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ON THE HOMOTOPY FIBRE OF THE INCLUSION MAP $F_n(X) \hookrightarrow \prod_1^n X$ FOR SOME ORBIT SPACES X

MAREK GOLASIŃSKI, DACIBERG LIMA GONÇALVES AND JOHN GUASCHI

ABSTRACT. *Under certain conditions, we describe the homotopy type of the homotopy fibre of the inclusion map $F_n(X) \hookrightarrow \prod_1^n X$ for the n^{th} configuration space $F_n(X)$ of a topological manifold X without boundary such that $\dim(X) \geq 3$. We then apply our results to the cases where either the universal covering of X is contractible or X is an orbit space \mathbb{S}^k/G of a tame, free action of a Lie group G on the k -sphere \mathbb{S}^k . If the group G is finite and k is odd, we give a full description of the long exact sequence in homotopy of the homotopy fibration of the inclusion map $F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$.*

This paper is dedicated to Professor S. Gitler, for his outstanding work in homotopy theory, his readiness to encourage the scientific progress of colleagues and of institutions, his unwavering intellectual honesty, and his warm friendship.

Introduction

Configuration spaces have a long history and continue to be an active topic of research, due to their interesting topological and geometric properties [6, 10]. They arise in various areas in mathematics, such as low-dimensional topology, homotopy theory, dynamical systems and hyperplane arrangements, and play an important rôle in the study of braid groups [4]. If $n \geq 1$, the n^{th} configuration space of a topological space X is defined to be the subspace of the n -fold Cartesian product $\prod_1^n X$ defined by:

$$(0.1) \quad F_n(X) = \left\{ (x_1, \dots, x_n) \in \prod_1^n X \mid x_i \neq x_j \text{ for all } i \neq j \right\}.$$

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The classical notion of configuration space has been generalised in several directions, one of them being the following. Given a free action $G \times X \rightarrow X$ of a group G on X , we define the n^{th} orbit configuration space of X (with respect to G) as follows [7]:

$$(0.2) \quad F_n^G(X) = \left\{ (x_1, \dots, x_n) \in \prod_1^n X \mid Gx_i \cap Gx_j = \emptyset \text{ for all } i \neq j \right\}.$$

Note that if G is the trivial group then $F_n^G(X)$ coincides with the usual configuration space $F_n(X)$.

One general question that is related to the study of $F_n(X)$ is the analysis of the inclusion map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$, and if one is interested in the homotopy groups of these spaces, the homotopy fibre $I_{i_n(X)}$ of this map (up to homotopy equivalence). The study of the homotopy fibre of this inclusion map has been carried out for certain manifolds X . If X is a one-dimensional manifold (the circle \mathbb{S}^1 or the real line \mathbb{R}^1) then by [20], the configuration space $F_n(\mathbb{R}^1)$ is homeomorphic to a disjoint union of $n!$ copies of the open n -simplex, and using [6, Example 2.6], there is a homeomorphism $\mathbb{S}^1 \times F_n(\mathbb{R}^1) \xrightarrow{\cong} F_{n+1}(\mathbb{S}^1)$. One may then describe the homotopy fibre of the inclusion map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$ for $X = \mathbb{R}^1$ or \mathbb{S}^1 .

The study of configuration spaces for surfaces was initiated in [11]. In the case, the question about the homotopy fibre of $i_n(X)$ was considered by Goldberg in [13] when the surface is different from \mathbb{S}^2 and $\mathbb{R}P^2$, in which case the homotopy fibre is aspherical. Recently the cases when X is either \mathbb{S}^2 or $\mathbb{R}P^2$ have been studied in [14, 15], and it was shown that the homotopy fibre of $i_n(X)$ is no longer aspherical.

In this paper, we study the homotopy fibre of the inclusion map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$ for topological manifolds X without boundary and of dimension ≥ 3 . As special cases, if the universal covering of X is contractible or X is an orbit space \mathbb{S}^k/G of a free, tame action of a Lie group G on the k -sphere \mathbb{S}^k then we describe the homotopy type of the homotopy fibre $I_{i_n(X)}$ of the map $i_n(X)$. In particular if $X = \mathbb{S}^k/G$, the group G is finite and $k \geq 3$ is odd, we give a full description of the long exact sequence in homotopy of the homotopy fibration of the inclusion map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$.

Let X and Y be topological spaces. We write $X \simeq Y$ if X and Y have the same homotopy type, and if $f, g : X \rightarrow Y$ are maps, we write $f \sim g$ if they are homotopic. The *connectivity* of X , denoted by $\text{conn}(X)$, is defined to be the largest non-negative integer n (or infinity) for which $\pi_i(X) = 0$ for all $i \leq n$. In Section 1, we recall and prove various results about pull-backs with a homotopical approach to fibrations that will be used in the rest of the paper. It is well known that a homotopy-commutative

diagram of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

gives rise to a map $\mathcal{I}(g, g') : I_f \longrightarrow I_{f'}$ between the homotopy fibres of f and f' that is compatible with this diagram (see Remark 1.2 for the precise definition of this map and some of its properties). Proposition 1.5 highlights various homotopy fibrations and homotopy equivalences that arise from such homotopy-commutative diagrams, and is main result of Section 1.

In Section 2, X will mostly be a topological manifold without boundary such that $\dim(X) \geq 3$, and we shall frequently assume that there exists an action of a topological group G on X . At certain points, we will suppose additionally that the action is tame [8], so that the quotient space X/G is also a topological manifold. In this case, if $Q_1 \in X$, the map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$ and the quotient map $q(X) : X \longrightarrow X/G$ induce a map $\psi_n(X) : F_n^G(X) \longrightarrow F_n(X/G)$ between the n^{th} orbit configuration space of X and the n^{th} configuration space of X/G , an inclusion $i_n''(X) : F_n^G(X \setminus GQ_1) \hookrightarrow \prod_1^n X$, and an inclusion $i_n''(X) : F_n^G((X/G) \setminus \bar{Q}_1) \hookrightarrow \prod_1^n X/G$, where $\bar{Q}_1 = q(X)(Q_1)$. These maps are defined at the beginning of Section 2. We also define maps $j_n(X) : \prod_1^{n-1} X \longrightarrow \prod_1^n X$ and $j_n'(X) : F_{n-1}(X \setminus Q_1) \longrightarrow F_n(X)$ that ‘insert’ the point Q_1 in a given position (see equation (2.4)).

In the first part of Section 2, we shall study these maps, the aim being to relate their homotopy fibres to that of the map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$, and to obtain (weak) homotopy equivalences between them. In Theorem 2.2, we prove a result concerning the connectivity of the map $i_n(X)$ that extends a result of Birman from the smooth category to the topological category [3, Theorem 1]. The following theorem, which is the main result of Section 2, describes certain properties of the long exact sequence in homotopy of the fibration associated with the inclusion map $F_n(X) \hookrightarrow \prod_1^n X$.

