Adding an integrator to backstepping: Output disturbances rejection for linear hyperbolic systems

Pierre-Olivier Lamare, Florent Di Meglio

Adding an Integrator to Backstepping: Output Disturbances Rejection for Linear Hyperbolic Systems

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***Abstract*— We consider the output disturbance rejection problem for linear first-order hyperbolic systems with anti-collocated boundary input and output. We employ backstepping to construct the controller. We incorporate an integral action into the target system, yielding a Delay Differential Equation (DDE) and use a classical result to obtain a sufficient condition for its stability in L^∞-norm. Then, we show that the full-state feedback control with the integral action rejects in-domain and boundary disturbances from the output. Besides, we show that when incorporating integral-action into the observer system, the resulting output feedback control rejects the disturbances too.**

I. INTRODUCTION

A wide number of physical networks may be represented by hyperbolic systems. Among them we can cite the hydraulic networks [3], [10], road traffic networks [11], oil well drilling [1], [8], or gas pipeline networks [12]. Due to the importance of such applications from an applicative point of view a large number of results concerning their control is emerging this last decade.

The boundary control of 2×2 linear hyperbolic systems has been analyzed in [1], [2], [10], [14], [16], [17]. The generalization for 2×2 quasilinear hyperbolic systems has been made in [5], for semilinear equations in [15], and for $n\times n$ systems in [9], [13]. The disturbance rejection problem has attracted the attention for this class of systems recently as in [1], [2], [10], [14], [16]. In [1], [2] the rejection of a perturbation affecting the left hand side of a 2×2 linear hyperbolic system is solved with backstepping. In [14], the stabilization of a reference trajectory is solved with a proportional-integral controller. Though [10] considered an integral action for output rejection its effectiveness is validated on experimental data. In [16], a sliding mode control approach is used to reject a boundary time-varying input disturbance. The results related to the present paper are the results in [14] for the integral control, [9], [17] for the control and observer by backstepping.

The construction of controller using backstepping for 2×2 linear hyperbolic system has been introduced in [17]. Since this seminal work, generalization of the method have been made for $n\times n$ systems [9], [13], for disturbance rejection [1], [2], for adaptive observer [8], for trajectory generation [13], [14].

This paper deals with output disturbance rejection for linear first-order hyperbolic systems with anti-collocated boundary input and output. We want to incorporate an integral action to a controller based on backstepping. The solution is obtained by mapping the original system into a target system where the integral term is added. The resulting system is a cascade of two Delay Differential Equations (DDE). Thus, the stability analysis can be led using the semigroup approach and a sufficient condition on the coefficient of the integrator is derived. This analysis imposes to consider the L^∞-norm for the stability of the PDE system (Section III-A). Then, the rejection of the perturbations by the controller obtained with the inverse transformation is proved (Section III-B). Since this controller requires the knowledge of the distributed states, its implementation is unrealistic. Thus, we consider an observer-controller structure (Section IV-A). For the proposed observer system to be stable in presence of perturbations, an integral action is added. We show that this action is able to stabilize the observer states around a perturbed version of the real states (Section IV-B). Then, we show that the resulting output feedback control law still rejects the output disturbances (Section V). Finally, we show that the obtained results may be generalized to systems (Section VI) of arbitrary numbers of PDEs.

II. PROBLEM STATEMENT

We consider the following system

$$u_t+\lambda_1(x)u_x=\gamma_1(x)v \tag{1}$$
$$v_t-\lambda_2(x)v_x=\gamma_2(x)u \tag{2}$$
$$u(0,t)=qv(0,t) \tag{3}$$
$$v(1,t)=\rho u(1,t)+U(t) \tag{4}$$
$$v(0,t)=y(t)\,, \tag{5}$$

where $t\in[0,+\infty)$ is the time variable, $x\in[0,1]$ is the spatial variable, y is the output of the system, $q\neq 0$, ρ are constant parameters, and U is the control input. The functions λ_1, λ_2 belong to $C^2([0,1])$ and satisfy $\lambda_1(x)$, $\lambda_2(x)>0$, for all $x\in[0,1]$, and the functions γ_i, $i=1,2$ belong to $C^1([0,1])$. Now let us assume that there exist some disturbances $d_1,d_2\in C^1([0,1])$ on the right-hand side of (1), (2) respectively and some disturbances d_3, $d_4\in\mathbb{R}$ on the right-hand side of (3), (4), respectively. The sys-

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tem (1)–(4) becomes

$$u_t + \lambda_1(x)u_x = \gamma_1(x)v + d_1(x) \tag{6}$$
$$v_t - \lambda_2(x)v_x = \gamma_2(x)u + d_2(x) \tag{7}$$
$$u(0,t) = qv(0,t) + d_3 \tag{8}$$
$$v(1,t) = \rho u(1,t) + U(t) + d_4 \,. \tag{9}$$

The aim of this paper is to propose a control U for which the perturbations are rejected in the output $y(t)$.

