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A strict Lyapunov function for non-holonomic systems under persistently-exciting controllers

Mohamed Maghenem* Antonio Loría** Elena Panteley**†

* *Univ. Paris-Saclay, Orsay, France. E-mail: mohamed.maghenem@l2s.centralesupelec.fr*

** *CNRS, Gif sur Yvette, France. E-mail: antonio.loria@l2s.centralesupelec.fr*

† *ITMO University, St. Petersburg, Russia. E-mail: elena.panteley@l2s.centralesupelec.fr*

Abstract: We study the stability of a non linear time-varying skew symmetric systems $\dot{x} = A(t, x)x$ with particular structures that appear in the study problems of non holonomic systems in chained form as well as adaptive control systems. Roughly, under the condition that each non diagonal element of $A(t, x)$ is persistently exciting or uniform δ persistently exciting with respect x . Although some stability results are known in this area, our main contribution lies in the construction of Lyapunov functions that allows a computation of convergence rate estimates for the class of non linear systems under study.

Keywords: Time-varying systems, Nonholonomic systems, adaptive systems.

1. INTRODUCTION

We revisit the stabilisation problem for non holonomic systems in chained form, defined by the equations:

$$\begin{cases} \dot{x}_1 = u_2 \\ \dot{x}_i = u_1 x_{i-1}, & i \in [2, n-1] \\ \dot{x}_n = u_1. \end{cases} \quad (1)$$

Such systems are used to model a variety of kinematic constraints appearing in a number of mechanical systems such as autonomous multiple-trailer vehicles, multi-body spacecrafts, *etc.*. See the survey Kolmanovsky and McClamroch [1995] for more details. Ever since the seminal work Brockett [1983] in which it is stated that chain-form systems cannot be stabilized at the origin by means of smooth time-invariant feedback, the stabilization problem attracted an exponentially-increasing interest in the community. Perhaps most of contributions in the field may be classified into discontinuous feedback controls, as in Astolfi [1996], Sordalen and Egeland [1995], and smooth time-varying, as in Morin and Samson [1997], Samson [1995].

Notably, in Samson [1995] the author proposed a class of smooth controllers which ensure global asymptotic stability. The controllers in Samson [1995] rely on a simple but powerful idea: to use exogenous signals of time, called “heating functions” in this reference, in order to excite all modes of the system. Another crucial property of the controllers in Samson [1995] is that they lead to a system in closed-loop with a so-called skew-symmetric structure, reminiscent of systems that appear in adaptive control via reference model.

The control design, as well as the underlying concepts used in Samson [1995] inspired our so-called δ -persistently exciting controllers, originally proposed in Loría et al. [1999]. See also the more evolved work Loría et al. [2002] where we established, for the first time, *uniform* global asymptotic stability via smooth time-varying control. Indeed, the method of proof in Samson [1995] does not allow to conclude uniformity of the origin’s attractivity.

In this paper we revisit the stabilisation problem for non-holonomic systems in chain form, retracing the steps of Samson [1995] and Loría et al. [2002]. As in these references, we use controllers with persistency of excitation (the term is not used in Samson [1995]). However, our main and novel contribution is to establish an estimate of the convergence rate in terms of the control parameters. Our analysis relies on constructing a strict Lyapunov function for skew-symmetric systems. Indeed, we have been able to locate in the literature strict Lyapunov functions for this type of systems.

2. PROBLEM STATEMENT AND MOTIVATION

To put our contributions in perspective we start by recalling the essential elements of the elegant control approach tailored in Samson [1995]. We start with the observation that the chain-form system (1) may be rewritten in the general form of a driftless system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-2} \\ 1 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2. \quad (2)$$

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Now, following Samson [1995], let us consider the following change of coordinates, defined starting with the n th variable down to the first, that is,

$$\bar{x}_{n-1} = x_{n-1} \quad (3a)$$

$$\bar{x}_{n-2} = x_{n-2} \quad (3b)$$

$$\bar{x}_j = k_{j+2}\bar{x}_{j+2} + L_{g_1}\bar{x}_{j+1} \quad 1 \leq j \leq n-3 \quad (3c)$$

where $k_{j+2} > 0$ for $1 \leq j \leq n-3$, and L_{g_1} denotes the Lie derivative, that is,

$$L_{g_1}\bar{x}_{j+1} := \frac{\partial \bar{x}_{j+1}}{\partial x} g_1(x)$$

Remark that the last change of coordinate has the following explicit form:

$$\bar{x}_j = x_j + \phi_j(x_{j+1}, \dots, x_{n-1}) \quad (4)$$

where $\phi_j(\cdot) : \mathbb{R}^{n-j-1} \rightarrow \mathbb{R}$, is sufficiently smooth function. We remark also that for $j \geq 1$,

$$\dot{\bar{x}}_{j+1} = L_{g_1}\bar{x}_{j+1}u_1 + L_{g_2}\bar{x}_{j+1}u_2 \quad (5)$$

