Central measures on multiplicative graphs, representations of Lie algebras and weight polytopes

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CENTRAL MEASURES ON MULTIPLICATIVE GRAPHS, REPRESENTATIONS OF LIE ALGEBRAS AND WEIGHT POLYTOPES

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Abstract. To each finite-dimensional representation of a simple Lie algebra is associated a multiplicative graph in the sense of Kerov and Vershik defined from the decomposition of its tensor powers into irreducible components. It was shown in [14] and [15] that the conditioning of natural random Littelmann paths to stay in their corresponding Weyl chamber is controlled by central measures on this type of graphs. Using the K-theory of associated C^*-algebras, Handelman [10] established a homeomorphism between the set of central measures on these multiplicative graphs and the weight polytope of the underlying representation. In the present paper, we make explicit this homeomorphism independently of Handelman's results by using Littelmann's path model. As a by-product we also get an explicit parametrization of the weight polytope in terms of drifts of random Littelmann paths. This explicit parametrization yields a complete description of harmonic and c-harmonic functions for this Littelmann paths model.

1. Introduction

Consider a simple Lie algebra $\mathfrak{g}$ of rank d over $\mathbb{C}$ and its root system in $\mathbb{R}^d$. Let P be the corresponding weight lattice and fix Δ a dominant Weyl chamber. Then $P_+ = P \cap \Delta$ is the cone of dominant weights of $\mathfrak{g}$. Denote by $S = \{\alpha_1, \dots, \alpha_d\}$ the underlying set of simple roots. To each dominant weight $\delta \in P_+$ corresponds a finite-dimensional representation $V(\delta)$ of $\mathfrak{g}$ of highest weight δ. In [17] Littelmann associated with $V(\delta)$ a set $B(\delta)$ of paths in $\mathbb{R}^d$ with length 1 starting at 0 with ends the set Π_δ of weights of $V(\delta)$. Random Littelmann paths can then be defined first by endowing $B(\delta)$ with a suitable probability distribution, next by considering random concatenations of paths in $B(\delta)$. In [14] and [15] distributions on the set $B(\delta)$ are defined from morphisms from P to $\mathbb{R}_{>0}$. This is equivalent to associate to each simple root α_i a real t_i in $]0, +\infty[$. It is then shown that these random paths and their conditioning to stay in the Weyl chamber Δ are controlled by the representation theory of $\mathfrak{g}$. In fact, one so obtains particular central distributions on the set $\Gamma_n(\mathbb{R}^d)$ of paths of any length $n \geq 1$ (obtained by concatenating n paths in $B(\delta)$). By central distributions we here mean that the probability of a finite path only depends on its length and its end. Equivalently, we get a central measure on the set of infinite concatenations $\Gamma(\mathbb{R}^d)$ of paths in $B(\delta)$ (see Section 2).

Write $\mathcal{H}_\infty(\mathbb{R}^d)$ for the set of central measures on $\Gamma(\mathbb{R}^d)$ and $\mathcal{H}_\infty(\Delta)$ for the subset of $\mathcal{H}_\infty(\mathbb{R}^d)$ of central measures on $\Gamma(\Delta)$, the set of infinite trajectories remaining in Δ. By Choquet Theorem both sets $\mathcal{H}_\infty(\mathbb{R}^d)$ and $\mathcal{H}_\infty(\Delta)$ are simplices so they are essentially determined by their minimal boundaries $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and $\partial\mathcal{H}_\infty(\Delta)$. Write $K(\delta)$ for the convex hull of Π_δ and set $K(\delta)^+ = \Delta \cap K(\delta)$. For walks in the Weyl chambers, the characterization of the sets $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and $\partial\mathcal{H}_\infty(\Delta)$ has been obtained by Handelman in [10] and [11] using a deepful work of Price [19, 20], by proving that they are respectively homeomorphic to $K(\delta)$ and $K(\delta)^+$. Nevertheless, the relevant homeomorphisms are not here made explicit. Their existence is established by considering the central measures as traces on certain fixed point C^*-algebras, and then using analytic tools. In particular, a central element of the proof is the

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extension of traces on C^*-algebras using K-theory (a short explanation of these arguments is given in Section 3.4).

The goal of this paper is essentially threefold: first we make explicit both homeomorphisms by using the Weyl characters of $\mathfrak{g}$ (see Theorem 3.1), next we give a purely algebraic self containing alternative proof of Handelman's results and finally we connect them with more recent studies on conditioned random walks or Brownian motions, Pitman transform generalizations and asymptotic Young tableaux (see [18], [1], [14, 15, 16], [5], [6], [23]). As a corollary of these results, we describe the set of harmonic and c-harmonic functions corresponding to the aforementioned random walks. Finally, we get a law of large numbers for random walks following the central measures obtained. Our two last results seem quite disconnected from the initial algebraic setting in representation theory, and we conjecture that they still hold for a very broad class of random paths. Our approach extends that of Kerov and Vershik to which it essentially reduces when $V(\delta)$ is the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$. Nevertheless, numerous difficulties arise when considering the general case of dominant weights of any simple algebra $\mathfrak{g}$, which explains the involved material used in the proof of Handelman. Our methods to determine $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and $\partial\mathcal{H}_\infty(\Delta)$ are quite similar. So we will now give its main steps only in the case of $\partial\mathcal{H}_\infty(\Delta)$.

We first need to show that the characterization of $\partial\mathcal{H}_\infty(\Delta)$ is equivalent to that of the extremal harmonic functions on the growth graph $\mathcal{G}(\Delta)$ associated with $\Gamma(\Delta)$. This growth diagram is rooted, graded and multiplicative: its vertices label the basis $\mathcal{B} = \{(s_\lambda, n) \mid V(\lambda)$ irreducible component of $V(\delta)^{\otimes n}$ and $n \geq 1\}$ of a commutative algebra $\hat{T}_\delta^+$ (here s_λ is the Weyl character of $V(\lambda)$). We then establish that the extremal nonnegative harmonic functions on $\mathcal{G}(\Delta)$ are in bijection with the algebra morphisms from $\hat{T}_\delta^+$ to $\mathbb{R}$ that are nonnegative on $\mathcal{B}$. Next, we prove that all these morphisms are obtained by associating to each simple root $\alpha_i, i = 1, \ldots, n$ a real in $[0, 1]$. The difficulty here comes from the fact that two such associations can yield the same morphism. So to obtain a genuine parametrization we need to restrict ourselves to a subset $[0, 1]_\delta^d$ (see (5) for a precise definition) of $[0, 1]^d$ whose combinatorial description is in terms of the δ-admissible subsets of S introduced in [22]. Finally, in Proposition 6.3, we show that our set $[0, 1]_\delta^d$ also parametrizes the simplex $K(\delta)^+$ by considering, for each d-tuple in $[0, 1]_\delta^d$, the drift of the corresponding random Littelmann path appearing in the construction of [14] and [15].

The paper is organized as follows. In Section 2, we recall some background on random chains, central measures and multiplicative graphs. We also give a generalization of a Theorem by Kerov and Vershik relating extremal harmonic functions on a multiplicative graph to nonnegative morphisms of the underlying algebra. The main result is written down in Section 3 where we also introduce the algebras $\hat{T}_\delta$ and $\hat{T}_\delta^+$; a sketch of Handelman's arguments is proposed at the end of Section 3. Section 4 gives the description of $\partial\mathcal{H}_\infty(\mathbb{R}^d)$. Here, we define our set $[0, 1]_\delta^d$ and relate it to the geometry of the polytope $K(\delta)$. The description of $\partial\mathcal{H}_\infty(\Delta)$ is deduced from that of $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ in Section 5. It is worth noticing that we need here (as in the result of Kerov and Vershik) a classical theorem relating polynomials with non positive roots to totally positive sequences. Another important ingredient in the proof is the use of certain plethyms of Schur and Weyl characters of $\mathfrak{g}$. Finally, Section 6 relates both descriptions of $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and $\partial\mathcal{H}_\infty(\Delta)$ to the drift of random Littelmann paths. Notably it explains how the polytope $K(\delta)$ can be simply parametrized by using the set $[0, 1]_\delta^d$.

2. General probabilistic framework

We present here a general probabilistic model of random paths in a domain, which is well suited to study probabilistic aspects of Littelmann paths and their asymptotics. We introduce first a discrete version of paths in a vector space.

2.1. **Random paths on a lattice.** Let $d \geq 0$ and let Λ be a lattice of $\mathbb{R}^d$.

Definition 2.1. *Let n be a nonnegative integer. A path γ on Λ is a piecewise linear function $\gamma : [0, n] \longrightarrow \mathbb{R}^d$ with $\gamma(0) = 0$, $\gamma(i) \in \Lambda$ for all $i \in \{0, ..., n\}$, and $\gamma(x) \in \Lambda$ for all x for which γ is not differentiable at x. The path γ is called infinitesimal if $t = 1$ and $\gamma(0) = 0$. The length of the path γ is defined as the length of the interval on which γ is defined and denoted by $l(\gamma)$: the path is said finite if its length is finite, and infinite otherwise.*

We denote by $\gamma.\tau$ the concatenation of two finite paths γ and τ. A path defines a sequence of vectors $(\gamma(0), \dots, \gamma(i), \dots)$ in Λ. Let $k \in \mathbb{N}$. When γ is a path of length $n \geq k$, we denote by $\gamma_{\downarrow k}$ the path $\gamma_{|[0,k]}$. Let X be a denumerable set of infinitesimal paths and let Ω be a domain of $\mathbb{R}^d$ such that $0 \in \Omega$; from now on, the set X is fixed and is not mentioned in the various notations. A path γ is called X-valued if γ is the concatenation of infinitesimal paths coming from X: equivalently, $\big(\gamma_{|[i,i+1]} - \gamma(i)\big) \in X$ for all $i \geq 1$. In the sequel, any path is always considered as X-valued. The set of X-valued paths (resp. finite X-valued paths, resp. X-valued paths of length n, with $n \in \mathbb{N} \cup \{\infty\}$) whose image is included in Ω is denoted by $\Gamma(\Omega)$ (resp. by $\Gamma_f(\Omega)$, resp. $\Gamma_n(\Omega)$). For $x, y \in \Lambda$, we denote by $\Gamma_\Omega(x, y)$ the set of infinitesimal paths $\gamma \in X$ such that $\gamma(1) = y - x$ and $x + \gamma \subset \Omega$. Finally, we denote by $\Gamma_\Omega(y, n)$ the set of finite paths of length n ending at y.

In order to consider random paths in Ω, we need to define a σ-algebra on $\Gamma(\Omega)$. Let τ be a finite rooted path of length n, and let $\Gamma_\Omega(\tau)$ be the set $\{\gamma \in \Gamma(\Omega) | l(\gamma) \geq n, \gamma_{\downarrow n} = \tau\}$. We define the σ-algebra $\mathcal{A}$ as the coarsest σ-algebra containing all the sets $\Gamma_\Omega(\tau)$ for $\tau \in \Gamma_f(\Omega)$. It is readily seen that $\Gamma_f(\Omega) \in \mathcal{A}$ and that the restriction of $\mathcal{A}$ to $\Gamma_f(\Omega)$ is the discrete σ-algebra. The set $M_1(\Gamma(\Omega))$ of probability measures on $\Gamma(\Omega)$ is considered with the initial topology with respect to the evaluation maps on the sets $\Gamma_\Omega(\tau), \tau \in \Gamma_f(\Omega)$. By Tychonov's Theorem, $M_1(\Gamma(\Omega))$ is a compact set with respect to this topology.

2.2. **Central random paths.**

Definition 2.2. *A random path ω in $\Gamma(\Omega)$ is called central if there is a function $p : \Lambda \times \Lambda \times \mathbb{N} \longrightarrow \mathbb{R}^+$ such that*

$$\mathbb{P}(\omega \in \Gamma_\Omega(\gamma)) = p(\gamma(0), \gamma(l(\gamma)), l(\gamma)),$$

for all $\gamma \in \Gamma_f(\Omega)$. A measure on $\Gamma(\Omega)$ is called central if the corresponding random path is central.

The set of central measures (resp. central measures supported on $\Gamma_f(\Omega)$, resp. central measures supported on $\Gamma_\infty(\Omega)$) is denoted by $\mathcal{H}(\Omega)$ (resp. $\mathcal{H}_f(\Omega), \mathcal{H}_\infty(\Omega)$). The sets $\mathcal{H}(\Omega)$, $\mathcal{H}_f(\Omega)$ and $\mathcal{H}_\infty(\Omega)$ are convex subsets of $M_1(\Gamma(\Omega))$. Conditioning elements of $\mathcal{H}(\Omega)$ on $\Gamma_f(\Omega)$ and $\Gamma_\infty(\Omega)$ yields that any central measure is a convex combination of central measures in $\mathcal{H}_f(\Omega)$ and $\mathcal{H}_\infty(\Omega)$. Therefore, the description of $\mathcal{H}(\Omega)$ is equivalent to the description of $\mathcal{H}_f(\Omega)$ and $\mathcal{H}_\infty(\Omega)$.

It is readily seen that there is an alternative equivalent definition of central random paths: a random path ω is central if and only if the law of $\omega_{\downarrow n}$ conditioned on the set $\{\gamma \in \Gamma(\Omega) | l(\gamma) \geq n, \gamma(n) = y\}$ is the uniform law on $\Gamma_\Omega(y, n)$. This equivalent definition gives a straightforward description of the set $\mathcal{H}_f(\Omega)$. Namely, conditioning on the last point of the random path yields that any central measure $P \in \mathcal{H}_f(\Omega)$ admits a unique decomposition

$$P = \sum_{\substack{y \in \Lambda \\ n \geq 1}} a_{y,n} P_{y,n},$$

where $a_{y,n} \geq 0$ and $P_{y,n}$ is the uniform distribution on the set $\Gamma_\Omega(y, n)$ for $y \in \Lambda$ and $n \geq 1$. On the other hand, the description of the set $\mathcal{H}_\infty(\Omega)$ is much more complicated. It is known (see the next section) that $\mathcal{H}_\infty(\Omega)$ is a convex set and even a Choquet simplex. Therefore,

there exists a subset $\partial\mathcal{H}_\infty(\Omega) \subset \mathcal{H}_\infty(\Omega)$, such that any central measure P_0 in $\mathcal{H}_\infty(\Omega)$ admits a unique integral representation

$$P_0 = \int_{\partial\mathcal{H}_\infty(\Omega)} P d\mu(P),$$

where μ is a probability measure on the set $\partial\mathcal{H}_\infty(\Omega)$. The set $\partial\mathcal{H}_\infty(\Omega)$ is called the minimal boundary of $\Gamma(\Omega)$.

2.3. **The graph embedding and Martin theory.** Let $P \in \mathcal{H}_\infty(\Omega)$. Then, by Definition 2.2 there exists a function $p : \Lambda \times \mathbb{N} \longrightarrow \mathbb{R}^+$ such that

$$P(\Gamma_\Omega(\gamma)) = p(\gamma(l(\gamma)), l(\gamma)),$$

for all $\gamma \in \Gamma_f(\Omega)$. Let $\lambda \in \Lambda$, and suppose that γ is a finite path of $\Gamma_f(\Omega)$ starting at x and ending at λ with length n. A path τ of length $n+1$ ending at $\mu \in \Lambda$ satisfies $\tau_{\downarrow n} = \gamma$ if and only if $\tau_{\downarrow n} = \gamma$ and $\tau_{[n,n+1]}$ is an infinitesimal path joining λ to μ. Therefore, $\Gamma_\Omega(\gamma)$ can be decomposed as

$$\Gamma_\Omega(\gamma) = \coprod_{\mu\in\Lambda} \coprod_{\tau\in\Gamma\lambda,\mu(\Omega)} \Gamma(\gamma.\tau).$$

Thus,

$$P\left(\Gamma_\Omega(\gamma)\right) = \sum_{\mu\in\lambda} \sum_{\tau\in\Gamma_\Omega(\lambda,\mu)} P\left(\Gamma(\gamma.\tau)\right),$$

which translates into the relation

$$p(\lambda, n) = \sum_{\mu\in\Lambda} \#\Gamma_\Omega(\lambda, \mu) p(\mu, n+1), \tag{1}$$

where in the latter equality and in the sequel of the paper the cardinality of a set X is denoted by $\#X$. The set $\mathcal{H}_\infty(\Omega)$ is in bijection with the set of nonnegative solutions of (1) with value 1 on $(0, 0)$. This equivalence leads to an alternative description of central random paths.

Definition 2.3. *The growth graph of* $\Gamma(\Omega)$ *is the rooted graded graph* $\mathcal{G}(\Omega)$ *defined recursively as follows:*

- *The root is denoted by* $(0, 0)$.
- *For each element* λ *of* Λ *such that there exists an infinitesimal path ending at* λ*, we define a vertex* $(\lambda, 1)$ *of rank* 1 *and an edge between* $(0, 0)$ *and* $(\lambda, 1)$ *with weight* $e(x, \lambda) = \#\Gamma_{0,\lambda}(\Omega)$.
- *Let* $n \geq 1$*, and suppose that the graded graph is defined up to rank* n*: the set* $\mathcal{G}_n(\Omega)$ *of vertices of rank* n *can be written as* $\{(\lambda, n)\}_{\lambda\in\Lambda_n}$*, where* Λ_n *is a subset of* Λ*. For each element* μ *of* Λ *such that there exists an infinitesimal path* γ *with* $\gamma(0) \in \Lambda_n$ *and* $\gamma(1) = \mu$*, we define a vertex* $(\mu, n+1)$ *of rank* $n+1$*. For each* $\lambda \in \Lambda_n$ *there is an edge from* (λ, n) *to* $(\mu, n+1)$ *with weight* $e(\lambda, \mu) = \#\Gamma_\Omega(\lambda, \mu)$.

We write $\lambda \nearrow \mu$ when $\#\Gamma_\Omega(\lambda, \mu) \neq 0$. It is readily seen that the number of paths between the root and (λ, n) is canonically equal to $\#\Gamma_\Omega(\lambda, n)$, and the set $\mathcal{H}_\infty(\Omega)$ is isomorphic to the set of nonnegative functions $p : \coprod_{n\geq 0} \Lambda_n \longrightarrow \mathbb{R}^+$ with $p(0, 0) = 1$ and $p(\lambda, n) = \sum_{\lambda\nearrow\mu} e(\gamma, \mu) p(\mu, n+1)$.

