Macroscopic evolution of mechanical and thermal energy in a harmonic chain with random flip of velocities

Tomasz Komorowski, Stefano Olla, Marielle Simon

▶ **To cite this version:**

Tomasz Komorowski, Stefano Olla, Marielle Simon. Macroscopic evolution of mechanical and thermal energy in a harmonic chain with random flip of velocities. 2016. hal-01358979v2

HAL Id: hal-01358979

https://hal.science/hal-01358979v2

Preprint submitted on 29 Mar 2017 (v2), last revised 20 Sep 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MACROSCOPIC EVOLUTION OF MECHANICAL AND THERMAL ENERGY IN A HARMONIC CHAIN WITH RANDOM FLIP OF VELOCITIES

TOMASZ KOMOROWSKI, STEFANO OLLA, AND MARIELLE SIMON

ABSTRACT. We consider an unpinned chain of harmonic oscillators with periodic boundary conditions, whose dynamics is perturbed by a random flip of the sign of the velocities. The dynamics conserves the total volume (or elongation) and the total energy of the system. We prove that in a diffusive space-time scaling limit the profiles corresponding to the two conserved quantities converge to the solution of a diffusive system of differential equations. While the elongation follows a simple autonomous linear diffusive equation, the evolution of the energy depends on the gradient of the square of the elongation.

1. INTRODUCTION

Harmonic chains with energy conserving random perturbations of the dynamics have recently received attention in the study of the macroscopic evolution of energy [1, 2, 5, 8, 10, 12]. They provide models that have a non-trivial macroscopic behavior which can be explicitly computed. We consider here the dynamics of an unpinned chain where the velocities of particles can randomly change sign. This random mechanism is equivalent to the deterministic collisions with independent *environment* particles of infinite mass. Since the chain is unpinned, the *relevant* conserved quantities of the dynamics are the *energy* and the *volume* (or elongation).

Under a diffusive space-time scaling, we prove that the profile of elongation evolves independently of the energy and follows the linear diffusive equation

$$\partial_t r(t,u) = \frac{1}{2\gamma}\partial^2_{uu} r(t,u). \tag{1.1}$$

Here u is the Lagrangian space coordinate of the system and $\gamma > 0$ is the intensity of the random mechanism of collisions. The energy profile can be decomposed into the sum of *mechanical* and *thermal* energy

$$e(t,u) = e_{\mathrm{mech}}(t,u) + e_{\mathrm{thm}}(t,u)$$

where the mechanical energy is given by $e_{\mathrm{mech}}(t,u) = \frac{1}{2} r(t,u)^2$, while the thermal part $e_{\mathrm{thm}}(t,u)$, that coincides with the temperature profile, evolves following the

Key words and phrases. Hydrodynamic limit, heat diffusion, Wigner distribution, thermalization.

non-linear equation:

$$\partial_t e_{\rm thm}(t,u) = \frac{1}{4\gamma}\partial^2_{uu} e_{\rm thm}(t,u) + \frac{1}{2\gamma}\left(\partial_u r(t,u)\right)^2 . \tag{1.2}$$

This is equivalent to the following conservation law for the total energy:

$$\partial_t e(t,u) = \frac{1}{4\gamma}\partial^2_{uu}\left(e(t,u) + \frac{r(t,u)^2}{2}\right). \tag{1.3}$$

The derivation of the macroscopic equations (1.1) and (1.2) from the microscopic dynamical system of particles, after a diffusive rescaling of space and time, is the goal of this paper. Concerning the distribution of the energy in the frequency modes: the mechanical energy $e_{\rm mech}(t,u)$ is concentrated on the modes corresponding to the largest wavelength, while the thermal energy $e_{\rm thm}(t,u)$ is distributed uniformly over all frequencies. Note that $\frac{1}{2\gamma}\left(\partial_u r(t,u)\right)^2$ is the rate of dissipation of the mechanical energy into thermal energy.

The presence of the non-linearity in the evolution of the energy makes the macroscopic limit non-trivial. Relative entropy methods (as introduced in [13]) identify correctly the limit equation (see [12]), but in order to make them rigorous one needs sharp bounds on higher moments than cannot be controlled by the relative entropy[1]. In this sense the proof in [12] is not complete.

We follow here a different approach based on Wigner distributions. The Wigner distributions permit to control the energy distribution over various frequency modes and provide a natural separation between mechanical and thermal energies. The initial positions and velocities of particles can be random, and the only condition we ask, besides to have definite mean asymptotic profiles of elongation and energy, is that the thermal energy spectrum has a square integrable density. In the macroscopic limit we prove that locally the thermal energy spectrum has a constant density equal to the local thermal energy (or temperature), i.e. that the system is, at macroscopic positive times, in local equilibrium, even though it is not at initial time. Also follows from our result that the mechanical energy is concentrated on the lowest modes. This is a stronger local equilibrium result than the one usually obtained with relative entropy techniques. The Wigner distribution approach had been successfully applied in different contexts for systems perturbed by noise with more conservation laws in [5, 8]. Here we need a particular *asymmetric* version of the Wigner distribution, in order to deal with a finite size discrete microscopic system.

When the system is pinned, only energy is conserved and its macroscopic evolution is linear, and much easier to be obtained. In this case the thermalization and the correlation structure have been studied in [9, 10].

[1]For more details related to these moments bounds, that are still conjectured but not proved (on the contrary to what is claimed in [12]), we refer the reader to an *erratum* which is available online at http://chercheurs.lille.inria.fr/masimon/erratum-v2.pdf.

When the chain of oscillators is *anharmonic*, still with velocity flip dynamics, the hydrodynamic limit is a difficult non-gradient problem, for the moment still open. In that case the macroscopic equations would be:

$$\begin{aligned} \partial_t r(t,u) &= \frac{1}{2\gamma}\partial^2_{uu}\big[\tau(r,e)\big], \\ \partial_t e(t,u) &= \partial_u\big[\mathcal{D}(r,e)\partial_u\beta^{-1}(r,e)\big] + \frac{1}{4\gamma}\partial^2_{uu}\left(\tau(r,e)^2\right). \end{aligned} \tag{1.4}$$

where $\tau(r,e)$ is the thermodynamic equilibrium tension as function of the volume r and of the energy e, and $\beta^{-1}(r,e)$ is the corresponding temperature, while $\mathcal{D}(r,e)$ is the thermal diffusivity defined by the usual Green-Kubo formula, as space-time variance of the energy current in the equilibrium infinite dynamics at average elongation r and energy e (see Section 4 for the definition of these quantities). The linear response and the existence of $\mathcal{D}(r,e)$ have been proven in [3].

2. Microscopic dynamics

2.1. **Periodic chain of oscillators.** In the following we denote by $\mathbb{T}_n := \mathbb{Z}/n\mathbb{Z} = \{0,\dots,n-1\}$ the discrete circle with n points, and, for any $L>0$, by $\mathbb{T}(L)$ the continuous circle of length L, and we set $\mathbb{T} := \mathbb{T}(1)$.

We consider a one-dimensional harmonic chain of n oscillators, all of mass 1, with periodic boundary conditions. The clearest way to describe this system is as a massive one dimensional discrete surface $\{\varphi_x \in \mathbb{R}, x \in \mathbb{T}_n\}$. The element (or particle) x of the surface is at height φ_x and has mass equal to 1. We call its velocity (that coincides with its momentum) $p_x \in \mathbb{R}$. Each particle x is connected to the particles $x-1$ and $x+1$ by harmonic springs, so that $n-1$ and 0 are connected in the same way. The total energy of the system is given by the Hamiltonian:

$$\mathcal{H}_n := \sum_{x\in\mathbb{T}_n} \mathcal{E}_x, \qquad \mathcal{E}_x := \frac{p_x^2}{2} + \frac{(\varphi_x-\varphi_{x-1})^2}{2}. \tag{2.1}$$

In addition to the Hamiltonian dynamics associated to the harmonic potentials, particles are subject to a random interaction with the environment: at independently distributed random Poissonian times, the momentum p_x is flipped into $-p_x$. The resulting equations of the motion are

$$\begin{cases} \mathrm{d}\varphi_x(t) = n^2 p_x(t)\,\mathrm{d}t, \\ \mathrm{d}p_x(t) = n^2\big(\varphi_{x+1}(t)+\varphi_{x-1}(t)-2\varphi_x(t)\big)\,\mathrm{d}t - 2p_x(t^-)\,\mathrm{d}\mathcal{N}_x(\gamma n^2 t), \end{cases} \tag{2.2}$$

for any $x \in \mathbb{T}_n$. Here $\{\mathcal{N}_x(t)\ ;\ t\geqslant 0,\ x\in\mathbb{T}_n\}$ are n independent Poisson processes of intensity 1, and the constant γ is positive. We have already accelerated the time scale by n^2, according to the diffusive scaling. Notice that the energy $\mathcal{H}_n$ is conserved by this dynamics. There is another important conservation law that is given by the sum of the elongations of the springs, that we define as follows.

We call $r_x = \varphi_x - \varphi_{x-1}$ the elongation of the spring between x and $x-1$, and since $x \in \mathbb{T}_n$ we have $r_0 = \varphi_0 - \varphi_{n-1}$. The equation of the dynamics in these coordinates are given by:

$$\begin{cases} \mathrm{d}r_x(t) = n^2\big(p_x(t) - p_{x-1}(t)\big)\,\mathrm{d}t \\ \mathrm{d}p_x(t) = n^2\big(r_{x+1}(t) - r_x(t)\big)\,\mathrm{d}t - 2p_x(t^-)\,\mathrm{d}\mathcal{N}_x(\gamma n^2 t), \qquad x \in \mathbb{T}_n. \end{cases} \tag{2.3}$$

This implies that the dynamics is completely defined giving the initial conditions $\{r_x(0), p_x(0), x \in \mathbb{T}_n\}$.

The periodicity in the φ_x variables would impose that $\sum_{x=0}^{n-1} r_x(0) = 0$. On the other hand the dynamics defined by (2.3) is well defined also if $\sum_{x=0}^{n-1} r_x(0) \neq 0$ and has the conservation law $\sum_{x=0}^{n-1} r_x(t) = \sum_{x=0}^{n-1} r_x(0) := R_n$. Note that R_n can also assume negative values. In this case we can picture the particles as n points $q_0, \dots, q_{n-1} \in \mathbb{T}(|R_n|)$, the circle of length $|R_n|$. These points can be defined as $q_x := \big[\sum_{y=0}^{x} r_y\big]_{\mathrm{mod}|R_n|}$, for $x = 0, \dots, n-1$. It follows that $q_n = q_0$. We will not use neither the q_x coordinates nor the φ_x coordinates, but we consider only the evolution defined by (2.3) with initial configurations $\sum_{x=0}^{n-1} r_x(0) = R_n \in \mathbb{R}$.

2.2. **Generator and invariant measures.** The generator of the stochastic dynamics ($\mathbf{r}(t) := \{r_x(t)\}_{x\in\mathbb{T}_n}$, $\mathbf{p}(t) := \{p_x(t)\}_{x\in\mathbb{T}_n}$), is given by

$$\mathcal{L}_n := n^2\mathcal{A}_n + n^2\gamma\,\mathcal{S}_n,$$

where the Liouville operator $\mathcal{A}_n$ is formally given by

$$\mathcal{A}_n = \sum_{x\in\mathbb{T}_n} \Big\{(p_x - p_{x-1})\frac{\partial}{\partial r_x} + (r_{x+1} - r_x)\frac{\partial}{\partial p_x}\Big\},$$

while, for $f : \Omega_n \to \mathbb{R}$,

$$\mathcal{S}_n f(\mathbf{r},\mathbf{p}) = \sum_{x\in\mathbb{T}_n} \big\{f(\mathbf{r},\mathbf{p}^x) - f(\mathbf{r},\mathbf{p})\big\}$$

where $\mathbf{p}^x$ is the configuration that is obtained from $\mathbf{p}$ by reversing the sign of the velocity at site x, namely: $(\mathbf{p}^x)_y = p_y$ if $y \neq x$ and $(\mathbf{p}^x)_x = -p_x$.

The two conserved quantities $\mathcal{H}_n = \sum_{x\in\mathbb{T}_n} \mathcal{E}_x$ and $R_n = \sum_{x\in\mathbb{T}_n} r_x$, are determined by the initial data (eventually random), and typically they should be proportional to n: $\mathcal{H}_n = ne, R_n = nr$, with $e \in \mathbb{R}_+$ the average energy per particle, and $r \in \mathbb{R}$ the average spring elongation. Consequently the system has a two parameters family of stationary measures given by the canonical Gibbs distributions

$$\mu^n_{\tau,\beta}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) = \prod_{x\in\mathbb{T}_n} \exp\big(-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}_{\tau,\beta}\big)\,\mathrm{d}r_x\mathrm{d}p_x, \qquad \beta > 0, \tau \in \mathbb{R},$$

where

$$\mathcal{G}_{\tau,\beta} = \log\Big[\sqrt{2\pi\beta^{-1}}\int_{\mathbb{R}} e^{-\frac{\beta}{2}(r^2 - 2\tau r)}\,\mathrm{d}r\Big] = \log\Big[2\pi\beta^{-1}\exp\Big(\frac{\tau^2\beta}{2}\Big)\Big].$$

As usual, the parameters $\beta^{-1} > 0$ and $\tau \in \mathbb{R}$ are called respectively *temperature* and *tension*. Observe that the function

$$r(\tau, \beta) = \beta^{-1}\, \partial_\tau \mathcal{G}_{\tau,\beta} = \tau \tag{2.4}$$

gives the average equilibrium length in function of the tension τ, and

$$\mathcal{E}(\tau, \beta) = \tau\, r(\tau, \beta) - \partial_\beta \mathcal{G}_{\tau,\beta} = \beta^{-1} + \frac{\tau^2}{2} \tag{2.5}$$

is the corresponding thermodynamic internal energy function. Note that the energy $\mathcal{E}(\tau, \beta)$ is composed by a *thermal* energy β^{-1} and a *mechanical* energy $\frac{\tau^2}{2}$.

2.3. **Hydrodynamic limits.** Let $\mu_n(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p})$ be an initial Borel probability distribution on Ω_n. We denote by $\mathbb{P}_n$ the law of the process $\{(\mathbf{r}(t), \mathbf{p}(t))\ ;\ t \geqslant 0\}$ starting from the measure μ_n and generating by $\mathcal{L}_n$, and by $\mathbb{E}_n$ its corresponding expectation. We are given initial continuous profiles of tension $\{\tau_0(u)\ ;\ u \in \mathbb{T}\}$ and of temperature $\{\beta_0^{-1}(u) > 0\ ;\ u \in \mathbb{T}\}$. The thermodynamic relations (2.4) and (2.5) give the corresponding initial profiles of elongation and energy as

$$r_0(u) := \tau_0(u) \quad \text{and} \quad e_0(u) := \frac{1}{\beta_0(u)} + \frac{\tau_0^2(u)}{2}, \qquad u \in \mathbb{T}. \tag{2.6}$$

The initial distributions μ_n are assumed to satisfy the following mean convergence statements:

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\Big(\frac{x}{n}\Big) \mathbb{E}_n\big[r_x(0)\big] \xrightarrow[n \to +\infty]{} \int_{\mathbb{T}} G(u)\, r_0(u)\, \mathrm{d}u, \tag{2.7}$$

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\Big(\frac{x}{n}\Big) \mathbb{E}_n\big[\mathcal{E}_x(0)\big] \xrightarrow[n \to +\infty]{} \int_{\mathbb{T}} G(u)\, e_0(u)\, \mathrm{d}u, \tag{2.8}$$

for any test function G that belongs to the set $\mathcal{C}^\infty(\mathbb{T})$ of smooth functions on the torus. We expect the same convergence to happen at the macroscopic time t:

$$\begin{aligned} &\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\Big(\frac{x}{n}\Big) \mathbb{E}_n\big[r_x(t)\big] \xrightarrow[n \to +\infty]{} \int_{\mathbb{T}} G(u)\, r(t,u)\, \mathrm{d}u, \\ &\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\Big(\frac{x}{n}\Big) \mathbb{E}_n\big[\mathcal{E}_x(t)\big] \xrightarrow[n \to +\infty]{} \int_{\mathbb{T}} G(u)\, e(t,u)\, \mathrm{d}u, \end{aligned} \tag{2.9}$$

where the macroscopic evolution for the volume and energy profiles follows the system of equations:

$$\begin{cases} \partial_t r(t,u) = \dfrac{1}{2\gamma} \partial^2_{uu} r(t,u), \\ \partial_t e(t,u) = \dfrac{1}{4\gamma} \partial^2_{uu} \Big(e + \dfrac{r^2}{2}\Big)(t,u), \end{cases} \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{T}, \tag{2.10}$$

with the initial condition

$$r(0,u) = r_0(u), \qquad e(0,u) = e_0(u).$$

The solutions $e(t,\cdot)$, $r(t,\cdot)$ of (2.10) are smooth when $t>0$ (the system of partial differential equations is parabolic). Note that the evolution of $r(t,u)$ is autonomous of $e(t,u)$. The precise assumptions that are needed for the convergence (2.9) are stated in Theorems 3.7 and 3.8 below.

