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Abstract: We present a smooth nonlinear time-varying controller for leader-follower tracking of non-holonomic mobile robots. Our design relies upon the standing assumption that *either* the rotational *or* the translational reference velocity is persistently exciting. Then, we extend our results to cover the problem of formation tracking for a swarm of vehicles interconnected under a spanning tree communication topology rooted at the virtual leader. In this case, we propose a simple distributed control law that establishes the convergence of the error coordinate of each agent, relatively to its neighbourhood, under the same condition of persistency of excitation. In addition, our proofs are based on Lyapunov's second method, that is, we provide a strict Lyapunov function.

Keywords: Consensus, Formation control, Autonomous vehicles, Nonholonomic systems

1. INTRODUCTION

Tracking control of non-holonomic mobile robots has been long addressed by the nonlinear control community starting, at least, with the seminal paper Kanayama et al. [1990] in which global stability was established using Lyapunov's first method. In dagger and Nijmeijer [1997] a backstepping approach was used to construct a controller that guarantees asymptotic stability for both, tracking and set-point stabilisation. The latter was generalised to the adaptive case in Fukao et al. [2000] –see also Huang et al. [2014]. In Wang et al. [2009] a finite-time tracking controller is designed using mainly two finite-time stabilising control laws; the stability analysis appeals to a cascades argument. In Panteley et al. [1998] a simple linear time-varying controller was proposed and uniform global asymptotic stability was established under the standing assumption is that the reference angular velocity is persistently exciting. In Fliess et al. [1995] a time scaling method was used to solve the the path following problem for a Driftless flat systems including nonholonomic mobile robots, a flatness based approach for path following control of mobile platforms was also studied in Woernle [1998] using a Frenet-Serret coordinates.

The follow-the-virtual-leader approach of Kanayama et al. [1990] is still used in the multi-robot tracking control problem, in which the goal is to conserve a certain formation while tracking a certain reference trajectory. Yet, in spite of the bulk of literature on tracking control for mobile robots the extension to the case of formation tracking control for swarms of robots is far from obvious. In Lin et al. [2005] the problem of reaching a certain geometric configuration using a distributed control was addressed; a necessary and sufficient graphical conditions were deduced. In den Broek et al. [2009], Consolini et al. [2008] and Guo et al. [2010] a virtual structure and a leader-follower approaches were investigated; a comparison between the tow methods can be found in den Broek et al. [2009].

In Do and Pan [2007], the authors solve the formation tracking problem using a combination of the virtual structure and path-tracking approaches to generate the reference for each agent, then an output feedback control law was designed to track each agent toward its reference, using an asymptotic observer to estimate the velocities. This work was extended in Do [2007], where the problem formation tracking with collision avoidance was considered, and under a limited sensing range. In Dong and Farrell [2008] a backstepping based approach is proposed such a group of nonholonomic mobile agents converges and tracks a virtual leader under the assumption that the leader rotational velocity is persistently exciting.

The control approach proposed in Loria et al. [2016] consists in applying, repeatedly, a follow-the-leader controller to each pair of vehicles interconnected in a spanning tree topology. In contrast to other schemes relying on persistency of excitation, the controller proposed in this reference applies to straight-path trajectories (zero-

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angular reference velocity); it relies on a relaxed form of persistency of excitation tailored for state-dependent regressors, called δ-persistency of excitation. However, the analysis is very complex as it is trajectory-based and no Lyapunov function is provided. Besides, the assumptions may be difficult to verify.

In this paper we solve the leader-follower formation tracking control problem for a group of mobile robots using distributed control. As in Loria et al. [2016], each robot communicates only with two neighbours. To one, a follower, it transmits its forward and angular velocities and, from the other, a leader, it receives the corresponding velocities. That is, the communications graph is considered to be a spanning tree, the root of which is a virtual robot moving with a reference forward and angular velocities, that are communicated only to the leader robot in the formation. We establish uniform global asymptotic stability for the error system under a simple condition of persistency of excitation on *either* of the reference velocities, angular or forward. Our proofs rely on Lyapunov's direct method; in the construction of our Lyapunov's functions we borrow inspiration from Mazenc [2003], Mazenc et al. [2009], and Malisoff and Mazenc [2009].