Now, from (4), we have $L_{g_2}\bar{x}_{j+1}u_2 = 0$, for all $j \geq 1$. Then, using (3) and (5), we obtain

$$\dot{\bar{x}}_{j+1} = u_1\bar{x}_j - k_{j+2}u_1\bar{x}_{j+2} \quad \forall j \geq 1 \quad (6)$$

and, for $j = 1$,

$$\begin{aligned} \dot{\bar{x}}_1 &= \dot{x}_1 + \dot{\phi}_1(x_2, \dots, x_{n-1}) \\ &= u_2 + \dot{\phi}_1(x_2, \dots, x_{n-1}). \end{aligned} \quad (7)$$

So, defining

$$u_2(t, x) = -k_1\bar{x}_1 - k_2\bar{x}_2 - \dot{\phi}_1(x_2, \dots, x_{n-1}), \quad (8)$$

with $k_1, k_2 > 0$, the closed-loop dynamics takes the convenient cascaded form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \vdots \\ \dot{\bar{x}}_{n-1} \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2u_1 & \cdots & 0 \\ u_1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -k_{n-1}u_1 \\ 0 & \cdots & u_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n-1} \end{bmatrix} \quad (9)$$

$$\dot{x}_n = u_1 \quad (10)$$

–cf. Samson [1995], Loría et al. [2002].

Next, consider for (9) the Lyapunov function candidate

$$V_1(\bar{x}) = \frac{1}{2} \left[\bar{x}_1^2 + \sum_{i=2}^{n-1} \left(\prod_{l=2}^i k_l \right) \bar{x}_i^2 \right] \quad (11)$$

which is positive definite and radially unbounded. Actually,

$$\min_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\} |\bar{x}|^2 \leq 2V_1(\bar{x}) \leq \max_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\} |\bar{x}|^2. \quad (12)$$

Moreover, in view of the “skew-symmetry” of the matrix in (9)

$$\dot{V}_1(\bar{x}) = -k_1\bar{x}_1^2. \quad (13)$$

Therefore, $\{\bar{x} = 0\}$ is uniformly globally stable for (9) that is, the solutions are uniformly globally bounded and the origin is uniformly stable. Moreover, this property holds

with linear gain; this follows from integrating $\dot{V}(\bar{x}(t)) \leq 0$ to obtain $|x(t)| \leq c|x(t_0)|$ with

$$c := \frac{\max_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\}}{\min_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\}}. \quad (14)$$

The challenge, then, is to design a smooth time-varying control law $u_1(t, x)$ that guarantees uniform global attractivity of the origin $(\bar{x}, x_n) = (0, 0)$ for the overall system (9) and (10).

In Loría et al. [2002] it was showed that

$$u_1(t, x) = -k_n x_n + h(t, \bar{x}) \quad (15)$$

with h satisfying certain property of persistency of excitation, achieves the control goal. The central idea, which is inspired by Samson [1995], is to design this function to render u_1 persistently exciting to render the origin $\{\bar{x} = 0\}$ of (9) uniformly globally attractive. Simultaneously, relative to the x_n equation (10), h must be a bounded perturbation vanishing with \bar{x} .

The property of persistency of excitation was coined in the context of systems identification. For the particular case of a locally integrable scalar function $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, it is defined as follows.

Definition 1. (Persistency of Excitation). The function a is persistently exciting if there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} |a(s)| ds > \mu, \quad \forall t \geq 0. \quad (16)$$

For nonlinear functions of the system’s state and time the following property was introduced in Loría et al. [1999], Panteley et al. [2001].

