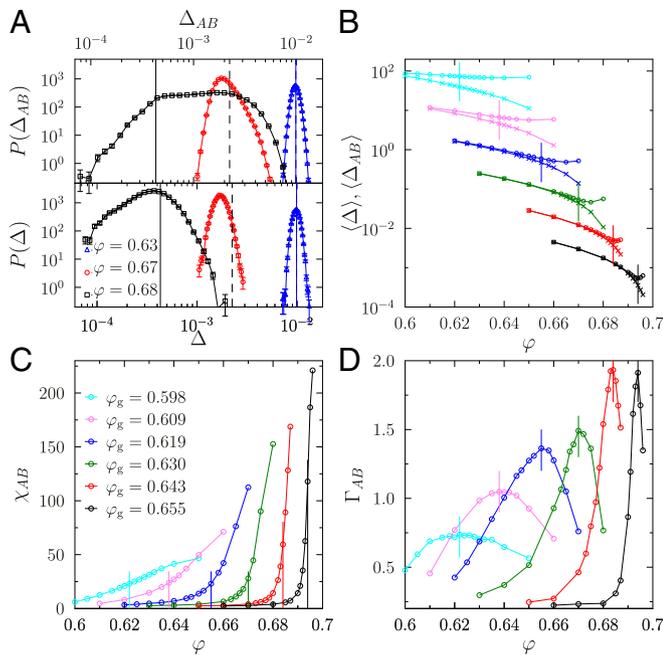
The authors declare no conflict of interest.

Data deposition: Data relevant to this work have been archived and can be accessed at [doi.org/10.7924/G8QN64NT](https://doi.org/10.7924/G8QN64NT).

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**Fig. 3.** Global fluctuations of the order parameter. (A) Probability distribution functions for  $\Delta_{AB}$  and  $\Delta$  above, at, and below the Gardner cross-over,  $\varphi_G = 0.670(2)$  for  $\varphi_g = 0.630$ . Vertical lines mark  $\langle \Delta \rangle$  (solid) and  $\langle \Delta_{AB} \rangle$  (dashed), which also represent the peak positions. (B) Comparing  $\langle \Delta \rangle$  and  $\langle \Delta_{AB} \rangle$  shows that the average values separate for  $\varphi \gtrsim \varphi_G$  (Data are multiplied by  $5^k$ , where  $k=0, 1, \dots, 5$  for  $\varphi = 0.655, 0.643, \dots, 0.598$ , respectively.) Around  $\varphi_G$ , (C) the global susceptibility  $\chi_{AB}$  grows very rapidly, and (D) the skewness  $\Gamma_{AB}$  peaks. Numerical estimates for  $\varphi_G$  are indicated by vertical segments.

When the glass is compressed beyond a certain  $\varphi_G$ , however,  $\Delta(t, t_w)$  displays both a strong dependence on the waiting time  $t_w$ , i.e., aging, and a slow dynamics, as captured by the emergence of two plateaus. These effects suggest a complex vibrational dynamics. Aging, in particular, provides a striking signature of a growing timescale associated with vibrations, revealing the existence of a “glass transition” deep within the glass phase.

To determine the timescale associated with this slowdown, we estimate the distance between independent pairs of configurations by first compressing two independent copies,  $A$  and  $B$ , from the same initial state at  $\varphi_g$  to the target  $\varphi$ , and then measuring their relative distance

$$\Delta_{AB}(t) = \frac{1}{N} \sum_{i=1}^N \left\langle |r_i^A(t) - r_i^B(t)|^2 \right\rangle, \quad [5]$$

so that  $\Delta_{AB}(t \rightarrow \infty) \simeq \Delta(t \rightarrow \infty, t_w \rightarrow \infty)$ , as shown in Fig. 2B. The two copies share the same positions of particles at  $\varphi_g$ , but are assigned different initial velocities drawn from the Maxwell-Boltzmann distribution. The time evolution of the difference  $\delta\Delta(t, t_w) = \Delta_{AB}(t_w + t) - \Delta(t, t_w)$  indicates that whereas the amplitude of particle motion naturally becomes smaller as  $\varphi$  increases, the corresponding dynamics becomes slower (Fig. 2C). In other words, as  $\varphi$  grows particles take longer to explore a smaller region of space. In a crystal, by contrast,  $\delta\Delta(t, t_w)$  decays faster under similar circumstances. A relaxation timescale,  $\tau$ , can be extracted from the decay of  $\delta\Delta(t, t_w)$  at large  $t$ , whose logarithmic form,  $\delta\Delta(t, t_w) \sim 1 - \ln t / \ln \tau$ , is characteristic of the glassiness of vibrations. As  $\varphi \rightarrow \varphi_G$ , we find that  $\tau$  dramatically increases (Fig. 2D), which provides direct evidence of a marked cross-over characterizing the evolution of the glass upon compression.