3. Main results

3.1. **Notations.**

3.1.1. *Discrete Fourier transform.* Let us denote by $\widehat{f}$ the Fourier transform of a finite sequence $\{f_x\}_{x\in\mathbb{T}_n}$ of numbers in $\mathbb{C}$, defined as follows:

$$\widehat{f}(k)=\sum_{x\in\mathbb{T}_n} f_x\, e^{-2i\pi xk}, \qquad k\in\widehat{\mathbb{T}}_n:=\big\{0,\tfrac{1}{n},...,\tfrac{n-1}{n}\big\}. \tag{3.1}$$

Reciprocally, for any $f:\widehat{\mathbb{T}}_n\to\mathbb{C}$, we denote by $\big\{\check{f}_x\big\}_{x\in\mathbb{T}_n}$ its inverse Fourier transform given by

$$\check{f}_x=\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} e^{2i\pi xk} f(k), \qquad x\in\mathbb{T}_n. \tag{3.2}$$

The Parseval identity reads

$$\|f\|^2_{\mathbf{L}^2}:=\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n}\big|\widehat{f}(k)\big|^2=\sum_{x\in\mathbb{T}_n}\big|f_x\big|^2. \tag{3.3}$$

If $\{f_x\}_{x\in\mathbb{T}_n}$ and $\{g_x\}_{x\in\mathbb{T}_n}$ are two sequences indexed by the discrete torus, their convolution is given by

$$(f*g)_x:=\sum_{y\in\mathbb{T}_n} f_y\, g_{x-y}, \qquad x\in\mathbb{T}_n.$$

3.1.2. *Continuous Fourier transform.* Let $\mathcal{C}(\mathbb{T})$ be the the space of continuous, complex valued functions on $\mathbb{T}$. For any function $G\in\mathcal{C}(\mathbb{T})$, let $\mathcal{F}G:\mathbb{Z}\to\mathbb{C}$ denote its Fourier transform given as follows:

$$\mathcal{F}G(\eta):=\int_{\mathbb{T}} G(u)\, e^{-2i\pi u\eta}\,\mathrm{d}u, \qquad \eta\in\mathbb{Z}. \tag{3.4}$$

Similar identities to (3.2) and (3.3) can easily be written: for instance, we shall repeatedly use the following

$$G(u)=\sum_{\eta\in\mathbb{Z}}\mathcal{F}G(\eta)\, e^{2i\pi\eta u}, \qquad u\in\mathbb{T}. \tag{3.5}$$

Note that when G is smooth the Fourier coefficients satisfy

$$\sup_{\eta\in\mathbb{Z}}\Big\{(1+\eta^2)^p|\mathcal{F}G(\eta)|\Big\}<+\infty, \qquad \text{for any } p\in\mathbb{N}. \tag{3.6}$$

If $J:\mathbb{T}\times\mathbb{T}\to\mathbb{C}$ is defined on a two-dimensional torus, we still denote by $\mathcal{F}J(\eta,v)$, $(\eta,v)\in\mathbb{Z}\times\mathbb{T}$, its Fourier transform with respect to the first variable. We equip

the set $\mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$ of smooth (with respect to the first variable) functions with the norm

$$\|J\|_0 := \sum_{\eta\in\mathbb{Z}} \sup_{v\in\mathbb{T}} \big|\mathcal{F}J(\eta,v)\big|. \tag{3.7}$$

Let $\mathcal{A}_0$ be the completion of $\mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$ in this norm and $(\mathcal{A}_0', \|\cdot\|_0')$ its dual space.

3.1.3. *A fundamental example.* In what follows, we often consider the discrete Fourier transform associated to a function $G\in\mathcal{C}(\mathbb{T})$, and to avoid any confusion we introduce a new notation: let $\mathcal{F}_nG : \widehat{\mathbb{T}}_n \to \mathbb{C}$ be the discrete Fourier transform of the finite sequence $\{G(\frac{x}{n})\}_{x\in\mathbb{T}_n}$ defined similarly to (3.1) as

$$\mathcal{F}_nG(k) := \sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)e^{-2i\pi xk}, \qquad k\in\widehat{\mathbb{T}}_n.$$

In particular, we have the Parseval identity

$$\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\, f_x^\star = \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} (\mathcal{F}_nG)(k)\, \widehat{f}^\star(k). \tag{3.8}$$

Furthermore, note that

$$\frac{1}{n}\mathcal{F}_nG\left(\frac{\eta}{n}\right) \underset{n\to\infty}{\longrightarrow} \mathcal{F}G(\eta), \qquad \text{for any } \eta\in\mathbb{Z}.$$

3.2. **Assumptions on initial data.** Without losing too much of generality, one can put natural assumptions on the initial probability measure $\mu_n(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p})$.

The first assumption concerns the mean of the initial configurations, and is sufficient in order to derive the first of the hydrodynamic equations (2.10) :

Assumption 3.1.

- *The initial total energy can be random but with uniformly bounded expectation:*

$$\sup_{n\geqslant 1} \mathbb{E}_n\left[\frac{1}{n}\sum_{x\in\mathbb{T}_n}\mathcal{E}_x(0)\right] < +\infty. \tag{3.9}$$

- *We assume that there exist continuous initial profiles $r_0 : \mathbb{T}\to\mathbb{R}$ and $e_0 : \mathbb{T}\to(0,+\infty)$ such that*

$$\mathbb{E}_n[p_x(0)] = 0, \qquad \mathbb{E}_n[r_x(0)] = r_0\Big(\frac{x}{n}\Big) \qquad \textit{for any } x\in\mathbb{T}_n \tag{3.10}$$

and for any $G\in\mathcal{C}^\infty(\mathbb{T})$

$$\frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[\mathcal{E}_x(0)\big] \xrightarrow[n\to+\infty]{} \int_{\mathbb{T}} G(u)\, e_0(u)\, \mathrm{d}u. \tag{3.11}$$

Identity (3.10), in particular, implies the mean convergence of the initial elongation:

$$\frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(0)\big] \xrightarrow[n\to+\infty]{} \int_{\mathbb{T}} G(u)\, r_0(u)\, \mathrm{d}u, \tag{3.12}$$

for any $G \in \mathcal{C}^\infty(\mathbb{T})$.

Remark 3.2. *By energy conservation* (3.9) *implies that*

$$\sup_{n\geqslant 1} \mathbb{E}_n\left[\frac{1}{n}\sum_{x\in\mathbb{T}_n} \mathcal{E}_x(t)\right] < +\infty, \qquad \text{for all } t \geqslant 0. \tag{3.13}$$

Remark 3.3. *Conditions in* (3.10) *are assumed in order to simplify the proof, but they can be easily relaxed.*

Next assumption is important to obtain the macroscopic equation for the energy in (2.10). It concerns the energy spectrum of fluctuations around the means at initial time. Define the *initial thermal energy spectrum* $\mathfrak{u}_n(0,k)$, $k \in \widehat{\mathbb{T}}_n$, as follows: let $\widehat{r}(0,k)$ and $\widehat{p}(0,k)$ denote respectively the Fourier transforms of the initial random configurations $\{r_x(0)\}_{x\in\mathbb{T}_n}$ and $\{p_x(0)\}_{x\in\mathbb{T}_n}$, and let

$$\mathfrak{u}_n(0,k) := \frac{1}{2n}\mathbb{E}_n\Big[\big|\widehat{p}(0,k)\big|^2 + \big|\widehat{r}(0,k) - \mathbb{E}_n[\widehat{r}(0,k)]\big|^2\Big], \quad k \in \widehat{\mathbb{T}}_n. \tag{3.14}$$

Due to the Parseval identity (3.3) we have

$$\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathfrak{u}_n(0,k) = \frac{1}{2n}\sum_{x\in\mathbb{T}_n} \mathbb{E}_n\Big[p_x^2(0) + \big(r_x(0) - \mathbb{E}_n[r_x(0)]\big)^2\Big].$$

Assumption 3.4. (Square integrable initial thermal energy spectrum)

$$\sup_{n\geqslant 1}\left\{\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathfrak{u}_n^2(0,k)\right\} < +\infty. \tag{3.15}$$

Remark 3.5. *Assumptions 3.1 and 3.4 are satisfied if the measures* μ_n *are given by local Gibbs measures (non homogeneous product), corresponding to the given initial profiles of tension and temperature* $\{\tau_0(u), \beta_0^{-1}(u)\ ;\ u \in \mathbb{T}\}$, *defined as follows:*

$$\mathrm{d}\mu^n_{\tau_0(\cdot),\beta_0(\cdot)} = \prod_{x\in\mathbb{T}_n} \exp\Big\{-\beta_0\Big(\frac{x}{n}\Big)\Big(\mathcal{E}_x - \tau_0\Big(\frac{x}{n}\Big)r_x\Big) - \mathcal{G}_{\tau_0(\frac{x}{n}),\beta_0(\frac{x}{n})}\Big\}\,\mathrm{d}r_x\mathrm{d}p_x. \tag{3.16}$$

with $r_0(u) = \tau_0(u)$ *and* $e_0(u) = \beta_0^{-1}(u) + \frac{r_0^2(u)}{2}$, *see* [7, Sections 9.2.3–9.2.5]. *Note that our assumptions are much more general, as we do not assume any specific condition on the correlation structure of* μ_n. *In particular microcanonical versions of* (3.16), *where total energy and total volumes are conditioned at fixed values ne and nr, are included by our assumptions.*

Remark 3.6. *Intuitively, our assumptions state that the initial energy has a mechanical part, related to* $\tau_0(\cdot)$ *that concentrates on the longest wavelength (i.e. around* $k = 0$*) while the thermal energy has a square integrable density w.r.t. k.*

3.3. **Formulation of mean convergence.** In this section we state two theorems dealing with the mean convergence of the two conserved quantities, namely the elongation and energy. The first one (Theorem 3.7) is proved straightforwardly in Section 3.4 below. The second one is more involved, and is the main subject of the present paper.

Theorem 3.7 (Mean convergence of the elongation profile)**.** *Assume that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures on Ω_n such that (3.10) is satisfied, with $r_0 \in \mathcal{C}(\mathbb{T})$. Let $r(t,u)$ be the solution defined on $\mathbb{R}_+ \times \mathbb{T}$ of the linear diffusive equation:*

$$\begin{cases} \partial_t r(t,u) = \dfrac{1}{2\gamma}\partial^2_{uu} r(t,u), & (t,u) \in \mathbb{R}_+ \times \mathbb{T}, \\ r(0,u) = r_0(u). \end{cases} \tag{3.17}$$

Then, for any $G \in \mathcal{C}^\infty(\mathbb{T})$ and $t \in \mathbb{R}_+$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(t)\big] = 0, \tag{3.18}$$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(t)\big] = \int_{\mathbb{T}} G(u)\, r(t,u)\, \mathrm{d}u. \tag{3.19}$$

Theorem 3.8 (Mean convergence of the empirical profile of energy)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on Ω_n such that Assumptions 3.1 and 3.4 are satisfied. Then, for any smooth function $G : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ compactly supported with respect to the time variable $t \in \mathbb{R}_+$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{x\in\mathbb{T}_n} \int_{\mathbb{R}_+} G\Big(t, \frac{x}{n}\Big)\mathbb{E}_n\big[\mathcal{E}_x(t)\big]\, \mathrm{d}t = \int_{\mathbb{R}_+\times\mathbb{T}} G(t,u)\, e(t,u)\, \mathrm{d}t\mathrm{d}u, \tag{3.20}$$

where $e(t,u) = e_{\mathrm{mech}}(t,u) + e_{\mathrm{thm}}(t,u)$, with

- *the* mechanical energy*, given by $e_{\mathrm{mech}}(t,u) := \frac{1}{2}\big(r(t,u)\big)^2$ and the function $r(t,u)$ being the solution of (3.17),*
- *the* thermal energy *$e_{\mathrm{thm}}(t,u)$, defined as the solution of*

$$\begin{cases} \partial_t e_{\mathrm{thm}}(t,u) = \dfrac{1}{4\gamma}\partial^2_{uu} e_{\mathrm{thm}}(t,u) + \dfrac{1}{2\gamma}\big(\partial_u r(t,u)\big)^2, \\ e_{\mathrm{thm}}(0,u) = \beta_0^{-1}(u) = e_0(u) - e_{\mathrm{mech}}(0,u) > 0. \end{cases} \tag{3.21}$$

The proof of Theorem 3.8 is presented in Sections 5 – 7.

