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# A computable bound of the essential spectral radius of finite range Metropolis-Hastings kernels

Loïc HERVÉ, and James LEDOUX \*

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## Abstract

Let  $\pi$  be a positive continuous target density on  $\mathbb{R}$ . Let  $P$  be the Metropolis-Hastings operator on the Lebesgue space  $\mathbb{L}^2(\pi)$  corresponding to a proposal Markov kernel  $Q$  on  $\mathbb{R}$ . When using the quasi-compactness method to estimate the spectral gap of  $P$ , a mandatory first step is to obtain an accurate bound of the essential spectral radius  $r_{ess}(P)$  of  $P$ . In this paper a computable bound of  $r_{ess}(P)$  is obtained under the following assumption on the proposal kernel:  $Q$  has a bounded continuous density  $q(x, y)$  on  $\mathbb{R}^2$  satisfying the following finite range assumption :  $|u| > s \Rightarrow q(x, x + u) = 0$  (for some  $s > 0$ ). This result is illustrated with Random Walk Metropolis-Hastings kernels.

AMS subject classification : 60J10, 47B07

Keywords : Markov chain operator, Metropolis-Hastings algorithms, Spectral gap

## 1 Introduction

Let  $\pi$  be a positive distribution density on  $\mathbb{R}$ . Let  $Q(x, dy) = q(x, y)dy$  be a Markov kernel on  $\mathbb{R}$ . Throughout the paper we assume that  $q(x, y)$  satisfies the following finite range assumption: there exists  $s > 0$  such that

$$|u| > s \implies q(x, x + u) = 0. \quad (1)$$

Let  $T(x, dy) = t(x, y)dy$  be the nonnegative kernel on  $\mathbb{R}$  given by

$$t(x, y) := \min \left( q(x, y), \frac{\pi(y)q(y, x)}{\pi(x)} \right) \quad (2)$$

and define the associated Metropolis-Hastings kernel:

$$P(x, dy) := r(x) \delta_x(dy) + T(x, dy) \quad \text{with} \quad r(x) := 1 - \int_{\mathbb{R}} t(x, y) dy, \quad (3)$$

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where  $\delta_x(dy)$  denotes the Dirac distribution at  $x$ . The associated Markov operator is still denoted by  $P$ , that is we set for every bounded measurable function  $f : \mathbb{R} \rightarrow \mathbb{C}$  :

$$\forall x \in \mathbb{R}, \quad (Pf)(x) = r(x)f(x) + \int_{\mathbb{R}} f(y)t(x,y) dy. \quad (4)$$

In the context of Monte Carlo Markov Chain methods, the kernel  $Q$  is called the proposal Markov kernel. We denote by  $(\mathbb{L}^2(\pi), \|\cdot\|_2)$  the usual Lebesgue space associated with the probability measure  $\pi(y)dy$ . For convenience,  $\|\cdot\|_2$  also denotes the operator norm on  $\mathbb{L}^2(\pi)$ , namely: if  $U$  is a bounded linear operator on  $\mathbb{L}^2(\pi)$ , then  $\|U\|_2 := \sup_{\|f\|_2=1} \|Uf\|_2$ . Since

$$t(x,y)\pi(x) = t(y,x)\pi(y), \quad (5)$$

we know that  $P$  is reversible with respect to  $\pi$  and that  $\pi$  is  $P$ -invariant (e.g. see [RR04]). Consequently  $P$  is a self-adjoint operator on  $\mathbb{L}^2(\pi)$  and  $\|P\|_2 = 1$ . Now define the rank-one projector  $\Pi$  on  $\mathbb{L}^2(\pi)$  by

$$\Pi f := \pi(f)1_{\mathbb{R}} \quad \text{with} \quad \pi(f) := \int_{\mathbb{R}} f(x)\pi(x) dx.$$

Then the spectral radius of  $P - \Pi$  equals to  $\|P - \Pi\|_2$  since  $P - \Pi$  is self-adjoint, and  $P$  is said to have the spectral gap property on  $\mathbb{L}^2(\pi)$  if

$$\varrho_2 \equiv \varrho_2(P) := \|P - \Pi\|_2 < 1.$$

In this case the following property holds:

$$\forall n \geq 1, \forall f \in \mathbb{L}^2(\pi), \quad \|P^n f - \Pi f\|_2 \leq \varrho_2^n \|f\|_2. \quad (\text{SG}_2)$$

The spectral gap property on  $\mathbb{L}^2(\pi)$  of a Metropolis-Hastings kernel is of great interest, not only due to the explicit geometrical rate given by  $(\text{SG}_2)$ , but also since it ensures that a central limit theorem (CLT) holds true for additive functional of the associated Metropolis-Hastings Markov chain under the expected second-order moment conditions, see [RR97]. Furthermore, the rate of convergence in the CLT is  $O(1/\sqrt{n})$  under third-order moment conditions (as for the independent and identically distributed models), see details in [HP10, FHL12].