We conclude this subsection by establishing some connections between central measures on random paths and Markov chains on lattices. From the growth graph of $\Gamma(\Omega)$, it is clear that any central measure $P \in \mathcal{H}_\infty(\Omega)$ yields a Markov chain $Z = (Z(0), Z(1), \dots)$ on the lattice $\Lambda \cap \Omega$ with initial state 0 and with a family of Markov kernels $(Q_n)_{n\geq 1}$: the kernel Q_n can be explicitly given from the associated function $p : \coprod \Lambda_n \longrightarrow \mathbb{R}^+$ as

$$Q_n(\mu, \nu) = \mathbf{1}_{\mu\nearrow\nu, p(\mu,n-1)\neq 0} \frac{e(\mu, \nu) p(\nu, n)}{p(\mu, n-1)}.$$

By the equality $p(\mu, n-1) = \sum_{\mu \nearrow \nu} e(\mu,\nu) p(\nu, n)$, Q_n is a well-defined Markov kernel, and it is readily seen that this family of Markov kernels generates the random walk Z. Note that this random walk is generally not homogeneous in times, since the kernel Q_n depends on n through p.

2.4. **Doob conditioning and central measure.** Let $(\omega_t)_{t\geq 0}$ be the random path in $\Gamma(\mathbb{R}^d)$ defined by the Markovian evolution

$$\mathbb{P}(\omega_{|[i,i+1]} - \omega(i) = \gamma) = \frac{1}{Z}$$

where Z is the cardinality of X. Under this perspective, the central measures defined on $\Gamma(\Omega)$ in the previous section are exactly the possible ways to condition the random path ω to stay in Ω, while keeping the uniform law on each set $\Gamma_\Omega(x,n)$.

The standard procedure to achieve this is the Doob-conditioning with c-harmonic functions. Assume that h is a c-harmonic function (with $c>0$) for the random path ω killed when exiting Ω. Namely, we have

$$h(x) = \frac{1}{cZ} \sum_{\substack{\gamma \in X \\ x+\gamma \subset \Omega}} h(x+\gamma(1)).$$

Then, the Doob conditioning ω^h of ω in Ω is given by the Markov kernel

$$\mathbb{P}(\omega^h_{|[i,i+1]} - \omega^h(i) = \gamma | \omega^h(i) = x) = \mathbf{1}_{x+\gamma\subset\Omega} \frac{1}{cZ} \frac{h(x+\gamma(1))}{h(x)},$$

and we have $\mathbb{P}(\omega^h \in \Gamma_\Omega(\gamma)) = \frac{1}{(cZ)^n} h(x)$ for all $\gamma \in \Gamma_\Omega(x,n)$. The resulting random path ω^h is thus central, and the associated function p_{ω^h} is exactly $p_{\omega^h}(x,n) = \frac{1}{(cZ)^n} h(x)$.

Reciprocally, suppose that ω is a central random path with an associated function p_ω which satisfies $p_\omega(x,n) = \frac{1}{K^n} p(x)$ for some function $p : \Lambda \cap \Omega \longrightarrow \mathbb{R}^+$ and $K>0$. Then, (1) yields

$$\frac{1}{K^n} p(x) = \sum_{y \in \Lambda} \#\Gamma_\Omega(x,y) \frac{1}{K^{n+1}} p(y),$$

which is equivalent to the relation

$$p(x) = \sum_{\substack{\gamma \in X \\ x+\gamma \subset \Omega}} \frac{1}{K} p(x+\gamma(1)).$$

Hence, p is a $\frac{K}{Z}$-harmonic function.

Thus, the set of c-harmonic functions is homeomorphic to the set of central random paths $\omega \in \mathcal{H}_\infty(\Omega)$ whose associated functions p_ω have the form $p_\omega(x,n) = \frac{p(x)}{(cZ)^n}$ with $p : \Lambda \cap \Omega \longrightarrow \mathbb{R}^+$. We denote by $\mathcal{H}_c(\Omega)$ the set of central measures coming from c-harmonic functions, and by $\partial\mathcal{H}_c(\Omega)$ the set of extreme points of $\mathcal{H}_c(\Omega)$. Up to our knowledge there is no general proof that $\partial\mathcal{H}_c(\Omega) = \partial\mathcal{H}_\infty(\Omega) \cap \mathcal{H}_c(\Omega)$; in our case of study, this equality is proven by explicitly describing both sets.

We remark that the random walk Z_ω associated with a central random path $\omega \in \mathcal{H}_c(\Omega)$ is homogeneous in time. A quick computation shows that Z_ω is homogeneous in time if and only if $\omega \in \mathcal{H}_c(\Omega)$ for some $c>0$.

2.5. **Central measures on multiplicative graphs.** A rooted graded graph $\mathcal{G} = \{*\} \sqcup \coprod_{n\geq 1} \mathcal{G}_n$ with weights $(e(\lambda,\mu))_{\mu,\lambda\in\mathcal{G}}$ is called multiplicative if there is a commutative algebra A and an injective map $i : \mathcal{G} \longrightarrow A$ such that $i(\lambda) i(*) = \sum_{\lambda \nearrow \mu} e(\lambda,\mu) i(\mu)$. We suppose that the graph is connected, which means that for all $\mu \in \mathcal{G}$, the number of paths between the root and μ is positive. The weight $w(\gamma)$ of a path γ between the root and a vertex μ is the

product of all the weights of the edges of γ. Let K be the positive cone spanned by $i(\mathcal{G})$, and let $A_{\mathcal{G}}$ be the unital subalgebra of A generated by K. The following result is an application of the Ring theorem of Kerov and Vershik (see for example [8, Section 8.4]) which characterizes the extreme points of the set $\mathcal{H}(\mathcal{G})$ of solutions to the following problem:

$$\left\{\begin{array}{c} p:\mathcal{G}\longrightarrow\mathbb{R}^+ \\ p(*)=1 \\ p(\lambda)=\sum_{\lambda\nearrow\mu} e(\gamma,\mu)p(\mu). \end{array}\right. \tag{2}$$

Denote by $\mathrm{Mult}^+(A_{\mathcal{G}})\subset A_{\mathcal{G}}^*$ the set of multiplicative functions on $A_{\mathcal{G}}$ which are nonnegative on K and equal to 1 on $i(*)$. Note that $i:\mathcal{G}\longrightarrow A_{\mathcal{G}}$ induces a map $i^*:A_{\mathcal{G}}^*\longrightarrow F(\mathcal{G},\mathbb{R})$.

Proposition 2.4. *Suppose that $K.K\subset K$. Then, the map i^* yields an homeomorphism between $\mathrm{Mult}^+(A_{\mathcal{G}})$ and the set of extreme points of $\mathcal{H}(\mathcal{G})$.*

The proof of this proposition is based on the following Theorem of Kerov and Vershik:

Theorem 2.5. [8, Section 8.4] *Let B be a unital commutative algebra over $\mathbb{R}$ and $K\subset B$ a convex cone satisfying the following conditions:*

- *$K-K=B$ (K generates B).*
- *$K.K\subset K$ (K is stable by multiplication).*
- *K is spanned by a countable set of elements.*
- *For all $a\in B$, there exists $\epsilon>0$ such that $1-\epsilon a\in K$.*

If L denotes the convex set of linear forms on B which are nonnegative on K and map 1_B to 1, then ϕ is an extreme point of L if and only if ϕ is multiplicative (meaning that $\phi(ab)=\phi(a)\phi(b)$ for all $a,b\in B$).

We give now the proof of Proposition 2.4.

Proof. Let $B=A_{\mathcal{G}}/\langle i(*)=1\rangle$ and let $\pi:A_{\mathcal{G}}\longrightarrow B$ be the canonical projection; denote by $\tilde{K}$ the projection of the cone $\mathbb{R}^+\mathrm{Id}+K$ in B. Since $K.K\subset K$ and $\{1,K\}$ spans $A_{\mathcal{G}}$, $\tilde{K}.\tilde{K}\subset\tilde{K}$ and $\tilde{K}$ spans B. Since $\mathcal{G}$ has a countable set of vertices, $\tilde{K}$ is spanned by a countable set of elements. Note that there is a bijection between the elements of $\mathcal{H}(\mathcal{G})$ and the linear forms on B which are nonnegative on $\tilde{K}$ and equal to 1 on 1: indeed $h\in\mathcal{H}(\mathcal{G})$ if and only if $h(\mu)=\sum_{\mu\nearrow\nu}e(\mu,\nu)h(\nu)$. Thus, for $f\in A_{\mathcal{G}}^*$, $i^*(f)\in\mathcal{H}(\mathcal{G})$ if and only if $f(i(*)i(\mu))=f(i(\mu))$; equivalently, this means that f factors through B. Non-negativeness on $\mathcal{G}$ for $i^*(f)$ is then equivalent to nonnegativeness on $\tilde{K}$ for f, and $[i^*(f)](*)=1$ if and only if $f(\pi\circ i(*))=f(1)=1$.

Let $a\in B$, and let us show that there exists ϵ such that $1-\epsilon a\in\tilde{K}$. Since $\tilde{K}-\tilde{K}=B$, and $1-b\in\tilde{K}$ for all $b\in-\tilde{K}$, we can suppose without loss of generality that $a\in\tilde{K}$. It is thus enough to prove that for $\mu\in\mathcal{G}$, there exists ϵ such that $1-\epsilon\pi\circ i(\mu)\in K$. Suppose that μ has rank n. Since the graph is connected, there exists a path γ_0 of weight $w(\gamma_0)$ between $*$ and μ. By iteration of the relation coming from the multiplicative structure of $\mathcal{G}$, $i(*)^n=\sum_{\substack{\nu\in\mathcal{G}\\ rk(\mu)=n}}(\sum_{\gamma:*\to\mu}w(\gamma))i(\nu)$. Thus $i(*)^n-w(\gamma_0)i(\mu)$ belongs to K. Since $\pi(i(*)^n)=1$, $1-w(\gamma_0)\pi\circ i(\mu)$ belongs to $\tilde{K}$. Therefore, we can apply Theorem 2.5 to $(B,\tilde{K})$, which yields that the extreme linear maps among the set of linear maps on B which are nonnegative on $\tilde{K}$ and equal to 1 on 1 are the multiplicative ones. Since there is a bijection between multiplicative maps on B which are nonnegative on $\tilde{K}$ and multiplicative maps on $A_{\mathcal{G}}$ which are nonnegative on K and equal to 1 on $i(*)$, the proof is complete. □

3. Littelmann paths in Weyl chambers

We describe a class of random paths coming from the representation theory of semi-simple Lie groups.

3.1. **Background.** We consider a simple Lie group G over $\mathbb{C}$ and its Lie algebra $\mathfrak{g}$. Let $R \subset V$ be the set of roots of $\mathfrak{g}$ regarded as a finite subset of the euclidean vector space V. We fix R_+ a subset of positive roots and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_d\} \subset R_+$ a basis of R. The Weyl group of $\mathfrak{g}$ is denoted by W.

Write P for the weight lattice of $\mathfrak{g}$ and $\omega_1, \ldots, \omega_d$ for its fundamental weights so that we have

$$P = \bigoplus_{i=1}^{d} \mathbb{Z}\omega_i.$$

We denote by Δ the fundamental Weyl chamber of $\mathfrak{g}$ with respect to S, which corresponds to the positive orthant on the weight space $\bigoplus_{i=1}^{d} \mathbb{R}\omega_i$. The cone of dominant weights is then

$$P^+ = P \cap \Delta = \bigoplus_{i=1}^{d} \mathbb{N}\omega_i.$$

Write Q^+ the subset of P spanned by linear combinations of the simple roots with nonnegative coefficients. We denote by $\mathbb{R}[P]$ the ring group of P over $\mathbb{R}$ with basis $\{e^\beta \mid \beta \in P\}$, and by $\mathbb{R}[Q^+]$ the subalgebra of $\mathbb{R}[P]$ generated by Q^+. Then

$$\mathbb{R}^W[P] = \{u \mid w(u) = u, w \in W\}$$

is the character ring of $\mathfrak{g}$. To each $\lambda \in P_+$ corresponds a simple finite-dimensional representation of $\mathfrak{g}$ we denote by $V(\lambda)$. The Weyl character of $V(\lambda)$ is

$$s_\lambda = \sum_{\gamma \in P} K_{\lambda,\gamma} e^\gamma$$

where $K_{\lambda,\gamma}$ is the dimension of the weight space γ in $V(\lambda)$. For $\vec{t} \in (\mathbb{R}^+)^d$ and $\gamma \in \mathbb{R}^d$, set $\vec{t}^{\,\gamma} = \prod_{1 \le i \le d} \exp(\gamma_i \log(t_i))$. It is then possible to define the evaluation of s_λ on $\vec{t} \in (\mathbb{R}^+)^d$ with the formula $s_\lambda(\vec{t}) = \sum_{\gamma \in P} K_{\lambda,\gamma} \vec{t}^{\,\gamma}$. For $\mu \ge \lambda$ (that is $\mu - \lambda$ is a sum of simple roots), denote by $S_{\lambda,\mu}$ the function

$$S_{\lambda,\mu} = e^{-\mu} s_\lambda = \sum_{\gamma \in P} K_{\lambda,\gamma} e^{\gamma - \mu}$$

where for any γ such that $K_{\lambda,\gamma} > 0$, $\gamma - \mu$ is a linear combination of the simple roots with nonpositive coefficients; for $\mu = \lambda$, we simply write S_λ, instead of $S_{\lambda,\lambda}$. By setting $T_i = e^{-\alpha_i}$ we thus obtain that $S_{\lambda,\mu} = S_{\lambda,\mu}(T_1, \ldots, T_d)$ is polynomial in the variables $T_1, \ldots, T_d$ with nonnegative integer coefficients. Recall also the Weyl dimension formula

$$\dim(V(\lambda)) = \prod_{\alpha \in R_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

In particular, $\dim(V(\lambda))$ is polynomial in the coordinates of λ on the basis of fundamental weights.

3.2. **Random Littelmann paths.** Now, fix a dominant weight $\delta \in P^+$ and denote by Π_δ the set of weights of the irreducible representation $V(\delta)$. Let P_δ be the sublattice of P generated by Π_δ. This defines subalgebras

$$\mathbb{R}[P_\delta] = \{e^\beta \mid \beta \in P_\delta\} \subset \mathbb{R}[P] \text{ and } \mathbb{R}^W[P_\delta] = \{u \in \mathbb{R}[P_\delta] \mid w(u) = u\} \subset \mathbb{R}^W[P].$$

Finally write T_δ^+ the subset of P^+ of weights λ such that $V(\lambda)$ appears as an irreducible component in a tensor power $V(\delta)^{\otimes n}, n \ge 0$. Given λ and μ in T_δ^+, we clearly have $\lambda + \mu$ in T_δ^+. Moreover the $\mathbb{Z}$-lattice T_δ generated by T_δ^+ is a sublattice of P_δ. We thus have the following inclusions of $\mathbb{Z}$-lattices

$$T_\delta \subset P_\delta \subset P.$$

Since $B = (\omega_1, \dots, \omega_d)$ is a $\mathbb{Z}$-basis of P_+ there exists $(q_1, \dots, q_d) \in \mathbb{Z}_{>0}$ such that $q_{i+1} \mid q_i$ for any $i = 1, \dots, d-1$ and

$$P_\delta = \bigoplus_{i=1}^{d} \mathbb{Z}_{\geq 0} q_i \omega_i.$$

Now let $\mathcal{A}_\delta$ be the subalgebra of $\mathbb{R}^W[P]$ generated by the Weyl character s_λ with $\lambda \in T_\delta^+$. We have the inclusions

$$\mathcal{A}_\delta \subset \mathbb{R}^W[P_\delta] \subset \mathbb{R}[P_\delta] \subset \mathbb{R}[P].$$

We denote by $K(\delta)$ the convex hull of the set Π_δ: $K(\delta)$ is a polytope whose extreme points are the elements $w(\delta)$ for $w \in W$. The intersection of $K(\delta)$ with the Weyl chamber Δ is denoted by $K(\delta)^+$. By Littelmann's paths theory, there is a set $B(\delta) = \{\gamma_i\}_{1 \leq i \leq \dim V(\delta)}$ of infinitesimal paths on P_δ, with the following properties:

- $\gamma_i(1) \in \Pi_\delta$ for all $1 \leq i \leq \dim V(\delta)$.
- The multiplicity of the weight μ in $V(\delta)^{\otimes n}$ is equal to $\#\Gamma_{\mathbb{R}^d}(\mu, n)$.
- The multiplicity of the irreducible representation $V(\nu)$ in $V(\mu) \otimes V(\delta)$ is equal to $\#\Gamma_\Delta(\mu, \nu)$ and the multiplicity of the irreducible representation $V(\nu)$ in $V(\delta)^{\otimes n}$ is equal to $\#\Gamma_\Delta(\nu, n)$ for all $\mu, \nu \in P^+$ and $n \geq 0$.

The set of infinite paths we are interested in is the set of infinite paths starting at 0 with set of infinitesimal paths $B(\delta)$.

3.3. **Statements of the result.** We recall that we consider the space of probability measures on each $\Gamma(\Omega)$ with the initial topology with respect to the evaluation maps on the cylinders $\Gamma_\Omega(\tau)$, $\tau \in \Gamma_f(\Omega)$. We give an algebraic proof of the identification of the minimal boundaries for random paths in $\Gamma(\mathbb{R}^d)$ and $\Gamma_\infty(\Delta)$ with the topological spaces $K(\delta)$ and $K(\delta)^+$, respectively. In both cases, the homeomorphism can be made explicit by the introduction of a natural parametrization $t : K(\delta) \longrightarrow [0,1]^d \times W$ of $K(\delta)$ such that $t(K(\delta)^+) \subset [0,1]^d \times \mathrm{Id}_W$ (this parametrization is explained in Section 5). For $m \in K(\delta)$, we denote by $(\vec{t}_m, w_m)$ the image of m through this parametrization. The main result of the paper is summarized in the following theorem:

Theorem 3.1. *A homeomorphism between the set of extremal measures $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and $K(\delta)$ is given by the map*

$$\mathbb{P} : \begin{cases} K(\delta) & \longrightarrow & \partial\mathcal{H}_\infty(\mathbb{R}^d) \\ m & \mapsto & \mathbb{P}_m \end{cases}$$

such that $\mathbb{P}_m(\Gamma_{\mathbb{R}^d}(\gamma)) = \frac{\vec{t}_m^{N\delta - w_m(\lambda)}}{S_\delta(\vec{t}_m)}$ for all $\gamma \in \Gamma_{\mathbb{R}^d}(\lambda, N)$. A homeomorphism between the set of extremal measures $\partial\mathcal{H}_\infty(\Delta)$ and $K(\delta)^+$ is given by the map

$$\mathbb{P}^+ \begin{cases} K(\delta)^+ & \longrightarrow & \partial\mathcal{H}_\infty(\Delta) \\ m & \mapsto & \mathbb{P}_m^+ \end{cases}$$

such that $\mathbb{P}_m^+(\Gamma_\Delta(\gamma)) = \frac{S_{\lambda, N\delta}(\vec{t}_m)}{S_\delta(\vec{t}_m)^N}$ for all $\gamma \in \Gamma_\Delta(\lambda, N)$.