Theorem 2.5. *Let $G \times X \longrightarrow X$ be a free, tame action of a Lie group G on a connected topological manifold X without boundary, and let $Q_1 \in X$. Suppose that the inclusion map $i_{n-1}''(X) : F_{n-1}^G(X \setminus GQ_1) \hookrightarrow \prod_1^{n-1} X$ is null homotopic (this is the case if for example the inclusion map $X \setminus GQ_1 \hookrightarrow X$ is null homotopic).*

- (1) *The maps $\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X}) : I_{i_{n-1}''(X)} \longrightarrow F_{n-1}^G(X \setminus GQ_1) \times \Omega(\prod_1^{n-1} X)$ and $\mathcal{I}(j_n'(X/G), j_n(X/G)) : I_{i_{n-1}'(X/G)} \longrightarrow I_{i_n(X/G)}$ are homotopy equivalences, and the map $\mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) : I_{i_{n-1}''(X)} \longrightarrow I_{i_{n-1}'(X/G)}$ is a weak homotopy equivalence. Furthermore, if G is discrete, X is 1-connected and*

$\dim(X/G) \geq 3$ then the map $\psi_{n-1}(X \setminus GQ_1) : F_{n-1}^G(X \setminus GQ_1) \longrightarrow F_{n-1}((X/G) \setminus \bar{Q}_1)$ is the universal covering.

(2) If $j \leq \min(\text{conn}(X), \dim(X/G) - 2)$ then $\pi_j(F_{n-1}^G(X \setminus GQ_1)) = 0$.

(3) Suppose that the homomorphism $\pi_j(X) \longrightarrow \pi_j(X/G)$ induced by the quotient map $X \longrightarrow X/G$ is injective for all $j \geq 1$ (this is the case if for example the inclusion map $GQ_1 \hookrightarrow X$ is null homotopic). For all $j \geq 2$, up to the identification of the groups $\pi_{j-1}(I_{in}(X/G))$ and $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ via the isomorphism

$$\pi_{j-1}(\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})) \circ (\pi_{j-1}(k_n(X)))^{-1},$$

the restriction to the subgroup $\pi_j(\prod_1^{n-1} X)$ of $\pi_j(\prod_1^n X/G)$ of the boundary homomorphism $\hat{\partial}_j : \pi_{j-1}(\Omega(\prod_1^n X/G)) \longrightarrow \pi_{j-1}(I_{in}(X/G))$ given by the long exact

sequence in homotopy of the homotopy fibration $I_{in}(X/G) \xrightarrow{p_{in}(X/G)} F_n(X/G) \xrightarrow{i_n(X/G)} \prod_1^n X/G$ of (2.7) coincides with the inclusion of $\pi_j(\prod_1^{n-1} X)$ in $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ via the usual identification of $\pi_j(X)$ with $\pi_{j-1}(\Omega(X))$.

The map $k_n(X) : I_{n-1}''(X) \longrightarrow I_{in}(X/G)$ mentioned in Theorem 2.5(3) is a weak homotopy equivalence that is defined just after the commutative diagram (2.7). By applying Theorem 2.5 to manifolds whose universal covering is contractible, in Proposition 2.7 we show that the homotopy fibre $I_{in}(X/G)$ is weakly homotopy equivalent to a number of different spaces that arise in the construction of the above-mentioned maps.

In Section 3, we apply the results of Section 2 to orbit manifolds of the form \mathbb{S}^k/G , G being a finite group that acts freely and tamely on the k -sphere \mathbb{S}^k , where $k \geq 3$. One of the two principal results of Section 3 is the following corollary that describes the homotopy type of the homotopy fibre of the inclusion $i_{n-1}''(\mathbb{S}^k) : F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \hookrightarrow \prod_1^{n-1} \mathbb{S}^k$.

Corollary 3.3. *Let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free, tame action of a compact Lie group G on \mathbb{S}^k , and let $Q_1 \in \mathbb{S}^k$.*

- (1) *There is a homotopy equivalence $I_{n-1}''(\mathbb{S}^k) \simeq F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \times \widetilde{\Omega(\prod_1^{n-1} \mathbb{S}^k)}$. If the group G is finite and $\dim(\mathbb{S}^k/G) \geq 3$ then $F_{n-1}(\mathbb{S}^k \setminus Q_1) \simeq F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)$.*
- (2) *If $j \leq k - \dim(G) - 2$ then $\pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)) = 0$.*

The second main result of Section 3 is the following proposition that gives a more detailed description of the long exact sequence in homotopy of the fibration associated with the inclusion map $i_n(\mathbb{S}^k/G) : F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$.

Proposition 3.6. *Let $k \geq 3$ be odd, let $j \geq 2$, and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of a finite group G on \mathbb{S}^k . Let $\Delta_j^n(\mathbb{S}^k/G)$ denote the diagonal subgroup of $\prod_1^n \pi_j(\mathbb{S}^k/G)$. Then:*

(1) the image of the homomorphism $\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \prod_1^n \pi_j(\mathbb{S}^k/G)$ is the diagonal subgroup $\Delta_j^n(\mathbb{S}^k/G)$.

(2) there are split short exact sequences of the form:

$$0 \longrightarrow \Delta_j^n(\mathbb{S}^k/G) \longrightarrow \pi_j\left(\prod_1^n \mathbb{S}^k/G\right) \longrightarrow \pi_{j-1}\left(\prod_1^{n-1} \Omega(\mathbb{S}^k/G)\right) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)) \longrightarrow \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \Delta_j^n(\mathbb{S}^k/G) \longrightarrow 0.$$

In particular, if G is the trivial group then there is a split short exact sequence:

$$0 \longrightarrow \pi_j(F_{n-1}(\mathbb{R}^k)) \longrightarrow \pi_j(F_n(\mathbb{S}^k)) \longrightarrow \Delta_j^n(\mathbb{S}^k) \longrightarrow 0.$$

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1. Preliminaries

In this section, we first prove a lemma concerning the pull-back of a covering map. We then go on to prove some results about homotopy fibres of homotopy commutative diagrams that will be useful in what follows.

Lemma 1.1. *Let (X, x_0) and (Y, y_0) be pointed topological spaces. Given a map $f : (X, x_0) \rightarrow (Y, y_0)$ and a covering map $p : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ for a path-connected space \tilde{Y} , consider the associated pull-back square:*

$$(1.1) \quad \begin{array}{ccc} X \times_Y \tilde{Y} & \xrightarrow{f'} & \tilde{Y} \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y, \end{array}$$

where p' and f' are the projections of $X \times_Y \tilde{Y}$ onto the first and second factors, respectively. Then $p' : X \times_Y \tilde{Y} \rightarrow X$ is a covering map, and $\pi_1(p')(\pi_1(X \times_Y \tilde{Y})) = \pi_1(f)^{-1}(\pi_1(p)\pi_1(\tilde{Y}))$. If further X is path connected and $p : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ is the universal covering map then the number $\#\pi_0(X \times_Y \tilde{Y})$ is equal to the index $[\pi_1(Y) : \pi_1(f)(\pi_1(X))]$ of the image $\pi_1(f)(\pi_1(X))$ in the group $\pi_1(Y)$. If in addition $\pi_1(f) : \pi_1(X) \rightarrow \pi_1(Y)$ is an epimorphism then $p' : X \times_Y \tilde{Y} \rightarrow X$ is also the universal covering map.

Proof. The fact that the induced map $p' : X \times_Y \tilde{Y} \rightarrow X$ is a covering is well known. Then the proof of [18, Chapter V, Proposition 11.1] leads to $\pi_1(p')(\pi_1(X \times_Y \tilde{Y})) = \pi_1(f)^{-1}(\pi_1(p)\pi_1(\tilde{Y}))$.

