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STABLE PERFECTLY MATCHED LAYERS FOR A CLASS OF ANISOTROPIC DISPERSIVE MODELS. PART I: NECESSARY AND SUFFICIENT CONDITIONS OF STABILITY. EXTENDED VERSION *

ÉLIANE BÉCACHE^{1,2} AND MARYNA KACHANOVSKA^{1,3}

Abstract. In this work we consider a problem of modelling of 2D anisotropic dispersive wave propagation in unbounded domains with the help of perfectly matched layers (PML). We study the Maxwell equations in passive media with the frequency-dependent diagonal tensor of dielectric permittivity and magnetic permeability. An application of the traditional PMLs to this kind of problems often results in instabilities. We provide a recipe for the construction of new, stable PMLs. For a particular case of non-dissipative materials, we show that a known necessary stability condition of the perfectly matched layers is also sufficient. The analysis is done for a constant absorption parameter of the PML. We illustrate our statements with extensive computational experiments.

1991 Mathematics Subject Classification. 65M12, 35Q60.

1. INTRODUCTION AND PROBLEM SETTING

The problem of wave propagation in anisotropic dispersive media, e.g. in negative index metamaterials or plasmas, is both of theoretical and practical interest. For some applications it is necessary to model the time-domain wave propagation in unbounded (or semi-bounded) domains [1]. One of the ways to bound the computational domain is offered by the use of the perfectly matched layers (PMLs), suggested by Bérenger in [2, 3], see e.g. [1, 4, 5] for plasmas and papers by Cummer [6] and Bécache et al. [7], as well as [8], for some types of metamaterials. However, it is well known that this technique often exhibits instabilities in the presence of dispersion and/or anisotropy [7, 9]. Multiple attempts had been made to overcome this problem, see e.g. [10], however, to our knowledge, no recipe to construct stable PMLs for an arbitrary hyperbolic system exists.

Indeed, there are other ways to tackle the problem of the unboundedness of the domain; a non-exhaustive list of those includes the FEM/BEM coupling [11, 12], methods based on pole condition [13, 14] or various absorbing boundary conditions [15–19] (see also references therein). However, we are not aware of any application of these methods to the class of problems we consider.

Keywords and phrases: perfectly matched layers, stability, Maxwell equations, passive metamaterials, Laplace transform

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In this work we will concentrate on a construction of stable in the time domain PMLs for a class of anisotropic dispersive models that can be described in the frequency domain as a wave equation with frequency-dependent coefficients, more precisely

$$\varepsilon_1(\omega)^{-1}\partial_x^2 u + \varepsilon_2(\omega)^{-1}\partial_y^2 u - \omega^2\mu(\omega)u = 0, \quad \omega \in \mathbb{R}, \quad (1)$$

where $\varepsilon_{1,2}$, μ have the meaning of dielectric permittivity and magnetic permeability. This class of models can be viewed as a generalization of isotropic materials considered in [7], where $\varepsilon_1(\omega) = \varepsilon_2(\omega)$. A notable example of such models includes a uniaxial cold plasma model in two dimensions [20]. In this work we will limit ourselves to the case when $\varepsilon_{1,2}(\omega)$, $\mu(\omega)$ satisfy $\text{Im}(\omega\varepsilon_{1,2}(\omega)) > 0$, $\text{Im}(\omega\mu(\omega)) \geq 0$ for $\text{Im}\omega > 0$, and are analytic in the upper complex half-plane (we will refer to this property as to passivity, see e.g. [21]).

This class of models is slightly different from the one considered in [7]. On one hand, we obtain results for anisotropic materials with arbitrary dielectric permittivity and magnetic permeability, covering both non-dissipative and dissipative classes of problems (see Definition 5.4), while only non-dissipative problems with meromorphic dielectric permittivity and magnetic permeability were considered in [7]. However, we limit our considerations to passive problems, while no such limitation is stated in [7].

Results obtained in this work, which, to our knowledge, have not appeared in the literature before, include:

- (1) a simple recipe of construction of stable perfectly matched layers for passive anisotropic systems (1), even in the case when the coefficients in (1) correspond to non-local in time operators; this is an extension of the ideas of Cummer [6] and Bécache et al. [7] to a more general class of models;
- (2) for a sub-class of problems (1) where $\varepsilon_{1,2}$ and μ satisfy additional requirements, we show that the necessary condition of the stability of the PML formulated in [7] in terms of the directions of phase and group velocity is also sufficient (the result for isotropic materials had been shown in [7]). This allows to construct possibly more efficient (in terms of the number of unknowns, and thus, computational costs) perfectly matched layers.

As a by-product of the above, we also obtain a mathematical reinterpretation of the physical passivity requirement on the coefficients $\varepsilon_{1,2}$, μ as a condition on the stability of the problem (1) in the time domain (Section 2.2).

An important difference between our work and [7] is that we perform the analysis by using the Laplace transform in time, rather than working with plane waves which come from the Fourier analysis. To our knowledge, in very few works the PML had been studied in the Laplace domain setting, see e.g. articles by Halpern et al. [22] or Chen [23]. Such an analysis allows us to obtain quite easily sufficient conditions of stability of the PML (while, in general, necessary stability conditions are easier to obtain based on plane wave techniques).

A word should be said about the limitations of the present article: first of all, at the moment we consider the stability of the PMLs in corners very briefly (thus looking at one direction), and we do not provide energy related results, postponing them to [24]. There we will show how the analysis in the Laplace domain pays off in obtaining explicit energy estimates. Moreover, we limit our considerations to two dimensions (although all the results can be extended to the 3D dispersive wave equation of the type (1), i.e. without mixed derivatives), and we analyze the PML with a constant absorption parameter in the whole space. An extension of the approach to a more general class of three-dimensional problems will be reported elsewhere. We do not discuss the primary cause of the instabilities of the perfectly matched layers, namely the presence of so-called backward propagating waves; an interested reader may consult [7] and [9].

This article is organized as follows. In Section 1.1 we discuss in more detail a class of problems of interest, providing some important examples. Next, we connect the passivity requirement to the stability of the problem (1) in the time domain (see Section 2). In Section 3 we discuss the construction of stable perfectly matched layers for general passive materials.

In the second part of the article, we concentrate on a subclass of materials, considered in [7], and describe some of their important properties, most of which are known from the literature (Section 4), and which we are going to use to prove the PML stability results. Based on these properties and the necessary stability condition

of [7], we provide a criterium of stability of PMLs for (1) with meromorphic even magnetic permeability and dielectric permittivity in Section 5.

We justify our claims with the numerical results in Sections 3.1, 3.3 and 5.4.

1.1. Problem Setting

We consider a problem of the wave propagation in dispersive, anisotropic media, which is described by the Maxwell equations. In particular, in the time domain (using the scaling $\varepsilon_0 = \mu_0 = c = 1$) in \mathbb{R}^2 it reads

$$\begin{aligned}\partial_t D_x &= \partial_y H_z, \\ \partial_t D_y &= -\partial_x H_z, \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x.\end{aligned}\tag{2}$$

The relation between the fields \mathbf{D} and \mathbf{E} , and H_z and B_z is not necessarily explicitly known in the time domain, but often is given in the Laplace domain. We will denote by s the Laplace variable, by $\hat{u} = \mathcal{L}u$ the Laplace transform of u in time, and by \mathbb{C}_+ the right complex half-plane: $\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. Additionally, given a function $f(s) : \mathbb{C}_+ \cup i\mathbb{R} \rightarrow \mathbb{C}$, we will denote

$$\tilde{f}(\omega) = f(-i\omega), \operatorname{Im} \omega \geq 0.\tag{3}$$

Then the following identities hold true: $\hat{\mathbf{D}} = \underline{\underline{\varepsilon}}(s)\hat{\mathbf{E}}$, $\hat{H}_z = \mu^{-1}(s)\hat{B}_z$. Generally, $\underline{\underline{\varepsilon}}$ and μ may become spatial operators, see [25, p.281], or depend on spatial variables, which is not covered by the article. In the latter case the application of the PMLs does not necessarily lead to a correct solution, and therefore we consider the wave propagation in homogeneous media. In particular, we concentrate on the case when the tensor of dielectric permittivity is diagonal and is given by $\underline{\underline{\varepsilon}}(s) = \begin{pmatrix} \varepsilon_1(s) & 0 \\ 0 & \varepsilon_2(s) \end{pmatrix}$. Then

$$\operatorname{curl} \mu^{-1}(s) \operatorname{curl} \hat{\mathbf{E}} - s^2 \begin{pmatrix} \varepsilon_1(s) & 0 \\ 0 & \varepsilon_2(s) \end{pmatrix} \hat{\mathbf{E}} = 0,$$

or, alternatively,

$$s^2 \mu(s) \hat{H}_z - \varepsilon_2(s)^{-1} \partial_{xx} \hat{H}_z - \varepsilon_1(s)^{-1} \partial_{yy} \hat{H}_z = 0.\tag{4}$$

Let us provide a few examples of such models, and comment on the state of the art of the stable PMLs for those:

(1) isotropic Drude materials:

$$\varepsilon_1(s) = \varepsilon_2(s) = 1 + \frac{\omega_e^2}{s^2}, \quad \mu(s) = 1 + \frac{\omega_m^2}{s^2}, \quad \omega_e, \omega_m \geq 0.\tag{5}$$

The corresponding system in the time domain reads, cf. [7],

$$\begin{aligned}\partial_t \mathbf{E} - \operatorname{curl} H_z + \omega_e^2 \mathbf{J} &= 0, \\ \partial_t H + \operatorname{curl} \mathbf{E} + \omega_m^2 K &= 0, \\ \partial_t \mathbf{J} &= \omega_e^2 \mathbf{E}, \\ \partial_t K &= H.\end{aligned}\tag{6}$$

The construction of the stable PMLs for this class of models has been suggested in [6]. In [7] the stability of these PMLs has been proven in the free space and for the constant absorption parameter.

(2) **uniaxial cold plasma model:**

$$\varepsilon_1(s) = 1 + \frac{\omega_p^2}{s^2}, \quad \omega_p > 0, \quad \varepsilon_2(s) = \mu(s) = 1. \quad (7)$$

In the time domain the corresponding system reads

$$\begin{aligned} \partial_t E_x - \partial_y H_z + j &= 0, \\ \partial_t E_y + \partial_x H_z &= 0, \\ \partial_t H_z + \partial_x E_y - \partial_y E_x &= 0, \\ \partial_t j &= \omega_p^2 E_x. \end{aligned}$$

This model was considered in [20] and [8]. In particular, the stable PMLs were constructed using an idea similar to that of [6].

(3) **Lorentz materials**, which generalize the previous two cases:

$$\begin{aligned} \varepsilon_1(s) &= 1 + \sum_{\ell=0}^{n_x} \frac{\varepsilon_{x\ell}}{s^2 + \omega_{x\ell}^2}, & \varepsilon_{x\ell} > 0, \quad \omega_{x\ell} \geq 0, & \ell = 0, \dots, n_x, \\ \varepsilon_2(s) &= 1 + \sum_{\ell=0}^{n_y} \frac{\varepsilon_{y\ell}}{s^2 + \omega_{y\ell}^2}, & \varepsilon_{y\ell} > 0, \quad \omega_{y\ell} \geq 0, & \ell = 0, \dots, n_y, \\ \mu(s) &= 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + \omega_{\mu\ell}^2}, & \mu_\ell > 0, \quad \omega_{\mu\ell} \geq 0, & \ell = 0, \dots, n_\mu. \end{aligned} \quad (8)$$

The corresponding system of equations can be constructed similarly to (6). For the case $\varepsilon_1(s) = \varepsilon_2(s)$, the stable PMLs for this system have been constructed in [7], in a manner, similar to [6]. More precisely, it was suggested to stabilize the perfectly matched layer with the help of the following change of variables

$$x \rightarrow x \left(1 + \frac{\sigma \psi(s)}{s} \right), \quad \sigma > 0,$$

with $\psi(s) \neq 1$ ($\psi(s) = 1$ would lead to a standard Bérenger's PML). It was shown how to choose such a function $\psi(s)$; moreover, it was proven that the necessary stability condition in terms of the group velocity, with an additional constraint on $\psi(s)$, is sufficient for the stability of the PMLs.

A major part of the present work will be devoted to extending this result to an anisotropic case, when $\varepsilon_1 \neq \varepsilon_2$.

Remark 1.1. *Under extra assumptions (causality, analyticity and certain bounds on the coefficients), (4) with zero initial conditions can be rewritten in the time domain as*

$$\int_0^t \mathcal{M}(t-\tau) \partial_{\tau\tau} H_z(\tau) d\tau - \int_0^t \mathcal{E}_2(t-\tau) \partial_{xx} H_z(\tau) d\tau - \int_0^t \mathcal{E}_1(t-\tau) \partial_{yy} H_z(\tau) d\tau = 0, \quad t \geq 0, \quad (9)$$

where $\mathcal{M} = \mathcal{L}^{-1} \mu$, $\mathcal{E}_{1,2} = \mathcal{L}^{-1} \varepsilon_{1,2}^{-1}$.

Remark 1.2. *In this work we concentrate on the construction of stable perfectly matched layers for the two-dimensional anisotropic dispersive wave equation (4); however, all the results can be extended to the 3D problem*

$$s^2\mu(s)u - \varepsilon_1(s)\partial_{xx}u - \varepsilon_2(s)\partial_{yy}u - \varepsilon_3(s)\partial_{zz}u = 0.$$

2. PROPERTIES OF THE DIELECTRIC PERMITTIVITY AND THE MAGNETIC PERMEABILITY

In the physics literature, when considering the Maxwell equations (2) in so-called passive media, one assumes that $\varepsilon_{1,2}$ and μ satisfy the following assumptions (here we list only those which are of interest for us) [21, 26]:

(P1) $\underline{\varepsilon}(s)$, $\mu(s)$ are analytic in $s \in \mathbb{C}_+$.

(P2) passivity, which we will use in the form

$$\operatorname{Re}(s\varepsilon_j(s)) > 0, \quad j = 1, 2, \quad \operatorname{Re}(s\mu(s)) > 0, \quad s \in \mathbb{C}_+. \quad (10)$$

Let us remark that in [21] this requirement is formulated (up to a complex rotation) as $\operatorname{Re}(s(\varepsilon_j(s) - 1)) \geq 0$ in \mathbb{C}_+ ; we use a slightly relaxed version.¹

Although we concentrate our presentation on the wave propagation problems (2), the model (4), which we will study in the present work, describes other phenomenae, for example, heat transfer (with a formal choice $\varepsilon_1(s) = \varepsilon_2(s) = 1$, $\mu(s) = s^{-1}$).

This motivates us to redefine separately the property of passivity.

Definition 2.1. We will call a function $c : \mathbb{C}_+ \rightarrow \mathbb{C}$ passive if it is analytic in \mathbb{C}_+ and satisfies $\operatorname{Re}(sc(s)) > 0$ in \mathbb{C}_+ .

We proceed as follows. First we recall some known properties of functions that satisfy (10), and next show that (P1) and (P2) are **sufficient** for the well-posedness of the problem (4).

2.1. Properties of Analytic Functions Satisfying the Passivity Requirement

Here we will provide a few known results on functions passive in the sense of Definition 2.1. Given such a function $c(s)$, notice that the function $f_c(z) := zc(-iz)$ with z satisfying $\operatorname{Im} z > 0$ is a Herglotz function (i.e. it is holomorphic in the upper half-plane $\operatorname{Im} z > 0$, and $\operatorname{Im} f(z) \geq 0$ there). This can be seen by setting $z = is$:

$$\operatorname{Im}(zc(-iz)) = \operatorname{Re}(-izc(-iz)) = \operatorname{Re}(sc(s)).$$

The class of analytic in \mathbb{C}_+ functions $g(s)$ which satisfy $\operatorname{Re} g(s) > 0$ in \mathbb{C}_+ , with some additional constraints (among them, in particular, $g(\bar{s}) = \overline{g(s)}$) is known also as positive real functions, see the works by Brune [27] or Richards [28], where a concise summary of their properties can be found.

The following lemma provides bounds on the functions satisfying the passivity requirement. The upper bound in this lemma follows from [29, Theorem 8', p.18], and the lower bound can be viewed as a minor improvement of [28, Theorem 3] (we consider a more general class of functions, however, the main idea of the proof is basically the same).

Lemma 2.2. *Let $c(s)$ be passive in the sense of Definition 2.1. Then*

(1) *there exists $C > 0$, s.t. for $s \in \mathbb{C}_+$, the function $c(s)$ satisfies*

$$|c(s)| \leq C|s|(\operatorname{Re} s)^{-1} \max\left(1, (\operatorname{Re} s)^{-2}\right). \quad (11)$$

¹In another reference [26], one considers materials with $\operatorname{Re}(s\varepsilon_j(s)) \geq 0$, $s \in \mathbb{C}_+$. Due to the maximum principle, applied to an analytic function $e^{-s\varepsilon_j(s)}$, if in some point s , the real part $\operatorname{Re}(s\varepsilon_j(s)) = 0$, then $s\varepsilon_j(s) = \text{const}$; one deduces $s\varepsilon_j(s) = i\alpha$, $\alpha \in \mathbb{R}$. This case is not of interest, since in the time domain the operator with the symbol $s\varepsilon_j(s)$ would map real fields into complex fields, which is non-physical.

(2) there exists $c > 0$, s.t. for all $s \in \mathbb{C}_+$, it holds that

$$\operatorname{Re}(sc(s)) > c|s|^{-2} \min(1, (\operatorname{Re} s)^2) \operatorname{Re} s. \quad (12)$$

The proof of the above lemma can be found in Appendix A.

Remark 2.3. Notice that passive functions $c(s)$ are not necessarily defined on the imaginary axis, a simple example being $c(s) = \frac{\operatorname{Log}(is)}{is}$, which has a branch cut on the positive imaginary semiaxis $\{s : \operatorname{Im} s > 0, \operatorname{Re} s = 0\}$.

From the above we immediately obtain a similar result for a function $b(s)$ which satisfies $\operatorname{Re}(\bar{s}b(s)) > 0$ in \mathbb{C}_+ . Notice that given an analytic in \mathbb{C}_+ function $b(s)$

$$\operatorname{Re}(\bar{s}b(s)) > 0, s \in \mathbb{C}_+ \iff \operatorname{Re}\left(\frac{b(s)}{s}\right) > 0, s \in \mathbb{C}_+ \iff \operatorname{Re}\left(\frac{s}{b(s)}\right) > 0, s \in \mathbb{C}_+. \quad (13)$$

The function $\frac{1}{b(s)}$ is well-defined in \mathbb{C}_+ , since $b(s)$ does not vanish there. With the help of the above, we obtain the following simple result on the behaviour of such a function $b(s)$.

Corollary 2.4. Let $b(s)$ be analytic in \mathbb{C}_+ and $\operatorname{Re}(\bar{s}b(s)) > 0$ there. Then

(1) there exists $C > 0$, s.t. for $s \in \mathbb{C}_+$, it holds

$$|b(s)| \leq C|s|^3 (\operatorname{Re} s)^{-1} \max\left(1, (\operatorname{Re} s)^{-2}\right). \quad (14)$$

(2) there exists $c > 0$, s.t. for $s \in \mathbb{C}_+$

$$\operatorname{Re}(\bar{s}b(s)) \geq c \min\left(1, (\operatorname{Re} s)^2\right) \operatorname{Re} s. \quad (15)$$

Proof. The upper and lower bounds can be obtained simply by noticing that $g_b(s) := b(s)s^{-2}$ satisfies $\operatorname{Re}(sg_b(s)) = \operatorname{Re}\left(\bar{s}\frac{b(s)}{|s|^2}\right) > 0$ in \mathbb{C}_+ . Then both bounds follow immediately from (11, 12) applied to $g_b(s)$. Namely,

$$|b(s)| = |s|^2 |g_b(s)| \leq C|s|^3 (\operatorname{Re} s)^{-1} \max\left(1, (\operatorname{Re} s)^{-2}\right), \quad C > 0, \quad \operatorname{Re}(\bar{s}b(s)) \geq c \min\left(1, (\operatorname{Re} s)^2\right) \operatorname{Re} s, \quad c > 0.$$

□

The following lemma summarizes some of the results on the behaviour of functions $h(s)$ such that $\operatorname{Re} h(s) > 0$ for all $s \in \mathbb{C}_+$ on the imaginary axis, and is essentially a reformulation of results of [27] (the first statement of the lemma below), [29, p.308, Theorem 1] (the second statement of the lemma for $h(s)$ purely imaginary on $i\mathbb{R}$), [30, Proposition 13.8.1, p.920] (the second statement of the lemma, in a slightly modified form).

Lemma 2.5. Let $h(s)$ be analytic in \mathbb{C}_+ and $\operatorname{Re} h(s) > 0$ there. Then

- (1) poles and zeros of $h(s)$ that lie on the imaginary axis are simple. Residues of the function $h(s)$ of its poles on the imaginary axis are real, positive, and so are the derivatives of $h(s)$ in its zeros on the imaginary axis.
- (2) if $h(s)$ is additionally meromorphic in \mathbb{C} , and $h(-\bar{s}) = -\overline{h(s)}$, all the poles and zeros of $h(s)$ lie on the imaginary axis and they alternate.

Proof. (1) let us first consider the case of zeros. Assume that $h(s)$ has a zero $s_0 = i\omega_0$ of multiplicity m on the imaginary axis. Then in a small vicinity of $i\omega_0$, with $s = i\omega_0 + \varepsilon e^{i\phi}$ and $\varepsilon \rightarrow 0$:

$$h(i\omega_0 + \varepsilon e^{i\phi}) = \frac{h^{(m)}(i\omega_0)}{m!} \varepsilon^m e^{im\phi} + O(\varepsilon^{m+1}), \quad \phi \in (-\pi, \pi].$$

Assume $h^{(m)}(i\omega_0) = re^{i\phi_0}$, $r > 0$, for some $\phi_0 \in (-\pi, \pi]$. For $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. when $\operatorname{Re} s > 0$, it holds

$$\operatorname{Re} h(s) = \frac{r\varepsilon^m}{m!} \cos(m\phi + \phi_0) + O(\varepsilon^{m+1}) > 0,$$

from which it follows that, necessarily, $\cos(m\phi + \phi_0) > 0$ for all $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This would be impossible if m were larger than 1, or $m = 1$ and $\phi_0 \neq 0$. Hence, $m = 1$ and $\phi_0 = 0$.

In particular, we immediately obtain $h'(i\omega_0) = r > 0$ in a zero $s_0 = i\omega_0$.

The condition on the poles, and thus, the residues of h , can be obtained by looking at the principal part of the Laurent series.

- (2) if $h(s) = -\bar{h}(-\bar{s})$, and $h(s)$ has a pole with $\operatorname{Re} s < 0$, $-\bar{s} \in \mathbb{C}_+$ is correspondingly its pole as well. This is a contradiction to the analyticity of $h(s)$. On the other hand, if $h(s)$ a zero with $\operatorname{Re} s < 0$, the existence of zero $-\bar{s} \in \mathbb{C}_+$ contradicts the condition $\operatorname{Re} h(s) > 0$ in \mathbb{C}_+ combined with the maximum principle, see the footnote on p. 5.

Next, assume that $h(s)$ has $m > 1$ zeros (for simplicity let $m = 2$), which lie on the imaginary axis and between which there is no pole. Let us take a contour γ , which is symmetric with respect to the imaginary axis and encloses these two zeros and no other zeros/poles of $h(s)$, and on which $h(s)$ is analytic. Thanks to the argument principle,

$$\int_{\gamma} \frac{h'(s)}{h(s)} ds = 4\pi i, \quad (16)$$

i. e. the change of the argument of $h(s)$ along the contour γ equals to 4π . Denoting by γ^- a part of the contour with $\operatorname{Re} s < 0$ and by γ^+ the part of the contour with $\operatorname{Re} s > 0$, we notice that for $s \in \gamma^+$, $-\bar{s} \in \gamma^-$, due to the symmetry. Moreover, $h(s) = -\bar{h}(-\bar{s})$, and thus the change of the argument of $h(s)$ along γ_+ equals 2π , and so does the change of the argument of $h(s)$ along γ_- .

Since $\gamma^+ \in \mathbb{C}_+$, this is a contradiction to $\operatorname{Re} h(s) > 0$ in \mathbb{C}_+ .

A similar argument can be applied to show that among two consecutive poles of $h(s)$ on the imaginary axis there is always a zero. □

Thus, a passive function $c(s)$ cannot vanish in zero (otherwise $sc(s)$ would have a zero of multiplicity two there).

2.2. Stability in the Time Domain

In this section we investigate a connection between the passivity requirement (P2) and the stability of the Maxwell system (4). For convenience, let us introduce a sesquilinear form $A(u, v) : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \rightarrow \mathbb{C}$:

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \quad s \in \mathbb{C}_+, \quad u, v \in H^1(\mathbb{R}^2). \quad (17)$$

Here $(u, v) = \int_{\mathbb{R}^2} u(x)\bar{v}(x)dx$, for $u, v \in L^2(\mathbb{R}^2)$. Given $f \in H^{-1}(\mathbb{R}^2)$, $u \in H^1(\mathbb{R}^2)$, we will denote by $\langle f, u \rangle$ the duality pairing induced by the above inner product (where $H^{-1}(\mathbb{R}^2)$ is a space of antilinear functionals on $H^1(\mathbb{R}^2)$).

Remark 2.6. *The operator corresponding to (17) is defined as follows*

$$\begin{aligned} \mathcal{A} : \quad & H^1(\mathbb{R}^2) \rightarrow H^{-1}(\mathbb{R}^2), \\ \mathcal{A}u = & -a(s)\partial_{xx}u - b(s)\partial_{yy}u + s^2c(s)u. \end{aligned} \quad (18)$$

Indeed, we recognize the problem (4) with

$$a(s) = \varepsilon_2(s)^{-1}, \quad b(s) = \varepsilon_1(s)^{-1}, \quad c(s) = \mu(s). \quad (19)$$

Remark 2.7. The case $a(s) = b(s)$ corresponds to an isotropic medium, see also [7].

Remark 2.8. All the results that follow will hold for the sesquilinear form $A(u, v) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{C}$ (with Ω being a bounded Lipschitz domain)

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \quad s \in \mathbb{C}_+, \quad u, v \in H_0^1(\Omega),$$

which would correspond to (4) posed in Ω with zero Dirichlet conditions. Similarly, one can look at the above sesquilinear form written for zero Neumann boundary conditions, and consider the well-posedness for this case.

An approach that we are going to adopt here is based on the Laplace transform, see e.g. [31]. Some of the results on the stability of the systems presented in the following sections are possible to obtain with the help of the plane-wave analysis [32, Chapter 2], see Section 2.4.

The results used here are based on the theory developed in [33, Section 2.1], which was, under stronger assumptions, made more precise in the following theorem of Dominguez et al. [34]. We present here a less refined version of this theorem. Let us remark that by causal we mean distributions ϕ defined on \mathbb{R} , s.t. $\langle \phi, \lambda \rangle = 0$, for all $\lambda : \text{supp } \lambda \in (-\infty, 0)$.

Theorem 2.9 (Theorem 7.1 [34], Proposition 3.2.2 [35]). *Let X, Y be Banach spaces, and f be an $\mathcal{L}(X, Y)$ -valued causal distribution whose Laplace transform $F(s)$ exists for all $s \in \mathbb{C}_+$ and satisfies*

$$\|F(s)\|_{\mathcal{L}(X, Y)} \leq C_F(s_r)|s|^\mu, \quad s \in \mathbb{C}_+, \quad (20)$$

where $\mu \geq 0$ and $C_F : (0, \infty) \rightarrow (0, \infty)$ is a non-increasing function, s.t. $C_F(\lambda) < \frac{C}{\sigma^m}$, $C > 0$, $m \geq 0$, for $\sigma \in (0, 1]$.

