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Algebraic and geometric properties of equilibria in cyclic switched dynamic systems

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SUMMARY

The analysis of some properties for the equilibria of switched dynamic systems is addressed. In particular, the geometric properties of the equilibrium region in state space and the algebraic properties of the equations defining it are studied. Based on fundamental results from algebraic geometry the equilibria properties of switched dynamic systems is analyzed. This alternative approach allows to obtain information about the set of equilibrium points without explicitly computing it. This study is developed for three different formulations of switched dynamic systems, revealing some interesting algebraic and geometric relations in their corresponding equilibria. Some examples, including the case of a power converter, are presented for illustration purposes. Copyright © 2016 &

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KEY WORDS: cyclic switched system; equilibrium; method of moments; Lagrange polynomials; Groebner bases

1. INTRODUCTION

A starting point for the study of dynamic systems usually involves studying the set of steady solutions that do not have a temporal evolution, which is called the equilibria. These steady solutions provide valuable information for performing further analysis such as evaluating the stability of the system. Thus, a unifying approach for analyzing the properties of the set of equilibrium points of switched systems, regardless of the dimension or the number of operation modes, is needed.

Many different results have been obtained with respect to the stability and convergence properties for equilibrium points for switched systems [1][2][3][4]. Most of these approaches rely on the definition of multiple Lyapunov functions. [5] presents a method based on geometric intuition for obtaining conic switching laws for stabilizing a switched system. [6] extends this method for different equilibrium types. However, these approaches are limited for the treatment of two

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dimensional problems with two subsystems. [7] introduces sufficiency conditions on the form of linear matrix inequalities for stability of switched systems. In [8], it is shown that by fast switching, convergence to a point in the convex hull of the equilibria can be achieved. [9] [10] present sufficient conditions for the stability of switched systems in a region of state space. Computing a strategy that stabilizes a switched system is a widely studied problem. However despite a lot of techniques, most of them do not deal with equilibrium point study because they assume the different modes of the switched system share a common equilibrium.

One of the main applications of equilibrium point analysis is to find admissible references in dynamic switched systems but also to address the problem of no common equilibrium points of some systems. Our contributions are related to the equilibrium study of the cyclic switched systems based on different relaxed representations of switched systems. Thanks to a density theorem in infinite time, it is possible to show that switching laws exist such that the trajectories of a relaxed switched system can be approached as close as desired by the one of a switched system [11]. Indeed, the relaxed representation of switched system in a quadratic optimal control problem yields a convex optimal control problem in the control signal which is easier to solve [12] [13]. The work proposed in this manuscript searches to find the equilibria proving that the original equilibrium points are part of those of the relaxed formulation. This allows to design a control law for the relaxed system and then apply it to the original system when an optimizer exists [14]. The analysis of the set of equilibrium points is achieved by defining it as a region in state space with some particular characteristics. The geometric structure of these regions and the algebraic properties of the defining equations are the central topics. Tools from algebraic geometry are used for making inferences on the geometry of the equilibrium regions by analyzing the corresponding algebraic parametrizations. In particular, existence and inclusion properties for the set of equilibrium points are studied. This method is general enough for the treatment of problems with more than two dimensions and more than two subsystems. It also allows to find relationships between the different switched system formulations introduced here, revealing an underlying common structure in them. With this new approach, information about the geometry of the equilibrium region is obtained without explicitly computing all equilibrium points.

This paper is organized as follows: section 2 introduces concepts and definitions for switched systems, in section 3 three different formulations for switched dynamic systems are defined, section 4 presents a parametric definition for the equilibria in each formulation and introduces an implicitization algorithm, section 5 studies some of the algebraic and geometric properties of equilibria for each formulation and section 6 presents some examples.

The following notation will be used throughout the paper: $\mathbf{x}(t)$ is an n dimensional vector representing the system state in continuous time, $\{x_1, \dots, x_n\}$ are the components of the state vector. $K[x_1, \dots, x_n]$ is the set of all polynomials defined in $\{x_1, \dots, x_n\}$ (a commutative ring), with coefficients in the field K . f, g, p, r, s are polynomials defined for different polynomial rings. $co(\cdot)$ denotes the convex hull of a set. If A is a matrix, $A \succ 0$ (respectively $A \succeq 0$) indicates that matrix A is positive definite (respectively positive semidefinite).

2. SWITCHED DYNAMIC SYSTEMS

We begin by introducing a general formulation for Switched Dynamic Systems (SDS), with a notation based on that presented in [15]. A SDS can be characterized by a vector $\mathbf{x}(t) \in \mathbb{R}^n$ which corresponds to the continuous state, and a scalar $q(t) \in \underline{Q} = \{q_1, q_2, \dots, q_Q\}$ that represents the discrete state. \underline{Q} is a set representing the different modes of operation. The continuous dynamics for each mode are defined by linear differential equations $\dot{\mathbf{x}}(t) = \mathbf{A}_{q_j} \mathbf{x}(t) + \mathbf{b}_{q_j}$, where $\mathbf{A}_{q_j} \in \mathbb{R}^{n \times n}$ and $\mathbf{b}_{q_j} \in \mathbb{R}^n$. Notice that continuous control signals are not defined for this system. Instead, the dynamics are determined by the discrete control signal $q(t)$. This is the meaning of having a system with controlled switching. The continuous dynamics are determined only by the selected mode of operation. Also, there is not an explicit dependence of time in the equations. We now proceed to define the particular class of switched systems studied in the present paper.

Definition 1 (Cyclic Switched Dynamic System). Given a finite set of system modes $\underline{Q} = \{q_1, q_2, \dots, q_Q\}$, a Cyclic Switched Dynamic System (CSDS) is defined by

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^Q \gamma_{q_j}(q(t)) [\mathbf{A}_{q_j} \mathbf{x}(t) + \mathbf{b}_{q_j}] \quad (1)$$

where $\mathbf{A}_{q_j} \in \mathbb{R}^{n \times n}$, $\mathbf{b}_{q_j} \in \mathbb{R}^n$. The switching function $\gamma_{q_j}(\cdot)$ is defined as

$$\gamma_{q_j}(q(t)) = \begin{cases} 1 & \text{if } q(t) = q_j \\ 0 & \text{if } q(t) \neq q_j \end{cases} \quad (2)$$

with $q(t) \in \underline{Q}$, such that $\mathbf{x}(t)$ converges to an *almost-periodic cycle*[†].