It is easy to see that the measures $\mathbb{P}_m$ and $\mathbb{P}_m^+$ are indeed central. Note moreover that for $m \in K(\delta)^+$, Littelmann's theory yields that for $\gamma \in \Gamma_\Delta(y, N)$,

$$\begin{aligned}\sum_{\tilde{\gamma}\in\Gamma_{N+1}(\Delta),\tilde{\gamma}_{\downarrow N}=\gamma} \mathbb{P}_m^+(\Gamma_\Delta(\tilde{\gamma})) =& \frac{\sum_{\mu\in B(\delta),\gamma.\mu\in\Gamma_\Delta(y,N+1)} S_{\gamma(N)+\mu(1),N\delta+x}(\vec{t}_m)}{S_\delta(\vec{t}_m)^{N+1}}\\ =& \frac{S_{\gamma(N),N\delta}(\vec{t}_m)S_\delta(\vec{t}_m)}{S_\delta(t_m)^{N+1}}\\ =& \frac{S_{\gamma(N),N\delta}(t_1,\dots,t_d)}{S_\delta(\vec{t}_m)^N}\\ =& \mathbb{P}_m^+(\Gamma_\Delta(\gamma)),\end{aligned}$$

so that $\mathbb{P}_m^+$ is a well defined probability measure on $\Gamma_\infty(\Delta)$. The main point of the result is to prove that $\mathbb{P}$ and $\mathbb{P}^+$ are bijective.

Remark 3.2. *In type A_d, when $\delta = \omega_1$ is the first fundamental weight, $V(\delta)$ can be regarded as the defining representation of $\mathfrak{sl}_{d+1}$ or more conveniently, of $\mathfrak{gl}_{d+1}$. The set $\partial\mathcal{H}_\infty(\Delta)$ is then homeomorphic to*

$$K(\delta)^+ = \{(p_1,\dots,p_{n+1}) \in \mathbb{R}^{d+1} \mid p_1 \geq \cdots \geq p_{n+1} \geq 0 \text{ and } p_1 + \cdots + p_{n+1} = 1\}$$

and we recover the finite-dimensional version of the Thoma simplex.

As a corollary of Theorem 3.1, we get the complete characterization of c-harmonic measures killed when exiting Δ. Define the function $\hat{s}_\delta : \partial\mathcal{H}_\infty(\Delta) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ by $\hat{s}_\delta(\mathbb{P}_m) = s_\delta(\vec{t}_m)$.

Corollary 3.3. *For $c > 0$, the set $\partial\mathcal{H}_c(\Delta)$ is homeomorphic to $\hat{s}_\delta^{-1}(\{c \dim V(\delta)\})$. In particular,*

- *$\mathcal{H}_1(\Delta)$ is a singleton corresponding to $\mathbb{P}_{\vec{0}}$,*
- *and for $c < 1$, $\mathcal{H}_c(\Delta) = \emptyset$.*

We prove Corollary 3.3 in Section 6.3. We discuss here a possible generalization of the latter result. Let X be an arbitrary set of infinitesimal paths, and let $wg : X \longrightarrow \mathbb{R}^+$ be a weight function. We denote by $\hat{Z}(\vec{t}) := \sum_{\gamma\in X} wg(\gamma)\vec{t}^{\gamma(1)}$ the partition function for this weighting, and we simply write Z for $\hat{Z}(1)$. Finally, let us fix a cone $\mathcal{C}$ centered at 0 and denote by $K_\mathcal{C}$ the set of elements $\vec{t} \in \mathbb{R}^d$ such that $\sum_{\gamma\in X} wg(\gamma)\vec{t}^{\gamma(1)}\gamma(1) \in \mathcal{C}$.
The definition of c-harmonic functions for weighted paths is similar to the one for unweighted paths; namely, a function f is c-harmonic if and only if

$$f(x) = \sum_{\substack{\gamma\in X\\ x+\gamma\subset\mathcal{C}}} \frac{1}{cZ} wg(\gamma) f(x+\gamma(1)).$$

We use the same notation as in the unweighted case, since the theory is the same in this broader situation. Then, we conjecture that the following general result holds:

Conjecture 3.4. *For $c > 0$, the set $\partial\mathcal{H}_c(\Delta)$ is homeomorphic to $\hat{Z}^{-1}(\{cZ\}) \cap K_\mathcal{C}$. In particular,*

- *for $u = \min_{K_\mathcal{C}} \hat{Z}$, $\mathcal{H}_{u/Z}(\Delta)$ is a singleton.*
- *and for $c < u/Z$, $\mathcal{H}_c(\Delta) = \emptyset$.*

This conjecture is a generalization of the conjecture of Raschel [21, Conjecture 1] for two dimensional random walks with bounded increments, which asserts that such a random walk admits a unique harmonic function killed on the boundary of a quarter plane. This special situation can be seen in the above conjecture, in which case the minimum of $\hat{Z}$ is exactly Z.

3.4. **The approach of Handelman and Price.** The existence of the homeomorphism of Theorem 3.1 can also be deduced from the main results of [10, 11], themselves based on fundamental results of [19, 20]. We review here their approach, and the reader could read the aforementioned articles and references therein for a detailed proof.

Let n denote the dimension of $V(\delta)$, and consider the adjoint representation $\pi : G \longrightarrow GL(M_n(\mathbb{C}))$ which is defined by $\pi(g)(M) = u_\delta(g)Mu_\delta(g)^{-1}$, where u_δ is the irreducible representation associated with δ. Form the infinite tensor product $A := \bigotimes M_n(\mathbb{C})$ as an inductive limit of the sequence of finite-dimensional C^*-algebras $(M_n(\mathbb{C})^{\otimes k})_{k\geq 1}$, where $M_n(\mathbb{C})^{\otimes k}$ embeds in $M_n(\mathbb{C})^{\otimes k+1}$ with the map $X \mapsto X \otimes \mathrm{Id}_n$. We can canonically associate a structure of C^*-algebra to this inductive limit of C^*-algebras. Then, G acts continuously on each $M_n(\mathbb{C})^{\otimes k}$ and on A with the map $\tilde{\pi}(g) := \bigotimes \pi(g)$ (which means that g acts as $\pi(g)$ on each component of the tensor product), and we can therefore consider the C^*-algebra A^δ (resp. A_k^δ) of elements of A (resp. $M_n(\mathbb{C})^{\otimes k}$) fixed by π. The algebra A^δ is the inductive limit of the finite-dimensional C^*-algebras $(A_k^\delta)_{k\geq 1}$, and the Bratteli diagram of this inductive limit is exactly the growth graph of $\Gamma_\infty(\Delta)$. Therefore, the set of central measures on $\Gamma_\infty(\Delta)$ is in bijection with the set of traces on A^δ.
Doing the same construction for the restriction of the representation δ to the maximal torus $T \subset G$, we get another sequence of finite dimensional C^*-algebras $(A_k^T)_{k\geq 1}$, whose inductive limit is denoted by A^T. Similarly, the Bratteli diagram of A^T is exactly the growth graph of $\Gamma_\infty(\mathbb{R}^d)$, and the set of central measures on $\Gamma_\infty(\mathbb{R}^d)$ is in bijection with the set of traces on A^T.
Note that we have the natural inclusion of C^*-algebras $A^\delta \subset A^T$. The main result of [10] is that any extreme trace on A^δ extends to an extreme trace on A^T. To prove this, the author uses the bijection between the set of traces on an approximately finite C^*-algebra A and the set of states on its associated dimension group $K_0(A)$. Let us quickly explain the nature of $K_0(A)$: a dimension group is a group with a notion of positive cone. By considering equivalence classes of projections on the $*$-algebra $\bigoplus_{k\geq 1} M_k(A)$, one can canonically associate a dimension group $K_0(A)$ to each C^*-algebra A; this dimension group is always a ring in our case. An important fact is that an inclusion of C^*-algebras induces an inclusion of the associated dimension groups, and therefore the problem reduces to extend any state on $K_0(A^\delta)$ to a state on $K_0(A^T)$. Handelman managed to prove this in [10], and the main ingredient of the proof is the non-trivial property that $K_0(A^T)$ is a finitely generated $K_0(A^\delta)$-module.
Once proven that any trace on A^δ extends to a trace on A^T, the problem amounts to describe the set of traces on A^T. In [11], the author achieves this by proving that the set of faithful traces on A^T is in bijection with the interior of $K(\delta)$. Then, the identification of the set of faithful traces on A^δ with the interior of $K(\delta)^+$ is done thanks to a result of [20], which asserts that the Weyl group W acts transitively on the set of traces extending a particular faithful trace on A^T. Finally, the case of non-faithful traces is done by considering parabolic subgroups of G.

3.5. **The extended algebra of characters.** Our proof of Theorem 3.1 will mainly use algebraic properties of the representations of the Lie algebra $\mathfrak{g}$. We define the extended algebra of characters $\hat{\mathcal{A}}_\delta$ as follows:

- $\hat{\mathcal{A}}_\delta$ is isomorphic to $\mathcal{A}_\delta \times \mathbb{R}[T]$ as a vector space; for $x \in A_\delta$, we simply denote by (x, n) the element (x, T^n). A basis of $\hat{\mathcal{A}}_\delta$ is given by the set $\mathcal{B} = \{(s_\lambda, n)\}_{n\geq 0, \lambda \in T_\delta^+}$.
- The multiplicative structure of $\hat{\mathcal{A}}_\delta$ is defined on $\mathcal{B}$ with the product

$$(s_\lambda, n) \times (s_\mu, m) = (s_\lambda s_\mu, n + m).$$

We denote by $\hat{T}_\delta^+$ the subalgebra of $\hat{\mathcal{A}}_\delta$ spanned by the set $\{(s_\lambda, n) | V(\lambda) \in V(\delta)^{\otimes n}\}$. Here $V(\lambda) \in V(\delta)^{\otimes n}$ means that $V(\lambda)$ is an irreducible component of $V(\delta)^{\otimes n}$. Likewise, we define the extended algebra of weights $\hat{P}_\delta$ as follows

- $\hat{P}_\delta$ is isomorphic to $\mathbb{R}[P_\delta] \times \mathbb{R}[T]$ as a vector space. A basis of $\hat{P}_\delta$ is given by the set $\mathcal{P} = \{(e^\gamma, n) | n \geq 0, \gamma \in P_\delta\}$.
- The multiplicative structure of $\hat{P}_\delta$ is defined on $\mathcal{P}$ with the product
$$(e^\gamma, n) \times (e^{\gamma'}, m) = (e^{\gamma+\gamma'}, n+m).$$

We denote by $\hat{T}_\delta$ the subalgebra of $\hat{P}_\delta$ spanned by the elements $\{(e^\gamma, n) | n \geq 1, K_{\delta^{\otimes n}, \gamma} > 0\}$. It is readily seen that $\hat{T}_\delta$ is the subalgebra of $\hat{P}_\delta$ generated by $\{(e^\gamma, 1) | \gamma \in \Pi_\delta\}$. Note that the inclusion $\mathcal{A}_\delta \subset \mathbb{R}[P_\delta]$ translates naturally into the inclusion $\hat{\mathcal{A}}_\delta \subset \hat{P}_\delta$ and $\hat{T}_\delta^+ \subset \hat{T}_\delta$. We can write the multiset of weights of δ in $\hat{T}_\delta$ as $\Pi_\delta = \{(e^{\gamma_1}, 1), \dots, (e^{\gamma_N}, 1)\}$ where each weight appears a number of times equal to its multiplicity. For any $k = 0, \dots, N$, let $e_k(X_1, \dots, X_N)$ be the k-th elementary symmetric function in the variables $X_1, \dots, X_N$. Define the polynomial $\Phi(X) \in \hat{T}_\delta[X]$ by
$$\Phi(X) = \prod_{\gamma \in \Pi_\delta} (X + (e^\gamma, 1)).$$

Proposition 3.5. *We have*
$$\Phi(X) = \sum_{k=0}^{N} (e_k(e^{\gamma_1}, \dots, e^{\gamma_N}), k) X^{N-k} \tag{3}$$
and for any $k = 0, \dots, N$, the expression $(e_k(e^{\gamma_1}, \dots, e^{\gamma_N}), k)$ decomposes as a sum of elements $(s_\lambda, n) \in \hat{T}_\delta^+$ with positive integer coefficients. In particular, we have $\Phi(X) \in \hat{T}_\delta^+[X]$.

Proof. The expression $e_k(e^{\gamma_1}, \dots, e^{\gamma_N})$ is the plethysm of the elementary symmetric function e_k by s_δ. This means that
$$e_k(e^{\gamma_1}, \dots, e^{\gamma_N}) = \mathrm{char}\left(\bigwedge^k V(\delta)\right)$$
is the character of the k-th exterior power of the representation $V(\delta)$. Since $\bigwedge^k V(\delta)$ is a submodule of $V(\delta)^{\otimes k}$, its character indeed decomposes as a sum of characters in $\{s_\lambda \mid V(\lambda) \in V(\delta)^{\otimes k}\}$ with positive integer coefficients. □

Corollary 3.6. *$\hat{T}_\delta^+$ is integrally closed in $\hat{T}_\delta$.*

Proof. Let $\overline{\hat{T}_\delta^+}$ denote the integral closure of $\hat{T}_\delta^+$ in $\hat{T}_\delta$. We have $\overline{\hat{T}_\delta^+} \subset \hat{T}_\delta$ by definition. Conversely, since $\overline{\hat{T}_\delta^+}$ is a ring and $\hat{T}_\delta$ is generated by the monomials $(e^\gamma, 1)$ with $\gamma \in \Pi_\delta$, it suffices to prove that each such $(e^\gamma, 1)$ belongs to $\overline{\hat{T}_\delta^+}$. But $-(e^\gamma, 1)$ is a root of $\Phi(X)$ which is, by the previous proposition, a monic polynomial with coefficients in $\overline{\hat{T}_\delta^+}$. Therefore $-(e^\gamma, 1)$ and $(e^\gamma, 1)$ are integers over $\hat{T}_\delta^+$ and thus belong to $\overline{\hat{T}_\delta^+}$. □

4. Minimal boundary of $\Gamma_\infty(\mathbb{R}^d)$

4.1. **Algebraic description of the growth graph.** Let $\mathcal{G}(\mathbb{R}^d)$ be the growth graph of $\Gamma(\mathbb{R}^d)$ and $\mathcal{G}(\Delta)$ be the one of $\Gamma(\Delta)$. Namely, the set Λ_n of vertices of rank n of the graph $\mathcal{G}(\mathbb{R}^d)$ are pairs (γ, n) where γ is a weight of P_δ such that $\Gamma_{\mathbb{R}^d}(\gamma, n) \neq \emptyset$, and the weight of the edge between (γ, n) and $(\gamma', n+1)$ is $e((\gamma, n), (\gamma', n+1)) = \#\Gamma_{\gamma,\gamma'}(\mathbb{R}^d)$. From the graph embedding of Section 1, the set of extreme central measures on $\Gamma_\infty(\mathbb{R}^d)$ is in bijection

with the set of extreme points of the convex set $\partial\mathcal{H}(\mathcal{G}(\mathbb{R}^d))$ of nonnegative functions $p : \coprod_{n\geq 0}\Lambda_n \longrightarrow \mathbb{R}^+$ with $p(0,0)=1$ and $p(\lambda,n)=\sum_{\lambda\nearrow\mu} e(\gamma,\mu)p(\mu,n+1)$, and the same holds for $\mathcal{G}(\Delta)$. An important feature of $\mathcal{G}(\mathbb{R}^d)$ is that this graded graph is multiplicative: it is related to the algebra $\hat{T}_\delta$ as follows.

Proposition 4.1. *$\mathcal{G}(\mathbb{R}^d)$ is a multiplicative graph associated with the algebra $\hat{P}_\delta$ with the injective map*

$$i : \begin{cases} \coprod_{n\geq 0}\Lambda_n & \longrightarrow & \hat{P}_\delta \\ (\gamma,n) & \mapsto & (e^\gamma,n), n\geq 1 \\ * & \mapsto & (s_\delta,1) \end{cases},$$

and $(\hat{P}_\delta)_{\mathcal{G}(\mathbb{R}^d)} = \hat{T}_\delta$. In particular, $\partial\mathcal{H}_\infty(\mathcal{G}(\mathbb{R}^d))$ is homeomorphic to $\mathrm{Mult}(\hat{T}_\delta)^+$ through the map

$$i^* : \begin{cases} \mathrm{Mult}(\hat{T}_\delta)^+ & \longrightarrow & \partial\mathcal{H}_\infty(\mathcal{G}(\mathbb{R}^d)) \\ f & \mapsto & f\circ i \end{cases}$$

Proof. Since $e((\gamma,n),(\gamma',n+1)) = \#\Gamma_{\mathbb{R}^d}(\gamma,\gamma') = K_{\delta,\gamma'-\gamma}$, the following equality holds for $(\gamma,n)\in\Lambda_n$:

$$\begin{aligned} i(\gamma,n)i(*) =& (e^\gamma,n)(\sum_{\kappa\in\Pi_\delta} K_{\delta,\kappa}e^{\gamma'},1) = \sum_{\kappa\in\Pi_\delta} K_{\delta,\kappa}(e^{\gamma+\kappa},n+1) \\ =& \sum_{\gamma'\in P_\delta,\gamma'-\gamma\in\Pi_\delta} K_{\delta,\gamma'-\gamma}(e^{\gamma'},n+1) \\ =& \sum_{\gamma'\in P_\delta} e((\gamma,n),(\gamma',n+1))i(\gamma',n+1). \end{aligned}$$

Thus, $\mathcal{G}(\mathbb{R}^d)$ is a multiplicative graph associated with $\hat{P}_\delta$ through the map i. Note that by construction, the sub-algebra of $\hat{P}_\delta$ generated by the elements $\{i(\gamma,n)\}_{(\gamma,n)\in\mathcal{G}(\mathbb{R}^d)}$ is precisely $\hat{T}_\delta$: the last part of the proposition is deduced from Proposition 2.4. □

4.2. **Characterization of the multiplicative maps on $\hat{T}_\delta$.** The set of extreme central measures on $\mathcal{G}(\mathbb{R}^d)$ is thus given by the set of positive morphisms from $\hat{T}_\delta$ to $\mathbb{R}$ which take the value 1 on $(s_\delta,1)$. We will prove in this subsection the following result:

Proposition 4.2. *Let $f\in\mathrm{Mult}(\hat{T}_\delta)^+$. There exists a multiplicative map $\phi : \mathbb{R}[Q^+]\longrightarrow\mathbb{R}^+$ and an element $w\in W$ such that*

$$f(e^\gamma,n) = \frac{1}{\phi(S_\delta)^n}\phi(e^{n\delta-w(\gamma)}),$$

for all $(e^\gamma,n)\in\hat{T}_\delta$.