The quasi-compactness approach can be used to compute the rate  $\varrho_2(P)$ . This method is based on the notion of essential spectral radius. Indeed, first recall that the essential spectral radius of  $P$  on  $\mathbb{L}^2(\pi)$ , denoted by  $r_{ess}(P)$ , is defined by (e.g. see [Wu04] for details):

$$r_{ess}(P) := \liminf_n (\|P^n - K\|_2)^{1/n} \quad (6)$$

where the above infimum is taken over the ideal of compact operators  $K$  on  $\mathbb{L}^2(\pi)$ . Note that the spectral radius of  $P$  is one. Then  $P$  is said to be quasi-compact on  $\mathbb{L}^2(\pi)$  if  $r_{ess}(P) < 1$ . Second, if  $r_{ess}(P) \leq \alpha$  for some  $\alpha \in (0, 1)$ , then  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$ , and the following properties hold: for every real number  $\kappa$  such that  $\alpha < \kappa < 1$ , the set  $\mathcal{U}_\kappa$  of the spectral values  $\lambda$  of  $P$  satisfying  $\kappa \leq |\lambda| \leq 1$  is composed of finitely many eigenvalues of  $P$ , each of them having a finite multiplicity (e.g. see [Hen93] for details). Third, if  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$  and satisfies usual aperiodicity and irreducibility conditions (e.g. see [MT93]), then  $\lambda = 1$  is the only spectral value of  $P$  with modulus one and  $\lambda = 1$  is a simple eigenvalue of  $P$ , so

that  $P$  has the spectral gap property on  $\mathbb{L}^2(\pi)$ . Finally the following property holds: either  $\varrho_2(P) = \max\{|\lambda|, \lambda \in \mathcal{U}_\kappa, \lambda \neq 1\}$  if  $\mathcal{U}_\kappa \neq \emptyset$ , or  $\varrho_2(P) \leq \kappa$  if  $\mathcal{U}_\kappa = \emptyset$ .

This paper only focusses on the preliminary central step of the previous spectral method, that is to find an accurate bound of  $r_{ess}(P)$ . More specifically, we prove that, if the target density  $\pi$  is positive and continuous on  $\mathbb{R}$ , and if the proposal kernel  $q(\cdot, \cdot)$  is bounded continuous on  $\mathbb{R}^2$  and satisfies (1) for some  $s > 0$ , then

$$r_{ess}(P) \leq \alpha_a \quad \text{with} \quad \alpha_a := \max(r_a, r'_a + \beta_a) \quad (7)$$

where, for every  $a > 0$ , the constants  $r_a, r'_a$  and  $\beta_a$  are defined by:

$$r_a := \sup_{|x| \leq a} r(x), \quad r'_a := \sup_{|x| > a} r(x), \quad \beta_a := \int_{-s}^s \sup_{|x| > a} \sqrt{t(x, x+u)t(x+u, x)} du. \quad (8)$$

This result is illustrated in Section 2 with Random Walk Metropolis-Hastings (RWMH) kernels for which the proposal Markov kernel is of the form  $Q(x, dy) := \Delta(|x - y|) dy$ , where  $\Delta : \mathbb{R} \rightarrow [0, +\infty)$  is an even continuous and compactly supported function.

In [AP07] the quasi-compactness of  $P$  on  $\mathbb{L}^2(\pi)$  is proved to hold provided that 1) the essential supremum of the rejection probability  $r(\cdot)$  with respect to  $\pi$  is bounded away from unity; 2) the operator  $T$  associated with the kernel  $t(x, y) dy$  is compact on  $\mathbb{L}^2(\pi)$ . Assumption 1) on the rejection probability  $r(\cdot)$  is a necessary condition for  $P$  to have the spectral gap property (SG<sub>2</sub>) (see [RT96]). But this condition, which is quite generic from the definition of  $r(\cdot)$  (see Remark 3), is far to be sufficient for  $P$  to satisfy (SG<sub>2</sub>). The compactness Assumption 2) of [AP07] is quite restrictive, for instance it is not adapted for random walk Metropolis-Hastings kernels. Here this compactness assumption is replaced by the condition  $r'_a + \beta_a < 1$ . As shown in the examples of Section 2, this condition is adapted to RWMH.

In the discrete state space case, a bound for  $r_{ess}(P)$  similar to (7) has been obtained in [HL16]. Next a bound of the spectral gap  $\varrho_2(P)$  has been derived in [HL16] from a truncation method for which the control of the essential spectral radius of  $P$  is a central step. It is expected that, in the continuous state space case, the bound (7) will provide a similar way to compute the spectral gap  $\varrho_2(P)$  of  $P$ . This issue, which is much more difficult than in the discrete case, is not addressed in this work.

## 2 An upper bound for the essential spectral radius of $P$

Let us state the main result of the paper.

**Theorem 1** *Assume that*

- (i)  $\pi$  is positive and continuous on  $\mathbb{R}$ ;
- (ii)  $q(\cdot, \cdot)$  is bounded and continuous on  $\mathbb{R}^2$ , and satisfies the finite range assumption (1).

For a  $a > 0$ , set  $\alpha_a := \max(r_a, r'_a + \beta_a)$ , where the constants  $r_a, r'_a$  and  $\beta_a$  are defined in (8). Then

$$\forall a > 0, \quad r_{ess}(P) \leq \alpha_a. \quad (9)$$

Theorem 1 is proved in Section 3 from Formula (6) by using a suitable decomposition of the iterates  $P^n$  involving some Hilbert-Schmidt operators.