III. Integral Action together with a Full-State Feedback Control Law

Let us denote the space $L^\infty(0,1) \times L^\infty(0,1) \times \mathbb{R}$ by E.

A. Target System

We consider the following target system

$$\alpha_t + \lambda_1(x)\alpha_x = 0 \tag{10}$$
$$\beta_t - \lambda_2(x)\beta_x = 0\,, \tag{11}$$

with the following boundary conditions

$$\alpha(0,t) = q\beta(0,t) \tag{12}$$
$$\beta(1,t) = -k_I\eta(t)\,, \tag{13}$$

where

$$\dot\eta(t) = \beta(0,t)\,. \tag{14}$$

The initial data α^0 and β^0 are supposed to be bounded, and therefore lie in $L^\infty(0,1)$. The following Lemma assesses the stability properties of this system.

Lemma 1: Let k_I be such that

$$0 < k_I < \frac{\pi}{2\tau} \tag{15}$$

where

$$\tau = \int_0^1 \frac{1}{\lambda_2(\xi)} d\xi\,, \tag{16}$$

then, the equilibrium $(\alpha,\beta,\eta)^\top \equiv (0,0,0)^\top$ of system (10), (11) with boundary conditions (12), (13), and (14) is exponentially stable for the L^∞-norm.

Proof: Differentiating (13) we get

$$\frac{d\beta(1,t)}{dt} = -k_I\beta(0,t)\,. \tag{17}$$

We have

$$\beta(0,t) = \beta(1,t-\tau)\,. \tag{18}$$

Hence, we have

$$\frac{d\beta(1,t)}{dt} + k_I\beta(1,t-\tau) = 0\,, \tag{19}$$

where τ is defined in (16). Using Theorem 5.1.7 of [6] we have that the semigroup corresponding to the delay differential equation (19) is exponentially stable if and only if the roots of the function

$$\Delta(s) = s + k_I e^{-\tau s} \tag{20}$$

lie in $\mathbb{C}^-$. Letting $z = \tau s$ and $q = -k_I\tau$ the equation

$$s + k_I e^{-\tau s} = 0 \tag{21}$$

becomes

$$q - ze^z = 0\,. \tag{22}$$

By Theorem 13.8 of [4] the roots of this equation lie in $\mathbb{C}^-$ if and only if (15) holds. Hence, under assumption (15), $\beta(1,t)$ is exponentially stable. It is not hard to check that it implies that β is exponentially stable for the L^∞-norm, and therefore that α too. Now by (18) we get

$$\dot\eta(t) = -k_I\eta(t-\tau)\,. \tag{23}$$

Thus, using the same argument as above η is exponentially stable for the L^∞-norm. This concludes the proof of Lemma 1. ■

To map the original system (1), (2) to the target system (10), (11) we use the following backstepping transformation, introduced in [17]

$$\alpha(x,t) = u(x,t) - \int_0^x K^{uu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)v(\xi,t)d\xi \tag{24}$$
$$\beta(x,t) = v(x,t) - \int_0^x K^{vu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)v(\xi,t)d\xi\,. \tag{25}$$

The kernels K^{uu}, K^{uv}, K^{vu}, and K^{vv} satisfy a well-posed set of linear first-order hyperbolic equations on a triangular domain $\mathscr{T}$, detailed in [17], that possesses a unique solution in $L^\infty(\mathscr{T})$. Besides, from the transformation (25) evaluated at $x = 1$, one gets

$$U(t) = -k_I\eta(t) - \rho u(1,t) + \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi\,, \tag{26}$$

with

$$\dot\eta(t) = y(t)\,. \tag{27}$$

The following proposition assesses the stability properties of the unperturbed system.

Proposition 1: Consider system (1), (2) with boundary conditions (3), (4) where U is given by (26), η satisfies (27), and with bounded initial conditions $(u^0, v^0, \eta^0) \in E$. Then, there exist two positive constants κ and λ such that the following holds for all $t \geq 0$

$$|(u,v,\eta)|_E \leq \kappa e^{-\lambda t} \left|(u^0,v^0,\eta^0)\right|_E\,. \tag{28}$$

Proof: Using Lemma 1 and the invertibility of the backstepping transformation (24), (25) as a Volterra equation of the second kind, the result holds. ■

B. Disturbance Rejection with the Full-State Feedback Control Law and Integral Action

The aim of this section is to prove that the perturbed system (6)–(9) with U given by (26) rejects in the output the perturbations. More precisely, we prove the following theorem.

