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Local time penalizations with various clocks for one-dimensional diffusions

Christophe Profeta[(1)], Kouji Yano[(2)]and Yuko Yano[(3)]

Abstract

For a generalized one-dimensional diffusion, we consider the measure weighted and normalized by a non-negative function of the local time evaluated at a parametrized family of random times, which we will call a *clock*. The aim is to give a systematic study of the *penalization* with the clock, i.e., its limit as the clock tends to infinity. We also discuss universal σ-finite measures which govern certain classes of penalizations, thus giving a path interpretation of these penalized processes.

1 Introduction

The systematic studies of penalizations started in 2003 with the works of Roynette, Vallois and Yor, essentially on Brownian motion; see for instance [9], [8], or [10] for a monograph on this subject. Since then, many authors have generalized their results to other processes. When dealing with weights involving local times $(L_t, t \geq 0)$, we may refer in particular to Debs [2] for random walks, Najnudel, Roynette and Yor [6] for Markov chains and Bessel processes, Yano, Yano and Yor [14] for stable processes, or Salminen and Vallois [11] and Profeta [7] for linear diffusions. In most of these papers, the authors focus on penalizations with a natural clock, letting the time t go to infinity in quantities such as $\dfrac{\mathbb{P}[F_s f(L_t)]}{\mathbb{P}[f(L_t)]}$ where f is a positive integrable function and (F_s) is a bounded adapted process. This in turn requires some assumptions on the considered processes, see for instance Salminen and Vallois [11], where the authors introduce a large family of diffusions for which local time penalization results apply.

In this paper, we shall rather study local time penalizations with different clocks, i.e. we shall study the limit of quantities such as $\dfrac{\mathbb{P}[F_s f(L_\tau)]}{\mathbb{P}[f(L_\tau)]}$ as τ tends to infinity in a certain sense along a parametrized family of random times. Examples of such results already appear in the literature, essentially when dealing with processes conditioned to avoid 0, i.e. with the function $f(u) = 1_{\{u=0\}}$. We refer to Knight [4] for Brownian motions, Chaumont and Doney [1] and Doney [1] for Lévy processes, and Yano and Yano [13] for diffusions. Note in particular that, in general, different choices of τ lead to different limits, hence different penalized processes.

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We consider here a generalized one-dimensional diffusion X (in the sense of Watanabe [12]) defined on an interval I whose left boundary is 0, with scale function $s(x) = x$ and speed measure $m(\mathrm{d}x)$. We assume that the function $m : [0, +\infty) \to [0, +\infty]$ is non-decreasing, right-continuous, null at 0 and such that

$$m \text{ is } \begin{cases} \text{strictly increasing on } [0, \ell'), \\ \text{flat and finite on } [\ell', \ell), \\ \text{infinite on } [\ell, +\infty) \end{cases} \tag{1.1}$$

where $0 < \ell' \le \ell \le +\infty$. The choice of the right boundary point of I will depend on m; see [13, Section 2] for the boundary classification; see also Section 7. As for the left boundary point, we assume that 0 is regular-reflecting for X. This implies in particular that X admits a local time at 0, which we shall denote $(L_t, t \ge 0)$, normalized so that

$$\mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}L_t\right] = \frac{r_q(x,0)}{r_q(0,0)}, \tag{1.2}$$

where $r_q(x, y)$ denotes the resolvent density of X with respect to $m(\mathrm{d}y)$. Let ϕ_q and ψ_q be the two classical eigenfunctions associated to X via the integral equations, for $x \in [0, \ell)$:

$$\phi_q(x) = 1 + q \int_0^x dy \int_{(0,y]} \phi_q(z) m(dz), \tag{1.3}$$

$$\psi_q(x) = x + q \int_0^x dy \int_{(0,y]} \phi_q(z) m(dz). \tag{1.4}$$

Set

$$H(q) = \lim_{x \uparrow \ell} \frac{\psi_q(x)}{\phi_q(x)}. \tag{1.5}$$

Denoting by $m(\infty)$ the limit $\lim\limits_{x \to +\infty} m(x)$, we have

$$\lim_{q \downarrow 0} H(q) = \ell \quad \text{and} \quad \lim_{q \downarrow 0} qH(q) = \frac{1}{m(\infty)} =: \pi_0. \tag{1.6}$$

With these notations, the resolvent density of X is given by

$$r_q(x, y) = r_q(y, x) = H(q)\phi_q(x)\left(\phi_q(y) - \frac{\psi_q(y)}{H(q)}\right), \quad 0 \le x \le y, \quad x, y \in I', \tag{1.7}$$

where I' is defined in [13, Section 2] (see also Section 7), We finally define, following [13],

$$h_q(x) = r_q(0,0) - r_q(0,x), \tag{1.8}$$

$$h_0(x) = \lim_{q \downarrow 0} h_q(x) = x - \pi_0 \int_0^x m(y) dy, \tag{1.9}$$

and we call h_0 the *normalized zero resolvent*.

Let us outline the main results of the paper. For simplicity, we assume $\ell' = \ell = \infty$ and we take up the following three cases:

the boundary ∞	$m(\infty)$	$\int_{(1,\infty)} xm(\mathrm{d}x)$
(i) type-1-natural	$= \infty$	$= \infty$
(ii) type-2-natural	$< \infty$	$= \infty$
(iii) entrance	$< \infty$	$< \infty$

The Brownian motion reflected at 0 is an example of case (i), where $\pi_0 = 0$ and $h_0(x) = x$. Some other examples will be given in Section 7. Let us now present our main results of the local time penalizations with various clocks.

1°) Let $\boldsymbol{e}_q$ denote an exponential random variable with parameter q which is independent of the diffusion considered. We may adopt $\{\boldsymbol{e}_q : q > 0\}$ as a clock since $\boldsymbol{e}_q \to \infty$ in law as $q \downarrow 0$.

Theorem 1.1. *Let $f \in L^1_+$ and $x \geq 0$. For any bounded stopping time T and any bounded adapted process (F_t),*

$$H(q)\mathbb{P}_x\big[F_T f(L_{\boldsymbol{e}_q}); T < \boldsymbol{e}_q\big] \underset{q\downarrow 0}{\longrightarrow} \mathbb{P}_x[F_T N_T^{h_0,f}] \quad \textit{and} \quad H(q)\mathbb{P}_x\big[F_T f(L_{\boldsymbol{e}_q})\big] \underset{q\downarrow 0}{\longrightarrow} \mathbb{P}_x[F_T M_T^{h_0,f}] \tag{1.10}$$

where the $\mathbb{P}_x$-supermartingale $N^{h_0,f}$ and the $\mathbb{P}_x$-martingale $M^{h_0,f}$ are defined by

$$N_t^{h_0,f} = h_0(X_t)f(L_t) + \int_0^{+\infty} f(L_t + u)du, \qquad t \geq 0 \tag{1.11}$$

and

$$M_t^{h_0,f} = N_t^{h_0,f} + \pi_0 \int_0^t f(L_u)du, \qquad t \geq 0. \tag{1.12}$$

2°) For $a \in I$, let $T_a = \inf\{t \geq 0 : X_t = a\}$ denote the first hitting time of a by X. We may adopt $\{T_a : a \geq 0\}$ as a clock since $T_a \to \infty$ a.s. as $a \to \infty$.

Theorem 1.2. *Assume that ∞ is natural. Let $f \in L^1_+$ and $x \geq 0$. For any bounded stopping time T and any bounded adapted process (F_t),*

$$a\mathbb{P}_x[F_T f(L_{T_a}); T < T_a] \underset{a\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{s,f}] \quad \textit{and} \quad a\mathbb{P}_x[F_T f(L_{T_a})] \underset{a\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{s,f}] \tag{1.13}$$

where $M^{s,f}$ is the $\mathbb{P}_x$-martingale defined by

$$M_t^{s,f} = X_t f(L_t) + \int_0^{+\infty} f(L_t + u)du, \qquad t \geq 0. \tag{1.14}$$

3°) For $a \geq 0$, let $(L_t^a, t \geq 0)$ denote the local time of X at level a, and define its right-continuous inverse:

$$\eta_u^a = \inf\{t \geq 0,\ L_t^a > u\}. \tag{1.15}$$

We may adopt as a clock $\{\eta_u^a : a \geq 0\}$ for a fixed $u > 0$, since $\eta_u^a \to \infty$ in law as $a \to \infty$.

Theorem 1.3. *Assume that ∞ is either type-1-natural or type-2-natural. Let $f \in L^1_+$, $x \geq 0$ and $u > 0$. For any bounded stopping time T and any bounded adapted process (F_t),*

$$a\mathbb{P}_x\left[F_T f(L_{\eta_u^a}); T < \eta_u^a\right] \underset{a\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{s,f}] \quad \text{and} \quad a\mathbb{P}_x\left[F_T f(L_{\eta_u^a})\right] \underset{a\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{s,f}] \tag{1.16}$$

where $M^{s,f}$ is the $\mathbb{P}_x$-martingale defined above.

We may also adopt as a clock $\{\eta_u^a : u \geq 0\}$ for a fixed $a > 0$, since $\eta_u^a \to \infty$ a.s. as $u \to \infty$.

Theorem 1.4. *Assume that ∞ is either entrance, type-1-natural or type-2-natural. Let $f \in L^1_+$, $x, a \geq 0$ and $\beta > 0$. For any bounded stopping time T and any bounded adapted process (F_t),*

$$\mathrm{e}^{\frac{\beta u}{1+\alpha\beta}}\mathbb{P}_x\left[F_T \mathrm{e}^{-\beta L_{\eta_u^a}}; T < \eta_u^a\right] \underset{u\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{\beta,\alpha}] \quad \text{and} \quad \mathrm{e}^{\frac{\beta u}{1+\alpha\beta}}\mathbb{P}_x\left[F_T \mathrm{e}^{-\beta L_{\eta_u^a}}\right] \underset{u\uparrow+\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{\beta,\alpha}] \tag{1.17}$$

where $M^{\beta,\alpha}$ is the $\mathbb{P}_x$-martingale defined by

$$M_t^{\beta,\alpha} = \frac{1 + \beta(X_t \wedge a)}{1 + \beta a} \exp\left(-\beta L_t + \frac{\beta}{1+\beta a} L_t^a\right), \qquad t \geq 0. \tag{1.18}$$

This paper is organized as follows. The local time penalizations are studied with an independent exponential clock in Section 2, then with a hitting time clock in Section 3 and finally with inverse local time clocks in Section 4. In Section 5, we study universal σ-finite measures. In Section 6, we characterize the limit measure for an exponential weight. The final section, Section 7, is an appendix on our boundary classification.

