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► **To cite this version:**

Tarik Chakkour. Inverse problem stability of a continuous-in-time financial model. 2016. hal-01358568v2

**HAL Id: hal-01358568**

**<https://hal.science/hal-01358568v2>**

Preprint submitted on 5 Sep 2016 (v2), last revised 21 Jul 2019 (v3)

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# Inverse problem stability of a continuous-in-time financial model \*

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September 5, 2016

**Abstract.** In this work, we study the inverse problem stability of the continuous-in-time model which is designed to be used for the finances of public institutions. We discuss this study with determining the Loan Measure from Algebraic Spending Measure in Radon measure space  $\mathcal{M}([t_I, \Theta_{\max}])$ , and in Hilbert space  $\mathbb{L}^2([t_I, \Theta_{\max}])$  when they are density measures. For this inverse problem we prove the uniqueness theorem, obtain a procedure for constructing the solution and provide necessary and sufficient conditions for the solvability of the inverse problem in  $\mathbb{L}^2([t_I, \Theta_{\max}])$ .

**Keywords:** Inverse problem; stability; mathematical model; Fredholm operator.

## 1 Introduction

In the last two decades, the theory and practice of inverse problems have been developed in many scientific domains. Consequently, it is rapidly growing, if not exploding. Moreover, document [3] shows how much researchers contribute to this field. Many inverse problems arising in scientific domains present numerical instability: the noise affecting the data may produce arbitrarily large errors in the solutions. In other words, these problems are ill-posed in the sense of Hadamard. The concept of ill-posedness was introduced by HADAMARD [7] in the field of partial differential equations. We mention the book on the mathematics of ill-posed problems by TIKHONOV and ARSENIN [9].

We constructed in previous work [6] the continuous-in-time model which is based on using the mathematical tools such convolution and integration. Indeed, this model uses measures over time interval to describe loan scheme, reimbursement scheme and interest payment scheme. The model contains some financial quantities. For instance, the Repayment Pattern Measure  $\tilde{\gamma}$  is a non-negative measure with total mass which equals 1, the Algebraic Spending Measure  $\tilde{\sigma}$  is defined such that the difference between spendings and incomes required to satisfy the current needs between times  $t_1$  and  $t_2$  is:

$$\int_{t_1}^{t_2} \tilde{\sigma}, \quad (1)$$

and the Loan Measure  $\tilde{\kappa}_E$  is defined such that the amount borrowed between times  $t_1$  and  $t_2$  is:

$$\int_{t_1}^{t_2} \tilde{\kappa}_E. \quad (2)$$

When measure  $\tilde{\gamma}$  is absolutely continuous with respect to the Lebesgue measure  $dt$ . This means that it read  $\gamma(t)dt$ , where  $t$  is the variable in  $\mathbb{R}$ . The work [6] proposes the resolution of the inverse problem over the space of square-integrable functions when density  $\gamma$  is equal to  $\frac{1}{\Theta_\gamma}$  over time period  $[0, \Theta_\gamma]$  and to 0 elsewhere. In paper [4], we use a mathematical framework to discuss an inverse problem of determining the Loan Measure  $\tilde{\kappa}_E$  from Algebraic Spending Measure  $\tilde{\sigma}$ . This inverse problem is

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\*This work is jointly funded by MGDIS company (<http://www.mgdis.fr/>) and the LMBA (<http://www.lmba-math.fr/>).

used in [6] on simplified examples in order to show its capability to be used to forecast a financial strategy.

We show in this paper the general results of this inverse problem for any density  $\gamma$ . We show the stability of the inverse problem. We continue to extend some results of this inverse problem in measure space with proving its stability. In other words, we describe a more complete numerical study for the inverse problem. The main result of this paper is the existence and uniqueness of solutions for the system modeling the financial multiyear planning. This result ensures the mathematical well-posedness under the balanced equation assumption.

The organization of the rest of this paper is as follows. Section 2 describes the inverse problem of the model in  $\mathbb{L}^2([t_I, \Theta_{\max}])$ . Section 3 shows the inverse problem of the model in  $\mathcal{M}([t_I, \Theta_{\max}])$ .

## 2 Inverse problem of the model in $\mathbb{L}^2([t_I, \Theta_{\max}])$

Denoting  $\mathbb{L}^2([t_I, \Theta_{\max}])$  the space of square-integrable functions over  $\mathbb{R}$  having their support in  $[t_I, \Theta_{\max}]$  and denoting  $\mathbb{L}^2([0, \Theta_\gamma])$  the space of square-integrable functions over  $\mathbb{R}$  having their support in  $[0, \Theta_\gamma]$ . We state the Repayment Pattern Density  $\gamma$  as follows:

$$\gamma \in \mathbb{L}^2([0, \Theta_\gamma]), \quad (3)$$

where  $\Theta_\gamma$  is a positive number such that:

$$\Theta_\gamma < \Theta_{\max} - t_I. \quad (4)$$

We justify relation (3) because the support of convolution of two compactly supported densities  $\kappa_E$  in  $[t_I, \Theta_{\max} - \Theta_\gamma]$  and  $\gamma$  in  $[0, \Theta_\gamma]$  is included in  $[t_I, \Theta_{\max}]$ .

