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Inverse problem stability of a continuous-in-time financial model *

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Abstract. In this paper, we study the inverse problem stability of the continuous-in-time model which is designed to be used for the finances of public institutions. We discuss this study with determining the Loan Measure from Algebraic Spending Measure in $\mathcal{M}([t_1, \Theta_{\max}])$, and in $\mathbb{L}^2([t_1, \Theta_{\max}])$ when they are density measures.

Keywords: Inverse problem; stability; mathematical model; Fredholm operator.

1 Introduction

In the last two decades, the theory and practice of inverse problems have been developed in many scientific domains. Consequently, it is rapidly growing, if not exploding. Moreover, document [3] shows how much researchers contribute to this field. Many inverse problems arising in scientific domains present numerical instability: the noise affecting the data may produce arbitrarily large errors in the solutions. In other words, these problems are ill-posed in the sense of Hadamard. The concept of ill-posedness was introduced by HADAMARD [6] in the field of partial differential equations. We mention the book on the mathematics of ill-posed problems by TIKHONOV and ARSENIN [8].

We constructed in previous work [5] the continuous-in-time model which is based on using the mathematical tools such convolution and integration. Indeed, this model uses measures over time interval to describe loan scheme, reimbursement scheme and interest payment scheme. The model contains some financial quantities. For instance, the Repayment Pattern Measure $\tilde{\gamma}$ is a non-negative measure with total mass which equals 1, the Algebraic Spending Measure $\tilde{\sigma}$ is defined such that the difference between spendings and incomes required to satisfy the current needs between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\sigma}, \quad (1)$$

and the Loan Measure $\tilde{\kappa}_E$ is defined such that the amount borrowed between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\kappa}_E. \quad (2)$$

When measure $\tilde{\gamma}$ is absolutely continuous with respect to the Lebesgue measure dt . This means that it read $\gamma(t)dt$, where t is the variable in \mathbb{R} . The work [5] proposes the resolution of the inverse problem over the space of square-integrable functions when density γ is equal to $\frac{1}{\Theta_\gamma}$ over interval $[0, \Theta_\gamma]$ and null elsewhere. In paper [4], we use a mathematical framework to discuss an inverse problem of determining the Loan Measure $\tilde{\kappa}_E$ from Algebraic Spending Measure $\tilde{\sigma}$ in measure space. The inverse problem is used in [5] on simplified examples in order to show its capability to be used to forecast a financial strategy.

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We show in this paper the general results of this inverse problem for any density γ . We show the stability of the inverse problem. We continue to extend some results of this inverse problem in measure space with proving its stability. We describe a more complete numerical study for the inverse problem.

The organization of the rest of this paper is as follows. Section 2 describes the inverse problem of the model in $\mathbb{L}^2([t_I, \Theta_{\max}])$. Section 3 shows the inverse problem of the model in $\mathcal{M}([t_I, \Theta_{\max}])$.

2 Inverse problem of the model in $\mathbb{L}^2([t_I, \Theta_{\max}])$

Denoting $\mathbb{L}^2([t_I, \Theta_{\max}])$ the space of square-integrable functions over \mathbb{R} having their support in $[t_I, \Theta_{\max}]$ and denoting $\mathbb{L}^2([0, \Theta_\gamma])$ the space of square-integrable functions over \mathbb{R} having their support in $[0, \Theta_\gamma]$. We state the Repayment Pattern Density γ as follows:

$$\gamma \in \mathbb{L}^2([0, \Theta_\gamma]), \quad (3)$$

where Θ_γ is a positive number such that:

$$\Theta_\gamma < \Theta_{\max} - t_I. \quad (4)$$

We justify relation (3) because the support of convolution of two compactly supported densities κ_E in $[t_I, \Theta_{\max} - \Theta_\gamma]$ and γ in $[0, \Theta_\gamma]$ is included in $[t_I, \Theta_{\max}]$.

Lemma 2.1. *Linear operator \mathcal{L} acting on Loan Density $\kappa_E \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$ defined as:*

$$\mathcal{L}[\kappa_E](t) = \kappa_E(t) - (\kappa_E \star \gamma)(t) - \alpha \int_{t_I}^t (\kappa_E - \kappa_E \star \gamma)(s) ds, \quad (5)$$

is compact operator from $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$ to $\mathbb{L}^2([t_I, \Theta_{\max}])$.

