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M4-5n numerical solution using the Mixed FEM, validation against the Finite Difference Method

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Abstract The final aim of this work is to build a tool dedicated to the calculation of the mechanical fields in pavements incorporating possible vertical cracks in some layers or partial debonding at the interface between layers. The development of this tool is based on a specific layer-wise modeling of the structure so-called M4-5n. In this model the stress fields are approached through polynomial approximations in the vertical direction for each layer. Its construction is based on the Hellinger-Reissner (H-R) variational principle of continuum mechanics. One advantage of the M4-5n is to reduce by one the dimension of the problem. Moreover this model leads to finite values of the generalized interface stresses at the crack lips of the structures studied. This approach is thus particularly adapted to parametric studies and might be considered for analyzing crack growth in layered structures such as pavements. The contribution of the present paper to this model is focused on the computation of its numerical solution by means of the mixed Finite Element Method (FEM). The developed method is based on the maximum of the complementary energy theorem using Lagrangian multipliers to ensure the equilibrium equations. The resulting formulation is equivalent to the H-R variational principle applied to the generalized displacement and stress fields. This approach is applied to a beam structure composed of four elastic homogenous layers resting on Winkler's springs. Vertical cracks across some layers are introduced. The results obtained are compared with those from an earlier approach using the Finite Difference Method (FDM).

Keywords M4-5n, Mixed FEM, Cracking, Debonding

1 Introduction

A pavement structure is a multilayer structure resting on a soil. It can be damaged due to many phenomena resulting in different types of degradation among which vertical cracking across layers or partial/total debonding at the interface between two layers. To study these mechanisms, the existing advanced models are general-

ly based on 3D continuum mechanics using the fracture or the damage theories (Pommier et al. 2009). Recently Chabot et al. (2005) proposed to apply one of the Multi-Particle Models of Multilayer Materials (M4) to the pavement field. Among the different formulations proposed for this family of models, the M4-5n is especially adapted to the analysis of delamination in composite materials. It is particularly adapted to parametric studies (Chabot et al. 2013). This specific formulation developed for linear elasticity considers five generalized displacement fields per layer (n : total number of layers). In particular the use of generalized stress fields with finite values avoids the problem of singularity near cracks. Hence this formulation seems also appropriate for modeling cracking and debonding in pavements. The use of M4-5n makes it possible to reduce the problem by one dimension avoiding the explicit presence of the z -coordinate in the equations. This is an interesting feature with regards to numerical modeling (mesh generation and computation time). As a consequence, a 3D pavement structure is represented by plate geometry. In each layer of the M4-5n, the usual stress tensor is replaced by the generalized stresses $M_{\alpha\beta}^i(x, y)$, $Q_{\alpha}^i(x, y)$, $N_{\alpha\beta}^i(x, y)$ (bending, shear and normal components) per layer i ($\alpha, \beta \in \{1, 2\}$, $i \in \{1, n\}$). The related generalized displacements are $\Phi_{\alpha}^i(x, y)$, $V^i(x, y)$, $U_{\alpha}^i(x, y)$ (rotation, vertical and horizontal displacements). The link between two consecutive layers, i and $i+1$, is ensured by the interface shear $\tau_{\alpha}^{i,i+1}(x, y)$ and the normal $v^{i,i+1}(x, y)$ stresses. Each M4-5n layer has its own set of equations: compatibility, equilibrium and elastic constitutive law. The introduction of a vertical crack across a layer is performed by zeroing some of the generalized stresses along the crack contour projected on the (x, y) plane, in accordance with the assumptions made in terms of load transfer. On the other hand the debonding of interface $(i, i+1)$ can be taken into account by zeroing, on the relevant area, $\tau_{\alpha}^{i,i+1}(x, y)$ and possibly $v^{i,i+1}(x, y)$ depending on the contact conditions considered in the vertical direction. Discontinuities in the model are then induced from one side to the other of the contour of cracks or delaminated areas. The direct derivation of the M4-5n mathematical equations leads to a linear system of partial differential equations of second order with boundary conditions. This system involves a large number of unknown variables. The whole set of equations related to the pavement modeling using the M4-5n and the Winkler soil can be found in (Nasser and Chabot 2015). In this paper we present the main guidelines for the development of a mixed FEM method to solve more generally this set of equations. This method is then applied to a beam geometry example including a vertical crack across a layer and its implementation is validated by comparison with earlier results obtained with the FDM.