Define $k = \lfloor \mu + 2 \rfloor$ (where $\lfloor k \rfloor$ denotes integer part of k). Then for all causal $C^{k-1}(\mathbb{R})$ functions $g : \mathbb{R} \rightarrow X$ with integrable k -th distributional derivative, the distribution $f * g$ is a causal continuous function $\mathbb{R} \rightarrow Y$ such that

$$\|(f * g)(t)\|_Y \leq \alpha_{\mu, m} \max(t^m, 1) \int_0^t \|(\mathcal{P}_k g)(\tau)\|_X d\tau, \quad t \geq 0, \quad (21)$$

with $\alpha_{\mu, m} > 0$ and $(\mathcal{P}_k g) = \sum_{\ell=0}^k \binom{k}{\ell} g^{(\ell)}(t)$.

Proof. The proof of the case $\mu \geq 0$ is given in [35, Proposition 3.2.2] or [34, Theorem 7.1]. Notice that in derivation of the above bounds we use $\frac{t^\varepsilon}{(1+t)^\varepsilon} \leq \text{const}$ for $\varepsilon \geq 0$ (see [35, Proposition 3.2.2] for the notation), and $C_F(1/t) \leq C \max(t^m, 1)$, for some $C > 0$. \square

Remark 2.10. As remarked in [35, p. 43], the bounds in the above result may appear to be non-optimal. Crucially, provided e.g. a compactly supported right hand side data, the bound in (21) grows not faster than a polynomial in the time domain, which implies the time-domain stability.

2.3. Connection between Passivity and Stability

Now let us come back to the question of the well-posedness of the variational formulation with the sesquilinear form (17). We will need the following assumption on coefficients of the sesquilinear form (17), namely

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \quad s \in \mathbb{C}_+, \quad u, v \in H_1(\mathbb{R}^2).$$

Definition 2.11. We will call the sesquilinear form (17) passive if the functions $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ are passive in the sense of Definition 2.1.

Our goal in this section is to show that passive sesquilinear forms define stable systems in the time domain. This motivates the name 'passive', since for the Maxwell system (2) the passivity requirement is connected to a positivity of the mean (over the period) of the energy of interaction of the matter with the electromagnetic field, see [36, pp.15-17]. The main result of this section is the following stability statement.

Proposition 2.12. *Let the sesquilinear form $A(u, v)$ given by (17) be passive in the sense of Definition 2.11. Then:*

(1) for some $C_0, C > 0$ independent of s , it holds:

$$|A(u, v)| \leq C_0 |s|^3 (\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2}) \|u\|_{H^1} \|v\|_{H^1}, \quad C_0 > 0, \quad u, v \in H^1(\mathbb{R}^2), \quad (22)$$

$$\operatorname{Re}(e^{-i \operatorname{Arg} s} A(u, u)) \geq C \operatorname{Re} s |s|^{-1} \min(1, (\operatorname{Re} s)^2) \|u\|_{H^1}^2, \quad C > 0, \quad u \in H^1(\mathbb{R}^2); \quad (23)$$

(2) for all $G \in H^{-1}(\mathbb{R}^2)$, for all $s \in \mathbb{C}_+$, there exists a unique solution $U(s) \in H^1(\mathbb{R}^2)$ to the variational formulation

$$A(U(s), v) = \langle G, v \rangle, \quad v \in H^1(\mathbb{R}^2). \quad (24)$$

Moreover, for $s \in \mathbb{C}_+$,

$$\|U(s)\|_{H^1(\mathbb{R}^2)} \leq C |s| (\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2}) \|G\|_{H^{-1}(\mathbb{R}^2)}, \quad C > 0; \quad (25)$$

(3) let $G(s) \in H^{-1}(\mathbb{R}^2)$ be the Laplace transform of a causal C^2 -function $g(t) : \mathbb{R} \rightarrow H^{-1}(\mathbb{R}^2)$, whose 3rd derivative in the sense of distributions $g^{(3)} \in L^1(\mathbb{R}, H^{-1}(\mathbb{R}^2))$. Then the solution $U(s)$, $s \in \mathbb{C}_+$, of (24) with $G = G(s)$ is a Laplace transform of a causal continuous function $u(t) : \mathbb{R} \rightarrow H^1(\mathbb{R}^2)$, which can be bounded as follows:

$$\|u(t)\|_{H^1} \leq \alpha \max(t^3, 1) \int_0^t \|(\mathcal{P}_3 g)(\tau)\|_{H^{-1}} d\tau, \quad t \geq 0, \quad \alpha > 0,$$

where $\mathcal{P}_3 g$ is defined as in Theorem 2.9.

Proof. First of all, let us show that $A(u, v)$ is bounded and coercive. Indeed, given $u, v \in H^1(\mathbb{R}^2)$, using (11) and (14), we obtain, for some $C_0 > 0$ independent of s and all $s \in \mathbb{C}_+$:

$$|A(u, v)| \leq \max(|a(s)|, |b(s)|, |c(s)||s|^2) \|u\|_{H^1} \|v\|_{H^1} \leq C_0 |s|^3 (\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2}) \|u\|_{H^1} \|v\|_{H^1}.$$

To show the coercivity, consider $u \in H^1(\mathbb{R}^2)$, and take $A(su, u) = \bar{s}A(u, u)$, for $s \in \mathbb{C}_+$:

$$\begin{aligned} \operatorname{Re}(\bar{s}A(u, u)) &= \operatorname{Re}(\bar{s}a(s)) \|\partial_x u\|_{L^2}^2 + \operatorname{Re}(\bar{s}b(s)) \|\partial_y u\|_{L^2}^2 + |s|^2 \operatorname{Re}(sc(s)) \|u\|_{L^2}^2 \\ &\geq \min(\operatorname{Re}(\bar{s}a(s)), \operatorname{Re}(\bar{s}b(s)), |s|^2 \operatorname{Re}(sc(s))) \|u\|_{H^1}^2 \geq C \min(1, (\operatorname{Re} s)^2) \operatorname{Re} s \|u\|_{H^1}^2, \quad C > 0, \end{aligned}$$

where the last inequality was obtained with the help of (12) and (15). Notice that the above coercivity estimate is obtained for $\bar{s}A(u, u)$.

The second statement is obtained by a direct application of the Lax-Milgram lemma.

To obtain the third statement, we use the invertibility of the operator (18). Due to [37, p.592, Lemma 13], \mathcal{A}_s^{-1} is analytic in \mathbb{C}_+ as an operator-valued function, which implies, thanks to [38, Propositions 3.1, 3.2], that \mathcal{A}_s^{-1} is a Laplace transform of a causal distribution that satisfies the bound $\|\mathcal{A}_s^{-1}\|_{H^{-1} \rightarrow H^1} \leq$

$C|s|(\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2})$, $C > 0$, for $s \in \mathbb{C}_+$. Then the statement of the proposition follows by the application of Theorem 2.9 to $U(s) = \mathcal{A}_s^{-1}G(s)$:

$$\|u(t)\|_{H^1} = \|(\mathcal{L}^{-1}U(s))(t)\|_{H^1} \leq \alpha \max(t^3, 1) \int_0^t \|(\mathcal{P}_3 g)(\tau)\|_{H^{-1}} d\tau, \quad t \geq 0, \quad \alpha > 0.$$

□

Let us consider the model (4) in the free space written in the time domain (there exists several ways of doing so, however, what is important for the discussion is that the corresponding time domain problem has the form (4) in the Laplace domain). Assume that the magnetic permittivity and dielectric permeability are passive in the sense of Definition 2.1. Then Proposition 2.12 shows that such a model, equipped with zero initial conditions and provided a sufficiently smooth right-hand side data, is

- (1) well-posed, in the sense that there exists a unique continuous function $H_z(t) : \mathbb{R} \rightarrow H^1(\mathbb{R}^2)$, which solves this system in a weak sense;
- (2) stable, in the sense that if the right-hand side data is sufficiently smooth and decays at infinity sufficiently fast, the solution H_z is bounded polynomially in time. The same result can be obtained for the other unknowns of the system (2), see e.g. Section 4.2.1 for the Lorentz materials.

A formulation of a stable system in the time domain corresponding to the Maxwell equations (2) with general passive diagonal dielectric permittivity and magnetic permeability can be found in e.g. [39] or [24].

Remark 2.13. *The above result can be re-interpreted as a result on the well-posedness of the convolution equation (9).*

It would be natural to ask whether the passivity requirement is **necessary** for the stability of the class of models (4). The answer is known for a very particular subclass of isotropic systems and will be reported in [39]; so far we do not know an answer for a more general case.

In [24] we show how the above simple analysis implies conservation (or even decay) of a certain energy associated to (4).

2.4. Connection Between the Plane-Wave Analysis and the Laplace Domain Bounds

In this section we will briefly discuss a connection between the plane-wave analysis and the Laplace domain analysis. A Fourier-based approach, see e.g. [32, Chapter 2], had been used to study the stability of the perfectly matched layers in isotropic dispersive media [7], or anisotropic non-dispersive media [9]. When applied to (4), such analysis consists in looking for the plane-wave solutions $\hat{H}_z e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}$, $\mathbf{k}, \mathbf{x} \in \mathbb{R}^2$, $\omega \in \mathbb{C}$, of (4) rewritten in the time domain. This requires examining the corresponding dispersion relation. Such a dispersion relation in general depends on the time domain formulation of the problem. However, some of its solutions are given by, see also (3),

$$\tilde{a}(\omega)k_x^2 + \tilde{b}(\omega)k_y^2 - \omega^2 \tilde{c}(\omega) = 0. \quad (26)$$

For the anisotropic Lorentz materials (8) written in the form (53) the solutions of the dispersion relation are given by non-propagative modes $\omega = \omega_j$, $\omega_j \in \mathbb{R}$, and the solutions of the above, see Appendix C.

In the time domain the system (4) is stable if and only if all the solutions $\omega_j(\mathbf{k})$ of the above satisfy

$$\operatorname{Im} \omega_j(\mathbf{k}) \leq 0, \quad \mathbf{k} \in \mathbb{R}^2. \quad (27)$$

This holds true for passive sesquilinear forms.

Lemma 2.14. *Let the sesquilinear form (17) be passive in the sense of Definition 2.11. Then all solutions $\omega(\mathbf{k})$ of (26) satisfy (27).*

Proof. Instead of looking at (26) for $\omega \in \mathbb{C}$, we set $s = -i\omega$; then our goal is to show that the equation

$$s^2 c(s) + b(s)k_y^2 + a(s)k_x^2 = 0, \quad \mathbf{k} \in \mathbb{R}^2,$$

has no solutions $s \in \mathbb{C}_+$. Multiplying the above by \bar{s} , we obtain:

$$|s|^2 s c(s) + \bar{s} b(s)k_y^2 + \bar{s} a(s)k_x^2 = 0, \quad \mathbf{k} \in \mathbb{R}^2.$$

This equation has a solution $s_0 \in \mathbb{C}_+$ if and only if the real and imaginary part of the left hand side of the above expression vanish in s_0 . However, the real part of this equation is strictly positive for $s \in \mathbb{C}_+$, thanks to Assumption 2.11, and hence the equation (26) has no solutions ω in the upper complex half-plane $\text{Im } \omega > 0$. \square

The following corollary will be of use later.

Corollary 2.15. *Let the sesquilinear form (17) be passive in the sense of Definition 2.11. Let additionally $a(s)$, $b(s)$, $c(s)$ be even functions of s (and thus have an analytic continuation in $\mathbb{C} \setminus \{i\mathbb{R}\}$). Then all the solutions $\omega(\mathbf{k})$ of the dispersion relation (26) are real.*

3. CONSTRUCTION OF STABLE PMLs FOR GENERAL PASSIVE MATERIALS

3.1. A Brief Introduction into PMLs. Motivation

In this section we will present, in a nutshell, perfectly matched layers; a concise introduction into this technique can be found e.g. in [40] or [41]. There are at least two ways to apply the perfectly matched layers to the system (2), in particular, via the splitting of the time-domain system, as in seminal works by Bérenger [2, 3], or via the change of variables in the frequency domain, as reinterpreted by Chew et al. in [42], see also [43]. We will adopt here the latter approach. More precisely, consider the following equation associated to (18) (see also (4) and Remark 2.6)

$$a(s)\partial_{xx}u + b(s)\partial_{yy}u - s^2c(s)u = 0, \quad s \in \mathbb{C}_+, (x, y) \in \mathbb{R}^2. \quad (28)$$

Let the perfectly matched layer be located in the region $x \geq 0$. We assume that the above equation is valid for $x = \tilde{x} \in \mathbb{C}_+$, and introduce an analytic continuation of u that we denote $U(\tilde{x}, y)$. Naturally, U will satisfy the above equation, however, for $(\tilde{x}, y) \in \mathbb{C}_+ \times \mathbb{R}$. Then choosing a special frequency-dependent parameterization of \tilde{x} , namely

$$\tilde{x} = \begin{cases} x + s^{-1} \int_0^x \sigma(x') dx', & x \geq 0, \\ x, & x < 0, \end{cases} \quad (29)$$

where $\sigma(x') \geq 0$ for $x' > 0$, we obtain the following system of equations for $x > 0$:

$$a(s) \left(1 + \frac{\sigma(x)}{s}\right)^{-1} \partial_x \left(\left(1 + \frac{\sigma(x)}{s}\right)^{-1} \partial_x U \right) + b(s)\partial_{yy}U - s^2c(s)U = 0.$$

For $x < 0$, the system coincides with (4); both systems are coupled via transmission conditions. Importantly, $u(x, y) = U(x, y)$ for $x < 0$ by analytic continuation. Finally, it is necessary to rewrite the system in the time domain. There exists many ways to do so, hence for the moment we omit the passage to the time domain, which can be done in practice similarly to the way it is done in Section 4.2. Our arguments will be valid for all the systems in the time domain that can be rewritten in the Laplace domain in the form (28).

Other PML directions can be treated similarly, and in the corners the change of multiple variables should be used. To truncate the perfectly matched layer at some $x = L > 0$, which in practice cannot be chosen infinitely long, zero Dirichlet or Neumann boundary conditions are used on the exterior boundary of the PML.

As demonstrated in [7, 9], in some cases (anisotropy and/or dispersion) the change of variables (29) can lead to instabilities in the time domain. We demonstrate this with the help of the following simple numerical experiment.

3.1.1. Numerical Illustration: Instability of PMLs

Before providing numerical data, let us introduce some auxiliary notation, which we will use in all the numerical experiments that follow, if not stated otherwise. First of all, we compute solutions to problems inside rectangular domains; their physical (i.e. without the PML) dimensions in the direction x and y we will denote respectively by L_x, L_y . The width of the PML in x -direction will be denoted by L_x^σ , and in y -direction by L_y^σ . The absorption parameters of the PMLs in directions x and y we will denote by $\sigma_x(x), \sigma_y(y)$, c.f. (29). Moreover, if no PML is used in one of the directions, we employ zero Dirichlet boundary conditions. To truncate the PML layer, we use vanishing Dirichlet BCs.

In all the experiments the PML system has a structure similar to (56), unless stated otherwise, and we use zero initial conditions and the right-hand side $f(t, x, y)$ in (56a). Unless stated otherwise, the solution of this problem is done with the help of the Yee scheme [44] adapted for dispersive models, where the dispersive terms are discretized with the help of the trapezoidal rule, and non-dispersive part is computed with a leapfrog. In the scheme, we discretize the field H_z on integer time and spatial steps. The time step we denote by Δt , and the space step is $\Delta x = \Delta y$.

In all the experiments where we show a solution at different time steps, or compare the stable and unstable PMLs, we use the same color scale in all relevant figures, in order to compare the behaviour of the solution.

Example 3.1 (Instability of Bérenger's PML for anisotropic Drude Material). We model an **anisotropic** Drude material, which extends the isotropic model (5), where

$$a(s) = \varepsilon_2(s)^{-1} = \left(1 + \frac{\omega_a^2}{s^2}\right)^{-1}, \quad b(s) = \varepsilon_1(s)^{-1} = \left(1 + \frac{\omega_b^2}{s^2}\right)^{-1}, \quad \mu(s) = 1. \quad (30)$$

In particular, we choose $\omega_a = 8, \omega_b = 4$. We use the Bérenger's perfectly matched layer in the direction x , i.e. $\psi(s) = 1$ in (31), and in the direction y we employ zero Dirichlet boundary conditions, in order to show that the instability occurs not necessarily in a corner of a PML, but in a certain direction. The rest of the parameters can be found in Table 1. We show the field H_z computed with the help of the Bérenger's PML in Figure 1. One

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
8	24	2	0	$20x^2$	0	0.05	0.025	$10(t - 1.6)e^{-15(t-1.6)^2}e^{-300x^2 - 300y^2}$

TABLE 1. Parameters for the experiment with dielectric permittivity and magnetic permeability (30). The notation can be found in the beginning of Section 3.1.1.

can see the instability developing in this case. In the rightmost plot we demonstrate an exponential blow-up of the L_2 -norm of the solution H_z computed inside the physical domain and the perfectly matched layer.

3.2. Construction of Stable PMLs

Let us consider a sesquilinear form (17), namely

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \quad s \in \mathbb{C}_+, \quad u, v \in H_1(\mathbb{R}^2).$$

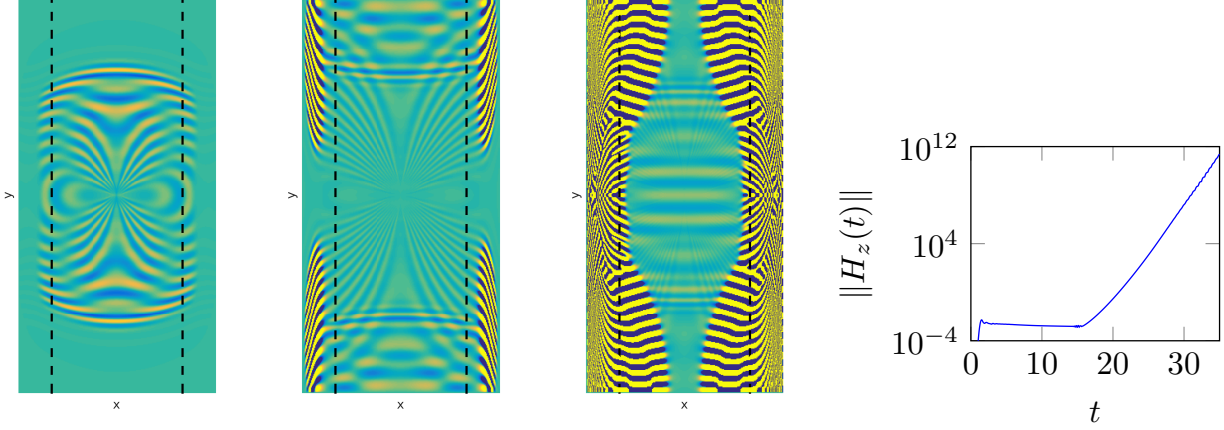


FIGURE 1. From left to right: a solution H_z of Example 3.1 computed with the Bérenger's PML at $t = 10, 20, 35$; the dependence of the L_2 -norm of H_z with respect to time. The boundary between the PML and the physical domain is marked in black.

and apply to it a (modified) PML change of variables

$$\tilde{x} = \begin{cases} x + s^{-1}\psi(s) \int_0^x \sigma(x') dx', & x \geq 0, \\ x, & x < 0, \end{cases} \quad (31)$$

where the absorption function $\sigma(x') > 0$. In the analysis, we will consider a simplified case, when the PML is applied in the whole space and the absorption function is a non-negative constant, i.e.

$$x \rightarrow x(1 + s^{-1}\psi(s)\sigma), \quad \sigma \geq 0. \quad (32)$$

In the Bérenger's PML, the function $\psi(s) = 1$. As shown in the previous section, as well as in [6, 7], in the dispersive media such a choice may lead to an instability. In recent works [6, 7, 20] it had been demonstrated that other choices of $\psi(s)$ can stabilize the PML.

Upon application of (32), the sesquilinear form (17) becomes

$$A_\sigma(u, v) = a(s) \left(1 + \frac{\sigma\psi(s)}{s}\right)^{-2} (\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \quad s \in \mathbb{C}_+, \quad u, v \in H_1(\mathbb{R}^2). \quad (33)$$

We suggest that the PML change of variables (32) with $\psi(s) = a(s)$ applied to the sesquilinear form (17) that is passive as in Definition 2.11 produces a coercive and bounded sesquilinear form. This result is generalized in the following very simple Theorem 3.2 and is explained in detail in Corollary 3.7. Although this theorem does not provide an explicit way to construct the function $\psi(s)$, it will play an important role in Section 5, where for a particular class of systems we will demonstrate the equivalence of this result (which gives a sufficient condition of stability) and the necessary condition of the stability of the PML derived in [7].

Theorem 3.2. *Let the sesquilinear form $A(u, v)$ given by (17) be passive in the sense of Definition 2.11. Let an analytic $\psi(s) : \mathbb{C}_+ \rightarrow \mathbb{C}$ satisfy for all $s \in \mathbb{C}_+$*

$$(S1) \quad \operatorname{Re}(\bar{s}\psi(s)) > 0,$$

- (S2) $\operatorname{Re}(\bar{s}\psi(s)a(s)^{-1}b(s)) > 0$,
(S3) $\operatorname{Re}(s\psi(s)a(s)^{-1}c(s)) > 0$.

Then for all $\sigma \geq 0$, the sesquilinear form

$$\tilde{A}_\sigma(u, v) = \frac{\psi(s)}{a(s)} \left(a(s) \left(1 + \frac{\sigma\psi(s)}{s} \right)^{-2} (\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v) \right), \quad s \in \mathbb{C}_+, \quad u, v \in H^1(\mathbb{R}^2), \quad (34)$$

is passive in the sense of Definition 2.11.

Proof. Notice that since $a(s)$ satisfies $\operatorname{Re}(\bar{s}a(s)) > 0$, $s \in \mathbb{C}_+$, it never vanishes in \mathbb{C}_+ . Let us verify the conditions on the coefficients of the sesquilinear form (34) required by passivity. The first coefficient, for $s \in \mathbb{C}_+$, satisfies

$$\operatorname{Re} \left(s\psi(s)^{-1} \left(1 + \frac{\sigma\psi(s)}{s} \right)^2 \right) = \operatorname{Re}(s\psi(s)^{-1}) + 2\sigma + \operatorname{Re}(s^{-1}\psi(s)) > 0,$$

thanks to (13) and (S1). Thus, it is analytic and its reciprocal is passive. The condition on the rest of the coefficients that is required by passivity is included explicitly into the conditions of the theorem. \square

Combining the above with Theorem 2.9 and Proposition 2.12, we obtain the following simple result of the time-domain stability of the PML system.

Corollary 3.3. *Let $\tilde{A}_\sigma(u, v)$ be defined as in Theorem 3.2 and be passive, and let $A_\sigma(u, v)$ be given in (33). Let an analytic $\psi(s) : \mathbb{C}_+ \rightarrow \mathbb{C}$ be s.t. $\operatorname{Re}(\bar{s}\psi(s)) > 0$, and additionally*

$$\left| \frac{\psi(s)}{a(s)} \right| \leq |s|^\mu C_f(\operatorname{Re} s), \quad s \in \mathbb{C}_+, \quad (35)$$

with $\mu \geq -1$ and $C_f(\lambda) : (0, \infty) \rightarrow (0, \infty)$ being non-increasing and satisfying $C_f(\lambda) < C\lambda^{-m}$, $C > 0$, $m \geq 0$. Then the following holds true for the sesquilinear form (33):

- (1) for all $G \in H^{-1}(\mathbb{R}^2)$, for all $s \in \mathbb{C}_+$, there exists a unique solution $U(s) \in H^1(\mathbb{R}^2)$ to the variational formulation

$$A_\sigma(U(s), v) = \langle G, v \rangle, \quad v \in H^1(\mathbb{R}^2), \quad (36)$$

which satisfies, for all $s \in \mathbb{C}_+$,

$$\|U(s)\|_{H^1(\mathbb{R}^2)} \leq C|s|^{\mu+1}(\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2})C_f(\operatorname{Re} s)\|G\|_{H^{-1}(\mathbb{R}^2)}, \quad C > 0; \quad (37)$$

- (2) let $k = \lfloor \mu + 3 \rfloor$, and $\ell = 3 + m$. Let $G(s) \in H^{-1}(\mathbb{R}^2)$ be the Laplace transform of a causal function $g(t) : \mathbb{R} \rightarrow H^{-1}(\mathbb{R}^2)$, which, for $k = 0$, is continuous and locally integrable, and for $k > 0$ belongs to $C^{k-1}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ and has an integrable k -th distributional derivative.

Then the solution $U(s)$, $s \in \mathbb{C}_+$, of (36) with $G = G(s)$ is a Laplace transform of a causal continuous function $u(t) : \mathbb{R} \rightarrow H^1(\mathbb{R}^2)$, which can be bounded as follows:

$$\|u(t)\|_{H^1} \leq C \max(t^\ell, 1) \int_0^t \|(\mathcal{P}_k g)(\tau)\|_{H^{-1}} d\tau, \quad t \geq 0, \quad C > 0. \quad (38)$$

Proof. The first statement follows directly from Proposition 2.12, applied to the following variational formulation, which is equivalent to (36):

$$\tilde{A}_\sigma(U(s), v) = \psi(s)a(s)^{-1}(G, v), \quad G \in H^1(\mathbb{R}^2), \quad s \in \mathbb{C}_+.$$

The second statement is obtained as in the proof of Proposition 2.12; it is important to notice that $\psi(s)a(s)^{-1}$ is analytic in \mathbb{C}_+ . The bound in the time domain follows by the application of Theorem 2.9. \square

Remark 3.4. *Indeed, if (35) holds with $\mu < -1$, we can always bound $|s|^\mu C_f(\operatorname{Re} s) = |s|^{-1}|s|^{1+\mu} C_f(\operatorname{Re} s) \leq |s|^{-1}(\operatorname{Re} s)^{\mu+1} C_f(\operatorname{Re} s)$, which satisfies the conditions of Corollary 3.3.*

Remark 3.5. *Like before, we have shown that if $u(t)$ satisfies a time domain system that in the Laplace domain can be rewritten in the form (33), and provided sufficiently smooth causal right-hand side data and zero initial conditions, the stability bound (38) holds true for $u(t)$. However, in the time domain $u(t)$ can be coupled with other unknowns, cf. e.g. Section 4.2, and this result does not necessarily imply the stability of the corresponding unknowns; however, it is always possible to choose auxiliary PML-related unknowns that would ensure the stability of the global system, see [24], or Section 4.2 for the PML for Lorentz materials.*

While the change of variables as suggested in Theorem 3.2 leads to a stable system, it is not obvious whether it leads to the energy decay (i.e. that the layer is absorbing). We will show this in [24]. Moreover, as seen from the proof of the above corollary, the introduction of the PML may require for obtaining the bounds of the solution (38) a smoother right-hand side data, compared to a non-PML system.

Let us also remark that the time-domain bounds in Corollary 3.3 are non-optimal, since they are based on the extensive use of Proposition 2.12 and non-optimal bounds of Theorem 2.9. Therefore this result cannot be used to compare the bounds on the solution of the PML and non-PML system. For example, we did not compute an explicit lower bound of $\operatorname{Re} \left(\bar{s}\psi(s) \left(1 + \frac{\sigma\psi(s)}{s} \right)^{-2} \right)$. The reason for such a simplification is mainly the fact that the bounds of Theorem 2.9, which is crucial for the derivation of all the results in this section, are not optimal.

Using a priori bounds on $\left| \frac{a(s)}{\psi(s)} \right|$ which can be computed with the help of Corollary 2.4, we obtain the following simple result.