Remark 3.9. *Note that (3.17) and (3.21) are equivalent to the system (2.10). This new way of seeing the macroscopic equations is more convenient, as it naturally arises from the proof. More precisely, using (3.17) we conclude that the mechanical energy $e_{\mathrm{mech}}(t,u)$ satisfies the equation*

$$\partial_t e_{\mathrm{mech}}(t,u) = \frac{1}{2\gamma}\Big(\partial^2_{uu} e_{\mathrm{mech}}(t,u) - \big(\partial_u r(t,u)\big)^2\Big)$$

and the macroscopic energy density function satisfies

$$\begin{cases} \partial_t e(t,u) = \dfrac{1}{4\gamma}\partial^2_{uu}\big(e(t,u) + e_{\mathrm{mech}}(t,u)\big), \\ e(0,u) = e_0(u). \end{cases}$$

Remark 3.10. *We actually prove a stronger result that includes a local equilibrium statement, see Theorem 7.5 below.*

3.4. **Proof of the hydrodynamic limit for the elongation.** Here we give a simple proof of Theorem 3.7. From the evolution equations (2.3) we have the following identities:

$$\frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(t) - r_x(0)\big] = n^2\int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(s) - p_{x-1}(s)\big]\mathrm{d}s$$

and

$$\begin{aligned} 2\gamma n^2\int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(s)\big]\mathrm{d}s = n^2\int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_{x+1}(s) - r_x(s)\big]\mathrm{d}s \\ + \frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(0) - p_x(t)\big]. \end{aligned}$$

Substituting from the second equation into the first one we conclude that

$$\begin{aligned} \frac{1}{n}\sum_{x\in\mathbb{T}_n} G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(t) - r_x(0)\big] = \int_0^t \frac{1}{2\gamma n}\sum_{x\in\mathbb{T}_n} \Delta_n G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(s)\big]\mathrm{d}s \\ - \frac{1}{2\gamma n^2}\sum_{x\in\mathbb{T}_n} \nabla_n G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(0) - p_x(t)\big], \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \nabla_n G\Big(\frac{x}{n}\Big) &= n\Big(G\Big(\frac{x+1}{n}\Big) - G\Big(\frac{x}{n}\Big)\Big) \\ \Delta_n G\Big(\frac{x}{n}\Big) &= n\Big(\nabla_n G\Big(\frac{x}{n}\Big) - \nabla_n G\Big(\frac{x-1}{n}\Big)\Big). \end{aligned}$$

By energy conservation and Assumption 3.1 it is easy to see that

$$\begin{aligned} \Big|\frac{1}{n^2}\sum_{x\in\mathbb{T}_n} \nabla_n G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(t)\big]\Big|^2 &\leqslant \frac{1}{n^2}\Big(\frac{1}{n}\sum_{x\in\mathbb{T}_n}\Big|\nabla_n G\Big(\frac{x}{n}\Big)\Big|^2\Big)\Big(\frac{1}{n}\sum_{x\in\mathbb{T}_n}\mathbb{E}_n\big[p_x^2(t)\big]\Big) \\ &\leqslant \frac{C(G)}{n}\Big(\frac{1}{n}\sum_{x\in\mathbb{T}_n}\mathbb{E}_n\big[\mathcal{E}_x^2(0)\big]\Big) \underset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.23}$$

Let us define

$$\bar{r}^{(n)}(t,u) := \mathbb{E}_n\big[r_x(t)\big], \qquad \text{for any } u \in \big[\tfrac{x}{n}, \tfrac{x+1}{n}\big), \quad n \geqslant 1.$$

Thanks to the energy conservation we know that there exists $R > 0$ such that

$$\sup_{n\geqslant 1}\sup_{t\in[0,T]}\|\bar{r}^{(n)}(t,\cdot)\|_{\mathbf{L}^2(\mathbb{T})} =: R < +\infty. \tag{3.24}$$

The above means that for each $t \in [0,T]$ the sequence $\{\bar{r}^{(n)}(t,\cdot)\}_{n\geqslant 1}$ is contained in $\bar{B}_R$ – the closed ball of radius $R > 0$ in $\mathbf{L}^2(\mathbb{T})$, centered at 0. The ball is compact in $\mathbf{L}^2_w(\mathbb{T})$ – the space of square integrable functions on the torus $\mathbb{T}$ equipped with the weak $\mathbf{L}^2$ topology. The topology restricted to $\bar{B}_R$ is metrizable, with the respective metric given e.g. by

$$d(f,g) := \sum_{n=1}^{+\infty}\frac{1}{2^n}\frac{|\langle f-g,\phi_n\rangle_{\mathbf{L}^2(\mathbb{T})}|}{1+|\langle f-g,\phi_n\rangle_{\mathbf{L}^2(\mathbb{T})}|}, \quad f,g\in\bar{B}_R,$$

where $\{\phi_n\}$ is a countable and dense subset of $\mathbf{L}^2(\mathbb{T})$ that can be chosen of elements of $\mathcal{C}^\infty(\mathbb{T})$. From (3.22) and (3.23) we conclude in particular that for each $T > 0$ the sequence $\{\bar{r}^{(n)}(\cdot)\}$ is equicontinuous in $\mathcal{C}\left([0,T],\bar{B}_R\right)$. Thus, according to the Arzela Theorem, see e.g. [6, p. 234], it is sequentially pre-compact in the space $\mathcal{C}\left([0,T],\mathbf{L}^2_w(\mathbb{T})\right)$ for any $T > 0$. Consequently, any limiting point of the sequence satisfies the partial differential equation (3.17) in a weak sense in the class of $\mathbf{L}^2(\mathbb{T})$ functions. Uniqueness of the weak solution of the heat equation gives the convergence claimed in (3.19) and the identification of the limit as the strong solution of (3.17).

Concerning (3.18), from (2.3) we have

$$\begin{aligned}\frac{1}{n}\sum_{x\in\mathbb{T}_n}G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(t)\big] =& \frac{e^{-2\gamma n^2 t}}{n}\sum_{x\in\mathbb{T}_n}G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[p_x(0)\big]\\ &+\int_0^t e^{-2\gamma n^2(t-s)}\sum_{x\in\mathbb{T}_n}\nabla_n^* G\Big(\frac{x}{n}\Big)\mathbb{E}_n\big[r_x(s)\big]\mathrm{d}s,\end{aligned}$$

where $\nabla_n^* G(\frac{x}{n}) = n\left(G(\frac{x}{n}) - G(\frac{x-1}{n})\right)$. Using again energy conservation and the Cauchy-Schwarz inequality, it is easy to see that the right hand side of the above vanishes as $n \to \infty$.

Remark 3.11. *Note that we have not used the fact that the initial average of the velocities vanishes. Additionally, by standard methods it is possible to obtain the convergence of elongation and momentum empirical distributions in probability (see (3.18) and (3.19)), but we shall not pursuit this here.*

4. Conjecture for anharmonic interaction and thermodynamic considerations

Our results concern only harmonic interactions, but we can state the expected macroscopic behavior for the anharmonic case. Consider a non-quadratic potential $V(r)$, of class $\mathcal{C}^1$ and growing fast enough to $+\infty$ as $|r| \to \infty$. The dynamics

is now defined by

$$\begin{cases} \mathrm{d}r_x(t) = n^2\big(p_x(t) - p_{x-1}(t)\big)\,\mathrm{d}t \\ \mathrm{d}p_x(t) = n^2\big(V'(r_{x+1}(t)) - V'(r_x(t))\big)\,\mathrm{d}t - 2p_x(t^-)\,\mathrm{d}\mathcal{N}_x(\gamma n^2 t),\ x \in \mathbb{T}_n. \end{cases} \tag{4.1}$$

The stationary measures are given by the canonical Gibbs distributions

$$d\mu^n_{\tau,\beta} = \prod_{x\in\mathbb{T}_n} e^{-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}_{\tau,\beta}}\,\mathrm{d}r_x\,\mathrm{d}p_x, \qquad \tau \in \mathbb{R}, \beta > 0, \tag{4.2}$$

where we denote

$$\mathcal{E}_x = \frac{p_x^2}{2} + V(r_x),$$

the energy that we attribute to the particle x, and

$$\mathcal{G}_{\tau,\beta} = \log\left[\sqrt{2\pi\beta^{-1}}\int e^{-\beta(V(r)-\tau r)}\,\mathrm{d}r\right]. \tag{4.3}$$

Thermodynamic entropy $S(r,e)$ is defined as

$$S(r,e) = \inf_{\tau\in\mathbb{R},\beta>0}\big\{\beta e - \beta\tau r + \mathcal{G}(\tau,\beta)\big\}. \tag{4.4}$$

Then we obtain the inverse temperature and tension as functions of the volume r and internal energy u:

$$\boldsymbol{\beta}^{-1}(r,e) = \partial_e S(r,e), \qquad \boldsymbol{\tau}(r,e) = -\boldsymbol{\beta}^{-1}(r,e)\partial_r S(r,e) \tag{4.5}$$

The macroscopic profiles of elongation $r(t,u)$ and energy $e(t,u)$ will satisfy the equations

$$\begin{aligned} \partial_t r &= \frac{1}{2\gamma}\partial^2_{uu}\big[\boldsymbol{\tau}(r,e)\big], \\ \partial_t e &= \partial_u\big[\mathcal{D}(r,e)\partial_u\boldsymbol{\beta}^{-1}\big] + \frac{1}{4\gamma}\partial^2_{uu}\left(\boldsymbol{\tau}(r,e)^2\right). \end{aligned} \tag{4.6}$$

Here the diffusivity $\mathcal{D}(r,e) > 0$ is defined by a Green-Kubo formula for the infinite dynamics in equilibrium at the given values (r,e). The precise definition and the proof of the convergence of Green-Kubo formula for this dynamics can be found in [3].

A straightforward calculation gives the expected increase of thermodynamic entropy:

$$\frac{d}{dt}\int_{\mathbb{T}} S(r(t,u),e(t,u))\,\mathrm{d}u = \int_{\mathbb{T}} \boldsymbol{\beta}\left(\frac{(\partial_u\boldsymbol{\tau})^2}{2\gamma} + \mathcal{D}(r,e)\big(\partial_u\boldsymbol{\beta}^{-1}\big)^2\right)\mathrm{d}u \geqslant 0. \tag{4.7}$$

5. Time-dependent Wigner distributions

5.1. **Wave function for the system of oscillators.** Let $\widehat{p}(t,k)$ and $\widehat{r}(t,k)$, for $k \in \widehat{\mathbb{T}}_n$, denote the Fourier transforms of, respectively, the momentum and elongation components of the microscopic configurations $\{p_x(t)\}_{x\in\mathbb{T}_n}$ and $\{r_x(t)\}_{x\in\mathbb{T}_n}$, as in (3.1). Since they are real valued we have, for any $k \in \widehat{\mathbb{T}}_n$,

$$\widehat{p}^\star(t,k) = \sum_{x\in\mathbb{T}_n} e^{2\pi i k x} p_x(t) = \widehat{p}(t,-k) \quad \text{and likewise} \quad \widehat{r}^\star(t,k) = \widehat{r}(t,-k). \tag{5.1}$$

The *wave function* associated to the dynamics is defined as

$$\psi_x(t) := r_x(t) + i p_x(t), \qquad x \in \mathbb{T}_n.$$

Its Fourier transform equals

$$\widehat{\psi}(t,k) := \widehat{r}(t,k) + i\widehat{p}(t,k), \qquad k \in \widehat{\mathbb{T}}_n.$$

Taking into account (5.1) we obtain

$$\widehat{p}(t,k) = \frac{1}{2i}\big(\widehat{\psi}(t,k) - \widehat{\psi}^\star(t,-k)\big),$$
$$\widehat{r}(t,k) = \frac{1}{2}\big(\widehat{\psi}(t,k) + \widehat{\psi}^\star(t,-k)\big).$$

With these definitions we have $|\psi_x|^2 = 2\mathcal{E}_x$ and the initial thermal energy spectrum, defined in (3.14), satisfies

$$\mathfrak{u}_n(0,k) = \widetilde{\mathfrak{u}}_n(0,k) + \mathrm{Im}\big[\mathrm{Cov}_n(\widehat{p}(0,k),\widehat{r}(0,k))\big], \qquad k \in \widehat{\mathbb{T}}_n. \tag{5.2}$$

Here $\mathrm{Cov}_n(X,Y) := \mathbb{E}_n[XY^\star] - \mathbb{E}_n[X]\,\mathbb{E}_n[Y^\star]$ is the covariance of complex random variables X and Y, and

$$\widetilde{\mathfrak{u}}_n(0,k) := \frac{1}{2n}\mathbb{E}_n\Big[\big|\widehat{\psi}(0,k) - \mathbb{E}_n[\widehat{\psi}(0,k)]\big|^2\Big]$$

Using Cauchy-Schwarz inequality we conclude easily that

$$\frac{1}{2}\widetilde{\mathfrak{u}}_n(0,k) \leqslant \mathfrak{u}_n(0,k) \leqslant 2\widetilde{\mathfrak{u}}_n(0,k), \quad k \in \widehat{\mathbb{T}}_n.$$

Therefore, (3.15) is equivalent with

$$\sup_{n\geqslant 1}\left\{\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n}\widetilde{\mathfrak{u}}_n^2(0,k)\right\} < +\infty. \tag{5.3}$$

After a straightforward calculation, the equation that governs the time evolution of the wave function can be deduced from (2.3) as follows:

$$\begin{aligned}\mathrm{d}\widehat{\psi}(t,k) = &-n^2\Big(2i\sin^2(\pi k)\widehat{\psi}(t,k) + \sin(2\pi k)\widehat{\psi}^\star(t,-k)\Big)\,\mathrm{d}t \\ &-\frac{1}{n}\sum_{k'\in\widehat{\mathbb{T}}_n}\Big\{\widehat{\psi}(t^-,k-k') - \widehat{\psi}^\star(t^-,k'-k)\Big\}\,\mathrm{d}\widehat{\mathcal{N}}(t,k'),\end{aligned} \tag{5.4}$$

with initial condition $\widehat{\psi}(0,k) = \widehat{r}(0,k)$. The semi-martingales $\big\{\widehat{\mathcal{N}}(t,k)\ ;\ t \geqslant 0\big\}$ are defined as

$$\widehat{\mathcal{N}}(t,k) := \sum_{x\in\mathbb{T}_n} \mathcal{N}_x(\gamma n^2 t) e^{-2i\pi xk}, \qquad k \in \widehat{\mathbb{T}}_n.$$

Observe that we have $\widehat{\mathcal{N}}^\star(t,k) = \widehat{\mathcal{N}}(t,-k)$. In addition, its mean and covariance equal respectively

$$\begin{aligned}
\big\langle \mathrm{d}\widehat{\mathcal{N}}(t,k)\big\rangle &= \gamma n^3\ \delta_{k,0}\ \mathrm{d}t,\\
\big\langle \mathrm{d}\widehat{\mathcal{N}}(t,k), \mathrm{d}\widehat{\mathcal{N}}(t,k')\big\rangle &= \gamma n^3 t\ \delta_{k,-k'}\ \mathrm{d}t,
\end{aligned}$$

where $\delta_{x,y}$ is the usual Kronecker delta function, which equals 1 if $x = y$ and 0 otherwise. The conservation of energy, and Parseval's identity, imply together that:

$$\big\|\widehat{\psi}(t)\big\|_{\mathbf{L}^2} = \big\|\widehat{\psi}(0)\big\|_{\mathbf{L}^2} \quad \text{for all } t \geqslant 0. \tag{5.5}$$

5.2. **Wigner distributions and Fourier transforms.** The *Wigner distribution* $\mathbf{W}_n^+(t)$ corresponding to the wave function $\psi(t)$ is a distribution defined by its action on smooth functions $G \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$ as

$$\big\langle \mathbf{W}_n^+(t),\ G\big\rangle := \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n}\sum_{\eta\in\mathbb{Z}} W_n^+(t,\eta,k)(\mathcal{F}G)^\star(\eta,k), \tag{5.6}$$

where the *Wigner function* $W_n^+(t)$ is given for any $(k,\eta) \in \widehat{\mathbb{T}}_n \times \mathbb{Z}$ by

$$W_n^+(t,\eta,k) := \frac{1}{2n}\mathbb{E}_n\Big[\widehat{\psi}\Big(t, k+\frac{\eta}{n}\Big)\widehat{\psi}^\star(t,k)\Big]. \tag{5.7}$$

Here, we use the mapping $\mathbb{Z} \ni \eta \mapsto \frac{\eta}{n} \in \widehat{\mathbb{T}}_n$, and $\mathcal{F}G(\eta,v)$ denotes the Fourier transform with respect to the first variable.