**Remark 1** Assume that the assumptions (i)-(ii) of Theorem 1 hold. Then, if there exists some  $a > 0$  such that  $\alpha_a < 1$ ,  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$ . Suppose moreover that the proposal Markov kernel  $Q(x, dy)$  satisfies usual irreducibility and aperiodicity conditions. Then  $P$  has the spectral gap property on  $\mathbb{L}^2(\pi)$ . Actually, if  $q$  is symmetric (i.e.  $q(x, y) = q(y, x)$ ), it can be easily proved that, under the condition  $r'_a + \beta_a < 1$ ,  $P$  satisfies the so-called drift condition with respect to  $V(x) := 1/\sqrt{\pi(x)}$ , so that  $P$  is  $V$ -geometrically ergodic, that is  $P$  has the spectral gap property on the space  $(\mathcal{B}_V, \|\cdot\|_V)$  composed of the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\|f\|_V := \sup_{x \in \mathbb{R}} |f(x)|/V(x) < \infty$ . If furthermore  $\int_{\mathbb{R}} \sqrt{\pi(x)} dx < \infty$ , then the spectral gap property of  $P$  on  $\mathbb{L}^2(\pi)$  can be deduced from the  $V$ -geometrical ergodicity since  $P$  is reversible (see [RR97, Bax05]). However this fact does not provide a priori any precise bound on the essential spectral radius of  $P$  on  $\mathbb{L}^2(\pi)$ . Indeed, mention that the results [Wu04, Th. 5.5] provide a comparison between  $r_{\text{ess}}(P)$  and  $r_{\text{ess}}(P|_{\mathcal{B}_V})$ , but unfortunately, to the best of our knowledge, no accurate bound of  $r_{\text{ess}}(P|_{\mathcal{B}_V})$  is known for Metropolis-Hasting kernels. In particular note that the general bound of  $r_{\text{ess}}(P|_{\mathcal{B}_V})$  given in [HL14, Th. 5.2] is of theoretical interest but is not precise, and that the more accurate bound of  $r_{\text{ess}}(P|_{\mathcal{B}_V})$  given in [HL14, Th. 5.4] cannot be used here since in general no iterate of  $P$  is compact from  $\mathcal{B}_0$  to  $\mathcal{B}_V$ , where  $\mathcal{B}_0$  denotes the space of bounded measurable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  equipped with the supremum norm. Therefore, the  $V$ -geometrical ergodicity of  $P$  is not discussed here since the purpose is to bound the essential spectral radius of  $P$  on  $\mathbb{L}^2(\pi)$ .

**Remark 2** If  $\pi$  and  $q$  satisfy the assumptions (i)-(ii) of Theorem 1, and if moreover  $q$  satisfies the following mild additional condition

$$\forall x \in \mathbb{R}, \exists y \in [x - s, x + s], \quad q(x, y)q(y, x) \neq 0, \quad (10)$$

then, for every  $a > 0$ , we have  $r_a < 1$ , so that the quasi-compactness of  $P$  on  $\mathbb{L}^2(\pi)$  holds provided that there exists some  $a > 0$  such that  $r'_a + \beta_a < 1$ . Note that Condition (10) is clearly fulfilled if  $q$  is symmetric. To prove the previous assertion on  $r_a$ , observe that  $r(\cdot)$  is continuous on  $\mathbb{R}$  (use Lebesgue's theorem). Consequently, if  $r_a = 1$  for some  $a > 0$ , then  $r(x_0) = 1$  for some  $x_0 \in [-a, a]$ , but this is impossible from the definition of  $r(x_0)$  and Condition (10).

**Remark 3** Actually, under the assumptions (i)-(ii) of Theorem 1, the fact that  $r_a < 1$  for every  $a > 0$ , and even the stronger property  $\sup_{x \in \mathbb{R}} r(x) < 1$ , seem to be quite generic. For instance, if  $q$  is of the form  $q(x, y) = \Delta(|x - y|)$  for some function  $\Delta$  and if there exists  $\theta > 0$  such that  $\pi$  is increasing on  $(-\infty, -\theta]$  and decreasing on  $[\theta, +\infty)$ , then  $\sup_{x \in \mathbb{R}} r(x) < 1$ . Thus, for every  $a > 0$ , we have  $r_a < 1$  and  $r'_a < 1$ . Indeed, first observe that  $r_a < 1$  for every  $a > 0$  from Remark 2. Consequently, if  $\sup_{x \in \mathbb{R}} r(x) = 1$ , then there exists  $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$  such that  $\lim_n |x_n| = +\infty$  and  $\lim_n r(x_n) = 1$ . Let us prove that this property is impossible under our assumptions. To simplify, suppose that  $\lim_n x_n = +\infty$ . Then, from the definition of  $r(\cdot)$ , from our assumptions on  $q$ , and finally from Fatou's Lemma, it follows that, for almost every  $u \in [-s, s]$  such that  $\Delta(u) \neq 0$ , we have  $\liminf_n \min(1, \pi(x_n + u)/\pi(x_n)) = 0$ . But this is impossible since, if  $u \in [-s, 0]$  and  $x_n \geq \theta + s$ , then  $\pi(x_n + u) \geq \pi(x_n)$ .