Corollary 3.6. *Let $a(s)^{-1}$, $\psi(s)^{-1}$ be passive. Then the estimate (35) holds true with $\mu \leq 4$, $m \leq 6$. If additionally $a(s)^{-1}$, $\psi(s)^{-1}$ are Lorentz materials, i.e. have an expansion as in (8), the estimate (35) holds true with $\mu \leq 1$, $m \leq 3$.*

Proof. First, let us prove the result for arbitrary materials. Notice that for any $a(s)$, $\psi(s)$ satisfying the conditions of the corollary, the bound (35) holds with the following parameters, see Corollary 2.4,

$$\left| \frac{\psi(s)}{a(s)} \right| = \left| \frac{\bar{s}\psi(s)}{\bar{s}a(s)} \right| \leq |s| |\psi(s)| (\operatorname{Re}(\bar{s}a(s)))^{-1} \leq C |s|^4 (\operatorname{Re} s)^{-2} \max(1, (\operatorname{Re} s)^{-4}), \quad C > 0.$$

For the Lorentz materials, let us bound $a(s)^{-1}$ which has an expansion (8), i.e. $a(s)^{-1} = 1 + \sum_{\ell=0}^{n_a} \frac{a_\ell}{s^2 + \omega_\ell^2}$, with $a_\ell > 0$ and $\omega_\ell \in \mathbb{R}$. Notice that for all $\omega \in \mathbb{R}$ and $s \in \mathbb{C}_+$, it holds that

$$\left| \frac{1}{s^2 + \omega^2} \right| = \frac{1}{|s - i\omega||s + i\omega|} \leq \frac{1}{|s| \operatorname{Re} s}. \quad (39)$$

Then, for some $C, \tilde{C} > 0$, the modulus $|a(s)^{-1}| \leq 1 + \frac{C}{|s| \operatorname{Re} s} \leq \tilde{C} \max(1, (\operatorname{Re} s)^{-2})$. Thanks to (44), we obtain

$$\left| \frac{\psi(s)}{a(s)} \right| \leq \left| \frac{sa(s)^{-1}}{s\psi(s)^{-1}} \right| \leq \tilde{C} |s| \max(1, (\operatorname{Re} s)^{-2}) (\operatorname{Re} s \psi(s)^{-1})^{-1} \leq \tilde{C} |s| \max(1, (\operatorname{Re} s)^{-3}).$$

□

The following corollary provides a simple way to construct stable perfectly matched layer under very relaxed assumptions, with a particular choice of $\psi(s)$.

Corollary 3.7. *Let a sesquilinear form $A(u, v)$ given by (17) be passive in the sense of Definition 2.11. Then for all $\sigma \geq 0$, the form*

$$A_\sigma(u, v) = a(s) \left(1 + \frac{\sigma a(s)}{s} \right)^{-2} (\partial_x u, \partial_x v) + b(s) (\partial_y u, \partial_y v) + s^2 c(s) (u, v),$$

obtained from (33) with a particular choice $\psi(s) = a(s)$, is passive in the sense of Definition 2.11.

Proof. Setting $\psi(s) = a(s)$ shows that the assumptions of Theorem 3.2 are satisfied. □

Example 3.8 (Uniaxial Cold Plasma). Consider the plasma model (7), with the PML in y -direction chosen as $\psi(s) = \varepsilon_1(s)^{-1}$. The stability of such a PML had been confirmed in [8, 20]. Notice that in this case Corollary 3.3 applies with $m = \mu = 0$.

This example can be generalized as follows.

Corollary 3.9. *Let a sesquilinear form $A(u, v)$ be given by (17). Assume that its coefficients $a(s) = \varepsilon_2(s)^{-1}$, $b(s) = \varepsilon_1(s)^{-1}$, $c(s) = \mu(s)$, where $\varepsilon_1(s)$, $\varepsilon_2(s)$, $\mu(s)$ are defined by (8), i.e. they correspond to the dielectric permittivity and magnetic permeability of the Lorentz materials.*

Then for all $\sigma \geq 0$, the form

$$A_\sigma(u, v) = \varepsilon_2(s)^{-1} \left(1 + \frac{\sigma \varepsilon_2(s)^{-1}}{s} \right)^{-2} (\partial_x u, \partial_x v) + \varepsilon_1(s)^{-1} (\partial_y u, \partial_y v) + s^2 \mu(s) (u, v),$$

obtained from (33) with a particular choice $\psi(s) = \varepsilon_2(s)^{-1}$, is passive.

Proof. One can show that the sesquilinear form $A(u, v)$ as above is passive in the sense of Definition 2.11. In particular, the lower bounds $\operatorname{Re}(s\mu(s)) > 0$, $\operatorname{Re}(s\varepsilon_j(s)) > 0$, $j = 1, 2$, for all $s \in \mathbb{C}_+$, follow from Theorem 4.3. Then the corollary is simply a reformulation of Corollary 3.7 for the Lorentz materials. □

Lemma 3.7 applied to the isotropic Drude model (5) suggests that the choice $\psi(s) = \varepsilon(s)^{-1}$ in the change of variables (32) leads to a stable perfectly matched layer. This is confirmed by the analysis and numerical experiments in [7]. However, in the same work it was demonstrated that a more general change of variables leads to a stable PML, more precisely, $\psi(s) = \left(1 + \frac{\omega_\ell^2}{s^2} \right)^{-1}$, where $\omega_\ell \in [\min(\omega_e, \omega_m), \max(\omega_e, \omega_m)]$. We explain this in the following lemma.

Lemma 3.10. *Let a sesquilinear form $A(u, v)$ given by (17) with $a(s) = b(s)$ be passive in the sense of Definition 2.11. Let $\psi(s) = (\alpha a(s)^{-1} + (1 - \alpha)c(s))^{-1}$, $0 \leq \alpha \leq 1$.*

Then for all $\sigma \geq 0$, the form $\tilde{A}_\sigma(u, v)$ defined in (34) is passive.

Proof. It is sufficient to verify conditions of Theorem 3.2.

- (1) thanks to (13), it is sufficient to check $\operatorname{Re}(s\psi(s)^{-1}) = \alpha \operatorname{Re}(sa(s)^{-1}) + (1 - \alpha) \operatorname{Re}(sc(s)) > 0$.
- (2) since $b(s) = a(s)$, $\psi(s)a(s)^{-1}b(s) = \psi(s)$.
- (3) to show the required bound for $a(s)^{-1}\psi(s)c(s)$, we use (13):

$$\operatorname{Re}(\bar{s}\psi(s)^{-1}a(s)c(s)^{-1}) = \alpha \operatorname{Re}(\bar{s}c(s)^{-1}) + (1 - \alpha) \operatorname{Re}(\bar{s}a(s)) > 0.$$

□

Finally, to construct a stable PML in corners, we extend the statement of Corollary 3.7.

Lemma 3.11. *Let a sesquilinear form $A(u, v)$ given by (17) be passive in the sense of Definition 2.11. Then for all $\sigma_x, \sigma_y \geq 0$, the form*

$$A_{\sigma_x, \sigma_y}(u, v) = a(s) \left(1 + \frac{\sigma_x a(s)}{s}\right)^{-2} (\partial_x u, \partial_x v) + b(s) \left(1 + \frac{\sigma_y b(s)}{s}\right)^{-2} (\partial_y u, \partial_y v) + s^2 c(s)(u, v),$$

obtained from (33) by the PML change of variables (32) in x -direction with a particular choice $\psi(s) = a(s)$, and in y -direction by choosing $\psi(s) = b(s)$, is passive in the sense of Definition 2.11.

The proof of this result follows the same arguments as the proof of Theorem 3.2.

Remark 3.12. *The stability of the PML as proved in this work implies the uniform stability of the PML as defined in [7], see Lemma 2.14.*

3.3. Numerical Verification of the Results

Example 3.13. (The system of Example 3.1, anisotropic Drude material) First of all, we apply the obtained results to the system described in Example 3.1. It is not difficult to check that such a system (without a PML) is passive, see also Theorem 4.3. We use exactly the same parameters, see (30) and Table 1, however, in this case we construct the PML system with the help of (31), choosing $\psi(s) = \varepsilon_2(s)^{-1}$. As before, in y -direction we use zero Dirichlet boundary conditions. The corresponding system in the time domain and its well-posedness are discussed in more detail in Section 4.2. Let us just remark that in the Laplace domain it is of the form (33).

The results of this experiment are shown in Figure 2. One clearly sees that the solution is stable. Interestingly, in this case the norm of the solution decays very lightly inside the domain and the PML. This is partially due to a dispersive, anisotropic nature of the problem (the solution remains non-zero inside a bounded domain for a fairly long time), and partially because of the fact that the PML was used only in one direction (cf. e.g. Figure 4, where the PML was used in two directions). Notice that the oscillations in the norm of the solution are likely due to the dispersive behaviour of the problem.

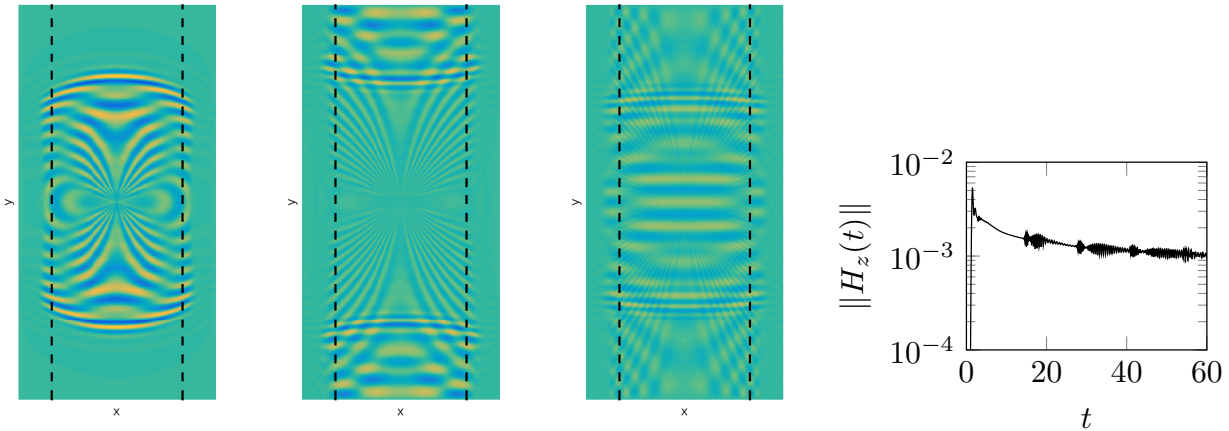


FIGURE 2. From left to right: a solution H_z of Example 3.13 computed with the new PML at $t = 10, 20, 35$; the dependence of the L_2 -norm of H_z with respect to time. The boundary between the domain and the layer is marked in black.

Example 3.14. (Anisotropic Lorentz Material) We consider an anisotropic Lorentz material (8) with the parameters

$$\varepsilon_1(s) = 1 + \frac{1}{12} \left(\frac{325}{s^2 + 4^2} + \frac{119}{s^2 + 8^2} \right), \quad \varepsilon_2(s) = 1 + \frac{16}{s^2 + 1} + \frac{16}{s^2 + 5^2}, \quad \mu(s) = 1 + \frac{3}{s^2 + 2^2}. \quad (40)$$

Notice that the corresponding time-domain system is stable, since $a = \varepsilon_2^{-1}$, $b = \varepsilon_1^{-1}$, $c = \mu$ satisfy conditions of Definition 2.11, cf. Theorem 4.3. This time we apply the PML change of variables in both directions, in particular in the direction x we use (31) with $\psi(s) = \varepsilon_2(s)^{-1}$ and in the direction y we employ (31) with $\psi(s) = \varepsilon_1(s)^{-1}$. The stability of such a PML is proven in Lemma 3.11.

The rest of the parameters for this experiment are provided in Table 2. The results of the experiment

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
8	8	2	2	$20x^2$	$20y^2$	0.025	0.0125	$10(t - 1.6)e^{-15(t-1.6)^2} e^{-20x^2 - 20y^2}$

TABLE 2. Parameters for the experiment with dielectric permittivity and magnetic permeability (40). The notation can be found in the beginning of Section 3.1.1.

are shown in Figure 4, and the comparison to the standard, Bérenger's PML is shown in Figure 3. Due to the anisotropy of the model, when using the Bérenger's PML, the instability in one of the directions is more pronounced and occurs earlier than the instability in another direction. Notice that the Bérenger's PML is unstable, and with the use of the new PML the norm of the solution decays, see Figure 4.

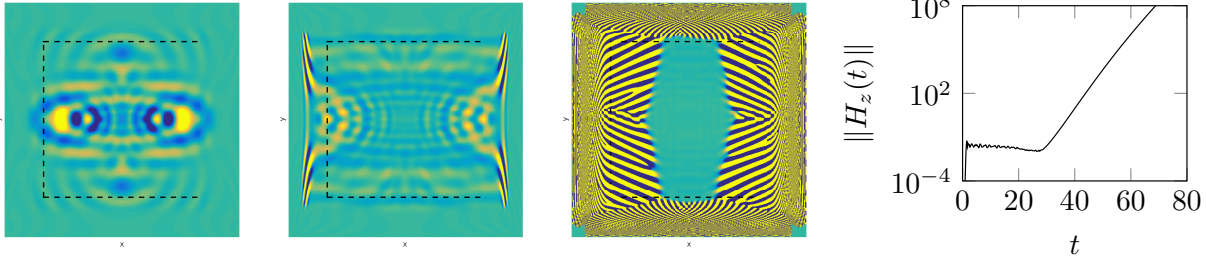


FIGURE 3. From left to right: a solution H_z of Example 3.14 at the times $t = 15, 30, 80$, computed with the help of the Bérenger's PML; the dependence on time of the L_2 -norm of this solution measured in the whole domain. The boundary between the physical domain and the PML is marked in black.

Example 3.15. (Non-Lorentz Materials) Let us consider the problem (4) with the parameters:

$$\varepsilon_1(s) = 1, \quad \varepsilon_2(s) = 1 + \frac{2}{3(s+3)} + \frac{25}{12s} + \frac{3}{4(s+2)}, \quad \mu(s) = 1. \quad (41)$$

It is not difficult to check that the corresponding sesquilinear form is passive as in Definition 2.11. Unlike the previous two examples, this problem does not fit the framework of the Lorentz materials (8). Moreover, the existing condition of the stability of perfectly matched layers [7] does not cover this class of models.

The parameters for this experiment are provided in Table 3. A numerical comparison of the new PML (31) with $\psi(s) = \varepsilon_2(s)^{-1}$ with the conventional perfectly matched layer (29) is shown in Figure 5. We employ the PML only in x -direction, and in y -direction we use zero Dirichlet data. Contrary to the Bérenger's PML, the new change of variables leads to a stable system.

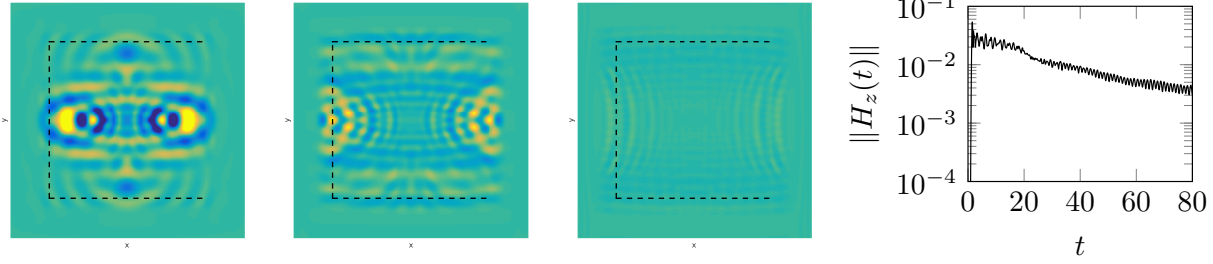


FIGURE 4. From left to right: a solution H_z of Example 3.14 at the times $t = 15, 30, 80$, computed with the help of the new PML; the dependence on time of the L_2 -norm of this solution measured in the whole domain. The boundary between the physical domain and the PML is marked in black.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
8	12	2	0	$50x^2$	0	0.05	0.025	$120e^{-16(t-2)^2}e^{-300x^2-800y^2}$

TABLE 3. Parameters for the experiment with dielectric permittivity and magnetic permeability (41). The notation can be found in the beginning of Section 3.1.1.

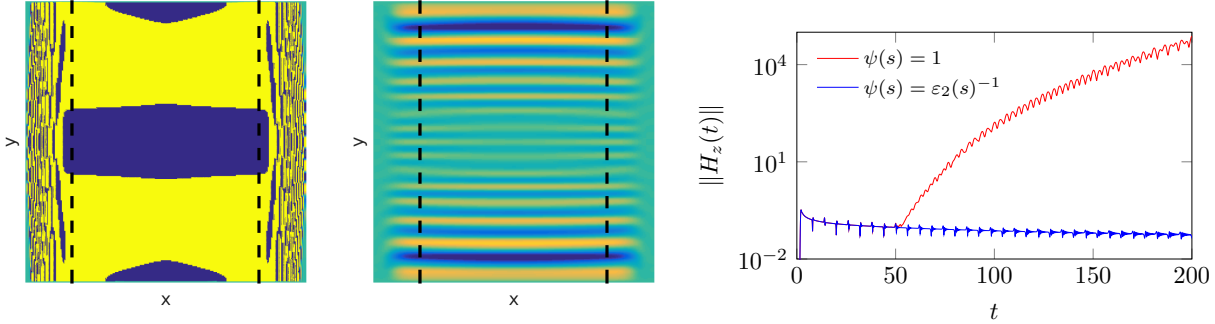


FIGURE 5. From left to right: a solution H_z to the problem of Example 3.15 at $t = 200$ computed with the Bérenger's PML; the same quantity computed with the new PML; the comparison of L_2 -norms of the solution H_z , computed with two different PMLs, measured in the whole domain. The boundary between the physical domain and the PML is marked in black.

Example 3.16. (Nonlocal Materials) Another (somewhat academic) example is provided by application of a PML to a problem with non-local operators. We choose the problem parameters as:

$$\varepsilon_1(s) = 1, \quad \varepsilon_2(s) = \left(1 + s(1 + e^{-0.5s})^{-1}\right)^{-1}, \quad \mu(s) = 1. \quad (42)$$

The above functions satisfy $\operatorname{Re}(s\mu(s)) > 0$ and $\operatorname{Re}(\bar{s}\varepsilon_j(s)^{-1}) > 0$, with $j = 1, 2$, for all $s \in \mathbb{C}_+$. Notice as well that $\mathcal{L}^{-1}(e^{-0.5s}\hat{u}(s))(t) = u(t-0.5)$ for causal functions. We compare the performance of the Bérenger's PML (29) employed in both directions, and the new PMLs (31) with $\psi(s) = \varepsilon_2(s)^{-1}$ in the direction x , and $\psi(s) = 1$ (i.e. Bérenger's PML) in the direction y .

To semi-discretize the corresponding equation (9) in time, we employ the Runge-Kutta (3-stage Radau IIA) convolution quadrature (CQ) method [45–47], and use the FFT-based algorithm described in [48]. A similar algorithm for volumetric problems can be found in [49]. To discretize the resulting problem in space we apply second order central finite differences. Notice that for stable perfectly matched layer there is no need to employ the CFL condition restriction [49], since the method we use is implicit and A -stable. However, a time step restriction may be required if the PML system is unstable. The parameters of the experiment are given in Table 4. The results of the comparison of two PMLs can be found in Figure 6. We can clearly see that the

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
12	12	2	2	$10x^2$	$10y^2$	0.025	0.1	$e^{-40x^2-40y^2-12(t-10)^2} (24^2(t-10)^2 - 24)$

TABLE 4. Parameters for the experiment with dielectric permittivity and magnetic permeability of Example 3.16. The notation can be found in the beginning of Section 3.1.1. We discretize directly the problem (4) where the PML change of variables (31) is used in directions x and y . We use zero initial conditions and the right hand side (in the time domain) $f(t, x, y)$.

solution computed with Bérenger’s PML is unstable. To investigate whether the instability problem comes the discretization, we computed as well the norms of the solution to the problem computed with a twice finer time step, see Figure 6. Refining the time step even aggravates the Bérenger’s PML instability, which indicates the convergence problem of the CQ in this case. However, refining the time step results in a stable, convergent solution when the new PML is used.

Indeed, the result produced by convolution quadrature discretization of the unstable PML has large stepwise errors. Moreover, the result is non-causal (in our experiments the right hand side $\|f(t, x, y)\|_\infty < 10^{-8}$ for $t \leq 8.5$, i.e. it almost vanishes, while this is not true for the solution). Such a global behaviour may be attributed to the algorithm that we used for the implementation of the CQ, which computes the solution of (4) in the Laplace domain for many values of frequencies $s \in \mathbb{C}_+$, and then comes back to the time domain. As a result, the instability of the PML propagates over all the time steps, which is shown in Figure 6, where this is expressed in the non-causal behaviour of the solution. Another problem comes from the fact that the discretizations of (17), to which the Bérenger’s PML change of variables (29) is applied, are highly ill-conditioned (in the algorithm we use, these matrices should be inverted to compute the solution). This is likely to be caused by the fact that the corresponding matrix has some of its eigenvalues in the right complex half-plane.

We checked the first assumption (i.e. propagation of the error globally) by computing the solution with the help of the unstable PML on the interval $[0, 12]$, before the instability occurs, see the left-most plot in Figure 6. It appeared that the norm of the solution on this interval did not grow exponentially, however the difference between the solutions computed with two PMLs was non-negligible. To check the second assumption (i.e. the effect of matrix ill-conditioning), we reduced the absorption parameter, by setting $\sigma_x(x) = x^2$, while keeping the rest of parameters as in Table 4, and again computed the solution on $[0, 12]$. This allowed to reduce the condition number of the matrices. The results displayed in the leftmost plot of Figure 6 show that the solutions computed with stable and unstable PMLs on the interval $[0, 12]$ almost coincide.

Summarizing all said, our experiments demonstrate that the solution remains stable when the new perfectly matched layer is applied, however, there are convergence and stability problems when the standard, Bérenger’s PML is used. Let us also remark that in general the discretization by the Runge-Kutta method Radau IIA introduces some dissipation into the problem (which can be controlled by decreasing the time step), however, even in this case we observe the instability of the standard, Bérenger’s PML.

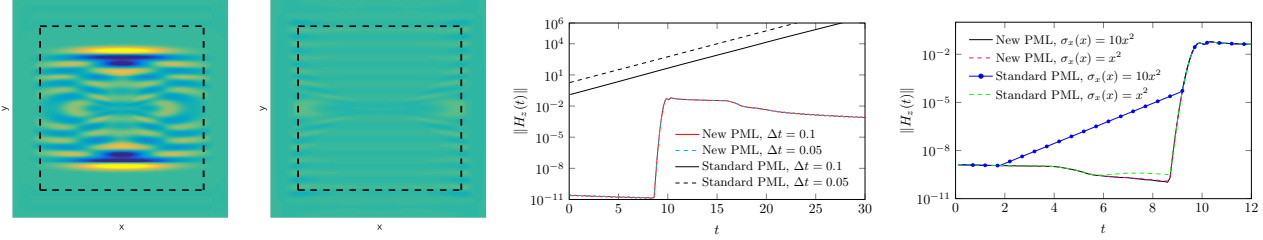


FIGURE 6. From left to right: a solution H_z of Example 3.16 at $t = 14$ and $t = 20$ computed with the help of the new PML; the dependence of the L_2 -norm of the solution H_z measured in the physical domain and the PML with respect to time, for the stable and unstable PMLs, for $t \leq 30$; the dependence of the L_2 -norm of the solution H_z measured in the physical domain and the PML with respect to time, for the stable and unstable PMLs, computed for $t \leq 12$ and different absorption parameters. The boundary between the physical domain and the PML is marked in black.

4. DISPERSIVE SYSTEMS WITH RATIONAL DIELECTRIC PERMITTIVITY AND MAGNETIC PERMEABILITY

In the previous section we have shown how to construct a stable PML for a model described by a passive sesquilinear form (17). In this section we will study a special case when the coefficients of the sesquilinear form are rational even functions (however, most of the results are valid as well for meromorphic even functions). For this particular case we will provide a simplified criterion that will allow to construct the function $\psi(s)$ in the PML change of variables (32) that would satisfy the stability conditions of Theorem 3.2.

This section is organized as follows. First of all, we will discuss some important properties of rational functions under consideration. We will state the corresponding results with sketches of proofs. Next, we will reformulate a necessary stability condition of the perfectly matched layers derived in the work [7]. Finally, we will show that this stability condition implies the stability condition of Theorem 3.2.

4.1. Properties of Even Rational Functions which Satisfy the Passivity Condition

We will consider a set of rational functions $r(z)$ satisfying the following assumptions (these are the same assumptions as in the work [7]).

- Assumption 4.1.** (1) $r(z)$ can be represented as $r(z) = 1 + \frac{p(z^2)}{q(z^2)}$, where $p(z)$ and $q(z)$ are polynomials with real coefficients that have no common roots, s.t. $\deg p < \deg q$.
 (2) all zeros and poles of $r(z)$ lie on the imaginary axis.

Recall that given $r(z)$, $z \in \mathbb{C}_+$, by $\tilde{r}(\omega)$ we denote the restriction $\tilde{r}(\omega) = r(-i\omega)$, $\text{Im } \omega \geq 0$, see (3).

Remark 4.2 (Physical Relevance). *Later on we will assume that $\varepsilon_j(s)$ for $j = 1, 2$, and $\mu(s)$ in (4) belong to the above class. These requirements are indeed physically meaningful:*

- (1) requiring that $\underline{\varepsilon}$ and μ are rational functions is equivalent to the fact that in the time domain the system (4) can be written with the help of local (i.e. linear combinations of $\partial_t^{(k)}$, for $k \in \mathbb{N}_0$) operators;
- (2) $\underline{\varepsilon}(\omega)$, $\tilde{\mu}(\omega) \rightarrow 1$ as $\omega \rightarrow \infty$ is a requirement that can be found in [50, §59, p. 251]; this means that for large frequencies the system becomes non-dispersive;
- (3) in a so-called transparency regime [50, §64, p. 260] the rational functions $\underline{\varepsilon}(s)$, $\mu(s)$ are real on the imaginary axis. Moreover, in this case $\underline{\varepsilon}$, $\tilde{\mu}$ are even functions of ω (see [50, p. 250]). From this it

follows that these rational functions should have real coefficients;

- (4) the passivity requirement, see Definition 2.1, implies that these functions cannot have poles and zeros in \mathbb{C}_+ , and hence, thanks to the fact that they are even, all their poles and zeros lie on the imaginary axis.

Let us remark that many of the results of this section are known, or follow from known theorems; however, we were not able to find a sole reference that would summarize all the properties we will make use of later. Some of the results that we describe in this section are summarized as well in [39]. The following result describes rational functions that satisfy the passivity requirement. It can be obtained from [28, Corollary 10.1], combined with [27, Theorem 5].

Theorem 4.3 ([28], [27]). *Let $r(s)$ be a rational function satisfying Assumption 4.1. Then this function is passive, i.e. $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$, if and only if*

$$r(s) = 1 + \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2}, \quad r_\ell > 0, \omega_\ell \in \mathbb{R}, \ell = 0, \dots, n. \quad (43)$$

Moreover, in this case

$$\operatorname{Re}(sr(s)) > \operatorname{Re} s. \quad (44)$$

Proof. Let us show that any passive function satisfying Assumption 4.1 satisfies (43). From Assumption 4.1 it follows that all the poles of $r(s)$ lie on the imaginary axis. Applying the first part of Lemma 2.5 to $sr(s)$, which shows that all purely imaginary poles of $r(s)$ are simple, and zero may be a pole of multiplicity either 1 or 2, and using the fact that $r(s)$ is even, we obtain the following partial fraction expansion for $r(s)$:

$$r(s) = 1 + \frac{r_0}{s^2} + \sum_{\ell=1}^n \frac{\mu_\ell^+}{s + i\omega_\ell} + \sum_{\ell=1}^n \frac{\mu_\ell^-}{s - i\omega_\ell}, \quad \omega_\ell > 0, \ell = 1, \dots, n. \quad (45)$$

Now, since $r(s) = r(-s)$ in \mathbb{C} , we obtain that $\frac{\mu_\ell^+}{s + i\omega_\ell} + \frac{\mu_\ell^-}{s - i\omega_\ell} = \frac{\mu_\ell^+}{-s + i\omega_\ell} + \frac{\mu_\ell^-}{-s - i\omega_\ell}$, for all $\ell = 1, \dots, n$, and thus $\mu_\ell^+ = -\mu_\ell^-$ for all $\ell = 1, \dots, n$. The positivity of residues of $sr(s)$, see Lemma 2.5, gives

$$r_0 > 0, \quad -i\omega_\ell \mu_\ell^+ > 0, \quad \ell = 1, \dots, n. \quad (46)$$

From this it immediately follows that $\mu_\ell^+ \in i\mathbb{R}_{>0}$. This allows us to obtain the expansion $r(s) = 1 + \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2}$, with $\omega_\ell \geq 0$ for $\ell = 0, \dots, n$, where $r_\ell = -2i\omega_\ell \mu_\ell^+$, with $\ell = 1, \dots, n$. The positivity of r_ℓ follows from (46).