Thus, the main contribution of this paper is twofold: first, we establish uniform global asymptotic stabilisation in the context of formation-tracking control under weak assumptions; secondly, as far as we know, this is the first paper in which a *strict* Lyapunov function [2] is proposed in the context of control of nonholonomic systems under a persistently excited reference velocities. Indeed, our stability proof is based on Lyapunov's direct method.

The rest of the paper is organised as follows. In Section 2 we present an original statement on leader-follower tracking control; in Section 3 we present our main results on formation-tracking for swarms of vehicles and we conclude with some remarks in Section 4.

2. A SINGLE AGENT CASE

Consider the kinematic model of a mobile robot, that is,

$$\begin{aligned}\dot{x} &= v\cos\theta\\ \dot{y} &= v\sin\theta\\ \dot{\theta} &= \omega\end{aligned}$$

where v denotes the forward velocity, ω corresponds to the angular velocity which are, also, the two control inputs. Given two velocity references $t \mapsto v_r$ and $t \mapsto \omega_r$ the tracking control problem consists in following a fictitious reference vehicle

$$\dot{x}_0 = v_r\cos\theta_r \tag{1a}$$

$$\dot{y}_0 = v_r\sin\theta_r \tag{1b}$$

$$\dot{\theta}_r = \omega_r. \tag{1c}$$

From a control viewpoint, the goal is to steer to zero the differences between the Cartesian coordinates of the two robots, as well as orientation angles,

$$\begin{aligned}p_x &= x_r - x\\ p_y &= y_r - y\\ p_\theta &= \theta_r - \theta.\end{aligned}$$

Then, according to the approach in Kanayama et al. [1990] we transform the error coordinates $[p_x,\ p_y,\ p_\theta]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot that is,

$$\begin{bmatrix}e_x\\ e_y\\ e_\theta\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}p_x\\ p_y\\ p_\theta\end{bmatrix}. \tag{2}$$

In the new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \tag{3a}$$

$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta) \tag{3b}$$

$$\dot{e}_\theta = \omega_r(t) - \omega. \tag{3c}$$

Therefore, the follow-the-leader tracking control problem of mobile robots amounts to a stabilisation problem, at the origin, for the system (3).

Our control approach is inspired by the cascades-based controllers originally presented in Panteley et al. [1998], in which persistency of excitation is used to guarantee exponential stabilisation of the origin for the error dynamics. In that reference the following simple linear time-varying controller was proposed:

$$v = v_r(t) + K_x e_x, \quad K_x > 0 \tag{4a}$$

$$\omega = \omega_r(t) + K_\theta e_\theta, \quad K_\theta > 0. \tag{4b}$$

Besides the obvious advantage that offers the simplicity of this controller, it is to be remarked that the closed-loop system has the attractive cascaded form

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-K_x & \omega_r(t)\\ -\omega_r(t) & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} + g(t,e) \tag{5a}$$

$$\dot{e}_\theta = -K_\theta e_\theta \tag{5b}$$

where $e = [e_x\ e_y\ e_\theta]^\top$ and we defined the interconnection term

$$g(t,e) := \begin{bmatrix}v_0(t)\big[\cos(e_\theta) - 1\big] + K_\theta e_\theta e_y\\ v_0(t)\sin(e_\theta) - K_\theta e_\theta e_x\end{bmatrix}. \tag{6}$$

As it is showed in Panteley et al. [1998], uniform global asymptotic stability of the origin of (5) is easily established upon the following cascades argument: first, we observe that because $K_\theta > 0$, e_θ converges exponentially fast; then, it is clear that $g(t,e)$ has linear growth in e_x and e_y and is uniformly bounded in t; finally, for the equations (5a) with $g \equiv 0$, the origin is exponentially stable provided that the reference angular velocity is persistently exciting that is, assuming that there exist μ, $T > 0$ such that

$$\int_t^{t+T} \omega_r(s)^2 ds \geq \mu, \qquad \forall\, t \geq 0. \tag{7}$$

Clearly, this simple argument relies on the bulk of literature on adaptive control systems. Notice that the nomi-

[2] That is positive definite with negative definite derivative.

nal system in (5a) has, precisely, the structure of model-reference-adaptive -control systems.