Remark 5.1. *Note that this definition of the Wigner function is not the standard symmetric one. Indeed, since the setting here is discrete, it turns out that* (5.7) *is the convenient way to identify the Fourier modes, otherwise we would have worked with ill-defined quantities, for instance* $\frac{\eta}{2n}$*, which are not always integers.*

The main interest of the Wigner distribution is that mean convergence of the empirical energy profile (3.20) can be restated in terms of convergence of Wigner functions (more precisely, their Laplace transforms, see Theorem 7.5 below), thanks to the following identity: if $G(u,v) \equiv G(u)$ does not depend on the second variable $v \in \mathbb{T}$, then

$$\big\langle \mathbf{W}_n^+(t), G\big\rangle = \frac{1}{n}\sum_{x\in\mathbb{T}_n} \mathbb{E}_n\big[\mathcal{E}_x(t)\big] G\Big(\frac{x}{n}\Big). \tag{5.8}$$

Indeed, from (5.6) we obtain then

$$\big\langle \mathbf{W}_n^+(t), G\big\rangle = \frac{1}{2n^2}\sum_{k\in\widehat{\mathbb{T}}_n}\sum_{x,x'\in\mathbb{T}_n}\sum_{\eta\in\mathbb{Z}}\mathbb{E}_n\big[\psi^\star_{x'}(t)\psi_x(t)\big]e^{2\pi i(x'-x)k}e^{2\pi i x\frac{\eta}{n}}(\mathcal{F}G)^\star(\eta). \tag{5.9}$$

Performing the summation over k we conclude that the right hand side equals

$$\frac{1}{2n}\sum_{x,x'\in\mathbb{T}_n}\sum_{\eta\in\mathbb{Z}}\mathbb{E}_n\big[\psi^\star_{x'}(t)\psi_x(t)\big]1_{\mathbb{Z}}\Big(\frac{x'-x}{n}\Big)e^{2\pi i x\frac{\eta}{n}}(\mathcal{F}G)^\star(\eta), \tag{5.10}$$

where $1_{\mathbb{Z}}$ is the indicator function of the set of integers. Since $0\leqslant x,x'\leqslant n-1$ and $|\psi_x(t)|^2 = 2\mathcal{E}_x(t)$ we conclude that the above expression equals

$$\frac{1}{n}\sum_{x\in\mathbb{T}_n}\sum_{\eta\in\mathbb{Z}}\mathbb{E}_n\big[\mathcal{E}_x(t)\big]e^{2\pi i x\frac{\eta}{n}}(\mathcal{F}G)^\star(\eta) = \frac{1}{n}\sum_{x\in\mathbb{T}_n}\mathbb{E}_n\big[\mathcal{E}_x(t)\big]G\left(\frac{x}{n}\right) \tag{5.11}$$

and (5.8) follows. Identity (5.8) can be also interpreted as follows: the k-average of the Wigner distribution gives the Fourier transform of the energy:

$$\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n}W_n^+(t,\eta,k) = \frac{1}{2n}\sum_{x\in\mathbb{T}_n}e^{-2\pi i x\frac{\eta}{n}}\,\mathbb{E}_n\big[|\psi_x(t)|^2\big] = \frac{1}{n}\,\mathbb{E}_n\left[\widehat{\mathcal{E}}\left(t,\frac{\eta}{n}\right)\right],\quad \eta\in\mathbb{Z}. \tag{5.12}$$

To close the equations governing the evolution of $\mathbf{W}_n^+(t)$, we need to define three other Wigner-type functions. We let

$$W_n^-(t,\eta,k) := \frac{1}{2n}\mathbb{E}_n\Big[\widehat{\psi}^\star\Big(t,-k-\frac{\eta}{n}\Big)\widehat{\psi}(t,-k)\Big] = (W_n^+)^\star(t,-\eta,-k), \tag{5.13}$$

$$Y_n^+(t,\eta,k) := \frac{1}{2n}\mathbb{E}_n\Big[\widehat{\psi}\Big(t,k+\frac{\eta}{n}\Big)\widehat{\psi}(t,-k)\Big], \tag{5.14}$$

$$Y_n^-(t,\eta,k) := \frac{1}{2n}\mathbb{E}_n\Big[\widehat{\psi}^\star\Big(t,-k-\frac{\eta}{n}\Big)\widehat{\psi}^\star(t,k)\Big] = \big(Y_n^+\big)^\star(t,-\eta,-k). \tag{5.15}$$

5.3. **Properties of the Wigner distributions.**

5.3.1. *Weak convergence.* From (3.13) we have that, for any $n\geqslant 1$ and $G\in\mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$,

$$\begin{aligned}
&\big|\big\langle \mathbf{W}_n^+(t), G\big\rangle\big| \qquad (5.16)\\
&\quad\leqslant \frac{1}{(2n)^2}\sup_{\eta\in\mathbb{Z}}\left\{\sum_{k\in\widehat{\mathbb{T}}_n}\mathbb{E}_n\Big[\Big|\widehat{\psi}\Big(t,k+\frac{\eta}{n}\Big)\Big|^2\Big]\right\}^{\frac12}\left\{\sum_{k\in\widehat{\mathbb{T}}_n}\mathbb{E}_n\Big[\big|\widehat{\psi}(t,k)\big|^2\Big]\right\}^{\frac12}\|G\|_0\\
&\quad\leqslant \frac{\|G\|_0}{2n}\,\mathbb{E}_n\Big[\sum_{x\in\mathbb{T}_n}\mathcal{E}_x(t)\Big]\leqslant C\|G\|_0,
\end{aligned}$$

where the norm $\|G\|_0$ is defined by (3.7). Hence, for the corresponding dual norm, we have the bound

$$\sup_{t\geqslant 0}\sup_{n\geqslant 1}\left\{\left\|\mathbf{W}_n^+(t)\right\|_0' + \left\|\mathbf{W}_n^-(t)\right\|_0' + \left\|\mathbf{Y}_n^+(t)\right\|_0' + \left\|\mathbf{Y}_n^-(t)\right\|_0'\right\} \leqslant 4C. \tag{5.17}$$

Condition (3.11) ensures that, if $G(u,v)\equiv G(u)$ at the initial time $t=0$ we have

$$\lim_{n\to+\infty}\left\langle \mathbf{W}_n^{\pm}(0), G\right\rangle = \int_{\mathbb{T}} e_0(u)\, G^\star(u)\, \mathrm{d}u,$$

5.3.2. *Decomposition into the mechanical and fluctuating part.* We decompose the wave function into its mean w.r.t. $\mathbb{E}_n$ and its fluctuating part, as follows:

$$\psi_x(t) = \overline{\psi}_x(t) + \widetilde{\psi}_x(t), \qquad x\in\mathbb{T}_n,\ t\geqslant 0.$$

Notice that for the initial data we have $\overline{\psi}_x(0) = r_0\left(\frac{x}{n}\right)$. It need not be true for $t>0$. The Fourier transform of the sequences $\{\overline{\psi}_x(t)\}$ and $\{\widetilde{\psi}_x(t)\}$ shall be denoted by $\widehat{\overline{\psi}}(t,k)$ and $\widehat{\widetilde{\psi}}(t,k)$. The deterministic function $\widehat{\overline{\psi}}(t,k)$ satisfies the autonomous equation:

$$\begin{aligned}\mathrm{d}\widehat{\overline{\psi}}(t,k) =& -n^2\Big(2i\sin^2(\pi k)\widehat{\overline{\psi}}(t,k) + \sin(2\pi k)\widehat{\overline{\psi}}^\star(t,-k)\Big)\,\mathrm{d}t\\ &- n^2\gamma\left\{\widehat{\overline{\psi}}(t,k) - \widehat{\overline{\psi}}^\star(t,-k)\right\}\mathrm{d}t,\end{aligned} \tag{5.18}$$

with initial condition $\widehat{\overline{\psi}}(0,k) = (\mathcal{F}_n r_0)(k)$.

The Wigner distribution $\mathbf{W}_n^+(t)$ can be decomposed accordingly as follows:

$$\mathbf{W}_n^+(t) = \overline{\mathbf{W}}_n^+(t) + \widetilde{\mathbf{W}}_n^+(t), \tag{5.19}$$

where the Fourier transforms of $\overline{\mathbf{W}}_n^+(t)$ and $\widetilde{\mathbf{W}}_n^+(t)$ are given by

$$\overline{W}_n^+(t,\eta,k) := \frac{1}{2n}\widehat{\overline{\psi}}\Big(t,k+\frac{\eta}{n}\Big)\,\widehat{\overline{\psi}}^\star(t,k) \tag{5.20}$$

$$\widetilde{W}_n^+(t,\eta,k) := \frac{1}{2n}\mathbb{E}_n\left[\widehat{\widetilde{\psi}}\Big(t,k+\frac{\eta}{n}\Big)\,\widehat{\widetilde{\psi}}^\star(t,k)\right], \tag{5.21}$$

for any $(\eta,k)\in\mathbb{Z}\times\widehat{\mathbb{T}}_n$.

At initial time $t=0$, to simplify notations we write $\overline{W}_n^+(\eta,k) := \overline{W}_n^+(0,\eta,k)$ and $\widetilde{W}_n^+(\eta,k) := \widetilde{W}_n^+(0,\eta,k)$. Note that the mean part is completely explicit: we have

$$\overline{W}_n^+(\eta,k) = \frac{1}{2n}(\mathcal{F}_n r_0)\left(k+\frac{\eta}{n}\right)(\mathcal{F}_n r_0)^\star(k), \tag{5.22}$$

Recalling (5.2), the initial thermal energy spectrum can be rewritten as

$$\mathfrak{u}_n(0,k) = \frac{1}{2n}\mathbb{E}_n\left[\left|\widehat{\widetilde{\psi}}(0,k)\right|^2\right].$$

Reproducing the decomposition (5.14) one can easily write similar definitions for $\overline{\mathbf{W}}_n^-$, $\widetilde{\mathbf{W}}_n^-$ and the respective $\overline{\mathbf{Y}}_n^\pm$, $\widetilde{\mathbf{Y}}_n^\pm$ distributions. Due to the fact that $\bar{\psi}_x(0) = r_0(\frac{x}{n})$ and is real-valued (and extending the convention of omitting the argument $t = 0$ for all Wigner-type distributions) we have

$$\overline{W}_n^-(\eta, k) = \overline{W}_n^+(\eta, k) = \overline{Y}_n^+(\eta, k) = \overline{Y}_n^-(\eta, k) = W_n(r_0\ ;\ \eta, k), \tag{5.23}$$

where we define

$$W_n(r\ ;\ \eta, k) := \frac{1}{2n}(\mathcal{F}_n r)\left(k + \frac{\eta}{n}\right)(\mathcal{F}_n r)^\star(k), \tag{5.24}$$

for any $r \in \mathcal{C}(\mathbb{T})$ and $(\eta, k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n$.

5.3.3. *Asymptotics of $\widetilde{W}_n^+$.* Throughout the remainder of the paper we shall use the following notation: given a function $f : \widehat{\mathbb{T}}_n \to \mathbb{C}$ we denote its k-average by

$$\big[f(\cdot)\big]_n := \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} f(k). \tag{5.25}$$

The initial fluctuating Wigner function is related, as $n \to +\infty$, to the initial thermal energy $e_{\mathrm{thm}}(0, u) = e_0(u) - r_0^2(u)/2$ as follows:

$$\Big[\widetilde{W}_n^+(\eta, \cdot)\Big]_n \xrightarrow[n\to\infty]{} \big(\mathcal{F}e_{\mathrm{thm}}(0, \cdot)\big)(\eta), \qquad \text{for any } \eta \in \mathbb{Z}. \tag{5.26}$$

The last convergence follows from Assumption 3.1 and from an explicit computation that yields:

$$\Big[\widetilde{W}_n^+(\eta, \cdot)\Big]_n = \frac{1}{n}\sum_{x\in\mathbb{T}_n} \mathbb{E}_n\big[\mathcal{E}_x(0)\big]e^{-2\pi i\eta\frac{x}{n}} - \mathcal{F}_n\Big(\frac{r_0^2}{2}\Big)\Big(\frac{\eta}{n}\Big).$$

In addition, Assumption 3.4 on the initial spectrum (see (3.15)) implies that

$$w_* := \sup_{n\geqslant 1}\sup_{\eta\in\mathbb{Z}}\left\{\sum_{\iota\in\{-,+\}}\left[\left|\widetilde{W}_n^\iota(\eta, \cdot)\right|^2 + \left|\widetilde{Y}_n^\iota(\eta, \cdot)\right|^2\right]_n\right\} < +\infty. \tag{5.27}$$

5.3.4. *Wigner distribution associated to a macroscopic smooth profile.* Similarly to (5.24), given a continuous real-valued function $\{r := r(u)\ ;\ u \in \mathbb{T}\}$, define

$$W(r\ ;\ \eta, \xi) := \frac{1}{2}(\mathcal{F}r)(\xi + \eta)(\mathcal{F}r)^\star(\xi), \qquad (\eta, \xi) \in \mathbb{Z}^2. \tag{5.28}$$

Observe that, for any $\eta \in \mathbb{Z}$,

$$\sum_{\xi\in\mathbb{Z}} W(r\ ;\ \eta, \xi) = \frac{1}{2}\mathcal{F}(r^2)(\eta). \tag{5.29}$$

Proposition 5.2. *Suppose that $r \in \mathcal{C}(\mathbb{T})$ and $G \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$. Then*

$$\begin{aligned}\lim_{n\to+\infty} \sum_{\eta\in\mathbb{Z}} \Big[W_n(r\ ;\ \eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\Big]_n &= \sum_{\eta\in\mathbb{Z}} \Big(\sum_{\xi\in\mathbb{Z}} W(r\ ;\ \eta,\xi)\Big)(\mathcal{F}G)^\star(\eta,0)\\ &= \frac{1}{2}\int_{\mathbb{T}} r^2(u)G^\star(u,0)\,\mathrm{d}u.\end{aligned} \tag{5.30}$$

Proof. We prove the proposition under the assumption that $r \in \mathcal{C}^\infty(\mathbb{T})$. The general case can be obtained by an approximation of a continuous initial profile by a sequence of smooth ones.

Using the dominated convergence theorem we conclude that the expression on the left hand side of (5.30) equals $\sum_{\eta\in\mathbb{Z}} \lim_{n\to+\infty} f_n(\eta)$, with

$$f_n(\eta) := \Big[\overline{W}_n^+(\eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\Big]_n, \qquad \eta\in\mathbb{Z}. \tag{5.31}$$

This can be written as

$$f_n(\eta) = \frac{1}{2n^2}\sum_{k\in\widehat{\mathbb{T}}_n}\sum_{x,x'\in\mathbb{T}_n} e^{-2\pi i\eta\frac{x}{n}}e^{2\pi ik(x'-x)}r\Big(\frac{x}{n}\Big)r\Big(\frac{x'}{n}\Big)(\mathcal{F}G)^\star(\eta,k).$$

Using Fourier representation (see (3.5)), we can write $f_n(\eta)$ as

$$\frac{1}{2n^2}\sum_{\xi,\xi'\in\mathbb{Z}}\sum_{m,x,x'\in\mathbb{T}_n} e^{-2\pi i(m+\xi+\eta)\frac{x}{n}}e^{2\pi i(m+\xi')\frac{x'}{n}}\ (\mathcal{F}r)^\star(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^\star\Big(\eta,\frac{m}{n}\Big).$$

Due to smoothness of $r(\cdot)$, its Fourier coefficients decay rapidly. Therefore for a fixed $\eta\in\mathbb{Z}$ and for any $\rho\in(0,1)$ we have that the limit $\lim_{n\to+\infty} f_n(\eta)$ equals

$$\lim_{n\to+\infty}\frac{1}{2n^2}\sum_{\substack{|\xi|\leqslant n^\rho\\ |\xi'|\leqslant n^\rho}}\sum_{m,x,x'\in\mathbb{T}_n} e^{-2\pi i(m+\xi+\eta)\frac{x}{n}}e^{2\pi i(m+\xi')\frac{x'}{n}}\ (\mathcal{F}r)^\star(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^\star\Big(\eta,\frac{m}{n}\Big).$$

Summing over x,x' we conclude that $\lim_{n\to+\infty} f_n(\eta)$ equals

$$\lim_{n\to+\infty}\frac{1}{2}\sum_{\substack{|\xi|\leqslant n^\rho\\ |\xi'|\leqslant n^\rho}}\sum_{m\in\mathbb{T}_n} 1_{\mathbb{Z}}\Big(\frac{m+\xi+\eta}{n}\Big)1_{\mathbb{Z}}\Big(\frac{m+\xi'}{n}\Big)(\mathcal{F}r)^\star(\xi)(\mathcal{F}r_0)(\xi')(\mathcal{F}G)^\star\Big(\eta,\frac{m}{n}\Big).$$