Notice that there is a switching function $\gamma_{q_j}(\cdot)$ dependent on the discrete control input for selecting the active mode for each time t . Thus, the complete system dynamics are determined by the mode selection given by the discrete control signal $q(t)$. Eq. (2) guarantees that only one mode is active at any time t since $\gamma_{q_j} \in \{0, 1\}$. For the trivial case when $q(t) = q_j$, $\forall t$, where q_j is any of the system modes, the behavior is that of the single selected mode with the system state approaching the unique equilibrium point $\bar{\mathbf{x}}$ determined by $\mathbf{A}_{q_j} \bar{\mathbf{x}} + \mathbf{b}_{q_j} = \mathbf{0}$ (provided \mathbf{A}_{q_j} is nonsingular). In general the equilibrium points for all individual system modes are different. Instead, if some or all the system modes are selected alternatively in a certain periodic sequence during some predetermined time intervals, the system state will show a cyclic behavior, approaching a steady cycle. We will refer to this periodic sequence of modes as a switching regime. The average value of \mathbf{x} over the cycle corresponds to the operation point of the cyclic SDS.

[†] *almost-periodic cycle* is used here as defined in [16], definition 4.6.3.

3. CONVEX, LAGRANGE AND RELAXED FORMULATIONS FOR SDS

Let us define the average of \mathbf{x} over this steady cycle, $\bar{\mathbf{x}}$, as follows:

$$\bar{\mathbf{x}}(t) = \frac{1}{T_p} \int_{t-T_p}^t \mathbf{x}(\tau) d\tau \quad (3)$$

where T_p is the period of the cycle. Dynamical model of $\bar{\mathbf{x}}(t)$ is obtained by differentiating (3). Nevertheless, this derivative is generally intractable or unusable because of its non-linear form. It can be proven that if T_p is small compared to the system's dynamics, the state \mathbf{x}_a corresponding to a convex SDS (explained in the next subsection) approximates $\bar{\mathbf{x}}$ and \mathbf{x} [11][17].

3.1. Convex SDS

The dynamics of the cyclic SDS can be approximated by an average model computed as the weighted sum of the SDS system modes. In this case the equilibrium points of the average model will correspond to the operation points of the cyclic system. A convex combination of subsystems will be obtained, where the convexity characteristic refers to weighting each system mode by its corresponding activation time interval. A definition for convex switched systems follows:

Definition 2 (Convex SDS). The Switched Dynamic System defined by differential equations

$$\dot{\mathbf{x}}_a(t) = \sum_{j=1}^Q \alpha_{q_j}(t) [\mathbf{A}_{q_j} \mathbf{x}_a(t) + \mathbf{b}_{q_j}] \quad (4)$$

is a convex Switched Dynamic System, where the switching functions $\gamma_{q_j}(\cdot)$ from (1) are replaced by parameters $\alpha_{q_j}(t)$ constrained by the convexity conditions $\sum_{j=1}^Q \alpha_{q_j}(t) = 1$, $\alpha_{q_j}(t) \in [0, 1]$, $\forall t$.

Remark 3.1. Notice that the value of each parameter α_{q_j} lies in the convex hull of the range set for the switching functions (2), $co(\{0, 1\})$, and equals the duty cycle of the switching function $\gamma_{q_j}(\cdot)$ for the corresponding mode. By changing the switching regime the system can be driven towards different regions in state space. Thus, these parameters are the control signals for the convex SDS.

3.2. Lagrange SDS

An alternative formulation for the SDS can be obtained in terms of a basis of Lagrange polynomials. These polynomials are commonly used for interpolation, but in this context they are used to formulate the switching functions. We present now the definition for this form.

Definition 3 (Lagrange SDS). Given a finite set of system modes $\underline{Q} = \{q_1, q_2, \dots, q_Q\}$, $q_j \in \mathbb{R}$, the system described by

$$\dot{\mathbf{x}}_L(t) = \sum_{j=1}^Q l_{q_j}(q(t)) [\mathbf{A}_{q_j} \mathbf{x}_L(t) + \mathbf{b}_{q_j}] \quad (5)$$

is called a Lagrange Switched Dynamic System, where the $l_{q_j}(\cdot)$ functions correspond to a basis of Lagrange polynomials

$$l_{q_j}(q(t)) = \prod_{\substack{i=1 \\ i \neq j}}^Q \frac{q(t) - q_i}{q_j - q_i}, \quad q(t) \in \text{co}(\underline{Q}) \quad (6)$$

The system is represented as a combination of the different modes, but in this case the weighting parameters are given by the basis of Lagrange polynomials. Notice that $q(t)$ corresponds to the control signal for the Lagrange SDS, since its value determines how the modes combine. It also determines the location in state space where the trajectory will reach a steady cycle. The control signal $q(t)$ is defined in the convex hull of the set \underline{Q} , allowing partial activation of system modes. The difference here is that only one parameter determines how the active modes combine. There are $Q - 1$ terms in each Lagrange basis polynomial. After distributing and reorganizing terms as powers of $q(t)$, Eq. (5) yields a $Q - 1$ polynomial in q as follows:

$$\dot{\mathbf{x}}_L(t) = \sum_{i=0}^{Q-1} \left[\sum_{j=1}^Q c_{ij} [\mathbf{A}_{q_j} \mathbf{x}_L(t) + \mathbf{b}_{q_j}] \right] q^i(t), \quad (7)$$

where c_{ij} are constants. Since the terms in the inner sum are functions depending only on the system state $\mathbf{x}_L(t)$, Eq. (7) can be rewritten as

$$\dot{\mathbf{x}}_L(t) = \sum_{i=0}^{Q-1} \Gamma_i(\mathbf{x}_L(t)) q^i(t) \quad (8)$$

Notice the nonlinearity due to the presence of powers of the control signal. A method for transforming a nonlinear, nonconvex optimal control problem into an equivalent linear, convex problem has been presented in [14]. This method is used here for obtaining a moment relaxation for the Lagrange SDS given in definition 3. The obtained form is more suitable for performing numerical computations such as obtaining an optimal control, as shown in [13]. The main ideas in the method of moments are presented [18][19][20].

3.3. Relaxed SDS

Theorem 3 in [14] is used for transforming the Lagrange SDS from definition 3 into a new relaxed formulation, where the nonlinear terms introduced by the powers q^i of the control signal are replaced by new variables m_i corresponding to the algebraic moments. This alternative form is presented next.

Definition 4 (Relaxed SDS). Given a finite set of system modes $\underline{Q} = \{q_1, q_2, \dots, q_Q\}$, $q_j \in \mathbb{R}$, the system described by

$$\dot{\mathbf{x}}_R(t) = \sum_{i=0}^{Q-1} \left[\sum_{j=1}^Q c_{ij} [\mathbf{A}_{q_j} \mathbf{x}_R(t) + \mathbf{b}_{q_j}] \right] m_i(t) \quad (9)$$

is called a Relaxed Switched Dynamic System, where the set $\{m_i\}$ forms a sequence of algebraic moments representing a probability measure supported on the real line. This sequence is constrained to form a positive semidefinite Hankel matrix $H_n(\mathbf{m}) \succeq 0$, $m_0(t) = 1$.