Theorem 1: Consider system (6), (7) with bounded initial conditions $\left(u^0, v^0, \eta^0\right) \in E$, boundary conditions (8), (9) where U is given by (26), η satisfies (27) and k_I satisfies

$$0 < k_I < \frac{\pi}{2\tau}, \tag{29}$$

where τ is given by (16). Then the following holds

$$\lim_{t\to+\infty} |v(0,t)| = 0. \tag{30}$$

Proof: The equilibrium $Z = (u_{ss}, v_{ss})^\top$ of the perturbed system (6)–(8) and (4) is the solution of the following ordinary differential equation

$$Z'(x) = F(x)Z(x) + G(x), \tag{31}$$

where $F(x) = \begin{pmatrix} 0 & \frac{\gamma_1(x)}{\lambda_1(x)} \\ -\frac{\gamma_2(x)}{\lambda_2(x)} & 0 \end{pmatrix}$, $G(x) = \begin{pmatrix} \frac{d_1(x)}{\lambda_1(x)} \\ -\frac{d_2(x)}{\lambda_2(x)} \end{pmatrix}$ with boundary conditions

$$Z_1(0) = d_3 \tag{32}$$

$$Z_2(0) = 0. \tag{33}$$

Besides, given (9), the equilibrium value of U, which we denote U_{ss}, satisfies

$$U_{ss} = v_{ss}(1; d_1, d_2, d_3) - \rho u_{ss}(1; d_1, d_2, d_3) - d_4. \tag{34}$$

The ordinary differential equation (31) together with the boundary conditions (32), (33) is a well-posed initial value problem for x. The equilibrium depends on d_1, d_2, and d_3. Let us denote this equilibrium by $u_{ss}(x; d_1, d_2, d_3)$, $v_{ss}(x; d_1, d_2, d_3)$. Using (26) it follows from (34) that the equilibrium value of η, namely η_{ss}, satisfies

$$\eta_{ss} = -\frac{v_{ss}(1; d_1, d_2, d_3) - d_4 - \theta_{ss}}{k_I}, \tag{35}$$

where

$$\theta_{ss} = \int_0^1 K^{vu}(1,\xi) u_{ss}(\xi; d_1, d_2, d_3)\, d\xi + \int_0^1 K^{vv}(1,\xi) v_{ss}(\xi; d_1, d_2, d_3)\, d\xi. \tag{36}$$

Let us define

$$\bar{u}(x,t) = u(x,t) - u_{ss}(x; d_1, d_2, d_3) \tag{37}$$

$$\bar{v}(x,t) = v(x,t) - v_{ss}(x; d_1, d_2, d_3) \tag{38}$$

$$\bar{\eta}(t) = \eta(t) - \eta_{ss}. \tag{39}$$

Using (31) with $Z = (u_{ss}, v_{ss})^\top$ together with (6), (7) it is shown that the variables $\bar{u}$ and $\bar{v}$ satisfy

$$\bar{u}_t + \lambda_1(x)\bar{u}_x = \gamma_1(x)\bar{v} \tag{40}$$

$$\bar{v}_t - \lambda_2(x)\bar{v}_x = \gamma_2(x)\bar{u}. \tag{41}$$

Setting $x = 0$ in (37), (38), and using (8), (32), and (33) we get that

$$\bar{u}(0,t) = q\bar{v}(0,t). \tag{42}$$

Setting $x = 1$ in (38) and using (9), (26), (35), and (39) we arrive at

$$\bar{v}(1,t) = \rho\bar{u}(1,t) - k_I\bar{\eta}(t) - \rho\bar{u}(1,t) + \int_0^1 K^{vu}(1,\xi)\bar{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\bar{v}(\xi,t)d\xi. \tag{43}$$

Using (39) and the fact that

$$\bar{v}(0,t) = v(0,t), \tag{44}$$

relation (27) becomes

$$\dot{\bar{\eta}}(t) = \bar{v}(0,t). \tag{45}$$

Under Proposition 1 the zero equilibrium of (40)–(43) and (45) is exponentially stable. It remains to show that relationship (30) holds. Since, our initial condition is bounded and lies in E, the solution $(v - v_{ss}, u - u_{ss}, \eta - \eta_{ss})^\top$ is bounded and lies in E. Thus, the following holds

$$|v(0,t)| \leq \left|(v - v_{ss}, u - u_{ss}, \eta - \eta_{ss})^\top\right|_E. \tag{46}$$

Using (28) we get that relationship (30) holds. This concludes the proof of Theorem 1. ■

Remark 1: A filtered integral action is proposed in [15] for a hyperbolic Lotka-Volterra system, meaning that the integrator is given by

$$\dot{\eta}(t) = -\varepsilon\eta(t) + v(t,0) \tag{47}$$

where $\varepsilon > 0$ is a small coefficient. This filtering has been introduced to be able to lead a Lyapunov analysis with a "diagonal" Lyapunov function analogous to a L^2-norm. Unfortunately, this approach does not enable proving that the disturbances are rejected for the output in the present case, although the closed-loop system does exhibit a stable behavior. In [14], a "non-diagonal" Lyapunov function has been introduced to avoid the filtering and to be able to prove the compensation of perturbations in the output using the L^2-norm.