Taking into account the fact that $m\in\mathbb{T}_n$ and the magnitude of ξ,ξ' is negligible when compared with n we conclude that the terms under the summation on the right hand side are non zero only if $m+\xi'=0$ and $m+\xi+\eta=0$, or $m+\xi'=n$

and $m+\xi+\eta=n$. Therefore,

$$\begin{aligned}\lim_{n\to+\infty} f_n(\eta) &= \lim_{n\to+\infty} \frac{1}{2} \sum_{\substack{|\xi|\leqslant n^\rho \\ |\xi'|\leqslant n^\rho}} \sum_{m\in\mathbb{T}_n} \delta_{n,m+\xi+\eta}\, \delta_{n,m+\xi'}\, (\mathcal{F}r)^\star(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^\star\left(\eta, \frac{m}{n}\right) \\ &\quad + \lim_{n\to+\infty} \frac{1}{2} \sum_{\substack{|\xi|\leqslant n^\rho \\ |\xi'|\leqslant n^\rho}} \sum_{m\in\mathbb{T}_n} \delta_{0,m+\xi+\eta}\, \delta_{0,m+\xi'}\, (\mathcal{F}r)^\star(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^\star\left(\eta, \frac{m}{n}\right) \\ &= \frac{1}{2}\, (\mathcal{F}G)^\star(\eta,0) \sum_{\xi\in\mathbb{Z}} (\mathcal{F}r)^\star(\xi)(\mathcal{F}r)(\xi+\eta)\end{aligned}$$

and formula (5.30) follows. □

In particular, for the initial conditions of our dynamics, Proposition 5.2 and (5.23) imply:

$$\begin{aligned}\lim_{n\to+\infty} \sum_{\eta\in\mathbb{Z}} \left[\overline{W}_n^+(\eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\right]_n &= \sum_{\xi,\eta\in\mathbb{Z}} W(r_0\,;\,\eta,\xi)(\mathcal{F}G)^\star(\eta,0) \\ &= \frac{1}{2}\int_{\mathbb{T}} r_0^2(u) G^\star(u,0)\,\mathrm{d}u.\end{aligned} \tag{5.32}$$

One of the main point of the proof of our theorem is to show that this convergence holds for any macroscopic time $t>0$, i.e. that for any compactly supported $G\in\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{T}^2)$ we have

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{\eta\in\mathbb{Z}} \int_0^{+\infty} \left[\overline{W}_n^+(t,\eta,\cdot)(\mathcal{F}G)^\star(t,\eta,\cdot)\right]_n \mathrm{d}t \\ &\qquad = \sum_{\xi,\eta\in\mathbb{Z}} \int_0^{+\infty} W(r_t\,;\,\eta,\xi)(\mathcal{F}G)^\star(t,\eta,0)\mathrm{d}t \\ &\qquad = \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{T}} r^2(t,u)G(t,u)\,\mathrm{d}t\mathrm{d}u.\end{aligned} \tag{5.33}$$

This would amount to showing that the Wigner distribution $\overline{W}_n^+(t)$ associated to $\overline{\psi}_x(t)$ is asymptotically equivalent to the one corresponding to the macroscopic profile $r(t,\cdot)$ via (5.28). This fact is not a consequence of Theorem 3.7, that implies only a weak convergence of $\overline{\psi}_x(t)$ to $r(t,\cdot)$. We will prove (5.33) in Proposition 7.1 below, showing the convergence of the corresponding Laplace transforms (see (7.2)).

6. Strategy of the proof and explicit resolutions

6.1. **Laplace transform of Wigner functions.** Since our subsequent argument is based on an application of the Laplace transform of the Wigner functions, we give some explicit formulas for the object that can be written in case of our

model. For any bounded complex-valued, Borel measurable function $\mathbb{R}_+ \ni t \mapsto f_t$ we define the Laplace operator $\mathcal{L}$ as:

$$\mathcal{L}(f_\cdot)(\lambda) := \int_0^{+\infty} e^{-\lambda t} f_t \, \mathrm{d}t, \qquad \lambda > 0.$$

Given the solution $r_t = r(t, \cdot)$ of (3.17), we define the Laplace transform of the Wigner distribution (5.28) associated to a macroscopic profile, as follows: for any $(\lambda, \eta, \xi) \in \mathbb{R}_+ \times \mathbb{Z}^2$,

$$w(r_\cdot \, ; \, \eta, \xi)(\lambda) = \mathcal{L}\big(W(r_\cdot \, ; \, \eta, \xi)\big)(\lambda) = \int_0^{+\infty} e^{-\lambda t} \, W(r_t \, ; \, \eta, \xi) \, \mathrm{d}t.$$

The following formulas are easily deduced, by a direct calculation, from (3.17), and are left to the reader:

Lemma 6.1. *For any $(\lambda, \eta, \xi) \in \mathbb{R}_+ \times \mathbb{Z}^2$ we have*

$$w(r_\cdot \, ; \, \eta, \xi)(\lambda) = \frac{W(r_0 \, ; \, \eta, \xi)}{\frac{2\pi^2}{\gamma} \left[\xi^2 + (\eta + \xi)^2\right] + \lambda}. \tag{6.1}$$

Consequently,

$$\begin{aligned} \frac{1}{2} \mathcal{L}\Big(\mathcal{F}\left((\partial_u r_t)^2\right)(\eta)\Big)(\lambda) &= \frac{1}{2} \int_0^{+\infty} e^{-\lambda t} \, \mathcal{F}\left((\partial_u r_t)^2\right)(\eta) \, \mathrm{d}t \\ &= 4\pi^2 \sum_{\xi \in \mathbb{Z}} (\eta + \xi)\xi \, w(r_\cdot \, ; \, \eta, \xi)(\lambda). \end{aligned} \tag{6.2}$$

Finally, we define $\mathbf{w}_n^+$ the Laplace transform of $\mathbf{W}_n^+$ as the tempered distribution given for any $G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ and $\lambda > 0$ by

$$\left\langle \mathbf{w}_n^+(\lambda), \, G \right\rangle = \int_0^{+\infty} e^{-\lambda t} \left\langle \mathbf{W}_n^+(t), \, G \right\rangle \mathrm{d}t := \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \sum_{\eta \in \mathbb{Z}} w_n^+(\lambda, \eta, k) (\mathcal{F}G)^\star(\eta, k), \tag{6.3}$$

where w_n^+ is the Laplace transform of W_n^+ as follows:

$$w_n^+(\lambda, \eta, k) := \int_0^{+\infty} e^{-\lambda t} \, W_n^+(t, \eta, k) \, \mathrm{d}t, \qquad (\lambda, \eta, k) \in \mathbb{R}_+ \times \mathbb{Z} \times \widehat{\mathbb{T}}_n.$$

In a similar fashion we can also define $\mathbf{w}_n^-(\lambda)$ and $\mathbf{y}_n^\pm(\lambda)$ the Laplace transforms of $\mathbf{W}_n^-(t)$ and $\mathbf{Y}_n^\pm(t)$, respectively, and their counterparts w_n^-, and $y_n^\pm$.

6.2. **Dynamics of the Wigner distributions.** Using the time evolution equations (5.4), one can first write a closed system of evolution equations for $W_n^\pm(t), Y_n^\pm(t)$ defined respectively in (5.7), (5.13), (5.14), (5.15).

For that purpose we first define two functions $\delta_n s$ and $\sigma_n s$ as follows: for any $(\eta, k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n$,

$$
\begin{aligned}
(\delta_n s)(\eta, k) &:= 2n\Big(\sin^2 \Big(\pi \Big(k + \frac{\eta}{n} \Big) \Big) - \sin^2(\pi k) \Big), \\
(\sigma_n s)(\eta, k) &:= 2\Big(\sin^2 \Big(\pi \Big(k + \frac{\eta}{n} \Big) \Big) + \sin^2(\pi k) \Big).
\end{aligned}
\tag{6.4}
$$

For the brevity sake, we drop the variables $(t, \eta, k) \in \mathbb{R}_+ \times \mathbb{Z} \times \widehat{\mathbb{T}}_n$ from the subsequent notation. From (5.4) one can easily check that:

$$
\left\{
\begin{aligned}
\partial_t W_n^+ &= -in(\delta_n s)\, W_n^+ - n^2 \sin(2\pi k)\, Y_n^+ - n^2 \sin\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big)\, Y_n^- \\
&\quad + \gamma n^2\, \mathbb{L}\big(2W_n^+ - Y_n^+ - Y_n^-\big), \\
\partial_t Y_n^+ &= n^2 \sin(2\pi k)\, W_n^+ - in^2(\sigma_n s)\, Y_n^+ - n^2 \sin\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big)\, W_n^- \\
&\quad + \gamma n^2\, \mathbb{L}\big(2Y_n^+ - W_n^+ - W_n^-\big) + \gamma n \sum_{k \in \widehat{\mathbb{T}}_n} (Y_n^- - Y_n^+), \\
\partial_t Y_n^- &= n^2 \sin\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big)\, W_n^+ + in^2(\sigma_n s)\, Y_n^- - n^2 \sin(2\pi k)\, W_n^- \\
&\quad + \gamma n^2\, \mathbb{L}\big(2Y_n^- - W_n^+ - W_n^-\big) + \gamma n \sum_{k \in \widehat{\mathbb{T}}_n} (Y_n^+ - Y_n^-), \\
\partial_t W_n^- &= in(\delta_n s)\, W_n^- + n^2 \sin\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big)\, Y_n^+ + n^2 \sin(2\pi k)\, Y_n^- \\
&\quad + \gamma n^2\, \mathbb{L}\big(2W_n^- - Y_n^+ - Y_n^-\big),
\end{aligned}
\right.
\tag{6.5}
$$

where $\mathbb{L}$ is the operator that is defined for any $f : \mathbb{Z} \times \widehat{\mathbb{T}}_n \to \mathbb{C}$ as

$$
(\mathbb{L}f)(\eta, k) := \big[f(\eta, \cdot) \big]_n - f(\eta, k), \qquad (\eta, k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n.
$$

Recalling the decomposition (5.19) and the evolution equations (5.18) for the mean part of the wave function, we have similarly that $\overline{W}_n^{\pm}(t), \overline{Y}_n^{\pm}(t)$ satisfy the

autonomous equations:

$$\begin{cases} \partial_t \overline{W}_n^+ = -in(\delta_n s)\,\overline{W}_n^+ - n^2\sin(2\pi k)\,\overline{Y}_n^+ - n^2\sin\big(2\pi\big(k+\frac{\eta}{n}\big)\big)\,\overline{Y}_n^- \\ \qquad\qquad - \gamma n^2\big(2\overline{W}_n^+ - \overline{Y}_n^+ - \overline{Y}_n^-\big), \\ \partial_t \overline{Y}_n^+ = n^2\sin(2\pi k)\,\overline{W}_n^+ - in^2(\sigma_n s)\,\overline{Y}_n^+ - n^2\sin\big(2\pi\big(k+\frac{\eta}{n}\big)\big)\,\overline{W}_n^- \\ \qquad\qquad - \gamma n^2\,\big(2\overline{Y}_n^+ - \overline{W}_n^+ - \overline{W}_n^-\big), \\ \partial_t \overline{Y}_n^- = n^2\sin\big(2\pi\big(k+\frac{\eta}{n}\big)\big)\,\overline{W}_n^+ + in^2(\sigma_n s)\,\overline{Y}_n^- - n^2\sin(2\pi k)\,\overline{W}_n^- \\ \qquad\qquad - \gamma n^2\,\big(2\overline{Y}_n^- - \overline{W}_n^+ - \overline{W}_n^-\big) \\ \partial_t \overline{W}_n^- = in(\delta_n s)\,\overline{W}_n^- + n^2\sin\big(2\pi\big(k+\frac{\eta}{n}\big)\big)\,\overline{Y}_n^+ + n^2\sin(2\pi k)\,\overline{Y}_n^- \\ \qquad\qquad - \gamma n^2\,\big(2\overline{W}_n^- - \overline{Y}_n^+ - \overline{Y}_n^-\big), \end{cases} \tag{6.6}$$

6.3. **Laplace transform of the dynamical system.** We deduce from (6.5) an equation satisfied by $\mathfrak{w}_n$ - the four-dimensional vector of Laplace transforms of the Wigner functions defined by $\mathfrak{w}_n := [w_n^+, y_n^+, y_n^-, w_n^-]^{\mathrm{T}}$. For the clarity sake we shall use the notation

$$\mathbf{1} := [1,1,1,1]^{\mathrm{T}}, \quad \mathrm{e} := [1,-1,-1,1]^{\mathrm{T}},$$

$$\mathfrak{v}_n^0 := \big[W_n^+(0), Y_n^+(0), Y_n^-(0), W_n^-(0)\big]^{\mathrm{T}},$$

$$\overline{\mathfrak{v}}_n^0 := \big[\overline{W}_n^+, \overline{Y}_n^+, \overline{Y}_n^-, \overline{W}_n^-\big]^{\mathrm{T}}, \qquad \widetilde{\mathfrak{v}}_n^0 := \big[\widetilde{W}_n^+, \widetilde{Y}_n^+, \widetilde{Y}_n^-, \widetilde{W}_n^-\big]^{\mathrm{T}},$$

and from (5.19) we have $\mathfrak{v}_n^0 = \overline{\mathfrak{v}}_n^0 + \widetilde{\mathfrak{v}}_n^0$. As before, we shall often drop the variables (λ,η,k) from the notations. Taking the Laplace transform of both sides of (6.5) we obtain a linear system that can be written for any (λ,η,k) in the form

$$(\mathbf{M}_n\,\mathfrak{w}_n)(\lambda,\eta,k) = \mathfrak{v}_n^0(\eta,k) + \gamma n^2\,\mathcal{I}_n(\lambda,\eta)\,\mathrm{e}, \tag{6.7}$$

where $\mathcal{I}_n$ is defined as the scalar product

$$\mathcal{I}_n(\lambda,\eta) := \mathrm{e}\cdot\Big[\mathfrak{w}_n(\lambda,\eta,\cdot)\Big]_n = \Big[\big(w_n^+ - y_n^+ - y_n^- + w_n^-\big)(\lambda,\eta,\cdot)\Big]_n.$$

In addition, the 2×2 block matrix $\mathbf{M}_n := \mathbf{M}_n(\lambda,\eta,k)$ is defined as follows:

$$\mathbf{M}_n := \begin{bmatrix} A_n & -n^2\,\gamma_n^-\,\mathrm{Id}_2 \\ -n^2\,\gamma_n^+\,\mathrm{Id}_2 & B_n \end{bmatrix}, \tag{6.8}$$

where, given a positive integer N, Id_N denotes the $N\times N$ identity matrix, and A_n, B_n are 2×2 matrices:

$$A_n := \begin{bmatrix} a_n & -n^2\gamma^- \\ -n^2\gamma^+ & b_n \end{bmatrix}, \qquad B_n := \begin{bmatrix} b_n^\star & -n^2\gamma^- \\ -n^2\gamma^+ & a_n^\star \end{bmatrix}.$$

Here and below,

$$\begin{cases} a_n := \lambda + in(\delta_n s) + 2\gamma n^2, \\ b_n := \lambda + in^2(\sigma_n s) + 2\gamma n^2, \end{cases} \qquad \begin{cases} \gamma_n^{\pm} := \gamma \pm \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right), \\ \gamma^{\pm} := \gamma \pm \sin(2\pi k). \end{cases} \tag{6.9}$$

Remark 6.2. *Note that the term $\gamma n^2\, \mathcal{I}_n(\lambda, \eta)\, \mathrm{e}$ (in the right hand side of (6.7)) contains all the effect of the random flips of velocities, which occur with intensity $\gamma > 0$. The other components of the system ($\mathbf{M}_n$ and $\mathfrak{v}_n^0$) only come from the Hamiltonian dynamics and the initial condition.*

An elementary observation yields the following symmetry properties

$$\gamma_n^{\pm}(-\eta, -k) = \gamma_n^{\mp}(\eta, k), \quad \gamma^{\pm}(-\eta, -k) = \gamma^{\mp}(\eta, k), \tag{6.10}$$

By the linearity of the Laplace transform we can write

$$\mathfrak{w}_n = \overline{\mathfrak{w}}_n + \widetilde{\mathfrak{w}}_n, \tag{6.11}$$

where $\overline{\mathfrak{w}}_n$ is the Laplace transform of $(\overline{W}_n^+(t), \overline{Y}_n^+(t), \overline{Y}_n^-(t), \overline{W}_n^-(t))$, and $\widetilde{\mathfrak{w}}_n$ is the Laplace transform of $(\widetilde{W}_n^+(t), \widetilde{Y}_n^+(t), \widetilde{Y}_n^-(t), \widetilde{W}_n^-(t))$.