Now suppose that X is path connected and that $p : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ is the universal covering map. We shall show that the set $\pi_0(X \times_Y \tilde{Y})$ of path-connected components of $X \times_Y \tilde{Y}$ is in bijection with the set of right $\pi_1(f)(\pi_1(X))$ -cosets in $\pi_1(Y)$. First, notice that the restriction of the map $f' : X \times_Y \tilde{Y} \rightarrow \tilde{Y}$ to $p'^{-1}(x_0)$ induces a bijection $\varphi : p'^{-1}(x_0) \rightarrow p^{-1}(y_0)$. In light of the canonical actions $\pi_1(X) \times p'^{-1}(x_0) \rightarrow p'^{-1}(x_0)$ and $\pi_1(Y) \times p^{-1}(y_0) \rightarrow p^{-1}(y_0)$, we have:

$$p'^{-1}(x_0) = \bigsqcup_{i \in \pi_0(X \times_Y \tilde{Y})} \pi_1(X)(x_0, \tilde{y}_i) \text{ and } p^{-1}(y_0) = \pi_1(Y)y_0.$$

Since the fibre $p'^{-1}(x_0)$ consists of the points of the form (x_0, \tilde{y}) , where $\tilde{y} \in p^{-1}(y_0)$, for any $(x_0, \tilde{y}) \in p'^{-1}(x_0)$ there exists $\alpha_{\tilde{y}} \in \pi_1(Y)$ such that $\tilde{y} = \alpha_{\tilde{y}}y_0$. Consequently, $\varphi(\pi_1(X)(x_0, \tilde{y}_i)) = ((\pi_1(f)(\pi_1(X)))\alpha_{\tilde{y}_i})y_0$ for $i \in \pi_0(X \times_Y \tilde{Y})$ which shows that two elements (x_0, \tilde{y}') , $(x_0, \tilde{y}'') \in p'^{-1}(x_0)$ belong to the same path component of $X \times_Y \tilde{Y}$ if and only if $(\pi_1(f)(\pi_1(X)))\alpha_{\tilde{y}'} = (\pi_1(f)(\pi_1(X)))\alpha_{\tilde{y}''}$.

To prove the last part of the statement, by the previous parts, $X \times_Y \tilde{Y}$ is a connected covering space of X , and $\pi_1(p')(\pi_1(X \times_Y \tilde{Y}))$ is trivial, so $\pi_1(X \times_Y \tilde{Y})$ is trivial because p' is a covering map. It follows that $X \times_Y \tilde{Y}$ is the universal covering space of X as required. \square

Let

$$E_f = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x)\} \text{ and } I_f = \{(x, \gamma) \in E_f \mid \gamma(1) = y_0\}$$

denote the *mapping path* and *homotopy fibre* of f , respectively. Notice that I_f is determined by the homotopy pull-back diagram,

$$(1.2) \quad \begin{array}{ccc} I_f & \xrightarrow{p_f} & X \\ \downarrow q & & \downarrow f \\ * & \xrightarrow{c_{y_0}} & Y, \end{array}$$

where $c_{y_0} : * \rightarrow Y$ is the constant map determined by a point $y_0 \in Y$, $q : I_f \rightarrow *$ is the constant map, and $p_f : I_f \rightarrow X$ is the projection map. It is well known that $I_f \hookrightarrow E_f \rightarrow Y$ is a fibration and that E_f and X have the same homotopy type [2, Proposition 3.5.8 and Remark 3.5.9]. We will refer to the sequence of maps $I_f \xrightarrow{p_f} X \xrightarrow{f} Y$ as a *homotopy fibration*. By [2, Proposition 3.3.17], the homotopy fibres of two homotopic maps have the same homotopy type.

Recall that a map $f : X \rightarrow Y$ of pointed spaces is said to be *n-connected* if the induced homomorphism $\pi_j(f) : \pi_j(X) \rightarrow \pi_j(Y)$ is surjective if $j = n$ and is an isomorphism for all $j \leq n - 1$. Note that the map f is *n-connected* if and only if its homotopy fibre I_f is $(n - 1)$ -connected, in other words, if $\pi_i(I_f) = 0$ for all $0 \leq i \leq n - 1$.

REMARK 1.2.

(a) Suppose that the square

$$(1.3) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

is homotopy-commutative and that $g'f(x_0) = f'g(x_0)$ for some $x_0 \in X$. Let $H : X \times I \rightarrow Y'$ be a homotopy from $g' \circ f$ to $f' \circ g$ such that $H(x_0, t) = g'f(x_0) = f'g(x_0)$ for all $t \in I$, and consider the associated map $\varphi : X \rightarrow (Y')^I$ is given by $\varphi(x)(t) = H(x, t)$ for all $x \in X$ and $t \in I$. In particular, $\varphi(x)(0) = g' \circ f(x)$

and $\varphi(x)(1) = f' \circ g(x)$. Then there is a homotopy-commutative diagram:

$$(1.4) \quad \begin{array}{ccccc} X & & & & \\ & \searrow h & & \nearrow g & \\ & & P & \xrightarrow{\bar{g}} & X' \\ & \searrow f & \downarrow \bar{f} & & \downarrow f' \\ & & Y & \xrightarrow{g'} & Y', \end{array}$$

where

$$P = \{(y, \gamma, x') \in Y \times (Y')^I \times X' \mid \gamma(0) = g'(y) \text{ and } \gamma(1) = f'(x')\}$$

is the standard homotopy pull-back P , the maps \bar{f} and \bar{g} are defined by projection onto the first and third factors respectively, and $h : X \rightarrow P$ is the corresponding whisker map given by $h(x) = (f(x), \varphi(x), g(x))$ for all $x \in X$. If $y_0 = f(x_0)$ and $y'_0 = g'f(x_0)$ then:

$$(1.5) \quad \begin{array}{ccc} I_f & \xrightarrow{p_f} & X \\ \downarrow q & & \downarrow f \\ * & \xrightarrow{c_{f(x_0)}} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} I_{f'} & \xrightarrow{p_{f'}} & X' \\ \downarrow q' & & \downarrow f' \\ * & \xrightarrow{c_{f'g(x_0)}} & Y' \end{array}$$

are the standard homotopy pull-backs. The map $H' : I_f \times I \rightarrow Y$ defined by $H'((x, \gamma), t) = \gamma(t)$ for all $(x, \gamma) \in I_f$ and $t \in I$ is a homotopy from $f p_f$ to $c_{f(x_0)} q$, and the associated path $\varphi' : I_f \rightarrow Y^I$ is given by $\varphi'(x, \gamma)(t) = \gamma(t)$. The square

$$(1.6) \quad \begin{array}{ccc} I_f & \xrightarrow{g p_f} & X' \\ \downarrow q & & \downarrow f' \\ * & \xrightarrow{c_{f'g(x_0)}} & Y' \end{array}$$