Now let us demonstrate that $r(s)$ given by (43) is passive, by showing that (44) holds true in this case:

$$\operatorname{Re}(sr(s)) = \operatorname{Re} s + \operatorname{Re} \left(\sum_{\ell=0}^n \frac{r_\ell s}{s^2 + \omega_\ell^2} \right) = \operatorname{Re} s + \operatorname{Re} \left(\sum_{\ell=0}^n \frac{r_\ell s(\bar{s}^2 + \omega_\ell^2)}{|s^2 + \omega_\ell^2|^2} \right) > \operatorname{Re} s,$$

when $s \in \mathbb{C}_+$. □

This result shows in particular that under Assumption 4.1 the passivity requirement can be satisfied **only** by the Lorentz materials (8). Naturally, we would also like to understand what the reciprocals of rational functions satisfying the passivity requirement look like.

Theorem 4.4. *Let $r(s)$ be a rational function satisfying Assumption 4.1. Then $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$, if and only if*

$$r(s) = 1 - \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2}, \quad r_\ell > 0, \omega_\ell \in \mathbb{R} \setminus \{0\}, \ell = 0, \dots, n, \quad r(0) = 1 - \sum_{\ell=0}^n \frac{r_\ell}{\omega_\ell^2} \geq 0. \quad (47)$$

Moreover,

$$\operatorname{Re}(\bar{s}r(s)) \geq r(0) \operatorname{Re} s + C \min(1, (\operatorname{Re} s)^2) \operatorname{Re} s, \quad C > 0. \quad (48)$$

Proof. First let us show that any rational function satisfying Assumption 4.1 and $\operatorname{Re}(\bar{s}r(s)) > 0$, when $s \in \mathbb{C}_+$, is of the form (47).

Thanks to Lemma 2.5, which says that $s^{-1}r(s)$ can have a simple pole in zero only, $r(s)$ cannot have a pole in zero; moreover, all the poles of $r(s)$ are imaginary, due to Assumption 4.1, and thus, due to Lemma 2.5, are simple. Like in Theorem 4.3, we write

$$r(s) = 1 + \sum_{\ell=0}^n \frac{\mu_\ell^+}{s + i\omega_\ell} + \sum_{\ell=0}^n \frac{\mu_\ell^-}{s - i\omega_\ell}, \quad \omega_\ell > 0, \ell = 1, \dots, n.$$

The same argument as in the proof of Theorem 4.3, shows that $\mu_\ell^+ = -\mu_\ell^-$, and $r(s) = 1 - \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2}$, where $\omega_\ell > 0$, $\ell = 0, \dots, n$, and $r_\ell = 2i\omega_\ell \mu_\ell^+ > 0$. Lemma 2.5 says that residues of $s^{-1}r(s)$ in all poles should be positive:

$$\lim_{s \rightarrow i\omega_\ell} (s - i\omega_\ell) s^{-1} r(s) = \frac{r_\ell}{2\omega_\ell^2} > 0.$$

If $r(0) \neq 0$, the positivity of the residue of $s^{-1}r(s)$ in zero implies that $r(0) > 0$.

To show that any $r(s)$ of the form (47) satisfies $\operatorname{Re}(\bar{s}r(s)) > 0$ in \mathbb{C}_+ , we will demonstrate the validity of the bound (48). First of all,

$$r(s) = 1 - \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2} = 1 - \sum_{\ell=0}^n \frac{r_\ell}{\omega_\ell^2} + \sum_{\ell=0}^n \frac{r_\ell}{\omega_\ell^2} - \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2} = r(0) + \sum_{\ell=0}^n \frac{r_\ell s^2}{\omega_\ell^2 (s^2 + \omega_\ell^2)} \quad (49)$$

Hence, for all $s \in \mathbb{C}_+$ it holds:

$$\begin{aligned} \operatorname{Re}(\bar{s}r(s)) &= r(0) \operatorname{Re} s + \operatorname{Re} \sum_{\ell=0}^n \frac{r_\ell |s|^2 s (\bar{s}^2 + \omega_\ell^2)}{\omega_\ell^2 |s^2 + \omega_\ell^2|^2} = r(0) \operatorname{Re} s + \operatorname{Re} s \sum_{\ell=0}^n r_\ell \omega_\ell^{-2} |s|^2 (|s|^2 + \omega_\ell^2) |s^2 + \omega_\ell^2|^{-2} \\ &\geq r(0) \operatorname{Re} s + \operatorname{Re} s \sum_{\ell=0}^n r_\ell \omega_\ell^{-2} |s|^2 (|s|^2 + \omega_\ell^2)^{-1}. \end{aligned} \quad (50)$$

For any $\omega_\ell \in \mathbb{R}^*$, the function $f(x) = \frac{x^2}{x^2 + \omega_\ell^2} = 1 - \frac{\omega_\ell^2}{x^2 + \omega_\ell^2}$ grows in x for $x > 0$. Since $0 < \operatorname{Re} s < |s|$, we obtain

$$\frac{|s|^2}{|s|^2 + \omega_\ell^2} \geq \frac{(\operatorname{Re} s)^2}{(\operatorname{Re} s)^2 + \omega_\ell^2} \geq \frac{(\operatorname{Re} s)^2}{2 \max((\operatorname{Re} s)^2, \omega_\ell^2)} = \frac{1}{2} (\operatorname{Re} s)^2 \min\left((\operatorname{Re} s)^{-2}, \omega_\ell^{-2}\right) \geq C \min(1, (\operatorname{Re} s)^2),$$

for some $C > 0$. This, combined with (50), gives (48). \square

Another property of passive materials, which is exploited in [7], is the so-called growing property. More precisely, a rational function $r(s)$ satisfies the growing property if for all $\omega \in \mathbb{R}$ except for poles of $\tilde{r}(\omega)$, it holds $(\omega\tilde{r}(\omega))' > 0$. It is interesting that in physical literature [51], [50, p.256] the above property is shown to be crucial for the positivity of the energy density.

We provide a minor generalization of this statement in the following two lemmas, stating before an auxiliary result that can be found in [27, Theorem V, Coroll. 1].

Lemma 4.5 ([27]). *Let $r(s)$ satisfy Assumption 4.1. Let $g_1(s) = sr(s)$, and $g_2(s) = s^{-1}r(s)$, $s \in \mathbb{C}_+$. If all zeros of $g_\alpha(s)$ are simple, and the derivative $\frac{d}{ds}g_\alpha(s)$ is real and positive in these zeros, then $\operatorname{Re}(g_\alpha(s)) > 0$ for all $s \in \mathbb{C}_+$. Here $\alpha = 1$ or $\alpha = 2$.*

To prove the validity of the statement, we start with the following trivial result.

Lemma 4.6. *Let $r(s)$ satisfy Assumption 4.1. Then:*

- (1) *if all the poles of $sr(s)$ are simple, and the corresponding residues are positive, then $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$.*
- (2) *if all the poles of $s^{-1}r(s)$ are simple, and the corresponding residues are positive, then $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$.*

Proof. The result can be proved following the lines of Theorems 4.3 and 4.4. More precisely, in the first case $r(s)$ is of the form (45); the positivity of residues and evenness of the function $r(s)$ results in the expansion (43), which, in turn, shows that $r(s)$ is passive, see Theorem 4.3. \square

Now we can give a proof of Lemma 4.5.

Proof. Let $\alpha = 1$ or 2. Notice that if all the zeros of $g_\alpha(s)$ are simple, and the derivative $\frac{d}{ds}g_\alpha(s)$ is real and positive in these zeros, then all the poles of $f_\alpha(s) = g_\alpha(s)^{-1}$ are simple, and the residues of $f_\alpha(s)$ in these poles are positive:

$$\operatorname{Res}_{s=s_0} f_\alpha(s) = \lim_{s \rightarrow s_0} (s - s_0)f_\alpha(s) = \lim_{s \rightarrow s_0} (s - s_0)g_\alpha(s)^{-1} = \lim_{s \rightarrow s_0} \frac{s - s_0}{g_\alpha(s) - g_\alpha(s_0)} = (g'_\alpha(s_0))^{-1} > 0.$$

Combined with Lemma 4.6, we obtain the desired result. \square

Finally, let us discuss the equivalence of the growing property and passivity.

Lemma 4.7. *Let $r(s)$ satisfy Assumption 4.1. Then the following conditions are equivalent:*

- (M1) *$(\omega\tilde{r}(\omega))' > 0$ in all points but poles of $\tilde{r}(\omega)$;*
- (M2) *the function $r(s)$ satisfies $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$.*

Proof. Thanks to Lemma 4.5, to show that (M1) implies (M2), it is sufficient to verify that from (M1) it follows that the zeros of $R(s) = sr(s)$ are simple, and $R'(s) > 0$ in these zeros. Notice that $\tilde{R}(\omega) = -i\omega\tilde{r}(\omega)$.

First let us show the implication (M1) \implies (M2). In zeros $s = -i\omega_0$ of $sr(s)$

$$\left. \frac{dR}{ds} \right|_{s=-i\omega_0} = r(-i\omega_0) - i\omega_0 r'(-i\omega_0) = \tilde{r}(\omega_0) + \tilde{r}'(\omega_0)\omega_0 = (\omega\tilde{r}(\omega))' \Big|_{\omega=\omega_0} > 0, \quad (51)$$

from which we obtain as well the simplicity of zeros of $R(s)$.

Notice that (M2) is equivalent to requiring that $\tilde{r}(\omega) = 1 - \sum_{\ell=0}^n \frac{r_\ell}{\omega^2 - \omega_\ell^2}$, $r_\ell > 0$, $\omega_\ell \in \mathbb{R}$, $\ell = 0, \dots, n$, see Theorem 4.3. Thus, (M2) \implies (M1) follows by a direct computation:

$$(\omega\tilde{r}(\omega))' = 1 + \sum_{\ell=0}^n r_\ell \left(\frac{2\omega^2}{(\omega^2 - \omega_\ell^2)^2} - \frac{1}{\omega^2 - \omega_\ell^2} \right) = 1 + \sum_{\ell=0}^n r_\ell \frac{\omega^2 + \omega_\ell^2}{(\omega^2 - \omega_\ell^2)^2} > 0$$

in all points but poles of $\tilde{r}(\omega)$. \square

A similar result can be formulated for functions $r(s)$, which satisfy $\operatorname{Re}(\bar{s}r(s)) > 0$ in \mathbb{C}_+ .

Lemma 4.8. *Let $r(s)$ satisfy Assumption 4.1. Then the following conditions are equivalent:*

- (M1) $(\omega^{-1}\tilde{r}(\omega))' < 0$ in all points $\omega \in \mathbb{R}$ but poles of $\omega^{-1}\tilde{r}(\omega)$;
- (M2) the function $r(s)$ satisfies $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$.

Proof. Thanks to Lemma 4.5, to show that (M1) implies (M2), it is sufficient to verify that from (M1) it follows that the zeros of $R(s) = s^{-1}r(s)$ are simple, and $R'(s) > 0$ in these zeros. Notice that $\tilde{R}(\omega) = i\omega^{-1}\tilde{r}(\omega)$. First let us show the implication (M1) \implies (M2). In zeros $s = -i\omega_0$ of $s^{-1}r(s)$

$$\left. \frac{dR}{ds} \right|_{s=-i\omega_0} = \omega_0^{-2}r(-i\omega_0) - (i\omega_0)^{-1}r'(-i\omega_0) = \tilde{r}(\omega_0)\omega_0^{-2} - \tilde{r}'(\omega_0)\omega_0^{-1} = -(\omega^{-1}\tilde{r}(\omega))|_{\omega=\omega_0} > 0, \quad (52)$$

from which we obtain as well the simplicity of zeros of $R(s)$.

Notice that (M2) is equivalent to requiring that $\tilde{r}(\omega) = r(0) + \sum_{\ell=0}^n \frac{r_\ell \omega^2}{\omega_\ell^2(\omega^2 - \omega_\ell^2)}$, $r_\ell > 0$, $\omega_\ell \in \mathbb{R} \setminus \{0\}$, $\ell = 0, \dots, n$, and $r(0) \geq 0$, see (49) in Theorem 4.4. Thus, (M2) \implies (M1) follows by a direct computation:

$$(\omega^{-1}\tilde{r}(\omega))' = -\frac{r(0)}{\omega^2} + \sum_{\ell=0}^n \frac{r_\ell}{\omega_\ell^2(\omega^2 - \omega_\ell^2)} - \sum_{\ell=0}^n \frac{2r_\ell \omega^2}{\omega_\ell^2(\omega^2 - \omega_\ell^2)^2} = -\frac{r(0)}{\omega^2} - \sum_{\ell=0}^n \frac{r_\ell}{\omega_\ell^2} \frac{(\omega^2 + \omega_\ell^2)}{(\omega^2 - \omega_\ell^2)^2}$$

The above expression is strictly negative in ω s.t. $\omega \neq 0$ and ω is not a pole of $\tilde{r}(\omega)$. If $r(0) = 0$, the above expression is strictly negative in $\omega = 0$. \square

The following statement is at least partially well known, cf. e.g. [29, p.308, Theorem 1].

Lemma 4.9. *Let $r(s)$ satisfy Assumption 4.1. Let $(\omega_\ell)_{\ell=1}^n$ be non-negative poles of $\tilde{r}(\omega)$ ordered in ascending order, and $(r_\ell)_{\ell=1}^n$ be its non-negative zeros, ordered in ascending order (counting multiplicities). Then*

- (I1) $r(s)$ is passive in the sense of Definition 2.1 if and only if $0 \leq \omega_1 < r_1 < \omega_2 < r_2 < \dots < \omega_n < r_n$.
- (I2) $r(s)^{-1}$ is passive in the sense of Definition 2.1 if and only if $0 \leq r_1 < \omega_1 < r_2 < \omega_2 < \dots < r_n < \omega_n$.

Proof. We explain the proof for (I1). The statement (I2) can be proved similarly.

Let us explain first " \implies "-part. The fact that zeros and poles of $r(s)$ are simple, with the exception of $s = 0$, and the condition on the multiplicity of the pole/zero in $s = 0$ follows from Lemmas 2.5. They interlace due to the same lemma (one notices that $h(s) = sr(s)$ satisfies $h(-\bar{s}) = -\bar{s}r(\bar{s}) = \overline{h(s)}$).

Finally, if $\omega_1 \neq 0$, then, necessarily $r(0) > 0$. Hence if r_1 were smaller than ω_1 , the sign change in r_1 would be from the positive to negative, and the function would have to decay on $(0, \omega_1)$. This is a contradiction to Lemma 4.7.

If $\omega_1 = 0$, the statement follows directly from the interlacing property of poles and roots.

The same argument applies to the case $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$.

Let us now show that the interlacing of roots and poles implies $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$. We will simply make use of Lemma 4.7.

First of all from the condition of the lemma we see that all zeros (and poles) of $sr(s)$ are simple. Next, we must verify whether the function $\omega\tilde{r}(\omega)$ grows.

We start by showing that $\tilde{r}(0) > 0$ or $\tilde{r}(\omega) \rightarrow -\infty$, when $\omega \rightarrow 0$. First notice that

$$\tilde{r}(\omega) = C \frac{(\omega^2 - r_1^2)(\omega^2 - r_2^2) \dots (\omega^2 - r_n^2)}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \dots (\omega^2 - \omega_n^2)},$$

where $C = 1$ due to $\tilde{r}(\omega) \rightarrow 1$ as $\omega \rightarrow \infty$. If $\omega_1 \neq 0$, $\tilde{r}(0) = \left(\prod_{k=1}^n (-r_k^2) \right) \left(\prod_{k=1}^n (-\omega_k^2) \right) > 0$. If $\omega_1 = 0$,

$$\tilde{r}(\varepsilon) = \frac{(\varepsilon^2 - r_1^2)(\varepsilon^2 - r_2^2) \dots (\varepsilon^2 - r_n^2)}{\varepsilon^2(\varepsilon^2 - \omega_2^2) \dots (\varepsilon^2 - \omega_n^2)} = \frac{(-1)^n (r_1^2 - \varepsilon^2)(r_2^2 - \varepsilon^2) \dots (r_n^2 - \varepsilon^2)}{\varepsilon^2(-1)^{n-1}(\omega_2^2 - \varepsilon^2) \dots (\omega_n^2 - \varepsilon^2)} \rightarrow -\infty, \text{ as } \varepsilon \rightarrow 0.$$

Consider two cases:

- $\tilde{r}(\omega) > 0$. In this case $\omega = 0$ is a zero of $\omega\tilde{r}(\omega)$. On the interval $(0, \omega_1)$ the function grows (otherwise it would have to intersect the real axis, which is a contradiction to $r_1 > \omega_1$); the same is valid on the interval $(-\omega_1, 0)$, due to evenness of $r(s)$. Therefore, $(\omega\tilde{r}(\omega))' > 0$ on this interval. In ω_1 the function $\tilde{r}(\omega)$ changes the sign from positive to negative, and in r_1 from negative to positive. Thanks to the interlacing of roots and poles, $(\omega\tilde{r}(\omega))' > 0$ for \mathbb{R} .
- $\tilde{r}(\omega) = -\infty$. Since $r_1 < \omega_2$, the function $\omega\tilde{r}(\omega)$ changes its sign in r_1 from negative to positive. The same argument as before allows to conclude that $(\omega\tilde{r}(\omega))' > 0$ on \mathbb{R} .

Similarly we can prove the second equivalence. □

4.2. Perfectly Matched Layers for Anisotropic Lorentz Materials: Time Domain Formulation

In the previous section we reminded that provided that the dielectric permittivity and magnetic permeability satisfy the conditions of passivity and Assumption 4.1, the corresponding materials are Lorentz (8). This class of models can be expressed in the time domain as a system of partial differential equations coupled with ordinary differential equations. In the following sections we will demonstrate that the stability of such a system, both with and without the PML, follows from the coercivity of the sesquilinear forms (17) and (33), see also the discussion after Proposition 2.12 and Remark 3.5.

First of all we will sketch this demonstration for the Lorentz materials in the time domain without the PML, and next we will argue that it can be extended to the formulation with the perfectly matched layers.

4.2.1. Lorentz Materials in the Time Domain

In this section we will apply previous results to the analysis of stability of the time-domain system describing the wave propagation in the Lorentz materials (8). Our goal is to show that the coercivity of the corresponding sesquilinear form (17) guarantees the stability of the whole system in the time domain (which is described by the PDEs coupled with ODEs).

Recall that the Lorentz dielectric permittivity and magnetic permeability are given by (8). To rewrite the Maxwell system (2) in the time domain, we first notice that one of the ways to express the relation $s\hat{D}_x = s\varepsilon_1(s)\hat{E}_x$ in the time domain is via the introduction of auxiliary unknowns:

$$\begin{aligned} \partial_t D_x &= \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell}, \\ \partial_t j_{x\ell} + \omega_{x\ell}^2 p_{x\ell} &= E_x, \quad \partial_t p_{x\ell} = j_{x\ell}, \quad \ell = 0, \dots, n_x. \end{aligned}$$

Similarly we deal with $s\hat{B}_z = s\mu(s)H_z$. Then the Maxwell system (2) can be rewritten in the following form (introducing a causal right-hand side data \mathbf{f} and zero initial conditions):

$$\begin{aligned} \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell} &= \partial_y H_z + f_{E_x}, \\ \partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} j_{y\ell} &= -\partial_x H_z + f_{E_y}, \end{aligned} \quad (53a)$$

$$\partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell j_{\mu\ell} = \partial_y E_x - \partial_x E_y + f_{H_z}, \quad (53b)$$

$$\begin{aligned} \partial_t j_{m\ell} + \omega_{m\ell}^2 p_{m\ell} &= E_m + f_{j_{m\ell}}, & \partial_t p_{m\ell} &= j_{m\ell} + f_{p_{m\ell}}, \quad \ell = 0, \dots, n_m, \quad m \in \{x, y\}, \\ \partial_t j_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} &= H_z + f_{j_{\mu\ell}}, & \partial_t p_{\mu\ell} &= j_{\mu\ell} + f_{p_{\mu\ell}}, \quad \ell = 0, \dots, n_\mu. \end{aligned}$$

We will proceed as follows. First, we show that the above system can be rewritten in the Laplace domain as the equation (4) for \hat{H}_z , with a right hand side data, and the rest of unknowns can be expressed via \hat{H}_z . Then the theory of Section 2.2 can be applied to (4). To bound the rest of the unknowns in the time domain, we will apply Theorem 2.9, making use of their expressions via \hat{H}_z and the Laplace domain bounds on \hat{H}_z .

In the Laplace domain the system (53) reads:

$$\begin{aligned} s\varepsilon_1(s)\hat{E}_x &= \partial_y \hat{H}_z + \hat{f}_{E_x} - \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(\frac{s\hat{j}_{x\ell}}{s^2 + \omega_{x\ell}^2} - \frac{\omega_{x\ell}^2}{s^2 + \omega_{x\ell}^2} \hat{f}_{p_{x\ell}} \right), \\ s\varepsilon_2(s)\hat{E}_y &= -\partial_x \hat{H}_z + \hat{f}_{E_y} - \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \left(\frac{s\hat{j}_{y\ell}}{s^2 + \omega_{y\ell}^2} - \frac{\omega_{y\ell}^2}{s^2 + \omega_{y\ell}^2} \hat{f}_{p_{y\ell}} \right), \\ \hat{j}_{m\ell} &= \left(s + \frac{\omega_{m\ell}^2}{s} \right)^{-1} \left(\hat{E}_m + \hat{f}_{j_{m\ell}} - \frac{\omega_{m\ell}^2}{s} \hat{f}_{p_{m\ell}} \right), & \hat{p}_{m\ell} &= s^{-1} \left(\hat{j}_{m\ell} + \hat{f}_{p_{m\ell}} \right), \quad \ell = 0, \dots, n_m, \quad m \in \{x, y\}, \\ \hat{j}_{\mu\ell} &= \left(s + \frac{\omega_{\mu\ell}^2}{s} \right)^{-1} \left(\hat{H}_z + \hat{f}_{j_{\mu\ell}} - \frac{\omega_{\mu\ell}^2}{s} \hat{f}_{p_{\mu\ell}} \right), & \hat{p}_{\mu\ell} &= s^{-1} \left(\hat{j}_{\mu\ell} + \hat{f}_{p_{\mu\ell}} \right), \quad \ell = 0, \dots, n_\mu, \\ s^2 \mu(s) \hat{H}_z &= \varepsilon_2(s)^{-1} \partial_{xx} \hat{H}_z + \varepsilon_1(s)^{-1} \partial_{yy} \hat{H}_z = F(\mathbf{f}, s), \end{aligned}$$

where the right hand side $F(\mathbf{f}, s) = F_e(\mathbf{f}, s) + F_p(\mathbf{f}, s)$,

$$\begin{aligned} F_e(\mathbf{f}, s) &= s\hat{f}_{H_z} - s \sum_{\ell=0}^{n_\mu} \mu_\ell \frac{s\hat{j}_{\mu\ell}}{s^2 + \omega_{\mu\ell}^2} + \varepsilon_1(s)^{-1} \partial_y \hat{f}_{E_x} - \varepsilon_2(s)^{-1} \partial_x \hat{f}_{E_y} \\ &\quad - \varepsilon_1(s)^{-1} \partial_y \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \frac{s\hat{j}_{x\ell}}{s^2 + \omega_{x\ell}^2} + \varepsilon_2(s)^{-1} \partial_x \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \frac{s\hat{j}_{y\ell}}{s^2 + \omega_{y\ell}^2}, \\ F_p(\mathbf{f}, s) &= s \sum_{\ell=0}^{n_\mu} \mu_\ell \frac{\hat{f}_{p_{\mu\ell}}}{s^2 + \omega_{\mu\ell}^2} + \varepsilon_1(s)^{-1} \partial_y \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \frac{\hat{f}_{p_{x\ell}}}{s^2 + \omega_{x\ell}^2} - \varepsilon_2(s)^{-1} \partial_x \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \frac{\hat{f}_{p_{y\ell}}}{s^2 + \omega_{y\ell}^2}. \end{aligned}$$

Notice that the equation for \hat{H}_z in the above is of the form (4), and corresponds to the sesquilinear form (17), see Remark 2.6. Thanks to Proposition 2.12, we obtain the unique solvability of the corresponding equation

and the bound, valid for all $s \in \mathbb{C}_+$,

$$\left\| \hat{H}_z \right\|_{H^1(\mathbb{R}^2)} \leq C |s| (\operatorname{Re} s)^{-1} \max \left(1, (\operatorname{Re} s)^{-2} \right) \|F(\mathbf{f}, s)\|_{H^{-1}(\mathbb{R}^2)}, \quad C > 0. \quad (54)$$

For convenience, let us rename the right-hand side data (with $N = n_x + n_y + n_\mu + 3$):

$$\begin{aligned} (\hat{g}_0, \dots, \hat{g}_{N+2}) &= \left(\hat{f}_{H_z}, \partial_y \hat{f}_{E_x}, \partial_x \hat{f}_{E_y}, \hat{f}_{j_{\mu 0}}, \dots, \hat{f}_{j_{\mu n_\mu}}, \partial_y \hat{f}_{j_{x0}}, \dots, \partial_y \hat{f}_{j_{x n_x}}, \partial_x \hat{f}_{j_{y0}}, \dots, \partial_x \hat{f}_{j_{y n_y}} \right), \\ (\hat{h}_0, \dots, \hat{h}_{N-1}) &= \left(\hat{p}_{\mu 0}, \dots, \hat{p}_{\mu n_\mu}, \partial_y \hat{f}_{p_{x0}}, \dots, \partial_y \hat{f}_{p_{x n_x}}, \partial_x \hat{f}_{p_{y0}}, \dots, \partial_x \hat{f}_{p_{y n_y}} \right). \end{aligned}$$

Thus, what remains is computing the bound on the right-hand side data. Notice that for any $s \in \mathbb{C}_+$ and $\omega \in \mathbb{R}$ it holds

$$|\varepsilon_\alpha(s)^{-1}| = \frac{|s|}{|s\varepsilon_\alpha(s)|} \leq \frac{|s|}{\operatorname{Re}(s\varepsilon_\alpha(s))} \leq C \frac{|s|}{\operatorname{Re} s}, \quad C > 0, \quad \alpha \in \{1, 2\}.$$

With the use of (39), we obtain, for some $C_e > 0$,

$$\begin{aligned} \|F_e(\mathbf{f}, s)\|_{H^{-1}(\mathbb{R}^2)} &\leq C_e \max \left(|s|, \frac{|s|}{\operatorname{Re} s}, |\varepsilon_2(s)^{-1}|, |\varepsilon_1(s)^{-1}|, \frac{|\varepsilon_2(s)^{-1}|}{\operatorname{Re} s}, \frac{|\varepsilon_1(s)^{-1}|}{\operatorname{Re} s} \right) \sum_{\ell=0}^{N+2} \|\hat{g}_\ell\|_{H^{-1}(\mathbb{R}^2)} \\ &\leq |s| \max \left(1, (\operatorname{Re} s)^{-2} \right) \sum_{\ell=0}^{N+2} \|\hat{g}_\ell\|_{H^{-1}(\mathbb{R}^2)}. \end{aligned}$$

Also, for some $C_p > 0$, it holds that

$$\|F_p(\mathbf{f}, s)\|_{H^{-1}(\mathbb{R}^2)} \leq C_p \max \left(\frac{1}{\operatorname{Re} s}, \frac{|\varepsilon_1(s)^{-1}|}{|s| \operatorname{Re} s}, \frac{|\varepsilon_2(s)^{-1}|}{|s| \operatorname{Re} s} \right) \sum_{\ell=0}^{N-1} \|\hat{h}_\ell\|_{H^{-1}(\mathbb{R}^2)} \leq \frac{1}{\operatorname{Re} s} \max \left(1, \frac{1}{\operatorname{Re} s} \right) \sum_{\ell=0}^{N-1} \|\hat{h}_\ell\|_{H^{-1}(\mathbb{R}^2)}.$$

To proceed, let us introduce the following spaces

$$\mathcal{W}_\ell = \left\{ f \in C^\ell(\mathbb{R}, H^{-1}(\mathbb{R}^2)), f(t) = 0 \text{ for } t < 0, \partial_t^{\ell+1} f \in L^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)) \right\}, \quad \ell \in \mathbb{N}_0.$$

Based on the same arguments as in Proposition 2.12, and making use of Theorem 2.9, let us assume that

$$g_\ell(t) \in \mathcal{W}_4, \quad \ell = 0, \dots, N+2, \quad h_k(t) \in \mathcal{W}_3, \quad k = 0, \dots, N-1.$$

Combined with (54) and the linearity of the problem, this allows us to obtain an explicit (though likely non-optimal) bound on the continuous function $H_z(t) : \mathbb{R} \rightarrow H^1(\mathbb{R}^2)$ in the time domain

$$\|H_z(t)\|_{H^1(\mathbb{R}^2)} \leq C \max(t^5, 1) \left(\sum_{\ell=0}^{N+2} \int_0^t \|\mathcal{P}_4 g_\ell(\tau)\|_{H^{-1}(\mathbb{R}^2)} d\tau + \sum_{\ell=0}^{N-1} \int_0^t \|\mathcal{P}_3 h_\ell(\tau)\|_{H^{-1}(\mathbb{R}^2)} d\tau \right).$$

Then with the help of the same technique it is not difficult to establish a similar bound for all other variables of the system (53). For example, the equation for $\hat{j}_{\mu\ell}$ for $\ell = 0, \dots, n_\mu$, gives, with the use of bounds (39) and (54),

$$\|\hat{j}_{\mu\ell}\|_{L^2(\mathbb{R}^2)} = (\operatorname{Re} s)^{-1} \left(|s| (\operatorname{Re} s)^{-1} \max \left(1, (\operatorname{Re} s)^{-2} \right) \|F(\mathbf{f}, s)\|_{H^{-1}(\mathbb{R}^2)} + \|\hat{f}_{j_{\mu\ell}}\|_{L^2(\mathbb{R}^2)} + \frac{\omega_{\mu\ell}^2}{|s|} \|\hat{f}_{p_{\mu\ell}}\|_{L^2(\mathbb{R}^2)} \right).$$

The same procedure as above allows to obtain the stability bound on $j_{\mu\ell}(t)$; for the rest of unknowns we proceed similarly. Therefore for the stability of the materials with rational dielectric permittivity and magnetic permeability it is sufficient to consider the well-posedness of (4) in the Laplace domain. Let us briefly demonstrate that this is also the case for the perfectly matched layer system.