Performing the Laplace transform of both sides of (6.6) we conclude that $\overline{\mathfrak{w}}_n$ solves the equation

$$\mathbf{M}_n \overline{\mathfrak{w}}_n = \overline{\mathfrak{v}}_n^0 = \overline{W}_n^+\, \mathbf{1}. \tag{6.12}$$

Using (6.7) we conclude that $\widetilde{\mathfrak{w}}_n$ solves

$$\mathbf{M}_n \widetilde{\mathfrak{w}}_n = \widetilde{\mathfrak{v}}_n^0 + \gamma n^2\, \mathcal{I}_n\, \mathrm{e}. \tag{6.13}$$

Following (6.11) we also write $\mathcal{I}_n = \overline{\mathcal{I}}_n + \widetilde{\mathcal{I}}_n$, where

$$\overline{\mathcal{I}}_n(\lambda, \eta) := \mathrm{e} \cdot \left[\overline{\mathfrak{w}}_n(\lambda, \eta, \cdot)\right]_n \quad \text{and} \quad \widetilde{\mathcal{I}}_n(\lambda, \eta) := \mathrm{e} \cdot \left[\widetilde{\mathfrak{w}}_n(\lambda, \eta, \cdot)\right]_n.$$

In Appendix 8.1, we show that the matrix $\mathbf{M}_n$ is invertible, therefore we can solve and rewrite (6.12) and (6.13) as:

$$\overline{\mathfrak{w}}_n = \overline{W}_n^+\, \mathbf{M}_n^{-1}\, \mathbf{1}, \tag{6.14}$$

$$\widetilde{\mathfrak{w}}_n = \mathbf{M}_n^{-1}\, \widetilde{\mathfrak{v}}_n^0 + \gamma n^2\, \mathcal{I}_n\, \mathbf{M}_n^{-1}\, \mathrm{e}. \tag{6.15}$$

In Section 7 we study the contribution of the terms appearing in the right hand sides of both (6.14) and (6.15) that reflect upon the evolution of the mechanical and fluctuating components of the energy functional.

7. Proof of the hydrodynamic behavior of the energy

In this section we conclude the proof of Theorem 3.8, up to technical lemmas that are proved in Section 8.

7.1. **Mechanical energy $\overline{\mathfrak{w}}_n$.** We start with the recollection of the results concerning the mechanical energy. The Laplace transform $\overline{\mathfrak{w}}_n$ is autonomous from the thermal part and satisfies (6.14). Let us introduce, for any $\lambda > 0$ and $\eta \in \mathbb{Z}$, the *mechanical Laplace-Wigner function*

$$\mathcal{W}^{+}_{\rm mech}(\lambda,\eta) := \sum_{\xi\in\mathbb{Z}} \frac{W(r_0\ ;\ \eta,\xi)}{\frac{2\pi^2}{\gamma}[\xi^2+(\xi+\eta)^2]+\lambda}. \tag{7.1}$$

From Lemma 6.1, it follows that $\mathcal{W}^{+}_{\rm mech}(\lambda,\eta)$ is the Fourier-Laplace transform of the mechanical energy density $e_{\rm mech}(t,u) = \frac{1}{2}\left(r(t,u)\right)^2$, where $r(t,u)$ is the solution of (3.17).

Given $M \in \mathbb{N}$ we denote by $\mathcal{P}_M$ the subspace of $\mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$ consisting of all trigonometric polynomials that are finite linear combinations of $e^{2\pi i\eta u}e^{2\pi i\xi v}$, with $\eta \in \{-M,\dots,M\}$, $\xi\in\mathbb{Z}$ and $u,v\in\mathbb{T}$.

Proposition 7.1 (Mechanical part)**.** *For any $M\in\mathbb{N}$ there exists $\lambda_M > 0$ such that for any $G\in\mathcal{P}_M$ and $\lambda > \lambda_M$ we have*

$$\begin{aligned}\lim_{n\to\infty}\sum_{\eta\in\mathbb{Z}}\Big[\overline{\mathfrak{w}}_n(\lambda,\eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\Big]_n &= \left(\sum_{\eta,\xi\in\mathbb{Z}}\frac{W(r_0\ ;\ \eta,\xi)}{\frac{2\pi^2}{\gamma}[\xi^2+(\xi+\eta)^2]+\lambda}(\mathcal{F}G)^\star(\eta,0)\right)\mathbf{1}\\ &= \left(\frac{1}{2}\int_{\mathbb{T}}\int_0^{+\infty} e^{-\lambda t}\ \left(r(t,u)\right)^2\ G^\star(u,0)\,\mathrm{d}t\,\mathrm{d}u\right)\mathbf{1}.\end{aligned} \tag{7.2}$$

Moreover, for any $\eta\in\{-M,...,M\}$

$$\begin{aligned}\lim_{n\to\infty}\Big\{\gamma n^2\,\overline{\mathcal{I}}_n(\lambda,\eta)\Big\} &= \frac{4\pi^2}{\gamma}\sum_{\xi\in\mathbb{Z}}\frac{\xi(\xi+\eta)W(r_0\ ;\ \eta,\xi)}{\frac{2\pi^2}{\gamma}[\xi^2+(\xi+\eta)^2]+\lambda}\\ &= \frac{1}{2\gamma}\mathcal{L}\Big(\mathcal{F}\left((\partial_u r)^2\right)(\eta)\Big)(\lambda).\end{aligned} \tag{7.3}$$

The proof of Proposition 7.1 is exposed in Section 8.3.

7.2. **The closing of thermal energy equation.** We now analyse equation (6.15) concerning the fluctuating part. After averaging (6.15) over $k\in\widehat{\mathbb{T}}_n$ and scalarly multiplying by e, we obtain the equation:

$$\widetilde{\mathcal{I}}_n = \widetilde{z}^{(0)}_n + \gamma n^2\left(\widetilde{\mathcal{I}}_n + \overline{\mathcal{I}}_n\right)\mathcal{M}_n, \tag{7.4}$$

where

$$\begin{aligned}\widetilde{z}^{(0)}_n &:= \Big[\mathrm{e}\cdot(\mathbf{M}_n^{-1}\ \widetilde{\mathfrak{v}}^0_n)\Big]_n\\ \mathcal{M}_n &:= \Big[\mathrm{e}\cdot(\mathbf{M}_n^{-1}\ \mathrm{e})\Big]_n.\end{aligned}$$

Therefore from (7.4) we solve explicitly

$$\widetilde{\mathcal{I}}_n(\lambda,\eta)=\frac{n^2\ \widetilde{z}_n^{(0)}(\lambda,\eta)+\big(\gamma n^2\ \overline{\mathcal{I}}_n(\lambda,\eta)\big)\ \big(n^2\mathcal{M}_n(\lambda,\eta)\big)}{n^2\big(1-\gamma n^2\ \mathcal{M}_n(\lambda,\eta)\big)}. \tag{7.5}$$

Asymptotics of $n^2\overline{\mathcal{I}}_n(\lambda,\eta)$ is given in (7.3). Below we describe the terms $n^2\ \widetilde{z}_n^{(0)}(\lambda,\eta)$ and $n^2\mathcal{M}_n(\lambda,\eta)$ that also appear in the right hand side of (7.5).

Lemma 7.2. *Fix $M\in\mathbb{N}$. There exists $\lambda_M>0$ such that, for any $\lambda>\lambda_M$ and $\eta\in\{-M,...,M\}$*

$$\lim_{n\to\infty}\Big\{\gamma n^2\ \mathcal{M}_n(\lambda,\eta)\Big\}=1, \tag{7.6}$$

$$\lim_{n\to\infty}\Big\{n^2\ \big(1-\gamma n^2\ \mathcal{M}_n(\lambda,\eta)\big)\Big\}=\frac{1}{2\gamma}\Big(\lambda+\frac{\eta^2\pi^2}{\gamma}\Big), \tag{7.7}$$

and

$$\lim_{n\to\infty}\Big\{\gamma n^2\ \widetilde{z}_n^{(0)}(\lambda,\eta)\Big\}=\big(\mathcal{F}e_{\mathrm{thm}}(0,\cdot)\big)(\eta). \tag{7.8}$$

As a direct consequence of the above lemma and (7.3), we obtain:

Corollary 7.3. *Fix $M\in\mathbb{N}$. There exists $\lambda_M>0$ such that, for any $\lambda>\lambda_M$, $\eta\in\{-M,...,M\}$*

$$\lim_{n\to\infty}\widetilde{\mathcal{I}}_n(\lambda,\eta)=2\mathcal{W}^{+}_{\mathrm{thm}}(\lambda,\eta),$$

where

$$\mathcal{W}^{+}_{\mathrm{thm}}(\lambda,\eta):=\Big(\lambda+\frac{\eta^2\pi^2}{\gamma}\Big)^{-1}\Big\{\big(\mathcal{F}e_{\mathrm{thm}}(0,\cdot)\big)(\eta)+\frac{1}{2\gamma}\mathcal{L}\Big(\mathcal{F}\left((\partial_u r)^2\right)(\eta)\Big)(\lambda)\Big\}.$$

The following lemma finalizes the identification of the limit for the Fourier transform of the thermal energy:

Proposition 7.4. *Fix $M\in\mathbb{N}$. There exists $\lambda_M>0$ such that, for any $\lambda>\lambda_M$, $\eta\in\{-M,...,M\}$*

$$\lim_{n\to\infty}\Big\{\widetilde{\mathcal{I}}_n(\lambda,\eta)-2\left[\widetilde{w}_n^{+}(\lambda,\eta,\cdot)\right]_n\Big\}=0. \tag{7.9}$$

The proofs of Lemma 7.2 and Proposition 7.4 go very much along the lines of the arguments presented in Section 8 and we will not present the details here. They are basically consequences of the following limit

$$\lim_{n\to\infty}\left\{n^2\ \mathrm{e}_1\cdot\mathbf{M}_n^{-1}(\lambda,\eta,k)\ \mathrm{e}\right\}=\frac{1}{2\gamma},$$

which is proved in Section 8.2.

7.3. **Asymptotics of $\widetilde{\mathfrak{w}}_n$ and $\mathfrak{w}_n$.** With a little more work one can prove the following *local equilibrium* result, which is an easy consequence of Proposition 7.1, Corollary 7.3 and Proposition 7.4 (recall also (6.15)).

Theorem 7.5. *Fix $M \in \mathbb{N}$. There exists $\lambda_M > 0$ such that, for any $\lambda > \lambda_M$ and $G \in \mathcal{P}_M$ we have*

$$\lim_{n\to+\infty} \sum_{\eta\in\mathbb{Z}} \Big[w_n^+(\lambda,\eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\Big]_n$$
$$= \sum_{\eta\in\mathbb{Z}} \Big\{ \mathcal{W}^+_{\rm thm}(\lambda,\eta) \int_{\mathbb{T}} (\mathcal{F}G)^\star(\eta,v)\, \mathrm{d}v + \mathcal{W}^+_{\rm mech}(\lambda,\eta)\, (\mathcal{F}G)^\star(\eta,0)\Big\} \quad (7.10)$$

and

$$\lim_{n\to+\infty} \sum_{\eta\in\mathbb{Z}} \Big[y_n^+(\lambda,\eta,\cdot)(\mathcal{F}G)^\star(\eta,\cdot)\Big]_n = \sum_{\eta\in\mathbb{Z}} \mathcal{W}^+_{\rm mech}(\lambda,\eta)\, (\mathcal{F}G)^\star(\eta,0), \quad (7.11)$$

We will not give the details for the proof of this last theorem, since the argument is very similar to Proposition 7.1.

7.4. **End of the proof of Theorem 3.8.** From the uniform bound (5.17), we know that the sequence of all Wigner distributions $\{\mathbf{W}_n^+(\cdot), \mathbf{Y}_n^+(\cdot), \mathbf{Y}_n^-(\cdot), \mathbf{W}_n^-(\cdot)\}_n$ is sequentially pre-compact with respect to the $\star$-weak topology in the dual space of $\mathbf{L}^1(\mathbb{R}_+, \mathcal{A}_0)$. More precisely, one can choose a subsequence n_m such that any of the components above, say for instance $\mathbf{W}^+_{n_m}(\cdot)$, $\star$-weakly converges in the dual space of $\mathbf{L}^1(\mathbb{R}_+, \mathcal{A}_0)$ to some $\mathbf{W}^+(\cdot)$.

To characterize its limit, we consider $\mathbf{w}^+_{n_m}(\lambda)$ obtained by taking the Laplace transforms of the respective $\mathbf{W}^+_{n_m}(\cdot)$. For any $\lambda > 0$, it converges $\star$-weakly, as $n_m \to +\infty$, in $\mathcal{A}_0'$ to some $\mathbf{w}^+(\lambda)$ that is the Laplace transform of $\mathbf{W}^+(\cdot)$. The latter is defined as

$$\big\langle \mathbf{w}^+(\lambda), G\big\rangle := \int_0^\infty \big\langle \mathbf{W}^+(t), e^{-\lambda t} G\big\rangle\, \mathrm{d}t \qquad \lambda > 0,\ G \in \mathcal{A}_0.$$

Given a trigonometric polynomial $G \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{T})$ we conclude, thanks to Theorem 7.5, that for any $\lambda > \lambda_M$,

$$\Big\langle \mathbf{w}^+(\lambda), G\Big\rangle = \int_{\mathbb{R}_+\times\mathbb{T}^2} e^{-\lambda t}\, e(t,u) G(u,v)\, \mathrm{d}t\, \mathrm{d}u\, \mathrm{d}v, \quad (7.12)$$

where $e(t,u)$ is defined as in Theorem 3.8 and $M \in \mathbb{N}$ is such that $\mathcal{F}G(\eta, v) \equiv 0$ for all $|\eta| > M$.

Due to the uniqueness of the Laplace transform (that can be argued by analytic continuation), this proves that in fact equality (7.12) holds for all $\lambda > 0$. By a density argument it can be then extended to all $G \in \mathcal{A}_0$ and shows that $\mathbf{W}^+(t,u,v) = e(t,u)$, for any $(t,u,v) \in \mathbb{R}_+ \times \mathbb{T}^2$. This ends the proof of (3.20), and thus Theorem 3.8.