Definition 2. (Uniform δ -PE along trajectories). The continuous function $a : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ is uniformly δ -persistently exciting (u δ -PE) with respect to x , if for each $\delta > 0$ there exist $\mu > 0$, $T > 0$ such that

$$\min_{s \in [t, t+T]} |x(s)| > \delta \Rightarrow \int_t^{t+T} |a(s, x(s))| ds > \mu \quad \forall t \geq 0. \quad (17)$$

In Panteley et al. [2001] it is showed that u δ -PE is a necessary and sufficient condition for uniform global asymptotic stability for a class of nonlinear time-varying systems which include (9) for $n = 2$ that is,

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2u_1(t, \bar{x}) \\ u_1(t, \bar{x}) & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}. \quad (18)$$

The rationale to conclude uniform global attractivity of the origin for (18) is the following. First, we observe that the origin is uniformly globally stable; indeed, V_1 in (11) for this system corresponds to

$$V_1(\bar{x}) = \frac{1}{2} [\bar{x}_1^2 + k_2\bar{x}_2^2].$$

whose derivative satisfies (13) and, therefore, $V(\bar{x}(t)) \leq V(\bar{x}(t_0))$ which implies that $|\bar{x}(t)| \leq c|\bar{x}(t_0)|$ for all $t \geq t_0$ with $c := [\max\{1, k_2\} / \min\{1, k_2\}]$. It also follows that for any $\sigma > 0$, defining $\delta := \sigma/c$, we have $|\bar{x}(t')| \leq \delta$ implies that

$$|\bar{x}(t' + T)| \leq \sigma \quad \forall T > 0. \quad (19)$$

This holds for any $t' \geq t_0$. The property (19) implies uniform global attractivity hence, to establish uniform global asymptotic stability, it is left to show that for any $\delta > 0$ there exists $t' \geq t_0$ such that $|\bar{x}(t')| \leq \delta$ for any initial states $|x_0| \leq r$ and any $r > 0$. To establish this consider u_1 along the trajectories $\bar{x}(t)$ that is, $u_1(t, \bar{x}(t))$. Then, the system (18) may be considered as linear time-varying with $a(t) := u_1(t, \bar{x}(t))$. It is well known that the origin of

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 a(t) \\ a(t) & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \quad (20)$$

is exponentially stable if and only if a is persistently exciting. Therefore, for the nonlinear system (18), we may conclude that under the condition that $u_1(t, \bar{x})$ is u δ -PE in the sense of Definition 2 the trajectories converge exponentially fast to zero hence, there exists a finite time T_δ , independent of t_0 such that $|x(t_0 + T_\delta)| \leq \delta$ and (19) holds with $t' := t_0 + T_\delta$.

In Loría et al. [2002] a similar argument is used to establish uniform global attractivity of the origin of (9) for any $n \geq 2$. However, the proof is based on an inductive argument following an intricate trajectory-based analysis. In this paper, we give an estimate of the time of convergence T_δ . Our analysis is constructive as it relies on an original strict Lyapunov function for linear “skew-symmetric” systems. This constitutes our first result.

3. LYAPUNOV ANALYSIS OF SKEW-SYMMETRIC SYSTEMS

3.1 The linear case

Fundamental to our main results is the following preliminary but original statement for so-called skew-symmetric systems,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 a(t) & 0 & \cdots & 0 \\ a(t) & 0 & -k_3 a(t) & 0 & \vdots \\ 0 & a(t) & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -k_{n-1} a(t) \\ 0 & \cdots & 0 & a(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix}. \quad (21)$$

Lemma 1. Consider the skew symmetric system (21), with $k_i > 0$ for all $i \in [1, n-1]$. For the function $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ assume that there exist positive real constants \bar{a} , μ and T such that

$$\max \left\{ \sup_{t \geq 0} |a(t)|, \sup_{t \geq 0} |\dot{a}(t)| \right\} \leq \bar{a} \quad a.e. \quad (22)$$

$$\int_t^{t+T} |a(s)| ds \geq \mu \quad \forall t \geq 0. \quad (23)$$

Then, the origin is uniformly exponentially stable.

Furthermore, for each $i \leq n$ let us define (in reverse order),

$$\alpha_n = 1, \quad \alpha_{n-1} = 1 + \alpha_n + \frac{9nT\bar{a}^2\alpha_n^2}{\mu k_n} \quad (24a)$$

$$\alpha_i = 1 + \alpha_{i+1} + \frac{9nT\bar{a}^2\alpha_{i+1}^2}{\mu k_{i+1}} + \frac{nT[k_{i+2}\alpha_{i+1} + \alpha_{i+2}k_{i+1}]^2}{\mu k_{i+1}k_{i+2}} \quad (24b)$$

as well as the constant

$$\gamma \geq \frac{\bar{a}^4[\alpha_2 + 1]}{k_1} + \frac{nT\alpha_2^2[3\bar{a}^3 + \bar{a}^4k_1]^2}{\mu k_2 k_1} + \frac{nT\bar{a}^8[\alpha_2 k_3 + k_2 \alpha_3]^2}{\mu k_2 k_3 k_1} + 2 \frac{\bar{a}^3 \sum_{i=1}^{n-1} \alpha_{i+1} \left(\prod_{j=2}^i k_j \right)}{\min \left\{ 1, \prod_{j=2}^i k_j \right\}} \quad (25)$$

and the function

$$Q_{a^4}(t) = 1 + \bar{a}^4 T - \frac{1}{T} \int_t^{t+T} \int_t^m a(s)^4 ds dm. \quad (26)$$