**Lemma 2.1.** *Linear operator  $\mathcal{L}$  acting on Loan Density  $\kappa_E \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$  defined as:*

$$\mathcal{L}[\kappa_E](t) = \kappa_E(t) - (\kappa_E \star \gamma)(t) - \alpha \int_{t_I}^t (\kappa_E - \kappa_E \star \gamma)(s) ds, \quad (5)$$

is compact operator from  $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$  to  $\mathbb{L}^2([t_I, \Theta_{\max}])$ .

*Proof.* Definition (5) of operator  $\mathcal{L}$  gives that for any two Loan Densities  $\kappa_{E_1}$  and  $\kappa_{E_2}$  the following equality:

$$\begin{aligned} \mathcal{L}[\kappa_{E_2}](t) - \mathcal{L}[\kappa_{E_1}](t) &= \kappa_{E_2}(t) - \kappa_{E_1}(t) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(t) \\ &\quad - \alpha \int_{t_I}^t \kappa_{E_2}(s) - \kappa_{E_1}(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds. \end{aligned} \quad (6)$$

Taking norm  $\mathbb{L}^2([t_I, \Theta_{\max}])$  and applying triangle inequality to relation (6), we obtain the following inequality:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + |\alpha| \times C_{E_1}^2, \quad (7)$$

where  $C_{E_1}^2$  is defined and is increased as follows:

$$\begin{aligned} C_{E_1}^2 &= \left\| \int_{t_I}^t (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \\ &= \sqrt{\int_{t_I}^{\Theta_{\max}} \left( \int_{t_I}^t (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds \right)^2 dt}, \\ &= \sqrt{\Theta_{\max} - t_I} \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}. \end{aligned} \quad (8)$$

We use Cauchy-Schwarz inequality to obtain:

$$\begin{aligned}
\|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} &= \int_{t_I}^{\Theta_{\max}} 1 \times |(\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s)| \, ds, \\
&\leq \sqrt{\int_{t_I}^{\Theta_{\max}} 1^2 \, ds} \\
&\times \sqrt{\int_{t_I}^{\Theta_{\max}} \left( (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) \right)^2 \, ds}, \\
&\leq \sqrt{\Theta_{\max} - t_I} \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{9}
\end{aligned}$$

Thanks to properties (8) and (9), we get:

$$C_{E_1^2} \leq (\Theta_{\max} - t_I) \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{10}$$

From this and according to relation (7), we get:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (1 + |\alpha| \times (\Theta_{\max} - t_I)) \times \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{11}$$

The triangle and the Young's Inequalities implice that:

$$\begin{aligned}
\|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} + \|((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \\
&\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} + \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \\
&\times \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}, \\
&\leq (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}) \times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])}. \tag{12}
\end{aligned}$$

From this and according to relation (11), we get:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (1 + |\alpha| \times (\Theta_{\max} - t_I)) \times (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}) \times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])}. \tag{13}$$

Consequently, linear operator  $\mathcal{L}$  is uniformly bounded and is a Hilbert-Schmidt operator in  $\mathbb{L}^2([t_I, \Theta_{\max}])$  of constant  $(1 + |\alpha| \times (\Theta_{\max} - t_I)) \times (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma] )})$ , achieving the proof of the lemma.  $\square$

**Lemma 2.2.** *Linear operator  $\mathcal{L}$  given by relation (5) is Fredholm operator such that:*

$$\text{codim } \text{Im}(\mathcal{L}) = \text{dim } \text{Ker}(\mathcal{L}) < \infty. \tag{14}$$

*Proof.* We consider  $\mathbf{L}$  the space of continuous linear applications of  $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$  in  $\mathbb{L}^2([t_I, \Theta_{\max}])$ . We define operator  $K$  in  $\mathbf{L}$  as an integral operator:

$$\forall \kappa_E \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma]), K[\kappa_E](x) = \int_{t_I}^{\Theta_{\max}} F(x, y) \kappa_E(y) \, dy. \tag{15}$$

We want to show that linear operator  $\mathcal{L}$  is a difference between the identity application from space  $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$  to  $\mathbb{L}^2([t_I, \Theta_{\max}])$  and the compact operator  $K$  given by (15) as the following form:

$$\mathcal{L}[\kappa_E](x) = Id_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma]) \rightarrow \mathbb{L}^2([t_I, \Theta_{\max}])}[\kappa_E](x) - K[\kappa_E](x), \quad (16)$$

where,  $F$  which defines operator  $K$ , is a function in  $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])$  to be determined. In what follows, the definition (5) of linear operator  $\mathcal{L}$  is used in order to introduce the indicator function  $\mathbb{1}_{\{y \leq x\}}$ . It allows to simplify integrals because of using Fubini-Tonelli Theorem to permut integrals. These simplifications can be detailed as follows:

$$\begin{aligned} \mathcal{L}[\kappa_E](x) &= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{y \leq x\}} (\kappa_E - \kappa_E \star \gamma)(y) dy, \\ &= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y \leq x\}} dy + \int_{t_I}^{\Theta_{\max}} \alpha \mathbb{1}_{\{y \leq x\}} (\kappa_E \star \gamma)(y) dy, \\ &= \kappa_E(x) - \int_{t_I}^{\Theta_{\max}} \kappa_E(y) \gamma(x - y) dy - \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y \leq x\}} dy + \\ &\quad \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \left( \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right) dy, \\ &= \kappa_E(x) - \int_{t_I}^{\Theta_{\max}} \kappa_E(y) \left( \gamma(x - y) + \alpha \mathbb{1}_{\{y \leq x\}} - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right) dy. \end{aligned} \quad (17)$$