2 Local time penalization with an exponential clock

Let L^1_+ denote the set of non-negative functions f on $[0, \infty)$ such that $\int_0^\infty f(u)\mathrm{d}u < \infty$.

Lemma 2.1. *Let $f \in L^1_+$, $q > 0$ and $x \in I$. Then*

$$\mathbb{P}_x[f(L_{\boldsymbol{e}_q})] = \frac{1}{H(q)}\left\{h_q(x)f(0) + \frac{r_q(x,0)}{r_q(0,0)}\int_0^\infty \mathrm{e}^{-u/H(q)} f(u)\mathrm{d}u\right\}. \tag{2.1}$$

Proof. Using the excursion theory, we have

$$\mathbb{P}_0\left[\int_0^\infty f(L_t)q\mathrm{e}^{-qt}\mathrm{d}t\right] = \mathbb{P}_0\left[\sum_u \int_{\eta_{u-}}^{\eta_u} f(u)q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{2.2}$$

$$= \mathbb{P}_0\left[\sum_u f(u)\mathrm{e}^{-q\eta_{u-}} \int_0^{T_0(p(u))} q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{2.3}$$

$$= \mathbb{P}_0\left[\int_0^\infty f(u)\mathrm{e}^{-q\eta_u}\mathrm{d}u\right] \boldsymbol{n}\left[1-\mathrm{e}^{-qT_0}\right] \tag{2.4}$$

$$= \int_0^\infty f(u)\mathrm{e}^{-u/H(q)}\mathrm{d}u \cdot \frac{1}{H(q)}, \tag{2.5}$$

where we write p for the excursion point process and set $\eta_u = \sum_{s \le u} T_0(p(s))$, the inverse local time at 0. We now obtain

$$\mathbb{P}_x[f(L_{\boldsymbol{e}_q})] = \mathbb{P}_x\left[\int_0^\infty f(L_t)q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{2.6}$$

$$= \mathbb{P}_x\left[\int_0^{T_0} f(L_t)q\mathrm{e}^{-qt}\mathrm{d}t\right] + \mathbb{P}_x\left[\mathrm{e}^{-qT_0}\right]\mathbb{P}_0\left[\int_0^\infty f(L_t)q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{2.7}$$

$$= f(0)\left\{1 - \frac{r_q(x,0)}{r_q(0,0)}\right\} + \frac{r_q(x,0)}{r_q(0,0)} \cdot \int_0^\infty f(u)\mathrm{e}^{-u/H(q)}\mathrm{d}u \cdot \frac{1}{H(q)}. \tag{2.8}$$

This yields (2.1). □

We have the following remarkable formulae.

Theorem 2.2. *Let $f \in L^1_+$ and $x \in I$. Then the following assertions hold:*

(i) If $\ell < \infty$, i.e., 0 is transient, then

$$\mathbb{P}_x[f(L_\infty)] = \frac{1}{\ell}\left\{xf(0) + \left(1 - \frac{x}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell}f(u)\mathrm{d}u\right\}. \tag{2.9}$$

(ii) If $\pi_0 > 0$, i.e., 0 is positive recurrent, then

$$\mathbb{P}_x\left[\int_0^\infty f(L_t)\mathrm{d}t\right] = \frac{1}{\pi_0}\left\{h_0(x)f(0) + \int_0^\infty f(u)\mathrm{d}u\right\}. \tag{2.10}$$

Proof. (i) Suppose f is bounded. Denote $g = \sup\{t : X_t = 0\}$. Then, at every sample point, we have $f(L_{(1/q)\boldsymbol{e}_1}) = f(L_g) = f(L_\infty)$ for $q > 0$ small enough. Hence we obtain $\mathbb{P}_x[f(L_{\boldsymbol{e}_q})] = \mathbb{P}_x[f(L_{(1/q)\boldsymbol{e}_1})] \to \mathbb{P}_x[f(L_\infty)]$ as $q \downarrow 0$ by the dominated convergence theorem. It is obvious that the right hand side of (2.1) converges to that of (2.9). Hence we obtain (2.9).

For the general case, we have the formula (2.9) for $f \wedge n$ and from this, letting $n \to \infty$, we obtain (2.9) for f.

(ii) We may rewrite (2.1) as

$$\mathbb{P}_x\left[\int_0^\infty f(L_t)\mathrm{e}^{-qt}\mathrm{d}t\right] = \frac{1}{qH(q)}\left\{h_q(x)f(0) + \frac{r_q(x,0)}{r_q(0,0)}\int_0^\infty \mathrm{e}^{-u/H(q)}f(u)\mathrm{d}u\right\}. \tag{2.11}$$

Letting $q \downarrow 0$, we obtain $\mathbb{P}_x\big[\int_0^\infty f(L_t)\mathrm{e}^{-qt}\mathrm{d}t\big] \to \mathbb{P}_x\big[\int_0^\infty f(L_t)\mathrm{d}t\big]$ by the monotone convergence theorem, and hence we obtain (2.10). $\square$

Let $\mathcal{F}_t^0 = \sigma(X_s : s \le t)$ and $\mathcal{F}_t = \mathcal{F}_{t+}^0$. Note that $\boldsymbol{e}_q$ is independent of $\mathcal{F}_\infty := \sigma(\bigcup_t \mathcal{F}_t)$.

Lemma 2.3. *Let $f \in L^1_+$ and $x \in I$. For $q > 0$, set*

$$N_t^q = H(q)\mathbb{P}_x\big[f(L_{\boldsymbol{e}_q})1_{\{t<\boldsymbol{e}_q\}}|\mathcal{F}_t\big], \quad M_t^q = H(q)\mathbb{P}_x\big[f(L_{\boldsymbol{e}_q})|\mathcal{F}_t\big] \tag{2.12}$$

and

$$N_t^{h_0,f} = h_0(X_t)f(L_t) + \left(1 - \frac{X_t}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell}f(L_t+u)\mathrm{d}u, \tag{2.13}$$

$$M_t^{h_0,f} = N_t^{h_0,f} + A_t^{h_0,f}, \tag{2.14}$$

$$A_t^{h_0,f} = \pi_0\int_0^t f(L_u)\mathrm{d}u. \tag{2.15}$$

Then the following assertions hold:

(i) $N_t^q \to N_t^{h_0,f}$ and $M_t^q \to M_t^{h_0,f}$, $\mathbb{P}_x$-a.s. as $q \downarrow 0$;

(ii) $(N_t^{h_0,f})$ is a $\mathbb{P}_x$-supermartingale.

Proof. In what follows in this section we sometimes write N_t, M_t and A_t simply for $N_t^{h_0,f}$, $M_t^{h_0,f}$ and $A_t^{h_0,f}$, respectively.

(i) Since $f(a+\cdot) \in L^1_+$, we have, applying the Markov property,

$$N_t^q = H(q)\mathrm{e}^{-qt}\,\mathbb{P}_{X_t}[f(a+L_{\boldsymbol{e}_q})]\big|_{a=L_t} \tag{2.16}$$

$$= \mathrm{e}^{-qt}\left\{h_q(X_t)f(L_t) + \frac{r_q(X_t,0)}{r_q(0,0)}\int_0^\infty \mathrm{e}^{-u/H(q)}f(L_t+u)\mathrm{d}u\right\}. \tag{2.17}$$

It is now clear that $N_t^q \to N_t$, $\mathbb{P}_x$-a.s. Since

$$A_t^q := M_t^q - N_t^q \tag{2.18}$$

$$= H(q)\mathbb{P}_x[f(L_{\boldsymbol{e}_q})1_{\{\boldsymbol{e}_q \le t\}}|\mathcal{F}_t] \tag{2.19}$$

$$= qH(q)\int_0^t f(L_u)\mathrm{e}^{-qu}\mathrm{d}u, \tag{2.20}$$

we obtain $A_t^q \to A_t$ and $M_t^q \to M_t$, $\mathbb{P}_x$-a.s.

(ii) Since for $s \le t$ we have $1_{\{t<\boldsymbol{e}_q\}} \le 1_{\{s<\boldsymbol{e}_q\}}$, we easily see that (N_t^q) is a $\mathbb{P}_x$-supermartingale. For $s \le t$, we apply Fatou's lemma to obtain

$$\mathbb{P}_x[N_t|\mathcal{F}_s] \le \liminf_{q\downarrow 0}\mathbb{P}_x[N_t^q|\mathcal{F}_s] \le \liminf_{q\downarrow 0} N_s^q = N_s, \tag{2.21}$$

which shows that (N_t) is a $\mathbb{P}_x$-supermartingale. $\square$

Theorem 2.4. *Let $f \in L^1_+$ and $x \in I$. Then, for any finite stopping time T, it holds that*

$$N_T^q \underset{q \downarrow 0}{\longrightarrow} N_T^{h_0,f} \quad \text{in } L^1(\mathbb{P}_x). \tag{2.22}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$\lim_{q\downarrow 0} H(q)\mathbb{P}_x[F_T f(L_{\boldsymbol{e}_q}); T < \boldsymbol{e}_q] = \mathbb{P}_x[F_T N_T^{h_0,f}]. \tag{2.23}$$

Proof. Observe first by Fatou's lemma that

$$\mathbb{P}_x[N_T] \le \liminf_{n\to\infty} \mathbb{P}_x[N_{T\wedge n}] \le \mathbb{P}_x[N_0] < \infty. \tag{2.24}$$

Let us compute N_T^q. We have

$$N_T^q = \mathrm{e}^{-qT} h_q(X_T) f(L_t) + \mathrm{e}^{-qT} \frac{r_q(X_T,0)}{r_q(0,0)} \int_0^\infty \mathrm{e}^{-u/H(q)} f(L_T+u)\mathrm{d}u \tag{2.25}$$

$$= (\mathrm{I})_q + (\mathrm{II})_q. \tag{2.26}$$

We write similarly

$$N_T = h_0(X_T) f(L_T) + \left(1 - \frac{X_T}{\ell}\right) \int_0^\infty \mathrm{e}^{-u/\ell} f(L_T+u)\mathrm{d}u \tag{2.27}$$

$$= (\mathrm{I}) + (\mathrm{II}). \tag{2.28}$$

Since $(\mathrm{II})_q \le \int_0^\infty f(u)\mathrm{d}u$, we may apply the dominated convergence theorem to obtain $(\mathrm{II})_q \to (\mathrm{II})$ in $L^1(\mathbb{P}_x)$.