4.2.2. Perfectly Matched Layers for the Lorentz Materials: Time-Domain Formulation

Our goal is to rewrite the system (53) in the time domain with the PML change of variables (31) and non-constant positive absorption parameter $\sigma(x)$. In this case, like in the previous section, we will show that for the analysis of the stability for $\sigma(x) = \text{const}$ it is sufficient to study solely the coercivity of the sesquilinear form (33). Let $\psi(s) = \psi(0) + \sum_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega_{\psi_\ell}^2} \frac{s^2}{s^2 + \omega_{\psi_\ell}^2}$, $\psi_\ell \geq 0$, $\omega_{\psi_\ell} > 0$, see Theorem 4.4. Naturally, we suppose that $\psi(s)$ satisfies assumptions of Theorem 3.2 in order to ensure the stability of the corresponding formulation. For instance, for the Bérenger's PML this may not be the case. To show the way we derive the PML system, let us consider the equation for E_y of (53) with zero right hand side data written in the Laplace domain, with the PML change of variables (31):

$$s\hat{E}_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \hat{j}_{y\ell} = - \left(1 + \frac{\sigma\psi(s)}{s}\right)^{-1} \partial_x \hat{H}_z. \quad (55)$$

We thus introduce in the Laplace domain an auxiliary variable $\hat{H}_{zx} = \left(1 + \frac{\sigma\psi(s)}{s}\right)^{-1} \partial_x \hat{H}_z$:

$$\begin{aligned} s\hat{H}_{zx} + \sigma\psi(0)\hat{H}_{zx} + \sigma \sum_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega_{\psi_\ell}^2} \hat{j}_{H_{zx}\ell} &= s\partial_x \hat{H}_z, \\ s\hat{j}_{H_{zx}\ell} + \omega_{\psi_\ell}^2 \hat{p}_{H_{zx}\ell} &= s\hat{H}_{zx}, \quad s\hat{p}_{H_{zx}\ell} = \hat{j}_{H_{zx}\ell}, \quad \ell = 0, \dots, n_\psi. \end{aligned}$$

Using the above strategy to derive all the remaining equations, the system (53) with the change of variables (31) and (new) right hand side data \mathbf{f} becomes in the time domain:

$$\begin{aligned} \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell} &= \partial_y H_z + f_{E_x}, \\ \partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} j_{y\ell} &= -H_{zx} + f_{E_y}, \\ \partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell j_{\mu\ell} &= \partial_y E_x - E_{yx} + f_{H_z}, \\ \partial_t j_{m\ell} + \omega_{m\ell}^2 p_{m\ell} &= E_m + f_{j_{m\ell}}, \quad \partial_t p_{m\ell} = j_{m\ell} + f_{p_{m\ell}}, \quad \ell = 0, \dots, n_m, \quad m \in \{x, y\}, \\ \partial_t j_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} &= H_z + f_{j_{\mu\ell}}, \quad \partial_t p_{\mu\ell} = j_{\mu\ell} + f_{p_{\mu\ell}}, \quad \ell = 0, \dots, n_\mu, \\ \partial_t H_{zx} + \sigma\psi(0)H_{zx} &= -\sigma \sum_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega_{\psi_\ell}^2} j_{H_{zx}\ell} + \partial_t \partial_x H_z + f_{H_{zx}}, \\ \partial_t j_{H_{zx}\ell} + \omega_{\psi_\ell}^2 p_{H_{zx}\ell} &= \partial_t H_{zx} + f_{j_{H_{zx}\ell}}, \quad \partial_t p_{H_{zx}\ell} = j_{H_{zx}\ell} + f_{p_{H_{zx}\ell}}, \quad \ell = 0, \dots, n_\psi, \\ \partial_t E_{yx} + \sigma\psi(0)E_{yx} &= -\sigma \sum_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega_{\psi_\ell}^2} j_{E_{yx}\ell} + \partial_t \partial_x E_y + f_{E_{yx}}, \\ \partial_t j_{E_{yx}\ell} + \omega_{\psi_\ell}^2 p_{E_{yx}\ell} &= \partial_t E_{yx} + f_{j_{E_{yx}\ell}}, \quad \partial_t p_{E_{yx}\ell} = j_{E_{yx}\ell} + f_{p_{E_{yx}\ell}}, \quad \ell = 0, \dots, n_\psi. \end{aligned} \quad (56a)$$

Remark 4.10. *For the numerical experiments we use a slightly different system, for which it is easier to derive explicit energy estimates for a constant absorption parameter; however, both formulations are equivalent (i.e. they lead to the same solution in terms of **physical** unknowns) provided properly chosen initial conditions and right hand side data. Another formulation, equivalent to (56), can be found in [24].*

We can see that the above system has a structure similar to (53); we can repeat almost verbatim the arguments of the previous section for the above system.

5. NECESSARY AND SUFFICIENT STABILITY CONDITION OF PERFECTLY MATCHED LAYERS

While Corollary 3.7 provides an explicit way to construct stable perfectly matched layers for anisotropic systems of type (17) by choosing $\psi(s) = a(s)$, such a choice of the function $\psi(s)$ may appear to be non-optimal, in a sense that it does not necessarily lead to the smallest number of unknowns in the resulting system in the time domain. On the other hand, while Theorem 3.2 provides a sufficient condition which should be satisfied by $\psi(s)$, it does not provide a constructive way of choosing such a function $\psi(s)$.

In the work [7], which is based on examining the behaviour of modes of the dispersion relation, a necessary condition of the stability of perfectly matched layers is formulated. Such a condition is easier to analyze, since instead of dealing with the right-half complex plane (as in Theorem 3.2), one studies the behaviour of functions restricted to the imaginary axis. It is however well-known [9] that such conditions are not necessarily sufficient for the stability. For example, when applied to non-dispersive cases, they account only for the behaviour in the high-frequency regimes. Nevertheless, in [7] the authors have demonstrated that for a class of isotropic dispersive models (described by the sesquilinear form (17) with $a(s) = b(s)$) such a necessary condition becomes sufficient.

The goal of this section is to demonstrate that the necessary condition derived in [7] is sufficient for the stability of the perfectly matched layers for a more general class of models of the form (17), which, unlike the models considered in [7], are anisotropic. However, we restrict our considerations to passive materials only.

First of all, we recall the necessary stability condition derived in [7], and then formulate the main result (Theorem 5.11) in Section 5.2. The proof of this result can be found in Section 5.3. An easy to use method to construct the function $\psi(s)$, which follows the procedure of [7], can be found in Appendix E. Finally, in Section 5.4 we show some numerical experiments with the new PML confirming the main results of Theorem 5.11.

5.1. The Necessary Stability Condition

A necessary stability condition of the perfectly matched layers for anisotropic dispersive medium had been deduced in [7] with the help of the plane-wave analysis. We do not repeat the derivation of this result for the system recast in the Laplace domain, but simply recall the result itself.

In this section we limit ourselves to passive (17) whose coefficients satisfy an extra condition. This is summarized in the following assumption, see also (13).

Assumption 5.1. *The coefficients of the sesquilinear form (17) $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ are passive in the sense of Definition 2.1 and satisfy Assumption 4.1.*

Remark 5.2. *Since $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ satisfy Assumption 5.1, we know, thanks to Theorem 4.3, that they have a partial fraction expansion (43) and therefore correspond to Lorentz materials. However, the results of this section can be extended to the case when $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ are even meromorphic functions **and** have the partial fraction expansion as in (43) with infinite number of terms ($n = \infty$), cf. e.g. [29, Theorem 1, p. 308]. Moreover, most of the results of this section do not rely on the (physical) assumption $a(s) = \varepsilon_1(s)^{-1}$, $b(s) = \varepsilon_2(s)^{-1}$, $c(s) = \mu(s) \rightarrow 1$ as $s \rightarrow \infty$. A numerical illustration of these two statements will be provided in Example 5.24.*

Remark 5.3. *If a passive function does not have poles and zeros s , s.t. $\operatorname{Re} s < 0$ (i.e. non-dissipative case, cf. Definition 5.4), it is necessarily meromorphic, see e.g. [52].*

Let us first recall a few definitions and crucial assumptions on the Maxwell system in the Laplace domain (4). First of all, the dispersion relation for this system is obtained when looking at the plane-wave solutions of (4). In particular, for Lorentz materials (53), it reads, see Appendix C:

$$F(\omega, \mathbf{k}) = \left(\tilde{a}(\omega)k_x^2 + \tilde{b}(\omega)k_y^2 - \omega^2\tilde{c}(\omega) \right) \prod_{\ell=0}^{n_a} (\omega^2 - \omega_{a\ell}^2) \prod_{\ell=0}^{n_b} (\omega^2 - \omega_{b\ell}^2) \prod_{\ell=0}^{n_c} (\omega^2 - \omega_{c\ell}^2) \omega \tilde{a}(\omega)^{-1} \tilde{b}(\omega)^{-1} = 0. \quad (57)$$

Here $(\omega_{a\ell})_{\ell=0}^{n_a}$, $(\omega_{b\ell})_{\ell=0}^{n_b}$, $(\omega_{c\ell})_{\ell=0}^{n_c}$ are the poles of correspondingly \tilde{a} , \tilde{b} and \tilde{c} . The solutions of the dispersion relation are called modes.

Definition 5.4 ([7]). A system is called non-dissipative if all solutions $\omega(\mathbf{k})$ of its dispersion relation are real for all $\mathbf{k} = (k_x, k_y) \in \mathbb{R}^2$.

Due to Corollary 2.15, the system (53) is non-dissipative.

The PML change of variables (32) leads to a system with a new dispersion relation, see Appendix D,

$$F_\sigma(\omega, \mathbf{k}) = \left(\tilde{a}(\omega) \left(1 - \frac{\sigma \tilde{\psi}(\omega)}{i\omega} \right)^{-2} k_x^2 + \tilde{b}(\omega)k_y^2 - \omega^2\tilde{c}(\omega) \right) \times \prod_{\ell} (\omega^2 - \omega_{c\ell}^2) \prod_{\ell} (\omega^2 - \omega_{\psi\ell}^2)^2 \prod_{\ell} (\omega^2 - \omega_{a\ell}^2) \prod_{\ell} (\omega^2 - \omega_{b\ell}^2) \left(-i\omega + \sigma \tilde{\psi}(\omega) \right)^2 \tilde{a}(\omega)^{-1} \tilde{b}(\omega)^{-1} \omega = 0. \quad (58)$$

Here $(\omega_{\psi\ell})_{\ell=0}^{n_\psi}$ are the poles of $\tilde{\psi}$.

Definition 5.5 (Definition 4.3.2 in [7]). A PML system constructed with the help of the change of variables (32) is called uniformly stable if all its modes $\omega(\mathbf{k}, \sigma)$ satisfy:

$$\text{Im} \omega(\mathbf{k}, \sigma) \leq 0, \quad \text{for all } \mathbf{k} \in \mathbb{R}^2 \text{ and } \sigma \geq 0.$$

To formulate the condition of the stability of the PMLs, we will make use of the concepts of the phase and the group velocities.

Definition 5.6. A phase velocity of a mode $\omega(\mathbf{k})$ is defined by $\mathbf{v}_p(\omega(\mathbf{k})) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|} \frac{\mathbf{k}}{|\mathbf{k}|}$, and the group velocity by $\mathbf{v}_g(\omega(\mathbf{k})) = \nabla_{\mathbf{k}} \omega(\mathbf{k})$, provided that this derivative is well-defined.

The choice of the function ψ in [7] is restricted to the class of functions $\tilde{\psi}(\omega) = h(\omega)$, where

$$h(\omega) = 1 + \sum_{\ell=1}^n \frac{h_\ell}{\omega^2 - \omega_\ell^2}, \quad h_\ell \in \mathbb{R}, \quad \omega_\ell > 0, \quad \ell = 1, \dots, n. \quad (59)$$

A necessary condition of a uniform stability of the modified perfectly matched layers in the direction x applied to the system (17) reads as follows.

Theorem 5.7 (Propositions 4.7, 4.8 in [7]). *Given $\tilde{\psi}(\omega) = 1 + \sum_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega^2 - \omega_{\psi\ell}^2}$, let the PML system corresponding to the dispersion relation (58) be uniformly stable. Then*

- (N1) *the coefficients of the expansion of $\tilde{\psi}(\omega)$ satisfy $\psi_\ell > 0, \ell = 0, \dots, n_\psi$. Additionally, $\tilde{\psi}(0) \geq 0$.*
- (N2) *for all the solutions $\omega_j(\mathbf{k}), j = 1, \dots, N$, of the original dispersion relation (57), it holds that*

$$\tilde{\psi}(\omega_j(\mathbf{k})) \mathbf{v}_{g,x}(\omega_j(\mathbf{k})) \mathbf{v}_{p,x}(\omega_j(\mathbf{k})) \geq 0, \quad \mathbf{k} \in \mathbb{R}^2.$$

Here the subscript x indexes the x -component of a vector.

One can notice that the condition (N1) coincides with the requirement $\operatorname{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$, see Theorem 4.4. In [7] it was derived by requiring that the function $-i\omega + \sigma\tilde{\psi}(\omega)$ has no zeros with $\operatorname{Im}\omega > 0$, for $\sigma \geq 0$, cf. one of the factors of the dispersion relation of the PML system (58). The condition (N2) was deduced for the case when $\omega_j(\mathbf{k}) \neq \omega_m(\mathbf{k})$, $j \neq m$, however, it was suggested that the result remains true when the equality holds [7].

Let us re-interpret the above result in terms that are more suitable for our problem. All the modes of (58) satisfy the following:

- (1) either these modes are non-propagating (i.e. independent of \mathbf{k}) and either are real $\omega(\mathbf{k}) = \operatorname{const} \in \mathbb{R}$ or solve $\sigma\tilde{\psi}(\omega) - i\omega = 0$. The stability of these modes is assured by $\operatorname{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$.
- (2) or they are propagating, i.e. solve, for some $k_x, k_y \in \mathbb{R}$:

$$\tilde{a}(\omega) \left(1 - \frac{\sigma\tilde{\psi}(\omega)}{i\omega} \right)^{-2} k_x^2 + \tilde{b}(\omega)k_y^2 - \omega^2\tilde{c}(\omega) = 0. \quad (60)$$

A necessary condition of the stability of the propagating modes is given by the condition (N2). Further we will concentrate on this problem.

Let us rewrite (N2) with the use of the implicit function theorem. First notice that the group velocity does not vanish only for propagative modes, which, in turn, solve the original dispersion relation, see (57):

$$\mathcal{F}(\omega, \mathbf{k}) = \omega^{-1}\tilde{a}(\omega)k_x^2 + \omega^{-1}\tilde{b}(\omega)k_y^2 - \omega\tilde{c}(\omega) = 0. \quad (61)$$

We denote the set of propagative modes as

$$\Omega_p := \{ \omega \in \mathbb{R} : \mathcal{F}(\omega, \mathbf{k}) = 0 \text{ for some } \mathbf{k} \in \mathbb{R}^2 \}. \quad (62)$$

Then, given $\mathcal{F}(\omega, \mathbf{k})$ as in (61), we deduce for the points where $\partial_\omega \mathcal{F}(\omega, \mathbf{k}) \neq 0$,

$$\partial_{k_x} \omega(k_x, k_y) = -(\partial_{k_x} \mathcal{F}(\omega, \mathbf{k})) (\partial_\omega \mathcal{F}(\omega, \mathbf{k}))^{-1} = -2k_x \omega^{-1} \tilde{a}(\omega) \left(k_x^2 (\omega^{-1} \tilde{a}(\omega))' + (\omega^{-1} \tilde{b}(\omega))' k_y^2 - (\omega \tilde{c}(\omega))' \right)^{-1}.$$

The condition (N2) in Theorem 5.7 can be rewritten in the form: for all $\omega \in \Omega_p$, $k_x, k_y \in \mathbb{R}$, the following condition holds true:

$$-k_x^2 \tilde{\psi}(\omega) \tilde{a}(\omega) \left(k_x^2 (\omega^{-1} \tilde{a}(\omega))' + k_y^2 (\omega^{-1} \tilde{b}(\omega))' - (\omega \tilde{c}(\omega))' \right)^{-1} \geq 0, \quad (63)$$

provided that the above expression is well-defined.

We will need to reformulate the above condition in a simpler form. First of all, we will simplify (62). Given a rational function $r(z)$ satisfying Assumption 4.1, let us introduce a set $\mathcal{D}_r = \{ \omega \in \mathbb{R} : \tilde{r}(\omega) = 0 \text{ or } \tilde{r}(\omega)^{-1} = 0 \}$.

Lemma 5.8. *Let $a(s)$, $b(s)$, $c(s)$ satisfy Assumption 4.1. Then Ω_p defined in (62) can be represented as a union of the following (possibly intersecting) sets:*

$$\Omega_p = \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p,$$

$$\Omega_p^{\tilde{a}\tilde{c}>0} = \{ \omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_c) : \tilde{a}(\omega)\tilde{c}(\omega) > 0 \}, \quad (64)$$

$$\Omega_p^{\tilde{a}\tilde{b}<0} = \left\{ \omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_b) : \tilde{a}(\omega)\tilde{b}(\omega) < 0 \right\}, \quad (65)$$

$$\mathcal{Z}_p = \left\{ \omega \in \mathcal{D}_a \cup \mathcal{D}_b \cup \mathcal{D}_c : \text{for some } k_x, k_y \in \mathbb{R}, \text{ it holds } \omega^{-1}\tilde{a}(\omega)k_x^2 + \omega^{-1}\tilde{b}(\omega)k_y^2 - \omega\tilde{c}(\omega) = 0 \right\}.$$

Proof. First we show $\Omega_p \subset \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p$. Let $\omega \notin \mathcal{Z}_p$. Then the modes $\omega \in \Omega_p$ satisfy: $k_x^2 = -\frac{\tilde{b}}{\tilde{a}}k_y^2 + \omega^2\frac{\tilde{c}}{\tilde{a}}$. This equation has at least one real solution k_x if and only if there exists $k_y \in \mathbb{R}$ s.t.

$$-\tilde{b}(\omega)\tilde{a}(\omega)^{-1}k_y^2 + \omega^2\tilde{c}(\omega)\tilde{a}(\omega)^{-1} \geq 0. \quad (66)$$

The above inequality has no real solutions k_y if $\tilde{b}(\omega)\tilde{a}(\omega)^{-1} \geq 0$ and $\tilde{c}(\omega)\tilde{a}(\omega)^{-1} < 0$. Hence, the inequality (66) has a real solution k_y if and only if $\omega \notin \mathcal{Z}_p$ satisfies at least one of the following conditions:

- $\tilde{b}(\omega)\tilde{a}(\omega)^{-1} < 0$, i.e. $\omega \in \Omega_p^{\tilde{a}\tilde{b}<0}$;
- $\tilde{c}(\omega)\tilde{a}(\omega)^{-1} > 0$, i.e. $\omega \in \Omega_p^{\tilde{a}\tilde{c}>0}$.

The inclusion $\Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p \subset \Omega_p$ can be proved similarly. \square

With the above we can reformulate the necessary stability condition (N2).

Lemma 5.9 (Necessary Stability Condition (N2)). *Let $a(s)$, $b(s)$, $c(s)$ satisfy Assumption 5.1, and let $\psi(s)$ satisfy Assumption 4.1 and $\text{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$. Let for $\omega \in \Omega_p$, the stability condition (63) hold true, provided that the latter expression is well-defined. Then for all $\omega \in \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0}$, see (64, 65), it holds*

$$\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0. \quad (67)$$

Proof. Consider the denominator of the left-hand side of (63): $\mathcal{H} := k_x^2(\omega^{-1}\tilde{a}(\omega))' + k_y^2(\omega^{-1}\tilde{b}(\omega))' - (\omega\tilde{c}(\omega))'$. Thanks to Lemmas 4.7, 4.8 about the signs of the derivatives of passive functions, for all $\omega \notin \mathcal{Z}_p$, it holds

$$(\omega^{-1}\tilde{a}(\omega))' < 0, \quad (\omega^{-1}\tilde{b}(\omega))' < 0, \quad (\omega\tilde{c}(\omega))' > 0,$$

Therefore, the denominator $\mathcal{H} < 0$ for all $k_x, k_y \in \mathbb{R}$ and $\omega \notin \mathcal{Z}_p$. We can thus rewrite the inequality (63) as

$$\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0.$$

\square

5.2. Equivalence of the Necessary and Sufficient Stability Conditions

In this section we prove the following result about the stability of the PML (67) stemming from the necessary condition of stability. To formulate it, we need the following assumption.

Assumption 5.10. *The functions $a(s)$, $b(s)$, $c(s)$ satisfy Assumption 5.1, and $\psi(s)$ satisfies Assumption 4.1 and $\text{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$.*

Theorem 5.11. *Let $a(s)$, $b(s)$, $c(s)$, $\psi(s)$ satisfy Assumption 5.10 and let*

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v)$$

be the corresponding passive sesquilinear form. Then the following two conditions are equivalent:

(NSC1) *for all $\omega \in \Omega_p^{\tilde{a}\tilde{b}<0} \cup \Omega_p^{\tilde{a}\tilde{c}>0}$ (see (64, 65)), it holds that $\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0$.*

(NSC2) *For any $\sigma \geq 0$, the sesquilinear form*

$$\tilde{A}_\sigma(u, v) = \frac{\psi(s)}{a(s)} \left(a(s) \left(1 + \frac{\sigma\psi(s)}{s} \right)^{-2} (\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v) \right), \quad s \in \mathbb{C}_+,$$

is passive in the sense of Definition 2.11, and thus, bounded and coercive (due to Proposition 2.12).

Proof. The proof (NSC2) \implies (NSC1) follows from the results of [7]. Consider one of the factors of the dispersion relation (58) given by (60). Thanks to Lemma 2.14, the passivity of the sesquilinear form $\tilde{A}_\sigma(u, v)$ implies that all the solutions $\omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^2$, of (60) have a non-positive imaginary part, i.e. $\text{Im } \omega(\mathbf{k}) \leq 0$. In [7, Proposition 4.8], the following implication was shown:

$$\text{all solutions } \omega(\mathbf{k}), \mathbf{k} \in \mathbb{R}^2 \text{ of (60) are s.t. } \text{Im } \omega(\mathbf{k}) \leq 0 \implies \text{for } \omega \in \Omega_p, \text{ (63) holds true.}$$

Thanks to Lemma 5.9, from the above (NSC1) follows.

To show the implication (NSC1) \implies (NSC2), we use Theorem 3.2. It is sufficient to prove that $\psi(s)$ that satisfies (NSC1) is such that the following two conditions hold:

$$\text{Re} \left(sc(s)a(s)^{-1}\psi(s) \right) > 0, \quad s \in \mathbb{C}_+, \quad (\text{NSC2})\text{-}(a)$$

$$\text{Re} \left(\bar{s}b(s)a(s)^{-1}\psi(s) \right) > 0, \quad s \in \mathbb{C}_+. \quad (\text{NSC2})\text{-}(b)$$

The proof of these passivity conditions require several technical lemmas, which we will formulate and prove in Section 5.3. \square

We readily obtain the following corollary for a specific class of anisotropic systems (17) with $c(s) = b(s)^{-1}$. A particular representative of this class is a uniaxial cold plasma model given in (7), with $b(s) = 1$.

Corollary 5.12. *Let $a(s), \psi(s), b(s)$ satisfy Assumption 5.10. Then the following two conditions are equivalent:*

(1) for all $\sigma \geq 0$, the sesquilinear form

$$\tilde{A}_\sigma(u, v) = \frac{\psi(s)}{a(s)} \left(a(s) \left(1 + \frac{\sigma\psi(s)}{s} \right)^{-2} (\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 b(s)^{-1}(u, v) \right), \quad s \in \mathbb{C}_+,$$

is passive in the sense of Definition 2.11.

(2) $\psi(s) = a(s)$.