Proof. Definition (5) of operator \mathcal{L} gives that for any two Loan Densities κ_{E_1} and κ_{E_2} the following equality:

$$\begin{aligned} \mathcal{L}[\kappa_{E_2}](t) - \mathcal{L}[\kappa_{E_1}](t) &= \kappa_{E_2}(t) - \kappa_{E_1}(t) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(t) \\ &\quad - \alpha \int_{t_I}^t \kappa_{E_2}(s) - \kappa_{E_1}(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds. \end{aligned} \quad (6)$$

Taking norm $\mathbb{L}^2([t_I, \Theta_{\max}])$ and applying triangle inequality to relation (6), we obtain the following inequality:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + |\alpha| \times C_{E_1}^2, \quad (7)$$

where $C_{E_1}^2$ is defined and is increased as follows:

$$\begin{aligned} C_{E_1}^2 &= \left\| \int_{t_I}^t (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \\ &= \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^t (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) ds \right)^2 dt}, \\ &= \sqrt{\Theta_{\max} - t_I} \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}. \end{aligned} \quad (8)$$

We use Cauchy-Schwarz inequality to obtain:

$$\begin{aligned}
\|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} &= \int_{t_I}^{\Theta_{\max}} 1 \times |(\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s)| \, ds, \\
&\leq \sqrt{\int_{t_I}^{\Theta_{\max}} 1^2 \, ds} \\
&\times \sqrt{\int_{t_I}^{\Theta_{\max}} \left((\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) \right)^2 \, ds}, \\
&\leq \sqrt{\Theta_{\max} - t_I} \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{9}
\end{aligned}$$

Thanks to properties (8) and (9), we get:

$$C_{E_1^2} \leq (\Theta_{\max} - t_I) \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{10}$$

From this and according to relation (7), we get:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (1 + |\alpha| \times (\Theta_{\max} - t_I)) \times \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{11}$$

The triangle and the Young's Inequalities imply that:

$$\begin{aligned}
\|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} + \|((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \\
&\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} + \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \\
&\times \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}, \\
&\leq (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}) \times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])}. \tag{12}
\end{aligned}$$

From this and according to relation (11), we get:

$$\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (1 + |\alpha| \times (\Theta_{\max} - t_I)) \times (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])}) \times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])}. \tag{13}$$

Consequently, linear operator \mathcal{L} is uniformly bounded and is a Hilbert-Schmidt operator $\mathbb{L}^2([t_I, \Theta_{\max}])$ of constant $(1 + |\alpha| \times (\Theta_{\max} - t_I)) \times (1 + \|\gamma\|_{\mathbb{L}^1([0, \Theta_\gamma])})$, achieving the proof of the lemma. \square

Lemma 2.2. *Linear operator \mathcal{L} given by relation (5) is Fredholm operator such that:*

$$\text{codim } \text{Im}(\mathcal{L}) = \text{dim } \text{Ker}(\mathcal{L}) < \infty. \tag{14}$$

Proof. We consider \mathbf{L} the space of continuous linear applications of $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$ in $\mathbb{L}^2([t_I, \Theta_{\max}])$. We define operator K in \mathbf{L} as an integral operator:

$$\forall \kappa_E \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma]), K[\kappa_E](x) = \int_{t_I}^{\Theta_{\max}} F(x, y) \kappa_E(y) \, dy. \tag{15}$$

We want to show that linear operator \mathcal{L} is a difference between the identity application from space $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$ to $\mathbb{L}^2([t_I, \Theta_{\max}])$ and the compact operator K given by (15) as the following form:

$$\mathcal{L}[\kappa_E](x) = Id_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma]) \rightarrow \mathbb{L}^2([t_I, \Theta_{\max}])}[\kappa_E](x) - K[\kappa_E](x), \quad (16)$$

where, F which defines operator K , is a function in $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])$ to be determined. In what follows, the definition (5) of linear operator \mathcal{L} is used in order to introduce in its second term the indicator function $\mathbb{1}_{\{y \leq x\}}$ to get first line (17). It allows to simplify integrals because of using Fubini-Tonelli Theorem to permut intergals. These simplifications can be detailed as follows:

$$\begin{aligned} \mathcal{L}[\kappa_E](x) &= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{y \leq x\}} (\kappa_E - \kappa_E \star \gamma)(y) dy, \\ &= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y \leq x\}} dy + \int_{t_I}^{\Theta_{\max}} \alpha \mathbb{1}_{\{y \leq x\}} (\kappa_E \star \gamma)(y) dy, \\ &= \kappa_E(x) - \int_{t_I}^{\Theta_{\max}} \kappa_E(y) \gamma(x - y) dy - \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y \leq x\}} dy + \\ &\quad \int_{t_I}^{\Theta_{\max}} \alpha \kappa_E(y) \left(\int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right) dy, \\ &= \kappa_E(x) - \int_{t_I}^{\Theta_{\max}} \kappa_E(y) (\gamma(x - y) + \alpha \mathbb{1}_{\{y \leq x\}} - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt) dy. \end{aligned} \quad (17)$$