IV. Boundary Observer

The control law (26) is a full-state feedback law. Its practical implementation as such would require measurements of the distributed states u and v, which is unrealistic in practice. In this section, we recall the observer design from [9] estimating states from boundary measurements and prove that the observer still asymptotically reaches an equilibrium profile in the presence of disturbances.

A. Observer Structure

Since the measured output is $v(0,t)$, we consider the following observer designed in [9]

$$\hat{u}_t + \lambda_1(x)\hat{u}_x = \gamma_1(x)\hat{v} - p_1(x)\left(y(t) - \hat{v}(0,t)\right) \tag{48}$$

$$\hat{v}_t - \lambda_2(x)\hat{v}_x = \gamma_2(x)\hat{u} - p_2(x)\left(y(t) - \hat{v}(0,t)\right), \tag{49}$$

with the following boundary conditions

$$\hat{u}(0,t) = qv(0,t) \tag{50}$$

$$\hat{v}(1,t) = -k_I\hat{\eta}(t) + \rho\hat{u}(1,t) + U(t), \tag{51}$$

where

$$\dot{\hat{\eta}}(t) = v(0,t) - \hat{v}(0,t)\,. \tag{52}$$

Defining the estimate error variables $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{u}(t,x) = v(t,x) - \hat{v}(t,x)$, this yields the following error system

$$\tilde{u}_t + \lambda_1(x)\tilde{u}_x = \gamma_1(x)\tilde{v} + p_1(x)\tilde{v}(0,t) + d_1(x) \tag{53}$$
$$\tilde{v}_t - \lambda_2(x)\tilde{v}_x = \gamma_2(x)\tilde{u} + p_2(x)\tilde{v}(0,t) + d_2(x)\,, \tag{54}$$

with boundary conditions

$$\tilde{u}(0,t) = d_3 \tag{55}$$
$$\tilde{v}(1,t) = -k_I\hat{\eta}(t) + \rho\tilde{u}(1,t) + d_4\,, \tag{56}$$

where

$$\dot{\hat{\eta}}(t) = \tilde{v}(t,0)\,. \tag{57}$$

Let us consider the solution $(u_{ss}, v_{ss})^\top$ of the ordinary differential equation (31) with the boundary conditions (32), (33). Using (56), it follows

$$\hat{\eta}_{ss} = -\frac{v_{ss}(1;d_1,d_2,d_3) - \rho u_{ss}(1;d_1,d_2,d_3) - d_4}{k_I}\,. \tag{58}$$

Let us define

$$\check{u}(x,t) = \tilde{u}(x,t) - u_{ss}(x;d_1,d_2,d_3) \tag{59}$$
$$\check{v}(x,t) = \tilde{v}(x,t) - v_{ss}(x;d_1,d_2,d_3) \tag{60}$$
$$\check{\eta}(t) = \hat{\eta}(t) - \hat{\eta}_{ss}\,. \tag{61}$$

Using (31) with $Z = (u_{ss}, v_{ss})^\top$ together with (53), (54) it is shown that the variables $\check{u}$ and $\check{v}$ satisfy

$$\check{u}_t + \lambda_1(x)\check{u}_x = \gamma_1(x)\check{v} + p_1(x)\check{v}(0,t) \tag{62}$$
$$\check{v}_t - \lambda_2(x)\check{v}_x = \gamma_2(x)\check{u} + p_2(x)\check{v}(0,t)\,. \tag{63}$$

Setting $x = 0$ in (59) and using (55), (32) we get that

$$\check{u}(0,t) = 0\,. \tag{64}$$

Setting $x = 1$ in (60) and using (56), (58), and (61) we get that

$$\check{v}(1,t) = -k_I\check{\eta}(t) + \rho\check{u}(1,t)\,. \tag{65}$$

Using (61) and the fact that

$$\check{v}(0,t) = \tilde{v}(0,t)\,, \tag{66}$$

relation (57) becomes

$$\dot{\check{\eta}}(0,t) = \check{v}(0,t)\,. \tag{67}$$

To find the injection gain $p_1(x)$ and $p_2(x)$ we map the system (62), (63) into the target system

$$\tilde{\alpha}_t + \lambda_1(x)\tilde{\alpha}_x = \int_0^x g(x,\xi)\tilde{\alpha}(\xi,t)d\xi \tag{68}$$
$$\tilde{\beta}_t - \lambda_2(x)\tilde{\beta}_x = \gamma_2(x)\tilde{\alpha} + \int_0^x h(x,\xi)\tilde{\alpha}(\xi,t)d\xi \tag{69}$$

with the boundary conditions

$$\tilde{\alpha}(0,t) = 0 \tag{70}$$
$$\tilde{\beta}(1,t) = -k_I\hat{\eta}(t)\,,\,, \tag{71}$$

where

$$\dot{\hat{\eta}}(t) = \tilde{\beta}(0,t)\,. \tag{72}$$

To this end, we consider the backstepping transformation introduced in [9]

$$\check{u}(x,t) = \tilde{\alpha}(x,t) + \int_0^x m_1(x,\xi)\tilde{\beta}(\xi,t)d\xi \tag{73}$$
$$\check{v}(x,t) = \tilde{\beta}(x,t) + \int_0^x m_2(x,\xi)\tilde{\beta}(\xi,t)d\xi\,. \tag{74}$$