Proof. Notice that the above sesquilinear form is as in Theorem 5.11 with $\tilde{c}(\omega) = \tilde{b}(\omega)^{-1}$. Using Theorem 5.11, we see that in points $\omega \in \mathbb{R}$ s.t. $\tilde{a}(\omega)\tilde{c}(\omega) = \tilde{a}(\omega)\tilde{b}(\omega)^{-1} \geq 0$, the functions $\tilde{\psi}(\omega)$ and $\tilde{a}(\omega)$ should be of the same sign. The same should in points where $\tilde{a}(\omega)\tilde{b}(\omega) \leq 0$. Since all the poles and zeros of the rational $s^{-1}\psi(s)$ are simple, see also Lemma 4.9, the assertion follows immediately. \square

Remark 5.13. *Thanks to Corollary 3.9, there always exists $\tilde{\psi}$ satisfying the condition (NSC1) of Theorem 5.11, namely $\psi(s) = a(s)$.*

5.3. Proof of Theorem 5.11

To prove Theorem 5.11, we proceed as follows. First of all, let us remark that the condition (NSC1) of Theorem 5.11 is equivalent to the following two conditions:

$$\text{For all } \omega \in \Omega_p^{\tilde{a}\tilde{c}>0} : \tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0, \quad (\text{NSC1})\text{-}(a)$$

$$\text{For all } \omega \in \Omega_p^{\tilde{a}\tilde{b}<0} : \tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0. \quad (\text{NSC1})\text{-}(b)$$

We will first demonstrate that (NSC1)-(a) implies the passivity condition (NSC2)-(a). To show that from (NSC1)-(b) the condition (NSC2)-(b) follows, we will make use of a logical argument (Lemma 5.20).

We proceed as follows:

- (1) rewrite the condition (NSC2)-(a) in terms of equivalent conditions on the signs of $\tilde{\psi}$ in poles (zeros) of functions \tilde{a} , \tilde{c} (Lemma 5.14);
- (2) show that if $\tilde{\psi}$ satisfies (NSC1)-(a), then above conditions on the signs of $\tilde{\psi}$ follows. This is explained in Lemma 5.15, whose proof relies largely on Lemmas 5.17, 5.19 and Corollary 5.18.

5.3.1. Reformulation of the Condition (NSC2)-(a)

We start with the following simple lemma that allows us to reformulate the passivity condition (NSC2)-(a) in a more suitable form.

Lemma 5.14. *Let $a(s)$, $c(s)$, $\psi(s)$ satisfy Assumption 5.10. Then (NSC2)-(a) holds if and only if the following limits exist and satisfy:*

- (a) in all poles $(\omega_{c_j})_{j=0}^{n_c}$ of $\tilde{c}(\omega)$, it holds that $\lim_{\omega \rightarrow \omega_{c_j}} \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) \geq 0$.
- (b) in all zeros $(\omega_{a_j})_{j=0}^{n_a}$ of $\tilde{a}(\omega)$, it holds that $\lim_{\omega \rightarrow \omega_{a_j}} \tilde{\psi}(\omega) \tilde{c}(\omega) \geq 0$.
- (c) in all poles $(\omega_{\psi_j})_{j=0}^{n_\psi}$ of $\tilde{\psi}(\omega)$, it holds that $\lim_{\omega \rightarrow \omega_{\psi_j}} \tilde{a}(\omega)^{-1} \tilde{c}(\omega) \leq 0$.

Proof. Thanks to Theorem 4.3, the function $c(s)a(s)^{-1}\psi(s)$ is passive if and only if it has the following partial fraction expansion:

$$\tilde{c}(\omega)\tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) = 1 - \sum_{\ell=0}^N \frac{p_\ell}{\omega^2 - \omega_\ell^2}, \quad p_\ell > 0, \quad \omega_\ell \in \mathbb{R}, \quad \ell = 0, \dots, N.$$

Hence to show the equivalence of the statements, it is sufficient to ensure that in each of the poles $\omega = \omega_k$, $k = 0, \dots, N$, the following limit exists and satisfies

$$\lim_{\omega \rightarrow \omega_k} (\omega^2 - \omega_k^2) \tilde{c}(\omega) \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) = -p_k < 0. \quad (68)$$

Since $(\omega_k)_{k=0}^N$ is a subset of poles of \tilde{c} , \tilde{a}^{-1} and $\tilde{\psi}$, we separately consider (68) for each of these poles. Let ω_{c_j} be a pole of $\tilde{c}(\omega)$. The latter function, thanks to Theorem 4.3, has an expansion $\tilde{c}(\omega) = 1 - \sum_{\ell=0}^{n_c} \frac{c_\ell}{\omega^2 - \omega_{c_\ell}^2}$, where $c_\ell > 0$ for $\ell = 0 \dots n_c$. Evaluating the limit (68) in $\omega_k = \omega_{c_j}$, we obtain:

$$\lim_{\omega \rightarrow \omega_{c_j}} (\omega^2 - \omega_{c_j}^2) \tilde{c}(\omega) \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) = \lim_{\omega \rightarrow \omega_{c_j}} \left((\omega^2 - \omega_{c_j}^2) \tilde{c}(\omega) \right) \lim_{\omega \rightarrow \omega_{c_j}} \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) = -c_j \lim_{\omega \rightarrow \omega_{c_j}} \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) \leq 0.$$

The equality sign would mean that $c(s)a(s)^{-1}\psi(s)$ has no pole in ω_{c_j} . Similarly we deduce the rest of the inequalities in the statement of the lemma. \square

5.3.2. Necessary Stability Condition (NSC1)-(a) Implies Passivity of $c(s)a(s)^{-1}\psi(s)$

Let us first present the main result of this section, which shows how from the necessary stability condition (NSC1)-(a) we obtain the passivity (NSC2)-(a).

Lemma 5.15. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 5.10. Assume that (NSC1)-(a) holds. Then (NSC2)-(a) holds true as well.*

Proof. Due to Lemma 5.14, it is sufficient to analyze step by step the condition on ψ in various cases: poles of c , zeros of a , as well as examine the values of a, c in poles of ψ . Let us thus look at the following cases according to Lemma 5.14:

- given a pole ω_{c_j} of $\tilde{c}(\omega)$, we must show that the following limit exists and satisfies: $\lim_{\omega \rightarrow \omega_{c_j}} \tilde{a}(\omega)^{-1} \tilde{\psi}(\omega) \geq 0$. This follows by a direction application of Lemma 5.17, see below, with $\alpha = a$, $\gamma = c$ and $\phi = \psi$.
- given a zero ω_{a_j} of $\tilde{a}(\omega)$, we must demonstrate $\lim_{\omega \rightarrow \omega_{a_j}} \tilde{c}(\omega) \tilde{\psi}(\omega) \geq 0$. This is shown in Corollary 5.18, see below.

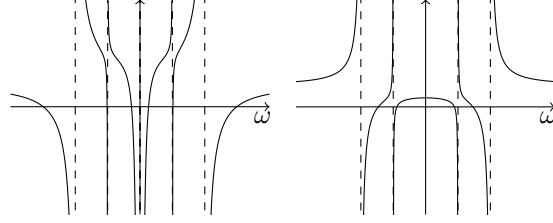


FIGURE 7. In the left figure we depict $\tilde{r}(\omega)$, s.t. $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$ and in the right one we plot $\tilde{r}(\omega)$, s.t. $r(s)$ satisfies $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$. In both cases $r(s)$ is a rational function satisfying Assumption 4.1.

- finally, given a pole of ω_{ψ_j} of the function $\tilde{\psi}$, we obtain from Lemma 5.19, see below, the existence of the following non-positive limit $\lim_{\omega \rightarrow \omega_{\psi_j}} \tilde{c}(\omega)\tilde{a}(\omega)^{-1} \leq 0$.

□

Remark 5.16. *The necessary condition in the above lemma is also sufficient, see Appendix B.*

Before formulating results mentioned in the proof of Lemma 5.15, let us summarize simple properties of passive functions and their reciprocals crucial for proving the result of Theorem 5.11, see also Figure 7 for illustration. More precisely, for a passive rational function $r(s)$ satisfying Assumption 4.1, it holds, thanks to Lemmas 4.9, 4.7 and Theorem 4.3:

- (P1) if $\omega = 0$ is a pole of \tilde{r} , then $\tilde{r}(\omega) < 0$ in a sufficiently small vicinity of $\omega = 0$. Otherwise $\tilde{r}(0) > 0$.
- (P2) in a pole $\omega > 0$, the function \tilde{r} changes its sign from positive to negative.
- (P3) in a zero $\omega > 0$, the function \tilde{r} changes its sign from negative to positive.

Similarly, for a rational function $r(s)$ satisfying Assumption 4.1, whose reciprocal is passive, i.e. $\operatorname{Re}(\bar{s}r(s)) > 0$, $s \in \mathbb{C}_+$, it holds, thanks to Lemmas 4.9, 4.8, Theorem 4.4:

- ($\bar{P}1$) if $\omega = 0$ is a zero of \tilde{r} , then $\tilde{r}(\omega) < 0$ in a sufficiently small vicinity of $\omega = 0$. Otherwise $\tilde{r}(0) > 0$.
- ($\bar{P}2$) in a pole $\omega > 0$, the function \tilde{r} changes its sign from negative to positive.
- ($\bar{P}3$) in a zero $\omega > 0$, the function \tilde{r} changes its sign from positive to negative.

Now let us state the auxiliary results used in the proof of Lemma 5.15.

Lemma 5.17. *Let $\alpha(s)$, $\phi(s)$, $\gamma(s)$ satisfy Assumption 4.1, and $\operatorname{Re}(\bar{s}\alpha(s)) > 0$, $\operatorname{Re}(\bar{s}\phi(s)) > 0$, and $\operatorname{Re}(s\gamma(s)) > 0$, for all $s \in \mathbb{C}_+$. Assume that for all $\omega \in \mathbb{R}$, possibly with the exception of poles and zeros of $\tilde{\gamma}$, $\tilde{\alpha}$, it holds*

$$\tilde{\gamma}(\omega)\tilde{\alpha}(\omega) > 0 \implies \tilde{\phi}(\omega)\tilde{\alpha}(\omega) \geq 0. \quad (69)$$

Then in all the poles $(\omega_{\gamma_j})_{j=0}^{n_\gamma}$ of $\tilde{\gamma}(\omega)$ the following limit exists and satisfies

$$\lim_{\omega \rightarrow \omega_{\gamma_j}} \tilde{\alpha}(\omega)^{-1}\tilde{\phi}(\omega) \geq 0. \quad (70)$$

Proof. In the proof, we will use extensively the fact that all the poles and zeros of α , ϕ , γ , with the exception of possibly $s = 0$, are simple, see Lemma 4.9.

Let ω_{γ_j} be a non-negative pole of $\tilde{\gamma}(\omega)$. The case of negative poles is treated similarly. Let us consider the following possible cases:

- $\omega_{\gamma_j} = 0$. Then, thanks to Theorem 4.4 applied to $\alpha(s)$, only one of the following can hold true:
 - (1) $\tilde{\alpha}(0) = 0$. Then, for sufficiently small $\omega > 0$:
 - $\tilde{\alpha}(\omega)^{-1} < 0$, see ($\bar{P}1$),

– $\tilde{\gamma}(\omega) < 0$, see (P1).

From this it follows together with (69) that in a small vicinity of zero $\tilde{\phi}(\omega) < 0$, which is possible only if $\tilde{\phi}(0) = 0$, see ($\bar{P}1$). Since $\tilde{\phi}$, $\tilde{\alpha}$ both have a zero of order 2 in $\omega = 0$, see Lemma 4.9, the above considerations give $\lim_{\omega \rightarrow 0} \tilde{\phi}(\omega)\tilde{\alpha}(\omega)^{-1} > 0$.

- (2) $\tilde{\alpha}(0)^{-1} > 0$. Applying Theorem 4.4 to $\phi(s)$, we notice $\tilde{\phi}(0) \geq 0$, hence (70) holds true.
- $\omega_{\gamma_j} > 0$. The following cases are possible:
 - (1) $0 < \tilde{\alpha}(\omega_{\gamma_j}) < \infty$ ($-\infty < \tilde{\alpha}(\omega_{\gamma_j}) < 0$). Due to (P2), the inequality $\tilde{\gamma}(\omega)\tilde{\alpha}(\omega) > 0$ holds for all $\omega = \omega_{\gamma_j} - \delta$ ($\omega = \omega_{\gamma_j} + \delta$), with $\delta > 0$ being sufficiently small. There are two possibilities in this case:
 - $\tilde{\phi}(\omega)$ does not have a pole in ω_{γ_j} . Then, due to (69), the limit (70) exists and satisfies the inequality (70) (this includes the case $\tilde{\phi}(\omega_{\gamma_j}) = 0$).
 - $\tilde{\phi}$ has a pole in ω_{γ_j} . This is impossible, since in this case (69) would require that $\tilde{\phi}(\omega) > 0$ ($\tilde{\phi}(\omega) < 0$) for $\omega = \omega_{\gamma_j} - \delta$ ($\omega = \omega_{\gamma_j} + \delta$), for $\delta > 0$; this would be a contradiction to the fact that in a pole $\tilde{\phi}(\omega)$ changes its sign from negative to positive, see ($\bar{P}2$).
 - (2) $\tilde{\alpha}(\omega)$ has a zero in ω_{γ_j} . Thanks to the fact that in $\omega_{\gamma_j} \neq 0$, $\tilde{\alpha}$ changes its sign from positive to negative, see ($\bar{P}3$), and so does \tilde{c} , see (P2), in a vicinity of ω_{γ_j} , the product $\tilde{\alpha}^{-1}\tilde{\gamma} > 0$. Therefore, (69) gives $\tilde{\phi}(\omega)\tilde{\alpha}(\omega) \geq 0$. Then $\tilde{\phi}$ should change its sign in ω_{γ_j} from positive to negative, see ($\bar{P}3$); this is possible only if $\tilde{\phi}(\omega_{\gamma_j}) = 0$ and, thanks to simplicity of non-vanishing zeros of $\tilde{\alpha}$, $\tilde{\psi}$, (70) holds true.
 - (3) $\tilde{\alpha}$ has a pole in ω_{γ_j} . Then there are two possibilities:
 - $\tilde{\phi}$ is finite in ω_{γ_j} ; this immediately implies (70).
 - $\tilde{\phi}$ has a pole in ω_{γ_j} . Due to ($\bar{P}3$) in zero both functions $\tilde{\phi}$, $\tilde{\alpha}$ change their signs from positive to negative. This, combined with the simplicity of poles of $\tilde{\alpha}$, $\tilde{\phi}$, shows that (70) holds true. \square

The following result is a direct corollary of the above.

Corollary 5.18. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 5.10. Assume that (NSC1)-(a) holds true. Then in all the zeros $(\omega_{a_j})_{j=0}^{n_a}$ of $\tilde{a}(\omega)$ the following limit exists and satisfies*

$$\lim_{\omega \rightarrow \omega_{a_j}} \tilde{c}(\omega)\tilde{\psi}(\omega) \geq 0. \quad (71)$$

Proof. Notice that the condition (NSC1)-(a) can be rewritten as follows: for all $\omega \in \mathbb{R}$, with the possible exception of poles and zeros of \tilde{c}, \tilde{a} , it holds: $\tilde{c}(\omega)^{-1}\tilde{a}(\omega)^{-1} > \implies \tilde{\psi}(\omega)\tilde{c}(\omega)^{-1} \geq 0$. Then the validity of the statement follows by application of Lemma 5.17 with $\alpha(s) = c(s)^{-1}$, $\gamma(s) = a(s)^{-1}$ and $\phi(s) = \psi(s)$, see also the relation between passive functions and their reciprocals (13). \square

Finally, what remains to show is the following.

Lemma 5.19. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 5.10. Assume that (NSC1)-(a) holds. Then in all the poles $(\omega_{\psi_j})_{j=0}^{n_\psi}$ of $\tilde{\psi}(\omega)$, the following limit exists and satisfies*

$$\lim_{\omega \rightarrow \omega_{\psi_j}} \tilde{c}(\omega)\tilde{a}(\omega)^{-1} \leq 0. \quad (72)$$

Proof. Recall that $\omega_{\psi_j} = 0$ cannot be a pole of $\tilde{\psi}$, see Theorem 4.4. Hence we consider the case $\omega_{\psi_j} > 0$ (negative values are treated similarly). We thus look at the following cases:

- (1) let $-\infty < \tilde{c}(\omega_{\psi_j}) < 0$ ($0 < \tilde{c}(\omega_{\psi_j}) < \infty$). There are few possibilities in this case:

- $\tilde{a}(\omega)$ is finite and does not change its sign in ω_{ψ_j} ; assume that (72) does not hold true, i.e. $\tilde{c}(\omega_{\psi_j})\tilde{a}(\omega_{\psi_j})^{-1} > 0$. By continuity this holds in a vicinity of ω_{ψ_j} . Then, due to (NSC1)-(a), $\tilde{a}(\omega)\tilde{\psi}(\omega) \geq 0$ in the vicinity of ω_{ψ_j} , hence we arrive at the contradiction ($\tilde{\psi}$ changes its sign in ω_{ψ_j} but not \tilde{a}). Hence, necessarily, $\tilde{c}(\omega_{\psi_j})\tilde{a}(\omega_{\psi_j})^{-1} \leq 0$.
 - ω_{ψ_j} is a pole of $\tilde{a}(\omega)$. Then obviously the equality in (72) holds.
 - $\tilde{a}(\omega_{\psi_j}) = 0$: due to ($\bar{P}3$), \tilde{a} changes its sign from positive to negative in ω_{ψ_j} , and the inequality $\tilde{c}(\omega)\tilde{a}(\omega) > 0$ holds for all $\omega = \omega_{\gamma_j} + \delta$ ($\omega = \omega_{\gamma_j} - \delta$), with $\delta > 0$ being sufficiently small. Then, due to (NSC1)-(a), the product $\tilde{a}(\omega)\tilde{\psi}(\omega) \geq 0$ for such values ω , which is impossible, since $\tilde{\psi}$ changes in ω_{ψ_j} its sign from negative and positive and \tilde{a} from positive to negative, see ($\bar{P}3$) and ($\bar{P}2$).
- (2) $\tilde{c}(\omega_{\psi_j}) = 0$. Then
- if $\tilde{a}(\omega_{\psi_j})$ is finite or has a pole in ω_{ψ_j} , (72) is obviously true (in the pole ω_{ψ_j} it would change the signs from negative to positive, and \tilde{c} changes its sign in the zero ω_{ψ_j} from negative to positive).
 - if $\tilde{a}(\omega_{\psi_j}) = 0$, then ($P3$) and ($\bar{P}2$) imply that $\tilde{a}(\omega)\tilde{c}(\omega) < 0$ in a vicinity of ω_{ψ_j} . Combined with the fact that positive zeros and poles of \tilde{a} and \tilde{c} are simple, we obtain the result.
- (3) \tilde{c} has a pole in ω_{ψ_j} . As shown in Lemma 5.17, in this case $\tilde{\psi}$ can have a pole in ω_{ψ_j} only if \tilde{a} has a pole in ω_{ψ_j} . The result follows from the changes of signs of \tilde{c} and \tilde{a} in ω_{ψ_j} , see ($P2$) and ($\bar{P}2$).

□

5.3.3. Necessary Stability Condition (NSC1)-(b) Implies Passivity of $b(s)^{-1}a(s)\psi(s)^{-1}$

Now our goal is to connect the passivity of $b(s)^{-1}a(s)\psi(s)^{-1}$ and the necessary stability condition. We will not make use of lemmas similar to Lemma 5.17, but rather use some trivial logic arguments to show that Lemma 5.15 implies the validity of the following result.

Lemma 5.20. *Let $a(s)$, $\psi(s)$, $b(s)$ satisfy Assumption 5.10. If (NSC1)-(b) holds true, then so does (NSC2)-(b).*

Proof. Given a , b , ψ , let us introduce the following new unknowns:

$$c_n := \psi^{-1}, \quad a_n := b, \quad \psi_n := a. \quad (73)$$

Notice that $c_n(s)$ has the same properties as $c(s)$, see (13). Due to Lemma 5.15, we know that the first condition below implies the second one:

- (1) for all $\omega \in \mathbb{R} \setminus (\mathcal{D}_{a_n} \cup \mathcal{D}_{c_n})$ it holds that

$$\tilde{c}_n \tilde{a}_n > 0 \implies \tilde{\psi}_n \tilde{a}_n \geq 0, \quad (74)$$

- (2) $\operatorname{Re}(sc_n(s)a_n(s)^{-1}\psi_n(s)) > 0$ in \mathbb{C}_+ , or, equivalently, see (13), and the notation (73), $\operatorname{Re}(\bar{s}b(s)\psi(s)a(s)^{-1}) > 0$ in \mathbb{C}_+ .

With the new notation (73), the expression (74) reads

$$\tilde{\psi}(\omega)^{-1}\tilde{b}(\omega) > 0 \implies \tilde{a}(\omega)\tilde{b}(\omega) \geq 0, \quad \omega \in \mathbb{R} \setminus (\mathcal{D}_\psi \cup \mathcal{D}_b),$$

However, the above is equivalent to

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \implies \tilde{\psi}(\omega)^{-1}\tilde{b}(\omega)^{-1} \leq 0, \quad \omega \in \mathbb{R} \setminus (\mathcal{D}_\psi \cup \mathcal{D}_b),$$

or, alternatively, due to the continuity of all the functions in ω ,

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \implies \tilde{\psi}(\omega)\tilde{a}(\omega) \leq 0, \quad \omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_b).$$

From this we obtain the desired statement. □

Remark 5.21. *As before, the necessary condition in the above lemma can be shown to be sufficient, see Remark 5.16.*

5.4. Numerical Experiments

In this section we will verify the results of Theorem 5.11, as well as discuss its applicability to the Lorentz materials with losses.

Example 5.22. (Anisotropic Lorentz material. Numerical verification of Theorem 5.11) We start with the system (4), where we choose the parameters as follows, see Remark 2.6,

$$\tilde{a}(\omega) = \tilde{\varepsilon}_2(\omega)^{-1} = \frac{\omega^2}{\omega^2 - \omega_p^2}, \quad \tilde{b}(\omega) = \tilde{\varepsilon}_1(\omega)^{-1} = \frac{\omega^2 - \frac{\omega_p^2}{4}}{\omega^2 - 4\omega_p^2}, \quad \tilde{c}(\omega) = \tilde{\mu}(\omega) = 1, \quad \omega_p = 5. \quad (75)$$

To construct the PML in x -direction, let us study the signs of $\tilde{a}(\omega)$, $\tilde{b}(\omega)$ and $\tilde{c}(\omega)$, and apply Theorem 5.11 to obtain a condition on the signs of $\tilde{\psi}$ that would generate a stable PML:

- $\omega \in (0, \frac{\omega_p}{2})$: $\tilde{a}(\omega) < 0$, $\tilde{b}(\omega) > 0$, $\tilde{c}(\omega) > 0$. Hence $\tilde{\psi}(\omega) < 0$.
- $\omega \in (\frac{\omega_p}{2}, \omega_p)$: $\tilde{a}(\omega) < 0$, $\tilde{b}(\omega) < 0$, $\tilde{c}(\omega) > 0$. Hence there is no condition on the sign of $\tilde{\psi}$ on this interval.
- $\omega \in (\omega_p, 2\omega_p)$: $\tilde{a}(\omega) > 0$, $\tilde{b}(\omega) < 0$, $\tilde{c}(\omega) > 0$. Hence $\tilde{\psi}(\omega) > 0$.
- $\omega > 2\omega_p$: $\tilde{\psi}(\omega) > 0$.

Let us choose $\psi(s)$ that would satisfy these requirements. We will consider four different functions $\psi(s) = \psi_j(s)$, defined as follows:

$$\psi_1(s) = 1, \quad \tilde{\psi}_2(\omega) = \frac{\omega^2}{\omega^2 - \frac{\omega_p^2}{4}}, \quad \tilde{\psi}_3(\omega) = \frac{\omega^2 \left(\omega^2 - \frac{\omega_p^2}{4} \right)}{\left(\omega^2 - \frac{\omega_p^2}{8} \right) \left(\omega^2 - \omega_p^2 \right)}, \quad \psi_4(s) = \varepsilon_2(s)^{-1}. \quad (76)$$

Notice that $\operatorname{Re}(\bar{s}\psi_j(s)) > 0$, $j = 1, \dots, 4$, in \mathbb{C}_+ thanks to Lemma 4.9. We expect the system to be unstable for the choice $\psi(s) = \psi_1(s)$ (Bérenger's PML) and stable in three other cases (where $\psi(s)$ satisfies the sign conditions stated above).

The parameters of the numerical experiment are given in Table 5. As before, in y -direction we use zero

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
8	12	2	0	$20x^2$	0	0.025	0.0125	$(t-2)e^{-10(t-2)^2 - 1000(x-3.8)^2 - 2z^2} H(3.98 - x)$

TABLE 5. Parameters for the experiment with dielectric permittivity and magnetic permeability (75). By $H(x)$ we denote the left-continuous Heaviside function. The notation can be found in the beginning of Section 3.1.1.

Dirichlet boundary conditions. The results of these experiments are shown in Figure 8. The new PMLs are stable, unlike the Bérenger's PML. The rate of the decrease of the solution norm for the PMLs with $\psi(s) = \psi_2(s)$, $\psi(s) = \psi_3(s)$, $\psi(s) = \psi_4(s)$ is almost indistinguishable on the scale used in the figure.

Remark 5.23. *The choice of stable $\psi(s)$ as in (76) may not necessarily minimize the number of unknowns in the resulting PML system (in the most general case each pole in $\psi(s)$ introduces new unknowns associated to it, cf. Section 4.2.2). In some cases the results of Theorem 5.11, however, may facilitate reducing the number of unknowns, by reducing the number of poles in a stable function $\psi(s)$. The simplest case is given by a system (2) with parameters*

$$\tilde{\varepsilon}_1(\omega) = \tilde{\varepsilon}_2(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \tilde{\mu}(\omega) = 1, \quad \omega_p > 0.$$

One readily checks that a Bérenger's PML with $\tilde{\psi}(\omega) = 1$ is stable in this case.

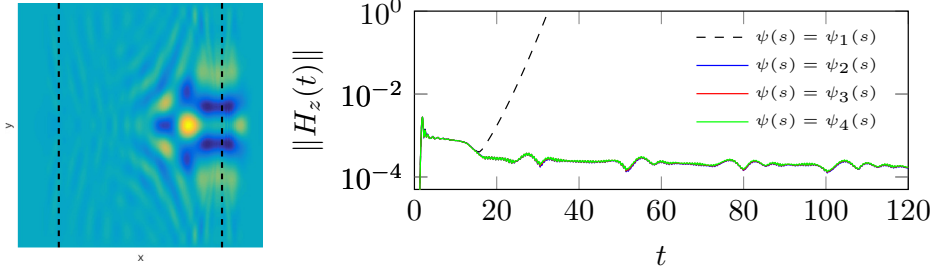


FIGURE 8. Left: H_z of Example 5.22 at $t = 60$ computed with the new PML with $\psi(s) = \psi_2(s)$, see (76). The boundary between the physical domain and the PML is marked in black. Right: the L_2 -norm of H_z in the whole domain (i.e. with the perfectly matched layer), computed with different PMLs, with respect to time.

Example 5.24. (Nonlocal Materials) One of the examples of a function that satisfies the passivity requirement $\text{Re}(sf(s)) > 0$, $s \in \mathbb{C}_+$, is [53, p.19] a function $s^{-1} \tanh(s)$, which has the following partial fraction expansion [54, 1.421.2, p.43]:

$$s^{-1} \tanh(s) = 2 \sum_{\ell=1}^{\infty} \left(s^2 + (2\ell - 1)^2 \left(\frac{\pi}{2} \right)^2 \right)^{-1}. \quad (77)$$

One can see that the above expression coincides, up to a constant, with the expansion (43) where $n = \infty$. We would like to apply the results of Theorem 5.11 to the system (4) with parameters, see also Remark 5.2:

$$\varepsilon_1(s) = s^{-1} \tanh(4s), \quad \varepsilon_2(s) = s^{-1} \tanh(2s), \quad \mu(s) = s^{-1} \tanh(s). \quad (78)$$

Notice that these functions do not satisfy the behaviour at the infinity as prescribed by Assumption 4.1. Our goal is to demonstrate numerically that the statement of Theorem 5.11 can be extended to this case. More precisely, we would like to construct the PML in the direction x ; in the direction y we will use zero boundary conditions.