Although simple, a drawback of this is that it relies on a property of persistency of excitation for the angular velocity. Therefore, straight-path trajectories are excluded. In Cao and Tian [2007], Lee et al. [2001] where complex nonlinear time varying controls are designed to allow for reference velocity trajectories that converge to zero. Furthermore, in Lee et al. [2001] the authors cover the case when also the forward velocity v_0 may converge to zero that is, tracking control towards a fixed point. In Cao and Tian [2007] the controller is designed so as to make the robot *converge* to the straight-line trajectory resulting in a path that makes it go back and forth. In Loria et al. [2016] we presented a controller which relies on a relaxed form of persistency of excitation, which solves the tracking control problem on straight paths. However, in view of the recursive design, verifying the assumptions in the latter reference may be a tedious and difficult task for large swarms of robots.

In this paper we propose the following nonlinear time-varying controller:

$$v = v_r(t)\cos(e_\theta) + K_x e_x \tag{8a}$$

$$\omega = \omega_r(t) + K_\theta e_\theta + v_r(t) K_y e_y \phi(e_\theta) \tag{8b}$$

where ϕ is the so-called 'sync' function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta} \tag{9}$$

which has several useful properties: it is smooth, bounded and locally positive, actually, $|\phi(s)| > 0$ for any $|s| < \pi$. Our standing assumption is that *either* the forward or the angular reference velocities are persistently exciting.

The control design is motivated by the resulting structure of the closed-loop system, which not only includes a convenient persistently-excited skew-symmetric matrix but for which uniqueness of the equilibrium point may be ensured. The closed-loop dynamics of system (3) with the controller (8) takes the form:

$$\dot{e} = A(t,e)e, \qquad e^\top := [e_x \; e_y \; e_\theta] \tag{10}$$

$$A(t,e) := \begin{bmatrix} -K_x & \omega(t,e) & 0 \\ -\omega(t,e) & 0 & v_r(t)\phi(e_\theta) \\ 0 & -v_r(t)K_y\phi(e_\theta) & -K_\theta \end{bmatrix}$$

Our first result is the following.

Theorem 1. Assume that v_r, ω_r, $\dot{v}_r$ and $\dot{\omega}_r$ are bounded. If, moreover, there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \left[v_r(s)^2 + \omega_r(s)^2 \right] ds \geq \mu \quad \forall\, t \geq 0 \tag{11}$$

then, the origin of (10) in closed loop with the controller velocities (8) is uniformly globally asymptotically stable, for any positive gains K_x, K_y and K_θ.

Proof. Consider first the Lyapunov function candidate $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ defined as

$$V_1(t,e) = \frac{1}{2}\left[e_x^2 + e_\theta^2 + \frac{1}{K_y} e_y^2 \right] \tag{12}$$

whose time derivative along trajectories of (10) is negative semidefinite, indeed,

$$\dot{V}_1(t,e) = -K_x e_x^2 - K_\theta e_\theta^2. \tag{13}$$

It follows from this and Barbălat's lemma that $e_x \to 0$ and $e_\theta \to 0$ and all solutions are uniformly globally bounded. Actually, integrating on both sides of $\dot{V}_1(t, e(t)) \leq 0$ and defining

$$c_1 := \min\{1/2,\; 1/2K_y\}$$
$$c_2 := \max\{1,\; 1/K_y\}, \quad c_3 := \sqrt{c_2/c_1}$$

we obtain

$$|e(t)| \leq c_3 |e(t_\circ)| \quad \forall\, t \geq t_\circ \geq 0. \tag{14}$$

That is, the origin is uniformly globally stable with linear growth.