If $\pi_0 = 0$, then we have $(\mathrm{I})_q \le X_T f(L_T) = h_0(X_T) f(L_T) \le N_T$. If $\pi_0 > 0$ and ℓ' is regular-reflecting, then we have $h_0(x) \ge cx$ with $c = h_0(\ell')/\ell' > 0$, since $h_0(x)$ is concave. We now have $(\mathrm{I})_q \le X_T f(L_T) \le c^{-1} h_0(X_T) f(L_T) \le c^{-1} N_T$. In both cases, since $\mathbb{P}_x[N_T] \le \mathbb{P}_x[N_0] < \infty$, we may apply the dominated convergence theorem to obtain $(\mathrm{I})_q \to (\mathrm{I})$ in $L^1(\mathbb{P}_x)$.

If $\pi_0 > 0$ and ℓ' is either entrance or natural, we have $(\mathrm{I})_q \le X_T f(L_T)$. Since we see by (ii) of Lemma 3.2 that

$$\mathbb{P}_x[X_T f(L_T)] \le \mathbb{P}_x[N_T^{s,f}] \le \mathbb{P}_x[N_0^{s,f}] = x f(0) + \left(1 - \frac{x}{\ell}\right) \int_0^\infty \mathrm{e}^{-u/\ell} f(u)\mathrm{d}u < \infty, \tag{2.29}$$

we may apply the dominated convergence theorem to obtain $(\mathrm{I})_q \to (\mathrm{I})$ in $L^1(\mathbb{P}_x)$.

Therefore we have obtained the former assertion.

For the latter assertion, we have

$$H(q)\mathbb{P}_x[F_T f(L_{\boldsymbol{e}_q}); T < \boldsymbol{e}_q] = \mathbb{P}_x[F_T N_T^q] \underset{q\downarrow 0}{\longrightarrow} \mathbb{P}_x[F_T N_T]. \tag{2.30}$$

□

Theorem 2.5. *Let $f \in L^1_+$ and $x \in I$. Let T be a finite stopping time such that*

$$\mathbb{P}_x\left[\int_0^T f(L_u)\mathrm{d}u\right] < \infty. \tag{2.31}$$

Then it holds that

$$M_T^q \to M_T^{h_0,f} \quad \text{in } L^1(\mathbb{P}_x). \tag{2.32}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$\lim_{q\downarrow 0} H(q)\mathbb{P}_x[F_T f(L_{\boldsymbol{e}_q})] = \mathbb{P}_x[F_T M_T^{h_0,f}]. \tag{2.33}$$

Proof. By (2.31), we have $\int_0^T f(L_u)\mathrm{e}^{-qu}\mathrm{d}u \to \int_0^T f(L_u)\mathrm{d}u$ in $L^1(\mathbb{P}_x)$. This shows that $A_T^q \to A_T$ in $L^1(\mathbb{P}_x)$, which implies $M_T^q \to M_T$ in $L^1(\mathbb{P}_x)$. The latter assertion is obvious. □

Theorem 2.6. *The condition* (2.31) *is satisfied whenever T is a bounded stopping time. For any $f \in L^1_+$, it holds that*

$$M_t^{h_0,f} = h_0(X_t)f(L_t) + \left(1 - \frac{X_t}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell} f(L_t+u)\mathrm{d}u + \pi_0 \int_0^t f(L_u)\mathrm{d}u \tag{2.34}$$

is a $\mathbb{P}_x$-martingale. Consequently, the identity $N_t^{h_0,f} = M_t^{h_0,f} - A_t^{h_0,f}$ may be regarded as the Doob–Meyer decomposition of the supermartingale $(N_t^{h_0,f})$.

Proof. Since

$$q^2\int_0^\infty \mathbb{P}_x\left[\int_0^t f(L_u)\mathrm{d}u\right]\mathrm{e}^{-qt}\mathrm{d}t = \mathbb{P}_x\left[\int_0^\infty f(L_u)q\mathrm{e}^{-qu}\mathrm{d}u\right] = \mathbb{P}_x\big[f(L_{\boldsymbol{e}_q})\big] < \infty \tag{2.35}$$

and since $t \mapsto \mathbb{P}_x\left[\int_0^t f(L_u)\mathrm{d}u\right]$ is increasing, we see that $\mathbb{P}_x\left[\int_0^t f(L_u)\mathrm{d}u\right] < \infty$ for all $t \geq 0$. In other words, the assumption (2.31) is satisfied when T is a bounded stopping time. This shows that (M_t) is a $\mathbb{P}_x$-martingale. □

Remark 2.7. If ℓ' is type-1-natural, then the identity (2.34) becomes

$$M_t^{h_0,f} = X_t f(L_t) + \int_0^\infty f(L_t+u)\mathrm{d}u, \tag{2.36}$$

which is nothing else but the Azema–Yor martingale. In this sense we may regard the identity (2.34) as a generalization of the Azema–Yor martingale. Another generalization will be given in Theorem 3.5.

Remark 2.8. If we take $f(u) = 1_{\{u=0\}}$, we have

$$M_t^{h_0,f} = h_0(X_t)1_{\{T_0>t\}} + \pi_0(T_0 \wedge t). \tag{2.37}$$

In particular, from the identity $\mathbb{P}_x[M_0^{h_0,f}] = \mathbb{P}_x[M_t^{h_0,f}]$, we obtain

$$h_0(x) = \mathbb{P}_x[h_0(X_t); T_0 > t] + \pi_0\mathbb{P}_x[T_0 \wedge t], \tag{2.38}$$

which verifies the first assertion of Theorem 6.4 of [13].

3 Local time penalization with a hitting time clock

In this section we assume that $\ell(=\ell')$ is either entrance or natural. Since any point in $[0,\ell)$ is accessible but ℓ is not, we have

$$\mathbb{P}_x(T_a \to \infty \text{ as } a \uparrow \ell) = 1. \tag{3.1}$$

Lemma 3.1. *Let $f \in L^1_+$ and $x \in I$. Then, for any $a \in I$ with $x < a$,*

$$\mathbb{P}_x[f(L_{T_a})] = \frac{1}{a}\left\{ xf(0) + \left(1 - \frac{x}{a}\right) \int_0^\infty \mathrm{e}^{-u/a} f(u)\mathrm{d}u \right\}. \tag{3.2}$$

Proof. Let $\mathbb{P}^a_x$ denote the law of $X_{\cdot \wedge T_a}$ under $\mathbb{P}_x$. Then we have

$$\mathbb{P}_x[f(L_{T_a})] = \mathbb{P}^a_x[f(L_\infty)]. \tag{3.3}$$

Since $\{X, \mathbb{P}^a_x\}$ is a diffusion process on $[0,a]$ where a is a regular-absorbing boundary, we may use (i) of Theorem 2.2 and obtain (3.2). □

Lemma 3.2. *Let $f \in L^1_+$ and $x \in I$. For any $a \in I$ with $x < a$, set*

$$N^a_t = a\mathbb{P}_x\left[f(L_{T_a})1_{\{t<T_a\}}|\mathcal{F}_t\right], \quad M^a_t = a\mathbb{P}_x[f(L_{T_a})|\mathcal{F}_t] \tag{3.4}$$

and

$$M^{s,f}_t = X_t f(L_t) + \left(1 - \frac{X_t}{\ell}\right) \int_0^\infty \mathrm{e}^{-u/\ell} f(L_t + u)\mathrm{d}u. \tag{3.5}$$

Then the following assertions hold:

(i) $N^a_t \to M^{s,f}_t$ and $M^a_t \to M^{s,f}_t$, $\mathbb{P}_x$-a.s. as $a \uparrow \ell$;

(ii) $(M^{s,f}_t)$ is a $\mathbb{P}_x$-supermartingale and is a local $\mathbb{P}_x$-martingale.

Proof. In what follows in this section we sometimes write M_t simply for $M^{s,f}_t$.

(i) Since $f(b + \cdot) \in L^1_+$, we have, by Lemma 3.1,

$$\begin{aligned} N^a_t &= a\, \mathbb{P}_{X_t}[f(b + L_{T_a})]|_{b=L_t}\, 1_{\{t<T_a\}} &(3.6)\\ &= \left\{ X_t f(L_t) + \left(1 - \frac{X_t}{a}\right) \int_0^\infty \mathrm{e}^{-u/a} f(L_t + u)\mathrm{d}u \right\} 1_{\{t<T_a\}}. &(3.7) \end{aligned}$$

Since $T_a \to \infty$ as $a \uparrow \ell$, we have $N^a_t \to M_t$, $\mathbb{P}_x$-a.s. Set

$$A^a_t = M^a_t - N^a_t = af(L_{T_a})1_{\{T_a \le t\}}. \tag{3.8}$$

Since $A^a_t \to 0$, $\mathbb{P}_x$-a.s., we have $M^a_t \to M_t$, $\mathbb{P}_x$-a.s.

(ii) In the same way as (ii) of Lemma 2.3, we can see that (M_t) is a $\mathbb{P}_x$-supermartingale.

It is obvious that (M^a_t) is a $\mathbb{P}_x$-martingale. Let $\{a_n\}$ be a sequence of I such that $a_n \uparrow \ell$. If we take $\sigma_n = \inf\{t : X_t > a_n\}$, we have $A^a_{\sigma_n \wedge t} = af(L_{T_a})1_{\{T_a \le \sigma_n \wedge t\}} = 0$ for any $a > a_n$, so that we have $M^a_{\sigma_n \wedge t} \to M_{\sigma_n \wedge t}$ in $L^1(\mathbb{P}_x)$ as $a \uparrow \ell$. This shows that (M_t) is a local $\mathbb{P}_x$-martingale. □

Theorem 3.3. *Let $f \in L^1_+$ and $x \in I$. Then, for any finite stopping time T, it holds that*

$$N^a_T \underset{a\uparrow\ell}{\longrightarrow} M^{s,f}_T \quad \text{in } L^1(\mathbb{P}_x). \tag{3.9}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$a\mathbb{P}_x[F_T f(L_{T_a}); T < T_a] \underset{a\uparrow\ell}{\longrightarrow} \mathbb{P}_x[F_T M^{s,f}_T]. \tag{3.10}$$

Proof. We have

$$N^a_T = X_T f(L_T) 1_{\{T<T_a\}} + \left(1 - \frac{X_T}{a}\right) \int_0^\infty \mathrm{e}^{-u/a} f(L_T + u) \mathrm{d}u 1_{\{T<T_a\}}, \tag{3.11}$$

$$M_T = X_T f(L_T) + \left(1 - \frac{X_T}{\ell}\right) \int_0^\infty \mathrm{e}^{-u/\ell} f(L_T + u) \mathrm{d}u. \tag{3.12}$$

Since $N^a_T \le M_T$ and since

$$\mathbb{P}_x[M_T] \le \liminf_{n\to\infty} \mathbb{P}_x[M_{T\wedge n}] \le \mathbb{P}_x[M_0] < \infty, \tag{3.13}$$

we may apply the dominated convergence theorem to obtain (3.9). The remaining assertion is obvious. $\square$

Lemma 3.4. *Suppose that ℓ is natural. Then*

$$a\mathbb{P}_x(T_a \le t) \underset{a\uparrow\ell}{\longrightarrow} 0 \quad \text{for all } t \ge 0. \tag{3.14}$$

Proof. If $\ell < \infty$, i.e., ℓ is type-3-natural, then (3.14) is obvious.