One can notice that on the interval $\omega \in \left(\frac{3\pi}{8}, \frac{\pi}{2} \right)$, it holds

$$\tilde{\varepsilon}_1(\omega) < 0, \quad \tilde{\varepsilon}_2(\omega) < 0, \quad \tilde{\mu}(\omega) > 0.$$

According to Theorem 3.2, the sign of the function $\tilde{\psi}(\omega)$ on this interval can be arbitrary (in particular, positive, so that it does not coincide with the sign of $\tilde{\varepsilon}_2(\omega)$). We thus suggest that the choices of $\psi(s) = \psi_{2,3}(s)$ for the PML (31) in the direction x should lead to a stable method, and the choice $\psi(s) = 1$ would be unstable:

$$\psi_1(s) = 1, \quad \psi_2(s) = \varepsilon_2(s)^{-1}, \quad \psi_3(s) = \varepsilon_2(s)^{-1} \left(s^2 + \left(\frac{3\pi}{8} \right)^2 \right)^{-1} \left(s^2 + \left(\frac{\pi}{2} \right)^2 \right). \quad (79)$$

One can check that $\tilde{\psi}_3(\omega)$ is positive on $\left(\frac{3\pi}{8}, \frac{\pi}{2} \right)$, and its sign coincides on the rest of the real axis with $\tilde{\varepsilon}_2(\omega)$.

To verify the condition that $\text{Re}(\bar{s}\psi_3(s)) > 0$ in \mathbb{C}_+ , we refer to an extension of Lemma 4.9 for a certain class of meromorphic functions, see [29, p.308, Theorem 1], as well as [55, pp.140-143] for a more detailed proof of the corresponding result.

To explain this in more detail, let us consider the aforementioned theorem, which is originally due to Krein.

Theorem 5.25 (Theorem of Krein, [29]). *In order that a meromorphic function $\phi(z)$, s.t. $\phi(z)$ is real on \mathbb{R} , map the upper half-plane into the upper half-plane, it is necessary and sufficient that this function be represented*

in the form:

$$\phi(z) = C \frac{z - a_0}{z - b_0} \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{a_k}\right) \left(1 - \frac{z}{b_k}\right)^{-1}, \quad (80)$$

with $C > 0$, $b_k < a_k < b_{k+1}$, for all $k \in \mathbb{Z}$ and $a_{-1} < 0 < b_1$.

It is easy to see that given any even passive meromorphic function $c(s)$, which is real on the imaginary axis, we can define $h_c(z) = zc(-iz)$, with $z \in \mathbb{C}$ which would satisfy the conditions of the above theorem. Moreover, $h_c(z)$ maps the upper half plane into the upper half plane if and only if $c(z)$ is passive. In particular, for z s.t. $\text{Im } z > 0$, it holds

$$\text{Im } h_c(z) = \text{Re}(-ih_c(z)) = \text{Re}(-izc(-iz)) > 0, \quad \text{Re}(-iz) = \text{Im } z > 0.$$

Moreover, such $h_c(z)$ is real on the real axis. The application of the Krein theorem gives the following representation:

$$\tilde{c}(z) = c(-iz) = Cz^{-1} \frac{z - a_0}{z - b_0} \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{a_k}\right) \left(1 - \frac{z}{b_k}\right)^{-1}. \quad (81)$$

If $\tilde{c}(z)$ is even, because of symmetry of the poles and roots of $\tilde{c}(z)$ with respect to zero, only two possibilities are possible:

- (1) either $a_0 = 0$; from this $b_k = b_{-k+1}$, when $k \in \mathbb{Z}$ and $a_k = a_{-k}$, $k \in \mathbb{Z} \setminus \{0\}$;
- (2) either $b_0 = 0$; from this $a_k = a_{-k-1}$, for $k \in \mathbb{Z}$, and $b_k = b_{-k}$ for $k \in \mathbb{Z} \setminus \{0\}$.

In all other cases it is impossible to place zeros and poles of $\tilde{c}(z)$ symmetrically with respect to zero.

In particular, for $\tilde{\varepsilon}_2(\omega)$, the root $a_0 = 0$, and the other roots $a_k = \frac{\pi k}{2}$, with $k \in \mathbb{Z} \setminus 0$. Similarly, the pole $b_0 = -\frac{\pi}{4}$, and $b_k = \frac{\pi}{4} + \frac{\pi}{2}(k-1) = \frac{\pi}{4}(2k-1)$, for $k \in \mathbb{Z} \setminus \{0\}$.

Finding C in (81) from the condition $\varepsilon_2(0) = 2$, we obtain the following expansion of $\varepsilon_2(s)$:

$$\varepsilon_2(s) = \frac{\pi}{2i(s - i\frac{\pi}{4})} \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 + \frac{2s}{i\pi k}\right) \left(1 + \frac{4s}{i(2k-1)\pi}\right)^{-1}. \quad (82)$$

From this expansion for $\varepsilon_2(s)$ we can obtain a similar product expansion for $\psi_3(s)$ (thus showing that $h_{\psi_3}(z)$ is of the form (80). Notice that the poles and roots of $\tilde{\psi}_3(\omega)$ interlace. The non-negative roots of $\tilde{\psi}_3(\omega)$ are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $(2\ell-1)\frac{\pi}{4}$, with $\ell = 4, 5, \dots$. The non-negative poles, in turn, are 0 (which is a pole of multiplicity 2), $\frac{3\pi}{8}$, π , $\frac{3\pi}{2}$, $\ell\frac{\pi}{2}$, where $\ell = 4, 5, \dots$. Moreover, thanks to (79), $(\psi_3(s))^{-1}$ possesses an expansion similar to (82):

$$\begin{aligned} (\psi_3(s))^{-1} &= \frac{\pi}{2i(s - i\frac{\pi}{4})} \prod_{k \in \mathbb{Z} \setminus \{0, \pm 1\}} \left(1 + \frac{2s}{i\pi k}\right) \left(1 + \frac{4s}{i(2k-1)\pi}\right)^{-1} \left(1 + \frac{2s}{i\pi}\right) \left(1 + \frac{4s}{i\pi}\right)^{-1} \\ &\times \left(1 - \frac{2s}{i\pi}\right) \left(1 - \frac{4s}{3i\pi}\right)^{-1} \left(1 - \frac{2s}{i\pi}\right)^{-1} \left(1 + \frac{2s}{i\pi}\right)^{-1} \frac{4}{\pi^2} \left(1 - \frac{8s}{3i\pi}\right) \left(1 + \frac{8s}{3i\pi}\right) \left(\frac{3\pi}{8}\right)^2 \\ &= \frac{9\pi}{32i(s - i\frac{\pi}{4})} \prod_{k \in \mathbb{Z} \setminus \{0, \pm 1\}} \left(1 + \frac{2s}{i\pi k}\right) \left(1 + \frac{4s}{i(2k-1)\pi}\right)^{-1} \left(1 + \frac{8s}{3i\pi}\right) \left(1 + \frac{4s}{i\pi}\right)^{-1} \left(1 - \frac{8s}{3i\pi}\right) \left(1 - \frac{4s}{3i\pi}\right)^{-1} \end{aligned}$$

This gives the form of the expansion (80) for $h_{\psi_3}(z)$, and shows that $h_{\psi_3}(z)$ maps the upper half plane into the upper half plane, thanks to the Krein theorem. Hence $\psi_3(s)$ is passive.

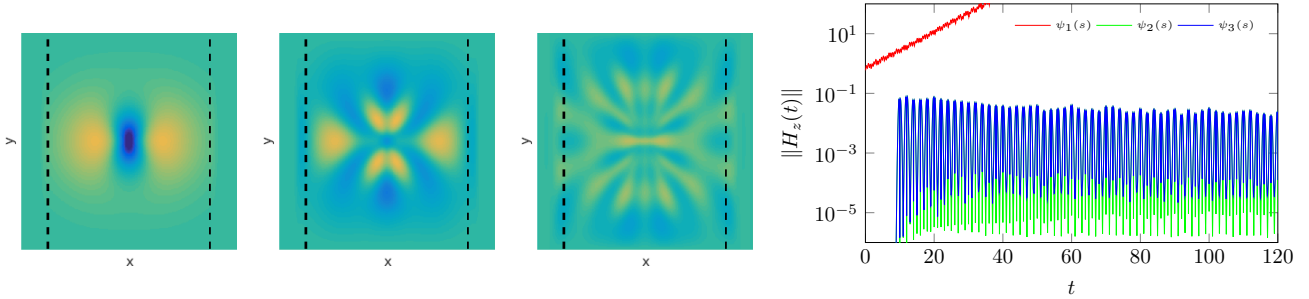


FIGURE 9. From left to right: a solution H_z of Example 5.24 computed with the new PML with $\psi(s) = \psi_2(s)$, see (79), at the times $t = 15.75$, 31.75 and 59.9 ; the norm of the solution H_z (inside the physical domain and the perfectly matched layer) with respect to time computed with different PMLs. The boundary between the domain and the perfectly matched layer is marked in black.

As before, see Example 3.16, we conducted an experiment with the help of the 3-stage Radau IIA convolution quadrature method. The rest of the parameters of the experiments are provided in Table 6. The results of the

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
12	16	2	0	$20x^2$	0	0.05	0.05	$e^{-40x^2 - 40y^2 - 12(t-10)^2} (24^2(t-10)^2 - 24) H(2 - x)H(2 - y)$

TABLE 6. Parameters for the experiment with a dielectric permittivity and magnetic permeability (79). We discretize directly the problem (4) where the PML change of variables (31) is used in direction x . We use zero initial conditions and the right hand side (in the time domain) $f(t, x, y)$. By $H(x)$ we denote the left-continuous Heaviside function. The notation can be found in the beginning of Section 3.1.1.

experiment are shown in Figure 9. We can observe the non-causality of the behaviour of the solution norm, when the Bérenger's PML is used, see Example 3.16 for a possible explanation, as well as its growth (the largest achieved value of $\|H_z\|$ not shown in the plot was close to $2 \cdot 10^7$), which evidences the instability of the PML. On the other hand, both PMLs (31) with $\psi(s) = \psi_2(s)$ and $\psi(s) = \psi_3(s)$ are stable, however, one can observe distinctively a small difference in their behaviour for this problem in the rightmost plot of Figure 9, contrary to what happens in Example 5.22, Figure 8; notice that the difference between the computed norms is fairly small. We measured the norms inside the domain (i.e. without the PML), and the difference between them is much less pronounced (it would not be visible on the same scale as in Figure 9).

Moreover, we can see that the solution norm in this case decreases very slowly, which is probably due to the dispersivity of this problem (we checked that this decay does not come from a slight dissipation of the Runge-Kutta method by computing the norm of the solution obtained with twice smaller time step; the difference between the norms is negligible). The stability of the PML (31) with $\psi(s) = \psi_{2,3}(s)$ is a numerical illustration of the possibility of the extension of the results of Theorem 5.11 to a slightly more general class of materials.

Example 5.26. (Anisotropic Lorentz Materials with Loss) Another interesting question is whether a perfectly matched layer which we have devised for the Lorentz materials (8) in this section can be applied to Lorentz materials *with losses*, more precisely, when the dielectric permittivity/permeability are of the form

$$r(s) = 1 + \sum_{\ell=0}^n \frac{r_\ell}{s^2 + 2\nu_\ell s + \omega_\ell^2}, \quad r_\ell > 0, \omega_\ell \geq 0, \nu_\ell \geq 0.$$

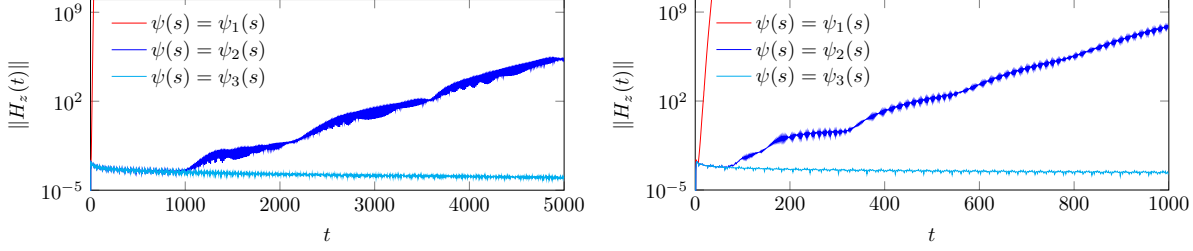


FIGURE 10. The dependence of the L_2 -norm of the solution H_z (computed inside the domain and the PML) to the problem of Example 5.26 on time, computed with different PMLs. In the left plot $\nu = 0.01$ in (83), and in the right plot $\nu = 0.1$.

It is easy to check that $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$; this shows that all the roots of $r(s)$ lie in $\mathbb{C} \setminus \mathbb{C}_+$. Moreover, all poles of $r(s)$ also lie in $\mathbb{C} \setminus \mathbb{C}_+$. To see this notice that the poles of $r(s)$ are given by $s_\ell^\pm = -\nu_\ell \pm \sqrt{\nu_\ell^2 - \omega_\ell^2}$, $\ell = 0, \dots, n$. If $\nu_\ell \geq \omega_\ell$, then $s_\ell^\pm \leq 0$; otherwise $\operatorname{Re} s_\ell^\pm = -\nu_\ell$.

We consider the problem (4) with the following parameters:

$$\varepsilon_1(s) = 1, \quad \varepsilon_2(s) = 1 + \frac{\varepsilon_{21}}{s^2 + 2\nu s + \omega_p^2}, \quad \mu(s) = 1, \quad \varepsilon_{21} = 12, \quad \omega_p = 2. \quad (83)$$

Although this system is dissipative, we would like to check whether it's possible to use for it the PML in x -direction, which was constructed for the analogical non-dissipative case, namely $\psi(s) = \varepsilon_2(s)^{-1}$ from (83) with $\nu = 0$. The existing theory of the stability of the PML does not provide necessary stability conditions for this case, and hence we would like to check it numerically. Thus, we test the behaviour of the perfectly matched layers with $\psi = \psi_j(s)$ in the direction x :

$$\psi_1(s) = 1, \quad \psi_2(s) = 1 + \frac{\varepsilon_{21}}{s^2 + \omega_p^2}, \quad \psi_3(s) = 1 + \frac{\varepsilon_{21}}{s^2 + 2\nu s + \omega_p^2}. \quad (84)$$

The choice $\psi(s) = \psi_2(s)$ corresponds to a stable PML in the case $\nu = 0$, whereas the choice $\psi(s) = \psi_3(s)$ is a stable PML of Corollary 3.7. The parameters of the experiment are presented in Table 7. The results of the

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t, x, y)$
8	12	2	0	$50x^2$	0	0.05	0.025	$e^{-1000(x-3.8)^2 - 300y^2 - 300z^2} H(3.98 - x) e^{-15(t-1.6)^2}$

TABLE 7. Parameters for the experiment of Example 5.26 with the dielectric permittivity and magnetic permeability (83). By $H(x)$ we denote the left-continuous Heaviside function. The notation can be found in the beginning of Section 3.1.1.

experiment are shown in Figure 10. In particular, we can see that for computations on long times, when the material is dissipative, one should rather use the PML change of variables with $\psi_3(s)$ as suggested by Corollary 3.7, than $\psi_2(s)$, which would stabilize the corresponding problem without dissipation. Moreover, the larger the dissipation is, the worse stability properties are exhibited by the latter PML. However, the PML suggested in Corollary 3.7 remains always stable, as predicted by theory.

6. CONCLUSIONS

In this work we have shown how to construct stable perfectly matched layers for a class of anisotropic dispersive models described in the Laplace domain by the wave equation with frequency-dependent coefficients

$$a(s)\partial_{xx}u + b(s)\partial_{yy}u - s^2c(s) = 0, \quad s \in \mathbb{C}_+,$$

where the coefficients $a(s)$, $b(s)$, $c(s)$ are analytic in \mathbb{C}_+ , and $a(s)$, $b(s)$ are s.t. $\operatorname{Re}(\bar{s}a(s)) > 0$, $\operatorname{Re}(\bar{s}b(s)) > 0$, $s \in \mathbb{C}_+$, and $c(s)$ is s.t. $\operatorname{Re}(sc(s)) > 0$, $s \in \mathbb{C}_+$. Following [7], in order to construct the PML in one direction (assuming the PML layer is located in a half-plane $x > 0$), we suggest to use the following change of variables:

$$x \rightarrow x + \frac{\psi(s)}{s} \int_0^x \sigma(x') dx', \quad x > 0,$$

where $\psi(s)$ satisfies $\operatorname{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$. This article provides choices of $\psi(s)$ that would stabilize the PML. Based on the Laplace domain analysis in the free space for $\sigma = \text{const}$, we suggest to use the following $\psi(s)$:

- (1) for any material: in the direction x use the PML change of variables with $\psi(s) = a(s)$. If the PML in corners needs to be constructed, use in the direction y a similar change of variables with $\psi(s) = b(s)$;
- (2) $\psi(s) = (\alpha c(s) + (1 - \alpha)a^{-1}(s))^{-1}$, $0 \leq \alpha \leq 1$, for isotropic materials with $a(s) = b(s)$;
- (3) in the case when $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ are Lorentz materials, we suggest to choose $\psi(s)$, s.t. $\psi(s)^{-1}$ is Lorentz and which would satisfy the following condition for all $\omega \in \mathbb{R}$:

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \text{ or } \tilde{a}(\omega)\tilde{c}(\omega) > 0 \implies \tilde{a}(\omega)\tilde{\psi}(\omega) \geq 0.$$

Moreover, if $b(s) = c(s)^{-1}$ (2D uniaxial cold plasma model (7)), the only possible choice among passive $\psi(s)^{-1}$ satisfying Assumption 4.1 is $\psi(s) = a(s)$.

We confirm the obtained results with the help of numerical experiments, including non-local in time and dissipative examples. Indeed, there are many open questions remaining which are the subject of a future work. In particular, they include the construction of stable PMLs for 3D anisotropic Maxwell's equations, even in the simplest case of a diagonal tensor of dielectric permittivity (where, as we show in [20], generally the instabilities cannot be overcome by the use of a special frequency-dependent change of variables). Another important naturally arising question is a construction of stable PMLs for nonpassive materials, which is also a subject of a future research.

APPENDIX A. PROOF OF LEMMA 2.2

Define $z = is$, $s \in \mathbb{C}_+$, and recall that $f_c(z) := zc(-iz)$ is a Herglotz function. The upper bound on $|f(z)|$ can be deduced e.g. from [29, Theorem 8', p. 18]. For consistency we show another way of computing it, using the methods of [56], which will serve as a basis to compute the lower bound for $\text{Im } f_c(z)$ as well.

The main idea is to construct a function on the unit circle to which the Schwarz's lemma [56, Chapter 4, Theorem 1] can be easily applied. Let us set $h(s) = sc(s)$. Taking $s_0 \in \mathbb{C}_+$, we define the Moebius transformation [56, p.43,44]

$$r := \frac{s - s_0}{s + \bar{s}_0}, \quad |r| < 1, \quad s \in \mathbb{C}_+,$$

$$g\left(\frac{s - s_0}{s + \bar{s}_0}\right) := \frac{h(s) - h(s_0)}{h(s) + \overline{h(s_0)}}.$$

The function $g(r)$ is analytic inside the unit circle $|r| < 1$; $g(0) = 0$ and $|g(r)| < 1$. Then, thanks to the Schwarz's lemma, $|g(r)| \leq |r|$ inside the unit circle. Hence,

$$\left| \frac{h(s) - h(s_0)}{h(s) + \overline{h(s_0)}} \right| \leq \left| \frac{s - s_0}{s + \bar{s}_0} \right|. \quad (85)$$

(1) The proof of the upper bound on $|h(s)|$. From (85) it follows

$$|h(s)h(s_0)^{-1} - 1| \leq \left| \frac{s - s_0}{s + \bar{s}_0} \right| \left| h(s)h(s_0)^{-1} - \overline{h(s_0)}h(s_0)^{-1} \right| \leq \left| \frac{s - s_0}{s + \bar{s}_0} \right| (|h(s)h(s_0)^{-1}| + 1).$$

Using $|h(s)h(s_0)^{-1} - 1| \geq |h(s)h(s_0)^{-1}| - 1$, and recalling that $r = \left| \frac{s - s_0}{s + \bar{s}_0} \right|$, and $|r| < 1$, we get the following bound

$$|h(s)| \leq |h(s_0)| (1 + r) (1 - r)^{-1} = |h(s_0)| (1 + r)^2 (1 - r^2)^{-1} \leq 4|h(s_0)| (1 - r^2)^{-1}. \quad (86)$$

Notice that the function $(1 - r^2)^{-1}$ grows in r ; hence, let us obtain the upper bound on r in terms of $\text{Re } s$ and $|s|$:

$$\left| \frac{s - s_0}{s + \bar{s}_0} \right|^2 = \frac{|s|^2 - 2 \text{Re}(s\bar{s}_0) + |s_0|^2}{|s|^2 + 2 \text{Re}(ss_0) + |s_0|^2} = 1 - \frac{2 \text{Re}(ss_0) + 2 \text{Re}(s\bar{s}_0)}{|s|^2 + 2 \text{Re}(ss_0) + |s_0|^2} = 1 - \frac{4 \text{Re } s \text{Re } s_0}{|s|^2 + 2 \text{Re}(ss_0) + |s_0|^2}. \quad (87)$$

The lower bound for

$$\frac{\text{Re } s \text{Re } s_0}{|s + \bar{s}_0|^2} \geq \frac{1}{2} \frac{\text{Re } s \text{Re } s_0}{|s|^2 + |s_0|^2}. \quad (88)$$

Therefore, using (86), we obtain

$$|h(s)| \leq 2|h(s_0)|(\text{Re } s \text{Re } s_0)^{-1} (|s|^2 + |s_0|^2) \leq C' (|s|^2 + |s_0|^2) (\text{Re } s)^{-1},$$

for some $C' > 0$. From this we immediately obtain, for some $C > 0$,

$$|sc(s)| \leq C'|s|^2 (1 + |s_0|^2|s|^{-2}) (\text{Re } s)^{-1} \leq C|s|^2 (\text{Re } s)^{-1} \max\left(1, (\text{Re } s)^{-2}\right), \quad s \in \mathbb{C}_+.$$

The same (up to a constant) bound can be obtained with the help of [29, Theorem 8', p. 18].

(2) The proof of the lower bound on $\operatorname{Re} h(s)$, $s \in \mathbb{C}_+$. We rewrite (85) as follows, with the use of (87):

$$\frac{|h(s)|^2 - 2 \operatorname{Re}(\overline{h(s_0)} h(s)) + |h(s_0)|^2}{|h(s)|^2 + 2 \operatorname{Re}(h(s_0) \overline{h(s)}) + |h(s_0)|^2} = 1 - \frac{4 \operatorname{Re} h(s) \operatorname{Re} h(s_0)}{|h(s) + \overline{h(s_0)}|^2} \leq 1 - \frac{4 \operatorname{Re} s \operatorname{Re} s_0}{|s|^2 + 2 \operatorname{Re}(s_0 s) + |s_0|^2}.$$

This gives, with the use of (88),

$$\frac{\operatorname{Re} h(s) \operatorname{Re} h(s_0)}{|h(s) + \overline{h(s_0)}|^2} \geq \frac{1}{2} \frac{\operatorname{Re} s \operatorname{Re} s_0}{|s|^2 + |s_0|^2}.$$

Next, notice that $|h(s) + \overline{h(s_0)}|^2 \geq (\operatorname{Re} h(s) + \operatorname{Re} \overline{h(s_0)})^2 \geq (\operatorname{Re} h(s_0))^2$. Using $|s|^2 + |s_0|^2 \leq C_0 \max(|s|^2, 1)$, $C_0 > 0$, we obtain the desired bound:

$$\operatorname{Re} h(s) \geq C \min(|s|^{-2}, 1) \operatorname{Re} s \geq C |s|^{-2} \operatorname{Re} s \min(1, (\operatorname{Re} s)^2).$$

APPENDIX B. PROOF OF EQUIVALENCE IN LEMMA 5.15

Lemma B.1. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 5.10. Assume (NSC2)-(a) holds true, i.e. $c(s)a(s)^{-1}\psi(s)$ is passive. Then (NSC1)-(a) holds, i.e. for all $\omega \in \Omega_p^{ac>0}$ the product $\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0$.*

Proof. First of all, in $\omega = 0$, due to Theorem 4.4, it holds that $\tilde{\psi}(0)\tilde{a}(0) \geq 0$. Due to the equivalence of the positivity of the derivative $(\omega\tilde{f}(\omega))' > 0$, $\omega \in \mathbb{R}$, and the property $\operatorname{Re}(sf(s)) > 0$ in \mathbb{C}_+ , the following derivative is strictly positive for $\omega \in \mathbb{R}$ with the exception of poles of $\tilde{c}\tilde{\psi}\tilde{a}^{-1}$:

$$\left(\omega\tilde{c}(\omega)\tilde{\psi}(\omega)\tilde{a}(\omega)^{-1}\right)' = (\omega\tilde{c}(\omega))'\tilde{\psi}(\omega)\tilde{a}(\omega)^{-1} + \omega\tilde{c}(\omega)\tilde{\psi}(\omega)'\tilde{a}(\omega)^{-1} + \omega\tilde{c}(\omega)\tilde{\psi}(\omega)(\tilde{a}(\omega)^{-1})' > 0. \quad (89)$$

Take ω s.t. $\tilde{c}(\omega)\tilde{a}(\omega) > 0$ and assume that $\tilde{a}(\omega)\tilde{\psi}(\omega) < 0$. Using the equation (13), which shows the passivity of $a(s)^{-1}$, and Lemmas 4.7 and 4.8 about the signs of the derivatives of the passive functions, we obtain that each term of the above is strictly negative, and thus the contradiction. \square

Remark B.2. *One could try proving Lemma 5.15 by examining the sign of the derivative (89); however, when $\tilde{c}(\omega)\tilde{a}(\omega) < 0$, and $\tilde{\psi}(\omega)\tilde{a}(\omega)$ is of any sign, one does not see immediately that the expression (89) is strictly positive.*

APPENDIX C. DISPERSION RELATION FOR ANISOTROPIC LORENTZ MATERIALS

In this section we provide some details on a computation of the dispersion relation of the Lorentz system (53). For simplicity, we will compute it for the corresponding 1D formulation (in the 2D case the computation remains the same):

$$\begin{aligned} \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell} &= \partial_y H_z, \\ \partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_{\ell} j_{\mu\ell} &= \partial_y E_x, \\ \partial_t j_{x\ell} + \omega_{x\ell}^2 p_{x\ell} &= E_x, & \partial_t p_{x\ell} &= j_{x\ell}, & \ell &= 0, \dots, n_x, \\ \partial_t j_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} &= H_z, & \partial_t p_{\mu\ell} &= j_{\mu\ell}, & \ell &= 0, \dots, n_\mu. \end{aligned}$$

We consider the plane-wave solutions of the above system in the form $\mathbf{V}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$, where $\mathbf{V} \in \mathbb{C}^{2(n_x+1)+2(n_\mu+1)+2}$ is a constant vector. There exists such a solution with $\mathbf{V} \neq 0$ if and only if the determinant of the following block matrix vanishes:

$$\begin{aligned} U &= \begin{pmatrix} U_{E_x E_x} & U_{E_x H_z} \\ U_{H_z E_x} & U_{H_z H_z} \end{pmatrix}, & U_{E_x H_z} &= \text{diag}(-ik_y, 0, \dots, 0) \in \mathbb{C}^{(2n_x+1) \times (2n_\mu+1)}, \\ U_{H_z E_x} &= \text{diag}(-ik_y, 0, \dots, 0) \in \mathbb{C}^{(2n_x+1) \times (2n_\mu+1)}, \\ U_{E_x E_x} &= \begin{pmatrix} -i\omega & \varepsilon_{x0} & 0 & \varepsilon_{x1} & 0 & \dots & \varepsilon_{xn_x} & 0 \\ \mathbf{v}_{2(n_x+1) \times 1} & & & \Omega_x & & & & \end{pmatrix}, \\ U_{H_z H_z} &= \begin{pmatrix} -i\omega & \mu_0 & 0 & \mu_1 & 0 & \dots & \mu_{n_\mu} & 0 \\ \mathbf{v}_{2(n_\mu+1) \times 1} & & & \Omega_y & & & & \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Omega_\alpha &= \begin{pmatrix} -i\omega & \omega_{\alpha 0}^2 & 0 & 0 & \dots & 0 & 0 \\ -1 & -i\omega & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -i\omega & \omega_{\alpha 1}^2 & \dots & 0 & 0 \\ 0 & 0 & -1 & -i\omega & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -i\omega & \omega_{\alpha n_\alpha}^2 \\ 0 & 0 & 0 & 0 & \dots & -1 & -i\omega \end{pmatrix}, & \alpha \in \{x, \mu\}, & (90) \\ \mathbf{v}_{2m \times 1} \text{ is a vector of size } 2m \times 1, & (\mathbf{v}_{2m \times 1})_k &= \begin{cases} -1 & k \text{ is odd,} \\ 0 & k \text{ is even.} \end{cases} \end{aligned}$$