From this, we get the expression of function  $F$ :

$$F(x, y) = \gamma(x - y) + \alpha \mathbb{1}_{\{y \leq x\}} - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt. \quad (18)$$

In order to show that function  $F$  is square-integrable over  $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$ , we will show that three functions they are. Indeed, these functions are  $(x, y) \rightarrow \gamma(x - y)$ ,  $(x, y) \rightarrow \mathbb{1}_{\{y \leq x\}}$  and  $(x, y) \rightarrow \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt$ . Otherwise, since we have:

$$\begin{aligned} \|\mathbb{1}_{\{y \leq x\}}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \left( \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{y \leq x\}}^2 dy \right) dx, \\ &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} (x - t_I) dx, \\ &= \frac{(\Theta_{\max} - \Theta_\gamma - t_I)^2}{2}, \end{aligned} \quad (19)$$

we have  $(x, y) \rightarrow \mathbb{1}_{\{y \leq x\}} \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])$ . In what to follows, we will show that function  $\int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt$  is square-integrable over  $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$ . For that, we set the following inequality:

$$\begin{aligned} \left\| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \int_{t_I}^{\Theta_{\max}} \left| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right|^2 dx dy, \\ &\leq \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \int_{t_I}^{\Theta_{\max}} \left( \int_{t_I}^x |\gamma(t - y)| dt \right)^2 dx dy. \end{aligned} \quad (20)$$

We use Cauchy-Schwarz inequality to obtain:

$$\forall x \in [t_I, \Theta_{\max} - \Theta_\gamma], \left( \int_{t_I}^x |\gamma(t-y)| dt \right)^2 \leq (x - t_I) \times \int_{t_I}^x |\gamma(t-y)|^2 dt. \quad (21)$$

Since density  $\gamma$  satisfies relation (3), there exists a constant  $C_\gamma$  such that relation (21) gives:

$$\left( \int_{t_I}^x |\gamma(t-y)| dt \right)^2 \leq (x - t_I) \times C_\gamma. \quad (22)$$

According to relations (20) and (22), we get the following inequality:

$$\left\| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t-y) dt \right\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 \leq C_\gamma \times (\Theta_{\max} - t_I) \times \frac{(\Theta_{\max} - \Theta_\gamma - t_I)^2}{2}, \quad (23)$$

which proves that function  $(x, y) \rightarrow \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t-y) dt$  is square-integrable over the interval  $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$ . Consequently, we get:

$$F \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]). \quad (24)$$

It is concluded that linear operator  $\mathcal{L}$  can be written in the form which is shown in relation (16). We showed that operator  $\mathcal{L}$  is Fredholm operator (see Theorem 3.45 page 206 in [8] or Fredholm Alternative Theorem 1.3.1 in [2] page 13). In addition, the Fredholm Alternative Theorem (Lemma 4.45 page 160 in [1]) is used to get that linear operator  $\mathcal{L}$  has a finite codimension, a closed image and a finite dimension of its kernel. The proof of lemma is achieved.  $\square$

**Lemma 2.3.** *Linear operator  $\mathcal{D}$  acting on Initial Debt Repayment Density  $\rho_K^I \in \mathbb{L}^2([t_I, \Theta_{\max}])$  defined as:*

$$\mathcal{D}[\rho_K^I](t) = -\alpha \int_t^{\Theta_{\max}} \rho_K^I(s) ds - \rho_K^I(t), \quad (25)$$

*is compact operator from  $\mathbb{L}^2([t_I, \Theta_{\max}])$  to  $\mathbb{L}^2([t_I, \Theta_{\max}])$ .*

*Proof.* The triangle inequality is applied to definition (25) of operator  $\mathcal{D}$  to get:

$$\|\mathcal{D}[\rho_K^I]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq |\alpha| \times \left\| \int_t^{\Theta_{\max}} \rho_K^I(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (26)$$

Since we have  $\left\| \int_t^{\Theta_{\max}} \rho_K^I(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (\Theta_{\max} - t_I) \times \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$  and according to relation (26), we have:

$$\|\mathcal{D}[\rho_K^I]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (|\alpha| \times (\Theta_{\max} - t_I) + 1) \times \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (27)$$

$\square$

We can decompose the Algebraic Spending Density  $\sigma$  defined in relation (1) as a sum of operators  $\mathcal{L}$  and  $\mathcal{D}$ :

$$\sigma(t) = \mathcal{L}[\kappa_E](t) + \mathcal{D}[\rho_K^I](t). \quad (28)$$

**Lemma 2.4.** *The singular point of function  $1 - \mathcal{F}(\gamma)$  is zero for any constant and affine density  $\gamma$ .*