Note that the element $\phi(e^{n\delta-w(\gamma)})$ is well-defined: indeed, if $(e^\gamma,n)\in\hat{T}_\delta$, then the weight γ appears in the representation $V(\delta)^{\otimes n}$ and $w(\gamma)$ is thus smaller than $n\delta$ with respect to the roots order relative to the set of simple roots S. Therefore, $n\delta-w(\gamma)\in Q^+$.

Let f be a multiplicative map on $\hat{T}_\delta$. Since f is multiplicative and $\hat{T}_\delta$ is generated by the set $\tilde{\Pi}_\delta := \{(e^\gamma,1),\gamma\in\Pi_\delta\}$, f is completely determined by its value on $\tilde{\Pi}_\delta$. We suppose from now on that $f\in\mathrm{Mult}(\hat{T}_\delta)^+$. Let $M_f = \sum_{\gamma\in\Pi_\delta} K_{\delta,\gamma}f(\gamma,1)\gamma$: M_f belongs to $\mathbb{R}^d$, thus there exists $w\in W$ such that $w(M_f)\in\Delta$. Replacing f by $f\circ w^{-1}$ gives another multiplicative map on $\hat{T}_\delta$ such that $M_{f\circ w^{-1}} = \sum_{\gamma\in\Pi_\delta} K_{\delta,\gamma}(f\circ w^{-1})(e^\gamma,1)\gamma\in\Delta$ and such that f can be expressed from $f\circ w^{-1}$ with the formula $f=(f\circ w^{-1})\circ w$.

Lemma 4.3. *Assume that $M_f \in \Delta$ and let $\alpha \in S$. For all $\gamma \in \Pi_\delta$ such that $\gamma - \alpha \in \Pi_\delta$,*

$$f(e^\gamma, 1) = 0 \Longrightarrow f(e^{\gamma-\alpha}, 1) = 0.$$

In particular, $f(e^\delta, 1) \neq 0$.

Proof. Let $\alpha \in S$, and suppose that there exists $\gamma \in \Pi_\delta$ such that $\gamma - \alpha \in \Pi_\delta, f(e^\gamma, 1) = 0$ and $f(e^{\gamma-\alpha}, 1) \neq 0$. If γ' is another vector of Π_δ such that $f(e^{\gamma'}, 1) \neq 0$, then necessarily $\gamma' - \alpha \notin \Pi_\delta$: indeed, if $\gamma' - \alpha \in \Pi_\delta$, then

$$f(e^{\gamma'-\alpha}, 1)f(e^\gamma, 1) = f(e^{\gamma+\gamma'-\alpha}, 2) = f(e^{\gamma-\alpha}, 1)f(e^{\gamma'}, 1) \neq 0,$$

which contradicts the fact that $f(e^\gamma, 1) = 0$. For all $\gamma' \in \Pi_\delta$, $\gamma' - \alpha \notin \Pi_\delta$ implies that $\langle \gamma', \alpha \rangle \leq 0$: thus, $f(e^{\gamma'}, 1) \neq 0$ implies that $\langle \gamma', \alpha \rangle \leq 0$. We get

$$\langle \sum_{\gamma' \neq \gamma - \alpha} K_{\delta,\gamma'} f(e^{\gamma'}, 1)\gamma', \alpha \rangle \leq 0.$$

Since $f(e^{\gamma-\alpha}, 1) \neq 0$ and from the previous argument, we get $\gamma - 2\alpha \notin \Pi_\delta$. Hence, $\frac{2\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle} \leq 1$, which yields $\langle \gamma - \alpha, \alpha \rangle < 0$. Finally,

$$\langle M, \alpha \rangle = K_{\delta,\gamma-\alpha} f(e^{\gamma-\alpha}, 1)\langle \gamma - \alpha, \alpha \rangle + \langle \sum_{\gamma' \neq \gamma - \alpha} K_{\delta,\gamma'} f(e^{\gamma'}, 1)\gamma', \alpha \rangle < 0,$$

which contradicts the fact that $M \in \Delta$. Let $\gamma \in \Pi_\delta$ be such that $f(e^\gamma, 1) \neq 0$. Since $\gamma \in \Pi_\delta$, there exists a finite sequence $(x_i)_{1 \leq i \leq r}$ in S such that $\delta - \sum_{i=1}^j x_i \in \Pi_\delta$ for all $1 \leq j \leq r$ and $\delta - \sum_{i=1}^r x_i = \gamma$. Thus, from the first part of the lemma, $f(e^{\delta - \sum_{i=1}^j x_i}, 1) \neq 0$ for all $1 \leq j \leq r$; in particular, $f(e^{\delta - x_1}, 1) \neq 0$, and applying again the first part of the lemma yields that $f(e^\delta, 1) \neq 0$. □

We can now prove Proposition 4.2:

Proof of Proposition 4.2. Let $f \in \mathrm{Mult}(\hat{T}_\delta)^+$ be such that $M_f \in \Delta$. Let $\alpha \in S$. If for all $\gamma \in \Pi_\delta$ such that $f(e^\gamma, 1) \neq 0$ we have $\gamma - \alpha \notin \Pi_\delta$, then set $\phi(e^\alpha) = 0$. Otherwise, let $\gamma \in \Pi_\delta$ be such that $f(e^\gamma, 1) \neq 0$ and such that $\gamma - \alpha \in \Pi_\delta$, and set $\phi(e^\alpha) = \frac{f(e^{\gamma-\alpha}, 1)}{f(e^\gamma, 1)}$. Then, $\phi(\alpha)$ is independent of the choice of γ. Indeed, if γ' is another weight satisfying the same hypothesis, then

$$f(e^\gamma, 1)f(e^{\gamma'-\alpha}, 1) = f(e^{\gamma+\gamma'-\alpha}, 2) = f(e^{\gamma-\alpha}, 1)f(e^{\gamma'}, 1),$$

so that finally,

$$\frac{f(e^{\gamma-\alpha}, 1)}{f(e^\gamma, 1)} = \frac{f(e^{\gamma'-\alpha}, 1)}{f(e^{\gamma'}, 1)}.$$

Note that we have in particular proven that for all $\gamma \in \Pi_\delta$ such that $\gamma + \alpha \in \Pi_\delta$ and $f(e^{\gamma+\alpha}, 1) \neq 0$, we have

$$\frac{f(e^\gamma, 1)}{f(e^{\gamma+\alpha}, 1)} = \phi(e^\alpha). \tag{4}$$

Let $\phi : \mathbb{R}[Q^+] \longrightarrow \mathbb{R}^+$ be the multiplicative map obtained by extending multiplicatively the map ϕ defined on $\{e^\alpha, \alpha \in S\}$ and by specifying the value $\phi(1) = 1$. Consider the roots order with respect to the set of simple roots S and let us prove by induction on the roots order that $f(e^\gamma, 1) = f(e^\delta, 1)\phi(e^{\delta-\gamma})$ for $\gamma \in \Pi_\delta$. For $\gamma = \delta$ the result is straightforward. Let $\gamma \in \Pi_\delta$ and suppose that the result is true for all $\gamma' > \gamma$. There exists $\alpha \in S$ such that $\gamma + \alpha \in \Pi_\delta$.

If $f(e^{\gamma+\alpha},1)=0$, then $f(e^{\gamma},1)=0$ by Lemma 4.3; in particular, $f(e^{\gamma},1)=\phi(e^{\alpha})f(e^{\gamma+\alpha},1)$. By the induction hypothesis, $f(e^{\gamma+\alpha},1)=f(e^{\delta},1)\phi(e^{\delta-(\gamma+\alpha)})$, and finally,

$$f(e^{\gamma},1)=\phi(e^{\alpha})f(e^{\delta},1)\phi(e^{\delta-(\gamma+\alpha)})=f(e^{\delta},1)\phi(e^{\delta-\gamma}).$$

If $f(e^{\gamma+\alpha},1)\neq 0$, then by (4) and by the induction hypothesis,

$$f(e^{\gamma},1)=\phi(e^{\alpha})f(e^{\gamma+\alpha},1)=\phi(e^{\alpha})f(e^{\delta},1)\phi(e^{\delta-(\gamma+\alpha)})=f(e^{\delta},1)\phi(e^{\delta-\gamma}).$$

Let $(\gamma,n)\in\hat{T}_{\delta}$, and let $\gamma_1,\dots,\gamma_n\in\Pi_{\delta}$ such that $\gamma=\sum_{i=1}^{n}\gamma_i$. Then, by multiplicativity of f and the result above, we have

$$\begin{aligned}f(e^{\gamma},n)=&f(e^{\sum_{i=1}^{n}\gamma_i},n)=\prod_{i=1}^{n}f(e^{\gamma_i},1)=\prod_{i=1}^{n}f(e^{\delta},1)\phi(e^{\delta-\gamma_i})\\=&f(e^{\delta},1)^n\phi(e^{n\delta-\sum_{i=1}^{n}\gamma_i})=f(e^{\delta},1)^n\phi(e^{n\delta-\gamma})\end{aligned}$$

Since $f(s_{\delta},1)=1$, we have on the one hand

$$\sum_{\gamma\in\Pi_{\delta}}K_{\delta,\gamma}f(e^{\gamma},1)=1.$$

On the other hand, from the result above,

$$\sum_{\gamma\in\Pi_{\delta}}K_{\delta,\gamma}f(e^{\gamma},1)=\sum_{\gamma\in\Pi_{\delta}}K_{\delta,\gamma}f(e^{\delta},1)\phi(e^{\delta-\gamma})=f(e^{\delta},1)\phi(S_{\delta}).$$

Thus, $f(e^{\delta},1)=\frac{1}{\phi(S_{\delta})}$, which ends the proof of the proposition in the case $M_f\in\Delta$. Suppose that f is a general nonnegative multiplicative function on $\hat{T}_{\delta}$. Let $w\in W$ be such that $M_{f\circ w^{-1}}\in\Delta$. By the first part of the proof, there exists $\phi\in\mathrm{Mult}(\mathbb{R}[Q+])^{+}$ such that $f\circ w^{-1}(e^{\gamma},n)=\frac{1}{\phi(S_{\delta})^n}\phi(e^{n\delta-\gamma})$. Thus, composing $f\circ w^{-1}$ with w yields that $f(e^{\gamma},n)=\frac{1}{\phi(S_{\delta})^n}\phi(e^{n\delta-w(\gamma)})$ for $(\gamma,n)\in\hat{T}_{\delta}$. □

Remark 4.4. *Suppose that $\phi(e^{\alpha})\neq 0$ for all $\alpha\in S$. Then, the map ϕ extends to a homomorphism $\phi:\mathbb{R}[P]\longrightarrow\mathbb{R}^{+}$ with the formula*

$$\phi(e^{\gamma})=\prod_{\alpha\in S}\phi(e^{\alpha})^{r_{\alpha}}\ \text{for}\ \gamma=\sum_{\alpha\in S}r_{\alpha}\alpha.$$

In this case,

$$f(e^{\gamma},n)=f(e^{\delta},1)^n\phi(e^{\lambda-n\delta})=\left(\frac{f(e^{\delta},1)}{\phi(e^{\delta})}\right)^{n}\phi(e^{\gamma}).$$

Since, $f(s_{\delta},1)=1$, $\frac{f(e^{\delta},1)}{\phi(e^{\delta})}=\phi(s_{\delta})^{-1}$. Hence, when $\phi(e^{\alpha})>0$ for all $\alpha\in S$, f can be written on $\hat{T}_{\delta}$ as

$$f(e^{\gamma},n)=\frac{\phi(e^{\gamma})}{\phi(s_{\delta})^n},$$

with $\phi:P\longrightarrow\mathbb{R}^{+}$ a multiplicative map.
In this case, for $(s_{\lambda},n)\in\hat{T}_{\delta}^{+}$, we have also

$$f(s_{\lambda},n)=\frac{\phi(s_{\lambda})}{\phi(s_{\delta})^n},$$

where the restriction $\phi:\mathcal{A}_{\delta}\longrightarrow\mathbb{R}^{+}$ is again a multiplicative map.

To summarize, let us define the map $\Phi : \mathrm{Mult}(\mathbb{R}[Q^+])^+ \times W \longrightarrow \mathrm{Mult}(\hat{T}_\delta)^+$ by

$$\Phi(\phi, w)(e^\gamma, n) = \frac{1}{\phi(S_\delta)}\phi(e^{n\delta - w(\gamma)}).$$

Proposition 4.2 yields that the map Φ is surjective. Since $\mathbb{R}[Q^+]$ is the free commutative algebra generated by $\{e^\alpha, \alpha \in S\}$, $\mathrm{Mult}(\mathbb{R}[Q^+])^+$ is isomorphic to $(\mathbb{R}^+)^d$ through the map $\theta : \mathrm{Mult}(\mathbb{R}[Q^+])^+ \longrightarrow (\mathbb{R}^+)^d$ given by $\theta(\phi) = (\phi(e^{\alpha_i}))_{1\leq i\leq d}$ for $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$. The composition of Φ with θ^{-1} yields thus a surjective map $(\mathbb{R}^+)^d \times W \longrightarrow \mathrm{Mult}(\hat{T}_\delta)^+$. Since Φ is not necessarily injective, the latter map is not bijective. The lack of injectivity comes from two facts: first, if M_f lies at the intersection of two Weyl chambers, then $M_{f\circ w^{-1}} \in \Delta$ for several $w \in W$. Secondly, some degeneracy may occur when δ is orthogonal to some simple roots. The goal of the next subsection is to overcome the second problem.

4.3. **Dominant faces of the weight polytope.** Let $f \in \mathrm{Mult}(\hat{T}_\delta)$ such that $M_f \in \Delta$; it is possible to give a geometric description of the set $\Pi_\delta(f) := \{\gamma \in \Pi_\delta \mid f(e^\gamma, 1) \neq 0)\}$. A dominant face F is a face of the polytope $K(\delta)$ such that $F \cap \Delta \neq 0$. We denote by Π_F the intersection of Π_δ with F. We say that a subset $S' \subset S$ of simple roots is δ-admissible if each indecomposable component of S' contains a root which is not orthogonal to δ; in particular, according to this definition, the empty set is a δ-admissible subset, since it has no indecomposable component. For each subset $S' \subset S$, denote by $W_{S'}$ the Weyl group generated by the elements $s_{\alpha'}, \alpha' \in S'$ (where $W_\emptyset$ is simply $\{\mathrm{Id}\}$). We will use the following results which comes from [22].

Theorem 4.5. *Assigning to each δ-admissible subset $S' \subset S$ the polytope $F_{S'} = \mathrm{Conv}(w'\delta \mid w' \in W_{S'})$ yields a one-to-one correspondence between δ-admissible subsets of S and dominant faces of the polytope $K(\delta)$. Moreover, the set $\Pi_{F_{S'}}$ coincides with the set $(\delta + \langle S'\rangle) \cap \Pi_\delta$ and $\dim F_{S'} = \#S'$*

This yields the following characterization $\Pi_\delta(f)$.

Proposition 4.6. *There exists a dominant face F of the weight polytope $K(\delta)$ such that $\Pi_\delta(f) = \Pi_F$.*

Before proving Proposition 4.6, let us prove the following lemma:

Lemma 4.7. *Let $S' \subset S$ and $\gamma \in \Pi_\delta$ such that $\delta - \gamma = \sum_{\alpha\in S'} k_\alpha\alpha$ with $k_\alpha > 0$ for all $\alpha \in S'$. Then, S' is δ-admissible and $\gamma \in F_{S'}$.*

The proof of this lemma uses ingredients similar to those of Vinberg in [22].