In this formulation $m_i(t)$ corresponds to the control signal, since its values determine how the different system modes q_j combine. The relaxed form (9) is represented as an affine function of the control signals m_i . Furthermore, the positivity constraint on the sequence of algebraic moments yields a convex set in parameter space $(m_0, m_1, \dots, m_{Q-1}) \in \mathbb{R}^Q$. The convexity of the feasible set is a desirable property for performing numerical computations such as those required for obtaining an optimal control. In the following section the equilibria for each SDS formulation presented here is studied.

4. EQUILIBRIA FOR SWITCHED DYNAMIC SYSTEMS

For each of the three formulations presented here a set of equations in explicit form will be obtained that describe the corresponding equilibria. These equations will be important for understanding the algebraic and geometric properties of the equilibria in the three SDS formulations. Results from algebraic geometry introduced in the next section will provide the tools for studying the set of equilibrium points based on these equations. Let us now introduce equilibrium points, as presented by [21].

4.1. Convex equilibria

The set of equilibrium points for the convex SDS is given by

$$\bar{X}^{[C]} = \left\{ \mathbf{x}_a \in \mathbb{R}^n : \sum_{j=1}^Q \alpha_{q_j} [\mathbf{A}_{q_j} \mathbf{x}_a + \mathbf{b}_{q_j}] = \mathbf{0} \right\} \quad (10)$$

with the α_{q_j} constrained by the convexity conditions given in definition 2. Distributing terms and reorganizing we can solve for \mathbf{x}_a

$$\bar{X}^{[C]} = \left\{ \mathbf{x}_a \in \mathbb{R}^n : \mathbf{x}_a = - \left[\sum_{j=1}^Q \alpha_{q_j} \mathbf{A}_{q_j} \right]^{-1} \sum_{j=1}^Q \alpha_{q_j} \mathbf{b}_{q_j} \right\}. \quad (11)$$

on the condition that $\mathbf{A} = \sum_{j=1}^Q \alpha_{q_j} \mathbf{A}_{q_j}$ is a full rank matrix. Eq. (11) corresponds to a parametric representation of $\bar{X}^{[C]}$ since each point in the set is determined by some values in the parameters α_{q_j} . Because of the matrix inversion, the parametric representation for the convex equilibria will take the general form of rational polynomial functions.

4.2. Lagrange equilibria

Similarly, for the Lagrange SDS presented in definition 3 the equilibria is given by

$$\bar{X}^{[L]} = \left\{ \mathbf{x}_L \in \mathbb{R}^n : \sum_{j=1}^Q l_{q_j}(q) [\mathbf{A}_{q_j} \mathbf{x}_L + \mathbf{b}_{q_j}] = \mathbf{0} \right\}. \quad (12)$$

Solving for \mathbf{x} we obtain the explicit representation

$$\bar{X}^{[L]} = \left\{ \mathbf{x}_L \in \mathbb{R}^n : \mathbf{x}_L = - \left[\sum_{j=1}^Q l_{q_j}(q) \mathbf{A}_{q_j} \right]^{-1} \sum_{j=1}^Q l_{q_j}(q) \mathbf{b}_{q_j} \right\} \quad (13)$$

where the $l_{q_j}(q)$ are a basis of Lagrange polynomials (2) whose values are determined by the single parameter $q \in co(Q)$, and the matrix $\mathbf{A} = \sum_{j=1}^Q l_{q_j}(q) \mathbf{A}_{q_j}$ is full rank. The intuition that a parametric representation defined by a single parameter yields a one dimensional curve will prove true for this case. This representation also has the form of rational polynomial functions.

4.3. Relaxed equilibria

In the case of the Relaxed SDS presented in definition 2, the equilibria set can be expressed as

$$\bar{X}^{[R]} = \left\{ \mathbf{x}_R \in \mathbb{R}^n : \sum_{i=0}^{Q-1} \left[\sum_{j=1}^Q c_{ij} [\mathbf{A}_{q_j} \mathbf{x}_R + \mathbf{b}_{q_j}] \right] m_i = \mathbf{0} \right\}. \quad (14)$$

The explicit representation obtained by solving for \mathbf{x} becomes

$$\bar{X}^{[R]} = \left\{ \mathbf{x}_R \in \mathbb{R}^n : \mathbf{x}_R = - \left[\sum_{i=0}^{Q-1} \sum_{j=1}^Q c_{ij} m_i \mathbf{A}_{q_j} \right]^{-1} \sum_{i=0}^{Q-1} \sum_{j=1}^Q c_{ij} m_i \mathbf{b}_{q_j} \right\}. \quad (15)$$

with the condition on matrix $\mathbf{A} = \sum_{i=0}^{Q-1} \sum_{j=1}^Q c_{ij} m_i \mathbf{A}_{q_j}$ being full rank. In this case the parameters defining the representation are the moments m_i , restricted by the semidefinite condition on the Hankel matrix. The constants c_{ij} depend only on the system modes.

5. ALGEBRAIC AND GEOMETRIC PROPERTIES OF EQUILIBRIA

5.1. Implicitization

Algorithms based on results from algebraic geometry can be applied on the rational parametric representations presented so far for studying the equilibria properties of each formulation. Algebraic geometry deals with the relation between certain classes of algebraic and geometric objects. In the context of this paper, algebraic geometry provides a way to study the geometric properties of equilibrium regions in state space for a SDS by performing computations on some particular sets of algebraic equations. The results obtained from such computations provide insights into the structure of the equilibrium regions. [22][23][24][25] present the basic concepts from algebraic geometry used next. They are also presented in the appendix.

The following result shows how implicitization can be done for a rational polynomial parametrization. This result will be used later in this article.

Theorem 5.1 (Polynomial Implicitization). *Given K an infinite field, let $F : K^m \rightarrow K^n$ represent the polynomial parametrization given by*

$$\begin{aligned} x_1 &= p_1(t_1, t_2, \dots, t_m) \\ x_2 &= p_2(t_1, t_2, \dots, t_m) \\ &\vdots \\ x_n &= p_n(t_1, t_2, \dots, t_m) \end{aligned}$$

Let I be the ideal $I = \langle x_1 - p_1, \dots, x_n - p_n \rangle \subset K[t_1, \dots, t_m, x_1, \dots, x_n]$ and let $I_m = I \cap K[x_1, \dots, x_n]$ be the m^{th} elimination ideal. Then $\mathbf{V}(I_m)$ is the smallest variety in K^n containing $F(K^m)$.