Then, there exist η_1, η_2 and $\eta_3 > 0$ such that the Lyapunov function

$$V_n(t, x) = [Q_{a^4}(t) + \gamma] V_1(x) + a^3 \sum_{i=1}^{n-1} \alpha_{i+1} \left[\prod_{j=2}^i k_j \right] x_i x_{i+1}, \quad (27)$$

where V_1 is defined in (11), satisfies

$$\eta_1 |x|^2 \leq V_n(t, x) \leq \eta_2 |x|^2 \quad (28)$$

$$\dot{V}_n(t, x) \leq -\frac{\mu}{2T} |x|^2. \quad (29)$$

□

Proof. We first show the existence of η_1 and η_2 . To that end, note that

$$1 \leq Q_{a^4}(t) \leq 1 + \bar{a}^4 T \quad (30)$$

while the cross terms in (27) satisfy

$$a^3 \sum_{i=1}^{n-1} \alpha_{i+1} \left[\prod_{j=2}^i k_j \right] x_i x_{i+1} \leq \gamma V_1(x)$$

that is, $V_1(x) \leq V_n(t, x) \leq [1 + \bar{a}^4 T + 2\gamma] V_1(x)$. The bound (28) follows from the latter and (12) with

$$\eta_1 := \frac{1}{2} \min_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\} \quad (31a)$$

$$\eta_2 := \max_{i \in [2, n]} \left\{ 1, \prod_{j=2}^i k_j \right\} [1 + \bar{a}^4 T + 2\gamma]. \quad (31b)$$

Next, we evaluate the total derivative of V_n along the trajectories of (21). To that end, we first note that

$$\dot{Q}_{a^4}(t) = -\frac{1}{T} \int_t^{t+T} a(s)^4 ds + a(t)^4 \quad (32)$$

therefore,

$$\begin{aligned}
\dot{V}_n(t, x) \leq & -\gamma k_1 x_1^2 - \frac{\mu}{T} V_1(x) + a^4 V_1(x) \\
& + a^4 \left[\sum_{i=2}^{n-1} \alpha_{i+1} \left(\prod_{j=2}^i k_j \right) [x_{i-1} x_{i+1} \right. \\
& \quad \left. - k_{i+1} x_{i+1}^2 + x_i^2 - k_{i+2} x_i x_{i+2}] \right] \\
& + a^4 \alpha_2 [-k_1 x_1 x_2 - k_2 x_2^2 + x_1^2 - k_3 x_1 x_3] \quad (33)
\end{aligned}$$

and, expanding terms, we obtain

$$\begin{aligned}
\dot{V}_n(t, x) \leq & -\frac{\mu}{2T} V_1(x) + \left[-\gamma k_1 x_1^2 + 3a^2 \dot{a} \alpha_2 x_1 x_2 \right. \\
& - a^4 \alpha_2 k_1 x_1 x_2 + (\alpha_2 + 1) a^4 x_1^2 \\
& - a^4 (\alpha_2 k_3 - k_2 \alpha_3) x_1 x_3 \\
& \quad \left. - \frac{\mu}{2nT} (k_2 x_2^2 + k_2 k_3 x_3^2) \right] \\
& + \sum_{i=2}^{n-2} \left[\prod_{l=2}^i k_l \right] \left[(1 - \alpha_i) a^4 x_i^2 + 3a^2 \dot{a} \alpha_{i+1} x_i x_{i+1} \right. \\
& \quad + \alpha_{i+1} a^4 x_i^2 \\
& \quad - a^4 (k_{i+2} \alpha_{i+1} - \alpha_{i+2} k_{i+1}) x_i x_{i+2} \\
& \quad \left. - \frac{\mu}{2nT} (k_{i+1} x_{i+1}^2 + k_{i+1} k_{i+2} x_{i+2}^2) \right] \\
& + \left[\prod_{l=2}^{n-1} k_l \right] \left[-(\alpha_{n-1} - 1) a^4 x_{n-1}^2 \right. \\
& \quad + 3a^2 \dot{a} \alpha_n x_{n-1} x_n \\
& \quad + \alpha_n a^4 x_{n-1}^2 - \frac{\mu}{2nT} k_n x_n^2 \\
& \quad \left. - (\alpha_n - 1) a^4 \left(\prod_{l=2}^n k_l \right) x_n^2 \right] \quad (34)
\end{aligned}$$