Proof. Suppose γ can be written as

$$\gamma = \delta - \sum_{\alpha\in S'} k_\alpha\alpha,$$

with S' a subset of S and $k_\alpha \in \mathbb{N}^*$ for $\alpha \in S'$. Since $\gamma \in \Pi_\delta$, there exists a sequence $(\gamma_i)_{0\leq i\leq t}$ with $t = \sum_{\alpha\in S'} k_\alpha$ such that $\gamma_i \in \Pi_\delta$, $\gamma_0 = \gamma$, $\gamma_t = \delta$ and $\gamma_{i+1} - \gamma_i \in S$. Since for all $\gamma \in \Pi_\delta$, $\delta - \gamma$ is a sum of simple roots with nonnegative coefficients, for all $0 \leq i \leq t-1$ we have $\gamma_{i+1} - \gamma_i \in S'$ and $\#\{0 \leq i \leq t-1 | \gamma_{i+1} - \gamma_i = \alpha\} = k_\alpha$ for $\alpha \in S'$. This implies in particular that $\gamma_i \in \delta + \langle S'\rangle$ for all $0 \leq i \leq t$. Let $\alpha \in S'$: since $k_\alpha > 0$, there exists $0 \leq i_\alpha \leq t-1$ such that $\gamma_{i+1} - \gamma_i = \alpha$. This yields that $\dim(K(\delta) \cap (\delta + \langle S'\rangle) = \#S'$. Let l be the linear form such that $l(\alpha) = 1$ for $\alpha \in S \setminus S'$ and $l(\alpha) = 0$ for $\alpha \in S$. For $\gamma \in \Pi_\delta$, $\delta - \gamma$ is a sum of simple roots with positive coefficients, thus $l(\gamma) \leq l(\delta)$, with equality if and only if $\gamma \in \delta + \langle S'\rangle$. Thus, $(K(\delta) \cap (\delta + \langle S'\rangle)$ is a face of the polytope K_δ; since $\dim(K(\delta) \cap (\delta + \langle S'\rangle) = \#S'$, S' is δ-admissible by [22, p.10]. Finally, $K(\delta) \cap (\delta + \langle S'\rangle) = F_{S'}$ and $\gamma \in F_{S'}$. □

Lemma 4.8. *$\Pi_F \subset \Pi_\delta(f)$ if and only if ϕ is nonzero on S_F.*

Proof. Suppose that $\Pi_F \subset \Pi_\delta(f)$. Let $\alpha_0 \in S_F$. Since $f(e^\gamma, 1)$ is nonzero for $\gamma \in \Pi_F$, by Lemma 4.3 and the definition of ϕ it suffices to prove that there exists $\gamma \in \Pi_F$ such that $\gamma + \alpha_0 \in \Pi_F$ or $\gamma - \alpha_0 \in \Pi_F$. Since S_F is δ-admissible, $\dim F = \#S_F$; $F = \mathrm{Conv}(w.\delta \mid w \in W_{S_F})$ and $\dim F = \#S_F$, thus there exists $w \in W_F$ such that $\delta - w.\delta = \sum_{\alpha \in S_F} k_\alpha \alpha$ with $k_{\alpha_0} > 0$. This implies the existence of $\gamma \in \Pi_F$ such that $\gamma + \alpha_0 \in \Pi_F$. Since $\Pi_F \subset \Pi_\delta(f)$, $f(e^{\gamma+\alpha_0}, 1) \neq 0$ and $f(e^\gamma, 1) \neq 0$, and thus

$$\phi(\alpha_0) = \frac{f(e^\gamma, 1)}{f(e^{\gamma+\alpha_0}, 1)} \neq 0.$$

Conversely, suppose that ϕ is nonzero on S_F. By Theorem 4.5, $\Pi_F = (\delta + \langle S_F \rangle) \cap \Pi_\delta$. Since $f(e^\delta, 1) \neq 0$ and ϕ is nonzero on S_F, f is nonzero Π_F by Proposition 4.2. □

We turn now to the proof of Proposition 4.6.

Proof. We order the set of dominant faces by the inclusion order; note that the set of dominant faces is a lattice with respect to this order, and we denote by $F \vee F'$ the supremum of two dominant faces F and F': $F \vee F'$ is the smallest dominant face containing both F and F'. Let $\gamma \in \Pi_\delta$ such that $f(e^\gamma, 1) \neq 0$, and let F be the smallest dominant face containing γ. We denote by S_F the δ-admissible subset of simple roots corresponding to F. Then, γ can be written as

$$\gamma = \delta - \sum_{\alpha \in S_F} k_\alpha \alpha$$

with $k_\alpha \in \mathbb{N}$. Necessarily, we have $k_\alpha > 0$ for all $\alpha \in S_F$. Otherwise, Lemma 4.7 would imply that γ belongs to a smaller dominant face of $K(\delta)$. Let $(\gamma_i)_{0 \leq i \leq t}$ with $t = \sum_{\alpha \in S'} k_\alpha$ be a sequence of Π_δ such that $\gamma_i \in \Pi_\delta$, $\gamma_0 = \gamma$, $\gamma_t = \delta$ and $\gamma_{i+1} - \gamma_i \in S_F$. Since $f(e^{\gamma_0}, 1) \neq 0$, Lemma 4.3 yields that $f(e^{\gamma_i}, 1) \neq 0$ for $0 \leq i \leq t$. Let $\alpha \in S_F$: since $k_\alpha > 0$, a similar deduction as in the proof of the previous lemma yields that there exists $0 \leq i \leq t-1$ such that $\gamma_{i+1} - \gamma_i = \alpha$. Therefore,

$$\phi(e^\alpha) = \frac{f(e^{\gamma_i}, 1)}{f(e^{\gamma_{i+1}}, 1)} \neq 0.$$

Since $\phi(e^\alpha) \neq 0$ for $\alpha \in S_F$, $f(e^\gamma, 1)$ is nonzero on $\Pi_\delta \cap (\delta - \langle S_F \rangle)$, and $\Pi_F \subset \Pi_\delta(f)$. We have thus proven that if a weight γ is in $\Pi_\delta(f)$, then the intersection of Π_δ with the smallest dominant face containing γ is also included in $\Pi_\delta(f)$; hence, $\Pi_\delta(f)$ is an union of sets Π_F, where F are dominant faces.

Let F and F' be two dominant faces such that $\Pi_F, \Pi_{F'} \subset \Pi_\delta(f)$, and let us show that $\Pi_{F \vee F'} \subset \Pi_\delta(f)$. Note first that $F \vee F' = F_{S_F \cup S_{F'}}$: on the first hand, the smallest vector space containing both $\langle S_F \rangle$ and $\langle S_{F'} \rangle$ is $\langle S_F \cup S_{F'} \rangle$. On the other hand, since S_F and $S_{F'}$ are δ-admissible, $S_F \cup S_{F'}$ is again δ-admissible. It suffices thus to show that $\Pi_{F_{S_F \cup S_{F'}}} \subset \Pi_\delta(f)$. But Lemma 4.8 yields that $\phi(e^\alpha)$ is nonzero for $\alpha \in S_F$ and $\alpha \in S_{F'}$. Thus, $\phi(e^\alpha)$ is nonzero for $\alpha \in S_F \cup S_{F'}$ and $\Pi_{F_{S_F \cup S_{F'}}} \subset \Pi_\delta(f)$. Let F_0 be the supremum of $\{F \text{ dominant face of } K(\delta), \Pi_F \subset \Pi_\delta(f)\}$. By the previous argument, $\Pi_{F_0} \subset \Pi_\delta(f)$. Let $\gamma \in \Pi_\delta(f)$ and let F be the smallest dominant face of $K(\delta)$ containing γ. By the first part of the proof, $\Pi_F \subset \Pi_\delta(f)$. Thus $F \subset F_0$ and $\gamma \in F_0$: this proves that $\Pi_\delta(f) \subset \Pi_{F_0}$, and finally $\Pi_\delta(f) = \Pi_{F_0}$. □

Corollary 4.9. *Let $f \in \mathrm{Mult}(\hat{T}_\delta)^+$ be such that $M_f \in \Delta$. There exists a unique $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ such that $\Phi(\phi, \mathrm{Id}) = f$ and $\{\alpha, \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset of S.*

Proof. Let ϕ be such that $\Phi(\phi, \mathrm{Id}) = f$. By Proposition 4.6, there exists a face F of $K(\delta)$ such that $\Pi_F = \Pi_\delta(f)$. Lemma 4.8 yields that ϕ is nonzero on S_F. Let $\alpha \in S_F$: then, there exists $\gamma \in \Pi_\delta$ such that $\gamma \in \Pi_F, \gamma - \alpha \in \Pi_F$; thus, $f(e^\gamma, 1) \neq 0$ and $f(e^{\gamma-\alpha}, 1) \neq 0$. Therefore, the value of ϕ on α has to be equal to $\frac{f(e^{\gamma-\alpha},1)}{f(e^\gamma,1)}$. Hence, there exists at most one ϕ such that $\{\alpha, \phi(e^\alpha) \neq 0\}$ is the δ-admissible subset S_F. Such a map ϕ exists, since f is zero on $\Pi_\delta \setminus \Pi_\delta(f)$. Suppose that there exists a bigger δ-admissible subset $S_F \subsetneq S'$ such that ϕ is nonzero on S'. Then by Lemma 4.8, $\Pi_{F_{S'}} \subset \Pi_\delta(f)$. But by Theorem 4.5, there is a bijection between dominant faces and δ-admissible subsets: therefore, $\Pi_\delta(f) = \Pi_F \subsetneq \Pi_{F_{S'}} \subset \Pi_\delta(f)$, which is a contradiction. Thus, there exists exactly one map ϕ such that $\Phi(\phi, \mathrm{Id}) = f$ and $\{\alpha \in S, \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset (and this δ-admissible subset has to be S_F). □

4.4. **Identification of the minimal boundary.** We give in this subsection a complete description of the minimal boundary by describing $\mathrm{Mult}(\hat{T}_\delta)^+$.

Lemma 4.10. *Let $f \in \mathrm{Mult}(\hat{T}_\delta)^+$ be such that $M_f \in \Delta$, and let $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ be such that $\Phi(\phi, \mathrm{Id}) = f$. Then $\phi(e^\alpha) \in [0, 1]$.*

Proof. Let $f \in \mathrm{Mult}(\hat{T}_\delta)^+$ be such that

$$M = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\gamma, 1)\gamma \in \Delta.$$

Let $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ be a morphism associated with f by Proposition 4.2, and let $\alpha \in S$. Since $M \in \Delta$, $\langle M, \alpha\rangle \geq 0$. Moreover,

$$\langle M, \alpha\rangle = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\delta, 1)\phi(e^{\delta-\gamma})\langle\gamma, \alpha\rangle.$$

By invariance of Π_δ under the symmetry s_α, $\gamma \in \Pi_\delta$ implies that $s_\alpha(\gamma) \in \Pi_\delta$. Since $s_\alpha^2 = \mathrm{Id}$ and since $s_\alpha(\gamma) = \gamma$ if and only if$\langle\alpha, \gamma\rangle = 0$, we have

$$\begin{aligned}\langle M, \alpha\rangle =& f(e^\delta, 1) \sum_{\substack{\gamma \in \Pi_\delta \\ \langle\gamma,\alpha\rangle > 0}} K_{\delta,\gamma}(\phi(e^{\delta-\gamma}) - \phi(e^{\delta - s_\alpha(\gamma)}))\langle\gamma, \alpha\rangle \\ =& f(e^\delta, 1) \sum_{\substack{\gamma \in \Pi_\delta \\ \langle\gamma,\alpha\rangle > 0}} K_{\delta,\gamma}\phi(e^{\delta-\gamma})(1 - \phi(e^\alpha)^{\frac{2\langle\gamma,\alpha\rangle}{\langle\alpha,\alpha\rangle}})\langle\gamma, \alpha\rangle.\end{aligned}$$

If $\phi(e^\alpha) > 1$, then $(1 - \phi(e^\alpha)^{\frac{2\langle\gamma,\alpha\rangle}{\langle\alpha,\alpha\rangle}}) < 0$ for all $\gamma \in \Pi_\delta$ such that $\langle\gamma, \alpha\rangle > 0$, and thus $\langle M, \alpha\rangle < 0$: this would contradict the choice of f. Therefore, $\phi(e^\alpha) \leq 1$. □

The set $\{1, \ldots, d\}$ is identified with S by ordering the set of simple roots, and for $S' \subset S$, we denote by $W^{S'}$ the set of minimal right-coset representatives with respect to S': namely,

$$W^{S'} = \{w \in W | l(sw) > l(w) \text{ for } s \in S'\}.$$

For $x \in \Delta$, we denote by S_x the set $\{\alpha \in S, \langle\alpha, x\rangle = 0\}$.

Lemma 4.11. *Let $x \in \mathbb{R}^d$ and let y be the unique element of Wx belonging to Δ. There exists a unique element $w \in W^{S_y}$ such that $wx = y$.*

Proof. Let W_y be the parabolic subgroup generated by S_y. Then, W_y is the stabilizer of y. In particular, the set $\{w \in W, w(y) = x\}$ is a left coset of W_y in W, and thus the set $\{w \in W, w(x) = y\}$ is a right coset of W_y in W. By [12, 1.10], there exists a unique $\tilde{w} \in W^{S_y}$ such that $\{w \in W, w(x) = y\}\tilde{w}wW_y$. Thus, there exists a unique $\tilde{w} \in W^{S_y}$ such that $\tilde{w}(x) = y$. □

For each d-tuple $\vec{t} = (t_1, \dots, t_d)$, denote by $\mathbf{0}^c(\vec{t})$ the set of indices i such that $t_i \neq 0$ and by $\mathbf{1}(\vec{t})$ the set of indices i such that $t_i = 1$. Now consider the set $[0,1]^d_\delta$ such that

$$[0,1]^d_\delta := \{\vec{t} \in [0,1]^d \mid \mathbf{0}^c(\vec{t}) \text{ is } \delta\text{-admissible}\}. \tag{5}$$

This set will turn out to be a natural parametrization of $K(\delta)^+$. Then, we will prove in Section 6 that there exists a natural map $t : K(\delta) \longrightarrow [0,1]^d_\delta \times W$, written as $t(m) = (\vec{t}_m, w_m)$, such that $t(K(\delta)^+) = [0,1]^d_\delta \times \mathrm{Id}$.

Proposition 4.12. *The map $\Phi \circ \theta^{-1}$ yields a bijection Ψ between* $\mathrm{Mult}(\hat{T}_\delta)^+$ *and*

$$\{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\}.$$

Proof. Let $f \in \mathrm{Mult}(\hat{T}_\delta)^+$. Let y be the unique point in $W(M_f) \cap \Delta$ and denote by S_y the set $\{\alpha \in S \mid \langle \alpha, y\rangle = 0\}$. By Lemma 4.11, there exists a unique $w \in W^{S_y}$ such that $w(M_f) = y$. Thus, by Proposition 4.2, Corollary 4.9 and Lemma 4.10, there exists a unique $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ such that $\Phi(\phi, w) = f$ and $\{\alpha \in S \mid \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset. In order to conclude, we just have to show that $\phi(e^\alpha) = 1$ if and only if $\langle \alpha, w(M_f)\rangle = 0$: but, as in the proof of Lemma 4.10, we have

$$\begin{aligned}
\langle \alpha, w(M_f)\rangle =& \langle w^{-1}(\alpha), M_f\rangle = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\gamma, 1)\langle w^{-1}(\alpha), \gamma\rangle \\
=& \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - w(\gamma)})\langle w^{-1}(\alpha), \gamma\rangle = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - w(\gamma)})\langle \alpha, w(\gamma)\rangle \\
=& \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - \gamma})\langle \alpha, \gamma\rangle = \frac{1}{\phi(S_\delta)} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha\rangle > 0}} K_{\delta,\gamma}(\phi(e^{\delta-\gamma}) - \phi(e^{\delta - s_\alpha(\gamma)}))\langle \alpha, \gamma\rangle \\
=& \frac{1}{\phi(S_\delta)} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha\rangle > 0}} K_{\delta,\gamma} \phi(e^{\delta-\gamma})(1 - \phi(e^\alpha)^{\frac{2\langle \gamma, \alpha\rangle}{\langle \alpha, \alpha\rangle}})\langle \alpha, \gamma\rangle,
\end{aligned}$$

where the fourth inequality is due to the fact that w yields a bijection on the set of weights which satisfies the relation $K_{\delta, w(\gamma)} = K_{\delta,\gamma}$ for each $\gamma \in \Pi_\delta$. Thus, $\langle M_f, \alpha\rangle = 0$ if and only $\phi(e^\alpha) = 1$. □

Note that the bijection Ψ in the above proposition is explicitly given by Proposition 4.2: for $\vec{t} \in [0,1]^d_\delta$, denote by $\phi_{\vec{t}}$ the unique element of $\mathrm{Mult}(\mathbb{R}[Q^+])^+$ such that $\{\phi_t(\alpha) \neq 0\}$ is δ-admissible and $\Phi(\phi_t, w) = \Psi(\vec{t}, w)$. Then,

$$\Psi(\vec{t}, w)(e^\gamma, n) = \frac{1}{\phi_{\vec{t}}(S_\delta)^n} \phi_{\vec{t}}(e^{n\delta - w(\gamma)}) = \frac{\vec{t}^{\,n\delta - w(\gamma)}}{S_\delta(\vec{t})^n},$$

for $(e^\gamma, n) \in \hat{T}_\delta$.

Remark 4.13. *The restriction of the set of parameters $(\vec{t}, w)$ from $[0,1]^d \times W$ to $\{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\}$ is only useful to ensure the injectivity of the map Ψ. It is however still possible to define an element of* $\mathrm{Mult}(\hat{T}_\delta)^+$ *by applying the map $\Phi \circ \theta^{-1}$ to any element $(\vec{t}, w)$. The lack of injectivity without the restriction of the parameters can be seen in the following example: consider the Lie algebra of type A_2 with set of simple roots $\{\alpha_1, \alpha_2\}$ and choose $\delta = \omega_1$, the first fundamental weight. Then, $(\omega_1, \alpha_2) = 0$, and thus any weight $\gamma \neq \omega_1$ of Π_{ω_1} is written $\gamma = \omega_1 - k_1\alpha_1 - k_2\alpha_2$ with $k_1 > 0$: hence, if $t_1 = 0$, we have $\phi(e^{\omega_1 - \gamma}) = \delta_{\gamma,\omega_1}$ for all value of t_2. On the other hand, the ω_1-admissible subsets of $\{\alpha_1, \alpha_2\}$ are $\emptyset$, $\{\alpha_1\}$ and $\{\alpha_1, \alpha_2\}$. Thus the empty ω_1-admissible subset $\emptyset$ yields the unique choice of t_2 such that*

$t_1 = 0$ and $\mathbf{0}^c(\vec{t})$ is ω_1-admissible, namely $t_2 = 0$. The latter procedure has singled out a particular choice of parameters $t_1 = 0, t_2 = 0$ among all the choices of $\vec{t}$ yielding the map $\phi(e^{\omega_1 - \gamma}) = \delta_{\gamma,\omega_1}$.

A straightforward application of Proposition 4.1 yields the following corollary:

Corollary 4.14. *The map $i \circ \Psi$ gives a bijection between the minimal boundary $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ and*

$$\{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\}.$$

5. Minimal boundary of $\Gamma_\infty(\Delta)$

In this section, we use the description of $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ to get the one of $\partial\mathcal{H}_\infty(\Delta)$.