Theorem 5.1 provides a procedure for finding the defining equations of the smallest variety containing the given polynomial parametrization. An analogous result provides this procedure for a rational parametrization. In order to find the smallest variety a possibility consists in controlling the denominators, the ideal I is slightly modified by using an extra dimension characterized by a variable, noted y . Let us consider the polynomial ring $K[y, t_1, \dots, t_m, x_1, \dots, x_n]$, the product $s = s_1 \cdot s_2 \cdot \dots \cdot s_n$ and the ideal $J = \langle s_1 x_1 - r_1, \dots, s_n x_n - r_n, 1 - sy \rangle$.

Theorem 5.2 (Rational Implicitization). *Given K an infinite field, let $F : K^m - W \rightarrow K^n$ represent the rational parametrization given by*

$$x_1 = \frac{r_1(t_1, t_2, \dots, t_m)}{s_1(t_1, t_2, \dots, t_m)}, \dots, x_n = \frac{r_n(t_1, t_2, \dots, t_m)}{s_n(t_1, t_2, \dots, t_m)}$$

where $W = \mathbf{V}(s_1 \cdot s_2 \cdot \dots \cdot s_n) \subset K^n$ defines the vanishing locus of denominators. Let J be the ideal $J = \langle s_1 x_1 - r_1, \dots, s_n x_n - r_n, 1 - sy \rangle \subset K[y, t_1, \dots, t_m, x_1, \dots, x_n]$ where $s = s_1 \cdot s_2 \cdot \dots \cdot s_n$ and let $J_{m+1} = J \cap K[x_1, \dots, x_n]$ be the $(m+1)^{\text{th}}$ elimination ideal. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in K^n containing $F(K^m - W)$.

Theorem 5.2 of rational implicitization implies a procedure for obtaining an implicit representation, which will be useful for analyzing the existing relations between the equilibria for the presented formulations. This procedure is presented in algorithm 1.

Algorithm 1 Implicitization for rational parametrization

- 1: Given a rational parametrization $x_1 = \frac{r_1(t_1, \dots, t_m)}{s(t_1, \dots, t_m)}, \dots, x_n = \frac{r_n(t_1, \dots, t_m)}{s(t_1, \dots, t_m)}$, where $\{r_1, \dots, r_n\}$ are polynomials and $[x_1, \dots, x_n]$ are the state components,
 - 2: Define ideal $J = \langle s x_1 - r_1, \dots, s x_n - r_n, 1 - sy \rangle$.
 - 3: Compute a Groebner basis for ideal J with respect to lexicographic ordering $y > t_1 > \dots > t_m > x_1 > \dots > x_n$.
 - 4: Let J_{m+1} be generated by the generating elements of J not involving y, t_1, \dots, t_m .
 - 5: $\mathbf{V}(J_{m+1})$ is the smallest variety containing the rational parametrization.
-

We are interested in the study of the geometric properties of the equilibrium regions in affine space determined by the set of defining equations. The strategy here is to describe such geometric properties by studying the structure of the algebraic counterpart. Ideals play a central role here as

they provide a language for computing with varieties. There are some geometric properties for the equilibrium region of a given SDS formulation that provide us with some insights on the system structure. Each of these properties will be analyzed by studying its algebraic structure.

5.2. Consistency

Definition 5 (Consistency of Equilibria). Given a SDS of the form (4) (respectively (5), (9)), equilibria is said to be consistent if the set of equilibrium points (10) (respectively (12), (14)) is nonempty.

The term consistency refers to the fact that a set of inconsistent defining equations does not vanish simultaneously for any point in affine space. The relevant question here from the point of view of control systems can be posed as: *given a SDS of the forms described before, does it have at least one point in state space about which a stable switching cycle can be established?* It will be shown that this is always the case.

Lemma 5.1. *Given a SDS of the form (4) (respectively (5), (9)) where a parametrization (11) (respectively (13), (15)) exists, the equilibria is always consistent; that is, the set of equilibrium points is a nonempty set.*

Proof

The proof comes by contradiction. Assume that the set of equilibrium points is empty; that is, it corresponds to the affine variety $X = \emptyset$. Also assume K is an algebraically closed field. By the *Weak Nullstellensatz*, the ideal satisfying $X = \mathbf{V}(I) = \emptyset$ corresponds to the whole ring $I = K[x_1, \dots, x_n]$, which implies that the constant polynomial $1 \in I$. By the definition of ideal, $I = \langle 1 \rangle$, which is clearly an inconsistent set of equations. On the other hand, from theorem 2, chapter 3 of [22], the smallest variety containing $X = F(K^m - W)$ (where the map F corresponds to Eq. (10), (12) or (14)) is $\mathbf{V}(J_{m+1})$. Because of the explicit form of the map F , the constant polynomial 1 can not be in the ideal: $1 \notin J_{m+1}$. Since I corresponds to the whole ring of polynomials, $J_{m+1} \subseteq I$ implying $1 \notin I$, which yields a contradiction. Therefore, the set of equilibrium points must be nonempty. \square

5.3. Inclusions

In section 4, different equilibria parametrizations have been presented. It was shown before that there is an important connection between algebraic and geometric objects. Thus we would expect that these parametrizations yield similarities in the geometry of the equilibria. This is indeed the case, as it is shown in the following result.

Lemma 5.2. *Given a Lagrange SDS as presented in definition 3, the set of its equilibrium points described by the defining equations (12) is a subset of the affine variety determined by the defining equations (10) of the corresponding convex SDS given by Definition 2.*

Proof

Let $\bar{X}^{[L]}$ be the set of equilibrium points of the Lagrange SDS, defined by the rational parametrization (13). For simplicity, let us unify the notation by expressing the switching functions

l_{q_j} as parameters t_j :

$$\bar{\mathbf{x}}^{[L]} = - \left[\sum_{j=1}^Q t_j \mathbf{A}_{q_j} \right]^{-1} \sum_{j=1}^Q t_j \mathbf{b}_{q_j}$$

with $t_1 = l_{q_1}(q), \dots, t_Q = l_{q_Q}(q)$. This is in the form of rational functions $\bar{x}_i^{[L]} = r_i^{[L]}/s_i^{[L]}$ defined by polynomials $r_i^{[L]}, s_i^{[L]} \in K[t_1, \dots, t_Q]$. This parametrization defines the ideal:

$$J^{[L]} = \langle s_1^{[L]}x_1 - r_1^{[L]}, \dots, s_n^{[L]}x_n - r_n^{[L]}, 1 - s^{[L]}y, t_1 - l_{q_1}, \dots, t_Q - l_{q_Q} \rangle \quad (16)$$

where $s^{[L]} = s_1^{[L]}s_2^{[L]} \dots s_n^{[L]}$. This ideal can be written in a more general form as:

$$J^{[L]} = \langle f_1(t_1, \dots, t_Q, x_1), \dots, f_n(t_1, \dots, t_Q, x_n), \\ f_{n+1}(t_1, \dots, t_Q, y), g_1(t_1, q), \dots, g_Q(t_Q, q) \rangle \quad (17)$$

a subset of the polynomial ring $K[q, y, t_1, \dots, t_Q, x_1, \dots, x_n]$. Similarly, for the set (11) of equilibrium points in the convex SDS $\bar{X}^{[C]}$, the switching parameters α_{q_j} can be expressed also as parameters t_j :

$$\bar{\mathbf{x}}^{[C]} = - \left[\sum_{j=1}^Q t_j \mathbf{A}_{q_j} \right]^{-1} \sum_{j=1}^Q t_j \mathbf{b}_{q_j}. \quad (18)$$