The gains p_1, p_2 are given by

$$p_1(x) = -\lambda_2(0)m_1(x,0) \tag{75}$$
$$p_2(x) = -\lambda_2(0)m_2(x,0)\,. \tag{76}$$

B. Stable Error Estimate System in presence of Disturbances

The following proposition asseses the stability properties of the observer error system in the presence of disturbances.

Proposition 2: Consider system (53), (54) with boundary conditions (55), (56) where $\hat{\eta}$ satisfies (57), bounded initial condition $\left(\tilde{u}^0, \tilde{v}^0, \hat{\eta}^0\right)^\top \in E$, such that k_I satisfies

$$0 < k_I < \frac{\pi}{2\tau}\,, \tag{77}$$

where τ is given by (16), and gains p_1, p_2 defined in (75) and (76) respectively. Then, there exist two positive constants λ and κ such that the following holds for all $t \geq 0$

$$|(\check{u},\check{v},\check{\eta})|_E \leq \kappa e^{-\lambda t}\left|\left(\check{u}^0, \check{v}^0, \check{\eta}^0\right)\right|_E\,. \tag{78}$$

Proof: Using the proof of Lemma 1, Lemma 1 and Proposition 3 from [7], and the invertibility of the backstepping transformations (24), (25) as a Volterra equation of the second kind, the result holds. ∎

Remark 2: An adaptive observer is designed in [8] for $(n+1)$–state heterodirectional hyperbolic systems in presence of uncertain parameters. Somehow these uncerain parameters may be seen as perturbations for the systems. The estimates converge to the actual value of the state. Here, the estimates do not converge to the actual value of the state but converge to a perturbed value of the state. The proposed method still performs *output* disturbance rejection, as shown in the next section.

V. Disturbance Rejection by the Output Feedback Control

Consider system (6), (7) with boundary conditions (8), (9) where U has the following form

$$U(t) = -k_I\eta(t) - \rho\hat{u}(1,t) + \int_0^x K^{vu}(x,\xi)\hat{u}(t,\xi)d\xi + \int_0^x K^{vv}(x,\xi)\hat{v}(t,\xi)d\xi\,, \tag{79}$$

where η satisfies (27). The aim of this section is to prove that the disturbance are rejected, in the sense of Theorem 1, for the output $v(0,t)$ with this control law. To prove the main result of this paper we need an ISS result for the following system

$$\alpha_t + \lambda_1(x)\alpha_x = 0 \tag{80}$$
$$\beta_t - \lambda_2(x)\beta_x = 0\,, \tag{81}$$

with the boundary conditions

$$\alpha(0,t) = q\beta(0,t) \tag{82}$$
$$\beta(1,t) = -k_I\eta(t) + g(t), \tag{83}$$

where η satisfies (14) and g is a bounded function in $L^\infty(\mathbb{R}^+)$. It is given in the following proposition.

Proposition 3: Consider system (80), (81) with boundary conditions (82), (83) where η satisfies (14). Let k_I be such that

$$0 < k_I < \frac{\pi}{2\tau}, \tag{84}$$

where τ is given by (16). Then, there exist two positive constants κ, λ, and a class $\mathscr{K}$ function γ such that

$$|(\alpha,\beta,\eta)|_E \le \kappa e^{-\lambda t}\left|(\alpha^0,\beta^0,\eta^0)\right|_E + \gamma\left(|g|_{L^\infty(0,t)}\right). \tag{85}$$

Proof: If β is the solution of equation (81) with the boundary condition (83), then the function

$$\rho(x,t) = \beta(x,t) - g(t) \tag{86}$$

satisfies the equation

$$\rho_t - \lambda_2(x)\rho_x = -\dot{g}(t), \tag{87}$$

with the boundary condition

$$\rho(1,t) = -k_I\tilde{\eta}(t), \tag{88}$$

where

$$\dot{\tilde{\eta}}(t) = \rho(0,t) + g(t). \tag{89}$$

We have

$$\frac{d\rho(1,t)}{dt} = -k_I\rho(0,t) - k_Ig(t). \tag{90}$$

Moreover, it may be shown that

$$\rho(x,t) = \rho\left(1, t - \int_x^1 \frac{1}{\lambda_2(\xi)}d\xi\right) + g\left(t - \int_x^1 \frac{1}{\lambda_2(\xi)}d\xi\right) - g(t), \tag{91}$$

is the solution to (87). Hence, we get

$$\frac{d\rho(1,t)}{dt} = -k_I\rho(1,t-\tau) - k_Ig(t-\tau). \tag{92}$$

Using the proof of Lemma 1 and Lemma 1, Proposition 3 from [7] the result holds. ■

We are now ready to state the main result of the paper assessing the output disturbance rejection by the output feedback controller.