Suppose $\ell = \infty$. Then we have

$$a\mathbb{P}_x(T_a \le t) \le a\mathrm{e}^t \mathbb{P}_x[\mathrm{e}^{-T_a}] = \mathrm{e}^t \phi_1(x) \cdot \frac{a}{\phi_1(a)}. \tag{3.15}$$

Since $\ell = \infty$ is natural, we have

$$\phi_1(a) = 1 + \int_0^a \mathrm{d}x \int_{(0,x]} \phi_1(y) \mathrm{d}m(y) \ge \int_0^a \mathrm{d}x \int_{(0,x]} \mathrm{d}m(y) \underset{a\uparrow\ell}{\longrightarrow} \infty \tag{3.16}$$

and, for $a > 1$,

$$\phi_1'(a) \ge \int_{(0,a]} \mathrm{d}m(x) \int_0^x \phi_1'(y) \mathrm{d}y \ge \phi_1'(1) \int_{(1,a]} \mathrm{d}m(x) \int_1^x \mathrm{d}y \underset{a\uparrow\ell}{\longrightarrow} \infty. \tag{3.17}$$

Thus, by the l'Hôpital rule, we obtain $a/\phi_1(a) \to 0$ as $a \uparrow \ell = \infty$. Therefore we obtain (3.14). $\square$

Theorem 3.5. *Suppose that ℓ is natural. Then, for any $f \in L^1_+$ and for any bounded stopping time T, it holds that*

$$M^a_T \underset{a\uparrow\ell}{\longrightarrow} M^{s,f}_T \quad \text{in } L^1(\mathbb{P}_x). \tag{3.18}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$a\mathbb{P}_x[F_T f(L_{T_a})] \underset{a\uparrow\ell}{\longrightarrow} \mathbb{P}_x\left[F_T M^{s,f}_T\right]. \tag{3.19}$$

It also holds that

$$M^{s,f}_t = X_t f(L_t) + \left(1 - \frac{X_t}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell} f(L_t + u)\mathrm{d}u \tag{3.20}$$

is a $\mathbb{P}_x$-martingale.

Proof. Suppose that $f \in L^1_+$ is bounded. Since $A^a_T \to 0$, $\mathbb{P}_x$-a.s. and since

$$\mathbb{P}_x[A^a_T] \leq a\|f\|_\infty \mathbb{P}_x(T_a \leq T) \underset{a\uparrow\ell}{\longrightarrow} 0, \tag{3.21}$$

we see that $A^a_T \to 0$ in $L^1(\mathbb{P}_x)$. Hence we obtain (3.18) and (3.19) in this special case.

We now see that $\mathbb{P}_x[M^{s,f}_t] = \mathbb{P}_x[M^{s,f}_0]$, i.e.,

$$\begin{aligned}&\mathbb{P}_x\left[X_t f(L_t) + \left(1 - \frac{X_t}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell} f(L_t + u)\mathrm{d}u\right] \\ =&x f(0) + \left(1 - \frac{x}{\ell}\right)\int_0^\infty \mathrm{e}^{-u/\ell} f(u)\mathrm{d}u\end{aligned} \tag{3.22}$$

holds for all $t \geq 0$ and all bounded $f \in L^1_+$. By considering $f \wedge n$, taking $n \to \infty$ and applying the monotone convergence theorem, we can drop the boundedness assumption and obtain (3.22) for all $t \geq 0$ and all $f \in L^1_+$. By (ii) of Lemma 3.2, we see, for any $f \in L^1_+$, that $(M^{s,f}_t)$ is a $\mathbb{P}_x$-supermartingale with constant expectation, which turns out to be a $\mathbb{P}_x$-martingale.

Let $f \in L^1_+$. Since $(M^{s,f}_t)$ is a $\mathbb{P}_x$-martingale, we may apply the optional stopping theorem to see that

$$\mathbb{P}_x[A^a_T] = \mathbb{P}_x[X_{T_a} f(L_{T_a}); T_a \leq T] \tag{3.23}$$
$$\leq \mathbb{P}_x[M_{T_a}; T_a \leq T] \tag{3.24}$$
$$= \mathbb{P}_x[M_T; T_a \leq T] \underset{a\uparrow\ell}{\longrightarrow} 0. \tag{3.25}$$

Since $A^a_T \to 0$, $\mathbb{P}_x$-a.s., we see that $A^a_T \to 0$ in $L^1(\mathbb{P}_x)$. Hence we obtain (3.18) and (3.19) in a general case. $\square$

Remark 3.6. Suppose ℓ is entrance. We claim that (M_t) is *not* a true $\mathbb{P}_x$-martingale if $f(0) > 0$. Suppose (M_t) were a $\mathbb{P}_x$-martingale. On one hand we would have

$$\mathbb{P}_x[M_{t\wedge T_0}] = M_0 = xf(0) + \int_0^\infty f(u)\mathrm{d}u. \tag{3.26}$$

On the other hand, we see that $\mathbb{P}_x[M_{t\wedge T_0}]$ is equal to

$$\mathbb{P}_x[M_t; t < T_0] + \mathbb{P}_x[M_{T_0}; t \geq T_0] = \mathbb{P}_x[X_t; t < T_0]f(0) + \int_0^\infty f(u)\mathrm{d}u. \tag{3.27}$$

Hence we would have $\mathbb{P}_x[X_t; t < T_0] = x$, which would contradict the fact that the scale function $s(x) = x$ is *not* $\mathbb{P}_x$-invariant (see Theorem 6.5 of [13]).

4 Local time penalization with inverse local time clocks

4.1 Limit as a tends to infinity with u being fixed

Suppose $\ell'(= \ell = \infty)$ is either entrance, type-1-natural or type-2-natural. We thus have, for any $x \in I$ and any $u > 0$,

$$\mathbb{P}_x(\eta_u^a < \infty) = 1 \quad \text{and} \quad \eta_u^a \underset{a\to\infty}{\longrightarrow} \infty \text{ in law under } \mathbb{P}_x; \tag{4.1}$$

in fact, we have

$$\mathbb{P}_x[\mathrm{e}^{-q\eta_u^a}] \to \begin{cases} 1 & \text{as } q \downarrow 0, \\ 0 & \text{as } a \to \infty, \end{cases} \tag{4.2}$$

since we have $r_q(a,a) \underset{q\downarrow 0}{\longrightarrow} \infty$, $\phi_q(a) \underset{a\to\infty}{\longrightarrow} \infty$ and

$$\mathbb{P}_x[\mathrm{e}^{-q\eta_u^a}] = \mathbb{P}_x[\mathrm{e}^{-qT_a}]\mathbb{P}_a[\mathrm{e}^{-q\eta_u^a}] = \frac{\phi_q(x)}{\phi_q(a)} \exp\left(-\frac{u}{r_q(a,a)}\right). \tag{4.3}$$

For $\nu \geq 0$, we denote by $I_\nu(x)$ the modified Bessel function of the first kind, which may be represented as a series expansion formula (see e.g. [5], eq. (5.7.1) on page 108) by

$$I_\nu(x) = \sum_{n=0}^\infty \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \quad x > 0. \tag{4.4}$$

We recall the asymptotic formulae (see e.g. [5], Section 5.16):

$$I_\nu(x) \underset{x\downarrow 0}{\sim} \frac{(x/2)^\nu}{\Gamma(1+\nu)}, \quad I_\nu(x) \underset{x\to\infty}{\sim} \frac{\mathrm{e}^x}{\sqrt{2\pi x}}. \tag{4.5}$$

Lemma 4.1. *Let $a \in (0,\infty)$. Then the process $\{(L_{\eta^a_u})_{u\ge 0}, \mathbb{P}_a\}$ is a compound Poisson process with Laplace transform*

$$\mathbb{P}_a\left[\mathrm{e}^{-\beta L_{\eta^a_u}}\right] = \exp\left\{-u\int_0^\infty (1-\mathrm{e}^{-\beta s})\frac{1}{a^2}\mathrm{e}^{-s/a}\mathrm{d}s\right\} = \mathrm{e}^{-\frac{u\beta}{1+\beta a}}. \tag{4.6}$$

For any $u>0$ and $f \in L^1_+$,

$$\mathbb{P}_a[f(L_{\eta^a_u})] = \mathrm{e}^{-u/a}f(0) + \int_0^\infty f(y)\rho^a_u(y)\mathrm{d}y, \tag{4.7}$$

where

$$\rho^a_u(y) = \mathrm{e}^{-(u+y)/a}\frac{\sqrt{u/y}}{a}I_1\left(\frac{2\sqrt{uy}}{a}\right). \tag{4.8}$$

Proof. Let $p^a(v)$ denote the point process of excursions away from a and $\boldsymbol{n}^a$ its excursion measure. Since L increases only on the intervals (η^a_{v-}, η^a_v), we have

$$L_{\eta^a_u} = \sum_{v\le u:\, p^a(v)\in\{T_0<\infty\}} (L_{\eta^a_v} - L_{\eta^a_{v-}}) = \sum_{v\le u:\, p^a(v)\in\{T_0<\infty\}} L_{T_a}(p^a(v)). \tag{4.9}$$

Since $\boldsymbol{n}^a(T_0 < T_a) = 1/a < \infty$, the sum of (4.9) is a finite sum, and so we see that $\{(L_{\eta^a_u})_{u\ge 0}, \mathbb{P}_a\}$ is a compound Poisson process with Lévy measure

$$\boldsymbol{n}^a(L_{T_a} \in \mathrm{d}s; T_0 < T_a). \tag{4.10}$$

By the strong Markov property of $\boldsymbol{n}^a$, we have

$$\boldsymbol{n}^a(L_{T_a} > s; T_0 < T_a) = \boldsymbol{n}^a(T_0<\infty)\mathbb{P}_0(L_{T_a} > s) = \frac{1}{a}\mathbb{P}_0(L_{T_a}>s). \tag{4.11}$$

Let $\lambda^0_a = \inf\{v : p^0(v) \in \{T_a < \infty\}\}$. Then we have

$$\mathbb{P}_0(L_{T_a} > s) = \mathbb{P}_0(T_a > \eta^0_s) = \mathbb{P}_0(\lambda^0_a > s) = \mathrm{e}^{-s\boldsymbol{n}^0(T_a<\infty)} = \mathrm{e}^{-s/a}. \tag{4.12}$$

Thus we obtain (4.6).