Next, we show that the origin is uniformly globally attractive. To that end, for any locally integrable function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\sup_{t \geq 0} |f(t)| \leq \bar{f}$, let us define

$$Q_f(t) := 1 + 2\bar{f}T - \frac{2}{T}\int_t^{t+T}\int_t^m f(s)\,ds\,dm \tag{15}$$

Note that this function, introduced first in Mazenc [2003], satisfies

$$1 \leq Q_f(t) < \bar{Q}_f := 1 + 2\bar{f}T$$

$$\dot{Q}_f(t) = -\frac{2}{T}\int_t^{t+T} f(s)\,ds + 2f(s).$$

Furthermore, let us introduce the function $V_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} V_2(t,e) = {} & \gamma_r V_1(t,e) - \omega_r(t) e_x e_y + \alpha_r v_r(t) e_\theta e_y \phi(e_\theta) \\ & + \frac{1}{2}\left[K_y \phi^2(e_\theta) Q_{v_r^2}(t) + Q_{\omega_r^2}(t) \right] e_y^2 \end{aligned} \tag{16}$$

where α_r and γ_r are positive constants such that V_2 is positive definite and radially unbounded. Notice, indeed, that in view of the boundedness of v_r and ω_r, for a suitable choice of the parameters α_r and γ_r, there exist $c_1' > 0$ and $c_2' > 0$ such that

$$c_1' |e|^2 \leq V_2(t,e) \leq c_2' |e|^2.$$

The time derivative of (3) along the closed-lopp trajectories of (10) is:

$$\begin{aligned} \dot{V}_2 = {} & -\gamma_r \left[K_x e_x^2 + K_\theta e_\theta^2 \right] + \omega_r^2 e_x^2 + v_r^2 \alpha_r \phi^2(e_\theta) e_\theta^2 \\ & + \Psi_{xy}(t,e) e_x e_y + \Psi_{\theta y}(t,e) e_y e_\theta + \Psi_{\theta x}(t,e) e_x e_\theta \\ & - \left[\int_t^{t+T} \frac{1}{T} v_r^2(s)\,ds \right] \phi^2(e_\theta) K_y e_y^2 \\ & - (\alpha_r - 1) v_r^2 K_y e_y^2 \phi^2(e_\theta) - \int_t^{t+T} \frac{1}{T}\omega_r^2(s)\,ds\, e_y^2 \\ & - \left[\omega_r + 2\phi(e_\theta) \frac{\cos(e_\theta) - \phi(e_\theta)}{e_\theta} Q_{v_r^2} K_y \right] K_y v_r \phi(e_\theta) e_y^3 \end{aligned} \tag{17}$$

where

$$
\begin{aligned}
\Psi_{xy} =& -\dot{\omega}_r + \omega_r K_x + K_y v_r \omega_r e_x \phi(e_\theta) \\
& - \left[K_y \phi^2(e_\theta) Q_{v_r^2} + Q_{\omega_r^2}\right] \\
& \quad \times \left[\omega_r + K_\theta e_\theta + e_y K_y v_r \phi(e_\theta)\right] \\
\Psi_{\theta x} =& \omega_r e_x K_\theta - v_r \omega_r \phi(e_\theta) - v_r \alpha_r \omega_r \\
& - v_r^2 \alpha_r y K_y \phi^2(e_\theta) - v_r \alpha_r K_\theta \phi(e_\theta) e_\theta \\
\Psi_{\theta y} =& -\omega_r K_\theta e_y - \alpha_r v_r \phi(e_\theta) K_\theta + \alpha_r \dot{v}_r \phi(e_\theta) \\
& + \left[K_y \phi^2(e_\theta) Q_{v_r^2} + Q_{\omega_r^2}\right] v_r \phi(e_\theta) \\
& - 2\phi(e_\theta) \left(\frac{\cos(e_\theta) - \phi(e_\theta)}{e_\theta}\right) Q_{v_r^2} K_y K_\theta e_y.
\end{aligned}
$$