obtained by concatenating the square (1.3) and the left-hand square of (1.5) is homotopy commutative, and a homotopy $G : I_f \times I \rightarrow Y'$ is determined by the concatenation of the homotopies $\bar{H} \circ (p_f \times \text{id}_I) : I_f \times I \rightarrow Y'$ and $g' \circ H' : I_f \times I \rightarrow Y'$, where $\bar{H}(x, t) = H(x, 1 - t)$ for all $x \in X$ and $t \in I$. The map $\psi : I_f \rightarrow (Y')^I$ associated to G is given by $\psi(x, \gamma) = \varphi(x)^{-1} * (g' \circ \gamma)$ for all $(x, \gamma) \in I_f$, and the whisker map $\mathcal{I}(g, g') : I_f \rightarrow I_{f'}$ corresponding to the right-hand homotopy pull-back of (1.5) and the square (1.6) and defined by $\mathcal{I}(g, g')(x, \gamma) = (g(x), \varphi(x)^{-1} * (g' \circ \gamma))$ for all $(x, \gamma) \in I_f$ makes the following

diagram

$$\begin{array}{ccccc}
I_f & \xrightarrow{p_f} & X & \xrightarrow{f} & Y \\
\downarrow \mathcal{I}(g, g') & & \downarrow g & & \downarrow g' \\
I_{f'} & \xrightarrow{p_{f'}} & X' & \xrightarrow{f'} & Y'
\end{array}$$

homotopy-commutative. Note that the construction of $\mathcal{I}(g, g')$ corresponds to that given in [19, pp. 225-226].

- (b) With the notation of part (a), if the maps $g : X \rightarrow X'$ and $g' : Y \rightarrow Y'$ are homotopy equivalences then by applying [19, Lemma 45] to the following homotopy-commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xleftarrow{\quad} & * \\
\downarrow g & & \downarrow g' & & \parallel \\
X' & \xrightarrow{f'} & Y' & \xleftarrow{\quad} & *,
\end{array}$$

we see that the map $\mathcal{I}(g, g') : I_f \rightarrow I_{f'}$ is also a homotopy equivalence. In particular, if $g : X \rightarrow X'$ (resp. $g' : Y \rightarrow Y'$) is a homotopy equivalence, it follows from the commutative diagrams

$$\begin{array}{ccc}
\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \parallel & & \downarrow g' \\ X & \xrightarrow{g'f} & Y' \end{array} & \text{and} & \begin{array}{ccc} X & \xrightarrow{f'g} & Y' \\ \downarrow g & & \parallel \\ X' & \xrightarrow{f'} & Y' \end{array}
\end{array}$$

that the map $\mathcal{I}(\text{id}_X, g') : I_f \rightarrow I_{g'f}$ (resp. $\mathcal{I}(g, \text{id}_Y) : I_{f'g} \rightarrow I_{f'}$) is a homotopy equivalence.

- (c) If diagram (1.3) is commutative and not just homotopy commutative then we will always take H to be constant *i.e.* $H(x, t) = f' \circ g(x)$ for all $(x, t) \in X \times I$, and in this case, $\mathcal{I}(g, g')(x, \gamma) = (g(x), g' \circ \gamma)$, from which it follows that $g \circ p_f = p_{f'} \circ \mathcal{I}(g, g')$. Further, $\mathcal{I}(g, g')$ is the restriction to the homotopy fibres I_f and $I_{f'}$ of the map $X \times Y^I \rightarrow X' \times (Y')^I$ that sends $(x, \gamma) \in X \times Y^I$ to $(g(x), g' \circ \gamma) \in X' \times (Y')^I$.
- (d) By [2, Lemma 6.4.11], a map $f : X \rightarrow Y$ is n -connected if and only if $\text{conn}(I_f) \geq n - 1$.
- (e) Let $f : X \rightarrow Y$ be a pointed map, and consider the associated homotopy fibration $I_f \xrightarrow{p_f} X \xrightarrow{f} Y$. The boundary map $d : \Omega(Y) \rightarrow I_f$ is defined by $d(\omega) = (x_0, \omega)$ for all $\omega \in \Omega(Y)$, where $\Omega(Y)$ denotes the loop space of Y . Suppose further that f is null homotopic. Setting $g = \text{id}_X$, $g' = \text{id}_Y$ and $f' = c_{y_0}$ in the construction of part (a), and taking the homotopy H to be such that $H(x_0, t) = y_0$ for all $t \in I$, we have $\varphi(x_0) = c_{y_0}$, and the map $\mathcal{I}(\text{id}_X, \text{id}_Y) : I_f \rightarrow$

$X \times \Omega(Y)$ is a homotopy equivalence defined by $\mathcal{I}(\text{id}_X, \text{id}_Y)(x, \gamma) = (x, \varphi(x)^{-1} * \gamma)$ for all $(x, \gamma) \in I_f$. Thus $\mathcal{I}(\text{id}_X, \text{id}_Y) \circ d : \Omega(Y) \rightarrow X \times \Omega(Y)$ is given by $\mathcal{I}(\text{id}_X, \text{id}_Y) \circ d(\omega) = (x_0, \omega)$ for $\omega \in \Omega(Y)$, and so coincides with inclusion of $\Omega(Y)$. This yields a homotopy equivalence:

$$(1.7) \quad I_f \xrightarrow{\simeq} X \times \Omega(Y).$$

If $F \hookrightarrow X \rightarrow Y$ is a fibration then it is well known that the homotopy fibre of the inclusion map $F \hookrightarrow X$ has the homotopy type of $\Omega(Y)$. Consequently, if the inclusion map $F \hookrightarrow X$ is null homotopic then by (1.7) there is a homotopy equivalence:

$$(1.8) \quad \Omega(Y) \xrightarrow{\simeq} F \times \Omega(X).$$

Lemma 1.3 ([15, Appendix]). *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be pointed maps, let $\mathcal{I}(f, \text{id}_Z) : I_{gf} \rightarrow I_g$ be the map defined in Remark 1.2(c), and let $p_{gf} : I_{gf} \rightarrow X$ and $p_g : I_g \rightarrow Y$ denote the respective projections. Then the induced map $\mathcal{I}(p_{gf}, p_g) : I_{\mathcal{I}(f, \text{id}_Z)} \rightarrow I_f$ is a homotopy equivalence.*

The statement of Lemma 1.3 may be summed up by the three rightmost columns of the following commutative diagram:

$$(1.9) \quad \begin{array}{ccccccc} I(f') & \xrightarrow[\simeq]{\mathcal{I}(\iota_{gf}, \iota_g)} & I_{\mathcal{I}(f, \text{id}_Z)} & \xrightarrow[\simeq]{\mathcal{I}(p_{gf}, p_g)} & I_f & & \\ \vdots \downarrow & & \vdots \downarrow & & \vdots \downarrow & & \\ F_{gf} & \xrightarrow[\simeq]{\iota_{gf}} & I_{gf} & \xrightarrow{\dots} & X & \xrightarrow{gf} & Z \\ \downarrow f' & & \downarrow \mathcal{I}(f, \text{id}_Z) & & \downarrow f & & \parallel \\ F_g & \xrightarrow[\simeq]{\iota_g} & I_g & \xrightarrow{\dots} & Y & \xrightarrow{g} & Z, \end{array}$$

where the dotted arrows indicate the corresponding homotopy fibres.