5.1. **Algebraic description of the growth graph of $\Gamma_\infty(\Delta)$.** The growth graph $\mathcal{G}(\Delta)$ of $\Gamma_\infty(\Delta)$ admits a similar description as the one of $\Gamma_\infty(\mathbb{R}^d)$. The set Λ_n^+ of vertices of rank n of the graph $\mathcal{G}(\Delta)$ are pairs (λ, n) where λ is a weight of P_δ^+ such that $\Gamma_\Delta(\lambda, n) \neq \emptyset$, and the weight of the edge between (λ, n) and $(\mu, n+1)$ is $e^+((\lambda, n), (\mu, n+1)) = \#\Gamma_\Delta(\lambda, \mu)$. Moreover, we have the following algebraic description of $\mathcal{G}(\Delta)$:

Proposition 5.1. *$\mathcal{G}(\Delta)$ is a multiplicative graph associated with the algebra $\hat{\mathcal{A}}_\delta$ with the injective map*

$$i : \begin{cases} \coprod_{n\geq 0} \Lambda_n & \longrightarrow & \hat{\mathcal{A}}_\delta \\ (\lambda, n) & \mapsto & (s_\lambda, n), n \geq 1 \\ * & \mapsto & (s_\delta, 1) \end{cases},$$

and $\left(\hat{\mathcal{A}}_\delta\right)_{\mathcal{G}(\Delta)} = \hat{T}_\delta^+$. In particular, $\partial\mathcal{H}(\mathcal{G}(\Delta))$ is isomorphic to $\mathrm{Mult}(\hat{T}_\delta^+)^+$ through the map

$$i^* : \begin{cases} \mathrm{Mult}(\hat{T}_\delta^+)^+ & \longrightarrow & \partial\mathcal{H}(\mathcal{G}(\Delta)) \\ f \mapsto f \circ i \end{cases}$$

Proof. By Littelmann's paths theory, the following equality holds for $(\lambda, n) \in \Lambda_n$:

$$\begin{aligned} i(\lambda, n) i(*) =& (s_\lambda, n)(s_\delta, 1) = \sum_{\mu \in \mathcal{A}_\delta} \#\Gamma_\Delta(\lambda, \mu)(s_\mu, n+1) \\ &= \sum_{\mu \in \mathcal{A}_\delta} e\big((\lambda, n), (\mu, n+1)\big) i(\mu, n+1). \end{aligned}$$

Thus, $\mathcal{G}(\Delta)$ is a multiplicative graph associated with $\hat{\mathcal{A}}_\delta$ with the map i. Note that by construction, $\left(\hat{\mathcal{A}}_\delta\right)_{\mathcal{G}(\Delta)} = \hat{T}_\delta^+$: the last part of the proposition is deduced from Proposition 2.4. □

Now we are going to connect the sets $\mathrm{Mult}(\hat{T}_\delta^+)^+$ and $\mathrm{Mult}(\hat{T}_\delta)^+$.

5.2. **Relation between $\mathrm{Mult}(\hat{T}_\delta)^+$ and $\mathrm{Mult}(\hat{T}_\delta^+)^+$.** Recall that $\hat{T}_\delta^+$ is a subalgebra of $\hat{T}_\delta$; therefore, any nonnegative morphism on $\hat{T}_\delta$ induces by restriction a nonnegative morphism on $\hat{T}_\delta^+$. This yields a map $\chi : \mathrm{Mult}(\hat{T}_\delta)^+ \longrightarrow \mathrm{Mult}(\hat{T}_\delta^+)^+$. The important step in the description of $\mathrm{Mult}(\hat{T}_\delta^+))^+$ is the following:

Proposition 5.2. *The map χ yields a surjection from $\{f \in \mathrm{Mult}(\hat{T}_\delta)^+ \mid M_f \in \Delta\}$ to $\mathrm{Mult}(\hat{T}_\delta^+)^+$.*

The proof of this proposition needs some preparation. Let f be a nonnegative morphism from $\hat{T}_\delta^+$ to $\mathbb{R}$. By Corollary 3.6 and by Corollary 4 page 35 in [3], f can be extended to a morphism $\tilde{f}$ from $\hat{T}_\delta$ to $\mathbb{C}$. The first task is to prove that $\tilde{f} \in \mathrm{Mult}(\hat{T}_\delta)^+$.

We need to recall a classical result by Aissen, Edrei, Schoenberg and White on polynomials with real coefficients having negative zeros.

Theorem 5.3. [2] *Consider a polynomial* $P(T) = a_m T^m + a_{m-1}T^{m-1} + \cdots + a_1 T + a_0 \in \mathbb{R}[T]$. *Then* P *has only real and nonpositive zeros if and only if the sequence* $a_0, a_1, \ldots, a_m, 0, 0, 0, \ldots$ *is totally positive, that is if and only if all the minors of the infinite matrix*

$$\begin{matrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \end{matrix}$$

are nonnegative.

Proposition 5.4. *Any morphism* $\tilde{f}$ *defined on* $\hat{T}_\delta$ *which extends the positive morphism* f *belongs to* $\mathrm{Mult}(\hat{T}_\delta)^+$.

Proof. Let $\tilde{f}$ be a morphism extending f. Set $\varphi(T) = \tilde{f}(\Phi)(T)$ that is

$$\varphi(T) = \prod_{\gamma \in \Pi_\delta} (T + \tilde{f}(e^\gamma, 1)).$$

By using the same arguments as in the proof of Proposition 3.5, we obtain that the coefficients of $\varphi(T)$ are the

$$\tilde{f}(e_k(e^{\gamma_1}, \ldots, e^{\gamma_N}), k) \in \mathbb{C}, k = 0, \ldots, N$$

Now it follows from the Jacobi-Trudi determinantal expression of the Schur functions that the minors of the matrix defined from the coefficients of $\varphi(T)$ as in Theorem 5.3 coincide with the complex numbers

$$\tilde{f}(\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N}), |\Lambda|), \Lambda \in \mathcal{P}_N$$

where $\mathcal{P}_N$ is the set of partitions with at most N parts and $\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N})$ is the plethysm of the Schur function $\boldsymbol{s}_\lambda$ in N variables $X_1, \ldots, X_N$ by the Weyl character s_δ. If we consider any young symmetrizer c_Λ of shape Λ in $\mathbb{R}[S_l]$, the group algebra of the symmetric group S_l (see [7]), the space

$$c_\Lambda \cdot V(\delta)^{\otimes l} \text{ such that } l = |\Lambda|$$

has indeed the structure of a G-module and

$$\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N}) = \mathrm{char}\left(c_\Lambda \cdot V(\delta)^{\otimes l}\right).$$

This shows that $\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N})$ decomposes as a sum of characters in $\{s_\lambda \mid \lambda \in \delta^{\otimes|\Lambda|}\}$ with nonnegative integer coefficients. In particular, $(\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N}), |\Lambda|)$ belongs to $\hat{T}_\delta^+$ and therefore we get that $\tilde{f}(\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N}), |\Lambda|) = f(\boldsymbol{s}_\Lambda(e^{\gamma_1}, \ldots, e^{\gamma_N}), |\Lambda|)$ is real nonnegative since f is assumed nonnegative. By Theorem 5.3 this shows that $-\tilde{f}(e^\gamma, 1)$ is real nonpositive for any $\gamma \in \Pi_\delta$. Finally we obtain that $\tilde{f}(e^\gamma, 1)$ is real nonnegative for any $\gamma \in \Pi_\delta$ and thus $\tilde{f}$ takes real nonnegative values on $\hat{T}_\delta$. □

Proof of Proposition 5.2. Let $f \in \mathrm{Mult}(\hat{T}_\delta^+)^+$. By Proposition 5.3, there exists $\tilde{f} \in \mathrm{Mult}(\hat{T}_\delta)^+$ such that $\tilde{f}(s_\lambda, n) = f(s_\lambda, n)$ for $(s_\lambda, n) \in \hat{T}_\delta^+$. Let $w \in W$ and $(s_\lambda, n) \in \hat{T}_\delta^+$: since w^{-1} yields a multiplicity preserving bijection on Π_λ,

$$\begin{aligned}(\tilde{f} \circ w)(s_\lambda, n) &= \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \tilde{f}(e^{w(\gamma)}, n) \\ &= \sum_{\gamma \in \Pi_\lambda} K_{\lambda, w^{-1}(\gamma)} \tilde{f}(e^{\gamma}, n) \\ &= \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \tilde{f}(e^{\gamma}, n) = \tilde{f}(s_\lambda, n).\end{aligned}$$

Thus, for all $w \in W$, $(\tilde{f} \circ w)_{|\hat{T}_\delta^+} = \tilde{f}_{|\hat{T}_\delta^+}$. Let $w \in W$ be such that $M_{\tilde{f} \circ w} \in \Delta$ and set $g = \tilde{f} \circ w$. Then, g is an element of $\mathrm{Mult}(\hat{T}_\delta)^+$ such that $\chi(g) = f$ and $M_g \in \Delta$. □

A straightforward application of Proposition 4.2 and Proposition 4.12 yields the following corollary:

Corollary 5.5. *There exists $\vec{t} \in [0,1]_\delta^d$ such that $\mathbf{0}^c(\vec{t})$ is δ-admissible and such that f is equal to*

$$f(s_\lambda, n) = \frac{S_{\lambda, n\delta}(\vec{t})}{(S_\delta(\vec{t}))^n}.$$

We denote this function by $f_{\vec{t}}$.

We denote by $\Psi^+ : [0,1]_\delta^d \longrightarrow \mathrm{Mult}(\hat{T}_\delta^+)^+$ the map sending $\vec{t}$ to the multiplicative map $f_{\vec{t}}$ of the latter corollary.

5.3. **Injectivity of the map Ψ^+.** It remains to show that the map Ψ^+ is injective.

Lemma 5.6. *Let $f \in \mathrm{Mult}(\hat{T}_\delta^+)$. Let $\vec{t} \in [0,1]_\delta^d$. For any $(\lambda, n) \in \hat{T}_\delta^+$ we have*

$$1 \leq \frac{S_{\lambda, n\delta}(\vec{t})}{\vec{t}^{n\delta - \lambda}} \leq \dim(V(\lambda)).$$

Proof. On the first hand,

$$S_{\lambda, n\delta}(\vec{t}) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \vec{t}^{n\delta - \gamma} \geq \vec{t}^{n\delta - \lambda},$$

which yields $1 \leq \frac{S_{\lambda, n\delta}(\vec{t})}{\vec{t}^{n\delta - \lambda}}$. On the other hand, since $t_i \leq 1$ for all $1 \leq i \leq d$,

$$S_{\lambda, n\delta}(\vec{t}) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \vec{t}^{n\delta - \gamma} \leq \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \vec{t}^{n\delta - \lambda} \leq \dim V(\lambda) \vec{t}^{n\delta - \lambda},$$

yielding the other inequality

$$\frac{S_{\lambda, n\delta}(\vec{t})}{\vec{t}^{n\delta - \lambda}} \leq \dim(V(\lambda)).$$

□

Corollary 5.7. *Let $t = (t_1, \ldots, t_d), \tau = (\tau_1, \ldots, \tau_d)$ be such that $\Psi^+(\vec{t}) = \Psi^+(\vec{\tau})$. Then $S_\delta(\vec{t}) = S_\delta(\vec{\tau})$.*

Proof. For all $n \geq 1$, $(n\delta, n) \in \hat{T}_\delta^+$. Thus, by Lemma 5.6, we have

$$1 \leq S_{n\delta,n\delta}(\vec{t}) \leq \dim(V(n\delta)) \text{ and } 1 \leq S_{n\delta,n\delta}(\vec{\tau}) \leq \dim(V(n\delta)).$$

This yields

$$\frac{1}{\dim(V(n\delta))} \leq \frac{S_{n\delta,n\delta}(\vec{t})}{S_{n\delta,n\delta}(\vec{\tau})} \leq \dim(V(n\delta)).$$

But

$$\frac{S_{n\delta,n\delta}(\vec{t})}{S_{n\delta,n\delta}(\vec{\tau})} = \frac{S_\delta(\vec{\tau})^n \Psi(t_1,\dots,t_d)(s_{n\delta},n)}{S_\delta(\vec{t})^n \Psi(\tau_1,\dots,\tau_d)(s_{n\delta},n)} = \frac{S_\delta(\vec{\tau})^n}{S_\delta(\vec{t})^n},$$

the last equality being due to the fact that $\Psi(t_1,\dots,t_d) = \Psi(\tau_1,\dots,\tau_d)$. Therefore, we have the inequality

$$\frac{1}{\dim(V(n\delta))} \leq \frac{S_\delta(\vec{\tau})^n}{S_\delta(\vec{t})^n} \leq \dim(V(n\delta)).$$

Since $\dim(V(n\delta))$ is polynomial in n, necessarily $S_\delta(\vec{t}) = S_\delta(\vec{\tau})$. □

The proof of the injectivity uses the combinatorics of Littelmann paths and we assume that the reader is familiar with this theory. We refer to [17] for an introduction to the operator $f_\alpha, \alpha \in S$ which are used in the following proofs. We recall that $B(\delta)$ denotes the set of Littelmann paths obtained from a path π_0 in Δ of weight δ. We introduce moreover the following decomposition of a δ-admissible subset $S' \subset S$.

Definition 5.8. *Let $S' \subset S$ be δ-admissible. A Dynkin subchain of type α and length r is a sequence $(\alpha_1,\dots,\alpha_r)$ of simple roots in S' such that $\alpha_1 = \alpha$, $\langle \alpha_r, \delta\rangle \neq 0$ and $\langle \alpha_i, \alpha_{i+1}\rangle \neq 0$ for $1 \leq i \leq r-1$. The depth $d_{S',\delta}(\alpha)$ of α relatively to S' is the smallest integer corresponding to the length of a Dynkin subchain of type α.*

Note that any simple root of a δ-admissible subset admits at least one Dynkin subchain, since it belongs to an indecomposable root system which is not orthogonal to δ.

Lemma 5.9. *Let $\lambda \in P^+$ and $\alpha \in S$ be such that $\langle \lambda, \alpha\rangle \neq 0$. Then there exists $(\mu, n) \in \hat{T}_\lambda^+$ such that $n\lambda - \mu = \alpha$.*

Proof. Suppose that $\langle \lambda, \alpha\rangle > 0$, and thus $\lambda - \alpha \in \Pi_\lambda$. We denote by π_0 the Littelmann path of $B(\lambda)$ with weight λ. Then, $wt(f_\alpha(\pi_0)) = \lambda - \alpha$. Moreover, since $f_\alpha(\pi_0) = \pi_0 - v\alpha$ with $v : [0,1] \longrightarrow [0,1]$, we have

$$\langle f_\alpha(\pi_0)(t), \alpha\rangle = \langle \pi_0(t), \alpha\rangle - v(t)\langle \alpha, \alpha\rangle \geq -\langle \alpha, \alpha\rangle,$$

and for all simple root $\alpha' \neq \alpha$ and $t \in [0,1]$, we have

$$\langle f_\alpha(\pi_0)(t), \alpha'\rangle = \langle \pi_0(t), \alpha'\rangle - v(t)\langle \alpha', \alpha\rangle \geq 0,$$

because π_0 lies in the Weyl chamber Δ and $\langle \alpha, \alpha'\rangle \leq 0$. Consider an integer $n \geq 2$ such that $\langle (n-1)\lambda, \alpha\rangle \geq \langle \alpha, \alpha\rangle$. Then, from the two previous inequalities, $\pi_0^{*(n-1))} * f_\alpha(\pi_0)$ lies in Δ: thus, $wt\big(\pi_0^{*(n-1)} * f_\alpha(\pi_0)\big) = (n-1)\lambda + (\lambda - \alpha)$ is the highest weight of an irreducible component of $V(\lambda)^{\otimes n}$, and $((n-1)\lambda + (\lambda - \alpha), n) = (n\lambda - \alpha, n) \in \hat{T}_\lambda^+$: setting $\mu = n\lambda - \alpha$ gives the result. □

The latter result can be generalized along a Dynkin subchain and yields the following Lemma:

Lemma 5.10. *Let $S' \subset S$ be δ-admissible and let $\alpha_0 \in S'$. There exists $(\lambda, n) \in \hat{P}_\delta^+$ such that $n\delta - \lambda = \alpha_0 + \sum_{\substack{\alpha' \in S' \\ d(\alpha') < d(\alpha_0)}} k_{\alpha'}\alpha'$.*

Proof. Let $S' \subset S$ be a δ-admissible subset. We will prove the result by induction on the depth of the simple root. For $d(\alpha) = 1$, the result is given by Lemma 5.9. Let $i \geq 2$. Suppose that the result is proven for all roots of depth at most $i-1$, and let α be a root in S' of depth i. Let $(\alpha, \alpha_2, \dots, \alpha_i)$ be a Dynkin chain of minimal length for α: by minimality, α_j has depth $i-j+1$ for $2 \leq j \leq i$. Since $d(\alpha_2) = i-1$, there exists $(\lambda', l) \in \hat{T}_\delta^+$ such that $l\delta - \lambda' = \alpha_2 + \sum_{d(\alpha')<d(\alpha_2)} k_{\alpha'}\alpha'$. If α' is such that $d(\alpha') < d(\alpha_2)$, then necessarily $\langle \alpha, \alpha' \rangle = 0$ (otherwise, there would exist a Dynkin subchain of type α and length smaller than i); likewise, since $d(\alpha) \geq 2$, $\langle \delta, \alpha \rangle = 0$. Thus,

$$\langle \lambda', \alpha \rangle = \left\langle l\delta - \alpha_2 - \sum_{d(\alpha')<d(\alpha_2)} k_{\alpha'}\alpha', \alpha \right\rangle = -\langle \alpha_2, \alpha \rangle > 0.$$

Let π be a Littelmann path in $B(\delta)^{\otimes l}$ lying in Δ and having weight λ'. We consider π as the Littelmann path of highest weight for the irreducible representation $V(\lambda')$. Since $\langle \alpha, \lambda' \rangle > 0$, $\lambda' - \alpha$ is a weight of $V(\lambda')$. Applying Lemma 5.9 yields the existence of $m \geq 1$ such that $\pi^{*m} * f_\alpha(\pi)$ lies in the Weyl chamber. Thus, $\pi^{*m} * f_\alpha(\pi)$ correspond to a highest weight vector in $(V(\delta)^{\otimes l})^{\otimes m}$. On the other hand,

$$wt(\pi^{*m} * f_\alpha(\pi)) = m\lambda' - \alpha = lm\delta - \alpha - m\alpha_2 - m\sum_{d(\alpha')<d(\alpha_2)} k_{\alpha'}\alpha' = lm\delta - \alpha - \sum_{d(\alpha')<d(\alpha)} k'_{\alpha'}\alpha',$$

with $k'_\alpha \geq 0$. Setting $\lambda = lm\delta - \alpha - \sum_{d(\alpha')<d(\alpha)} k'_{\alpha'}\alpha'$ and $n = lm$, we get an element $(\lambda, n) \in \hat{T_\delta}^+$ satisfying the hypothesis of the Lemma. □