2 The mixed FEM to solve the M4-5n equations

The developed approach is inspired from the derivation of the H-R principle (Reissner 1950) for 3D elastic problems that we transpose to the generalized fields

of stresses and displacements of M4-5n. To accomplish this, we start from the theorem of maximum complementary energy, using the elastic density energy related to the generalized stresses of the M4-5n layers. The equilibrium equations and the stress boundary conditions are taken into account through fields of Lagrangian variables which, for an appropriate choice, appear to be the generalized displacements fields of M4-5n. The expression of a Lagrangian (L) is thus obtained. This can be integrated by parts to lower the derivation order of the generalized stresses. Then the mixed formulation is obtained by expressing the stationarity of L with respect to all fields ($\delta L = 0$). The resulting variational formulation is similar to that of H-R principle expressed in terms of the generalized displacement and stress fields. Then this formulation can be solved using a mixed FEM in which the generalized stress and displacement fields are discretized. A special attention must be paid to the dimensions of the interpolation spaces of the different fields to avoid undetermined system of algebraic equations (Lagrangian spaces to be chosen with lower dimension than those of the dual generalized stresses).

3 Example of 2D cracked pavement structure

To test the method described in the previous section, a specific FE numerical program along the x direction was developed for the computation of the M4-5n structures located in the real 2D (x, z) plane. All the M4-5n fields depend on x only (13n-4 for this “beam” geometry). Hereafter the results obtained with this numerical program for some given data set are compared to those from simulations based on FDM (Nasser and Chabot 2015). The structure under consideration is taken similar to a 2D composite pavement composed of three layers: a surface layer of 0.08m of semi-coarse asphalt concrete (BBSG, $E^1 = 5400\text{MPa}$, $\nu^1 = 0.35$); a base layer of 0.15m of gravel-bitumen (GB3, $E^2 = 9300\text{MPa}$, $\nu^2 = 0.35$); a foundation layer composed of 0.23m of cement-treated base (GC3, $E^3 = 23000\text{MPa}$, $\nu^3 = 0.25$). A fourth layer resting on Winkler springs (k) represents the soil ($E^4 = 120\text{MPa}$, $\nu^4=0.35$, $k=45.6\text{ MPa/m}$) (Figure 1).

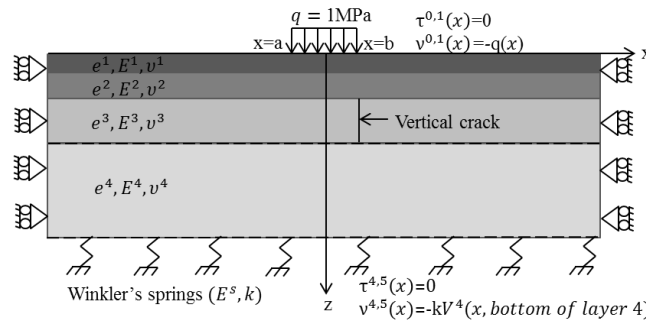


Fig. 1 Sketch of the structure considered for the comparison between the mixed FEM and FDM

The third layer is assumed to be crossed by a vertical crack simulating the shrinkage of the cement material. A unit pressure load (1MPa), $q(x)=H(x-a)-H(x-b)$, is applied at the surface of the structure over interval $[a,b]$ with $(b-a) = 0.15m$. The right edge of the load is taken coincident with the x location of the crack. The length of the structure is taken equal to 6m. The horizontal displacements $U^i(x)$ and rotations $\Phi^i(x)$ are blocked on the lateral boundaries, whereas the shear efforts per layer i , $Q^i(x)$ are set to zero. For this example, after integration by parts and here omitting the contribution of the crack for the sake of simplification, the Lagrangian writes as (Eq. 1):

$$\begin{aligned}
L(\widehat{N}^1, \widehat{M}^1, \widehat{Q}^1, \underbrace{\tau_{=0}^{0,1}}, \underbrace{\nu_{-q(x)}^{0,1}}, \widehat{U}^1, \widehat{\Phi}^1, \widehat{V}^1, \widehat{N}^2, \dots, \hat{\tau}^{1,2}, \hat{\nu}^{1,2}, \dots, \underbrace{\tau_{=0}^{4,5}}, \hat{\nu}^{4,5}, \dots, \widehat{V}^4) = \\
- \int_{-l}^l (w_{5n}^* + w_{Winkler}^*) dx + \sum_{i=1}^4 \int_{-l}^l [\widehat{N}^i \widehat{U}^i + \widehat{M}^i \widehat{\Phi}^i + \widehat{Q}^i \widehat{V}^i + \widehat{Q}^i \widehat{\Phi}^i] dx - \\
\sum_{i=2}^3 \int_{-l}^l [(\hat{\tau}^{i,i+1} - \hat{\tau}^{i-1,i}) \widehat{U}^i + (\hat{\nu}^{i,i+1} - \hat{\nu}^{i-1,i}) \widehat{V}^i + \frac{e^i}{2} (\hat{\tau}^{i-1,i} + \hat{\tau}^{i,i+1}) \widehat{\Phi}^i] dx - \\
\int_{-l}^l [\hat{\tau}^{1,2} (\widehat{U}^1 - \frac{e^1}{2} \widehat{\Phi}^1) - \widehat{V}^1 (q(x) + \hat{\nu}^{1,2}) + \hat{\tau}^{3,4} (\widehat{U}^4 + \frac{e^4}{2} \widehat{\Phi}^4) + \widehat{V}^4 (\hat{\nu}^{4,5} - \\
\hat{\nu}^{3,4})] dx - \sum_{i=1}^4 [\widehat{N}^i \widehat{U}^i + \widehat{M}^i \widehat{\Phi}^i]_{-l}^l \quad (1)
\end{aligned}$$