8. Proofs of the technical results stated in Section 7

In what follows we shall adopt the following notation: we say that the sequence $C_n(\lambda,\eta,k) \preceq 1$ if for any given integer $M \in \mathbb{N}$, there exist $\lambda_M > 0$ and $n_M \in \mathbb{N}$ such that

$$\sup\Big\{C_n(\lambda,\eta,k)\ ;\ \lambda > \lambda_M,\ \eta \in \{-M,...,M\},\ n > n_M,\ k \in \widehat{\mathbb{T}}_n\Big\} < +\infty.$$

8.1. Invertibility of $\mathbf{M}_n(\lambda,\eta,k)$.

Proposition 8.1. *The matrix $\mathbf{M}_n(\lambda,\eta,k)$ defined in (6.8) is invertible for all $n \geqslant 1$, $\lambda > 0$ and $(\eta,k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n$.*

Proof. The block entries of the matrix $\mathbf{M}_n$ defined in (6.9) satisfy the commutation relation

$$[A_n, B_n] = A_nB_n - B_nA_n = \begin{bmatrix} 0 & -2\gamma^- n^2\ \mathrm{Re}[a_n - b_n] \\ -2\gamma^+ n^2\ \mathrm{Re}[b_n - a_n] & 0 \end{bmatrix} = 0.$$

Thanks to the well known formula for the determinants of block matrices with commuting entries we have (see e.g. [4, formula (Ib), p. 46])

$$\det(\mathbf{M}_n) = \det(A_nB_n - \gamma_n^+\gamma_n^- n^4\ \mathrm{Id}_2) = \big|a_nb_n^\star + n^3(\delta\gamma_n)\big|^2 - 4n^4\gamma^+\gamma^-(\mathrm{Re}[a_n])^2$$

and, substituting from (6.9), we get

$$\begin{aligned}\det(\mathbf{M}_n) = n^6 &\Big((\delta_n s)\ (\sigma_n s) + (\delta\gamma_n)\Big)^2 \\ &+ \big(\lambda + 2\gamma n^2\big)^2 \Big\{\lambda^2 + n^2\big(4\gamma\lambda + (\delta_n s)^2\big) \\ &\quad + n^4\Big(2\sin^2(2\pi k) + 2\sin^2\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big) + (\sigma_n s)^2\Big)\Big\}. \end{aligned} \tag{8.1}$$

Here $\delta_n s$, $\sigma_n s$ are given by (6.4) and

$$\delta\gamma_n(\eta,k) := n(\gamma^+\gamma^- - \gamma_n^+\gamma_n^-) = n\Big(\sin^2\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big) - \sin^2(2\pi k)\Big).$$

The proposition is a direct conclusion of (8.1). □

It is also clear that

$$\det(\mathbf{M}_n) = n^8\Delta_n, \tag{8.2}$$

where

$$\Delta_n = \frac{1}{n^2}\Big\{4\gamma^2\ \Gamma_n + 4\gamma^2\big(4\lambda\gamma + (\delta_n s)^2\big) + \big((\delta_n s)(\sigma_n s) + (\delta\gamma_n)\big)^2\Big\} + 4\gamma\lambda\ \frac{\Gamma_n}{n^4} + \frac{C_n}{n^3}. \tag{8.3}$$

for some $|C_n| \preceq 1$ and

$$\Gamma_n(\eta,k) := n^2\Big(2\sin^2(2\pi k) + 2\sin^2\big(2\pi\big(k + \tfrac{\eta}{n}\big)\big) + (\sigma_n s)^2\Big). \tag{8.4}$$

On the one hand, note that for k sufficiently *far* from 0, the dominant term is $(4\gamma^2\ \Gamma_n)/n^2$ and then

$$\Delta_n \sim 4\gamma^2 \left(4\sin^2(2\pi k) + 16\sin^4(\pi k)\right).$$

On the other hand, for $k = \frac{\xi}{n}$ and fixed $\xi \in \mathbb{Z}$ we have

$$n^2\ \Delta_n\Big(\lambda, \eta, \frac{\xi}{n}\Big) = \frac{1}{n^6}\det(\mathbf{M}_n)\Big(\lambda, \eta, \frac{\xi}{n}\Big) \sim 16\gamma^2\Big[\lambda\gamma + 2\pi^2\left(\xi^2 + (\eta+\xi)^2\right)\Big]. \quad (8.5)$$

Since the block entries of $\mathbf{M}_n$ commute we can also write

$$\mathbf{M}_n^{-1} = \begin{bmatrix} \Big[A_nB_n - (\gamma_n^+\gamma_n^- n^4)\mathrm{Id}_2\Big]^{-1} & 0 \\ 0 & \Big[A_nB_n - (\gamma_n^+\gamma_n^- n^4)\mathrm{Id}_2\Big]^{-1} \end{bmatrix} \begin{bmatrix} B_n & \gamma_n^- n^2\ \mathrm{Id}_2 \\ \gamma_n^+ n^2\ \mathrm{Id}_2 & A_n \end{bmatrix}.$$

Note that

$$\Big[A_nB_n - (\gamma_n^+\gamma_n^- n^4)\mathrm{Id}_2\Big]^{-1} = \frac{1}{\det(\mathbf{M}_n)} \begin{bmatrix} a_n^\star b_n + n^3\ (\delta\gamma_n) & 2\gamma^- n^2\ \mathrm{Re}[a_n] \\ 2\gamma^+ n^2\ \mathrm{Re}[b_n] & a_n b_n^\star + n^3\ (\delta\gamma_n) \end{bmatrix}.$$

With these formulas we conclude that

$$\mathbf{M}_n^{-1} = \frac{1}{\det(\mathbf{M}_n)} \begin{bmatrix} d_n^+ & \gamma^- d_n & \gamma_n^- c_n^\star & \gamma^-\gamma_n^- c_n^0 \\ \gamma^+ d_n^0 & d_n^- & \gamma^+\gamma_n^- c_n^0 & \gamma_n^- c_n \\ \gamma_n^+ c_n^\star & \gamma^-\gamma_n^+ c_n^0 & (d_n^-)^\star & \gamma^-(d_n^0)^\star \\ \gamma^+\gamma_n^+ c_n^0 & \gamma_n^+ c_n & \gamma^+ d_n^\star & (d_n^+)^\star \end{bmatrix} \quad (8.6)$$

where all the constants are explicit and given by

$$\begin{cases} d_n^+ := a_n^\star\big|b_n\big|^2 + n^3 b_n^\star(\delta\gamma_n) - 2\gamma^-\gamma^+ n^4\ \mathrm{Re}[a_n], \\ d_n^- := b_n^\star\big|a_n\big|^2 + n^3 a_n^\star(\delta\gamma_n) - 2\gamma^-\gamma^+ n^4\ \mathrm{Re}[b_n], \\ d_n := 2n^2 a_n^\star\ \mathrm{Re}[a_n] - n^2 a_n^\star b_n - n^5(\delta\gamma_n), \\ d_n^0 := 2n^2 b_n^\star\ \mathrm{Re}[b_n] - n^2 a_n b_n^\star - n^5(\delta\gamma_n), \\ c_n := n^2 a_n b_n^\star + n^5(\delta\gamma_n), \\ c_n^0 := 2n^4\ \mathrm{Re}[a_n]. \end{cases} \quad (8.7)$$

8.2. **Asymptotics of the coefficients.** Substituting from (6.9) into the respective formulas of (8.7) and then identifying the order of magnitude of the appearing terms we conclude the following:

Lemma 8.2. *The following asymptotic equalities hold:*

$$\frac{d_n^+}{n^6} = 4\gamma^3 + 4\gamma\sin^2(2\pi k) + 2\gamma(\sigma_n s)^2 - \frac{i(\delta_n s)}{n}\left(4\gamma^2 + (\sigma_n s)^2\right) + \frac{C_n}{n^2},$$

$$\frac{c_n}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2\left(2\pi\left(k + \tfrac{\eta}{n}\right)\right) - \sin^2(2\pi k) + \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{d_n}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2(2\pi k) - \sin^2\left(2\pi\left(k + \tfrac{\eta}{n}\right)\right) - 2i\gamma\frac{(\delta_n s)}{n} - \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{c_n^0}{n^6} = 4\gamma + \frac{C_n}{n^2},$$

$$\frac{d_n^0}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2(2\pi k) - \sin^2\left(2\pi\left(k + \tfrac{\eta}{n}\right)\right) - 2i\gamma\frac{(\delta_n s)}{n} - \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{d_n^-}{n^6} = 4\gamma^3 + 4\gamma\sin^2(2\pi k) - 4i\gamma^2(\sigma_n s) + 2\gamma\Big(\sin^2\left(2\pi\left(k + \tfrac{\eta}{n}\right)\right) - \sin^2(2\pi k)\Big) + \frac{C_n}{n^2},$$

with $C_n \preceq 1$.

From the above asymptotics it is clear that $\mathbf{M}_n^{-1}(\lambda, \eta, k) \to 0$ for a fixed $k \neq 0$. In addition, $\mathbf{M}_n^{-1}(\lambda, \eta, \frac{\xi}{n})$ and $n^2\ \mathrm{e}^{\mathrm{T}}\ \mathbf{M}_n^{-1}(\lambda, \eta, k)$ tend to finite limits that we need to compute explicitly in order to complete the proof.

Using (8.6), (8.2) and the formulas for the asymptotics of the entries of $\mathbf{M}_n^{-1}$, provided by Lemma 8.2, we conclude that

$$\lim_{n\to\infty} \mathbf{M}_n^{-1}\Big(\lambda, \eta, \frac{\xi}{n}\Big) = \frac{\gamma}{4\lambda\gamma + 8\pi^2\left[\xi^2 + (\xi+\eta)^2\right]}\,\mathbf{1}\otimes\mathbf{1}. \tag{8.8}$$

The above in particular implies that $\mathrm{e}^{\mathrm{T}} \cdot \mathbf{M}_n^{-1}(\lambda, \eta, \frac{\xi}{n})\mathbf{1} \to 0$, for any $\xi \in \mathbb{Z}$. By the same token we can also compute

$$\lim_{n\to+\infty}\Big\{n^2\ \mathrm{e}^{\mathrm{T}} \cdot \mathbf{M}_n^{-1}\Big(\lambda, \eta, \frac{\xi}{n}\Big)\mathbf{1}\Big\} = \frac{4\pi^2\ \xi(\xi+\eta)}{\lambda\gamma^2 + 2\gamma\pi^2\ \left[\xi^2 + (\xi+\eta)^2\right]}, \tag{8.9}$$

and

$$\lim_{n\to\infty}\Big\{n^2\ \mathrm{e}^{\mathrm{T}} \cdot\ \mathbf{M}_n^{-1}(\lambda, \eta, k)\Big\} = \frac{1}{2\gamma}\left[1, 0, 0, 1\right], \quad k \neq 0. \tag{8.10}$$

We prove here only (8.9) and we let the reader verify (8.8) and (8.10) using similar computations. An explicit calculation gives

$$n^2\ \mathrm{e}^{\mathrm{T}} \cdot (\mathbf{M}_n^{-1}\ \mathbf{1}) = \frac{n^2\ \Xi_n}{\det(\mathbf{M}_n)}, \tag{8.11}$$

where

$$\Xi_n(\lambda,\eta,k) := 2\mathrm{Re}\big[d_n^+ - d_n^-\big] + \gamma^-(d_n - (d_n^0)^\star) + \gamma^+(d_n^\star - d_n^0) \\ + 4i\sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big)\mathrm{Im}[c_n] + 4\sin(2\pi k)\sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big)c_n^0.$$

Substituting from (8.7) yields:

$$\begin{aligned}\Xi_n(\lambda,\eta,k) =&(\lambda+2\gamma n^2)\Big\{2\big[n^4(\sigma_n s)^2 - n^2(\delta_n s)^2\big] \\ &+ 4in^4\Big(\sin(2\pi k) - \sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big)\Big)(\sigma_n s) \\ &+ 4in^3\Big(\sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big) + \sin(2\pi k)\Big)(\delta_n s) \\ &+ 8n^4\sin(2\pi k)\sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big)\Big\}. \end{aligned} \tag{8.12}$$

From the above and a direct calculation we obtain (8.9).

8.3. **Proof of Proposition 7.1.** Basically the argument follows the same idea as the proof of Proposition 5.2. In fact (8.8) implies that $\overline{\mathfrak{w}}_n(\lambda,\eta,k)$ concentrates on small k-s like $\overline{W}_n^+(\eta,k)$. The main difficulty is to deal with the averaging $[\cdot]_n$. For that purpose, for $\rho\in\big(0,\frac{1}{2}\big)$, define

$$\widehat{\mathbb{T}}_{n,\rho} := \Big\{k\in\widehat{\mathbb{T}}_n : \ |\sin(\pi k)| \geqslant n^{-\rho}\Big\} \tag{8.13}$$

and its complement $\widehat{\mathbb{T}}^c_{n,\rho} := \widehat{\mathbb{T}}_n\backslash\widehat{\mathbb{T}}_{n,\rho}$. Recall the left hand side of (7.2): it can be written as $\mathrm{I}_n + \mathrm{II}_n$, where I_n and II_n correspond to the summations over $\widehat{\mathbb{T}}_{n,\rho}$ and $\widehat{\mathbb{T}}^c_{n,\rho}$ respectively. First we show that for $G\in\mathcal{P}_M$

$$\mathrm{I}_n = \sum_{\eta\in\mathbb{Z}}\frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_{n,\rho}}\big(\mathbf{M}_n^{-1}(\lambda,\eta,k)\mathbf{1}\big)\,\overline{W}_n^+(\eta,k)(\mathcal{F}G)^\star(\eta,k) \underset{n\to\infty}{\longrightarrow} 0. \tag{8.14}$$

In fact we have

$$\Big\|\sum_{\eta\in\mathbb{Z}}\frac{1}{n}\sum_{|\xi|\geqslant n^\rho}\Big(\mathbf{M}_n^{-1}\Big(\lambda,\eta,\frac{\xi}{n}\Big)\mathbf{1}\Big)\overline{W}_n^+\Big(\eta,\frac{\xi}{n}\Big)(\mathcal{F}G)^\star\Big(\eta,\frac{\xi}{n}\Big)\Big\|_\infty \\ \leqslant C\|G\|_0\sum_{|\eta|\leqslant M}\sum_{|\xi|\geqslant n^\rho}\Big\|\mathbf{M}_n^{-1}\Big(\lambda,\eta,\frac{\xi}{n}\Big)\mathbf{1}\Big\|_\infty \underset{n\to\infty}{\longrightarrow} 0,$$

where the constant C depends only on the initial mechanical energy. Then by the same argument as the one used in the proof of Proposition 5.2, and from (5.30)

and (8.8) we have

$$\lim_{n\to\infty} \sum_{|\eta|\leqslant M} \frac{1}{n} \sum_{|\xi|<n^\rho} \Big(\mathbf{M}_n^{-1}\Big(\lambda,\eta,\frac{\xi}{n}\Big)\mathbf{1}\Big)\overline{W}_n^+\Big(\eta,\frac{\xi}{n}\Big)(\mathcal{F}G)^\star\Big(\eta,\frac{\xi}{n}\Big)$$
$$= \sum_{\eta,\xi\in\mathbb{Z}} \frac{\gamma W(r_0\,;\,\eta,\xi)}{\lambda\gamma + 2\pi^2\left[\xi^2+(\xi+\eta)^2\right]}(\mathcal{F}G)^\star(\eta,0)\;\mathbf{1}.$$