From this, we get the expression of function F :

$$F(x, y) = \gamma(x - y) + \alpha \mathbb{1}_{\{y \leq x\}} - \alpha \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt. \quad (18)$$

In order to show that function F is square-integrable over $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$, we will show that three functions they are. Indeed, these functions are $(x, y) \rightarrow \gamma(x - y)$, $(x, y) \rightarrow \mathbb{1}_{\{y \leq x\}}$ and $(x, y) \rightarrow \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt$. Otherwise, since we have:

$$\begin{aligned} \|\mathbb{1}_{\{y \leq x\}}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \left(\int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{y \leq x\}}^2 dy \right) dx, \\ &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} (x - t_I) dx, \\ &= \frac{(\Theta_{\max} - \Theta_\gamma - t_I)^2}{2}, \end{aligned} \quad (19)$$

we have $(x, y) \rightarrow \mathbb{1}_{\{y \leq x\}} \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])$. In what follows, we will show that function $\int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt$ is square-integrable over $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$. For that, we set the following inequality:

$$\begin{aligned} \left\| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 &= \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \int_{t_I}^{\Theta_{\max}} \left| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t - y) dt \right|^2 dx dy, \\ &\leq \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^x |\gamma(t - y)| dt \right)^2 dx dy. \end{aligned} \quad (20)$$

We use Cauchy-Schwarz inequality to obtain:

$$\forall x \in [t_I, \Theta_{\max} - \Theta_\gamma], \left(\int_{t_I}^x |\gamma(t - y)| dt \right)^2 \leq (x - t_I) \times \int_{t_I}^x |\gamma(t - y)|^2 dt. \quad (21)$$

Since density γ satisfies relation (3), there exists a constant C_γ such that relation (21) gives:

$$\left(\int_{t_I}^x |\gamma(t-y)| dt \right)^2 \leq (x-t_I) \times C_\gamma. \quad (22)$$

According to relations (20) and (22), we get the following inequality:

$$\left\| \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t-y) dt \right\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}])}^2 \leq C_\gamma \times (\Theta_{\max} - t_I) \times \frac{(\Theta_{\max} - \Theta_\gamma - t_I)^2}{2}, \quad (23)$$

which proves that function $(x, y) \rightarrow \int_{t_I}^{\Theta_{\max}} \mathbb{1}_{\{t \leq x\}} \gamma(t-y) dt$ is square-integrable over $[t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]$. Consequently, we get:

$$F \in \mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma] \times [t_I, \Theta_{\max}]). \quad (24)$$

It is concluded that linear operator \mathcal{L} can be written in the form which is shown in relation (16). We showed that operator \mathcal{L} is Fredholm operator (see Theorem 3.45 page 206 in [7] or The Fredholm Alternative Theorem 1.3.1 in [2] page 13). In addition, the Fredholm Alternative Theorem (Lemma 4.45 page 160 in [1]) is used to get that linear operator \mathcal{L} has a finite codimension, a closed image and a finite dimension of its kernel. The proof of lemma is achieved. \square

Lemma 2.3. *Linear operator \mathcal{D} acting on Initial Debt Repayment Density $\rho_K^I \in \mathbb{L}^2([t_I, \Theta_{\max}])$ defined as:*

$$\mathcal{D}[\rho_K^I](t) = -\alpha \int_t^{\Theta_{\max}} \rho_K^I(s) ds - \rho_K^I(t), \quad (25)$$

is compact operator from $\mathbb{L}^2([t_I, \Theta_{\max}])$ to $\mathbb{L}^2([t_I, \Theta_{\max}])$.

Proof. The triangle inequality is applied to definition (25) of operator \mathcal{D} to get:

$$\|\mathcal{D}[\rho_K^I]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq |\alpha| \times \left\| \int_t^{\Theta_{\max}} \rho_K^I(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (26)$$

Since we have $\left\| \int_t^{\Theta_{\max}} \rho_K^I(s) ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (\Theta_{\max} - t_I) \times \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$ and according to relation (26), we have:

$$\|\mathcal{D}[\rho_K^I]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (|\alpha| \times (\Theta_{\max} - t_I) + 1) \times \|\rho_K^I\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (27)$$

\square

We can decompose Algebraic Spending Density σ as a sum of operators \mathcal{L} and \mathcal{D} :

$$\sigma(t) = \mathcal{L}[\kappa_E](t) + \mathcal{D}[\rho_K^I](t). \quad (28)$$

Lemma 2.4. *The singular point of function $1 - \mathcal{F}(\gamma)$ is zero for any constant and affine density γ .*