Theorem 1 is illustrated with symmetric proposal Markov kernels of the form

$$Q(x, dy) := \Delta(x - y) dy$$

where  $\Delta : \mathbb{R} \rightarrow [0, +\infty)$  is : 1) an even continuous function ; 2) assumed to be compactly supported on  $[-s, s]$  and positive on  $(-s, s)$  for some  $s > 0$ . Then  $q(x, y) := \Delta(x - y)$  satisfies (1) and  $t(\cdot, \cdot)$  is given by

$$\forall u \in [-s, s], \quad t(x, x + u) := \Delta(u) \min \left( 1, \frac{\pi(x + u)}{\pi(x)} \right).$$

**Corollary 1** *Assume that  $q(x, y) := \Delta(x - y)$  with  $\Delta(\cdot)$  satisfying the above assumptions and that  $\pi$  is an even positive continuous distribution density such that the following limit exists:*

$$\forall u \in [0, s], \quad \tau(u) := \lim_{x \rightarrow +\infty} \frac{\pi(x + u)}{\pi(x)} \in [0, 1]. \quad (11)$$

*Assume that the set  $\{u \in [0, s] : \tau(u) \neq 1\}$  has a positive Lebesgue-measure. Then  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$  with*

$$r_{ess}(P) \leq \alpha_\infty := \max(r_\infty, \gamma_\infty) < 1 \quad \text{where} \quad \gamma_\infty := 1 - \int_0^s \Delta(u)(1 - \tau(u)^{1/2})^2 du.$$

*Proof.* We know from Theorem 1 that, for any  $a > 0$ ,  $r_{ess}(P) \leq \max(r_a, r'_a + \beta_a)$  with  $r_a := \sup_{|x| \leq a} r(x)$ ,  $r'_a := \sup_{|x| > a} r(x)$ . It is easily checked that

$$\beta_a = \int_{-s}^s \Delta(u) \sup_{|x| > a} \min \left( \sqrt{\frac{\pi(x + u)}{\pi(x)}}, \sqrt{\frac{\pi(x)}{\pi(x + u)}} \right) du.$$

Note that

$$\forall x \in \mathbb{R}, \quad r(x) = 1 - \int_{-s}^s \Delta(u) \min \left( 1, \frac{\pi(x + u)}{\pi(x)} \right) du.$$

For  $u \in [-s, 0]$ ,  $\tau(u)$  is defined as in (11). Then

$$\forall u \in [-s, s], \quad \tau(u) = \lim_{y \rightarrow +\infty} \frac{\pi(y)}{\pi(y - u)} = \frac{1}{\tau(-u)}$$

with the convention  $1/0 = +\infty$ . Thus, for every  $u \in [-s, 0]$ , we have  $\tau(u) \in [1, +\infty]$ . Moreover we obtain for every  $u \in [-s, s]$ :

$$\lim_{x \rightarrow -\infty} \frac{\pi(x + u)}{\pi(x)} = \tau(-u).$$

since  $\pi$  is an even function. We have for every  $a > 0$

$$r'_a = 1 - \min \left( \inf_{x < -a} \int_{-s}^s \Delta(u) \min \left( 1, \frac{\pi(x + u)}{\pi(x)} \right) du, \inf_{x > a} \int_{-s}^s \Delta(u) \min \left( 1, \frac{\pi(x + u)}{\pi(x)} \right) du \right).$$

Moreover it follows from dominated convergence theorem and from the above remarks that

$$\lim_{x \rightarrow \pm\infty} \int_{-s}^s \Delta(u) \min \left( 1, \frac{\pi(x+u)}{\pi(x)} \right) du = \int_{-s}^s \Delta(u) \min(1, \tau(\pm u)) du$$

from which we deduce that

$$\begin{aligned} r'_\infty &:= \lim_{a \rightarrow +\infty} r'_a \\ &= 1 - \min \left( \int_{-s}^s \Delta(u) \min(1, \tau(-u)) du, \int_{-s}^s \Delta(u) \min(1, \tau(u)) du \right) \\ &= 1 - \int_{-s}^s \Delta(u) \min(1, \tau(u)) du \quad (\text{since } \Delta \text{ is an even function}) \\ &= 1 - \int_{-s}^0 \Delta(u) du - \int_0^s \Delta(u) \tau(u) du \quad (\text{since } \tau(u) \leq 1 \text{ for } u \in [0, s], \tau(u) \geq 1 \text{ for } u \in [-s, 0]) \\ &= 1 - \int_0^s \Delta(u) [1 + \tau(u)] du. \end{aligned}$$