The symbol $(\widehat{\quad})$ is used to distinguish the virtual generalized fields from the solution of the problem. w_{5n}^* and $w_{Winkler}^*$ are the density of the elastic stress energy of the M4-5n and the energy of Winkler's springs. These are expressed as follows (Eq. 2-3):

$$\begin{aligned}
w_{5n}^* \left(\widehat{N}^1, \widehat{M}^1, \widehat{Q}^1, \underbrace{\tau_{=0}^{0,1}}, \underbrace{\nu_{-q(x)}^{0,1}}, \widehat{N}^2, \dots, \hat{\tau}^{1,2}, \hat{\nu}^{1,2}, \dots, \underbrace{\tau_{=0}^{4,5}}, \hat{\nu}^{4,5} \right) = \frac{1}{2} \sum_{i=1}^4 \left[\frac{1-\nu^i}{e^i E^i} \widehat{N}^i{}^2 + \right. \\
\left. \frac{12(1-\nu^i)}{e^{i^3} E^i} \widehat{M}^i{}^2 + \frac{12(1+\nu^i)}{5e^i E^i} \widehat{Q}^i{}^2 \right] + \sum_{i=2}^3 \left[\frac{13e^i}{35E^i} (\hat{\nu}^{i-1,i^2} + \hat{\nu}^{i,i+1^2}) + \frac{9e^i}{35E^i} \hat{\nu}^{i-1,i} \hat{\nu}^{i,i+1} + \right. \\
\left. \frac{4e^i(1+\nu^i)}{15E^i} (\hat{\tau}^{i-1,i^2} + \hat{\tau}^{i,i+1^2}) - \frac{2e^i(1+\nu^i)}{15E^i} \hat{\tau}^{i-1,i} \hat{\tau}^{i,i+1} - \frac{2(1+\nu^i)}{5E^i} \widehat{Q}^i (\hat{\tau}^{i-1,i} + \hat{\tau}^{i,i+1}) \right] + \\
\frac{13e^1}{35E^1} (q(x)^2 + (\hat{\nu}^{1,2})^2) + \frac{9e^1}{35E^1} q(x) \hat{\nu}^{1,2} + \frac{4e^1(1+\nu^1)}{15E^1} (\hat{\tau}^{1,2})^2 - \frac{2(1+\nu^1)}{5E^1} \widehat{Q}^1 \hat{\tau}^{1,2} + \\
\frac{13e^4}{35E^4} ((\hat{\nu}^{3,4})^2 + (\hat{\nu}^{4,5})^2) + \frac{9e^4}{35E^4} \hat{\nu}^{3,4} \hat{\nu}^{4,5} + \frac{4e^4(1+\nu^4)}{15E^4} (\hat{\tau}^{3,4})^2 - \frac{2(1+\nu^4)}{5E^4} \widehat{Q}^4 \hat{\tau}^{3,4} \quad (2)
\end{aligned}$$

$$w_{Winkler}^* = \frac{1}{2} \frac{(\widehat{\nu}^{4,5})^2}{k} \quad (3)$$

The solution fields are those which make the Lagrangian be stationary and check the condition $\delta L = 0$ for all variations of the virtual fields. The discretization of this condition by the FEM leads to a mixed algebraic system having the typical shape given in (Eq. 4). Σ and U denote the vectors of nodal values of the generalized stresses and displacements, respectively. F_Σ, F_U are the vectors of nodal forces related to $q(x)$.

$$\begin{bmatrix} K_{\Sigma\Sigma} & K_{\Sigma U} \\ {}^t K_{\Sigma U} & 0 \end{bmatrix} \begin{pmatrix} \Sigma \\ U \end{pmatrix} = \begin{pmatrix} F_{\Sigma} \\ F_U \end{pmatrix} \quad (4)$$

As expected this linear system is symmetric since it derives from a quadratic form for the Lagrangian (excepted the terms including the pressure load $q(x)$). All the fields are interpolated by Lagrange polynomials ($\varphi(x)$) with the following orders (Eq. 5) ensuring the determination of the system:

$$\begin{aligned} \dim(\varphi_N) &= 5, \dim(\varphi_M) = 3, \dim(\varphi_Q) = 4, \dim(\varphi_{\tau}) = 4, \dim(\varphi_{\nu}) = 4, \\ \dim(\varphi_U) &= 4, \dim(\varphi_{\phi}) = 2, \dim(\varphi_V) = 3 \end{aligned} \quad (5)$$

Figure 2 shows the typical 1D finite element used to calculate the elementary matrix between two consecutive nodes j and $j+1$. A local condensation method is used to eliminate the internal nodes and keep only the end nodes.