All cross terms of undefined sign on the right-hand side of the previous inequality are quadratic while V_1 is quadratic positive definite in $[x_1 \cdots x_{n-1}]$. Therefore, we can always choose the design parameters α_i and γ to render \dot{V}_n negative definite.

To start with, for any $\alpha_2, k_1, k_2, k_3, \mu$ and $T > 0$, we pick γ such that

$$\begin{aligned}
& [(\alpha_2 + 1) a^4 - \gamma k_1] x_1^2 + 2\alpha_2 (3a^2 \dot{a} - a^4 k_1) x_1 x_2 - \frac{\mu k_2}{nT} x_2^2 \leq 0 \\
& \text{and} \\
& [(\alpha_2 + 1) a^4 - \gamma k_1] x_1^2 - 2a^4 (\alpha_2 k_3 - k_2 \alpha_3) x_1 x_3 - \frac{\mu k_2 k_3}{nT} x_3^2 \leq 0.
\end{aligned}$$

Next, we choose $\alpha_n = 1$ and α_{n-1} , such that:

$$-[\alpha_{n-1} - 1 - \alpha_n] a^4 x_{n-1}^2 + 3a^2 \dot{a} \alpha_n x_{n-1} x_n - \frac{\mu k_n}{2nT} x_n^2 \leq 0.$$

Finally, for each $i \leq n - 2$ down to $i = 1$ we choose α_i , such that

$$-[\alpha_i - 1 - \alpha_{i+1}] a^4 x_i^2 + 6a^2 \dot{a} \alpha_{i+1} x_i x_{i+1} - \frac{\mu k_{i+1}}{nT} x_{i+1}^2 \leq 0$$

and

$$\begin{aligned}
& -[\alpha_i - 1 - \alpha_{i+1}] a^4 x_i^2 - 2a^4 [k_{i+2} \alpha_{i+1} \\
& \quad - \alpha_{i+2} k_{i+1}] x_i x_{i+2} - \frac{\mu}{nT} k_{i+1} k_{i+2} x_{i+2}^2 \leq 0.
\end{aligned}$$

All of the inequalities above hold in view of (24) and (25) hence (29) holds. \blacksquare

The advantage of Lemma 1 with respect to other statements on stability for the system (9) –cf. Loría and Panteley [2002] is that the Lyapunov function V_n leads directly to an expression for the system's trajectories. Indeed, from (28), (29) and (31) we have

$$|x(t)|^2 \leq \frac{\eta_2}{\eta_1} |x(t_0)|^2 \exp\left(-\frac{\mu}{2T\eta_3} [t - t_0]\right). \quad (35)$$

3.2 The nonlinear case

Let us consider now the nonlinear skew-symmetric system (9) under a condition of uniform δ -persistency of excitation on the control law u_1 , in the sense of Def. 2. Along the system's trajectories, $a(t) := u_1(t, \bar{x}(t))$ is persistently exciting for all t such that $|\bar{x}(t)| \geq \delta$, for any $\delta > 0$ therefore, the solutions of (9) converge exponentially to zero according to (35) —note that an oscillatory behaviour by which $|\bar{x}(t)|$ might cross the boundary $|\bar{x}(t)| = \delta$ multiple times is excluded since the origin is uniformly stable. Even though the stabilizing mechanism of uniform δ -persistency of excitation in the sense of Definition 2 is intuitive, the inconvenience of this property is that it is formulated as a property of $a : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ and the system's trajectories. The following property which was introduced in Loría et al. [2005] has the advantage of being stated in terms of the system's state variable.

Definition 3. (Uniform δ -PE). The scalar function $a : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ is uniformly δ -persistently exciting with respect to x , if for each $\delta > 0$ there exist $\mu > 0$ and $T > 0$ such that

$$|x| > \delta \implies \int_t^{t+T} |a(s, x)| ds > \mu \quad \forall t \geq 0. \quad (36)$$

In general, for multivariable functions, the two properties, in Defs. 2 and 3, are different. Neither one implies the other –see Loría et al. [2005] however, for the type of functions of interest here, the following statement establishes a link between the two properties.