Let $f : X \rightarrow Y$ be a fibration, let $F_f = f^{-1}(y_0)$ be the topological fibre over a point $y_0 \in Y$, and let I_f be the homotopy fibre of f . Then, by [2, Proposition 3.5.10], the injective map $\iota_f : F_f \hookrightarrow I_f$ defined by $\iota_f(x) = (x, c_{y_0})$ is a homotopy equivalence. Let $j_f : F_f \hookrightarrow X$ denote the inclusion of the topological fibre in the total space X . Note that $j_f = p_f \circ \iota_f$. We obtain the following corollary (summarised in diagram (1.9)) by combining this with Remark 1.2(b) and Lemma 1.3.

Corollary 1.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be pointed maps, and suppose that $X \xrightarrow{gf} Z$ and $Y \xrightarrow{g} Z$ are fibrations with respective topological fibres F_{gf} and F_g . If $f' : F_{gf} \rightarrow F_g$ is the restriction of the map $f : X \rightarrow Y$ to F_{gf} and F_g then the map $\mathcal{I}(j_{gf}, j_g) : I_{f'} \rightarrow I_f$ is a homotopy equivalence.*

As a consequence of Lemma 1.3, we have the following result (cf. [16, Chapter IX]).

Proposition 1.5. *Given the homotopy-commutative square (1.3), consider the associated homotopy-commutative diagram (1.4). Then there exist fibrations $\hat{I}_h \rightarrow \hat{I}_f \rightarrow \hat{I}_{f'}$ and $\hat{I}_h \rightarrow \hat{I}_g \rightarrow \hat{I}_{g'}$, where for each $k \in \{f, g, f', g', h\}$, \hat{I}_k is a space that has the same homotopy type as the homotopy fibre I_k of the map k . Further, the homotopy fibres $I_{\mathcal{I}(g, g')}$, $I_{\mathcal{I}(f, f')}$ and I_h have the same homotopy type.*

Proof. We use the notation of Remark 1.2(a). Since $f = \bar{f}h$, applying Lemma 1.3 to the composition $X \xrightarrow{h} P \xrightarrow{\bar{f}} Y$, we obtain a fibration $\hat{I}_h \rightarrow \hat{I}_f \rightarrow I_{\bar{f}}$, where $\hat{I}_h = I_{\mathcal{I}(h, \text{id}_Y)} \simeq I_h$ and $\hat{I}_f = E_{\mathcal{I}(h, \text{id}_Y)} \simeq I_{\bar{f}h} = I_f$, which proves the first part of the statement for the first fibration.

We now prove that the homotopy fibres $I_{\mathcal{I}(g, g')}$ and I_h have the same homotopy type. Given a point $y_0 \in Y$, the two squares of the following diagram:

$$\begin{array}{ccccc} I_{\bar{f}} & \xrightarrow{p_{\bar{f}}} & P & \xrightarrow{\bar{g}} & X' \\ \downarrow & & \downarrow \bar{f} & & \downarrow f' \\ * & \xrightarrow{c_{y_0}} & Y & \xrightarrow{g'} & Y' \end{array}$$

are homotopy pull-backs, and thus the whole rectangle is a homotopy pull-back [19, Lemma 12]. But the square

$$\begin{array}{ccc} I_{f'} & \xrightarrow{p_{f'}} & X' \\ \downarrow & & \downarrow f' \\ * & \xrightarrow{c_{g'(y_0)}} & Y' \end{array}$$

is the standard homotopy pull-back, and $\mathcal{I}(\bar{g}, g') : I_{\bar{f}} \rightarrow I_{f'}$ is a whisker map by Remark 1.2(a). It follows from [19, p. 226] that $\mathcal{I}(\bar{g}, g')$ is a homotopy equivalence. Now $\mathcal{I}(g, g') = \mathcal{I}(\bar{g}h, g') = \mathcal{I}(\bar{g}, g') \circ \mathcal{I}(h, \text{id}_Y)$, and using the fact that $\mathcal{I}(\bar{g}, g')$ is a homotopy equivalence, it follows from the last part of Remark 1.2(b) that $\mathcal{I}(\text{id}_{I_f}, \mathcal{I}(\bar{g}, g')) : I_{\mathcal{I}(h, \text{id}_Y)} \rightarrow I_{\mathcal{I}(g, g')}$ is a homotopy equivalence. Further, $I_{\mathcal{I}(h, \text{id}_Y)} \simeq I_h$ by Lemma 1.3, from which we conclude that $I_{\mathcal{I}(g, g')} \simeq I_h$. The remaining assertions of the proposition follow by exchanging the rôles of f, \bar{f} and f' with those of g, \bar{g} and g' respectively. \square

To prove the final result of this section, we require the following lemma.

Lemma 1.6. *Let $f : X \rightarrow Y$ be a pointed map.*

(1) *If f is a homotopy equivalence then the homotopy fibre I_f of f is contractible.*

(2) If I_f is contractible then the induced map $\Omega f : \Omega X \longrightarrow \Omega Y$ is a homotopy equivalence. In particular, if the induced map $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ is surjective then $f : X \longrightarrow Y$ is a weak homotopy equivalence.

The first part of the lemma follows by applying [19, Lemma 45] to the homotopy pull-back diagram (1.2). The second part may be obtained by noting that if Z is a pointed space then the Puppe fibre sequence gives rise to a bijection $(\Omega f)_* : [Z, \Omega X] \longrightarrow [Z, \Omega Y]$, which implies that $\Omega f : \Omega X \longrightarrow \Omega Y$ is a homotopy equivalence.

REMARK 1.7. Let $f : X \longrightarrow Y$ be a pointed map of path-connected spaces such that $\Omega f : \Omega X \longrightarrow \Omega Y$ is a homotopy equivalence. Then f is a weak homotopy equivalence, but it is not necessarily a homotopy equivalence. Indeed, the loop space of the Warsaw circle is contractible, but the Warsaw circle is not. Nevertheless, under certain hypotheses, f is a homotopy equivalence. For example, if $f : X \longrightarrow Y$ is a pointed map of path-connected Dold spaces (the class of such spaces contains CW -spaces) for which $\Omega f : \Omega X \longrightarrow \Omega Y$ is a homotopy equivalence, Allaud [1] proved that f is a homotopy equivalence.

Combining diagram (1.9) with Lemma 1.6, we obtain the following result.

Corollary 1.8. *Given a fibration $f : X \longrightarrow Y$ and a map $g : Y' \longrightarrow Y$, consider the following strict pull-back diagram*

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then the induced map $\Omega \mathcal{I}(f', f) : \Omega I_{g'} \longrightarrow \Omega I_g$ is a homotopy equivalence. Further, if the map $g : Y' \longrightarrow Y$ is 1-connected then the map $\mathcal{I}(f', f) : I_{g'} \longrightarrow I_g$ is a weak homotopy equivalence.