*Proof.* If Repayment Pattern Density  $\gamma$  is a constant function given by:

$$\gamma(t) = \frac{1}{\Theta_\gamma} \mathbb{1}_{[0, \Theta_\gamma]}(t), \quad (29)$$

then we have:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - \frac{i}{\xi \Theta_\gamma} (e^{-i\xi \Theta_\gamma} - 1). \quad (30)$$

From this, we get:

$$\begin{aligned} \forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 0 &\Rightarrow \xi \Theta_\gamma = \sin(\xi \Theta_\gamma) \text{ et } \cos(\xi \Theta_\gamma) = 1, \\ &\Rightarrow (\xi \Theta_\gamma)^2 + 1 = 1, \\ &\Rightarrow \xi \Theta_\gamma = 0. \end{aligned} \quad (31)$$

Since real  $\Theta_\gamma$  is positive, function  $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$  is not zero function over  $\mathbb{R}^*$ . Inversely, if a real  $\xi$  is zero, function  $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$  is also a zero function. Indeed, the Fourier Transform of any density  $\gamma$  at the origine is defined as:

$$\mathcal{F}(\gamma)(0) = \int_{-\infty}^{+\infty} \gamma(t) dt. \quad (32)$$

Furthermore, since density  $\gamma$  is with total mass which equals 1, relation (32) implies that:

$$\mathcal{F}(\gamma)(0) = 1. \quad (33)$$

Now we will show that the singular point of function  $1 - \mathcal{F}(\gamma)$  is zero for an affine density  $\gamma$  given by:

$$\gamma(t) = \left( \frac{t}{\Theta_\gamma^2} + \frac{1}{2\Theta_\gamma} \right) \mathbb{1}_{[0, \Theta_\gamma]}(t). \quad (34)$$

The integration by parts states that:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - \left( \frac{1}{(\xi \Theta_\gamma)^2} + \frac{i}{2\xi \Theta_\gamma} \right) (e^{-i\xi \Theta_\gamma} - 1) - \frac{i}{\xi \Theta_\gamma} e^{-i\xi \Theta_\gamma}. \quad (35)$$

From this, we get the following system of equations:

$$\begin{cases} (\cos(\xi \Theta_\gamma) - 1) + \frac{3\xi \Theta_\gamma}{2} \sin(\xi \Theta_\gamma) = (\xi \Theta_\gamma)^2, \\ -\sin(\xi \Theta_\gamma) + \xi \Theta_\gamma \cos(\xi \Theta_\gamma) + \frac{\xi \Theta_\gamma}{2} (\cos(\xi \Theta_\gamma) - 1) = 0. \end{cases} \quad (36)$$

From this, we get the following equality:

$$\xi \Theta_\gamma \left( 1 + \frac{3(\xi \Theta_\gamma)^2}{2} \right) - \left( 1 + \frac{9(\xi \Theta_\gamma)^2}{4} \right) \sin(\xi \Theta_\gamma) = 0. \quad (37)$$

According to Figure 1, we state that equation (37) doesn't have no solution on  $\mathbb{R}^*$ . Consequently, function  $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$  is not a zero function over  $\mathbb{R}^*$ . Inversly, assuming that real  $\xi$  is zero, we get:

$$\begin{aligned}
1 - \mathcal{F}(\gamma)(0) &= 1 - \int_{-\infty}^{+\infty} \gamma, \\
&= 1 - \frac{1}{\Theta_\gamma^2} \int_0^{\Theta_\gamma} t \, dt - \frac{1}{2\Theta_\gamma} \int_0^{\Theta_\gamma} dt, \\
&= 0.
\end{aligned} \tag{38}$$

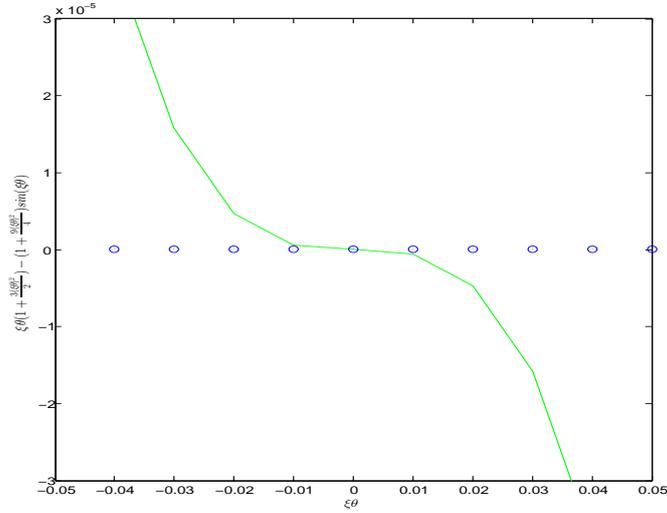


Figure 1: Graph of the function  $\xi\Theta_\gamma \rightarrow \xi\Theta_\gamma \left(1 + \frac{3(\xi\Theta_\gamma)^2}{2}\right) - \left(1 + \frac{9(\xi\Theta_\gamma)^2}{4}\right) \sin(\xi\Theta_\gamma)$  over interval  $[-0.05, 0.05]$ . Also shown that zero is its singular point.