Lemma 2. Let the function $a : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Definition 3. In addition, assume that the existence of ρ_1 and $x \in \mathbb{R}$ such that $|x| > \delta$ and $|a(t, x)| > \rho_1$, implies the existence of $\rho_2 \in (0, \rho_1)$ such that $|a(t, x)| > \rho_2$ for all x such that $|x| > \delta$. Then, $|a|$ is uniformly δ persistently exciting along trajectories, i.e., it satisfies Definition 2. \square

Proof. By assumption, the function $a : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Definition 3. Let the latter generate μ, T and $\delta > 0$ such that (36) holds. Let $x \in \mathbb{R}$ be arbitrarily fixed, such that $|x| > \delta$. Then, $a(t, x)$, for such fixed x , is persistently exciting that is, it satisfies (16). By [Loría and Panteley, 2002, Lemma 2] it follows that there exists $\rho_1 > 0$ such that, for each t , the set $I_t := \{\tau \in [t, t+T] : |a(\tau, x)| \geq \rho_1\}$ has strictly positive uniform measure that is, $\text{meas}(I_t) \geq \Delta > 0$ with Δ independent of t . By assumption, there exists $\rho_2 \in (0, \rho_1)$ such that, for all $t \in I_t$ and all x such that $|x| > \delta$, we have $|a(t, x)| > \rho_2$. In turn, this implies that

$$\int_t^{t+T} |a(s, x(s))| ds > \Delta \rho_2 \quad \forall t \geq 0$$

that is, (17) holds with $\mu := \Delta\rho_2$. \blacksquare

Thus, from the previous analysis, we draw the following conclusion.

Lemma 3. Consider the system (9). Let $k_i > 0$ for all $i \leq n$ and let u_1 satisfy the uniform continuity condition of Lemma 2 and be uniformly δ -persistently exciting in the sense of Definition 3. Then, the origin $\{\bar{x} = 0\}$ is uniformly globally asymptotically stable. Moreover, for any $r > 0$ and $\sigma > 0$, we have

$$|x(t_0)| \leq r \implies |x(t)| \leq \sigma \quad \forall t \geq T_{r,\sigma}$$

with

$$T_{r,\sigma} = -\frac{2T\eta_3}{\mu} \ln \left(\frac{\eta_1[\sigma/c]^2}{\eta_2 r^2} \right) \quad (37)$$

and c is defined in (14). \square

Proof. The origin of the system is uniformly globally stable and satisfies $|x(t)| \leq c|x(t_0)|$ for all $t \geq t_0$ and all $t_0 \geq 0$ —see Section 2. Let $\delta := \sigma/c$. In view of Lemma 2, u_1 is uniformly δ -persistently exciting along the system's trajectories. Let $a(t; t_0, x_0) := u_1(t, x(t; t_0, x_0))$. Then, for all t such that $|x(t)| \geq \delta$, the trajectories of (9) coincide with those of (21). It follows that the solutions of the former satisfy (35) at least for a finite time, that is, there exists $T' > 0$ such that (35) holds for all $t \in [t_0, t_0 + T']$ and, at $t' := t_0 + T'$, $|x(t')| = \delta$. Then, we have

$$\delta^2 = \frac{\eta_2}{\eta_1} r^2 \exp \left(-\frac{\mu T'}{2T\eta_3} \right)$$

which is equivalent to:

$$T' = -\frac{2T\eta_3}{\mu} \ln \left(\frac{\eta_1 \delta^2}{\eta_2 r^2} \right).$$

Now, in view of uniform global stability, $|x(t)| \leq c|x(t')|$ for all $t \geq t'$ that is, using $\delta = \sigma/c$, we verify that $|x(t)| \leq \sigma$ for all $t \geq t_0 + T_{r,\sigma}$ with $T_{r,\sigma} = T'$. \blacksquare

4. MAIN RESULTS

In the previous section we presented a strict Lyapunov function for linear time-varying skew-symmetric systems which may be used to compute an estimate of the convergence rate of the trajectories of the nonlinear time-varying system (9). Based on this statement we may now present our main result for the nonholonomic chain-form system (1).