Proof. Let $y'_0 \in Y'$, and set $y_0 = g(y'_0) \in Y$. Since $f : X \longrightarrow Y$ is a fibration, the fact that the above diagram is a strict pull-back implies that $f' : Y' \times_Y X \longrightarrow Y'$ is also a fibration. Further, the restriction of g' to the topological fibres $F_{f'} = f'^{-1}(y'_0)$ and $F_f = f^{-1}(y_0)$ induces a homeomorphism $F_{f'} \xrightarrow{\cong} F_f$. Since the injective maps $\iota_f : F_f \hookrightarrow I_f$ and $\iota_{f'} : F_{f'} \hookrightarrow I_{f'}$ (as defined just before the statement of Corollary 1.4) are homotopy equivalences, the map $\mathcal{I}(g', g) : I_{f'} \longrightarrow I_f$ is a homotopy equivalence, and so Lemma 1.6(1) implies that the homotopy fibre $I_{\mathcal{I}(g', g)}$ is contractible. So by Proposition 1.5, the homotopy fibre $I_{\mathcal{I}(f', f)}$ of the map $\mathcal{I}(f', f) : I_{g'} \longrightarrow I_g$ is also contractible, and we conclude from Lemma 1.6(2) that the induced map $\Omega \mathcal{I}(f', f) : \Omega I_{g'} \longrightarrow \Omega I_g$ is a homotopy equivalence, which proves the first part

of the statement. If further $g : Y' \rightarrow Y$ is 1-connected then $g' : Y' \times_Y X \rightarrow X$ is also 1-connected, and so by Remark 1.2(d), the spaces I_g and $I_{g'}$ are path connected. Lemma 1.6(2) implies that the map $\mathcal{I}(f', f) : I_{g'} \rightarrow I_g$ is a weak homotopy equivalence as required. \square

2. Configuration spaces and the inclusion map $F_n(X) \hookrightarrow \prod_1^n X$

Let X be a topological space, and let $n \geq 1$. Given a free action $G \times X \rightarrow X$ of a group G on X , recall from (0.1) (resp. from (0.2)) that $F_n(X)$ is the n^{th} configuration space (resp. $F_n^G(X)$ is the orbit configuration space) of X . Let $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$ denote the inclusion map. We assume from now on that G is a topological group. Let $q(X) : X \rightarrow X/G$ denote the associated quotient map, and let $\bar{x} = q(X)(x)$ for all $x \in X$. Then $q(X)$ induces a map $\psi_n(X) : F_n^G(X) \rightarrow F_n(X/G)$ given by $\psi_n(X)(x_1, \dots, x_n) = (\bar{x}_1, \dots, \bar{x}_n)$. This map is well defined, since if $(x_1, \dots, x_n) \in F_n^G(X)$ then for all $i \neq j$, $Gx_i \cap Gx_j = \emptyset$, so $\bar{x}_i \neq \bar{x}_j$. We thus obtain the following commutative diagram:

$$(2.1) \quad \begin{array}{ccc} F_n^G(X) & \xhookrightarrow{i_n(X)|_{F_n^G(X)}} & \prod_1^n X \\ \psi_n(X) \downarrow & & \downarrow \prod_1^n q(X) \\ F_n(X/G) & \xhookrightarrow{i_n(X/G)} & \prod_1^n X/G. \end{array}$$

Further, the associated (strict) pull-back $F_n(X/G) \times_{\prod_1^n X/G} \prod_1^n X$ is given by:

$$\left\{ ((\bar{x}_1, \dots, \bar{x}_n), (y_1, \dots, y_n)) \in F_n(X/G) \times \prod_1^n X \mid \bar{x}_i = \bar{y}_i \text{ for all } 1 \leq i \leq n \right\}.$$

One may then check that the map $F_n(X/G) \times_{\prod_1^n X/G} \prod_1^n X \rightarrow F_n^G(X)$ given by $((\bar{x}_1, \dots, \bar{x}_n), (x_1, \dots, x_n)) \mapsto (x_1, \dots, x_n)$ is a (well-defined) homeomorphism. In particular, (2.1) is a (strict) pull-back diagram.

Now suppose that X is a topological manifold, let $Q_r = \{q_1, \dots, q_r\}$ be a finite, non-empty subset of X whose elements belong to distinct orbits, and let $\bar{Q}_r = q(X)(Q_r)$. Let $i'_n(X) : F_n(X \setminus Q_r) \rightarrow \prod_1^n X$ (resp. $i''_n(X) : F_n^G(X \setminus GQ_r) \rightarrow \prod_1^n X$) denote the composition of $i_n(X \setminus Q_r)$ (resp. of $i_n(X \setminus Q_r)|_{F_n^G(X \setminus GQ_r)}$) with the inclusion map $\prod_1^n X \setminus Q_r \hookrightarrow \prod_1^n X$. Since the diagram

$$\begin{array}{ccc} \prod_1^n (X \setminus GQ_r) & \xhookrightarrow{\quad} & \prod_1^n X \\ q(X \setminus GQ_r) \downarrow & & \downarrow \prod_1^n q(X) \\ \prod_1^n ((X/G) \setminus \bar{Q}_r) & \xhookrightarrow{\quad} & \prod_1^n X/G \end{array}$$

is a (strict) pull-back, taking $X \setminus GQ_r$ in place of X in diagram (2.1) and composing with the above diagram, we obtain the following (strict) pull-back:

$$(2.2) \quad \begin{array}{ccc} F_n^G(X \setminus GQ_r) & \xleftarrow{i''_n(X)} & \prod_1^n X \\ \psi_n(X \setminus GQ_r) \downarrow & & \downarrow \prod_1^n q(X) \\ F_n((X/G) \setminus \bar{Q}_r) & \xleftarrow{i'_n(X/G)} & \prod_1^n X/G. \end{array}$$

Now the map $\prod_1^n q(X) : \prod_1^n X \rightarrow \prod_1^n X/G$ is a fibration, and the application of Corollary 1.8 to the map $\mathcal{I}(\psi_n(X \setminus GQ_r), \prod_1^n q(X)) : I_{i''_n(X)} \rightarrow I_{i'_n(X/G)}$ gives rise to the following homotopy equivalence:

$$(2.3) \quad \Omega \mathcal{I}(\psi_n(X \setminus GQ_r), \prod_1^n q(X)) : \Omega I_{i''_n(X)} \xrightarrow{\simeq} \Omega I_{i'_n(X/G)}.$$

If $1 \leq i \leq n$, let $p_i(X) : \prod_1^n X \rightarrow \prod_1^{n-1} X$ denote the projection map given by forgetting the i^{th} coordinate. Let $J = (i_1, \dots, i_r)$ be a sequence of integers that satisfy $1 \leq i_1 < i_2 < \dots < i_r \leq n$, and let $p_J(X) : \prod_1^n X \rightarrow \prod_1^{n-r} X$ be the map defined by $p_J(X) = p_{i_1}(X) \circ \dots \circ p_{i_r}(X)$. The map $p_i(X)$ restricts to the map $p_i(X)|_{F_n(X)} : F_n(X) \rightarrow F_{n-1}(X)$ that is a fibration whose fibre may be identified with $X \setminus Q_{n-1}$, where $Q_{n-1} = \{q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n\}$. This is a special case of the following result.