We conclude that function  $1 - \mathcal{F}(\gamma)$  is zero at the origine for density  $\gamma$  given by (29) (or by (34)). In what to follows, we want to extend this conclusion for any affine density  $\gamma$  given by:

$$\gamma(t) = (c_2 t + c_1) \mathbb{1}_{[0, \Theta_\gamma]}(t), \tag{39}$$

where coefficients  $c_1$  and  $c_2$  satisfy:

$$c_1 + \frac{c_2 \Theta_\gamma}{2} = \frac{1}{\Theta_\gamma}. \tag{40}$$

We obtain using the integration by parts:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - c_2 \left( \frac{e^{-i\xi\Theta_\gamma} - 1}{\xi^2} + \frac{i\Theta_\gamma e^{-i\xi\Theta_\gamma}}{\xi} \right) - \frac{ic_1(e^{-i\xi\Theta_\gamma} - 1)}{\xi} = 0. \tag{41}$$

Next, we obtain by separating the real and the imaginary parts of function  $1 - \mathcal{F}(\gamma)$  the following system of equations:

$$\begin{cases} 1 - c_2 \left( \frac{\cos(\xi\Theta_\gamma) - 1}{\xi^2} + \frac{\Theta_\gamma \sin(\xi\Theta_\gamma)}{\xi} \right) - \frac{c_1 \sin(\xi\Theta_\gamma)}{\xi} = 0, \\ c_2 \left( -\frac{\sin(\xi\Theta_\gamma)}{\xi^2} + \frac{\Theta_\gamma \cos(\xi\Theta_\gamma)}{\xi} \right) + \frac{c_1 (\cos(\xi\Theta_\gamma) - 1)}{\xi} = 0. \end{cases} \quad (42)$$

Let us check the consistency of the system of equations given by (42). Indeed, if coefficients  $c_1$  and  $c_2$  are respectively equal to  $\frac{1}{2\Theta_\gamma}$  and  $\frac{1}{\Theta_\gamma^2}$ , then we get relation (36). Thanks to (42), we get the following equality:

$$\frac{c_2 \Theta_\gamma}{\xi} + \xi \Theta_\gamma + \frac{c_1 \xi}{c_2} = \left( \frac{c_2}{\xi^2} + \Theta_\gamma (c_2 \Theta_\gamma + c_1) + \frac{c_1 (c_2 \Theta_\gamma + c_1)}{c_2} \right) \sin(\xi \Theta_\gamma). \quad (43)$$

According to relations (42) and (43), we get:

$$\frac{c_2 \Theta_\gamma}{\xi} + \xi \Theta_\gamma + \frac{c_1 \xi}{c_2} = \left( \frac{c_2}{\xi^2} + 2 + \frac{c_1^2}{c_2} \right) \sin(\xi \Theta_\gamma). \quad (44)$$

We will show that relation (44) is consistent. Indeed, assuming that coefficients  $c_1$  and  $c_2$  are respectively equal to  $\frac{1}{2\Theta_\gamma}$  and  $\frac{1}{\Theta_\gamma^2}$ , we get relation (37). Otherwise, relation (44) gives:

$$\sin(\xi \Theta_\gamma) - \xi \Theta_\gamma \left( \frac{c_2 \Theta_\gamma^2 + (\xi \Theta_\gamma)^2 (1 + \frac{c_1}{c_2 \Theta_\gamma})}{c_2 \Theta_\gamma^2 + (\xi \Theta_\gamma)^2 (2 + \frac{c_1^2}{c_2})} \right) = 0. \quad (45)$$

We use equalities (45) and (40) which is multiplied by  $\frac{c_1}{c_2}$  to give the following system of equations:

$$\begin{cases} \frac{c_1^2}{c_2} = -1 + \frac{1}{c_2 \Theta_\gamma^2} + \frac{c_2 \Theta_\gamma^2}{4}, \\ \frac{c_1}{c_2 \Theta_\gamma} = \frac{1}{c_2 \Theta_\gamma^2} - \frac{1}{2}. \end{cases} \quad (46)$$

Replacing quantities  $\frac{c_1^2}{c_2}$  and  $\frac{c_1}{c_2 \Theta_\gamma}$  defined in relation (46) in equality (45), we obtain the following equality:

$$\sin(\xi \Theta_\gamma) - \xi \Theta_\gamma \left( \frac{c_2 \Theta_\gamma^2 + (\xi \Theta_\gamma)^2 (\frac{1}{2} + \frac{1}{c_2 \Theta_\gamma^2})}{c_2 \Theta_\gamma^2 + (\xi \Theta_\gamma)^2 (1 + \frac{1}{c_2 \Theta_\gamma^2} + \frac{c_2 \Theta_\gamma^2}{4})} \right) = 0. \quad (47)$$

As

$$\forall \xi \in \mathbb{R}_*, \left| \frac{\sin(\xi \Theta_\gamma)}{\xi \Theta_\gamma} \right| \leq 1, \quad (48)$$

using inequality (48), we obtain according to (47) the following inequality:

$$\frac{1}{2} + \frac{1}{c_2 \Theta_\gamma^2} \leq 1 + \frac{1}{c_2 \Theta_\gamma^2} + \frac{c_2 \Theta_\gamma^2}{4}, \quad (49)$$

which is simplified to give:

$$-2 \leq c_2 \Theta_\gamma^2. \quad (50)$$

Figure 2 shows that equation (47) doesn't have no solution on  $\mathbb{R}^*$ . Consequently, the proof of the lemma is achieved.