## Global Fluctuations of the Order Parameter

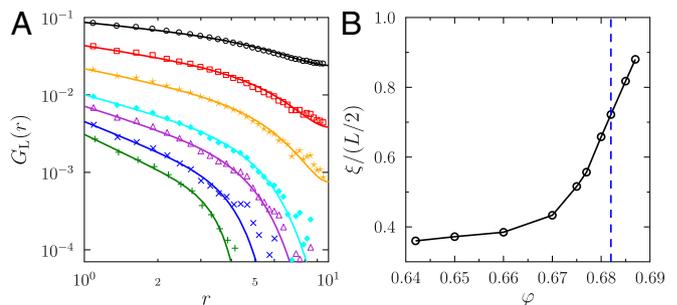
This sharp dynamical cross-over corresponds to a loss of ergodicity inside the glass, i.e., time and ensemble averages yield different results. To better characterize this cross-over, we define a time-scale  $\tau_{\text{cage}}$  for the onset of caging [ $\tau_{\text{cage}} \approx \mathcal{O}(1)$ ; SI Appendix], and the corresponding order parameters  $\Delta_{AB} \equiv \Delta_{AB}(\tau_{\text{cage}})$  and  $\Delta \equiv \Delta(\tau_{\text{cage}}, t_w = 0)$ .

The evolution of the probability distribution functions,  $P(\Delta_{AB})$  and  $P(\Delta)$ , as well as their first moments,  $\langle \Delta_{AB} \rangle$  and  $\langle \Delta \rangle$ , are presented in Fig. 3A and B for a range of densities across  $\varphi_G$ . For  $\varphi < \varphi_G$ , dynamics is fast,  $\langle \Delta_{AB} \rangle$  and  $\langle \Delta \rangle$  coincide, and  $P(\Delta_{AB})$  and  $P(\Delta)$  are narrow and Gaussian-like. For  $\varphi > \varphi_G$ , however, the MSD does not converge to its long-time limit,  $\langle \Delta \rangle < \langle \Delta_{AB} \rangle$ , which indicates that configuration space explored by vibrational motion is now broken into mutually inaccessible regions. Interestingly, the slight increase of  $\langle \Delta_{AB} \rangle$  with  $\varphi$  in this regime (Fig. 3B) suggests that states are then pushed further apart in phase space, which is consistent with theoretical predictions (11). When compressing a system across  $\varphi_G$ , its dynamics explores only a restricted part of phase space. As a result,  $\Delta_{AB}$  displays pronounced, non-Gaussian fluctuations (Fig. 3A). Repeated compressions from a same initial state at  $\varphi_g$  may end up in distinct states, which explains why  $\Delta_{AB}$  is typically much larger and more broadly fluctuating than  $\Delta$  (Fig. 3A). These results are essentially consistent with theoretical predictions (9, 11), which suggest that for  $\varphi > \varphi_G$ ,  $P(\Delta_{AB})$  should separate into two peaks connected by a wide continuous band with the left-hand peak continuing the single peak of  $P(\Delta)$ . The very broad distribution of  $\Delta_{AB}$  further suggests that spatial correlations develop as  $\varphi \rightarrow \varphi_G$ , yielding strongly correlated states at larger densities.

To quantify these fluctuations we measure the variance  $\chi_{AB}$  and skewness  $\Gamma_{AB}$  (SI Appendix and ref. 11) of  $P(\Delta_{AB})$  (Fig. 3C and D). The global susceptibility  $\chi_{AB}$  is very small for  $\varphi < \varphi_G$  and grows rapidly as  $\varphi_G$  is approached, increasing by about two decades for the largest  $\varphi_g$  considered (Fig. 3C). Whereas  $\chi_{AB}$  quantifies the increasing width of the distributions,  $\Gamma_{AB}$  reveals a change in their shapes. For each  $\varphi_g$  we find that  $\Gamma_{AB}$  is small on both sides of  $\varphi_G$  with a pronounced maximum at  $\varphi = \varphi_G$  (Fig. 3D). This reflects the roughly symmetric shape of  $P(\Delta_{AB})$  around  $\langle \Delta_{AB} \rangle$  on both sides of  $\varphi_G$  and the development of an asymmetric tail for large  $\Delta_{AB}$  around the cross-over, a known signature of sample-to-sample fluctuations in spin glasses (22) and mean-field glass models (11). Note that because the skewness maximum gives the clearest numerical estimate of  $\varphi_G$ , we use it to determine the values reported in Fig. 1.

## Growing Correlation Length

The rapid growth of  $\chi_{AB}$  in the vicinity of  $\varphi_G$  suggests the concomitant growth of a spatial correlation length,  $\xi$ . Its measurement



**Fig. 4.** Growing correlation length. (A) Spatial correlator  $G_L(r)$  (Eq. 6) for different  $\varphi$  (from bottom to top,  $\varphi = 0.65, 0.67, 0.675, 0.677, 0.68, 0.682, 0.687$ ) annealed from  $\varphi_g = 0.640$ , with  $N = 8,000$  (larger systems are used here to significantly measure the growth of  $\xi$ ). (B) Fitting  $G_L(r)$  to Eq. 7 (lines in A) provides the correlation length  $\xi$ , which grows with  $\varphi$  and becomes comparable to the linear system size upon approaching  $\varphi_G = 0.682(2)$  (dashed line).