Let $\{S_n\}$ be a process with i.i.d. increments $\mathbb{P}(S_n - S_{n-1} > s) = \mathrm{e}^{-s/a}$ such that $S_0 = 0$ and let N be a Poisson variable with mean u/a which is independent of $\{S_n\}$. Then we have $L_{\eta^a_u} \overset{\text{law}}{=} S_N$, and hence

$$\begin{aligned}\mathbb{P}_a[f(L_{\eta^a_u})] &= \mathbb{P}(N=0)f(0) + \sum_{n=1}^\infty \mathbb{P}(N=n)\mathbb{P}[f(S_n)] && (4.13)\\ &= \mathrm{e}^{-u/a}f(0) + \sum_{n=1}^\infty \mathrm{e}^{-u/a}\frac{(u/a)^n}{n!}\int_0^\infty f(y)\frac{(y/a)^{n-1}}{(n-1)!}\mathrm{e}^{-y/a}\frac{\mathrm{d}y}{a}. && (4.14)\end{aligned}$$

Thus, using (4.4), we obtain (4.7). □

Lemma 4.2. *For $u > 0$, $x, a \in I$ and $f \in L^1_+$, it holds that*

$$\mathbb{P}_x[f(L_{\eta_u^a})] = \frac{x \wedge a}{a}\mathbb{P}_a[f(L_{\eta_u^a})] + \left(1 - \frac{x}{a}\right)_+ \mathbb{P}_a[f(\boldsymbol{e}_{1/a} + L_{\eta_u^a})] \tag{4.15}$$

$$= \frac{x \wedge a}{a}\mathbb{P}_a[f(L_{\eta_u^a})] + \frac{1}{a}\left(1 - \frac{x}{a}\right)_+ \int_0^\infty f(y)\widetilde{\rho}_u^a(y)\mathrm{d}y, \tag{4.16}$$

where

$$\widetilde{\rho}_u^a(y) = \mathrm{e}^{-(u+y)/a} I_0\left(\frac{2\sqrt{uy}}{a}\right). \tag{4.17}$$

Proof. When $a \le x$, we have $\mathbb{P}_x[f(L_{\eta_u^a})] = \mathbb{P}_x[f(L_{T_a} + L_{\eta_u^a} \circ \theta_{T_a})] = \mathbb{P}_a[f(L_{\eta_u^a})]$, which proves identity (4.15).

Suppose $x < a$. Using Lemma 3.1, we have

$$\mathbb{P}_x[f(L_{\eta_u^a})] = \mathbb{P}_x\left[f\left(L_{T_a} + L_{\eta_u^a} \circ \theta_{T_a}\right)\right] \tag{4.18}$$

$$= \frac{x}{a}\mathbb{P}_a[f(L_{\eta_u^a})] + \frac{1}{a}\left(1 - \frac{x}{a}\right)\mathbb{P}_a\left[\int_0^\infty \mathrm{e}^{-v/a} f(v + L_{\eta_u^a})\mathrm{d}v\right], \tag{4.19}$$

which coincides with (4.15). Using the same notation as that of the proof of Lemma 4.1, we obtain

$$\mathbb{P}_a[f(\boldsymbol{e}_{1/a} + L_{\eta_u^a})] = \sum_{n=0}^\infty \mathbb{P}(N = n)\mathbb{P}[f(S_{n+1})] \tag{4.20}$$

$$= \sum_{n=0}^\infty \mathrm{e}^{-u/a}\frac{(u/a)^n}{n!}\int_0^\infty f(y)\frac{(y/a)^n}{n!}\mathrm{e}^{-y/a}\frac{\mathrm{d}y}{a}. \tag{4.21}$$

Thus, using (4.4), we obtain (4.16). $\square$

By (4.5), there exists a constant C such that

$$\begin{cases} I_\nu(x) \le Cx^\nu & \text{for } 0 < x \le 1, \\ I_\nu(x) \le C\mathrm{e}^x & \text{for } x \ge 1. \end{cases} \tag{4.22}$$

Lemma 4.3. *For any $u > 0$, $a > 0$ and $y > 0$, it holds that*

$$\rho_u^a(y) \le \frac{2Cu}{a^2}, \quad \widetilde{\rho}_u^a(y) \le C. \tag{4.23}$$

For any fixed $u > 0$ and $y > 0$, it holds that

$$\widetilde{\rho}_u^a(y) \underset{a\to\infty}{\longrightarrow} 1. \tag{4.24}$$

Proof. Using (4.5), we easily have (4.24).

If $2\sqrt{uy}/a \le 1$, we have

$$\rho^a_u(y) \le C\frac{2u}{a^2}, \quad \widetilde{\rho}^a_u(y) \le C. \tag{4.25}$$

If $2\sqrt{uy}/a > 1$, we have

$$\rho^a_u(y) \le C\mathrm{e}^{-(\sqrt{u}+\sqrt{y})^2/a}\frac{\sqrt{u/y}}{a} \le C\frac{2u}{a^2}, \tag{4.26}$$

$$\widetilde{\rho}^a_u(y) \le C\mathrm{e}^{-(\sqrt{u}+\sqrt{y})^2/a} \le C. \tag{4.27}$$

Therefore we obtain (4.23). □

Lemma 4.4. *Let $f \in L^1_+$, $x \in I$ and $u > 0$. For any $a \in I$, set*

$$N^{a,u}_t = a\mathbb{P}_x\left[f(L_{\eta^a_u})1_{\{t<\eta^a_u\}} \mid \mathcal{F}_t\right], \tag{4.28}$$

$$M^{a,u}_t = a\mathbb{P}_x\left[f(L_{\eta^a_u}) \mid \mathcal{F}_t\right]. \tag{4.29}$$

Then it holds that $N^{a,u}_t \to M^{s,f}_t$ and $M^{a,u}_t \to M^{s,f}_t$ in probability with respect to $\mathbb{P}_x$ as $a \to \infty$, where $M^{s,f}_t$ has been defined in (3.5).

Proof. In what follows in this section we sometimes write M_t simply for $M^{s,f}_t$.

(i) By the strong Markov property and by Lemma 4.2, we have, for $a > X_t$,

$$N^{a,u}_t = a\,\mathbb{P}_{X_t}[f(b + L_{\eta^a_{u-c}})]\Big|_{\substack{b=L_t\\ c=L^a_t}} 1_{\{t<\eta^a_u\}} = (\mathrm{I})_a + (\mathrm{II})_a, \tag{4.30}$$

where

$$(\mathrm{I})_a = X_t\left\{\mathrm{e}^{-\frac{u-c}{a}}f(b) + \int_0^\infty f(b+y)\rho^a_{u-c}(y)\mathrm{d}y\right\}\Bigg|_{\substack{b=L_t\\ c=L^a_t}} 1_{\{t<\eta^a_u\}}, \tag{4.31}$$

$$(\mathrm{II})_a = \left(1 - \frac{X_t}{a}\right)\int_0^\infty f(b+y)\widetilde{\rho}^a_{u-c}(y)\mathrm{d}y\Bigg|_{\substack{b=L_t\\ c=L^a_t}} 1_{\{t<\eta^a_u\}}. \tag{4.32}$$

Letting $a \to \infty$, we deduce from Lemma 4.3 that in probability with respect to $\mathbb{P}_x$

$$(\mathrm{I})_a \underset{a\to\infty}{\longrightarrow} X_t f(L_t), \tag{4.33}$$

$$(\mathrm{II})_a \underset{a\to\infty}{\longrightarrow} \int_0^\infty f(L_t + y)\mathrm{d}y. \tag{4.34}$$

We thus obtain $N^{a,u}_t \to M^{s,f}_t$ in probability with respect to $\mathbb{P}_x$. Set

$$A^{a,u}_t = M^{a,u}_t - N^{a,u}_t = af(L_{\eta^a_u})1_{\{\eta^a_u \le t\}}. \tag{4.35}$$

Since $A^{a,u}_t \to 0$, we obtain $M^{a,u}_t \to M^{s,f}_t$ in probability with respect to $\mathbb{P}_x$. □

Theorem 4.5. *Let $f \in L^1_+$, $x \in I$ and $u > 0$. Then, for any finite stopping time T, it holds that*

$$N_T^{a,u} \underset{a\to\infty}{\longrightarrow} M_T^{s,f} \quad \text{in } L^1(\mathbb{P}_x). \tag{4.36}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$a\mathbb{P}_x[F_T f(L_{\eta_u^a}); T < \eta_u^a] \underset{a\to\infty}{\longrightarrow} \mathbb{P}_x[F_T M_T^{s,f}]. \tag{4.37}$$

Proof. By the proof of Lemma 4.4 and by Lemma 4.3, we obtain, for $a > 1$,

$$N_t^{a,u} \leq X_t f(L_t) + \left(\frac{2Cu}{a} + C\right) \int_0^\infty f(L_t + y)\mathrm{d}y \tag{4.38}$$

$$\leq M_t^{s,f} + (2Cu + C) \int_0^\infty f(y)\mathrm{d}y, \tag{4.39}$$

where the last quantity is integrable with respect to $\mathbb{P}_x$. Thus we obtain the desired result by the dominated convergence theorem. □

Theorem 4.6. *Suppose that $\ell'(= \ell = \infty)$ is either type-1-natural or type-2-natural. Let $f \in L^1_+$, $x \in I$ and $u > 0$. Then, for any bounded stopping time T, it holds that*

$$M_T^{a,u} \underset{a\to\infty}{\longrightarrow} M_T^{s,f} \quad \text{in } L^1(\mathbb{P}_x). \tag{4.40}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$a\mathbb{P}_x\left[F_T f(L_{\eta_u^a})\right] \underset{a\to\infty}{\longrightarrow} \mathbb{P}_x\left[F_T M_T^{s,f}\right]. \tag{4.41}$$