Corollary 5.11. *Let $(t_1, \dots, t_d), (\tau_1, \dots, \tau_d) \in [0,1]_\delta^d$ and $1 \leq i \leq d$ be such that $t_i = 0$ and $\tau_i \neq 0$. Then, $\Psi(t_1, \dots, t_d) \neq \Psi(\tau_1, \dots, \tau_d)$.*

Proof. Note that $\mathbf{0}_{\vec{t}}^c$ and $\mathbf{0}_{\vec{\tau}}^c$ are δ-admissible subsets by definition of $[0,1]_\delta^d$. Since $\tau_i \neq 0$, $i \in \mathbf{0}_{\vec{\tau}}^c$. Thus, by Lemma 5.10, there exists $(\lambda, n) \in \hat{T}_\delta^+$ such that $\lambda = n\delta - \alpha_i - \sum_{\substack{j \in \mathbf{0}_{\vec{\tau}}^c \\ d(\alpha_j)<d(\alpha_i)}} k_{\alpha_j}\alpha_j$. Since $\tau_j > 0$ for all $j \in \mathbf{0}_\tau^c$,

$$\Psi(\tau_1, \dots, \tau_d)(s_\lambda, n) = \frac{S_{\lambda, n\delta}(\vec{\tau})}{S_\delta(\vec{\tau})^n} \geq \frac{\vec{\tau}^{n\delta-\lambda}}{S_\delta(\vec{\tau})^n} = \frac{1}{S_\delta(\vec{\tau})^n}\tau_i \prod_{\substack{j \in \mathbf{0}_{\vec{\tau}}^c \\ d(\alpha_j)<d(\alpha_i)}} \tau_j^{k_{\alpha_j}} > 0.$$

On the other hand, any weight of $V(\lambda)$ has the form $\lambda - \sum_{\alpha \in S} r_\alpha \alpha$ for some integer coefficients $r_\alpha \geq 0$; thus, since $t_i = 0$, for any weight $\mu = \lambda - \sum_{\alpha \in S} r_\alpha \alpha$ of $V(\lambda)$ we have

$$\vec{t}^{n\delta-\mu} = t_i \prod_{\substack{j \in \mathbf{0}_{\vec{\tau}}^c \\ d(\alpha_j)<d(\alpha_i)}} t_j^{k_{\alpha_j}} \prod_{\alpha_j \in S} t_j^{r_{\alpha_j}} = 0.$$

Thus, $\Psi(t_1, \dots, t_d)(s_\lambda, n) = 0 \neq \Psi(\tau_1, \dots, \tau_d)$. This yields that $\Psi(t_1, \dots, t_d) \neq \Psi(\tau_1, \dots, \tau_d)$. □

Proposition 5.12. *The map Ψ^+ is injective.*

Proof. Let $(t_1, \dots, t_d), (\tau_1, \dots, \tau_d) \in [0,1]_\delta^d$ be such that $\Psi(t_1, \dots, t_d) = \Psi(\tau_1, \dots, \tau_d)$. In this case, Corollary 5.7 yields that $S_\delta(\vec{\tau}) = S_\delta(\vec{t})$. By Corollary 5.11, we can assume that $\mathbf{0}_{\vec{t}}^c = \mathbf{0}_{\vec{\tau}}^c$, and we will denote this set S': we recall that the set of simple roots is identified with $\{1, \dots, d\}$, so that S' corresponds to a δ-admissible subset of S. We will prove by induction on the depth of the simple root α_j with respect to S' that $t_j = \tau_j$. Suppose that $\alpha_j \in S'$ is

such that $d(\alpha_j)=1$. By Lemma 5.10, there exists $n\geq 1$ such that $(n\delta-\alpha_j,n)\in\hat{T}_\delta^+$. Thus, $(nk\delta-k\alpha_j,kn)\in\hat{T}_\delta^+$ for all $k\geq 1$. Since $\Psi(t_1,\dots,t_d)=\Psi(\tau_1,\dots,\tau_d)$, we have

$$\frac{1}{S_\delta(\vec{t})^{kn}}S_{kn\delta-k\alpha_j,kn\delta}(\vec{t})=\frac{1}{S_\delta(\vec{\tau})^{kn}}S_{kn\delta-k\alpha_j,kn\delta}(\vec{\tau}),$$

which simplifies into $S_{kn\delta-k\alpha_j,kn\delta}(\vec{t})=S_{kn\delta-k\alpha_j,kn\delta}(\vec{\tau})$ because $S_\delta(\vec{t})=S_\delta(\vec{\tau})$. By Lemma 5.6, we have

$$1\leq\frac{S_{kn\delta-k\alpha_j,kn\delta}(\vec{t})}{t_j^k}\leq\dim V(kn\delta-k\alpha_j)\text{ and }1\leq\frac{S_{kn\delta-k\alpha_j,kn\delta}(\vec{\tau})}{\tau_j^k}\leq\dim V(kn\delta-k\alpha_j).$$

Thus,

$$\frac{1}{\dim V(kn\delta-k\alpha_j)}\leq\frac{t_j^k}{\tau_j^k}\leq\dim V(kn\delta-k\alpha_j).$$

Since $\dim V(kn\delta-k\alpha_j)$ is polynomial in k, necessarily $t_j=\tau_j$. Let $i\geq 2$, and suppose that we have proven that $t_j=\tau_j$ for all j such that $d(\alpha_j)<i$. Let α_l be such that $d(\alpha_l)=i$. By Lemma 5.10, there exists $(\lambda,n)\in\hat{T}_\delta^+$ such that $n\delta-\lambda=\alpha_l+\sum_{\substack{\alpha'\in S'\\ d(\alpha')<d(\alpha)}}k_{\alpha'}\alpha'$, with $k_{\alpha'}\geq 0$. Thus, for all $k\geq 1$, $(k\lambda,kn)\in\hat{T}_\delta^+$. As in the initial case, this implies that

$$S_{k\lambda,kn\delta}(\vec{t})=S_{k\lambda,kn\delta}(\vec{\tau}),$$

yielding together with Lemma 5.6 the inequality

$$\frac{1}{dimV(k\lambda)}\leq\frac{\vec{t}^{nk\delta-\lambda}}{\vec{\tau}^{nk\delta-\lambda}}\leq\dim V(k\lambda). \tag{6}$$

But $nk\delta-k\lambda=k\alpha_l+k\sum_{\substack{j\in\mathbf{0}_{\vec{t}}^c\\ d(\alpha_j)<d(\alpha_l)}}k_{\alpha_j}\alpha_j$, and by the induction hypothesis, $t_j=\tau_j$ for all $j\in\mathbf{0}_{\vec{t}}^c, d(\alpha_j)<d(\alpha_l)$. Thus

$$\frac{\vec{t}^{nk\delta-\lambda}}{\vec{\tau}^{nk\delta-\lambda}}=\frac{t_l^k}{\tau_l^k}.$$

Since $\dim V(k\lambda)$ is polynomial in k, (6) yields that $\frac{t_l}{\tau_l}=1$. This concludes the proof of Proposition 5.12. □

Corollary 5.13. *The map Ψ^+ is a bijection from $[0,1]_\delta^d$ to $\mathrm{Mult}(\hat{T}_\delta^+)^+$. In particular, $\partial\mathcal{H}_\infty(\Delta)$ is isomorphic to $[0,1]_\delta^d$.*

6. Drift of a path following a central measure

In this section, we identify the set $\{(\vec{t},w)\in[0,1]_\delta^d\times W|w\in W^{\mathbf{1}(\vec{t})}\}$ with $K(\delta)$ in order to complete the proof of Theorem 3.1: this identification is done by considering the mean vector of the random walk given by the map Ψ. At the end of this section we prove Corollary 3.3.

6.1. The mean vector $\vec{M}$.

Let us introduce the map

$$\vec{M}:\begin{cases}\{(\vec{t},w)\in[0,1]_\delta^d\times W|w\in W^{\mathbf{1}(\vec{t})}\} & \longrightarrow \quad K(\delta)\\ \vec{t}\times w & \mapsto \quad M_{\Psi(\vec{t},w)}\end{cases},$$

where the mean vector M_f has been introduced in Section 3.2 for any multiplicative map $f\in\mathrm{Mult}(\hat{T}_\delta)^+$. For $I\subset\{1,\dots,d\}$, denote by W_I the parabolic subgroup generated by the simple roots α_i for $i\in I$.

Lemma 6.1. *Let $(\vec{t},w)\in\{(\vec{t},w)\in[0,1]_\delta^d\times W|w\in W^{\mathbf{1}(\vec{t})}\}$. Then $M_{\Psi(\vec{t},w)}\in w'^{-1}(\Delta)$ if and only if $w'\in W_{\mathbf{1}(\vec{t})}w$.*

Proof. Let $\alpha_i \in S$. We have

$$M_{\Psi(\vec{t},w)} = \frac{1}{S_\delta(\vec{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \vec{t}^{\delta - w(\gamma)} \gamma.$$

Thus,

$$\begin{aligned}\langle M_{\Psi(\vec{t},w)}, w^{-1}(\alpha_i)\rangle =& \frac{1}{S_\delta(\vec{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \vec{t}^{\delta - w(\gamma)} \langle \gamma, w^{-1}(\alpha_i)\rangle = \frac{1}{S_\delta(\vec{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,w(\gamma)} \vec{t}^{\delta - w(\gamma)} \langle w(\gamma), \alpha_i\rangle \\ =& \frac{1}{S_\delta(\vec{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \vec{t}^{\delta - \gamma} \langle \gamma, \alpha_i\rangle = \frac{1}{S_\delta(\vec{t})} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha_i\rangle > 0}} K_{\delta,\gamma} \vec{t}^{\delta - \gamma} (1 - t_i^{2\frac{\langle \gamma, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}}) \langle \gamma, \alpha_i\rangle.\end{aligned}$$

Since each t_i is in $[0,1]$, $\langle M_{\Psi(\vec{t},w)}, w^{-1}(\alpha_i)\rangle \geq 0$ for $1 \leq i \leq d$ and hence $w(M_{\Psi(\vec{t},w)}) \in \Delta$. Moreover, $\langle M_{\Psi(\vec{t},w)}, w^{-1}(\alpha_i)\rangle = 0$ if and only if $t_i = 1$. Therefore, $w(M_{\Psi(\vec{t},w)}) \in w'(\Delta)$ if and only if w' is a product of reflections s_{α_i} such that $t_i = 1$. Applying w^{-1} to the latter result yields the proof of the Lemma. □

Proposition 6.2. *The map $\vec{M}$ is injective.*

Proof. Let $(\vec{t}, w)$ and $(\vec{t}', w')$ be two elements of $\{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\}$ such that $\vec{M}(\vec{t}, w) = \vec{M}(\vec{t}', w')$. We simply denote by M this common value. Lemma 6.1 implies that $W_{\mathbf{1}(\vec{t})} w = W_{\mathbf{1}(\vec{t}')} w'$. Thus, $W_{\mathbf{1}(\vec{t})} = W_{\mathbf{1}(\vec{t}')}$, which implies that $\mathbf{1}(\vec{t}) = \mathbf{1}(\vec{t}')$; since w and w' are both a minimal right coset representative of $W_{\mathbf{1}(\vec{t}')} w$, we have $w = w'$. Let F be the dominant face corresponding to the δ-admissible set $\mathbf{0}^c_{\vec{t}}$ and let F' be the one corresponding to the δ-admissible set $\mathbf{0}^c_{\vec{t}'}$. By the results of Section 4.3, $\vec{M}(\vec{t}, w) \in w^{-1}(\overset{\circ}{F})$ (where $\overset{\circ}{F}$ the interior of the face F) and $\vec{M}(\vec{t}', w) \in w'^{-1}(\overset{\circ}{F'})$. Since $w = w'$, we must have $F = F'$ and thus $\mathbf{0}^c_{\vec{t}} = \mathbf{0}^c_{\vec{t}'}$. Let $(X_l)_{l \geq 0}$, $(X'_l)_{l \geq 0}$ be two random walks with initial position $X_0 = X'_0 = 0$ and respective transition matrices

$$\mathbb{P}(X_{l+1} = \gamma | X_l = \gamma') = K_{\delta,\gamma-\gamma'} \frac{\vec{t}^{\delta - w(\gamma - \gamma')}}{S_\delta(\vec{t})}, \mathbb{P}(X'_{l+1} = \gamma | X'_l = \gamma') = K_{\delta,\gamma-\gamma'} \frac{\vec{t}'^{\delta - w(\gamma - \gamma')}}{S_\delta(\vec{t}')}.$$

Both random walks have mean M, thus it follows by the local limit theorem for large deviations (see for instance Theorem 4.2.1 in [14]) that for any sequences of weights $(\gamma_l)_{l \geq 1}, (\gamma'_l)_{l \geq 1}$ such that $\gamma_l - lM = o(l^{2/3}), \gamma'_l - lM = o(l^{2/3})$, and $\mathbb{P}(X_l = \gamma_l) \neq 0, \mathbb{P}(X_l = \gamma'_l) \neq 0$, we have

$$\mathbb{P}(X_l = \gamma_l) \sim \mathbb{P}(X_l = \gamma'_l), \tag{7}$$

and the same relation holds for $(X'_l)_{l \geq 1}$. Let $i \in \mathbf{0}^c_{\vec{t}}$. For $l \geq 1$, let $(\gamma_l, l) \in \hat{T}^+_\delta$ be such that γ_l is an element of $P_\delta \cap lF$ at minimal distance from lM and set $\gamma'_l = \gamma_l - \alpha_i$. Then, $\mathbb{P}(X_l = \gamma_l) \neq 0$. Since M belongs to the interior of M, $\gamma'_l \in P_\delta \cap lF$ for l large enough: thus, $\mathbb{P}(X_l = \gamma'_l) \neq 0$ for l large enough. The sequences $(\gamma_l - lM)_{l \geq 1}$ and $(\gamma'_l - lM)_{l \geq 1}$ are bounded, thus the local limit Theorem applies and

$$\mathbb{P}(X_l = \gamma_l) \sim \mathbb{P}(X_l = \gamma'_l) \tag{8}$$

as l goes to infinity. Since X comes from a central measure,

$$\mathbb{P}(X_l = \gamma_l) = \#\Gamma_{\mathbb{R}^d}(\gamma_l, l) \frac{\vec{t}^{n\delta - \gamma_l}}{\big(S_\delta(\vec{t})\big)^l}. \tag{9}$$

Using (7) with (9) yields that $\frac{\#\Gamma_{\mathbb{R}^d}(\gamma_l, l)}{\#\Gamma_{\mathbb{R}^d}(\gamma'_l, l)} \sim \vec{t}^{\gamma'_l - \gamma_l} = t_i^{-1}$. But the same holds for X', yielding that $\frac{\#\Gamma_{\mathbb{R}^d}(\gamma_l, l)}{\#\Gamma_{\mathbb{R}^d}(\gamma'_l, l)} \sim \vec{t}'^{\gamma'_l - \gamma_l} = t'^{-1}_i$. Finally, $t_i = t'_i$. □

We can now prove the main result of this subsection:

Proposition 6.3. *The map $\vec{M}$ is a bijective map from $\{(\vec{t},w)\in[0,1]^d_\delta\times W|w\in W^{\mathbf{1}(\vec{t})}\}$ to $K(\delta)$ such that $\vec{M}([0,1]^d_\delta\times \mathrm{Id})=K(\delta)^+$.*

Proof. The injectivity of $\vec{M}$ has already been proven in Proposition 6.2. Let us prove that $\vec{M}$ is surjective. Recall that Ψ is a restriction of the map $\Phi\circ\theta^{-1}:[0,1]^d\times W\longrightarrow \mathrm{Mult}(\hat{T}_\delta)^+$ defined at the end of Section 4.2, and that both maps have the same image; thus, it is enough to prove that the map $\vec{M}$ extended to the domain $[0,1]^d\times W$ by the formula $\vec{M}(\vec{t},w)=M_{\Phi\circ(\theta^{-1}(\vec{t}),w)}$ is surjective. Let us first prove that $\vec{M}_{|[0,1]^d\times\mathrm{Id}}$ is surjective onto $K(\delta)^+$. Let $1\leq i\leq d$ be such that $\langle\delta,\alpha_i\rangle\neq 0$: then, α_i is a δ-admissible set, and the dominant face associated with α_i is one-dimensional. Let $x_i=F_i\cap\partial\Delta$ (that is, x_i is the projection of δ on $\alpha_i^\perp$). Then, $K(\delta)^+$ is a convex polytope whose extreme points are the elements $\delta,0$ and $\{x_i\}_{\substack{1\leq i\leq d\\ \langle\alpha_i,\delta\rangle>0}}$. Let $\Sigma:(\mathbb{R}^+)^d\longrightarrow\mathbb{R}$ be the function defined by

$$\Sigma(\vec{u})=\log(S_\delta(e^{u_1},\dots,e^{u_d}))=\log(\sum_{\gamma\in\Pi_\delta}K_{\delta,\gamma}e^{\vec{u}.(\delta-\gamma)}).$$

Then,

$$\nabla\Sigma(\vec{u})=\left(\frac{1}{S_\delta(e^{u_1},\dots,e^{u_d})}\sum_{\gamma\in\Pi_\delta}K_{\delta,\gamma}(\delta_i-\gamma_i)e^{\vec{u}.(\delta-\gamma)}\right)_{1\leq i\leq d}=\delta-\vec{M}((e^{u_1},\dots,e^{u_d}),\mathrm{Id}).$$