Now, in view of the bound (14), these terms are bounded along the error trajectories, that is, for any $r > 0$ there exists $\bar{\Psi}_r > 0$ such that

$$
|e(t_\circ)| \leq r \implies \max\left\{\|\Psi_{xy}\|_\infty, \|\Psi_{\theta x}\|_\infty, \|\Psi_{\theta y}\|_\infty\right\} \leq \bar{\Psi}_r. \tag{18}
$$

On the other hand, for all e such that $|e| \leq c_3 r$, the derivative of V_2 satisfies

$$
\begin{aligned}
\dot{V}_2(t,e) \leq & -\gamma_r[K_x e_x^2 + K_\theta e_\theta^2] + \omega^2 e_x^2 + v_r^2 \alpha_r \phi^2(e_\theta) e_\theta^2 \\
& + \bar{\Psi}_r \left[\,|e_x e_y| + |e_y e_\theta| + |x e_\theta|\,\right] \\
& - (\alpha - 1) v_r^2 K_y e_y^2 \phi^2(e_\theta) \\
& + M_r K_y \left|v_r \phi(e_\theta) e_y^3\right| - \left[\int_t^{t+T} \frac{1}{T} \omega_r^2(s) ds \right. \\
& \left. + K_y \phi^2(e_\theta) \int_t^{t+T} \frac{1}{T} v_r^2(s) ds \right] e_y^2
\end{aligned} \tag{19}
$$

where we defined

$$
M_r = \left|\omega_r + \phi(e_\theta)\left(\frac{\cos(e_\theta) - \phi(e_\theta)}{e_\theta}\right) Q_{v_r^2} K_y\right|_\infty.
$$

To continue further, constructing a suitable bound on $\dot{V}_2$, we need to stress some useful inequalities. Firstly, for any given $\delta > 0$ let $\gamma_r := \gamma_{r1} + \gamma_{r2}$, with γ_{r1} verifying the inequality:

$$
\begin{aligned}
& -\gamma_{r1}[K_x e_x^2 + K_\theta e_\theta^2] + \omega^2 e_x^2 + v_r^2 \alpha_r \phi^2(e_\theta) e_\theta^2 + \\
& \qquad \bar{\Psi}_r \left[\,|e_x e_y| + |e_y e_\theta| + |e_x e_\theta|\,\right] \leq \frac{\delta}{2} e_y^2.
\end{aligned} \tag{20}
$$

Furthermore, note that for any $\delta > 0$,

$$
M_r \left|v_r \phi(e_\theta) e_y e_y^2\right| \leq \frac{\delta}{2} e_y^2 + \frac{M_r^2}{2\delta} e_y^2 \phi^2(e_\theta) v_r^2 e_y^2
$$

hence,

$$
\begin{aligned}
\dot{V}_2(t,e) \leq & -\gamma_{r2}[K_x e_x^2 + K_\theta e_\theta^2] \\
& - (\alpha - 1 - \frac{M_r^2}{2\delta} e_y^2) v_r^2 K_y e_y^2 \phi^2(e_\theta) \\
& + \frac{\delta K_y + \delta}{2} e_y^2 - \min\{1, K_y\} \left[\int_t^{t+T} \frac{1}{T} \omega_r^2(s) ds \right. \\
& \left. + \phi^2(e_\theta) \int_t^{t+T} \frac{1}{T} v_r^2(s) ds \right] e_y^2.
\end{aligned} \tag{21}
$$

Now, taking

$$
\alpha \geq 1 + \frac{M_r^2}{2\delta} r^2,
$$

we obtain,

$$
\begin{aligned}
\dot{V}_2(t,e) \leq & -\gamma_{r2}[K_x e_x^2 + K_\theta e_\theta^2] + \frac{\delta K_y + \delta}{2} e_y^2 \\
& - \frac{\min\{1, K_y\}}{T} \left[\int_t^{t+T} \left(\omega_r^2(s) + v_r^2(s)\right) ds\right] e_y^2 \\
& + \frac{\min\{1, K_y\}}{T} \left(1 - \phi^2(e_\theta)\right) \left[\int_t^{t+T} v_r^2(s) ds\right] e_y^2.
\end{aligned} \tag{22}
$$