Proof. Since $(M_t^{s,f})$ is a $\mathbb{P}_x$-martingale, we may apply the optional stopping theorem to see that

$$\mathbb{P}_x[A_T^{a,u}] = \mathbb{P}_x\left[X_{\eta_u^a} f(L_{\eta_u^a}); \eta_u^a \leq T\right] \tag{4.42}$$

$$\leq \mathbb{P}_x\left[M_{\eta_u^a}; \eta_u^a \leq T\right] \tag{4.43}$$

$$= \mathbb{P}_x[M_T; \eta_u^a \leq T] \underset{a\to\infty}{\longrightarrow} 0. \tag{4.44}$$

Since $A_T^{a,u} \to 0$, $\mathbb{P}_x$-a.s., we see that $A_T^{a,u} \to 0$ in $L^1(\mathbb{P}_x)$. Hence we obtain (4.40) and (4.41). □

4.2 Limit as u tends to infinity with a being fixed

Suppose $\ell'(= \ell = \infty)$ is either entrance, type-1-natural or type-2-natural. We thus have, for any $x, a \in I$,

$$\mathbb{P}_x(\eta_u^a < \infty) = 1 \quad \text{and} \quad \eta_u^a \underset{u\to\infty}{\longrightarrow} \infty \ \mathbb{P}_x\text{-a.s.} \tag{4.45}$$

In fact, η_u^a increases to a limit η_∞^a which must be infinite $\mathbb{P}_x$-a.s. by (4.3). For the clock $\tau = \eta_u^a$ in $u > 0$, we only consider the weights $f(L_{\eta_u^a})$ for $f(u) = \mathrm{e}^{-\beta u}$ and $f(u) = 1_{\{u=0\}}$.

Lemma 4.7. *Let $x, a \in I$, $\beta > 0$ and $t > 0$. For $u > 0$, set*

$$N_t^{u,\beta,a} = \mathrm{e}^{\frac{\beta u}{1+\beta a}} \mathbb{P}_x\left[\mathrm{e}^{-\beta L_{\eta_u^a}} 1_{\{t<\eta_u^a\}} \,\middle|\, \mathcal{F}_t\right], \quad M_t^{u,\beta,a} = \mathrm{e}^{\frac{\beta u}{1+\beta a}} \mathbb{P}_x\left[\mathrm{e}^{-\beta L_{\eta_u^a}} \,\middle|\, \mathcal{F}_t\right] \tag{4.46}$$

and

$$M_t^{\beta,a} = \frac{1+\beta(X_t \wedge a)}{1+\beta a} \exp\left(-\beta L_t + \frac{\beta}{1+\beta a} L_t^a\right). \tag{4.47}$$

Then it holds that $N_t^{u,\beta,a} \to M_t^{\beta,a}$ and $M_t^{u,\beta,a} \to M_t^{\beta,a}$, $\mathbb{P}_x$-a.s. as $u \to \infty$.

Proof. By the strong Markov property and by Lemmas 4.1 and 4.2, we have, for u large enough to have $\eta_u^a > t$,

$$\begin{aligned}
N_t^{u,\beta,a} =& \mathrm{e}^{\frac{\beta u}{1+\beta a}} \exp(-\beta L_t)\, \mathbb{P}_{X_t}\left[\exp\left(-\beta L_{\eta_{u-c}^a}\right)\right]\Big|_{c=L_t^a} && (4.48)\\
=& \left\{1_{\{a\le X_t\}} + \frac{1+\beta X_t}{1+\beta a} 1_{\{X_t<a\}}\right\} \exp\left(-\beta L_t + \frac{\beta}{1+\beta a} L_t^a\right) && (4.49)\\
=& M_t^{\beta,a}. && (4.50)
\end{aligned}$$

Thus we obtain $N_t^{u,\beta,a} \to M_t^{\beta,a}$, $\mathbb{P}_x$-a.s. as $u \to \infty$. Since

$$A_t^{u,\beta,a} := M_t^{u,\beta,a} - N_t^{u,\beta,a} = \mathrm{e}^{\frac{\beta u}{1+\beta a}} \mathrm{e}^{-\beta L_{\eta_u^a}} 1_{\{\eta_u^a \le t\}}, \tag{4.51}$$

we have $A_t^{u,\beta,a} \to 0$, $\mathbb{P}_x$-a.s., and thus we obtain $M_t^{u,\beta,a} \to M_t^{\beta,a}$, $\mathbb{P}_x$-a.s. □

Theorem 4.8. *Let $x, a \in I$ and $\beta > 0$. Then, for any $t > 0$, it holds that*

$$N_t^{u,\beta,a} \underset{u\to\infty}{\longrightarrow} M_t^{\beta,a} \quad \textit{and} \quad M_t^{u,\beta,a} \underset{u\to\infty}{\longrightarrow} M_t^{\beta,a} \quad \textit{in } L^1(\mathbb{P}_x). \tag{4.52}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$\lim_{u\to\infty} \mathrm{e}^{\frac{\beta u}{1+\beta a}} \mathbb{P}_x[F_t \mathrm{e}^{-\beta L_{\eta_u^a}}; t < \eta_u^a] = \lim_{u\to\infty} \mathrm{e}^{\frac{\beta u}{1+\beta a}} \mathbb{P}_x[F_t \mathrm{e}^{-\beta L_{\eta_u^a}}] = \mathbb{P}_x[F_t M_t^{\beta,a}]. \tag{4.53}$$

It also holds that $(M_t^{\beta,a})$ is a $\mathbb{P}_x$-martingale.

Proof. Let us first prove that $\mathbb{P}_x[\mathrm{e}^{cL_t^a}] < \infty$ for all $c > 0$ and $t > 0$. Following the same argument as in the proof of Lemma 2.1, we obtain

$$\mathbb{P}_a\left[\exp\left(cL_{\boldsymbol{e}_q}^a\right)\right] = \frac{1}{r_q(a,a)} \int_0^\infty \mathrm{e}^{cu} \mathrm{e}^{-u/r_q(a,a)} \mathrm{d}u. \tag{4.54}$$

Since $r_q(a,a) \to 0$ as $q \to \infty$, we may take $q > 0$ large enough so that $r_q(a,a) < 1/c$. This shows that $\mathbb{P}_a\left[\exp\left(cL_{\boldsymbol{e}_q}^a\right)\right] < \infty$. By the monotonicity, we see that $\mathbb{P}_x[\mathrm{e}^{cL_t^a}] < \infty$ for all $t > 0$.

The fact that L_t^a admits exponential moments implies that $M_t^{\beta,a} \in L^1(\mathbb{P}_x)$ for all $t > 0$. Thus, by the dominated convergence theorem, we see that $N_t^{u,\beta,a} \underset{u\to\infty}{\longrightarrow} M_t^{\beta,a}$ in $L^1(\mathbb{P}_x)$ for all $t > 0$.

We second note that, for $q > 0$,

$$\mathbb{P}_x(\eta_u^a \le t) \le \mathrm{e}^{qt}\mathbb{P}_x[\mathrm{e}^{-q\eta_u^a}] \le \mathrm{e}^{qt}\mathbb{P}_a[\mathrm{e}^{-q\eta_u^a}] = \mathrm{e}^{qt}\mathrm{e}^{-u/r_q(a,a)}. \tag{4.55}$$

We may take $q > 0$ large enough so that $r_q(a,a) < (1+\beta a)/\beta$. Then we obtain

$$\mathbb{P}_x[A_t^{u,\beta,a}] \le \mathrm{e}^{\frac{\beta u}{1+\beta a}}\mathbb{P}_x(\eta_u^a \le t) \le \mathrm{e}^{qt}\exp\left\{-\left(\frac{1}{r_q(a,a)} - \frac{\beta}{1+\beta a}\right)u\right\} \underset{u\to\infty}{\longrightarrow} 0. \tag{4.56}$$

Thus we obtain $A_t^{u,\beta,a} \underset{u\to\infty}{\longrightarrow} 0$ in $L^1(\mathbb{P}_x)$ for all $t > 0$, which implies $M_t^{u,\beta,a} \underset{u\to\infty}{\longrightarrow} M_t^{\beta,a}$ in $L^1(\mathbb{P}_x)$ for all $t > 0$. □

Theorem 4.9. *Let $x, a \in I$. For $u > 0$ and $t > 0$, set*

$$N_t^{u,\infty,a} = \mathrm{e}^{u/a}\mathbb{P}_x(t < \eta_u^a < T_0 \mid \mathcal{F}_t) \tag{4.57}$$
$$M_t^{u,\infty,a} = \mathrm{e}^{u/a}\mathbb{P}_x(\eta_u^a < T_0 \mid \mathcal{F}_t) \tag{4.58}$$
$$M_t^{\infty,a} = \frac{X_t \wedge a}{a}\mathrm{e}^{L_t^a/a}1_{\{t<T_0\}}. \tag{4.59}$$

Then it holds that

$$N_t^{u,\infty,a} \underset{u\to\infty}{\longrightarrow} M_t^{\infty,a} \quad \text{and} \quad M_t^{u,\infty,a} \underset{u\to\infty}{\longrightarrow} M_t^{\infty,a} \quad \mathbb{P}_x\text{-a.s. and in } L^1(\mathbb{P}_x). \tag{4.60}$$

Consequently, for any bounded adapted process (F_t), it holds that

$$\lim_{u\to\infty} \mathrm{e}^{u/a}\mathbb{P}_x[F_t; t < \eta_u^a < T_0] = \lim_{u\to\infty} \mathrm{e}^{u/a}\mathbb{P}_x[F_t; \eta_u^a < T_0] = \mathbb{P}_x[F_t M_t^{\infty,a}]. \tag{4.61}$$

It also holds that $(M_t^{\infty,a})$ is a $\mathbb{P}_x$-martingale.

Proof. Letting $\beta \to \infty$, we see, from Lemma 4.7 and Theorem 4.8, that

$$N_t^{u,\infty,a} = M_t^{\infty,a}1_{\{t<\eta_u^a\}} \tag{4.62}$$

and

$$A_t^{u,\infty,a} := M_t^{u,\infty,a} - N_t^{u,\infty,a} = \mathrm{e}^{u/a}1_{\{\eta_u^a<T_0\}}1_{\{\eta_u^a\le t\}}. \tag{4.63}$$

The remainder of the proof is the same as that of Theorem 4.8. □

5 Universal σ-finite measures

In this section we find universal σ-finite measures for the local time penalizations.