Proof. If Repayment Pattern Density γ is a constant function given by:

$$\gamma = \frac{1}{\Theta_\gamma} \mathbb{1}_{[0, \Theta_\gamma]}, \quad (29)$$

then we have:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - \frac{i}{\xi \Theta_\gamma} (e^{-i\xi \Theta_\gamma} - 1). \quad (30)$$

From this, we get:

$$\begin{aligned} \forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 0 &\Rightarrow \xi \Theta_\gamma = \sin(\xi \Theta_\gamma) \text{ et } \cos(\xi \Theta_\gamma) = 1, \\ &\Rightarrow (\xi \Theta_\gamma)^2 + 1 = 1, \\ &\Rightarrow \xi \Theta_\gamma = 0. \end{aligned} \quad (31)$$

Since real Θ_γ is positive, function $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$ is not zero function over \mathbb{R}^* . Inversely, if a real ξ is zero, function $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$ is also a zero function. Indeed, the Fourier Transform of any density γ at the origine is defined as:

$$\mathcal{F}(\gamma)(0) = \int_{-\infty}^{+\infty} \gamma(t) dt. \quad (32)$$

Furthermore, since density γ is with total mass which equals 1, relation (32) implies:

$$\mathcal{F}(\gamma)(0) = 1. \quad (33)$$

Now we will show that the singular point of function $1 - \mathcal{F}(\gamma)$ is zero for an affine density γ given by:

$$\gamma(t) = \left(\frac{t}{\Theta_\gamma^2} + \frac{1}{2\Theta_\gamma} \right) \mathbb{1}_{[0, \Theta_\gamma]}. \quad (34)$$

Integrating by parts states that:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - \left(\frac{1}{(\xi \Theta_\gamma)^2} + \frac{i}{2\xi \Theta_\gamma} \right) (e^{-i\xi \Theta_\gamma} - 1) - \frac{i}{\xi \Theta_\gamma} e^{-i\xi \Theta_\gamma}. \quad (35)$$

From this, we get the following system of equations:

$$\begin{cases} (\cos(\xi \Theta_\gamma) - 1) + \frac{3\xi \Theta_\gamma}{2} \sin(\xi \Theta_\gamma) = (\xi \Theta_\gamma)^2, \\ -\sin(\xi \Theta_\gamma) + \xi \Theta_\gamma \cos(\xi \Theta_\gamma) + \frac{\xi \Theta_\gamma}{2} (\cos(\xi \Theta_\gamma) - 1) = 0. \end{cases} \quad (36)$$

From this, we get the following equality:

$$\xi \Theta_\gamma \left(1 + \frac{3(\xi \Theta_\gamma)^2}{2} \right) - \left(1 + \frac{9(\xi \Theta_\gamma)^2}{4} \right) \sin(\xi \Theta_\gamma) = 0. \quad (37)$$

According to Figure 1, we state that equation (37) doesn't have no solution on \mathbb{R}^* . Consequently, function $\xi \rightarrow 1 - \mathcal{F}(\gamma)(\xi)$ is not a zero function over \mathbb{R}^* . Inversly, assuming that real ξ is zero, we get:

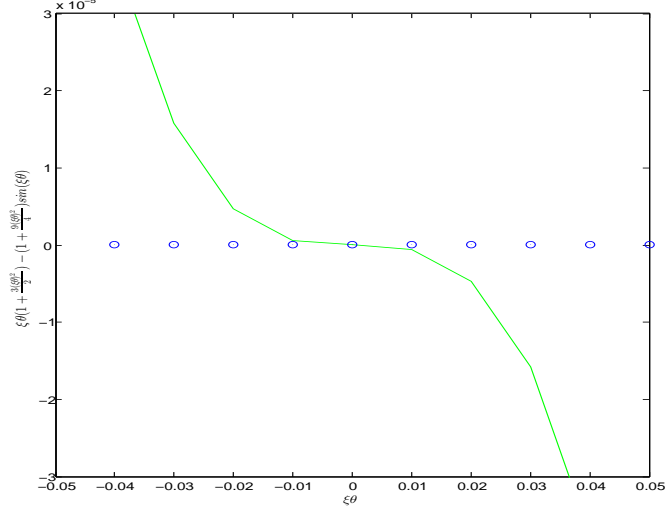


Figure 1: Graph of the function $\xi \rightarrow \xi\Theta_\gamma \left(1 + \frac{3(\xi\Theta_\gamma)^2}{2}\right) - \left(1 + \frac{9(\xi\Theta_\gamma)^2}{4}\right) \sin(\xi\Theta_\gamma)$ over interval $[-0.05, 0.05]$. Also shown that zero is its singular point.