requires spatial resolution of the fluctuations of  $\Delta_{AB}$ , hence for each particle  $i$  we define  $u_i = (|r_i^A - r_i^B|^2 / \langle \Delta_{AB} \rangle) - 1$  to capture its contribution to deviations around the average  $\langle \Delta_{AB} \rangle$ . A first glimpse of these spatial fluctuations is offered by snapshots of the  $u_i$  field (Fig. 1), which appear featureless for  $\varphi < \varphi_G$ , but highly structured and spatially correlated for  $\varphi \gtrsim \varphi_G$ . More quantitatively, we define the spatial correlator

$$G_L(r) = \frac{\left\langle \sum_{\mu=1}^3 \sum_{i \neq j} u_i u_j \delta\left(r - \left| r_{i,\mu}^A - r_{j,\mu}^A \right| \right) \right\rangle}{\left\langle \sum_{\mu=1}^3 \sum_{i \neq j} \delta\left(r - \left| r_{i,\mu}^A - r_{j,\mu}^A \right| \right) \right\rangle}, \quad [6]$$

where  $r_{i,\mu}$  is the projection of the particle position along direction  $\mu$ . Even for the larger system size considered, measuring  $G_L(r)$  is challenging because spatial correlations quickly become long-ranged as  $\varphi \rightarrow \varphi_G$  (Fig. 4A). Fitting the results to an empirical form that takes into account the periodic boundary conditions in a system of linear size  $L$ ,

$$G_L(r) \sim \frac{1}{r^a} e^{-(r/\xi)^b} + \frac{1}{(L-r)^a} e^{-(L-r)/\xi^b}, \quad [7]$$

where  $a$  and  $b$  are fitting parameters, nonetheless confirms that  $\xi$  grows rapidly with  $\varphi$  and becomes of the order of the simulation box at  $\varphi > \varphi_G$  (Fig. 4B). Note that although probed using a dynamical observable, the spatial correlations captured by  $G_L(r)$  are conceptually distinct from the dynamical heterogeneity observed in supercooled liquids (23), which is transient and disappears once the diffusive regime is reached.

## Experimental Consequences

The system analyzed in this work is a canonical model for colloidal suspensions and granular media. Hence, experiments along the lines presented here could be performed to investigate more closely vibrational dynamics in colloidal and granular glasses, using a series of compressions to extract  $\Delta$  and  $\Delta_{AB}$ . Experiments are also possible in molecular and polymeric glasses, for which the natural control parameter is temperature  $T$  instead of density. Let us therefore rephrase our findings from this viewpoint. As the system is cooled, the supercooled liquid dynamics is arrested at the laboratory glass transition temperature  $T_g$ . As the resulting glass is further cooled its phase space transforms, around a well-defined Gardner temperature  $T_G < T_g$ , from a simple state (akin to that of a crystal) into a more complex phase composed of a large number of glassy states (see *SI Appendix* for a discussion of the phase diagram as a function of  $T$ ).

Around  $T_G$ , vibrational dynamics becomes increasingly heterogeneous (Fig. 1), slow (Fig. 2), fluctuating from realization to realization (Fig. 3), and spatially correlated (Fig. 4). The  $\beta$ -relaxation dynamics inside the glass thus becomes highly cooperative (24, 25) and ages (26). The fragmentation of

phase space below  $T_G$  also gives rise to a complex response to mechanical perturbations in the form of plastic irreversible events, in which the system jumps from one configuration to another (4, 6, 27). This expectation stems from the theoretical prediction that the complex phase at  $T < T_G$  is marginally stable (9), which implies that glass states are connected by very low energy barriers, resulting in strong responses to weak perturbations (7).

A key prediction is that the aforementioned anomalies appear simultaneously around a  $T_G$  that is strongly dependent on the scale  $T_g$  selected by the glass preparation protocol. Annealed glasses with lower  $T_g$  are expected to present a sharper Gardner-like cross-over, at an increasingly lower temperature. Numerically, we produced a substantial variation of  $\varphi_g$  by using an efficient Monte Carlo algorithm to bypass the need for a broad range of compression rates. In experiments a similar or even larger range of  $T_g$  can be explored (28), using poorly annealed glasses from hyperquenching (29) and ultrastable glasses from vapor deposition (30–32). We expect ultrastable glasses, in particular, to display strongly enhanced glass anomalies, consistent with recent experimental reports (33–35). Interestingly, a Gardner-like regime may also underlie the anomalous aging recently observed in individual proteins (36).

## Conclusion

Since its prediction in the mean-field limit, the Gardner transition has been regarded as a key ingredient to understand the physical properties of amorphous solids. Understanding the role of finite-dimensional fluctuations is a difficult theoretical problem (37). Our work shows that clear signs of an apparent critical behavior can be observed in three dimensions, at least in a finite-size system, which shows that the correlation length becomes at least comparable to the system size as  $\varphi$  approaches  $\varphi_G$ . Although the fate of these findings in the thermodynamic limit remains an open question, the remarkably large signature of the effect strongly suggests that the Gardner phase transition paradigm is a promising theoretical framework for a universal understanding of the anomalies of solid amorphous materials, from granular materials to glasses, foams, and proteins.

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