Note that, for every  $a > 0$ , we have  $r_a < 1$  from Remark 2. Moreover  $r'_\infty \leq 1/2$  from the last equality. Thus  $r_\infty := \sup_{x \in \mathbb{R}} r(x) < 1$ . Next we obtain for every  $a > 0$

$$\beta_a = \int_{-s}^s \Delta(u) \max \left[ \sup_{x < -a} \min \left( \sqrt{\frac{\pi(x+u)}{\pi(x)}}, \sqrt{\frac{\pi(x)}{\pi(x+u)}} \right), \sup_{x > a} \min \left( \sqrt{\frac{\pi(x+u)}{\pi(x)}}, \sqrt{\frac{\pi(x)}{\pi(x+u)}} \right) \right] du$$

and again we deduce from dominated convergence theorem and from the above remarks that

$$\begin{aligned} \beta_\infty &:= \lim_{a \rightarrow +\infty} \beta_a = \int_{-s}^s \Delta(u) \max \left[ \min \left( \tau(-u)^{1/2}, \frac{1}{\tau(-u)^{1/2}} \right), \min \left( \tau(u)^{1/2}, \frac{1}{\tau(u)^{1/2}} \right) \right] du \\ &= \int_{-s}^s \Delta(u) \min \left( \tau(u)^{1/2}, \frac{1}{\tau(u)^{1/2}} \right) du \quad (\text{since } \tau(-u) = \frac{1}{\tau(u)}) \\ &= \int_{-s}^0 \Delta(u) \tau(u)^{-1/2} du + \int_0^s \Delta(u) \tau(u)^{1/2} du \\ &= \int_{-s}^0 \Delta(u) \tau(-u)^{1/2} du + \int_0^s \Delta(u) \tau(u)^{1/2} du \\ &= 2 \int_0^s \Delta(u) \tau(u)^{1/2} du. \end{aligned}$$

Thus

$$r'_\infty + \beta_\infty = 1 - \int_0^s \Delta(u) [1 + \tau(u) - 2\tau(u)^{1/2}] du = 1 - \int_0^s \Delta(u) (1 - \sqrt{\tau(u)})^2 du < 1$$

since by hypothesis the set  $\{u \in [0, s] : \tau(u) \neq 1\}$  has a positive Lebesgue-measure.

Since  $r_{\text{ess}}(P) \leq \max(r_a, r'_a + \beta_a)$  holds for every  $a > 0$ , we obtain that  $r_{\text{ess}}(P) \leq \max(r_\infty, r'_\infty + \beta_\infty) < 1$ . Thus  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$ .  $\square$

**Example 2.1 (Laplace distribution)** Let  $\pi(x) = e^{-|x|}/2$  be the Laplace distribution density, and set  $q(x, y) := \Delta(x - y)$  with  $\Delta(u) := (1 - |u|) 1_{[-1,1]}(u)$ . Then

$$\forall u \in [0, 1], \quad \tau(u) := \lim_{x \rightarrow +\infty} \frac{\pi(x+u)}{\pi(x)} = e^{-u}.$$

Then

$$\gamma_\infty = 1 - \int_0^1 (1-u)(1 - e^{-u/2})^2 du = 8e^{-1/2} - e^{-1} - 7/2.$$

From Corollary 1,  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$  with  $r_{ess}(P) \leq \max(1 - 1/e, 8e^{-1/2} - e^{-1} - 7/2) = 8e^{-1/2} - e^{-1} - 7/2 \approx 0.9843$  since  $r_\infty := \sup_{x \in \mathbb{R}} r(x) \leq 1 - 1/e$ .

**Example 2.2 (Gauss distribution)** Let  $\pi(x) = e^{-x^2/2}/\sqrt{2\pi}$  be the Gauss distribution density, and set  $q(x, y) := \Delta(|x - y|)$  with  $\Delta(u) := (1 - |u|) 1_{[-1,1]}(u)$ . Then

$$\forall u \in (0, 1], \quad \tau(u) := \lim_{x \rightarrow +\infty} \frac{\pi(x+u)}{\pi(x)} = 0,$$

so that

$$\gamma_\infty = 1 - \int_0^1 (1-u) du = \frac{1}{2}.$$

From Corollary 1,  $P$  is quasi-compact on  $\mathbb{L}^2(\pi)$  with  $r_{ess}(P) \leq \max(0.156, 0.5) = 0.5$  since  $r_\infty \leq 1 - e^{-1/2} - e^{1/8} \int_0^1 (1-u)e^{-(u+1)^2/2} du \leq 0.156$ .

In view of the quasi-compactness approach presented in Introduction for computing the rate  $\varrho_2(P)$  in  $(SG_2)$ , the bound  $r_{ess}(P) \leq 0.5$  obtained for Gauss distribution (for instance) implies that, for every  $\kappa \in (0.5, 1)$ , the set of the spectral values  $\lambda$  of  $P$  on  $\mathbb{L}^2(\pi)$  satisfying  $\kappa \leq |\lambda| \leq 1$  is composed of finitely many eigenvalues of finite multiplicity. Moreover, from aperiodicity and irreducibility,  $\lambda = 1$  is the only eigenvalue of  $P$  with modulus one and it is a simple eigenvalue of  $P$ . Consequently the spectral gap property  $(SG_2)$  holds with  $\varrho_2(P)$  given by

- $\varrho_2(P) = \max\{|\lambda|, \lambda \in \mathcal{U}_\kappa, \lambda \neq 1\}$  if  $\mathcal{U}_\kappa \neq \emptyset$ ,
- $\varrho_2(P) \leq \kappa$  if  $\mathcal{U}_\kappa = \emptyset$  (in particular, if for every  $\kappa \in (0.5, 1)$  we have  $\mathcal{U}_\kappa = \emptyset$ , then we could conclude that  $\varrho_2(P) \leq 0.5$ ).