$$\begin{aligned}
1 - \mathcal{F}(\gamma)(0) &= 1 - \int_{-\infty}^{+\infty} \gamma, \\
&= 1 - \frac{1}{\Theta_\gamma^2} \int_0^{\Theta_\gamma} t \, dt - \frac{1}{2\Theta_\gamma} \int_0^{\Theta_\gamma} dt, \\
&= 0.
\end{aligned} \tag{38}$$

We conclude that function $1 - \mathcal{F}(\gamma)$ is zero at the origine for density γ given by (29) or by (34). In what to follows, we want to extend this conclusion for any affine density γ given by:

$$\gamma(t) = (c_2 t + c_1) \mathbf{1}_{[0, \Theta_\gamma]}, \tag{39}$$

where coefficients c_1 and c_2 satisfy:

$$c_1 + \frac{c_2 \Theta_\gamma}{2} = \frac{1}{\Theta_\gamma}. \tag{40}$$

We obtain using the integration by parts:

$$\forall \xi \in \mathbb{R}^*, 1 - \mathcal{F}(\gamma)(\xi) = 1 - c_2 \left(\frac{e^{-i\xi\Theta_\gamma} - 1}{\xi^2} + \frac{i\Theta_\gamma e^{-i\xi\Theta_\gamma}}{\xi} \right) - \frac{ic_1(e^{-i\xi\Theta_\gamma} - 1)}{\xi} = 0. \tag{41}$$

We obtain by separating the real and the imaginary parts of function $1 - \mathcal{F}(\gamma)$ the following system of equations:

$$\begin{cases} 1 - c_2 \left(\frac{\cos(\xi\Theta_\gamma) - 1}{\xi^2} + \frac{\Theta_\gamma \sin(\xi\Theta_\gamma)}{\xi} \right) - \frac{c_1 \sin(\xi\Theta_\gamma)}{\xi} = 0, \\ c_2 \left(-\frac{\sin(\xi\Theta_\gamma)}{\xi^2} + \frac{\Theta_\gamma \cos(\xi\Theta_\gamma)}{\xi} \right) + \frac{c_1 (\cos(\xi\Theta_\gamma) - 1)}{\xi} = 0. \end{cases} \quad (42)$$

Let us check the consistency of the system of equations given by (42). Indeed, if coefficients c_1 and c_2 are respectively equal to $\frac{1}{2\Theta_\gamma}$ and $\frac{1}{\Theta_\gamma^2}$, then we get relation (36). Thanks to (42), we get the following equality:

$$\frac{c_2\Theta_\gamma}{\xi} + \xi\Theta_\gamma + \frac{c_1\xi}{c_2} = \left(\frac{c_2}{\xi^2} + \Theta_\gamma(c_2\Theta_\gamma + c_1) + \frac{c_1(c_2\Theta_\gamma + c_1)}{c_2} \right) \sin(\xi\Theta_\gamma). \quad (43)$$

According to relations (42) and (43), we get:

$$\frac{c_2\Theta_\gamma}{\xi} + \xi\Theta_\gamma + \frac{c_1\xi}{c_2} = \left(\frac{c_2}{\xi^2} + 2 + \frac{c_1^2}{c_2} \right) \sin(\xi\Theta_\gamma). \quad (44)$$

We will show that relation (44) is consistent. Indeed, assuming that coefficients c_1 and c_2 are respectively equal to $\frac{1}{2\Theta_\gamma}$ and $\frac{1}{\Theta_\gamma^2}$, we get relation (37). Otherwise, relation (44) gives:

$$\sin(\xi\Theta_\gamma) - \xi\Theta_\gamma \left(\frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1 + \frac{c_1}{c_2\Theta_\gamma})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(2 + \frac{c_1^2}{c_2})} \right) = 0. \quad (45)$$

We use equality (45) and equality in (40) which is multiplied by $\frac{c_1}{c_2}$ in order to give:

$$\begin{cases} \frac{c_1^2}{c_2} = -1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4}, \\ \frac{c_1}{c_2\Theta_\gamma} = \frac{1}{c_2\Theta_\gamma^2} - \frac{1}{2}. \end{cases} \quad (46)$$

Replacing equality (46) in relation (45), we obtain the following equality:

$$\sin(\xi\Theta_\gamma) - \xi\Theta_\gamma \left(\frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(\frac{1}{2} + \frac{1}{c_2\Theta_\gamma^2})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4})} \right) = 0. \quad (47)$$

As

$$\forall \xi \in \mathbb{R}_*, \left| \frac{\sin(\xi\Theta_\gamma)}{\xi\Theta_\gamma} \right| \leq 1, \quad (48)$$

using inequality (48), we obtain the following inequality:

$$\frac{1}{2} + \frac{1}{c_2\Theta_\gamma^2} \leq 1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4}. \quad (49)$$