This equation also has the form of rational functions $\bar{x}_i^{[C]} = r_i^{[C]}/s_i^{[C]}$ defined by polynomials $r_i^{[C]}, s_i^{[C]} \in K[t_1, \dots, t_Q]$. The ideal corresponding to parametrization (18) is

$$J^{[C]} = \langle s_1^{[C]}x_1 - r_1^{[C]}, \dots, s_n^{[C]}x_n - r_n^{[C]}, 1 - s^{[C]}y \rangle \quad (19)$$

where $s^{[C]} = s_1^{[C]}s_2^{[C]} \dots s_n^{[C]}$ just as in the previous case. Also, this ideal can be presented more generally as:

$$J^{[C]} = \langle f_1(t_1, \dots, t_Q, x_1), \dots, f_n(t_1, \dots, t_Q, x_n), \\ f_{n+1}(t_1, \dots, t_Q, y) \rangle \quad (20)$$

a subset of the polynomial ring $K[y, t_1, \dots, t_Q, x_1, \dots, x_n]$.

Any polynomial p_1 in the ideal $J^{[C]}$ can be written as $p_1 = h_1f_1 + \dots + h_{n+1}f_{n+1}$, where $h_1, \dots, h_{n+1} \in K[y, t_1, \dots, t_Q, x_1, \dots, x_n]$, and the set of generating polynomials for ideal $J^{[C]}$ is $\{h_1, \dots, h_{n+1}\}$. Since the first $n+1$ generating polynomials for ideals (17) and (20) are equal, p_1 can be also be written in terms of the representing equations from ideal $J^{[L]}$ as $p_1 = h_1f_1 + \dots + h_{n+1}f_{n+1} + h'_1g_1 + \dots + h'_Qg_Q$ where $h_1, \dots, h_Q \in K[q, y, t_1, \dots, t_Q, x_1, \dots, x_n]$, and all h'_1, \dots, h'_Q are the zero polynomial. Therefore, $J^{[C]} \subset J^{[L]}$.

On the other hand, let p_2 be any polynomial in the ideal $J^{[L]}$. It can be written as $p_2 = h_1f_1 + \dots + h_{n+1}f_{n+1} + h'_1g_1 + \dots + h'_Qg_Q$ where $h_1, \dots, h_{n+1}, h'_1, \dots, h'_Q \in K[q, y, t_1, \dots, t_Q, x_1, \dots, x_n]$. Assume p_2 can also be written in terms of the generating polynomials from ideal $J^{[C]}$ as $p_2 = h_1f_1 + \dots + h_{n+1}f_{n+1}$ with $f_1, \dots, f_{n+1}, h_1, \dots, h_{n+1} \in K[y, t_1, \dots, t_Q, x_1, \dots, x_n]$. But since $p_2 \in K[q, y, t_1, \dots, t_Q, x_1, \dots, x_n]$, it may have terms depending on q . Then, it can not be written as sums of products of polynomials in

$K[y, t_1, \dots, t_Q, x_1, \dots, x_n]$. Therefore $J^{[C]} \not\supset J^{[L]}$. Thus from the ideal-variety correspondence (theorem 7 in chapter 4 of [22]), since $J^{[L]} \supset J^{[C]}$, then $\mathbf{V}(J^{[L]}) \subset \mathbf{V}(J^{[C]})$. \square

6. ILLUSTRATING EXAMPLES

6.1. Buck-Boost power converter

Power converters are a particular case of SDS and are important, for instance, in portable electronic devices such as laptop computers or mobile phones for supplying power to electronic circuits from batteries. Several approaches have been proposed for stability and control of power converters such as [26] [27] [28]. Fig. 1 presents the schematic of a Buck-Boost power converter. This system is a type of DC-DC converter where the output voltage can take a range of different values by the action of the switch q , drawing power from input voltage source E . Depending on the state of the switch, the system will have two modes of operation. The dynamics for this system are given by:

$$\dot{\mathbf{x}}(t) = \begin{cases} \mathbf{A}_{q_1} \mathbf{x}(t) + \mathbf{b}_{q_1} & \text{if } q(t) = q_1 \text{ (Closed switch)} \\ \mathbf{A}_{q_2} \mathbf{x}(t) + \mathbf{b}_{q_2} & \text{if } q(t) = q_2 \text{ (Open switch)} \end{cases} \quad (21)$$

with $\mathbf{x}(t) = [i_L \ v_C]^T$ representing the state variables: inductor current and capacitor voltage. The system matrices are given by

$$\mathbf{A}_{q_1} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \mathbf{b}_{q_1} = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix},$$

$$\mathbf{A}_{q_2} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \mathbf{b}_{q_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is assumed that the power converter will operate in continuous conduction mode all the time. For simplicity, the system constants are defined as $R = L = C = E = 1$.

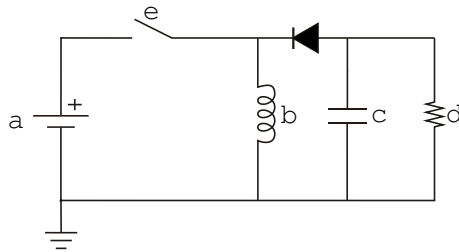


Figure 1. Buck-Boost converter

The parametric description of equilibria in the form of Eq. (11) is:

$$\bar{X}^{[C]} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = \frac{\alpha_{q_1}^2 + \alpha_{q_1} \alpha_{q_2}}{\alpha_{q_2}^2}, x_2 = -\frac{\alpha_{q_1} \alpha_{q_2}}{\alpha_{q_2}^2} \right\} \quad (22)$$

Fig. 2 presents the trajectories followed by the switched power converter for three different switching regimes, with initial condition $\mathbf{x}(0) = [0 \ 0]^T$. Since the switching regimes are constant,

the system reaches a stable cycle around a point in the equilibrium region. As mentioned in section 3.1, the cyclic SDS can be approximated by an average model (Convex SDS) computed from the weighted combination of system modes. Fig. 2 also presents the trajectories for this average model, which converge to the same points in the equilibrium region as the cyclic SDS. In the figure detail it can be observed that the trajectory for the cyclic SDS oscillates around the average model trajectory. The parametric description for the equilibria as represented by Eq. (13) is:

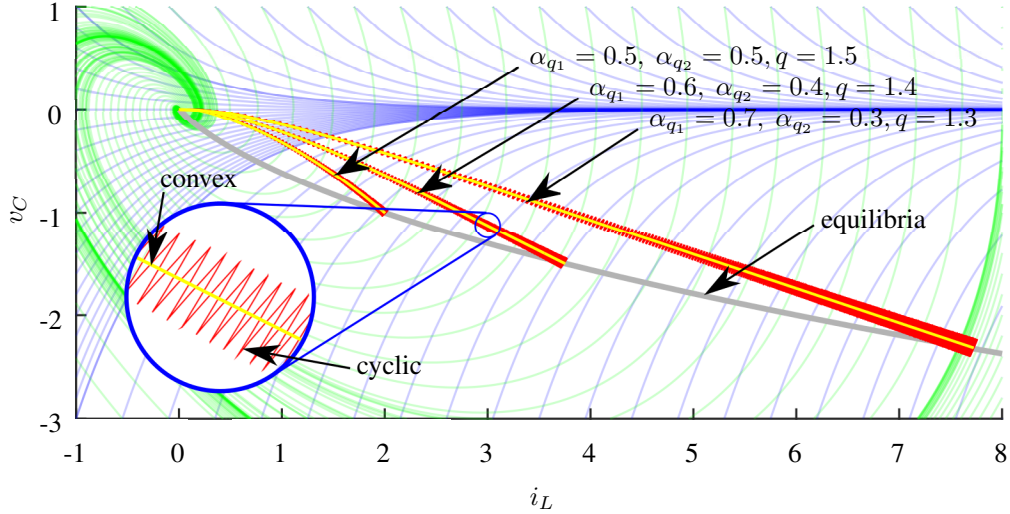


Figure 2. Phase portraits for Buck-Boost power converter: mode q_1 (blue), mode q_2 (green). Trajectories of the cyclic SDS (red) and convex SDS (yellow) for different values in the control parameters. Set of equilibrium points (gray). Switching frequency 10 Hz.

$$\bar{X}^{[L]} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = \frac{-q + 2}{q^2 - 2q + 1}, x_2 = \frac{q^2 - 3q + 2}{q^2 - 2q + 1} \right\} \quad (23)$$

Even though parametrizations (22) and (23) seem different, they represent the same curve in state space. It is easy to check that for any SDS with only two operation modes, the corresponding convex and Lagrange formulations have the same implicit representation. Fig. 2 also shows the equilibria for the Lagrange parametrization, and specifies the value for the control parameter q which achieves the equilibrium points specified in the previous case. It is also interesting to notice that the equilibrium points are located in regions of state space where the trajectories for the two phase portraits have opposite directions. This is a consequence of having the field \dot{x} vanish at these points. The magnitude will be “canceled out” by the weighting given by the α_{q_j} parameters.

If the Buck-Boost converter is represented using the Lagrange formulation in the form of Eq. (7), the powers of the control signal obtained are $q^i(t)$, $q = \{0, 1\}$. In this case, a relaxed formulation in the form of Eq. (9) can not be obtained, since the smallest nontrivial Hankel matrix that can be constructed requires moments up to the second order. So, a limitation in this method for obtaining a relaxed formulation such as presented in definition 4 is that a system with at least three operation modes is required.

6.2. A general three mode cyclic SDS

The following example presents a general cyclic SDS as defined in **1** with three modes $\{q_1, q_2, q_3\}$ and three state variables $\{x_1, x_2, x_3\}$. The matrices defining it are the following:

$$\mathbf{A}_{q_1} = \begin{bmatrix} -5 & -2 & -2 \\ -3 & -4 & 1 \\ 4 & 1 & -2 \end{bmatrix}, \mathbf{b}_{q_1} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}, \mathbf{A}_{q_2} = \begin{bmatrix} -1 & 3 & -3 \\ -2 & -5 & 4 \\ -2 & -5 & -1 \end{bmatrix},$$

$$\mathbf{b}_{q_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{A}_{q_3} = \begin{bmatrix} -2 & -5 & 4 \\ 3 & -5 & -1 \\ 0 & 2 & -5 \end{bmatrix}, \mathbf{b}_{q_3} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The set of equilibrium points for the convex form of this system is as presented in Eq. (11) for $\alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3}$. Recall from theorem 2, chapter 3 of [22], that the ideal defining the smallest variety that contains a given parametrization can be computed by elimination. Algorithm 1 presents the specific procedure for performing this computation.

The most important step in the implicitization algorithm is the computation of a Groebner basis for the ideal J . Because of the properties of Groebner bases and the monomial ordering used when computing it, the polynomials found gradually depend on fewer variables, eliminating the auxiliary variable y and the parameters $\alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3}$. Thus, some of the polynomials in the set only depend on the variables x_1, x_2, x_3 . These polynomials will form the generating set for the ideal $J_4^{[C]}$. The region of affine space where these polynomials vanish corresponds to the affine variety containing the initial parametrization. The ideal corresponding to the convex parametrization is $J^{[C]} = \langle s^{[C]}x_1 - r_1^{[C]}, s^{[C]}x_2 - r_2^{[C]}, s^{[C]}x_3 - r_3^{[C]}, 1 - s^{[C]}y \rangle$. Using a computer algebra system, the elimination ideal $J_4^{[C]}$ obtained by computing the Groebner basis consists of only one generating polynomial: $J_4^{[C]} = \langle g_1(x_1, x_2, x_3) \rangle$. Now, formulating the system as a Lagrange SDS, the set of equilibrium points has a parametrization of the form (13). The ideal associated with this parametrization is $J^{[L]} = \langle s^{[L]}x_1 - r_1^{[L]}, s^{[L]}x_2 - r_2^{[L]}, s^{[L]}x_3 - r_3^{[L]}, 1 - s^{[L]}y \rangle$. When algorithm 1 is applied for the Lagrange case, the elimination ideal $J_2^{[L]}$ obtained has a generating set with seven polynomials, which are not dependent on variables y and q .

$$J_2^{[L]} = \langle g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3),$$

$$g_4(x_1, x_2, x_3), g_5(x_1, x_2, x_3), g_6(x_1, x_2, x_3),$$

$$g_7(x_1, x_2, x_3) \rangle$$

The polynomials which generate $J_2^{[L]}$ are not reproduced here since the coefficients are too big to fit the page. However, what matters is that $g_1(x_1, x_2, x_3)$ equals the only polynomial that generates the elimination ideal $J_4^{[C]}$. As discussed in lemma 5.2, if $J_4^{[C]}$ is a subset of $J_2^{[L]}$ then $\mathbf{V}(J_2^{[L]})$ is a subset of $\mathbf{V}(J_4^{[C]})$. Then, the affine variety corresponding to the Lagrange SDS equilibria lies in the affine variety associated with the convex equilibria. Fig. 3 presents both the convex and Lagrange equilibria for this particular case.