5.1 The transient case

Theorem 5.1. *Suppose $\ell < \infty$, i.e., 0 is transient. Let $f \in L^1_+$ and $x \in I$. Let t be a constant time and let F_t be a bounded $\mathcal{F}_t$-measurable functional. Then*

$$\lim_{q \downarrow 0} \mathbb{P}_x[F_t f(L_{\boldsymbol{e}_q}); t < \boldsymbol{e}_q] = \lim_{q \downarrow 0} \mathbb{P}_x[F_t f(L_{\boldsymbol{e}_q})] = \mathbb{P}_x[F_t f(L_\infty)]. \tag{5.1}$$

If, in particular, ℓ is type-3-natural, then

$$\lim_{a \uparrow \ell} \mathbb{P}_x[F_t f(L_{T_a}); t < T_a] = \lim_{a \uparrow \ell} \mathbb{P}_x[F_t f(L_{T_a})] = \mathbb{P}_x[F_t f(L_\infty)]. \tag{5.2}$$

Proof. By Theorems 2.4 and 2.5, we see that (5.1) is equivalent to

$$\mathbb{P}_x[F_t f(L_\infty)] = \mathbb{P}_x[F_t M_t], \tag{5.3}$$

where

$$M_t = \frac{1}{\ell}\left\{ X_t f(L_t) + \left(1 - \frac{X_t}{\ell}\right) \int_0^\infty \mathrm{e}^{-u/\ell} f(L_t + u)\mathrm{d}u \right\}. \tag{5.4}$$

On the other hand, we use (i) of Theorem 2.2 and obtain

$$\mathbb{P}_x[f(L_\infty) | \mathcal{F}_t] = \mathbb{P}_{X_t}[f(a + L_\infty)]|_{a = L_t} = M_t. \tag{5.5}$$

Thus we obtain (5.3).

Using Theorems 3.3 and 3.5 instead of Theorems 2.4 and 2.5, we can obtain (5.2) in the same way as above. □

5.2 The recurrent case

Let $\mathbb{P}^{(u)}_{x,y}$ denote the law of the bridge with duration u starting from x and ending at y. Following [3], this measure can be characterized by

$$\mathbb{P}^{(u)}_{x,y}(A) = \mathbb{P}_x\left[1_A \frac{p_{u-t}(X_t, y)}{p_u(x, y)}\right], \quad A \in \mathcal{F}_t,\ 0 < t < u, \tag{5.6}$$

where $p_u(x, y)$ denotes the transition density of the process X with respect to $m(\mathrm{d}y)$. We write symbolically

$$\mathbb{P}_x[\mathrm{d}L_u] = p_u(x, 0)\mathrm{d}u. \tag{5.7}$$

We have the conditioning formula:

$$\mathbb{P}_x\left[\int_0^\infty F_u \mathrm{d}L_u\right] = \int_0^\infty \mathbb{P}_x[\mathrm{d}L_u] \mathbb{P}^{(u)}_{x,0}[F_u] \tag{5.8}$$

for all non-negative predictable processes (F_u). We also have the last exit decomposition formula:

$$\mathbb{P}_x[F_t; T_0 \leq t] = \int_0^t \mathbb{P}_x[\mathrm{d}L_u]\left(\mathbb{P}_{x,0}^{(u)} \bullet \boldsymbol{n}^{[t-u]}\right)[F_t] \tag{5.9}$$

for all non-negative $\mathcal{F}_t$-measurable functionals (F_t), where we denote

$$\boldsymbol{n}^{[t]}(\cdot) = \boldsymbol{n}(\cdot \cap \{t < T_0\}). \tag{5.10}$$

For $h = h_0$ or $h = s$, let $\mathbb{P}_x^h$ denote the law of h-transform:

$$\mathbb{P}_x^h(A; t < \zeta) = \frac{1}{h(x)} \mathbb{P}_x^0[1_A h(X_t)] \quad (x > 0), \tag{5.11}$$

$$\mathbb{P}_0^h(A; t < \zeta) = \boldsymbol{n}[1_A h(X_t)] \tag{5.12}$$

for $A \in \mathcal{F}_t$. Note that, when $h = h_0$ or $h = s$, the coordinate process under $\mathbb{P}_x^h$ never hits zero; see [13, Theorems 7.6 and 7.3]. We now define

$$\mathcal{P}_x^h = \int_0^\infty \mathbb{P}_x[\mathrm{d}L_u]\left(\mathbb{P}_{x,0}^{(u)} \bullet \mathbb{P}_0^h\right) + h(x)\mathbb{P}_x^h. \tag{5.13}$$

Theorem 5.2. *Suppose 0 is recurrent. Let $f \in L_+^1$ and $x \in I$. Let t be a constant time and let F_t be a bounded $\mathcal{F}_t$-measurable functional. Then*

$$\lim_{q \downarrow 0} H(q)\mathbb{P}_x[F_t f(L_{\boldsymbol{e}_q}); t < \boldsymbol{e}_q] = \mathcal{P}_x^{h_0}[F_t f(L_\zeta); t < \zeta]. \tag{5.14}$$

Proof. By Theorem 3.3, it suffices to show

$$\mathcal{P}_x^{h_0}[F_t f(L_\zeta); t < \zeta] = \mathbb{P}_x[F_t N_t^{h_0,f}]. \tag{5.15}$$

Denote $g = \sup\{t < \zeta : X_t = 0\}$, where $\sup \emptyset = 0$. On the set $\{0 = g \leq t < \zeta\}$, we have

$$\mathcal{P}_x^{h_0}[F_t f(L_\zeta); 0 = g \leq t < \zeta] = h_0(x)\mathbb{P}_x^{h_0}[F_t f(L_t); t < \zeta] \tag{5.16}$$

$$= \mathbb{P}_x[F_t f(L_t) h_0(X_t); t < T_0]. \tag{5.17}$$

On the set $\{0 < g \leq t < \zeta\}$, we have

$$\mathcal{P}_x^{h_0}[F_t f(L_\zeta); 0 < g \leq t < \zeta] = \int_0^t \mathbb{P}_x[\mathrm{d}L_u]\left(\mathbb{P}_{x,0}^{(u)} \bullet \mathbb{P}_0^h\right)[F_t f(L_t); t < \zeta] \tag{5.18}$$

$$= \int_0^t \mathbb{P}_x[\mathrm{d}L_u]\left(\mathbb{P}_{x,0}^{(u)} \bullet \boldsymbol{n}\right)[F_t f(L_t) h_0(X_t)] \tag{5.19}$$

$$= \mathbb{P}_x[F_t f(L_t) h_0(X_t); T_0 \leq t]. \tag{5.20}$$

On the set $\{t < g < \zeta\}$, we have

$$\mathcal{P}_x^{h_0}[F_t f(L_\zeta); T_0 \le t < g] = \int_t^\infty \mathbb{P}_x[\mathrm{d}L_u]\mathbb{P}_{x,0}^{(u)}[F_t f(L_u)] \tag{5.21}$$

$$= \mathbb{P}_x\left[F_t \int_t^\infty f(L_u)\mathrm{d}L_u\right] \tag{5.22}$$

$$= \mathbb{P}_x\left[F_t \int_{L_t}^\infty f(u)\mathrm{d}u\right]. \tag{5.23}$$

Therefore we obtain (5.15). $\square$

Theorem 5.3. *Suppose ℓ' is either entrance, type-1-natural or type-2-natural. Let $f \in L^1_+$ and $x \in I$. Let t be a constant time and let F_t be a bounded $\mathcal{F}_t$-measurable functional. Then*

$$\lim_{a \uparrow \ell} a\mathbb{P}_x[F_t f(L_{T_a}); t < T_a] = \mathcal{P}_x^s[F_t f(L_\zeta); t < \zeta]. \tag{5.24}$$

The proof is parallel to that of Theorem 5.2, where we use Theorem 3.5 instead of Theorem 3.3. So we omit it.

6 Exponential weights

Let us investigate the example where we take

$$f(x) = \mathrm{e}^{-cx}, \quad c > 0. \tag{6.1}$$

For $h = h_0$ or s, the supermartingale $N_t = N_t^{h,f}$ is given as

$$N_t = h^c(X_t)\mathrm{e}^{-cL_t} \tag{6.2}$$

where

$$h^c(x) = h(x) + \frac{1 - \frac{x}{\ell}}{c + \frac{1}{\ell}}. \tag{6.3}$$

Since (N_t) is a supermartingale, we may define the subprobability measure $\mathbb{Q}_x^{h,c}$ by

$$\mathbb{Q}_x^{h,c}(A; t < \zeta) = \mathbb{P}_x\left[\frac{h^c(X_t)}{h^c(x)}\mathrm{e}^{-cL_t}; A\right] \quad \text{for } A \in \mathcal{F}_t \text{ and } t \ge 0. \tag{6.4}$$

Then the process $\{X, (\mathbb{Q}_x^{h,c})_{x \in I}\}$ is a diffusion on I. The corresponding speed measure and the scale function are given as

$$m^{h,c}(x) = \int_{(0,x]} h^c(y)^2 \mathrm{d}m(y), \quad s^{h,c}(x) = \int_0^x \frac{\mathrm{d}y}{h^c(y)^2}. \tag{6.5}$$

Denote

$$\phi_q^{h,c} = h^c(0) \cdot \frac{\phi_q + c\psi_q}{h^c}, \quad \rho_q^{h,c} = h^c(0) \cdot \frac{\rho_q}{h^c}. \tag{6.6}$$

Then we see that $\varphi = \phi_q^{h,c}$ (resp. $\rho_q^{h,c}$) is a positive increasing (resp. decreasing) solution to the differential equation

$$\left(D_{m^{h,c}} D_{s^{h,c}} - \frac{\pi_0}{h^c} \right) \varphi = q\varphi \tag{6.7}$$

which satisfies the boundary condition

$$f(0) = 1 \quad \text{and} \quad D_{s^{h,c}} \phi_q^{h,c}(0) = 0, \tag{6.8}$$

where we have used

$$h^c(0) = \frac{1}{c + \frac{1}{\ell}}, \quad (h^c)'(0) = \frac{c}{c + \frac{1}{\ell}}. \tag{6.9}$$

Theorem 6.1. *The resolvent operator for the diffusion $\{X, (\mathbb{Q}_x^{h,c})_{x \in I}\}$ is given as*

$$\mathbb{Q}_x^{h,c} \left[\int_0^\infty \mathrm{e}^{-qt} f(X_t) \mathrm{d}t \right] = \int_I r_q^{h,c}(x,y) f(y) \mathrm{d}m^{h,c}(y), \quad q > 0, \tag{6.10}$$

where

$$r_q^{h,c}(x,y) = r_q^{h,c}(y,x) = \frac{H(q)}{h^c(0)^2 (cH(q) + 1)} \phi_q^{h,c}(x) \rho_q^{h,c}(y), \quad x, y \in I, \ x \le y. \tag{6.11}$$

Consequently, 0 for $\{X, (\mathbb{Q}_x^{h,c})_{x \in I}\}$ is regular-reflecting.