Theorem 1. Consider the system (1) in closed loop with (8) and

$$u_1(t, x) = -k_n x_n + h(t, y), \quad y := [\bar{x}_1, \dots, \bar{x}_{n-1}]^\top \quad (38)$$

where $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ is bounded and smooth, more precisely,

B1. (*Boundedness*) there exists a function $\rho \in \mathcal{K}$, such that:

$$\max \left\{ |h(\cdot)|, \left| \frac{\partial h(\cdot)}{\partial t} \right|, \left| \frac{\partial h(\cdot)}{\partial y} \right| \right\} \leq \rho(|y|). \quad (39)$$

B2. (*U δ -PE*) The function $[\partial h / \partial t](t, y(\bar{x}))$ is uniformly continuous (it satisfies the conditions of Lemma 2) and is uniformly δ -persistently exciting with respect to \bar{x} that is, in the sense of Definition 3. \blacksquare

B3. (*Integrability*) For all $\|x(t_0)\| \leq r$, there exists $\omega_r \geq 0$, such that:

$$\int_{t_0}^{\infty} \left| \frac{\partial h}{\partial \bar{x}_1}(s, y(s)) \bar{x}_1(s) \right| ds \leq \omega_r \quad (40)$$

Then, the origin is uniformly globally asymptotically stable and, for any $r > 0$ and $\sigma > 0$, we have

$$|x(t_0)| \leq r \implies |x(t)| \leq \sigma \quad \forall t \geq T_{r,\sigma}$$

Proof. The total derivative of the quadratic function

$$W(x) := V_1(x) + \frac{1}{2} x_n^2$$

along the system's trajectories yields

$$\begin{aligned} \dot{W}(x(t)) &\leq -k_n x_n(t)^2 - k_1 x_1(t)^2 + |x_n(t)| |h(t, y(t))| \\ &\leq -k_n x_n(t)^2 - k_1 x_1(t)^2 + \rho(|y(t)|) |x_n(t)|. \end{aligned} \quad (41)$$

By Lemma 3 $|y(t)|$ satisfies, on the maximal interval of definition of the solutions, $|y(t)| \leq c|x(t_0)|$. By continuity of the solutions, however, this interval may be extended to infinity hence, for all $t \geq t_0$,

$$\dot{W}(x(t)) \leq -k_n x_n(t)^2 - k_1 x_1(t)^2 + \rho(cr) |x_n(t)| \quad (42)$$

hence, for “large” values of $x_n(t)^2$ we see that $\dot{W}(x(t)) \leq 0$ and the solutions are uniformly globally bounded with linear bound that is, $|x(t)| \leq c|x(t_0)|$ for all $t \geq t_0$.

On the other hand, the control u_1 satisfies

$$\dot{u}_1 = - \left[k_n - \frac{\partial h}{\partial y} A y \right] u_1 - k_1 \frac{\partial h}{\partial \bar{x}_1} \bar{x}_1 + \frac{\partial h}{\partial t}(t, y) \quad (43)$$

where

$$A = \begin{bmatrix} 0 & -k_2 & \dots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -k_{n-1} \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

By assumption, $\frac{\partial h}{\partial t}(t, y(\bar{x}))$ is uniformly δ -PE with respect to \bar{x} . Therefore, by Lemma 4 from the Appendix, u_1 is uniformly δ -persistently exciting. It follows from Lemma 3, for any $r \geq 0$ and $\sigma > 0$,

$$|x(t_0)| \leq r \implies |y(t)| \leq \sigma \quad \forall t \geq t_0 + T_{r,\sigma}$$

with $T_{r,\sigma}$ as in (37). Resetting the initial time to $t' := t_0 + T_{r,\sigma}$ and solving the differential equation

$$\dot{x}_n = -k_n x_n + h(t, y),$$

we obtain

$$\begin{aligned} |x_n(t)| &\leq |x_n(t')| \exp(-k_n(t-t')) \\ &\quad + \frac{1}{k_n} \left[1 - \exp(-k_n(t-t')) \right] \rho(cr) \end{aligned}$$

for all $t \geq t_0 + T_{r,\sigma}$ that is, for all such t ,

$$|x_n(t)| \leq \exp(-k_n(t-t')) \left[cr - \frac{\rho(cr)}{k_n} \right] + \frac{\rho(cr)}{k_n}.$$

We wrap up the paper with a concise statement that gives an interesting particular choice of the function h such \blacksquare

that the control law satisfies the required condition on persistency of excitation. Let

$$h(t, y(\bar{x})) := \frac{\varphi(t)}{2} \left[k_n x_n^2 + \sum_{i=1}^{n-2} \frac{\bar{x}_i^2}{k_{i+1} \cdots k_{n-1}} + x_{n-1}^2 \right].$$

Then, in view of the “skew-symmetry” of A we have $\frac{\partial h}{\partial \bar{x}} A \bar{x} = 0$, and $\dot{u}_1 = -k_n u_1 - k_1 \bar{x}_1^2 + \frac{\partial h}{\partial t}(t, y)$, so, by the filtering property of PE functions, it is trivial to see that u_1 is uniformly δ persistently exciting provided that so is $\dot{\phi}$.