Next, we use the inequality,

$$
1 - \phi^2(e_\theta) \leq 2 e_\theta^2 \tag{23}
$$

and we invoke the persistency-of-excitation condition (11) to obtain

$$
\begin{aligned}
\dot{V}_2(t,e) \leq & -\gamma_{r2}[K_x e_x^2 + K_\theta e_\theta^2] \\
& - \left[\min\{1, K_y\} \frac{\mu}{T} - \frac{\delta K_y + \delta}{2}\right] e_y^2 \\
& + \min\{1, K_y\} \frac{2}{T} \left[\int_t^{t+T} v_r^2(s) ds\right] e_y^2 e_\theta^2.
\end{aligned} \tag{24}
$$

Finally, we see that by setting

$$
\begin{aligned}
\delta &= \frac{\mu}{T(1 + K_y)} \min\{1, K_y\} \\
\gamma_{r2} &= \frac{2 \min\{1, K_y\}}{T K_\theta} \bar{v}_r^2 T r^2
\end{aligned}
$$

we obtain,

$$
\dot{V}_2(t,e) \leq -\frac{\gamma_{r2}}{2}[K_x e_x^2 + K_\theta e_\theta^2] - \min\{1, K_y\} \frac{\mu}{2T} e_y^2
$$

for all $t \geq 0$ and all $|e| \leq c_3 r$.

That is, V_2 is positive definite, radially unbounded and its derivative is negative definite on any compact of the state. Uniform global attractivity of the origin follows. In addition, since the system is also uniformly globally stable, uniform global asymptotic stability follows.

3. FORMATION-TRACKING CONTROL

Let us consider, now, the case when a swarm of robots must advance in formation and follow a reference trajectory. We assume that only one robot possesses the information of the reference virtual vehicle and transmits is to one neighbour. The latter transmits its own velocities to one vehicle in the communication graph and so on. That is, vis-a-vis of the interconnections, the graph forms a spanning tree in which each robots has only one parent and one child except for the reference vehicle (root) and the leaf node. The control approach is simple. It consists in using a decentralised follow-the-leader tracking controller for each vehicle, whose model is given by

$$
\dot{x}_i = v_i \cos(\theta_i) \tag{25a}
$$

$$
\dot{y}_i = v_i \sin(\theta_i) \tag{25b}
$$

$$
\dot{\theta}_i = w_i, \qquad i \in [1,\, n] \tag{25c}
$$

The fictitious vehicle, which serves as reference to the swarm, describes the reference trajectory defined by (1); the desired linear and angular velocities v_r and ω_r are communicated to the leader robot only. Similarly to the case of tracking control we define the errors

$$\begin{aligned} p_{ix} &= x_{i-1} - x_i - d_{xi-1,i} \\ p_{iy} &= y_{i-1} - y_i - d_{yi-1,i} \\ p_{i\theta} &= \theta_{i-1} - \theta_i, \qquad i \in [1, n] \end{aligned}$$

where d_x and d_y are (piecewise-)constant design parameters imposed by the topology and path planner and, by definition, we set $(\cdot)_0 := (\cdot)_r$.

According to the spanning-tree communication topology, and following the setting for tracking control, the formation control problem reduces to that of stabilisation of the error dynamics between any pair of leader-follower robots. Then, for each $i \leq n$, we have

$$\dot{e}_{xi} = w_i e_{iy} - v_i + v_{i-1}\cos(e_{\theta i}) \tag{26a}$$
$$\dot{e}_{yi} = -w_i e_{ix} + v_{i-1}\sin(e_{\theta i}) \tag{26b}$$
$$\dot{e}_{\theta i} = w_{i-1} - w_i. \tag{26c}$$

The formation-tracking control problem for n robots reduces to the stabilisation of the origin in the space of $e := [e_x^\top, e_y^\top, e_\theta^\top]^\top$ where we redefined $e_{(\cdot)} := [e_{(\cdot)1}, \cdots e_{(\cdot)n}]^\top$. Then, for each $i \leq n$ we propose the controller defined by

$$v_i = v_{i-1}\cos(e_{\theta i}) + K_{xi} e_{xi} \tag{27a}$$
$$\omega_i = \omega_{i-1} + K_{\theta i} e_{\theta i} + v_{i-1} K_{y_i} e_{yi} \phi(e_{\theta i}) \tag{27b}$$

Theorem 2. For the multiagent error system (26) in closed loop with the controller (27), the origin is uniformly globally asymptotically stable if $\sqrt{\omega_r^2 + v_r^2}$ is persistently exciting *i.e.*, (11) holds, and K_{xi}, K_{y_i} and $K_{\theta i}$ are positive.