Theorem 2.1 ([11, Theorem 3]). *Let X be a topological manifold without boundary. With the above notation, the map $p_J(X)|_{F_n(X)} : F_n(X) \rightarrow F_{n-r}(X)$ is a fibration whose fibre may be identified with $F_r(X \setminus Q_{n-r})$, where $Q_{n-r} = Q_n \setminus \{q_{i_1}, \dots, q_{i_r}\}$.*

For $1 \leq i \leq n$, let $J_i = (1, \dots, i-1, i+1, \dots, n)$. From now on, we assume that X is a pointed topological manifold without boundary. We thus obtain the following commutative diagram:

$$(2.4) \quad \begin{array}{ccccc} F_{n-1}(X \setminus Q_1) & \xleftarrow{j'_n(X)} & F_n(X) & \xrightarrow{p_{J_i(X)}|_{F_n(X)}} & X \\ \downarrow i'_{n-1}(X) & & \downarrow i_n(X) & & \parallel \\ \prod_1^{n-1} X & \xleftarrow{j_n(X)} & \prod_1^n X & \xrightarrow{p_{J_i(X)}} & X, \end{array}$$

where the two inclusions $j'_n(X) : F_{n-1}(X \setminus Q_1) \hookrightarrow F_n(X)$ and $j_n(X) : \prod_1^{n-1} X \hookrightarrow \prod_1^n X$ are given by inserting the point q_i in the i^{th} position. Now the rows of (2.4) are fibrations by Theorem 2.1, so Corollary 1.4 implies that the map:

$$(2.5) \quad \mathcal{I}(j'_n(X), j_n(X)) : I_{i'_{n-1}(X)} \xrightarrow{\simeq} I_{i_n(X)}$$

is a homotopy equivalence for all choices of $1 \leq i \leq n$ above.

Our next goal is to prove the following theorem that is a stronger version of [3, Theorem 1], without the hypothesis that X is smooth.

Theorem 2.2. *Let X be a connected, topological manifold without boundary. Then the inclusion map $i_n(X) : F_n(X) \hookrightarrow \prod_1^n X$ is $(\dim(X) - 1)$ -connected.*

Clearly, Theorem 2.2 implies [3, Theorem 1]. In order to prove Theorem 2.2, we shall make use of the following lemma.

Lemma 2.3. *Let X be a connected, topological manifold without boundary, and let Q be a finite non-empty subset of X . Then the inclusion map $i(X) : X \setminus Q \hookrightarrow X$ is $(\dim(X) - 1)$ -connected.*

Proof. If $\dim(X) = 1$, the result is clear since X is connected. So assume that $\dim(X) \geq 2$, and suppose first that Q consists of a single point q . Write X as the union of $X \setminus Q$ and D_q , where D_q is a small open disc whose centre is q (such a disc exists because X is a topological manifold), and let x be a basepoint lying in $(X \setminus Q) \cap D_q$. Now $(X \setminus Q) \cap D_q$ has the homotopy type of a sphere whose dimension is equal to $\dim(X) - 1$, which is strictly positive, so by Van Kampen's Theorem, $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(X \setminus Q)$ by the normal closure in $\pi_1(X \setminus Q)$ of the image of $\pi_1((X \setminus Q) \cap D_q)$ by the homomorphism induced by the inclusion $(X \setminus Q) \cap D_q \hookrightarrow X \setminus Q$, which implies in particular that the homomorphism $\pi_1(i(X)) : \pi_1(X \setminus Q) \rightarrow \pi_1(X)$ is surjective.

If $\dim(X) = 2$, the result follows using the connectedness of X . Now assume that $\dim(X) > 2$. Then $\pi_1((X \setminus Q) \cap D_q)$ is trivial, and we conclude from the above arguments that the homomorphism $\pi_1(i(X)) : \pi_1(X \setminus Q) \rightarrow \pi_1(X)$ is an isomorphism. To compare the higher homotopy groups, consider the universal covering $p : \tilde{X} \rightarrow X$ of X , and set $X_1 = p^{-1}(X \setminus Q)$. Identifying X_1 with the pull-back $(X \setminus Q) \times_X \tilde{X}$, it follows from Lemma 1.1 that the map $p|_{X_1} : X_1 \rightarrow X \setminus Q$ is the universal covering of $X \setminus Q$, in analogy with the method of [9, Section 2]. By the relative form of the Mayer-Vietoris sequence, the relative homology groups $H_i(\tilde{X}, X_1)$ are zero for all $0 \leq i \leq \dim(X) - 1$. The Hurewicz Theorem then implies that the homomorphism $\pi_j(X_1) \rightarrow \pi_j(\tilde{X})$ is an isomorphism if $j + 1 < \dim(\tilde{X})$ and is surjective if $j = \dim(\tilde{X}) - 1$. So if Q consists of a single point, using the fact that $\pi_j(X_1) \cong \pi_j(X \setminus Q)$ if $j \geq 2$, the homomorphism $\pi_j(i(X)) : \pi_j(X \setminus Q) \rightarrow \pi_j(X)$ is an isomorphism if $j + 1 < \dim(\tilde{X})$ and is surjective if $j = \dim(\tilde{X}) - 1$, which proves the result in this case. In the general case, if $Q = \{q_1, \dots, q_m\}$, where $m \geq 2$, the result follows by induction on m and by writing the manifold $X \setminus Q$ as $(X \setminus \{q_1, \dots, q_{m-1}\}) \setminus \{q_m\}$. \square

Proof of Theorem 2.2. The proof is by induction on n . If $n = 1$ then the statement clearly holds. So assume that the result is true for some $n \geq 1$ and for any connected topological manifold without boundary. By taking the long exact sequence in homotopy of the fibrations of the commutative diagram (2.4), where we replace n by $n + 1$, we obtain the following commutative diagram of exact sequences:

$$(2.6) \quad \begin{array}{ccccccccc} \pi_{j+1}(X) & \longrightarrow & \pi_j(F_n(X \setminus Q_1)) & \longrightarrow & \pi_j(F_{n+1}(X)) & \longrightarrow & \pi_j(X) & \longrightarrow & \pi_{j-1}(F_n(X \setminus Q_1)) \\ \parallel & & \downarrow \pi_j(i'_n(X)) & & \downarrow \pi_j(i_{n+1}(X)) & & \parallel & & \downarrow \pi_{j-1}(i'_n(X)) \\ \pi_{j+1}(X) & \longrightarrow & \pi_j(\prod_1^n X) & \longrightarrow & \pi_j(\prod_1^{n+1} X) & \longrightarrow & \pi_j(X) & \longrightarrow & \pi_{j-1}(\prod_1^n X) \end{array}$$

for all $j \geq 1$. If $k \geq 0$, the homomorphism $\pi_k(i'_n(X))$ may be written as the composition $\pi_k(F_n(X \setminus Q_1)) \xrightarrow{\pi_k(i_n(X \setminus Q_1))} \pi_k(\prod_1^n X \setminus Q_1) \xrightarrow{\pi_k(\prod_1^n i(X))} \pi_k(\prod_1^n X)$ of the homomorphisms induced by the composition $F_n(X \setminus Q_1) \xrightarrow{i_n(X \setminus Q_1)} \prod_1^n X \setminus Q_1 \xrightarrow{\prod_1^n i(X)} \prod_1^n X$. If further $k \leq j$, the first (resp. second) homomorphism $\pi_k(i_n(X \setminus Q_1)) : \pi_k(F_n(X \setminus Q_1)) \rightarrow \pi_k(\prod_1^n X \setminus Q_1)$ (resp. $\pi_k(\prod_1^n i(X)) : \prod_1^n \pi_k(X \setminus Q_1) \rightarrow \prod_1^n \pi_k(X)$) is surjective if $j < \dim(X)$, and is injective if $j + 1 < \dim(X)$ by the induction hypothesis applied to the manifold $X \setminus Q_1$ (resp. by Lemma 2.3). Hence $\pi_j(i'_n(X))$ is surjective if $j < \dim(X)$, and is an isomorphism if $j + 1 < \dim(X)$. Applying the 5-Lemma to (2.6), it follows that $\pi_j(i_{n+1}(X))$ is surjective if $j < \dim(X)$ (resp. is an isomorphism if $j + 1 < \dim(X)$) as required. \square

Theorem 2.2 gives rise to the following corollary.