To compute the determinant of U , we multiply each of its $2(k+1) + 1$ st column, with $k = 0, \dots, n_x$, by $(-i\omega)^{-1}$ and add it to the $2(k+1)$ st column. Next, we repeat the same procedure for $2(n_x+1) + 4 + 2m$ and

$2(n_x + 1) + 3 + 2m$ th columns correspondingly, where $m = 0, \dots, n_\mu$. This results in the following expression for the determinant of the matrix U :

$$\det U = \det \begin{pmatrix} \tilde{U}_{E_x E_x} & U_{E_x H_z} \\ U_{H_z E_x} & \tilde{U}_{H_z H_z} \end{pmatrix},$$

$$\tilde{U}_{E_x E_x} = \begin{pmatrix} -i\omega & \varepsilon_{x0} & 0 & \varepsilon_{x1} & 0 & \dots & \varepsilon_{xn_x} & 0 \\ \mathbf{v}_{2(n_x+1) \times 1} & & & \tilde{\Omega}_x & & & & \end{pmatrix},$$

$$\tilde{U}_{H_z H_z} = \begin{pmatrix} -i\omega & \mu_0 & 0 & \mu_1 & 0 & \dots & \mu_{n_\mu} & 0 \\ \mathbf{v}_{2(n_y+1) \times 1} & & & \tilde{\Omega}_y & & & & \end{pmatrix},$$

$$\tilde{\Omega}_\alpha = \begin{pmatrix} -i\omega \left(1 - \frac{\omega_{\alpha 0}^2}{\omega^2}\right) & \omega_{\alpha 0}^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & -i\omega & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -i\omega \left(1 - \frac{\omega_{\alpha 1}^2}{\omega^2}\right) & \omega_{\alpha 1}^2 & \dots & 0 & 0 \\ 0 & 0 & 0 & -i\omega & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -i\omega \left(1 - \frac{\omega_{\alpha n_\alpha}^2}{\omega^2}\right) & \omega_{\alpha n_\alpha}^2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -i\omega \end{pmatrix}, \quad \alpha \in \{x, \mu\}.$$

This, in turn can be rewritten as

$$|\det U| = |(-i\omega)^{n_x+n_\mu+2}| \left| \det \begin{pmatrix} \tilde{U}'_{E_x E_x} & \tilde{U}'_{E_x H_z} \\ \tilde{U}'_{H_z E_x} & \tilde{U}'_{H_z H_z} \end{pmatrix} \right|, \quad (91)$$

$$\tilde{U}'_{E_x E_x} = \begin{pmatrix} -i\omega & \varepsilon_{x0} & \varepsilon_{x1} & \dots & \varepsilon_{xn_x} \\ -\mathbf{1}_{(n_x+1) \times 1} & & \tilde{\Omega}'_x & & \end{pmatrix}, \quad \tilde{U}'_{H_z H_z} = \begin{pmatrix} -i\omega & \mu_0 & \mu_1 & \dots & \mu_{n_\mu} \\ -\mathbf{1}_{(n_\mu+1) \times 1} & & \tilde{\Omega}'_y & & \end{pmatrix},$$

where

$$\tilde{\Omega}'_\alpha = \begin{pmatrix} -i\omega \left(1 - \frac{\omega_{\alpha 0}^2}{\omega^2}\right) & 0 & \dots & 0 \\ 0 & -i\omega \left(1 - \frac{\omega_{\alpha 1}^2}{\omega^2}\right) & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & -i\omega \left(1 - \frac{\omega_{\alpha n_\alpha}^2}{\omega^2}\right) \end{pmatrix}, \quad \alpha \in \{x, \mu\}, \quad (92)$$

$$\mathbf{1}_{m \times 1} \text{ is a vector of size } m \times 1, \quad \mathbf{1}_{m \times 1} = (1, \dots, 1)^T, \quad (93)$$

and $\tilde{U}'_{E_x H_z} \in \mathbb{C}^{(n_x+2) \times (n_\mu+2)}$, $\tilde{U}'_{H_z E_x} \in \mathbb{C}^{(n_\mu+2) \times (n_x+2)}$ are such that $(\tilde{U}'_{E_x H_z})_{11} = (\tilde{U}'_{H_z E_x})_{11} = -ik_y$. Next, for $k = 0, \dots, n_x$, scale $k + 1$ st row by $-\left(-i\omega \left(1 - \frac{\omega_{xk}^2}{\omega^2}\right)\right)^{-1} \varepsilon_{xk}$ and add the result to the first row. Every $n_x + 4 + m$ th row, for $m = 0, \dots, n_\mu$ scaled by $-\left(-i\omega \left(1 - \frac{\omega_{\mu m}^2}{\omega^2}\right)\right)^{-1} \mu_m$ we add to the $n_x + 4$ th

row. Notice that the entry (1,1) of the matrix $\tilde{U}'_{E_x E_x}$ becomes $-i\omega + \sum_{k=1}^{n_x+1} \left(-i\omega \left(1 - \frac{\omega_{xk-1}^2}{\omega^2}\right)\right)^{-1} \varepsilon_{xk-1} = -i\omega \left(1 - \sum_{k=1}^{n_x+1} (\omega^2 - \omega_{xk-1}^2)^{-1} \varepsilon_{xk-1}\right) = -i\omega \tilde{\varepsilon}_1(\omega)$. Similarly, the entry (1,1) of the matrix $\tilde{U}'_{H_z H_z}$ becomes $-i\omega \tilde{\mu}(\omega)$. After this procedure, it remains to compute the following:

$$|\det U| = |(-i\omega)^{n_x+n_\mu+2}| \left| \det \begin{pmatrix} \tilde{U}'_{E_x E_x} & \tilde{U}'_{E_x H_z} \\ \tilde{U}'_{H_z E_x} & \tilde{U}'_{H_z H_z} \end{pmatrix} \right|,$$

$$\tilde{U}'_{E_x E_x} = \begin{pmatrix} -i\omega \tilde{\varepsilon}_1(\omega) & 0 & 0 & \dots & 0 \\ -\mathbf{1}_{(n_x+1) \times 1} & \tilde{\Omega}'_x & & & \end{pmatrix}, \quad \tilde{U}'_{H_z H_z} = \begin{pmatrix} -i\omega \tilde{\mu}(\omega) & 0 & 0 & \dots & 0 \\ -\mathbf{1}_{(n_\mu+1) \times 1} & \tilde{\Omega}'_\mu & & & \end{pmatrix},$$

from which it immediately follows that

$$|\det U| = \left| \omega^{n_x+n_\mu+2} \omega^{n_x+n_\mu+2} \prod_{\ell=0}^{n_\mu} \left(1 - \frac{\omega_{\mu\ell}^2}{\omega^2}\right) \prod_{\ell=0}^{n_x} \left(1 - \frac{\omega_{x\ell}^2}{\omega^2}\right) \right| \times \left| \det \begin{pmatrix} -i\omega \tilde{\varepsilon}_1(\omega) & -ik_y \\ -ik_y & -i\omega \tilde{\mu}(\omega) \end{pmatrix} \right| \quad (94)$$

$$= \left| \prod_{\ell=0}^{n_\mu} (\omega^2 - \omega_{\mu\ell}^2) \prod_{\ell=0}^{n_x} (\omega^2 - \omega_{x\ell}^2) \right| \times |-\omega^2 \tilde{\varepsilon}_1 \tilde{\mu} + k_y^2|$$

$$= \left| \prod_{\ell=0}^{n_\mu} (\omega^2 - \omega_{\mu\ell}^2) \prod_{\ell=0}^{n_x} (\omega^2 - \omega_{x\ell}^2) \tilde{\varepsilon}_1 \right| \times |-\omega^2 \tilde{\mu} + k_y^2 \tilde{\varepsilon}_1^{-1}|$$

We can generalize the above result to two dimensions; the dispersion relation for the system (53) reads (an extra factor ω comes from $i\omega \tilde{\varepsilon}_2(\omega)$ in the corresponding matrix in the final computation (94):

$$F(\omega, \mathbf{k}) = \prod_{\ell=0}^{n_\mu} (\omega^2 - \omega_{\mu\ell}^2) \prod_{\ell=0}^{n_x} (\omega^2 - \omega_{x\ell}^2) \prod_{\ell=0}^{n_y} (\omega^2 - \omega_{y\ell}^2) \tilde{\varepsilon}_1(\omega) \tilde{\varepsilon}_2(\omega) \omega (k_x^2 \tilde{\varepsilon}_2^{-1} + k_y^2 \tilde{\varepsilon}_1^{-1} - \omega^2 \tilde{\mu}(\omega)) = 0.$$

APPENDIX D. DISPERSION RELATION FOR ANISOTROPIC LORENTZ MATERIALS WITH THE PML

In this appendix we briefly comment on how to compute the dispersion relation for the system (56). Following the procedure of Appendix C, we require that the determinant of the following matrix vanishes:

$$\det V = \begin{pmatrix} V_{E_y E_y} & V_{E_y E_{yx}} & V_{E_y H_z} & V_{E_y H_{zx}} & V_{E_y E_x} \\ V_{E_{yx} E_y} & V_{E_{yx} E_{yx}} & V_{E_{yx} H_z} & V_{E_{yx} H_{zx}} & V_{E_{yx} E_x} \\ V_{H_z E_y} & V_{H_z E_{yx}} & V_{H_z H_z} & V_{H_z H_{zx}} & V_{H_z E_x} \\ V_{H_{zx} E_y} & V_{H_{zx} E_{yx}} & V_{H_{zx} H_z} & V_{H_{zx} H_{zx}} & V_{H_{zx} E_x} \\ V_{E_x E_y} & V_{E_x E_{yx}} & V_{E_x H_z} & V_{E_x H_{zx}} & V_{E_x E_x} \end{pmatrix},$$

where

$$V_{E_\alpha E_\alpha} = \begin{pmatrix} -i\omega & \varepsilon_{\alpha 0} & 0 & \varepsilon_{\alpha 1} & 0 & \dots & \varepsilon_{\alpha n_\alpha} & 0 \\ \mathbf{v}_{2(n_\alpha+1) \times 1} & & & \Omega_\alpha & & & & \end{pmatrix}, \quad \alpha \in \{x, y\},$$

$$V_{H_z H_z} = \begin{pmatrix} -i\omega & \mu_0 & 0 & \mu_1 & 0 & \dots & \mu_{n_\mu} & 0 \\ \mathbf{v}_{2(n_\mu+1) \times 1} & & & \Omega_\mu & & & & \end{pmatrix},$$

with Ω_α , $\alpha \in \{x, y, \mu\}$ defined as in (90), and the following matrices are empty or diagonal with a non-vanishing entry (1, 1) (the size of these matrices can be deduced from the size of the diagonal matrices which are given in a more explicit manner further):

$$\begin{aligned} V_{E_y E_{yx}} &= 0, \quad V_{E_y H_z} = 0, \quad V_{E_y E_x} = 0, \quad V_{E_{yx} H_z} = 0, \quad V_{E_{yx} H_{zx}} = 0, \quad V_{E_{yx} E_x} = 0, \\ V_{H_z E_y} &= 0, \quad V_{H_z H_{zx}} = 0, \quad V_{H_{zx} E_y} = 0, \quad V_{H_{zx} E_{yx}} = 0, \quad V_{H_{zx} E_x} = 0, \\ V_{E_x E_y} &= 0, \quad V_{E_x E_{yx}} = 0, \quad V_{E_x H_{zx}} = 0, \\ V_{E_y H_{zx}} &= \text{diag}(1, 0, \dots, 0), \quad V_{E_{yx} E_y} = \text{diag}(-\omega k_x, 0, \dots, 0), \\ V_{H_z E_{yx}} &= \text{diag}(1, 0, \dots, 0), \quad V_{H_z E_x} = \text{diag}(-ik_y, 0, \dots, 0), \quad V_{H_{zx} H_z} = \text{diag}(-\omega k_x, 0, \dots, 0), \\ V_{E_x H_z} &= \text{diag}(-ik_y, 0, \dots, 0). \end{aligned}$$

Finally, the following two diagonal matrix blocks are identical:

$$V_{E_{yx} E_{yx}} = V_{H_{zx} H_{zx}} = \begin{pmatrix} -i\omega + \sigma\psi(0) & \sigma\psi_0\omega_{\psi 0}^{-2} & 0 & \sigma\psi_1\omega_{\psi 1}^{-2} & 0 & \dots & \sigma\psi_{n_\psi}\omega_{\psi n_\psi}^{-2} & 0 \\ -i\omega\mathbf{v}_{2(n_\psi+1) \times 1} & & & \Omega_\psi & & & & \end{pmatrix}.$$

To compute the determinant of the above matrix, we first apply a linear transformation to Ω_α , $\alpha \in \{x, y, \mu, \psi\}$, using the same procedure as in Appendix C, that would make these matrix blocks upper-diagonal. The computation of the determinant results in the following identity, cf. (91):

$$|\det V| = \left| (-i\omega)^{n_x+n_\mu+n_y+3+2(n_\psi+1)} \right| \det \begin{pmatrix} \tilde{V}_{E_y E_y} & \tilde{V}_{E_y E_{yx}} & \tilde{V}_{E_y H_z} & \tilde{V}_{E_y H_{zx}} & \tilde{V}_{E_y E_x} \\ \tilde{V}_{E_{yx} E_y} & \tilde{V}_{E_{yx} E_{yx}} & \tilde{V}_{E_{yx} H_z} & \tilde{V}_{E_{yx} H_{zx}} & \tilde{V}_{E_{yx} E_x} \\ \tilde{V}_{H_z E_y} & \tilde{V}_{H_z E_{yx}} & \tilde{V}_{H_z H_z} & \tilde{V}_{H_z H_{zx}} & \tilde{V}_{H_z E_x} \\ \tilde{V}_{H_{zx} E_y} & \tilde{V}_{H_{zx} E_{yx}} & \tilde{V}_{H_{zx} H_z} & \tilde{V}_{H_{zx} H_{zx}} & \tilde{V}_{H_{zx} E_x} \\ \tilde{V}_{E_x E_y} & \tilde{V}_{E_x E_{yx}} & \tilde{V}_{E_x H_z} & \tilde{V}_{E_x H_{zx}} & \tilde{V}_{E_x E_x} \end{pmatrix},$$

with off-diagonal matrix blocks being of the same form as before, but of smaller dimensions. The diagonal matrix blocks in turn are defined as follows:

$$\tilde{V}_{E_\alpha E_\alpha} = \begin{pmatrix} -i\omega & \varepsilon_{\alpha 0} & \varepsilon_{\alpha 1} & \cdots & \varepsilon_{\alpha n_\alpha} \\ -\mathbf{1}_{(n_x+1) \times 1} & & \tilde{\Omega}'_\alpha & & \end{pmatrix}, \quad \tilde{V}_{H_z H_z} = \begin{pmatrix} -i\omega & \mu_0 & \mu_1 & \cdots & \mu_{n_\mu} \\ -\mathbf{1}_{(n_\mu+1) \times 1} & & \tilde{\Omega}'_\mu & & \end{pmatrix}, \quad \alpha \in \{x, y\},$$

$$\tilde{V}_{E_{yx} E_{yx}} = \tilde{V}_{H_{zx} H_{zx}} = \begin{pmatrix} -i\omega + \sigma\tilde{\psi}(0) & \sigma\psi_0\omega_{\psi_0}^{-2} & \sigma\psi_1\omega_{\psi_1}^{-2} & \cdots & \sigma\psi_{n_\psi}\omega_{\psi_{n_\psi}}^{-2} \\ i\omega\mathbf{1}_{(n_\mu+1) \times 1} & & \tilde{\Omega}'_\psi & & \end{pmatrix},$$

where $\tilde{\Omega}'_\beta$, $\beta \in \{x, y, \mu, \psi\}$, are the diagonal matrices defined as in (92). We proceed as in Appendix C, performing a linear transformation that would nullify all the off-diagonal elements in the first row of the matrices $\tilde{V}_{E_\alpha E_\alpha}$, $\alpha \in \{x, y\}$, $\tilde{V}_{H_z H_z}$, $\tilde{V}_{E_{yx} E_{yx}}$, $\tilde{V}_{H_{zx} H_{zx}}$ (thus making these matrices lower-triangular). Notice that the element (1,1) of the matrices $\tilde{V}_{E_{yx} E_{yx}}$ and $\tilde{V}_{H_{zx} H_{zx}}$ becomes:

$$-i\omega + \sigma \left(\tilde{\psi}(0) - i\omega \sum_{\ell=0}^{n_\psi} \psi_\ell \omega_{\psi_\ell}^{-2} \left(-i\omega \left(1 - \frac{\omega_{\psi_\ell}^2}{\omega^2} \right) \right)^{-1} \right) = -i\omega + \sigma\tilde{\psi}(\omega).$$

Thus, we obtain the following:

$$|\det V| = \left| (-i\omega)^{n_x+n_\mu+n_y+3+2(n_\psi+1)} \right| \det \begin{pmatrix} \tilde{V}'_{E_y E_y} & \tilde{V}_{E_y E_{yx}} & \tilde{V}_{E_y H_z} & \tilde{V}_{E_y H_{zx}} & \tilde{V}_{E_y E_x} \\ \tilde{V}_{E_{yx} E_y} & \tilde{V}'_{E_{yx} E_{yx}} & \tilde{V}_{E_{yx} H_z} & \tilde{V}_{E_{yx} H_{zx}} & \tilde{V}_{E_{yx} E_x} \\ \tilde{V}_{H_z E_y} & \tilde{V}_{H_z E_{yx}} & \tilde{V}'_{H_z H_z} & \tilde{V}_{H_z H_{zx}} & \tilde{V}_{H_z E_x} \\ \tilde{V}_{H_{zx} E_y} & \tilde{V}_{H_{zx} E_{yx}} & \tilde{V}_{H_{zx} H_z} & \tilde{V}'_{H_{zx} H_{zx}} & \tilde{V}_{H_{zx} E_x} \\ \tilde{V}_{E_x E_y} & \tilde{V}_{E_x E_{yx}} & \tilde{V}_{E_x H_z} & \tilde{V}_{E_x H_{zx}} & \tilde{V}'_{E_x E_x} \end{pmatrix},$$

where

$$\tilde{V}'_{E_\alpha E_\alpha} = \begin{pmatrix} -i\omega\tilde{\varepsilon}_\alpha(\omega) & 0 & 0 & \cdots & 0 \\ -\mathbf{1}_{(n_x+1) \times 1} & & \tilde{\Omega}'_\alpha & & \end{pmatrix}, \quad \tilde{V}_{H_z H_z} = \begin{pmatrix} -i\omega\tilde{\mu}(\omega) & 0 & 0 & \cdots & 0 \\ -\mathbf{1}_{(n_\mu+1) \times 1} & & \tilde{\Omega}'_\mu & & \end{pmatrix}, \quad \alpha \in \{x, y\},$$

$$\tilde{V}_{E_{yx} E_{yx}} = \tilde{V}_{H_{zx} H_{zx}} = \begin{pmatrix} -i\omega + \sigma\tilde{\psi}(\omega) & 0 & 0 & \cdots & 0 \\ i\omega\mathbf{1}_{(n_\mu+1) \times 1} & & \tilde{\Omega}'_\psi & & \end{pmatrix}.$$

The direct computation of the determinant gives:

$$|\det V| = |(-i\omega)^{n_x+n_\mu+n_y+2n_\psi+5}| |\det(\Omega'_\psi)|^2 |\det(\Omega'_x)| |\det(\Omega'_y)| |\det(\Omega'_\mu)|$$

$$\times \det \begin{pmatrix} -i\omega\tilde{\epsilon}_y & 0 & 0 & 1 & 0 \\ -\omega k_x & -i\omega + \sigma\tilde{\psi}(\omega) & 0 & 0 & 0 \\ 0 & 1 & -i\omega\tilde{\mu} & 0 & -ik_y \\ 0 & 0 & -\omega k_x & -i\omega + \sigma\tilde{\psi}(\omega) & 0 \\ 0 & 0 & -ik_y & 0 & -i\omega\tilde{\epsilon}_x \end{pmatrix}.$$

Evaluating the last determinant, we obtain the following expression for the dispersion relation:

$$F(\omega, \mathbf{k}) = \prod_{\ell=0}^{n_\psi} (\omega^2 - \omega_{\psi\ell}^2)^2 \prod_{\ell=0}^{n_y} (\omega^2 - \omega_{y\ell}^2) \prod_{\ell=0}^{n_x} (\omega^2 - \omega_{x\ell}^2) \prod_{\ell=0}^{n_\mu} (\omega^2 - \omega_{\mu\ell}^2) \omega^3 \tilde{\epsilon}_x \tilde{\epsilon}_y$$

$$\times \left(1 - \frac{\sigma\tilde{\psi}(\omega)}{i\omega} \right)^2 \left(k_x^2 \tilde{\epsilon}_y^{-1} \left(1 - \frac{\sigma\tilde{\psi}(\omega)}{i\omega} \right)^{-2} + k_y^2 \tilde{\epsilon}_x^{-1} - \omega^2 \tilde{\mu}(\omega) \right) = 0.$$

APPENDIX E. CONSTRUCTION OF A FUNCTION $\psi(s)$ SATISFYING CONDITIONS OF
THEOREM 5.11

In [7], the authors have proposed a procedure to construct the function $\psi(s)$ that results in stable perfectly matched layer for isotropic dispersive systems, based on a result similar to Theorem 5.11. Let us briefly comment on the extension (and a minor modification) of their procedure to anisotropic dispersive systems in question. Consider a passive sesquilinear form $A(u, v)$ that satisfies Assumption 5.1. Let us describe a procedure of choosing a function $\psi(s)$ for the PML change of variables that satisfies conditions of Theorem 5.11.

The condition (NSC1) of Theorem 5.11 can be rewritten alternatively in the following form. Either the sign of $\tilde{\psi}$ is constant on the real axis (and thus is positive, since $\tilde{\psi}(\omega) \rightarrow 1$ as $\omega \rightarrow \infty$), or there exists a set of intervals $\mathcal{I}_k = (\omega_{2k-1}, \omega_{2k})$, $k = 1, \dots, m$, with $0 \leq \omega_1 < \omega_2 \leq \omega_3 < \dots \leq \omega_{2k-1} < \omega_{2m} = +\infty$, s.t. the sign of the function $\tilde{\psi}(\omega)$ on an interval \mathcal{I}_k is opposite to the sign of $\tilde{\psi}(\omega)$ on the intervals \mathcal{I}_{k-1} and \mathcal{I}_{k+1} . These signs are defined by the signs of $\tilde{a}(\omega)$. Let us remark that $\omega_{2m} = +\infty$, since as $\omega \rightarrow +\infty$, $\tilde{c}(\omega)\tilde{a}(\omega) > 0$, and thus, $\tilde{\psi}(\omega) > 0$.

The case when the sign of $\tilde{\psi}$ is constant on \mathbb{R} , is trivially reduced to using the Bérenger's PML $\tilde{\psi} = 1$. Otherwise, following [7], we consider two cases:

- (1) $m = 2\ell - 1$, $\ell > 1$. In this case the sign of $\tilde{\psi}(\omega)$ on \mathcal{I}_1 coincides with the sign of $\tilde{\psi}(\omega)$ on $(\omega_{2m}, +\infty)$, i.e. is positive. One can verify that the following holds true:

$$\begin{aligned} \omega \in (\omega_{4k-3}, \omega_{4k-2}) &\implies \tilde{\psi}(\omega) \geq 0, \\ \omega \in (\omega_{4k-1}, \omega_{4k}) &\implies \tilde{\psi}(\omega) \leq 0. \end{aligned} \tag{95}$$

Thus, one of the choices of $\tilde{\psi}(\omega)$ is given by

$$\tilde{\psi}(\omega) = \prod_{k=1}^{\ell-1} \frac{(\omega^2 - r_k^2)}{(\omega^2 - p_k^2)}, \quad r_k \in [\omega_{4k-2}, \omega_{4k-1}], \quad p_k \in [\omega_{4k}, \omega_{4k+1}].$$

First of all, notice that for all $k = 1, \dots, \ell-2$, the following interlacing property holds true: $r_k \leq \omega_{4k-1} < \omega_{4k} \leq p_k \leq \omega_{4k+1} < \omega_{4k+2} \leq r_{k+1}$. Applying Lemma 4.9, we see that $\tilde{\psi}(\omega)$ satisfies $\operatorname{Re}(\bar{s}\psi(s)) > 0$ in \mathbb{C}_+ . It remains only to verify that $\tilde{\psi}$ defined as above satisfies the sign conditions (95). Taking $\omega_* \in (\omega_{4j-3}, \omega_{4j-2})$, we can split

$$\tilde{\psi}(\omega) = \prod_{k=1}^{j-1} \frac{(\omega^2 - r_k^2)}{(\omega^2 - p_k^2)} \prod_{k=j}^{\ell-2} \frac{(\omega^2 - r_k^2)}{(\omega^2 - p_k^2)} = \tilde{\psi}_j^{(1)}(\omega) \tilde{\psi}_j^{(2)}(\omega).$$

One sees that all the poles and zeros of $\tilde{\psi}_j^{(1)}(\omega)$ are smaller than ω_* (notice that $p_{j-1} \leq \omega_{4j-3}$), hence $\tilde{\psi}_j^{(1)}(\omega_*) > 0$. Similarly, all the poles and zeros of $\tilde{\psi}_j^{(2)}(\omega)$ are larger than ω_* (since $r_j \geq \omega_{4j-2}$), hence by direct computation $\tilde{\psi}_j^{(2)}(\omega_*) > 0$. Therefore, $\tilde{\psi}(\omega_*) > 0$.

Similarly one can verify the validity of (95) for $\omega_* \in (\omega_{4j-1}, \omega_{4j})$.

- (2) $m = 2\ell$, $\ell > 1$. In this case the sign of $\tilde{\psi}(\omega)$ on \mathcal{I}_1 is negative. One can verify that the following holds true:

$$\begin{aligned} \omega \in (\omega_{4k-3}, \omega_{4k-2}) &\implies \tilde{\psi}(\omega) \leq 0, \\ \omega \in (\omega_{4k-1}, \omega_{4k}) &\implies \tilde{\psi}(\omega) \geq 0. \end{aligned} \tag{96}$$

One of the choices of $\tilde{\psi}(\omega)$ is provided by

$$\tilde{\psi}(\omega) = \prod_{k=1}^{\ell-1} \frac{(\omega^2 - r_k^2)}{(\omega^2 - p_k^2)}, \quad r_1 \in [0, \omega_1], r_k \in [\omega_{4k}, \omega_{4k+1}], p_1 \in [\omega_2, \omega_3], p_k \in [\omega_{4k+2}, \omega_{4k+3}].$$

As before, the condition $\operatorname{Re}(\bar{s}\psi(s)) > 0$ for $s \in \mathbb{C}_+$, and the sign conditions (96) can be ensured by the direct use of Lemma 4.9 and computations as in the previous case.

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