This concludes the proof of (7.2). Concerning the proof of (7.3), recall that

$$\overline{\mathcal{I}}_n = \Big[\mathrm{e}\cdot(\mathbf{M}_n^{-1}\,\mathbf{1})\,\overline{W}_n^+\Big]_n,$$

and that we have already computed the limit (8.9). Consequently, the result will follow if we are able to show that the contribution to the k-averaging from the higher frequencies is negligible. The quantity $n^2\,\overline{\mathcal{I}}_n$ can be written as $\mathrm{I}_n + \mathrm{II}_n$, where I_n and II_n correspond to the summations over $\widehat{\mathbb{T}}_{n,\rho}$ and $\widehat{\mathbb{T}}^c_{n,\rho}$ respectively. Using the explicit computations (8.11), (8.12) and (8.3) we can write

$$n^2\,\mathrm{e}\cdot(\mathbf{M}_n^{-1}\,\mathbf{1}) - \gamma^{-1} = \gamma^{-1}\left\{2\Big(\sin(2\pi k) - \sin\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big)\Big)^2 + \frac{C_n}{n}\right\}$$
$$\times\left\{2\sin^2(2\pi k) + 2\sin^2\big(2\pi\big(k+\tfrac{\eta}{n}\big)\big) + (\sigma_n s)^2 + \frac{C'_n}{n^2}\right\}^{-1}.$$

It is clear from the above equality that

$$\lim_{n\to+\infty} \sup_{|\eta|\leqslant M} \sup_{k\in\widehat{\mathbb{T}}_{n,\rho}} \left| n^2\,\mathrm{e}\cdot(\mathbf{M}_n^{-1}\,\mathbf{1})(\lambda,\eta,k) - \frac{1}{\gamma}\right| = 0. \tag{8.15}$$

Thanks to (8.15) we conclude that $\lim_{n\to+\infty}(\mathrm{I}_n - \mathrm{I}'_n) = 0$, where

$$\mathrm{I}'_n := \frac{1}{\gamma n}\sum_{k\in\widehat{\mathbb{T}}_{n,\rho}} \overline{W}_n^+(\eta,k). \tag{8.16}$$

After a straightforward calculation using the definition of $\overline{W}_n^+(\eta,k)$ (see (5.20) and (5.22)) we conclude that (8.16) equals

$$\frac{1}{2\gamma}\sum_{\xi,\xi'\in\mathbb{Z}}\sum_{k\in\widehat{\mathbb{T}}_{n,\rho}} 1_{\mathbb{Z}}\Big(k-\frac{\xi'}{n}\Big)1_{\mathbb{Z}}\Big(-k+\frac{\xi-\eta}{n}\Big)(\mathcal{F}r_0)(\xi)(\mathcal{F}r_0)^\star(\xi'),$$

where $1_{\mathbb{Z}}$ is the indicator function of the integer lattice. Due to the assumed separation of k from 0, see (8.13), and the decay of the Fourier coefficients of $r_0(\cdot)$ (that belongs to $\mathcal{C}^\infty(\mathbb{T})$) we conclude from the above that $\lim_{n\to+\infty}\mathrm{I}'_n = 0$, thus also $\lim_{n\to+\infty}\mathrm{I}_n = 0$.

Moreover, a similar calculation also yields

$$\mathrm{II}_n = \frac{1}{2}\sum_{\xi\in\mathcal{N}_{\rho,n}} n^2\,\mathrm{e}\cdot(\mathbf{M}_n^{-1}\,\mathbf{1})\Big(\lambda,\eta,\frac{\xi}{n}\Big)(\mathcal{F}r_0)(\xi+\eta)(\mathcal{F}r_0)^\star(\xi),$$

where

$$\mathcal{N}_{\rho,n} := \Big\{ \xi \in \mathbb{Z} \ : \ |\xi| \leqslant \tfrac{n}{2}, \quad \big|\sin\big(\tfrac{\pi\xi}{n}\big)\big| \leqslant n^{-\rho} \Big\}.$$

Using the dominated convergence theorem we conclude from (8.9) that

$$\lim_{n\to+\infty} \mathrm{II}_n = \sum_{\xi\in\mathbb{Z}} \frac{4\pi^2\xi(\xi+\eta)\, W(r_0\,;\,\eta,\xi)}{\lambda\gamma^2 + 2\gamma\pi^2\,[\xi^2+(\xi+\eta)^2]}. \tag{8.17}$$

8.4. **Proof of Lemma 7.2 and Proposition 7.4.** Since (7.6) is a direct consequence of (7.7), we prove directly (7.7), that is a consequence of the following lemma.

Lemma 8.3. *The following asymptotic equality holds:*

$$\mathcal{S}_n(\lambda,\eta) := n^2\,\big(1-\gamma n^2\,\mathcal{M}_n(\lambda,\eta)\big) = \frac{1}{2\gamma}\Big(\lambda + \frac{\eta^2\pi^2}{\gamma}\Big) + \frac{C_n}{n}, \tag{8.18}$$

where $|C_n| \leq 1$.

Proof. After a direct calculation, we obtain

$$\mathcal{S}_n(\lambda,\eta) = \left[\frac{n^2}{\det(\mathbf{M}_n)}\,\big(\det(\mathbf{M}_n) - \gamma n^2\Theta_n\big)\right]_n,$$

where

$$\Theta_n := 2\mathrm{Re}\Big[d_n^+ + d_n^- + 2\gamma^2 c_n^0 - 2\gamma c_n\Big] - \gamma^-\big(d_n + (d_n^0)^\star\big) - \gamma^+\big(d_n^\star + d_n^0\big).$$

Therefore, from (8.7) and Lemma 8.2, we have

$$\mathcal{S}_n(\lambda,\eta) = \frac{\lambda}{2\gamma n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathrm{I}_n(\eta,k) + \frac{1}{4\gamma^2 n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathrm{II}_n(\eta,k).$$

where

$$\mathrm{I}_n := \frac{\Gamma_n + 2\lambda\gamma + (\delta_n s)^2 + \frac{C_n}{n}}{\Gamma_n + 4\lambda\gamma + (\delta_n s)^2 + \gamma^{-2}\big((\delta_n s)(\sigma_n s) + (\delta\gamma_n)\big)^2 + \frac{C_n'}{n}},$$

$$\mathrm{II}_n := \frac{n^2\,\big((\sigma_n s)(\delta_n s) + (\delta\gamma_n)\big)^2}{\Gamma_n + 4\lambda\gamma + (\delta_n s)^2 + \gamma^{-2}\big((\delta_n s)(\sigma_n s) + (\delta\gamma_n)\big)^2 + \frac{C_n'}{n}}.$$

The expressions C_n, C_n' satisfy $|C_n| + |C_n'| \leq 1$. Directly from the definition of I_n and II_n we conclude that $|\mathrm{I}_n| + |\mathrm{II}_n| \leq 1$ and

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathrm{I}_n = 1,$$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} \mathrm{II}_n = (2\pi\eta)^2 \int_{\mathbb{T}} \frac{\big[4\sin(2\pi v)\sin^2(\pi v) + \sin(4\pi v)\big]^2}{4\sin^2(2\pi v) + 16\sin^4(\pi v)}\mathrm{d}v.$$

Using trigonometric identities

$$2\sin^2(\pi v) = 1 - \cos(2\pi v) \quad \text{and} \quad \sin(4\pi v) = 2\sin(2\pi v)\cos(2\pi v)$$

we conclude that the last integral equals

$$\int_{\mathbb{T}} \frac{\left[2\sin(2\pi v)(1-\cos(2\pi v)) + \sin(4\pi v)\right]^2}{4\sin^2(2\pi v) + 4[1-\cos(2\pi v)]^2} \mathrm{d}v = \int_{\mathbb{T}} \frac{\sin^2(2\pi v)}{2(1-\cos(2\pi v))} \mathrm{d}v = \frac{1}{2}.$$

Thus, we obtain

$$\mathcal{S}_n(\lambda,\eta) = \frac{1}{2\gamma}\Big(\lambda + \frac{\pi^2\eta^2}{\gamma}\Big) + \frac{C_n}{n},$$

with $|C_n| \preceq 1$. □

It remains to prove (7.8).This would be a direct consequence of (8.10), but we need some care in exchanging the limit with the $[\cdot]_n$ averaging.

Choose $\rho \in (0,1)$, then we can decompose

$$n^2\Big[\mathrm{e}\cdot\big(\mathbf{M}_n^{-1}\,\widetilde{\mathfrak{v}}_n^0\big)\Big]_n = \frac{1}{2\gamma}\left(\Big[\widetilde{W}_n^+\Big]_n + \Big[\widetilde{W}_n^-\Big]_n\right) + K_n^{(1)} + K_n^{(2)}$$

where

$$K_n^{(1)}(\lambda,\eta) = \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_{n,\rho}} \Big(n^2\mathrm{e}^{\mathrm{T}}\,\mathbf{M}_n^{-1}(\lambda,\eta,k) - \frac{1}{2\gamma}\,\mathrm{u}\Big)\cdot\widetilde{\mathfrak{v}}_n^0(\eta,k)$$

with $\mathrm{u} = [1,0,0,1]^{\mathrm{T}}$, and the definition of $K_n^{(2)}$ differs from $K_n^{(1)}$ only in that the range of the summation in k extends over $\widehat{\mathbb{T}}_{n,\rho}^c$.

From (5.26) we have

$$\lim_{n\to\infty}\Big[\widetilde{W}_n^+(\eta,\cdot)\Big]_n = \lim_{n\to\infty}\Big[\widetilde{W}_n^-(\eta,\cdot)\Big]_n = \big(\mathcal{F}e_{\mathrm{thm}}(0,\cdot)\big)(\eta),$$

and therefore we only have to prove that $K_n^{(1)}$ and $K_n^{(2)}$ vanish as $n\to\infty$. Concerning $K_n^{(1)}$ we write

$$\begin{aligned}\big|K_n^{(1)}(\lambda,\eta)\big| &\leqslant \sup_{k\in\widehat{\mathbb{T}}_{n,\rho}}\left\|n^2\mathrm{e}^{\mathrm{T}}\,\mathbf{M}_n^{-1}(\lambda,\eta,k) - \frac{1}{2\gamma}\,\mathrm{u}\right\|_\infty \left\|\,\big[\widetilde{\mathfrak{v}}_n^0(\eta,\cdot)\big]_n\,\right\|_\infty \\ &\leqslant C\sup_{k\in\widehat{\mathbb{T}}_{n,\rho}}\left\|n^2\mathrm{e}^{\mathrm{T}}\,\mathbf{M}_n^{-1}(\lambda,\eta,k) - \frac{1}{2\gamma}\,\mathrm{u}\right\|_\infty, \end{aligned} \tag{8.19}$$

where C depends on the bound on the initial energy. By direct estimation, using the information on the asymptotic behavior for the coefficients of $\mathbf{M}_n^{-1}$, provided by (8.10), we conclude that the right hand side of (8.19) converges to 0 as $n\to\infty$, for any given λ and η.

Concerning $K_n^{(2)}$, since $n^2\,\mathrm{e}^{\mathrm{T}}\cdot\,\mathbf{M}_n^{-1}(\lambda,\eta,k)$ are uniformly bounded in k, for any integer M there exists a constant $C_M > 0$ such that, for all n, $\lambda > \lambda_M$, $|\eta| \leqslant M$,

$$\big|K_n^{(2)}(\lambda,\eta)\big| \leqslant \frac{C_M}{n}\sum_{k\in\widehat{\mathbb{T}}_{n,\rho}^c}\big|\widetilde{\mathfrak{v}}_n^0(\eta,k)\big|.$$

Using the Cauchy-Schwarz inequality we have

$$\left|K_n^{(2)}(\lambda,\eta)\right| \leqslant \frac{C'_M}{n}\left|\widehat{\mathbb{T}}^c_{n,\rho}\right|^{\frac{1}{2}}\left(\sum_{k\in\widehat{\mathbb{T}}_n}\mathbb{E}\left|\widetilde{\mathfrak{v}}_n^0(\eta,k)\right|^2\right)^{\frac{1}{2}}$$
$$\leqslant C'_M\left(\frac{\left|\widehat{\mathbb{T}}^c_{n,\rho}\right|}{n}\right)^{\frac{1}{2}} w_*^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0,$$

where w_* is given by (5.27). This concludes the proof of Proposition 7.2.

Proposition 7.4 is also a direct consequence of (8.10): instead of computing the limit of

$$\widetilde{\mathcal{I}}_n(\lambda,\eta) = \left[\left(\widetilde{w}_n^+ - \widetilde{y}_n^+ - \widetilde{y}_n^- + \widetilde{w}_n^-\right)(\lambda,\eta,\cdot)\right]_n$$

we can compute the limit of

$$2\left[\widetilde{w}_n^+(\lambda,\eta,\cdot)\right]_n$$

by using very similar arguments.

Acknowledgments

T. K. acknowledges the support of the Polish National Science Center grant DEC-2012/07/B/ST1/03320.

The work of M.S. was supported by the ANR-14-CE25-0011 project (EDNHS) of the French National Research Agency (ANR), and by the Labex CEMPI (ANR-11-LABX-0007-01), and finally by CAPES (Brazil) and IMPA (Instituto de Matematica Pura e Aplicada, Rio de Janeiro) through a post-doctoral fellowship.

S.O. has been partially supported by the ANR-15-CE40-0020-01 grant LSD.

References

1. G. Basile, C. Bernardin, M. Jara, T. Komorowski, S. Olla, *Thermal conductivity in harmonic lattices with random collisions*, in "Thermal transport in low dimensions: from statistical physics to nanoscale heat transfer", S. Lepri ed., "Lecture Notes in Physics" **921**, chapter 5, Springer 2016. DOI: 10.10007/978-3-319-29261-8-5.
2. C. Bernardin, S. Olla, *Fourier law and fluctuations for a microscopic model of heat conduction*, J. Stat. Phys. **118**(3,4) (2005), 271–289.
3. C. Bernardin, S. Olla, *Transport properties of a chain of anharmonic oscillators with random flip of velocities*, J. Stat. Phys. **145** (2011), 1224–1255.
4. F. R. Gantmakher, *The theory of matrices* Vol. 1, AMS Chelsea, 2000.
5. M. Jara, T. Komorowski, S. Olla, *Superdiffusion of energy in a system of harmonic oscillators with noise*, Commun. Math. Phys. **339** (2015), 407–453.
6. Kelley, J. L. (1991), *General topology*, Springer-Verlag, ISBN 978-0-387-90125-1.
7. T. Komorowski, S. Olla, *Ballistic and superdiffusive scales in macroscopic evolution of a chain of oscillators.* Nonlinearity **29** (2016) 962-999.
8. T. Komorowski, S. Olla, *Diffusive propagation of energy in a non-acoustic chain*, Arch. Ration. Mech. Anal. (2016). DOI:10.1007/s00205-016-1032-9.
9. J. Lukkarinen, *Thermalization in harmonic particle chains with velocity flips*, J. Stat. Phys. **155**(6) (2014) 1143–1177.

10. J. Lukkarinen, M. Marcozzi, A. Nota, *Harmonic chain with velocity flips: thermalization and kinetic theory*, ArXiv:1604.08135v1, 2016.
11. J. Lukkarinen, H. Spohn, *Kinetic limit for wave propagation in a random medium*, Arch. Ration. Mech. Anal. **183**(1) (2006), 93–162.
12. M. Simon, Hydrodynamic limit for the velocity-flip model, Stochastic Processes and their Applications **123** (2013) 3623–3662.
13. H. T. Yau, *Relative entropy and hydrodynamics of Ginzburg-Landau models*, Lett. Math. Phys. **22**(1) (1991), 63–80.

Tomasz Komorowski, Institute of Mathematics, Polish Academy Of Sciences, Warsaw, Poland.
E-mail address: komorow@hektor.umcs.lublin.pl

Université Paris-Dauphine, PSL Research University, CNRS, CEREMADE, 75016 Paris, France
E-mail address: olla@ceremade.dauphine.fr

Inria Lille – Nord Europe, 40 avenue du Halley, 59650 Villeneuve d'Ascq, France, *and* Laboratoire Paul Painlevé, UMR CNRS 8524, Cité Scientifique, 59655 Villeneuve d'Ascq, France
E-mail address: marielle.simon@inria.fr