Proof. The closed-loop dynamics is

$$\begin{bmatrix} \dot{e}_{xi} \\ \dot{e}_{yi} \\ \dot{e}_{\theta_i} \end{bmatrix} = \underbrace{\begin{bmatrix} -K_{xi} & \omega_i(t, e_i) & 0 \\ -\omega_i(t, e_i) & 0 & v_{i-1}\phi(e_{\theta i}) \\ 0 & -v_{i-1}K_{y_i}\phi(e_{\theta i}) & -K_{\theta i} \end{bmatrix}}_{A_i(e_i, v_{i-1}, \omega_i)} \begin{bmatrix} e_{xi} \\ e_{yi} \\ e_{\theta i} \end{bmatrix} \tag{28}$$

which has exactly the same structure as (10). For each $i \leq n$, the Lyapunov function

$$V_{1i} := \frac{1}{2}\Big[e_{xi}^2 + e_{\theta i}^2 + \frac{1}{K_{y_i}} e_{yi}^2\Big] \tag{29}$$

satisfies

$$\dot{V}_{1i} = -\Big[K_{xi}|e_{xi}|^2 + K_{\theta i}|e_{\theta i}|^2\Big] \tag{30}$$

hence the origin is uniformly globally stable and $|e_x(t)|$, $|e_\theta(t)|$ converge asymptotically to zero. In particular, (14) holds for an appropriate redefinition of c_3.

Next, let us introduce the variables $\tilde{v}_i = v_i - v_{i-1}$ and $\tilde{\omega}_i = \omega_i - \omega_{i-1}$. So for each $i \geq 1$, the closed-loop system takes the form:

$$\dot{e}_{xi} = \varpi_i y_i - K_{xi} e_{xi} + \left[\sum_{k=1}^{i-1} \tilde{\omega}_k\right] e_{yi} + \left[\sum_{k=1}^{i-1} \tilde{v}_k\right] K_{y_i}\phi(e_{\theta i}) e_{yi}^2$$

$$\begin{aligned} \dot{e}_{yi} = &-\varpi_i e_{xi} + v_r \sin(e_{\theta i}) - \left[\sum_{k=1}^{i-1} \tilde{\omega}_k\right] e_{xi} \\ &- \left[\sum_{k=1}^{i-1} \tilde{v}_k\right] \left[K_{y_i}\phi(e_{\theta i}) e_{xi} e_{yi} + \sin(e_{\theta i})\right] \end{aligned}$$

$$\dot{e}_{\theta_i} = -\varpi_i + \omega_r - \left[\sum_{k=1}^{i-1} \tilde{v}_k\right] K_{y_i}\phi(e_{\theta i}) e_{yi} + \left[\sum_{k=1}^{i-1} \tilde{\omega}_k\right]$$

where

$$\varpi_i = K_{\theta i} e_{\theta i} + \omega_r + v_r K_{y_i}\phi(e_{\theta i}) e_{yi} \tag{31}$$

and

$$\tilde{v}_i = v_{i-1}\left[\cos(e_{\theta i}) - 1\right] + K_{xi} e_{xi} \tag{32a}$$
$$\tilde{\omega}_i = K_{\theta i} e_{\theta i} + v_{i-1} K_{y_i}\phi(e_{\theta_i}) e_{yi} \tag{32b}$$