Inequality (49) is simplified using relation (40) to give:

$$-2 \leq c_2\Theta_\gamma^2. \quad (50)$$

□

Theorem 2.5. *If Repayment Pattern γ satisfies relation (3) and following relation:*

$$\exists \epsilon > 0, \frac{1}{1 - \mathcal{F}(\gamma)} \Big|_{-\infty, -\epsilon[0], \epsilon, +\infty[} \in \mathbb{L}^\infty(\mathbb{R}), \quad (51)$$

where \mathcal{F} stands for the Fourier Transform Operator. And if Initial Debt Repayment Density $\rho_{\mathcal{K}}^I$ is in $\mathbb{L}^2([t_I, \Theta_{\max}])$, then for any Algebraic Spending Density σ in $\mathbb{L}^2([t_I, \Theta_{\max}])$ is satisfying the following equality:

$$\int_{t_I}^{\Theta_{\max}} \left(\sigma(y) - \mathcal{D}[\rho_{\mathcal{K}}^I](y) + \alpha \int_{t_I}^y (\sigma(s) - \mathcal{D}[\rho_{\mathcal{K}}^I](s)) e^{\alpha(y-s)} ds \right) dy = 0, \quad (52)$$

there exists an unique Loan Density κ_E stable in $\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])$ which is given in terms of σ by:

$$\kappa_E = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I] + \alpha \int_{t_I}^{\bullet} (\sigma(s) - \mathcal{D}[\rho_{\mathcal{K}}^I](s)) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right), \quad (53)$$

where \mathcal{F}^{-1} stands for Inverse Fourier Transform, such that (28) holds.

Proof of Theorem 2.5. Besides, Lemma 3.5, in [4] showed that operator \mathcal{L} given by relation (5) is a one-to-one application. From this, we obtain the uniqueness of κ_E .

If we assume that our noise (the error between measurement σ_2 and measurement σ_1) is small in the \mathbb{L}^2 -norm, so that $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq \delta$, and we are happy with a small error in the parameter in the $\mathbb{L}^2([t_I, \Theta_{\max}])$ sense, then there is "no problem". The reconstruction will be accurate in the sense that $\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \leq C_\alpha \delta$, where C_α is a real constant to be determined.

$$\kappa_{E_2} - \kappa_{E_1} = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\sigma_2 - \sigma_1 + \alpha \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right). \quad (54)$$

Since Inverse Fourier Transform \mathcal{F}^{-1} preserves norm from $\mathbb{L}^2(\mathbb{R})$ to $\mathbb{L}^2([t_I, \Theta_{\max}])$, we obtain from (54)

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} = \left\| \frac{\mathcal{F} \left(\sigma_2 - \sigma_1 + \alpha \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (55)$$

Under the assumption (51) and the fact that Fourier Transform \mathcal{F} preserves norm from $\mathbb{L}^2([t_I, \Theta_{\max}])$ to $\mathbb{L}^2(\mathbb{R})$, we have:

$$\begin{aligned} \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} + \\ &|\alpha| \times \left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \end{aligned} \quad (56)$$

It follows, we want to increase quantity $\left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2}$ by a constant to be determined times $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$

$$\begin{aligned}
\left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &= \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^t (\sigma_2 - \sigma_1)(s) e^{\alpha(t-s)} ds \right)^2 dt} \\
&\leq \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^{\Theta_{\max}} (\sigma_2 - \sigma_1)(s) e^{\alpha(\Theta_{\max}-s)} ds \right)^2 dt} \\
&\leq \sqrt{\Theta_{\max} - t_I} \times \sup_{s \in [t_I, \Theta_{\max}]} \{e^{\alpha(\Theta_{\max}-s)}\} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \\
&\leq \sqrt{\Theta_{\max} - t_I} \times e^{\alpha(\Theta_{\max}-t_I)} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}.
\end{aligned} \tag{57}$$

Since we have with using Cauchy-Schwarz inequality:

$$\|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \leq \sqrt{\Theta_{\max} - t_I} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \tag{58}$$

and according to relations (56) and (57), we get:

$$\begin{aligned}
\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq \left(\sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} + |\alpha| \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max}-t_I)} \right) \times \\
&\quad \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}.
\end{aligned} \tag{59}$$

If we set $C_\alpha^\gamma = \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\gamma)(\xi)|} \right\} + |\alpha| \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max}-t_I)}$, then we get:

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \leq C_\alpha^\gamma \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{60}$$