5. CONCLUSION

We have presented new results on stabilization of non-holonomic systems via smooth time-varying feedback. Our controllers rely on a property of persistency of excitation that implies the exponential convergence to any compact containing the origin. The formulation and the analysis tools that we employ, notably based on Lyapunov’s direct method, allow to compute estimates on the speed of convergence of the solutions.

Appendix A

We present a technical lemma that generalizes a well-known property of persistently exciting signals $a(t)$ which establishes that the output of a strictly proper stable filter driven by a PE input conserves such property. The lemma is actually reminiscent of a similar statement originally presented in Panteley et al. [2001]. However, for the purposes of this paper we paraphrase the statement and present an alternative proof.

Lemma 4. (Filtration property). Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \rightarrow \mathbb{R}$ and consider the system:

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} f(t, x, \omega) \\ f_2(t, x, \omega) + \psi(t, x) + \phi(t, x) \end{bmatrix} \quad (\text{A.1})$$

with $f_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$ locally Lipschitz in x , uniformly in t and measurable in t . Assume that $\phi(t, x)$ is $U\delta PE$ with respect to x . If ϕ and ψ are locally Lipschitz and there exists a non decreasing function $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that, for almost all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$:

$$\max \left\{ |\phi(\cdot)|, |\psi(\cdot)|, |f(\cdot)|, |f_2(\cdot)|, \left| \frac{\partial \phi(\cdot)}{\partial t} \right|, \left| \frac{\partial \phi(\cdot)}{\partial x_i} \right| \right\} \leq \rho_1(|x|).$$

Assume, further, that all solutions $t \mapsto x_\phi$, with $x_\phi = [x^\top, \omega]^\top$, are defined in $[t_0, \infty)$ and satisfy:

$$|x_\phi(t, t_0, x_{\phi_0})| \leq r \quad \forall t \geq t_0 \quad (\text{A.2})$$

and, there exists $\psi_r > 0$, such that:

$$\int_{t_0}^t |\psi(s, x(s))|^2 ds \leq \psi_r \quad \forall t \geq t_0 \quad (\text{A.3})$$

then ω is uniformly δ -persistently exciting with respect to x . Moreover,

$$T_{r\delta} = T \left(1 + \frac{2\psi_r + 4c(r)r}{\mu} \right), \quad \mu_{r\delta} = \frac{T_{r\delta}\mu}{4Tc(r)^2}. \quad (\text{A.4})$$

where $c(r) := 2\rho_1^2(r) + \rho_1(r)$. \square

Proof. The total derivative of the product $\omega\phi$ satisfies, in view of the boundedness of trajectories of (A.1),

$$\begin{aligned} \frac{d}{dt} \left\{ \omega\phi \right\} &= \phi^2 + \psi\phi + \omega \left[f_2\phi + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f \right] \\ &\geq -|\omega| \left[2\rho_1^2(r) + \rho_1(r) \right] + \frac{1}{2} (|\phi|^2 - |\psi|^2) \\ &\geq -c(r) |\omega| + \frac{1}{2} (|\phi|^2 - |\psi|^2) \\ \frac{d}{dt} (\omega\phi) &\geq -c(r) |\omega| + \frac{1}{2} (|\phi|^2 - |\psi|^2) \end{aligned} \quad (\text{A.5})$$

Then, since ϕ is uniformly δ -persistently exciting, there exist $\mu > 0$, such that:

$$\int_t^{t+(k+1)T} (\phi^2(\tau) - \psi^2(\tau)) d\tau \geq (k+1)\mu - \psi_r$$

Integrating (A.5) between $[t, t + (k+1)T]$ both with the Cauchy-Schwartz inequality applied to $\int_t^{t+(k+1)T} |\omega(\tau)| d\tau$, we get

$$\int_t^{t+(k+1)T} \omega^2(\tau) d\tau \geq \frac{\left[\frac{1}{2}(k+1)\mu - \frac{1}{2}\psi_r - c(r)r \right]^2}{c^2(r)T_{r\delta}} = \mu_{r\delta}.$$

Finally it remains to choose k such that we get $\mu_{r\delta} > 0$. \blacksquare

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