The numerical computation of the eigenvalues  $\lambda \in \mathcal{U}_\kappa$ ,  $\lambda \neq 1$ , is a difficult issue. Even to know whether the set  $\mathcal{U}_\kappa \setminus \{1\}$  is empty or not seems to be difficult. In the discrete state space case (i.e  $P = (P(i, j))_{i, j \in \mathbb{N}}$ ), this problem has been solved by using a weak perturbation method involving some finite truncated matrices derived from  $P$  (see [HL16]). In the continuous state space case, a perturbation method could be also considered, but it raises a priori difficult theoretical and numerical issues.



### 3 Proof of Theorem 1

For any bounded linear operator  $U$  on  $\mathbb{L}^2(\pi)$  we define

$$\forall f \in \mathbb{L}^2(\pi), \quad U_a f := 1_{[-a,a]} \cdot U f \quad \text{and} \quad U_{a^c} f := 1_{\mathbb{R} \setminus [-a,a]} \cdot U f.$$

Obviously  $U_a$  and  $U_{a^c}$  are bounded linear operators on  $\mathbb{L}^2(\pi)$ , and  $U = U_a + U_{a^c}$ . Define  $Rf = rf$  with function  $r(\cdot)$  given in (3). Recall that  $T$  is the operator associated with kernel  $T(x, dy) = t(x, y)dy$ . Then the M-H kernel  $P$  defined in (4) writes as follows:

$$P = R + T = R_a + R_{a^c} + T_a + T_{a^c}$$

with  $R_{a^c}R_a = R_aR_{a^c} = 0$  and  $R_aT_{a^c} = 0$ .

**Lemma 1** *The operators  $T_a, T_{a^c}R_a$  and  $(R_{a^c} + T_{a^c})^n R_a$  for any  $n \geq 1$  are compact on  $\mathbb{L}^2(\pi)$ .*

*Proof.* Using the detailed balance equation (5), we obtain for any  $f \in \mathbb{L}^2(\pi)$

$$\begin{aligned} (T_a f)(x) &= 1_{[-a,a]}(x) \int_{\mathbb{R}} f(y) t(x, y) dy = \int_{\mathbb{R}} f(y) 1_{[-a,a]}(x) \frac{t(x, y)}{\pi(y)} \pi(y) dy \\ &= \int_{\mathbb{R}} f(y) t_a(x, y) \pi(y) dy \quad \text{with} \quad t_a(x, y) := 1_{[-a,a]}(x) \frac{t(y, x)}{\pi(x)}. \end{aligned}$$

Function  $q(\cdot, \cdot)$  is supposed to be bounded on  $\mathbb{R}^2$ , so is  $t(\cdot, \cdot)$ . From  $\inf_{|x| \leq a} \pi(x) > 0$  it follows that  $t_a(\cdot, \cdot)$  is bounded on  $\mathbb{R}^2$ . Consequently  $t_a \in \mathbb{L}^2(\pi \otimes \pi)$ , so that  $T_a$  is a Hilbert-Schmidt operator on  $\mathbb{L}^2(\pi)$ . In particular  $T_a$  is compact on  $\mathbb{L}^2(\pi)$ .

Now observe that

$$(T_{a^c} R_a f)(x) = 1_{\mathbb{R} \setminus [-a,a]}(x) \int_{\mathbb{R}} 1_{[-a,a]}(y) r(y) f(y) t(x, y) dy = \int_{\mathbb{R}} f(y) k_a(x, y) \pi(y) dy$$

where  $k_a(x, y) := 1_{\mathbb{R} \setminus [-a,a]}(x) 1_{[-a,a]}(y) r(y) t(x, y) \pi(y)^{-1}$ . Then  $T_{a^c} R_a$  is a Hilbert-Schmidt operator on  $\mathbb{L}^2(\pi)$  since  $k_a(\cdot, \cdot)$  is bounded on  $\mathbb{R}^2$  from our assumptions. Thus  $T_{a^c} R_a$  is compact on  $\mathbb{L}^2(\pi)$ .

Let us prove by induction that  $(R_{a^c} + T_{a^c})^n R_a$  is compact on  $\mathbb{L}^2(\pi)$  for any  $n \geq 1$ . For  $n = 1$ ,  $(R_{a^c} + T_{a^c})R_a = T_{a^c}R_a$  is compact. Next,

$$(R_{a^c} + T_{a^c})^n R_a = (R_{a^c} + T_{a^c})^{n-1} (R_{a^c} + T_{a^c}) R_a = (R_{a^c} + T_{a^c})^{n-1} T_{a^c} R_a.$$

Since  $T_{a^c} R_a$  is compact on  $\mathbb{L}^2(\pi)$  and the set of compact operators on  $\mathbb{L}^2(\pi)$  is an ideal,  $(R_{a^c} + T_{a^c})^n R_a$  is compact on  $\mathbb{L}^2(\pi)$ .  $\square$