□

3 Inverse problem of the model in $\mathcal{M}([t_I, \Theta_{\max}])$

The aim of this section is to study the inverse problem in measure space. Denoting $\mathcal{M}([t_I, \Theta_{\max}])$ the Radon measure space which is a continuous and linear form acting on continuous functions space $\mathcal{C}_c^o([t_I, \Theta_{\max}])$ defined over a time interval $[t_I, \Theta_{\max}]$. The usual norm on $\mathcal{M}([t_I, \Theta_{\max}])$ is:

$$\|\mu\|_{\mathcal{M}([t_I, \Theta_{\max}])} = \sup_{\psi \in \mathcal{C}_c^o([t_I, \Theta_{\max}]), \psi \neq 0} \left\{ \frac{|\langle \mu, \psi \rangle|}{\|\psi\|_{L^\infty([t_I, \Theta_{\max}])}} \right\}, \tag{61}$$

where $\|\cdot\|_{L^\infty([t_I, \Theta_{\max}])}$ is the usual norm on $\mathcal{C}_c^o([t_I, \Theta_{\max}])$ defined as:

$$\|\psi\|_{L^\infty([t_I, \Theta_{\max}])} = \sup_{t \in [t_I, \Theta_{\max}]} \{|\psi(t)|\}. \tag{62}$$

We set the Repayment Pattern Measure $\tilde{\gamma}$ such that:

$$\tilde{\gamma} \in \mathcal{M}([0, \Theta_\gamma]), \tag{63}$$

where Θ_γ is positive number satisfying relation (4). By relation (63), the support of convolution of two compactly supported measures $\tilde{\kappa}_E$ in $[t_I, \Theta_{\max} - \Theta_\gamma]$ and $\tilde{\gamma}$ in $[0, \Theta_\gamma]$ is included in $[t_I, \Theta_{\max}]$. Indeed, formally:

$$Supp(\tilde{\kappa}_E \star \tilde{\gamma}) \subset Supp(\tilde{\kappa}_E) + Supp(\tilde{\gamma}). \quad (64)$$

Let \mathcal{L}_1 be a linear operator defined from $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$ to $\mathcal{M}([t_I, \Theta_{\max}])$ acting on Loan Measure $\tilde{\kappa}_E$ by

$$\mathcal{L}_1[\tilde{\kappa}_E] = \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma} - \alpha \left\langle \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma}, \mathbb{1}_{|[t_I, t[} \right\rangle dt. \quad (65)$$

Let \mathcal{D}_1 be an operator defined in $\mathcal{M}([t_I, \Theta_{\max}])$ acting on Initial Debt Repayment Measure $\tilde{\rho}_{\mathcal{K}}^I$ by

$$\mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^I] = -\alpha \left\langle \tilde{\rho}_{\mathcal{K}}^I, \mathbb{1}_{|[t, \Theta_{\max}[} \right\rangle dt - \tilde{\rho}_{\mathcal{K}}^I. \quad (66)$$

Algebraic Spending Measure $\tilde{\sigma}$ is defined such that the difference between spendings and incomes required to satisfy the current needs. Measure $\tilde{\sigma}$ is decomposed as a sum of operators \mathcal{L}_1 and \mathcal{D}_1 given by relations (65) and (66), respectively:

$$\tilde{\sigma} = \mathcal{L}_1[\tilde{\kappa}_E] + \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^I]. \quad (67)$$

Theorem 3.1. *If Repayment Pattern Measure $\tilde{\gamma}$ is satisfying relation (63) and following relation*

$$\frac{1}{1 - \mathcal{F}(\tilde{\gamma})} \Big|_{]-\infty, -\epsilon[\cup]\epsilon, +\infty[} \in \mathbb{L}^\infty(\mathbb{R}), \quad (68)$$

for any positive real ϵ and if Loan Measure $\tilde{\kappa}_E$ in $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$ exists for any Initial Debt Repayment Measure $\tilde{\rho}_{\mathcal{K}}^I$ and for any Algebraic Spending Measure $\tilde{\sigma}$ in $\mathcal{M}([t_I, \Theta_{\max}])$ satisfying the following equality:

$$\tilde{\kappa}_E = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^I] + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^I], e_{-\alpha|[t_I, t[} \right\rangle \right)}{1 - \mathcal{F}(\tilde{\gamma})} \right)}{\right), \quad (69)$$

then, Loan Measure $\tilde{\kappa}_E$ is unique and stable in space $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$.

Proof. We notice that since that the inverse Fourier transform \mathcal{F}^{-1} is not surjective from $\mathcal{M}(\mathbb{R})$ to $\mathcal{M}([t_I, \Theta_{\max}])$, the solution $\tilde{\kappa}_E$ doesn't exist for the inverse problem. Moreover, if Loan Measure $\tilde{\kappa}_E$ is supposed satisfying relation (69), then it is unique due to the injectivity of operator \mathcal{L}_1 . For that, we refer the reader to Lemma 3.4.