Moreover, we can show that Σ is a convex function: introduce the random variable X such that $\mathbb{P}(X=\delta-\gamma)=\frac{K_{\delta,\gamma}}{S_\delta(e^{u_1},\dots,e^{u_d})}e^{\vec{u}.(\delta-\gamma)}$. The Hessian matrix of Σ at $\vec{u}$ is exactly the covariance matrix of the random variable X, which is nonnegative: since this is true for all vector $\vec{u}\in(\mathbb{R}^+)^d$, Σ is indeed convex. Since Σ is a convex function and $(\mathbb{R}^+)^d$ is convex, the set $\nabla\Sigma(\mathbb{R}^+)^d)$ is a convex set. We have thus proven that the set $\{\delta-\vec{M}(e^{u_1},\dots,e^{u_d})|\vec{u}\in(\mathbb{R}^+)^d\}=\vec{M}(]0,1]^d,\mathrm{Id})$ is convex. Note first that $M(\mathbf{1},\mathrm{Id})=0$, yielding that $0\in\vec{M}(]0,1]^d\times\mathrm{Id})$. Let $1\leq i\leq d$ be such that $\langle\alpha_i,\delta\rangle\neq 0$. Since the map S_δ is continuous and nonzero on $[0,1]^d$, the map $\vec{M}$ is continuous on $[0,1]^d\times\mathrm{Id}$: thus, if $(\vec{t}_l)_{l\geq1}$ is a sequence of $]0,1]^d$ such that $\vec{t}_l\to(\delta_{ij})_{1\leq j\leq d}$ as l goes to infinity, then $\vec{M}(\vec{t}_l,\mathrm{Id})$ converges to $\vec{M}((\delta_{ij})_{1\leq j\leq d},\mathrm{Id})$. Let us set $(\alpha_i,\delta):=\frac{2\langle\alpha_i,\delta\rangle}{\langle\alpha_i,\alpha_i\rangle}$. For $1\leq l\leq(\delta,\alpha_i)$, $K_{\delta,\delta-l\alpha_i}=1$, since the only element of $B(\delta)$ ending at $\delta-l\alpha_i$ is $f^l_{\alpha_i}(\pi_0)$; thus, we have

$$\begin{aligned}\vec{M}((\delta_{ij})_{1\leq j\leq d},\mathrm{Id})=&\frac{1}{S_\delta((\delta_{ij})_{1\leq j\leq d})}\sum_{l=0}^{(\alpha_i,\delta)}K_{\delta,\delta-l\alpha_i}(\delta-l\alpha_i)\\ =&\frac{1}{\sum_{l=0}^{(\alpha_i,\delta)}K_{\delta,\delta-l\alpha_i}}\left(\sum_{l=0}^{(\alpha_i,\delta)}\delta-\sum_{l=0}^{(\alpha_i,\delta)}l\alpha_i\right)\\ =&\frac{1}{(\alpha_i,\delta)+1}\left(\big((\alpha_i,\delta)+1\big)\delta-\frac{(\alpha_i,\delta).\big((\alpha_i,\delta)+1\big)}{2}\alpha_i\right)\\ =&\delta-\frac{(\alpha_i,\delta)}{2}\alpha_i=x_i,\end{aligned}$$

and x_i belongs to $\overline{\vec{M}(]0,1]^d\times\mathrm{Id})}$, the closure of $\vec{M}(]0,1]^d\times\mathrm{Id})$. Similarly, if $(\vec{t}_l)$ is a sequence of $[0,1]^d$ converging to $\mathbf{0}$, then $\vec{M}(\vec{t}_l,\mathrm{Id})$ converges to $\vec{M}(\vec{0},\mathrm{Id})$. Since $\vec{M}(\vec{0},\mathrm{Id})=\delta$, $\delta\in\overline{\vec{M}(]0,1]^d\times\mathrm{Id})}$. Hence, $0,\delta$ and $\{x_i\}_{\substack{1\leq i\leq d\\ \langle\alpha_i,\delta\rangle>0}}$ are in the closure of $\vec{M}(]0,1]^d\times\mathrm{Id})$. Since

$\vec{M}(]0,1]^d \times \mathrm{Id})$ is convex, this yields that $K(\delta)^+ \subset \overline{\vec{M}(]0,1]^d \times \mathrm{Id})}$. Since $[0,1]^d$ is compact, $\vec{M}([0,1]^d, \mathrm{Id})$ is compact and thus $\overline{\vec{M}(]0,1]^d \times \mathrm{Id})} \subset \vec{M}([0,1]^d, \mathrm{Id})$: this yields that $K(\delta)^+ \subset \vec{M}([0,1]^d, \mathrm{Id})$. By Lemma 6.1, $\vec{M}([0,1]^d, \mathrm{Id}) \subset K(\delta)^+$, so that finally $\vec{M}([0,1]^d, \mathrm{Id}) = K(\delta)^+$. Since $\vec{M}(\vec{t}, w) = w^{-1}\vec{M}(\vec{t}, \mathrm{Id})$, $\vec{M}([0,1]^d \times W) = \bigcup_{w \in W} w(K(\delta)^+) = K(\delta)$. □

6.2. **Proof of Theorem 3.1.** We give the proof of Theorem 3.1 by gathering the different results of the paper. Let us prove the result only for $\partial\mathcal{H}_\infty(\mathbb{R}^d)$, since the proof for $\partial\mathcal{H}_\infty(\Delta)$ is similar.

- By Corollary 4.1, $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ is homeomorphic to $\mathrm{Mult}(\hat{T}_\delta)^+$ through the map $j : \mathrm{Mult}(\hat{T}_\delta)^+ \longrightarrow \partial\mathcal{H}_\infty(\mathbb{R}^d)$ defined by $j(f)(\Gamma(\tau)) = f(\gamma, n)$ for any path $\tau \in \Gamma_\infty(\mathbb{R}^d)$ of length n ending at γ. Since $\mathrm{Mult}(\hat{T}_\delta)^+$ is compact, $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ is a compact space.
- By Proposition 4.12 the map $\Psi : \{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\} \longrightarrow \mathrm{Mult}(\hat{T}_\delta)^+$ given by $\Psi(\vec{t}, w)(\gamma, n) = \frac{1}{S_\delta(\vec{t})^n}\vec{t}^{n\delta - w(\gamma)}$ is a bijection.
- Finally, by Proposition 6.3, the map $\vec{M} : \{(\vec{t}, w) \in [0,1]^d_\delta \times W | w \in W^{\mathbf{1}(\vec{t})}\} \longrightarrow K(\delta)$ given by $\vec{M}(\vec{t}, w) = \frac{1}{S_\delta(\vec{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma}\vec{t}^{\delta - w(\gamma)}$ is bijective.

Therefore, the map $\mathbb{P} : K(\delta) \longrightarrow \partial\mathcal{H}_\infty(\mathbb{R}^d)$ given by $\mathbb{P} = j \circ \Psi \circ (\vec{M}^{-1})$ is a bijection. Note that from the previous results, for $m \in K(\delta)$,

$$\mathbb{P}_m(\Gamma(\tau)) = \frac{1}{S_\delta(\vec{t}_m)^n}\vec{t}_m^{n\delta - w(\gamma)},$$

for all paths τ of length n ending at γ. It remains to show that $\mathbb{P}$ is indeed an homeomorphism. Since $K(\delta)$ and $\partial\mathcal{H}_\infty(\mathbb{R}^d)$ are compact, it suffices to prove that $\mathbb{P}$ or $\mathbb{P}^{-1}$ is continuous. But for $P \in \partial\mathcal{H}_\infty(\mathbb{R}^d)$,

$$\mathbb{P}^{-1}(P) = \sum_{\tau \in B(\delta)} P\big(\Gamma_{\mathbb{R}^d}(\tau)\big)\tau(1).$$

Thus $\mathbb{P}^{-1}$ is continuous, which concludes the proof of Theorem 3.1. The same proof holds for $\partial\mathcal{H}_\infty(\Delta)$ with $K(\delta)^+$ and the map $\mathbb{P}^+$ introduced in the statement of the Theorem. For a metric space X, denote by $M_1(X)$ the set of probability measures on X with respect to its Borel σ-algebra; we consider $M_1(X)$ as a topological space with the weak convergence topology. As a straightforward corollary of Theorem 3.1, we get the following integral representation of $\mathcal{H}_\infty(\mathbb{R}^d)$ and $\mathcal{H}_\infty(\Delta)$.

Corollary 6.4. *The topological spaces $\mathcal{H}_\infty(\mathbb{R}^d)$ and $\mathcal{H}_\infty(\Delta)$ are homeomorphic to $M_1(K(\delta))$ and $M_1(K(\delta)^+)$, respectively through the maps*

$$\mathcal{P} : \begin{cases} M_1(K(\delta)) & \longrightarrow & \mathcal{H}_\infty(\mathbb{R}^d) \\ \mu & \mapsto & \int_{K(\delta)} \mathbb{P}_m d\mu(m) \end{cases}$$

and

$$\mathcal{P} : \begin{cases} M_1(K(\delta)^+) & \longrightarrow & \mathcal{H}_\infty(\Delta) \\ \mu & \mapsto & \int_{K(\delta)^+} \mathbb{P}^+_m d\mu(m) \end{cases}.$$

We prove now that a random path in $\Gamma_\infty(\Delta)$ following the harmonic measure $\mathbb{P}^+_m$ admits a law of large numbers with drift m. In the case of a random path in $\Gamma_\infty(\mathbb{R}^d)$ following the harmonic measure $\mathbb{P}_m$, the result is clear from the definition of $\mathbb{P}_m$ and the classical law of large numbers for random walks. The case of $\mathbb{P}^+_m$ is more complicated, since the random path is constrained to remain in a domain. However, the result is still true:

Proposition 6.5. *Let γ_m be a random path in $\Gamma_\infty(\Delta)$ following the harmonic measure $\mathbb{P}_m^+$. Denote by $\tau_m(n)$ the position of the path after n steps. Then, almost surely,*

$$\frac{1}{n}\tau_m(n) \longrightarrow m,$$

as n goes to infinity.

Proof. Denote by $\tilde{\tau}_m$ the random path in $\Gamma_\infty(\mathbb{R}^d)$ following the harmonic measure $\mathbb{P}_m$. By [16, Theorem 4.12], we have the equality in law

$$\tau_m = \mathcal{P}_{\alpha_{i_1}} \dots \mathcal{P}_{\alpha_{i_r}}(\tilde{\tau}_m),$$

where $w_0 = s_{\alpha_{i_1}} \dots s_{\alpha_{i_r}}$ is a minimal length decomposition of the longest element of W, and each operator $\mathcal{P}_\alpha$ is the Pitman transformation associated with the root α. We recall that the definition of the operator $\mathcal{P}_\alpha$ on a path $\tau \in \Gamma_\infty(\mathbb{R}^d)$ is given by

$$\mathcal{P}_\alpha(\tau)(t) = \tau(t) - (\inf_{s\in[0,t]} \frac{2\langle \tau(s), \alpha\rangle}{\langle \alpha, \alpha\rangle})\alpha.$$

By a large deviation principle,

$$\|\frac{1}{t}(\tilde{\tau}_m)_{|[0,t]} - m\,\mathrm{Id}_{|[0,t]}\|_\infty \underset{t\to+\infty}{\longrightarrow} 0$$

with probability one. Thus, for $s \in [0,t]$ and $\alpha \in S$,

$$\left|\frac{1}{t}\frac{2\langle \tilde{\tau}_m(s), \alpha\rangle}{\langle \alpha, \alpha\rangle} - \frac{2\langle ms, \alpha\rangle}{\langle \alpha, \alpha\rangle}\right| \leq \epsilon(t),$$

with $\epsilon(t)$ converging to 0 when t goes to infinity with probability one. Since $m \in K(\delta)^+$, $\langle m, \alpha\rangle \geq 0$, and thus $\inf_{s\in[0,t]} s\frac{2\langle m,\alpha\rangle}{\langle \alpha,\alpha\rangle} = 0$. Hence,

$$\left|\frac{1}{t}\inf_{s\in[0,t]} \frac{2\langle \tilde{\tau}_m(s), \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha\right| \leq \epsilon(t)|\alpha| \underset{t\to+\infty}{\longrightarrow} 0,$$

and finally $\frac{1}{t}\mathcal{P}_\alpha(\tilde{\tau}_m)(t) \sim \frac{1}{t}\tilde{\tau}_m(t) \longrightarrow m$ as t goes to $+\infty$, with probability one. Iterating this result for $\mathcal{P}_{\alpha_{i_1}}, \dots, \mathcal{P}_{\alpha_{i_r}}$ yields that

$$\frac{1}{t}\mathcal{P}(\tilde{\tau}_m)(t) \underset{t\to+\infty}{\longrightarrow} m$$

with probability one. Since $\mathcal{P}(\tilde{\tau}_m)$ is equal in law to τ_m, the proof is done. □

6.3. **c-harmonic function killed on the boundary of Δ.** We end this section by proving Corollary 3.3. We recall that $\hat{s}$ is the map from $\partial\mathcal{H}_\infty(\Delta)$ to $\mathbb{R}^+ \cup \{\infty\}$ defined by

$$\hat{s}(\mathbb{P}_m^+) = \sum_{\gamma\in P} K_{\delta,\gamma}\vec{t}_m^{\,\gamma}.$$

Note that the range of $\vec{t}_m$ is exactly $[0,1]_\delta^d$ by Proposition 6.3.

Proof of Corollary 3.3. Suppose that $P \in \mathcal{H}_c(\Delta)$ and set $Z = \dim V(\delta)$. By Corollary 6.4, there exists $\mu \in \mathcal{M}_1(K(\delta)^+)$ such that $P = \int_{K(\delta)^+} \mathbb{P}_m^+ d\mu(m)$. Since $P \in \mathcal{H}_c(\Delta)$, there exists $p : \Lambda \cap \Delta \longrightarrow \mathbb{R}^+$ such that

$$P(\Gamma_\Delta(\gamma)) = \frac{p(x)}{(cZ)^n},$$

for $\gamma \in \Gamma_\Delta(x,n)$. Let $\gamma = (\gamma_n)_{n\geq 1}$ be a sequence of paths such that $\gamma_n \in \Gamma_\Delta(x,\phi(n))$ with $\phi(n) \longrightarrow +\infty$ when n goes to infinity. Then, on the one hand, $(cZ)^{\phi(n)}P(\Gamma_\Delta(\gamma_n))$ is constant and equal to $p(x)$. On the other hand, by the expression of P and Theorem 3.1,

$$(cZ)^{\phi(n)}P(\Gamma_\Delta(\gamma_n)) = \int_{K(\delta)^+} (cZ)^{\phi(n)} \frac{S_{x,\phi(n)\delta}(\vec{t}_m)}{S_\delta(\vec{t}_m)^{\phi(n)}} d\mu(m). \tag{10}$$

Suppose that m is such that $\hat{s}_\delta(\mathbb{P}_m^+) = +\infty$. This means that there exists $1 \leq i \leq d$ such that $(\vec{t}_m)_i = 0$. Thus, by Proposition 4.6, $\mathbb{P}_m^+(\Gamma_\Delta(\gamma))$ is nonzero only if $\gamma \in \Gamma_\Delta(y,n)$ for $y \in n\delta + n\Pi_F$, where F is a fixed dominant face which is strictly smaller than K_δ. Hence, for n large enough, $\mathbb{P}_m^+(\Gamma_\Delta(\gamma_n)) = 0$, and we can assume that the support of μ is included in $\{\mathbb{P}_m^+ | \vec{t}_m \in]0,1]^d\}$. For $\vec{t}_m \in]0,1]^d$, $s_x(\vec{t}_m)$ is well defined for all $x \in P^+$ and we have

$$\frac{S_{x,\phi(n)\delta}(\vec{t}_m)}{S_\delta(\vec{t}_m)^{\phi(n)}} = \frac{\vec{t}_m^{\phi(n)\delta} s_x(\vec{t}_m)}{\vec{t}_m^{\phi(n)\delta}(s_\delta(\vec{t}_m))^{\phi(n)}} = \frac{s_x(\vec{t}_m)}{\hat{s}_\delta(\mathbb{P}_m^+)^{\phi(n)}}.$$

Thus, (10) becomes

$$(cZ)^{\phi(n)}P(\Gamma_\Delta(\gamma_n)) = \int_{K(\delta)^+} (cZ)^{\phi(n)} \frac{s_x(\vec{t}_m)}{\hat{s}_\delta(\mathbb{P}_m^+)^{\phi(n)}} d\mu(m).$$

In order that the right hand-side of the latter expression does not go to infinity, we must have $\mu(\hat{s}_\delta^{-1}[0,cZ[) = 0$. Then, we have

$$p(x) = \lim_{n\to+\infty} \int_{K(\delta)^+} \left(\frac{tZ}{\hat{s}_\delta(\mathbb{P}_m^+)}\right)^{\phi(n)} s_x(\vec{t}_m) d\mu(m) = \int_{\hat{s}_\delta^{-1}(\{cZ\})} s_x(\vec{t}_m) d\mu(m),$$

and the support of μ is $\hat{s}_\delta^{-1}(\{cZ\})$. Finally, $\partial\mathcal{H}_c(\Delta) \subset \hat{s}_\delta^{-1}(\{cZ\})$.
Its is readily seen that $\hat{s}_\delta^{-1}(\{cZ\}) \subset \mathcal{H}_c(\Delta) \cap \partial\mathcal{H}_\infty(\Delta)$ by the expression of $\mathbb{P}_m^+$ from Theorem 3.1, and by the characterization of c-harmonic functions from Section 2.4; thus $\partial\mathcal{H}_c(\Delta) = \hat{s}_\delta^{-1}(\{cZ\})$, which proves the first part of the corollary.
For the second part of the corollary, a quick computation yields that $\log(s_\delta)$ is strictly convex on $\mathbb{R}^d$: the hessian matrix of $\log(s_\delta)$ at $\vec{t}$ is actually the covariance matrix of a non-degenerate random variable. Thus, $\log(s_\delta)$ admits a unique minimum on $\mathbb{R}^d$, which is located at the unique vector $\vec{t}_0$ such that $\nabla\log(s_\delta)(\vec{t}_0) = \vec{0}$. Since

$$\nabla\log(s_\delta)(\vec{t}) = \frac{1}{s_\delta(t)} \sum_{\gamma\in P} K_{\delta,\gamma}\vec{t}^\gamma\gamma = \vec{M}(\vec{t})$$

and $\vec{M}(\vec{1}) = \vec{0}$, the minimum of $\log(s_\delta)$ is at $\vec{1}$. The same holds for s_δ, and thus $\min_{\partial_\infty(\Delta,0)} \hat{s}_\delta = \hat{s}_\delta(\mathbb{P}_{\vec{1}}) = Z$. The second part of the corollary is a straightforward deduction of this fact.
□

Note that the proof of the latter corollary gives an explicit expression of the harmonic measure associated with the central measure $\mathbb{P}_m$.

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