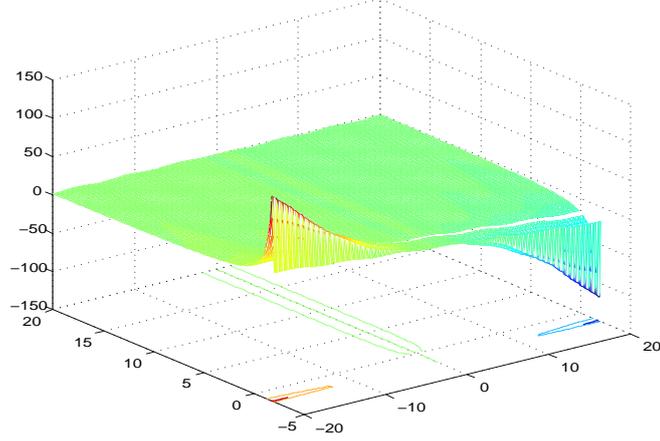


Figure 2: Graph of the function  $(\xi\Theta_\gamma, c_2\Theta_\gamma^2) \rightarrow \sin(\xi\Theta_\gamma) - \xi\Theta_\gamma \left( \frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(\frac{1}{2} + \frac{1}{c_2\Theta_\gamma^2})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4})} \right)$  over interval  $[-20, 20] \times [-2, 20]$ . Also shown that zero is its singular point.

□

Many inverse problems in finance involve its study with regular singularities. For instance, paper [5] deals with inverting the differential operators on the half-line having a discontinuity in an interior. For studying the inverse problem we agree that together with function  $1 - \mathcal{F}(\gamma)$  we suppose that zero is only its singularity for any density  $\gamma$ . This assumption gives a constructive procedure for the validity of the following theorem.

**Theorem 2.5.** *If Repayment Pattern  $\gamma$  satisfies relation (3) and following relation:*

$$\exists \epsilon > 0, \frac{1}{1 - \mathcal{F}(\gamma)} \Big|_{-\infty, -\epsilon] \cup [\epsilon, +\infty[} \in \mathbb{L}^\infty(\mathbb{R}), \quad (51)$$

where  $\mathcal{F}$  stands for the Fourier Transform Operator. And if Initial Debt Repayment Density  $\rho_{\mathcal{K}}^I$  is in  $\mathbb{L}^2([t_I, \Theta_{\max}])$ , then for any Algebraic Spending Density  $\sigma$  in  $\mathbb{L}^2([t_I, \Theta_{\max}])$  is satisfying the following equality:

$$\int_{t_I}^{\Theta_{\max}} \left( \sigma(y) - \mathcal{D}[\rho_{\mathcal{K}}^I](y) + \alpha \int_{t_I}^y (\sigma(s) - \mathcal{D}[\rho_{\mathcal{K}}^I](s)) e^{\alpha(y-s)} ds \right) dy = 0, \quad (52)$$

there exists an unique Loan Density  $\kappa_E$  stable in  $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$  which is given in terms of  $\sigma$  by:

$$\kappa_E = \mathcal{F}^{-1} \left( \frac{\mathcal{F} \left( \sigma - \mathcal{D}[\rho_k^I] + \alpha \int_{t_I}^{\bullet} (\sigma(s) - \mathcal{D}[\rho_k^I](s)) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right), \quad (53)$$

where  $\mathcal{F}^{-1}$  stands for Inverse Fourier Transform, such that (28) holds.

*Proof of Theorem 2.5.* Besides, Lemma 3.5 in [4] shows that operator  $\mathcal{L}$  given by relation (5) is a one-to-one application. From this, we obtain the uniqueness of  $\kappa_E$ .

If we assume that our noise (the error between measurement  $\sigma_2$  and measurement  $\sigma_1$ ) is small in the  $\mathbb{L}^2$ -norm, so that  $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq \delta$ , and we are happy with a small error in the parameter in the  $\mathbb{L}^2([t_I, \Theta_{\max}])$  sense, then there is no problem. The reconstruction will be accurate in the sense that  $\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \leq C_\alpha^\gamma \delta$ , where  $C_\alpha^\gamma$  is a real constant to be determined.

$$\kappa_{E_2} - \kappa_{E_1} = \mathcal{F}^{-1} \left( \frac{\mathcal{F} \left( \sigma_2 - \sigma_1 + \alpha \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right). \quad (54)$$