Now we will show that the solution Loan Measure $\tilde{\kappa}_E$ is stable. Definition (65) of operator \mathcal{L}_1 gives that for any two Loan Densities $\tilde{\kappa}_{E_1}$ and $\tilde{\kappa}_{E_2}$ the following equality:

$$\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1} = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|[t_I, t[} \right\rangle \right)}{1 - \mathcal{F}(\tilde{\gamma})} \right)}{\right). \quad (70)$$

The usual Radon norm on $\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])$ of quantity $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$ is defined by:

$$\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} = \sup_{\phi \in \mathcal{C}_c^0([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}, \phi \rangle|}{\|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \quad (71)$$

We replace measure $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$ given by definition (70) in relation (71) in order to obtain the following equality:

$$\begin{aligned} & \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} = \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})|} \times \\ & \sup_{\phi \in \mathcal{C}_c^o([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|_{[t_I, t]}} \rangle, \mathcal{F}(\phi) \rangle|}{\|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \end{aligned} \quad (72)$$

By the Fourier Transform of function $\phi \in \mathcal{C}_c^o([t_I, \Theta_{\max} - \Theta_\gamma])$, we get the following inequality:

$$\begin{aligned} |\mathcal{F}(\phi)(\xi)| &= \left| \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} \phi(x) e^{-ix\xi} dx \right|, \\ &\leq \|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])} \int_{t_I}^{\Theta_{\max} - \Theta_\gamma} |e^{-ix\xi}| dx, \\ &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \|\phi\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}. \end{aligned} \quad (73)$$

Using relations (72) and (73), we obtain:

$$\begin{aligned} \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})(\xi)|} \right\} \times \\ &\sup_{\phi \in \mathcal{C}_c^o([t_I, \Theta_{\max} - \Theta_\gamma]), \phi \neq 0} \left\{ \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|_{[t_I, t]}} \rangle, \mathcal{F}(\phi) \rangle|}{\|\mathcal{F}(\phi)\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \right\}. \end{aligned} \quad (74)$$

Since we have:

$$\begin{aligned} |\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|_{[t_I, t]}} \rangle| &= \frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|_{[t_I, t]}} \rangle|}{\|e_{-\alpha|_{[t_I, t]}}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}} \times \|e_{-\alpha|_{[t_I, t]}}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}, \\ &\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times \|e_{-\alpha|_{[t_I, t]}}\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}, \\ &\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times e^{-\alpha|t_I|}, \end{aligned} \quad (75)$$

we obtain:

$$\begin{aligned} |\alpha \tilde{e}_\alpha \langle \tilde{\sigma}_2 - \tilde{\sigma}_1, e_{-\alpha|_{[t_I, t]}} \rangle, \mathcal{F}(\phi) \rangle| &\leq |\alpha| \times (\Theta_{\max} - t_I) \times e^{|\alpha|(\Theta_{\max} - t_I)} \times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])} \times \\ &\|\mathcal{F}(\phi)\|_{L^\infty([t_I, \Theta_{\max} - \Theta_\gamma])}. \end{aligned} \quad (76)$$

According to relations (74) and (76), we obtain the following inequality:

$$\begin{aligned} \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_I, \Theta_{\max} - \Theta_\gamma])} &\leq (\Theta_{\max} - \Theta_\gamma - t_I) \times \sup_{\xi \in \mathbb{R}} \left\{ \frac{1}{|1 - \mathcal{F}(\tilde{\gamma})(\xi)|} \right\} \times \\ &(1 + |\alpha| \times (\Theta_{\max} - t_I) \times e^{|\alpha|(\Theta_{\max} - t_I)}) \times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_I, \Theta_{\max}])}. \end{aligned} \quad (77)$$

□

References

- [1] C. D. Aliprantis. *An Invitation to Operator Theory*, volume 50. American Mathematical Society, 2002.

- [2] Kendall E. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge University Press, First published 1997.
- [3] HP Baltés. Progress in inverse optical problems. In *Inverse Scattering Problems in Optics*, pages 1–13. Springer, 1980.
- [4] Tarik Chakkour and Emmanuel Frénod. Inverse problem and concentration method of a continuous-in-time financial model. *International Journal of Financial Engineering*, 3(No 2):20, 2016.
- [5] Emmanuel Frénod and Tarik Chakkour. A continuous-in-time financial model. *Mathematical Finance Letters*, 2016.
- [6] J Hadamard. Lectures on the cauchy problem in linear partial differential equations, 1923.
- [7] George W. Hanson and Alexander B. Yakovlev. *Operator Theory for Electromagnetics*. Springer Science & Business Media, 2002.
- [8] Andrej Nikolaevich Tikhonov and Vasilij Yakovlevich Arsenin. Solutions of ill-posed problems. 1977.