Theorem 2: Consider system (6), (7) with boundary conditions (8), (9) where U is given by (79), η satisfies (27), bounded initial condition $\left(u^0, v^0, \eta^0\right)^\top \in E$, such that k_I satisfies

$$0 < k_I < \frac{\pi}{2\tau}, \tag{93}$$

where τ is given by (16). Then the following holds

$$\lim_{t\to+\infty} |v(t,0)| = 0. \tag{94}$$

Proof: The equilibrium $\tilde{Z} = (\hat{u}_{ss}, \hat{v}_{ss})^\top$ of the system (48)–(50) with

$$\hat{v}(1,t) = U(t), \tag{95}$$

is zero. We are still considering the equilibrium $u_{ss}(x; d_1, d_2, d_3)$ and $v_{ss}(x; d_1, d_2, d_3)$ defined as the solution of the ordinary differential equation (31) with the boundary conditions (32), (33). Note that $(\hat{u}_{ss}, \hat{v}_{ss})^\top \neq (u_{ss}, v_{ss})^\top$ in general. From (9) it follows that the equilibrium value of U, namely U_{ss} satisfies

$$U_{ss} = v_{ss}(1; d_1, d_2, d_3) - \rho u_{ss}(1; d_1, d_2, d_3) - d_4. \tag{96}$$

Using (79) and (33) with $Z = (u_{ss}, v_{ss})^\top$, it follows from (96) that the equilibrium value of η, namely η'_{ss}, satisfies

$$\eta'_{ss} = -\frac{v_{ss}(1; d_1, d_2, d_3) - \rho u_{ss}(1; d_1, d_2, d_3) - d_4}{k_I}. \tag{97}$$

Let us define

$$\bar{u}(x,t) = u(x,t) - u_{ss}(x; d_1, d_2, d_3) \tag{98}$$
$$\bar{v}(x,t) = v(x,t) - v_{ss}(x; d_1, d_2, d_3) \tag{99}$$
$$\bar{\eta}(t) = \eta(t) - \eta'_{ss}. \tag{100}$$

Using (31) with $Z = (u_{ss}, v_{ss})^\top$ together with (6), (7) it is shown that the variables $\bar{u}$ and $\bar{v}$ satisfy

$$\bar{u}_t + \lambda_1(x)\bar{u}_x = \gamma_1(x)\bar{v} \tag{101}$$
$$\bar{v}_t - \lambda_2(x)\bar{v}_x = \gamma_2(x)\bar{u}. \tag{102}$$

Setting $x = 0$ in (98), (99), and using (8), (32), and (33) we get that

$$\bar{u}(0,t) = q\bar{v}(0,t). \tag{103}$$

Setting $x = 1$ in (99) and using (9), (79), (97), (100) together with the fact that

$$\hat{u}(x,t) = -(\tilde{u}(x,t) - u_{ss}(x)) + \bar{u}(x,t) \tag{104}$$
$$\hat{v}(x,t) = -(\tilde{v}(x,t) - v_{ss}(x)) + \bar{v}(x,t), \tag{105}$$

we get that

$$\begin{aligned}\bar{v}(1,t) = {} & \rho\bar{u}(1,t) - k_I\bar{\eta}(t) - \rho\bar{u}(1,t) \\ & + \int_0^1 K^{vu}(1,\xi)\bar{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\bar{v}(\xi,t)d\xi \\ & - \int_0^1 K^{uv}(1,\xi)\left(\tilde{u}(\xi,t) - u_{ss}(\xi; d_1, d_2, d_3)\right)d\xi \\ & - \int_0^1 K^{vv}(1,\xi)\left(\tilde{v}(\xi,t) - v_{ss}(\xi; d_1, d_2, d_3)\right)d\xi \\ & + \rho\left(\tilde{u}(1,t) - u_{ss}(1; d_1, d_2, d_3)\right). \end{aligned} \tag{106}$$

Boundary condition (106) may be rewritten formally as

$$\begin{aligned}\bar{v}(1,t) = {} & \rho\bar{u}(1,t) - k_I\bar{\eta}(t) - \rho\bar{u}(1,t) + \int_0^1 K^{vu}(1,\xi)\bar{u}(\xi,t)d\xi \\ & + \int_0^1 K^{vv}(1,\xi)\bar{v}(\xi,t)d\xi - g(t), \end{aligned} \tag{107}$$

where

$$\begin{aligned} g(t) = {} & -\rho\left(\tilde{u}(1,t) - u_{ss}(1; d_1, d_2, d_3)\right) \\ & + \int_0^1 K^{vu}(1,\xi)\left(\tilde{u}(\xi,t) - u_{ss}(\xi; d_1, d_2, d_3)\right)d\xi \\ & + \int_0^1 K^{vv}(1,\xi)\left(\tilde{v}(\xi,t) - v_{ss}(\xi; d_1, d_2, d_3)\right)d\xi. \end{aligned} \tag{108}$$