The surface represents the equilibrium region for the convex SDS with $\alpha_{q_j} \in [0, 1]$, $j = 1, 2, 3$ and the curve corresponds to the equilibrium region for the Lagrange SDS with $q \in [1, 3]$. Notice that both regions have three points in common, which are the equilibrium points for the individual system modes. The curve lies in a region “beyond” the surface but seems to be somehow related

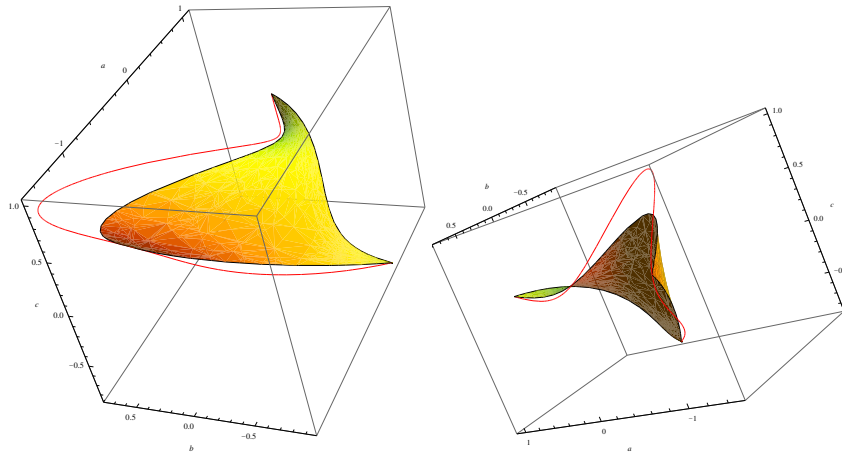


Figure 3. Two different views for the equilibrium region defined for convex (yellow surface) and Lagrange (red curve) parametrizations.

to it. The dimension for the convex equilibria is 2 since it corresponds to a surface in state space. This happens because the parametrization defining it is determined by two free parameters (the convexity constraint from definition 2 determines the value for the third). The Lagrange equilibria has dimension 1 as it corresponds to a curve in state space, which is determined by a single parameter.

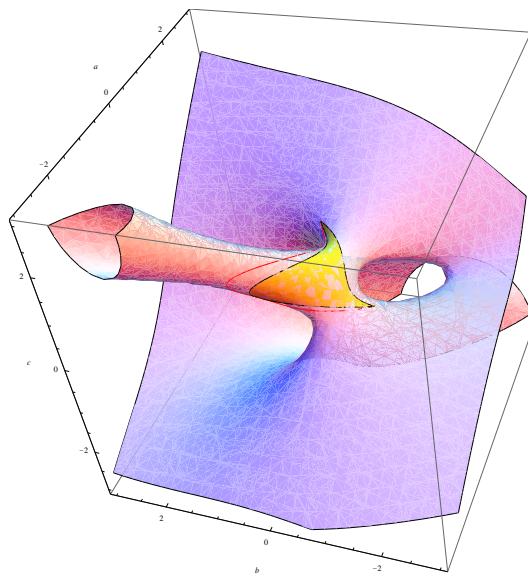


Figure 4. Affine variety corresponding to ideal $J_4^{[C]}$. Embedded convex and Lagrange equilibria.

Plotting the implicit equation found by applying algorithm 1 to the convex parametrization reveals the complete geometric structure for the equilibria, beyond the region limited by the α_{q_j} taking values on the interval $[0, 1]$, as shown in fig. 4. An interesting form emerges, which shows that both the convex and the Lagrange equilibria are embedded in the same affine variety. In this particular case, two conic branches and one sheet extend towards infinity.

This surface corresponds to all equilibrium points determined by the constraint $\sum_j \alpha_{q_j} = 1$. The polynomials in a Lagrange basis also satisfy this constraint, but are not required to have positive values. The region beyond the convex equilibria represents equilibrium points reachable by allowing some of the switching parameters to take negative values.

Since the relaxed formulation is derived from the Lagrange SDS, there are also different equilibria parametrizations for the relaxed case, depending on the original order relation defined. Here we present only the first one, where the modes correspond to $q_1 = 1, q_2 = 2, q_3 = 3$. In order to find the implicit equations for each parametric representation of the equilibria for the relaxed SDS, algorithm 1 must be applied. The elimination ideal obtained after the implicitization process is $J_3^{[R]} = \langle g_1(x_1, x_2, x_3) \rangle$, for any of the three different relaxed parametrizations. This ideal is the same that was obtained for the implicitization of the convex parametrization. The importance of this result is that since all three representations are algebraically related, they are also geometrically related. In particular, three different parametrizations for the equilibria yield the same implicit representation. This is illustrated in figure 5.

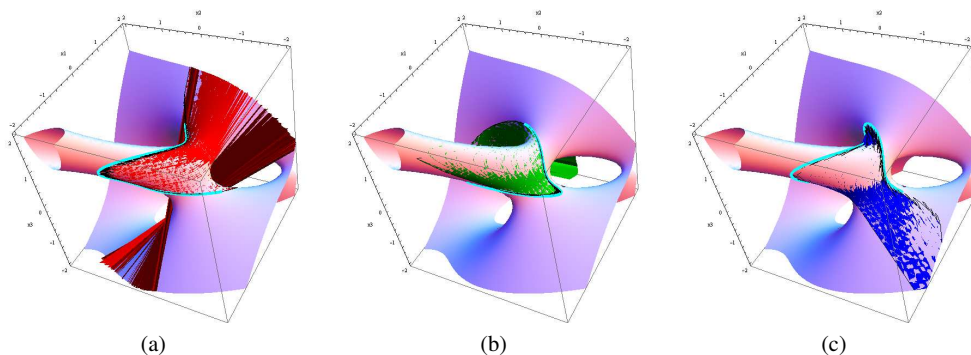


Figure 5. Relaxed SDS equilibria for three different system mode orderings: (a) $q_1 = 1, q_2 = 2, q_3 = 3$, (b) $q_1 = 2, q_2 = 3, q_3 = 1$, (c) $q_1 = 3, q_2 = 1, q_3 = 2$, with corresponding Lagrange equilibria (cyan).

7. CONCLUSIONS

The structure of a SDS imposes difficulties in its treatment, due to the nonlinearity and nonconvexity associated with the presence of a switching behavior. By finding convex relaxations of these nonlinear dynamics, more manageable forms can be found. Furthermore, a relaxed approximation found by the method of moments allows to find an affine form determined by convexity constraints. These characteristics are desirable for performing computations based on semidefinite programming, such as finding an optimal control.