**Lemma 2** *For every  $n \geq 1$ , there exists a compact operator  $K_n$  on  $\mathbb{L}^2(\pi)$  such that*

$$P^n = K_n + R_a^n + (R_{a^c} + T_{a^c})^n.$$

*Proof.* For  $n = 1$  we have  $P = K_1 + R_a + (R_{a^c} + T_{a^c})$  with  $K_1 := T_a$  compact by Lemma 1. Now assume that the conclusion of Lemma 2 holds for some  $n \geq 1$ . Since the set of compact operators on  $\mathbb{L}^2(\pi)$  forms a two-sided operator ideal, we obtain the following equalities for some compact operator  $K'_{n+1}$  on  $\mathbb{L}^2(\pi)$ :

$$\begin{aligned} P^{n+1} = P^n P &= (K_n + R_a^n + (R_{a^c} + T_{a^c})^n)(K_1 + R_a + (R_{a^c} + T_{a^c})) \\ &= K'_{n+1} + R_a^{n+1} + R_a^n R_{a^c} + R_a^n T_{a^c} + (R_{a^c} + T_{a^c})^n R_a + (R_{a^c} + T_{a^c})^{n+1} \\ &= [K'_{n+1} + (R_{a^c} + T_{a^c})^n R_a] + R_a^{n+1} + (R_{a^c} + T_{a^c})^{n+1}. \end{aligned}$$

Then the expected conclusion holds true for  $P^{n+1}$  since  $K_{n+1} := K'_{n+1} + (R_{a^c} + T_{a^c})^n R_a$  is compact on  $\mathbb{L}^2(\pi)$  from Lemma 1.  $\square$

Theorem 1 is deduced from the next proposition which states that  $\|T_{a^c}\|_2 \leq \beta_a$ . Indeed, observe that  $\|R_a\|_2 \leq r_a$  and  $\|R_{a^c}\|_2 \leq r'_a$ . Set  $\alpha_a := \max(r_a, r'_a + \beta_a)$ . Then Lemma 2 and  $\|T_{a^c}\|_2 \leq \beta_a$  give

$$\|P^n - K_n\|_2 \leq \|R_a\|_2^n + (\|R_{a^c}\|_2 + \|T_{a^c}\|_2)^n \leq 2\alpha_a^n.$$

The expected inequality  $r_{ess}(P) \leq \alpha_a$  in Theorem 1 then follows from Formula (6).

**Proposition 1** *For any  $a > 0$ , we have  $\|T_{a^c}\|_2 \leq \beta_a$ .*

*Proof.* Lemma 3 below shows that, for any bounded and continuous function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , we have  $\|T_{a^c} f\|_{\mathbb{L}^2(\pi)} \leq \beta_a \|f\|_{\mathbb{L}^2(\pi)}$ . Then Inequality  $\|T_{a^c}\|_2 \leq \beta_a$  of Proposition 1 follows from a standard density argument using that  $\|T\|_2 \leq \|P\|_2 = 1$  and that the space of bounded and continuous functions from  $\mathbb{R}$  to  $\mathbb{C}$  is dense in  $\mathbb{L}^2(\pi)$ .  $\square$

**Lemma 3** *For any bounded and continuous function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , we have*

$$\|T_{a^c} f\|_{\mathbb{L}^2(\pi)} \leq \int_{-s}^s \left[ \int_{\{|x|>a\}} |f(x+u)|^2 t(x, x+u)^2 \pi(x) dx \right]^{\frac{1}{2}} du \leq \beta_a \|f\|_{\mathbb{L}^2(\pi)}. \quad (12)$$

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a bounded and continuous function. Set  $B := \mathbb{R} \setminus [-a, a]$ . Then it follows from (1) that

$$(T_{a^c} f)(x) = 1_B(x) \int_{\mathbb{R}} f(y) t(x, y) dy = 1_B(x) \int_{-s}^s f(x+u) t(x, x+u) du. \quad (13)$$

For  $n \geq 1$  and for  $k = 0, \dots, n$ , set  $u_k := -s + 2sk/n$  and define the following functions:  $h_k(x) := 1_B(x) f(x+u_k) t(x, x+u_k)$ . Then

$$\begin{aligned} \left[ \int_B \left| \frac{2s}{n} \sum_{k=1}^n h_k(x) \right|^2 \pi(x) dx \right]^{\frac{1}{2}} &= \left\| \frac{2s}{n} \sum_{k=1}^n h_k \right\|_{\mathbb{L}^2(\pi)} \leq \frac{2s}{n} \sum_{k=1}^n \|h_k\|_{\mathbb{L}^2(\pi)} \\ &\leq \frac{2s}{n} \sum_{k=1}^n \left[ \int_B |f(x+u_k)|^2 t(x, x+u_k)^2 \pi(x) dx \right]^{\frac{1}{2}}. \end{aligned} \quad (14)$$

Below we prove that, when  $n \rightarrow +\infty$ , the left hand side of (14) converges to  $\|T_{a^c} f\|_{\mathbb{L}^2(\pi)}$  and that the right hand side of (14) converges to the right hand side of the first inequality in (12). Define

$$\forall x \in B, \quad \chi_n(x) := \frac{2s}{n} \sum_{k=1}^n h_k(x) = \frac{2s}{n} \sum_{k=1}^n f(x + u_k) t(x, x + u_k).$$