Using (100) and the fact that

$$\bar{v}(0,t) = v(0,t)\,, \tag{109}$$

relation (27) becomes

$$\dot{\bar{\eta}}(t) = \bar{v}(0,t)\,. \tag{110}$$

The function $g(t)$ is exponentially stable by Proposition 2. Using transformation (25) and Proposition 3, the zero equilibrium of (101)–(103), (107), and (110) is exponentially stable. Relationship (94) is proved analogously to the proof for relationship (30) in Theorem 1. This concludes the proof of Theorem 2. ■

VI. The Case of General Linear Systems

For some positive integers n, m, let us denote by E', the space $\underbrace{L^\infty(0,1)\times\cdots\times L^\infty(0,1)}_{n+m}\times\mathbb{R}^m$.

The methodology presented above may be generalized for systems of higher dimension. Let us consider the case of a general system, that is

$$u_t + \Lambda^+(x)u_x = \Gamma^{++}(x)u + \Gamma^{+-}(x)v + D_1(x) \tag{111}$$
$$v_t - \Lambda^-(x)v_x = \Gamma^{-+}(x)u + \Gamma^{--}(x)v + D_2(x)\,, \tag{112}$$

with the boundary conditions

$$u(0,t) = Qv(0,t) + D_3 \tag{113}$$
$$v(1,t) = Ru(1,t) + U(t) + D_4\,, \tag{114}$$

where Λ^+, Λ^- are diagonal positive definite matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{m\times m}$ respectively, Γ^{++}, Γ^{+-}, Γ^{-+}, Γ^{++}, Q, and R are some matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$, $\mathbb{R}^{m\times n}$, $\mathbb{R}^{m\times m}$, $\mathbb{R}^{n\times m}$, and $\mathbb{R}^{m\times n}$ respectively. The perturbations D_1, D_2, D_3, and D_4 are some vectors in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^m$ respectively.

By adding integrators in the target system introduced in [13], and using the backstepping transformations proposed in the latter reference, the following theorem may be proved.

Theorem 3: Consider system (111), (112) with bounded initial condition $(u,v,\eta^0)\in E'$, boundary condition (113), (114), with

$$U(t) = -K_I\eta(t) - Ru(1,t) + \int_0^1 \left[K^{vu}(1,\xi)u(\xi,t) + K^{vv}(1,\xi)v(\xi,t)\right]d\xi\,, \tag{115}$$

where $K\in\mathbb{R}^{n\times n}$, $L\in\mathbb{R}^{m\times m}$ are the kernels of the backstepping transformations (see [13]), and

$$\dot{\eta}(t) = (v_1(0,t),\ldots,v_m(0,t))^\top \tag{116}$$
$$K_I = \begin{pmatrix} k_I^1 & & 0 \\ & \ddots & \\ 0 & & k_I^m \end{pmatrix}, \tag{117}$$

such that

$$0 < k_I^i < \frac{\pi}{2\tau_i}\,,\quad i = 1,\ldots,m, \tag{118}$$

with

$$\tau_i = \int_0^1 \frac{1}{\mu_i(\xi)}d\xi\,, \tag{119}$$

where the $\mu_i(\cdot)$, $i=1,\ldots,m$ are the diagonal entries of $\Lambda^-(\cdot)$. Then, for all $i=1,\ldots,m$, the following holds

$$\lim_{t\to+\infty}|v_i(0,t)| = 0\,. \tag{120}$$

Proof: [Sketch of proof] We only briefly sketch the proof due to lack of space, and the fact that it follows the same steps as the proof of Theorem 2. The backstepping transformation introduced in [13] maps the unperturbed system to the following target system

$$\alpha_t + \Lambda^+(x)\alpha_x = \Gamma^{++}(x)u + \Gamma^{+-}(x)v + \int_0^x C(x,\xi)\alpha(\xi)d\xi + \int_0^x C^-(x,\xi)\beta(\xi)d\xi \tag{121}$$
$$\beta_t - \Lambda^-(x)\beta_x = G(x)\beta(t,0) \tag{122}$$

with the boundary conditions

$$\alpha(t,0) = Q\beta(t,0) \tag{123}$$
$$\beta(t,1) = -K_I\eta(t) \tag{124}$$

and

$$\dot{\eta}(t) = \beta(t,0)\,, \tag{125}$$

where C^+ and C^- are L^∞ functions on a triangular domain $\mathscr{T}\subset\mathbb{R}^2$ and G is a lower triangular matrix with zero diagonal, i.e.