Since Inverse Fourier Transform  $\mathcal{F}^{-1}$  preserves norm from  $\mathbb{L}^2(\mathbb{R})$  to  $\mathbb{L}^2([t_I, \Theta_{\max}])$ , we obtain from relation (54) the following equality:

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} = \left\| \frac{\mathcal{F} \left( \sigma_2 - \sigma_1 + \alpha \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (55)$$

Under assumption (51) and the fact that Fourier Transform  $\mathcal{F}$  preserves norm from  $\mathbb{L}^2([t_I, \Theta_{\max}])$  to  $\mathbb{L}^2(\mathbb{R})$ , we have:

$$\begin{aligned} \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + \\ &|\alpha| \times \left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \end{aligned} \quad (56)$$

It follows that we want to increase quantity  $\left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2}$  by a constant to be determined which is multiplied by  $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$

$$\begin{aligned} \left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &= \sqrt{\int_{t_I}^{\Theta_{\max}} \left( \int_{t_I}^t (\sigma_2 - \sigma_1)(s) e^{\alpha(t-s)} ds \right)^2 dt}, \\ &\leq \sqrt{\int_{t_I}^{\Theta_{\max}} \left( \int_{t_I}^{\Theta_{\max}} (\sigma_2 - \sigma_1)(s) e^{\alpha(\Theta_{\max}-s)} ds \right)^2 dt}, \\ &\leq \sqrt{\Theta_{\max} - t_I} \times \sup_{s \in [t_I, \Theta_{\max}]} \{e^{\alpha(\Theta_{\max}-s)}\} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}, \\ &\leq \sqrt{\Theta_{\max} - t_I} \times e^{\alpha(\Theta_{\max}-t_I)} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}. \end{aligned} \quad (57)$$

Using Cauchy-Schwarz inequality:

$$\|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \leq \sqrt{\Theta_{\max} - t_I} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \quad (58)$$

and according to relations (56) and (57), we get:

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \leq \left( \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} + |\alpha| \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max} - t_I)} \right) \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (59)$$

If we set  $C_\alpha^\gamma = \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} + |\alpha| \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max} - t_I)}$ , then we get:

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \leq C_\alpha^\gamma \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (60)$$

□

### 3 Inverse problem of the model in $\mathcal{M}([t_I, \Theta_{\max}])$

The aim of this section is to study the inverse problem in measure space. Denoting  $\mathcal{M}([t_I, \Theta_{\max}])$  the Radon measure space which is a continuous and linear form acting on continuous functions space  $\mathcal{C}_c^o([t_I, \Theta_{\max}])$  defined over a time interval  $[t_I, \Theta_{\max}]$ . The usual norm on  $\mathcal{M}([t_I, \Theta_{\max}])$  is:

$$\|\mu\|_{\mathcal{M}([t_I, \Theta_{\max}])} = \sup_{\psi \in \mathcal{C}_c^o([t_I, \Theta_{\max}]), \psi \neq 0} \left\{ \frac{|\langle \mu, \psi \rangle|}{\|\psi\|_{L^\infty([t_I, \Theta_{\max}])}} \right\}, \quad (61)$$

where  $\|\cdot\|_{L^\infty([t_I, \Theta_{\max}])}$  is the usual norm on  $\mathcal{C}_c^o([t_I, \Theta_{\max}])$  defined as:

$$\|\psi\|_{L^\infty([t_I, \Theta_{\max}])} = \sup_{t \in [t_I, \Theta_{\max}]} \{|\psi(t)|\}. \quad (62)$$

We set the Repayment Pattern Measure  $\tilde{\gamma}$  such that:

$$\tilde{\gamma} \in \mathcal{M}([0, \Theta_\gamma]), \quad (63)$$

where  $\Theta_\gamma$  is positive number satisfying relation (4). By relation (63), the support of convolution of two compactly supported measures  $\tilde{\kappa}_E$  in  $[t_I, \Theta_{\max} - \Theta_\gamma]$  and  $\tilde{\gamma}$  in  $[0, \Theta_\gamma]$  is included in  $[t_I, \Theta_{\max}]$ . Indeed, formally:

$$Supp(\tilde{\kappa}_E \star \tilde{\gamma}) \subset Supp(\tilde{\kappa}_E) + Supp(\tilde{\gamma}). \quad (64)$$

Let  $\mathcal{L}_1$  be a linear operator defined from  $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$  to  $\mathcal{M}([t_I, \Theta_{\max}])$  acting on Loan Measure  $\tilde{\kappa}_E$  by

$$\mathcal{L}_1[\tilde{\kappa}_E] = \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma} - \alpha \left\langle \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma}, \mathbf{1}_{[t_I, t]} \right\rangle dt. \quad (65)$$

Let  $\mathcal{D}_1$  be an operator defined in  $\mathcal{M}([t_I, \Theta_{\max}])$  acting on Initial Debt Repayment Measure  $\tilde{\rho}_K^I$  by

$$\mathcal{D}_1[\tilde{\rho}_K^I] = -\alpha \left\langle \tilde{\rho}_K^I, \mathbf{1}_{[t, \Theta_{\max}]} \right\rangle dt - \tilde{\rho}_K^I. \quad (66)$$