Proof. Let $\varphi^c(x) = \varphi(x) h^c(x)$. Then we have

$$\mathbb{P}_x \left[\int_0^\infty \mathrm{e}^{-qt} \varphi^c(X_t) \mathrm{e}^{-cL_t} \mathrm{d}t \right] \tag{6.12}$$

$$= \mathbb{P}_x \left[\int_0^{T_0} \mathrm{e}^{-qt} \varphi^c(X_t) \mathrm{d}t \right] + \mathbb{P}_x[\mathrm{e}^{-qT_0}] \mathbb{P}_0 \left[\int_0^\infty \mathrm{e}^{-qt} \varphi^c(X_t) \mathrm{e}^{-cL_t} \mathrm{d}t \right] \tag{6.13}$$

$$= R_q^0 \varphi^c(x) + \mathbb{P}_x[\mathrm{e}^{-qT_0}] \mathbb{P}_0 \left[\sum_u \mathrm{e}^{-cu - q\eta(u-)} \int_0^{T_0(p(u))} \mathrm{e}^{-qt} \varphi^c(p(u)_t) \mathrm{d}t \right] \tag{6.14}$$

$$= R_q^0 \varphi^c(x) + \mathbb{P}_x[\mathrm{e}^{-qT_0}] \mathbb{P}_0 \left[\int_0^\infty \mathrm{e}^{-cu - q\eta(u)} \mathrm{d}u \right] \boldsymbol{n} \left[\int_0^{T_0} \mathrm{e}^{-qt} \varphi^c(X_t) \mathrm{d}t \right] \tag{6.15}$$

$$= R_q^0 \varphi^c(x) + \mathbb{P}_x[\mathrm{e}^{-qT_0}] \cdot \frac{1}{c + \frac{1}{H(q)}} \cdot \frac{R_q \varphi^c(0)}{H(q)}. \tag{6.16}$$

Since $\mathbb{P}_x[\mathrm{e}^{-qT_0}] R_q \varphi^c(0) = R_q \varphi^c(x) - R_q^0 \varphi^c(x)$, we obtain

$$\mathbb{Q}_x^{h,c} \left[\int_0^\infty \mathrm{e}^{-qt} \varphi(X_t) \mathrm{d}t \right] = \frac{1}{h^c(x)} \left\{ \frac{1}{cH(q) + 1} R_q \varphi^c(x) + \frac{cH(q)}{cH(q) + 1} R_q^0 \varphi^c(x) \right\}. \tag{6.17}$$

From this we obtain (6.11). □

Remark 6.2. The boundary classification at ℓ' is the same as that for the h-transform of the stopped process; see Theorems 7.3 and 7.6 of [13].

7 Appendix: the boundary classification

The following tables explain the boundary classification which we take from [13] and the recurrence property of the corresponding diffusion to each class:

$x = \ell'$		I'	I	$x = 0$
regular-reflecting	$\ell' < \ell = \infty$	$[0, \ell']$	$= I'$	positive recurrent
regular-elastic	$\ell' < \ell < \infty$	$[0, \ell']$	$[0, \ell'] \cup \{\ell\}$	transient
regular-absorbing	$\ell' = \ell < \infty$	$[0, \ell)$	$[0, \ell]$	transient
exit	$\ell' = \ell < \infty$	$[0, \ell)$	$[0, \ell]$	transient
entrance	$\ell' = \ell = \infty$	$[0, \infty)$	$= I'$	positive recurrent
type-1-natural	$\ell' = \ell = \infty$	$[0, \infty)$	$= I'$	null recurrent
type-2-natural	$\ell' = \ell = \infty$	$[0, \infty)$	$= I'$	positive recurrent
type-3-natural	$\ell' = \ell < \infty$	$[0, \ell)$	$= I'$	transient

	$l = \infty$	$l < \infty$
$m(\infty) = \infty$ $\pi_0 = 0$	**(1)** 0 is null-recurrent $[\ell' = \ell = \infty]$ ℓ' is type-1-natural	**(3)** 0 is transient $[\ell' < \ell < \infty]$ ℓ' is regular-elastic $[\ell' = \ell < \infty]$ ℓ' is regular-absorbing exit type-3-natural
$m(\infty) < \infty$ $\pi_0 > 0$	**(2)** 0 is positive recurrent $[\ell' < \ell = \infty]$ ℓ' is regular-reflecting $[\ell' = \ell = \infty]$ ℓ' is entrance type-2-natural	[impossible]

Let us give some examples. Let $\widetilde{X}$ be a diffusion on $[0, \infty)$ where 0 is the reflecting boundary and whose local generator on $(0, \infty)$ is given by

$$\widetilde{L}f = \frac{1}{2}(f'' - bf') = \frac{\mathrm{d}}{\mathrm{d}\widetilde{m}}\frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}f \quad \text{on } C_c((0, \infty)) \tag{7.1}$$

for some function b. Then its scale change $X = \widetilde{s}(\widetilde{X})$ is a diffusion with natural scale $s(x) = x$ and the speed measure $m(\mathrm{d}x)$ defined by $m = \widetilde{m} \circ \widetilde{s}^{-1}$.

(i) Let $0 < \alpha < 1$ and let $\widetilde{L}f = \frac{1}{2}f'' - \frac{2\alpha-1}{2x}f' = \frac{\mathrm{d}}{\mathrm{d}\widetilde{m}}\frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}f$ for $f \in C_c((0, \infty))$, where $\widetilde{m}(x) = \frac{2}{2-2\alpha}x^{2-2\alpha}$ and $\widetilde{s}(x) = \frac{1}{2\alpha}x^{2\alpha}$. The corresponding diffusion is called the *reflecting Bessel process of index α*. If we take $m = \widetilde{m} \circ \widetilde{s}^{-1}$, then it falls into the case **(1)** above.

(ii) Let us study the case $b(x) = c\nu x^{\nu-1}$ with $c > 0$ and $\nu > 0$, i.e.,

$$Lf = \frac{1}{2}\big(f'' - c\nu x^{\nu-1} f'\big) \quad \text{on } C_c((0,\infty)), \tag{7.2}$$

which we may call the *power drift*. If $\nu = 1$, then it is a Brownian motion with constant negative drift. If $\nu = 2$, then it is an Ornstein–Uhlenbeck process. In this case

$$s' = \mathrm{e}^{cx^\nu}, \quad s = \int_0^x \mathrm{e}^{cy^\nu}\mathrm{d}y \tag{7.3}$$

and

$$m' = 2\mathrm{e}^{-cx^\nu}, \quad m = 2\int_0^x \mathrm{e}^{-cy^\nu}\mathrm{d}y. \tag{7.4}$$

In particular, we have $m(\infty) < \infty$. Note that

$$J := \frac{1}{2}\int_1^\infty \{s(x) - s(1)\}\mathrm{d}m(x) = \int_1^\infty \left(\int_1^x \mathrm{e}^{cy^\nu}\mathrm{d}y\right)\mathrm{e}^{-cx^\nu}\mathrm{d}x \tag{7.5}$$

$$= \int_1^\infty \left(\int_y^\infty \mathrm{e}^{-cx^\nu}\mathrm{d}x\right)\mathrm{e}^{cy^\nu}\mathrm{d}y. \tag{7.6}$$

We shall prove that

$$\infty \text{ is } \begin{cases} \text{type-2-natural if } 0 < \nu \le 2, \\ \text{entrance if } 2 < \nu < \infty. \end{cases} \tag{7.7}$$

If $1 \le \nu \le 2$, then

$$\int_1^x \mathrm{e}^{cy^\nu}\mathrm{d}y = \int_1^x (\mathrm{e}^{cy^\nu})' \frac{y^{1-\nu}}{c\nu}\mathrm{d}y \tag{7.8}$$

$$= \left[\mathrm{e}^{cy^\nu}\frac{y^{1-\nu}}{c\nu}\right]_1^x + \frac{\nu-1}{c\nu}\int_1^x \mathrm{e}^{cy^\nu}y^{-\nu}\mathrm{d}y \tag{7.9}$$

$$\ge \mathrm{e}^{cx^\nu}\frac{x^{1-\nu}}{c\nu} - c' \tag{7.10}$$

for some constant $c' > 0$. Hence we have

$$J \ge \frac{1}{c\nu}\int_1^\infty x^{1-\nu}\mathrm{d}x - c'\int_1^\infty \mathrm{e}^{-cx^\nu}\mathrm{d}x = \infty. \tag{7.11}$$

For $\nu > 0$, we have

$$\int_y^\infty \mathrm{e}^{-cx^\nu}\mathrm{d}x = -\int_y^\infty (\mathrm{e}^{-cx^\nu})' \frac{x^{1-\nu}}{c\nu}\mathrm{d}y \tag{7.12}$$

$$= -\left[\mathrm{e}^{-cx^\nu}\frac{x^{1-\nu}}{c\nu}\right]_y^\infty + \frac{1-\nu}{c\nu}\int_y^\infty \mathrm{e}^{-cx^\nu}x^{-\nu}\mathrm{d}x. \tag{7.13}$$

If $0 < \nu < 1$, then

$$\int_y^\infty \mathrm{e}^{-cx^\nu}\mathrm{d}x \geq \mathrm{e}^{-cy^\nu}\frac{y^{1-\nu}}{c\nu} \quad \text{and} \quad J \geq \frac{1}{c\nu}\int_1^\infty y^{1-\nu}\mathrm{d}y = \infty. \tag{7.14}$$

If $\nu > 2$, then

$$\int_y^\infty \mathrm{e}^{-cx^\nu}\mathrm{d}x \leq \mathrm{e}^{-cy^\nu}\frac{y^{1-\nu}}{c\nu} \quad \text{and} \quad J \leq \frac{1}{c\nu}\int_1^\infty y^{1-\nu}\mathrm{d}y < \infty. \tag{7.15}$$

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