With these notations, the error dynamics take the form

$$\dot{e}_i = \bar{A}(t, e_i) e_i + M_i(t, e_i) \sum_{k=1}^{i-1} \begin{bmatrix} \tilde{v}_k \\ \tilde{\omega}_k \end{bmatrix} \tag{33}$$

where

$$\bar{A}_i(t, e_i) := \begin{bmatrix} -K_{xi} & \varpi_i(t, e_i) & 0 \\ -\varpi_i(t, e_i) & 0 & v_r(t)\phi(e_{\theta i}) \\ 0 & -v_r(t) K_{yi}\phi(e_{\theta i}) & -K_{\theta i} \end{bmatrix}$$

and

$$M_i(t, e_i) := \begin{bmatrix} K_{y_i}\phi(e_{\theta i}) e_{yi}^2 & e_{yi} \\ K_{y_i}\phi(e_{\theta i}) e_{xi} e_{yi} + \sin(e_{\theta i}) & -e_{xi} \\ -K_{y_i}\phi(e_{\theta i}) e_{yi} & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{v}_k \\ \tilde{\omega}_k \end{bmatrix} = \underbrace{\begin{bmatrix} K_{xk} & 0 & v_{k-1} & 0 \\ 0 & v_{k-1} K_{y_k}\phi(e_{\theta k}) & 0 & K_{\theta k} \end{bmatrix}}_{B_k(t,e)} \underbrace{\begin{bmatrix} e_{xk} \\ e_{yk} \\ \cos(e_{\theta k}) - 1 \\ e_{\theta k} \end{bmatrix}}_{\xi(e_k)}$$

For each $i \leq n$, the system $\dot{e}_i := \bar{A}_i(t, e_i)$ is exactly of the form (10). Hence, from the proof of Theorem 1 we deduce that, for each corresponding i and any $r > 0$, the functions

$$\begin{aligned} V_{2i} = \gamma_{ri} V_{1i} &+ \frac{1}{2}\left[K_{y_i}\phi^2(e_{\theta i}) Q_{v_r^2} + Q_{\omega_r^2}\right] e_{yi}^2 \\ &- \omega_r e_{xi} e_{yi} + v_r \alpha_{ri} e_{\theta i} e_{yi}, \end{aligned} \tag{34}$$

satisfy

$$\frac{\partial V_{2_i}}{\partial t} + \frac{\partial V_{2_i}}{\partial e_i}\bar{A}(t, e_i) \leq -\,\sigma_i |e_i|^2 \tag{35}$$

for all $t \geq 0$, $|e_i| \leq c_3 r$ and appropriate values of β_{ri}, α_{ri}, γ_{ri}.

On the other hand, by continuity of the systems' dynamics, $|M_i(t, e_i) B_i(t, e)| \leq \eta_r$ for all e such that $|e_i| \leq c_3 r$ and, moreover, $|\xi(e_i)| \leq 2|e_i|$. Therefore, one can construct a Lyapunov function candidate of the form

$$V_2(t,e) = \sum_{i=1}^{n} -\psi_{ri} V_{2i}(t,e) \quad (36)$$

such that, for an appropriate choice of the constants ψ_{ri}, there exists $\varrho > 0$ such that the total derivative, along the trajectories of (33) for all $i \leq n$, satisfies

$$\dot{V}_2(t,e) \leq -\varrho|e|^2 \quad (37)$$

for all $t \geq 0$, and $|e| \leq c_3 r$. In addition, from uniformly globally stability, all trajectories generated by initial conditions $t_\circ \geq 0$, $|e_\circ| \leq r$ satisfy $|e(t)| \leq c_3 r$ for all $t \geq t_\circ$. Therefore, the origin is uniformly globally attractive.

Corollary 3. Under the conditions of Theorem 2 the origin of the system (26) in closed loop with (27) is uniformly exponentially stable at large on any compact.

4. CONCLUSIONS

We have presented a simple distributed control approach for the formation tracking control of swarms of velocity-controlled mobile robots. Our controllers ensure uniform global asymptotic stability under a simple condition of persistency of excitation of either of the reference velocities, forward or angular. In particular, our controller applies to the difficult problem of following straight paths: null angular velocity and constant forward velocity. Finally, our proofs are direct as they are based on Lyapunov's second method.

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