Algebraic Spending Measure  $\tilde{\sigma}$  is defined such that the difference between spendings and incomes required to satisfy the current needs. Measure  $\tilde{\sigma}$  is decomposed as a sum of operators  $\mathcal{L}_1$  and  $\mathcal{D}_1$  given by relations (65) and (66), respectively:

$$\tilde{\sigma} = \mathcal{L}_1[\tilde{\kappa}_E] + \mathcal{D}_1[\tilde{\rho}_K^I]. \quad (67)$$

**Theorem 3.1.** *If Repayment Pattern Measure  $\tilde{\gamma}$  is satisfying relation (63) and following relation*

$$\frac{1}{1 - \mathcal{F}(\tilde{\gamma})} \Big|_{]-\infty, -\epsilon[ \cup ]\epsilon, +\infty[} \in \mathbb{L}^\infty(\mathbb{R}), \quad (68)$$

for any positive real  $\epsilon$  and if Loan Measure  $\tilde{\kappa}_E$  exists in  $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$  for any Initial Debt Repayment Measure  $\tilde{\rho}_K^I$  and for any Algebraic Spending Measure  $\tilde{\sigma}$  in  $\mathcal{M}([t_I, \Theta_{\max}])$  satisfying the following equality:

$$\tilde{\kappa}_E = \mathcal{F}^{-1} \left( \frac{\mathcal{F} \left( \tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_K^I] + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_K^I], e_{-\alpha} \Big|_{[t_I, t]} \right\rangle \right)}{1 - \mathcal{F}(\tilde{\gamma})} \right), \quad (69)$$

then, Loan Measure  $\tilde{\kappa}_E$  is unique and stable in space  $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$ .

*Proof.* We notice that since that the inverse Fourier transform  $\mathcal{F}^{-1}$  is not surjective from  $\mathcal{M}(\mathbb{R})$  to  $\mathcal{M}([t_I, \Theta_{\max}])$ , the solution  $\tilde{\kappa}_E$  doesn't exist for the inverse problem. Moreover, if Loan Measure  $\tilde{\kappa}_E$  is supposed satisfying relation (69), then it is unique due to the injectivity of operator  $\mathcal{L}_1$  (see Lemma 3.4 in [4]).

Now we will show that the solution Loan Measure  $\tilde{\kappa}_E$  is stable. Definition (65) of operator  $\mathcal{L}_1$  gives that for any two Loan Densities  $\tilde{\kappa}_{E_1}$  and  $\tilde{\kappa}_{E_2}$  the following equality:

$$\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1} = \mathcal{F}^{-1} \left( \frac{\mathcal{F} \left( \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha} \Big|_{[t_I, t]} \right\rangle \right)}{1 - \mathcal{F}(\tilde{\gamma})} \right). \quad (70)$$

The usual Radon norm on  $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$  of quantity  $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$  is defined by:

$$\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} = \sup_{\phi \in \mathcal{C}_c^0([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}, \phi \rangle|}{\|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \quad (71)$$

We replace measure  $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$  given by (70) in right equality (71) in order to obtain the following equality:

$$\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} = \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})|} \times \sup_{\phi \in \mathcal{C}_c^0([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha} \Big|_{[t_I, t]} \right\rangle, \mathcal{F}(\phi) \rangle|}{\|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \quad (72)$$

By the Fourier Transform of function  $\phi \in \mathcal{C}_c^0([t_I, \Theta_{\max} - \Theta_\gamma])$ , we get the following inequality:

$$\begin{aligned} |\mathcal{F}(\phi)(\xi)| &= \left| \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \phi(x) e^{-ix\xi} dx \right|, \\ &\leq \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} |e^{-ix\xi}| dx \times \|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}, \\ &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}. \end{aligned} \quad (73)$$

Using relations (72) and (73), we obtain:

$$\begin{aligned} \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})(\xi)|} \right\} \times \\ &\sup_{\phi \in C_c^\infty([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|\cdot|} \rangle, \mathcal{F}(\phi) \rangle|}{\|\mathcal{F}(\phi)\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \end{aligned} \quad (74)$$

Since we have:

$$\begin{aligned} |\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|\cdot|} \rangle| &= \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|\cdot|} \rangle|}{\|e_{-\alpha|\cdot|}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \times \|e_{-\alpha|\cdot|}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}, \\ &\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times \|e_{-\alpha|\cdot|}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}, \\ &\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times e^{-\alpha t_I}, \end{aligned} \quad (75)$$

we obtain:

$$|\alpha \tilde{e}_\alpha \langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|\cdot|} \rangle, \mathcal{F}(\phi) \rangle| \leq |\alpha| \times (\Theta_{\max} - t_I) \times e^{|\alpha|(\Theta_{\max} - t_I)} \times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times \|\mathcal{F}(\phi)\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}. \quad (76)$$

According to relations (74) and (76), we obtain the following inequality:

$$\begin{aligned} \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})(\xi)|} \right\} \times \\ &(1 + |\alpha| \times (\Theta_{\max} - t_I) \times e^{|\alpha|(\Theta_{\max} - t_I)}) \times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])}. \end{aligned} \quad (77)$$

□

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