The implicitization algorithm is the specific tool that establishes a procedure for computing an implicit representation for the equilibria regions in each SDS formulation, independent on the values of the defining parameters. This approach allows to perform a direct comparison on these regions. All three formulations have a common geometric structure, suggesting the possibility of transforming a convex system into a relaxed form that may be more suitable for performing mathematical programming computations.

Two different examples are presented for illustrating the geometrical interpretation of the results. A Buck-Boost power converter is introduced for presenting the relationship between the convex and Lagrange SDS formulations. It would be interesting to simulate and implement in a real power converter the different switching regimes and compare the equilibrium points reached by the system with the parametrizations computed using the corresponding system parameters. It is shown that a relaxed formulation can not be presented in this case, since the power converter has only two operation modes. In the second example, a three mode, three dimensional system is presented for illustrating how the relaxed formulation is related to the convex and Lagrange formulations.

The ideas presented here set a starting point for further investigations on other relevant geometric properties of the equilibria such as the dimension of the equilibrium region for higher order systems, the presence of singularities (equilibrium points reachable for different switching regimes) and the relationship between the space dimension and the number of system modes required to reach certain regions. Also, the development of methods for computing an optimal control law for the relaxed SDS introduced here is a path that can be explored. Finally, it would be important to investigate if the information gained regarding the properties of the equilibrium region is useful for computing an optimal control.

APPENDIX A SOME CONCEPTS FROM ALGEBRAIC GEOMETRY

In this appendix, some basic definitions are presented.

Definition 6 (Affine Variety). Let K be a field, f_1, \dots, f_s be polynomials in the ring $K[x_1, \dots, x_n]$. The set

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in K^n : f_i(a_1, \dots, a_n) = 0, \forall i \in \{1, \dots, s\}\}$$

is called the affine variety defined by f_1, \dots, f_s .

Thus, an affine variety corresponds to the set of points in n -dimensional affine space K^n where all f_i polynomial functions vanish simultaneously. Some familiar geometric objects such as the line, the plane, the circle and the ellipse correspond to affine varieties. The set of defining equations $f_1 = \dots = f_s = 0$ define an implicit representation of the affine variety. An alternative representation for the set of solutions known as a parametrization may also be available. In this representation the variables x_1, \dots, x_n are defined explicitly as functions of some parameters: $x_1 = p_1(t_1, t_2, \dots, t_m), \dots, x_n = p_n(t_1, t_2, \dots, t_m)$, where each p_i is a polynomial or rational polynomial function defined for the set of parameters $\{t_1, t_2, \dots, t_m\}$. It is not always possible to obtain a parametric representation for an affine variety, given its defining implicit equations. However, the opposite process of finding the implicit representation given a parametrization can be done using elimination theory. Before reviewing this, the main algebraic object must be introduced, which is related to the parametric and implicit representations just discussed.

Definition 7 (Ideal). A subset $I \subset K[x_1, \dots, x_n]$ is an ideal if it satisfies:

1. $0 \in I$ (The zero polynomial is in the ideal)
2. If $f, g \in I$, then $f + g \in I$ (Closure property under addition for the elements of the ideal)

3. If $f \in I$ and $h \in K[x_1, \dots, x_n]$, then $hf \in I$ (Absorbent property for multiplication by an element in the ideal)

The ideal generated by the set of polynomials f_1, \dots, f_s is represented by

$$\langle f_1, \dots, f_s \rangle = \left\{ \sum_{i=1}^s h_i f_i : h_1, \dots, h_s \in K[x_1, \dots, x_n] \right\}. \quad (24)$$

Notice here that a set of polynomials f_1, \dots, f_s can be used both for defining the locus of an affine variety (def 6) or for specifying the generating set for an ideal (def 7). This fact represents just a small part in the deep relationship existing between affine varieties and ideals.

Theorem A.1 (Ideal-Variety Correspondence). *Let K be an arbitrary field. If $I_1 \subset I_2$ are ideals, then $\mathbf{V}(I_1) \supset \mathbf{V}(I_2)$ and similarly, if $V_1 \subset V_2$ are varieties, then $\mathbf{I}(V_1) \supset \mathbf{I}(V_2)$.*

The properties presented in this lemma are a direct consequence of the *Nullstellensatz* [22], and will be useful in the following sections for comparing the geometric structure of equilibria for the different SDS described before.

Another important result known as the Hilbert Basis Theorem states that every ideal in $K[x_1, \dots, x_n]$ is generated by a finite set (a basis) of polynomials. There can be many different bases for a given ideal, but there is a special one known as the *Groebner basis* whose properties are very useful for characterizations and computations with ideals.

Definition 8 (Groebner Basis). Given a monomial order on $K[x_1, \dots, x_n]$, a finite subset $G = \{g_1, \dots, g_t\}$ of an ideal I is said to be a Groebner basis if and only if the leading term on any element of I is divisible by the leading term of one of the g_i .

This definition may not clearly illustrate the importance of these bases as good generating sets for ideals. When a polynomial f is divided by a Groebner basis as $f = a_1 g_1 + \dots + a_t g_t + r$, the remainder r is uniquely determined. The condition that the remainder r is zero if and only if $f \in I$ is useful for testing the membership of the polynomial f to the ideal I . Another application for Groebner bases is in finding the implicit representation of a set of parametric equations by eliminating the parameter variables. The concept of an elimination ideal becomes relevant here. An important application for Groebner bases is in finding the implicit representation of a set of parametric equations by eliminating the parameter variables. The concept of an elimination ideal becomes relevant here.

Definition 9 (Elimination ideal). Given $I = \langle f_1, \dots, f_s \rangle \subset K[x_1, \dots, x_n]$, the l^{th} elimination ideal I_l is the ideal of $K[x_{l+1}, \dots, x_n]$ defined by $I_l = I \cap K[x_{l+1}, \dots, x_n]$.

We present now the theorem that allows to perform the elimination step for obtaining the implicit representation for an affine variety.

Theorem A.2 (Elimination Theorem). *Let $I \subset K[x_1, \dots, x_n]$ be an ideal and let G be a Groebner basis of I with respect to lexicographic order where $x_1 > x_2 > \dots > x_n$. Then, for every $0 \leq l \leq n$, the set $G_l = G \cap K[x_{l+1}, \dots, x_n]$ is a Groebner basis of the l^{th} elimination ideal I_l .*

The relevance of this theorem is that by computing a Groebner basis an elimination ideal can be found that does not involve any of the parameters that are to be eliminated. Eliminating the

parameters is called implicitization, that is, obtaining a set of implicit defining equations given a parametric representation.

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