From Riemann's integral it follows that

$$\forall x \in B, \quad \lim_{n \rightarrow +\infty} \chi_n(x) = \int_{-s}^s f(x + u) t(x, x + u) du$$

since the function  $u \mapsto f(x + u) t(x, x + u)$  is continuous on  $[-s, s]$  from the assumptions of Theorem 1. Note that  $\sup_n \sup_{x \in B} |\chi_n(x)| < \infty$  since  $f$  and  $t$  are bounded functions. From Lebesgue's theorem and from (13), it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_B \left| \frac{2s}{n} \sum_{k=1}^n h_k(x) \right|^2 \pi(x) dx &= \int_B \left| \int_{-s}^s f(x + u) t(x, x + u) du \right|^2 \pi(x) dx \\ &= \|T_{a^c} f\|_{\mathbb{L}^2(\pi)}^2. \end{aligned} \quad (15)$$

Next, observe that

$$\frac{2s}{n} \sum_{k=1}^n \left[ \int_B |f(x + u_k)|^2 t(x, x + u_k)^2 \pi(x) dx \right]^{\frac{1}{2}} = \frac{2s}{n} \sum_{k=1}^n \psi(u_k)$$

with  $\psi$  defined by

$$\psi(u) := \left[ \int_B |f(x + u)|^2 t(x, x + u)^2 \pi(x) dx \right]^{\frac{1}{2}}.$$

Using the assumptions Theorem 1, it follows from Lebesgue's theorem that  $\psi$  is continuous. Consequently Riemann integral gives

$$\lim_{n \rightarrow +\infty} \frac{2s}{n} \sum_{k=1}^n \left[ \int_B |f(x + u_k)|^2 t(x, x + u_k)^2 \pi(x) dx \right]^{\frac{1}{2}} = \int_{-s}^s \psi(u) du. \quad (16)$$

The first inequality in (12) follows from (14) by using (15) and (16).

Let us prove the second inequality in (12). The detailed balance equation (5) gives

$$\begin{aligned} & \int_{-s}^s \left[ \int_{\{|x|>a\}} |f(x + u)|^2 t(x, x + u)^2 \pi(x) dx \right]^{\frac{1}{2}} du \\ &= \int_{-s}^s \left[ \int_{\{|x|>a\}} |f(x + u)|^2 t(x, x + u) t(x, x + u) \pi(x) dx \right]^{\frac{1}{2}} du \\ &= \int_{-s}^s \left[ \int_{\{|x|>a\}} t(x, x + u) t(x + u, x) |f(x + u)|^2 \pi(x + u) dx \right]^{\frac{1}{2}} du \\ &\leq \|f\|_{\mathbb{L}^2(\pi)} \int_{-s}^s \sup_{|x|>a} \sqrt{t(x, x + u) t(x + u, x)} du = \|f\|_{\mathbb{L}^2(\pi)} \beta_a. \end{aligned}$$

□

## 4 Conclusion

The study of the iterates of a Metropolis-Hasting kernel  $P$  is of great interest to estimate the numbers of iterations required to achieve the convergence in the Metropolis-Hasting algorithm. In conclusion we discuss this issue by comparing the expected results depending on whether  $P$  acts on  $\mathcal{B}_V$  or on  $\mathbb{L}^2(\pi)$ . Recall that the  $V$ -geometrical ergodicity for  $P$  (see Remark 1) writes as: there exist  $\rho \in (0, 1)$  and  $C_\rho > 0$  such that

$$\forall n \geq 1, \forall f \in \mathcal{B}_V, \quad \|P^n f - \pi(f)\|_V \leq C_\rho \rho^n \|f\|_V. \quad (\text{SG}_V)$$

Let  $\varrho_V(P)$  be the infimum bound of the real numbers  $\rho$  such that  $(\text{SG}_V)$  holds true.

1. In most of cases, the number  $\varrho_V(P)$  is not known for Metropolis-Hasting kernels. The upper bounds of  $\varrho_V(P)$  derived from drift and minorization inequalities seem to be poor and difficult to improve, excepted in stochastically monotone case (e.g. see [MT96, Sec. 6] and [Bax05]). Consequently the inequality  $\varrho_2(P) \leq \varrho_V(P)$  (see [Bax05, Th. 6.1]) is not relevant here. Observe that applying the quasi-compactness approach on  $\mathcal{B}_V$  would allow us to estimate the value of  $\varrho_V(P)$ , but in practice this method cannot be efficient since no accurate bound of the essential spectral radius of  $P$  on  $\mathcal{B}_V$  is known.
2. The present paper shows that considering the action on  $\mathbb{L}^2(\pi)$  rather than on  $\mathcal{B}_V$  of a Metropolis-Hasting kernel  $P$  enables us to benefit from the richness of Hilbert spaces. The notion of Hilbert-Schmidt operators plays an important role for obtaining our bound (9). The reversibility of  $P$ , that is  $P$  is self-adjoint on  $\mathbb{L}^2(\pi)$ , implies that any upper bound  $\rho$  of  $\varrho_2(P)$  gives the inequality  $\|P^n f - \Pi f\|_2 \leq \rho^n \|f\|_2$  for every  $n \geq 1$  and every  $f \in \mathbb{L}^2(\pi)$ . Consequently any such  $\rho$  provides an efficient information to estimate the numbers of iterations required to achieve the convergence in the Metropolis-Hasting algorithm.

From our bound (7) it can be expected that the quasi-compactness method (cf. Introduction) will give a numerical procedure for estimating  $\varrho_2(P)$  in the continuous state space case.

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