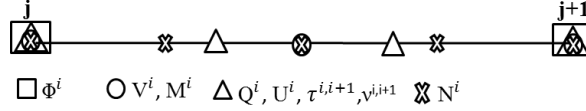


Fig. 2 1D M4-5n mixed finite element for 2D problems with multiple nodes related to the order of interpolation of each field and multiple degree of freedom related to the number of layers

The global matrix is then obtained as usual by assembly of the elementary condensed matrices for the whole 1D mesh. To introduce the vertical crack in a systematic way, we first double the nodes at the crack location and then all the degrees of freedom (dof) for these nodes. The continuity of stresses and displacements in the non-cracked layers is ensured by first penalty terms. The nil values of $\hat{N}^3, \hat{M}^3, \hat{Q}^3$ in layer 3 at the crack edge (Figure 1) are accounted for by other penalty terms. The nodal values of the interfacial stresses $\hat{\tau}^{2,3}, \hat{\tau}^{3,4}, \hat{\nu}^{2,3}, \hat{\nu}^{3,4}$ as well as the displacements of the cracked layer $\hat{U}^3, \hat{\Phi}^3, \hat{V}^3$ are left free without any special treatment and thus can reflect discontinuities. By doing so the Lagrangian of Eq. 1 becomes an augmented Lagrangian leading to some additional coefficients with large values in the final algebraic system.

4 Results and Validation

We compared the mixed FEM described in this paper with the FDM defined in (Nasser and Chabot 2015) on the 2D composite pavement example presented in section 3 with a vertical crack across layer 3 (Figure 1). The structure is discretized into 160 finite elements of the same length (0.0375m). Figure 3 shows the good agreement obtained between the two methods. We can also note on these graphs the discontinuities induced by the crack (Figure 3.b). Moreover, as illustrated in Figure 3.c-d, the M4-5nW leads to finite values of the shear and the nor-

mal stress at interfaces and near cracks. These values could be used directly to evaluate the risk of a debonding phenomenon in a mixed mode fracture condition.

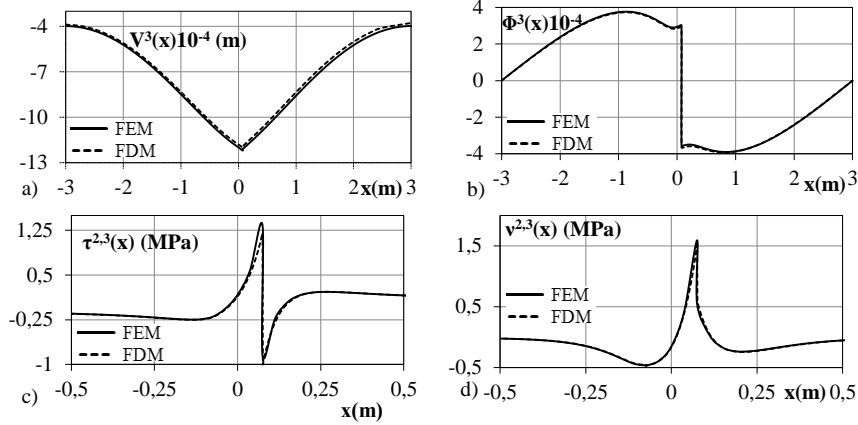


Fig. 3 Comparison between the mixed FEM and FDM: a) deflection $V^3(x)$; b) rotation $\Phi^3(x)$; c) interfacial shear stress $\tau^{2,3}(x)$; d) interfacial normal stress $\nu^{2,3}(x)$

5 Conclusion

This paper shows the advantages of the mixed variational H-R principle applied to M4-5n in terms of generalized stresses and displacements. This leads to a compact writing of the problem, making it possible to derive systematic numerical solving methods which can be extended in particular to plate geometry domains with cracks and partial delamination areas between two layers. The implementation of this extension will be our next step in order to apply the M4-5n to the modeling of 3D pavement structures (27n-6 unknowns for the plate geometry). The numerical developments will be based on the use of specialized open source software.

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