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1} & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}. \tag{126}$$

The equilibrium $(\alpha,\beta,\eta)^\top = (0,0,0)^\top$ of this system is exponentially stable. Indeed, given the structure of $G(\cdot)$, the first component of β satisfies

$$\frac{d\beta_1(t,1)}{dt} + k_I^1\beta_1(t-\tau_1,1) = 0\,, \tag{127}$$

which is exponentially stable provided k_I^1 satisfies (118) as shown in the proof of Lemma 1. Then, the second component of β satisfies

$$\frac{d\beta_2(t,1)}{dt} + k_I^2\beta_2(t-\tau_1,1) = g_{21}\beta_1(t,0)\,. \tag{128}$$

Considering the right-hand-side term of (128) as an exponentially decaying input, and noticing that the unforced system is exponentially stable, thus ISS, yields that β_2 also exponentially converges to zero. By induction, β exponentially converges to zero, and so does α using a similar ISS argument. The rest of the proof follows the same steps as in the case of two equations. ■

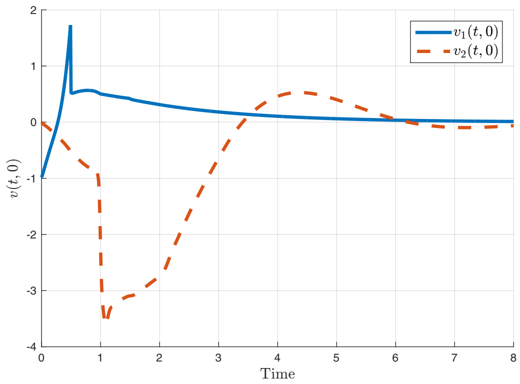

Fig. 1. Time evolution of the controlled output $v(t,0)$.

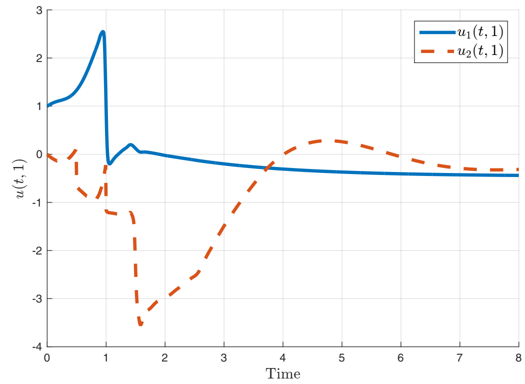

Fig. 2. Time evolution of $u(t,1)$.

VII. Numerical simulations

In this section, we illustrate our result with numerical simulations. We consider the following system parameters

$$\Lambda^+ = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Lambda^- = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{129}$$

$$\Gamma^{++} = \begin{pmatrix} 0 & 0 \\ 1 & -0.5 \end{pmatrix}, \qquad \Gamma^{+-} = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix}, \tag{130}$$

$$\Gamma^{-+} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \Gamma^{--} = \begin{pmatrix} 1 & 0 \\ -0.5 & 1 \end{pmatrix}, \tag{131}$$

$$Q_0 = \begin{pmatrix} 0.1 & 0 \\ 1 & 1.1 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0.5 & 0 \\ 2 & -0.5 \end{pmatrix}. \tag{132}$$

One can readily check that the subsystem $(u_1, v_1)^\top$ cannot be stabilized using static output feedback. More precisely, as proved in [3], there does not exist an L^2 Control Lyapunov Function. This justifies the use of backstepping to asymptotically stabilize the unperturbed system. Besides, we add the following disturbances

$$d_1(x) = \begin{pmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{pmatrix}, \qquad d_2(x) = \begin{pmatrix} -0.5 + \cos(2\pi x) \\ x \end{pmatrix}, \tag{133}$$

$$d_3 = \begin{pmatrix} -2 & -0.5 \end{pmatrix}^\top, \qquad d_4 = \begin{pmatrix} 1 & 2 \end{pmatrix}^\top. \tag{134}$$

We implement the full-state control law (115) with the following parameters

$$k_I^1 = \frac{1}{3}\frac{\pi}{2\tau_1}, \qquad k_I^1 = \frac{1}{6}\frac{\pi}{2\tau_2}. \tag{135}$$

The response in boundary outputs $v(t,0)$ and $u(t,1)$ are depicted on Figures 1 and 2, respectively. As expected from the theory, the outputs $v(t,0)$ converge to zero, whereas $u(t,1)$ converges to its perturbed steady-state value, which is different from zero.

VIII. Conclusions

We have presented a simple way to reject unknown constant output disturbances when backstepping is used to stabilize a system. The proposed method guarantees stability of the closed-loop system in the L^∞ norm. In future works, we plan to study the robustness of the proposed method to uncertainties on model parameters.

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