Corollary 2.4. *Let X be a connected topological manifold without boundary. Then the inclusion map $i'_r(X) : F_r(X \setminus Q_{n-r}) \hookrightarrow \prod_1^r X$ is $(\dim(X) - 1)$ -connected.*

Proof. By definition, $i'_r(X)$ is the composition of $i_r(X \setminus Q_{n-r}) : F_r(X \setminus Q_{n-r}) \hookrightarrow \prod_1^r X \setminus Q_{n-r}$ and the inclusion $\prod_1^r X \setminus Q_{n-r} \hookrightarrow \prod_1^r X$. It then suffices to apply Theorem 2.2 (resp. Lemma 2.3) to the first (resp. second) map. \square

Following [8], we say that the action of a topological group G on X is *tame* if the orbit space X/G is a topological manifold. For instance, [17, Theorem 7.10] implies that any free, proper and smooth action of a compact Lie group G on a smooth manifold X without boundary is tame.

Now let $G \times X \rightarrow X$ be a free, tame action of a group G on a topological manifold X without boundary, and consider the commutative diagram (2.2) and the leftmost commutative square of (2.4), where in the second case, we replace X by X/G and Q_1 by \bar{Q}_1 . Applying the construction of Remark 1.2(c) to each of these squares gives

rise to the following commutative diagram, where the rows are homotopy fibrations:

$$(2.7) \quad \begin{array}{ccccc} I_{n-1}''(X) & \xrightarrow{p_{n-1}''(X)} & F_{n-1}^G(X \setminus GQ_1) & \xhookrightarrow{i_{n-1}''(X)} & \prod_1^{n-1} X \\ \downarrow \mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) & & \downarrow \psi_{n-1}(X \setminus GQ_1) & & \downarrow \prod_1^{n-1} q(X) \\ I_{n-1}'(X/G) & \xrightarrow{p_{n-1}'(X/G)} & F_{n-1}((X/G) \setminus \bar{Q}_1) & \xhookrightarrow{i_{n-1}'(X/G)} & \prod_1^{n-1} X/G \\ \downarrow \mathcal{I}(j_n'(X/G), j_n(X/G)) & & \downarrow j_n'(X/G) & & \downarrow j_n(X/G) \\ I_n(X/G) & \xrightarrow{p_n(X/G)} & F_n(X/G) & \xhookrightarrow{i_n(X/G)} & \prod_1^n X/G, \end{array}$$

the map $\mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) : I_{n-1}''(X) \rightarrow I_{n-1}'(X/G)$ gives rise to a homotopy equivalence $\Omega \mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) : \Omega I_{n-1}''(X) \rightarrow \Omega I_{n-1}'(X/G)$ by (2.3), and the map $\mathcal{I}(j_n'(X/G), j_n(X/G)) : I_{n-1}'(X/G) \rightarrow I_n(X/G)$ is a homotopy equivalence by (2.5). Let $k_n(X) : I_{n-1}''(X) \rightarrow I_n(X/G)$ be the weak homotopy equivalence defined by $k_n(X) = \mathcal{I}(j_n'(X/G), j_n(X/G)) \circ \mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X))$.

Let $G \times X \rightarrow X$ be a tame, free action of a Lie group G on a topological manifold X without boundary. This action gives rise to a fibration $G \hookrightarrow X \rightarrow X/G$ of manifolds, where the fibre over the point $\bar{Q}_1 \in X/G$ is identified with GQ_1 . If the inclusion map $GQ_1 \hookrightarrow X$ is null homotopic then the homomorphism $\pi_j(X) \rightarrow \pi_j(X/G)$ induced by the quotient map $X \rightarrow X/G$ is injective for all $j \geq 0$, so we may regard $\pi_j(X)$ as a subgroup of $\pi_j(X/G)$. Furthermore, by (1.8), there is a homotopy equivalence $h : \Omega(X/G) \xrightarrow{\cong} G \times \Omega(X)$ that may be described in terms of a homotopy between the inclusion map $GQ_1 \hookrightarrow X$ and the constant map. If $e \in G$ denotes the unit element of G , the restriction of the inverse of the homotopy equivalence $h : \Omega(X/G) \xrightarrow{\cong} G \times \Omega(X)$ to $\{e\} \times \Omega(X)$ is homotopic to the loop of the projection map $X \rightarrow X/G$.

For the purpose of the following theorem, which is the main result of this section, for all $j \geq 0$, we identify $\pi_{j+1}(X/G)$ with $\pi_j(G) \times \pi_{j+1}(X)$ via the homotopy equivalence $h : \Omega(X/G) \xrightarrow{\cong} G \times \Omega(X)$, and $\pi_j(\prod_1^n X/G)$ with $\pi_j(\prod_1^{n-1} X/G) \times \pi_j(X/G)$. For this particular identification, we are taking $i = n$ in (2.4).

Theorem 2.5. *Let $G \times X \rightarrow X$ be a free, tame action of a Lie group G on a connected topological manifold X without boundary, and let $Q_1 \in X$. Suppose that the inclusion map $i_{n-1}''(X) : F_{n-1}^G(X \setminus GQ_1) \hookrightarrow \prod_1^{n-1} X$ is null homotopic (this is the case if for example the inclusion map $X \setminus GQ_1 \hookrightarrow X$ is null homotopic).*

- (1) *The maps $\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X}) : I_{n-1}''(X) \rightarrow F_{n-1}^G(X \setminus GQ_1) \times \Omega(\prod_1^{n-1} X)$ and $\mathcal{I}(j_n'(X/G), j_n(X/G)) : I_{n-1}'(X/G) \rightarrow I_n(X/G)$ are homotopy equivalences, and the map $\mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) : I_{n-1}''(X) \rightarrow I_{n-1}'(X/G)$ is a weak homotopy equivalence. Furthermore, if G is discrete, X is 1-connected and*

$\dim(X/G) \geq 3$ then the map $\psi_{n-1}(X \setminus GQ_1) : F_{n-1}^G(X \setminus GQ_1) \longrightarrow F_{n-1}((X/G) \setminus \bar{Q}_1)$ is the universal covering.

- (2) If $j \leq \min(\text{conn}(X), \dim(X/G) - 2)$ then $\pi_j(F_{n-1}^G(X \setminus GQ_1)) = 0$.
- (3) Suppose that the homomorphism $\pi_j(X) \longrightarrow \pi_j(X/G)$ induced by the quotient map $X \longrightarrow X/G$ is injective for all $j \geq 1$ (this is the case if for example the inclusion map $GQ_1 \hookrightarrow X$ is null homotopic). For all $j \geq 2$, up to the identification of the groups $\pi_{j-1}(I_{i_n(X/G)})$ and $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ via the isomorphism

$$\pi_{j-1}(\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})) \circ (\pi_{j-1}(k_n(X)))^{-1},$$

the restriction to the subgroup $\pi_j(\prod_1^{n-1} X)$ of $\pi_j(\prod_1^n X/G)$ of the boundary homomorphism $\hat{\partial}_j : \pi_{j-1}(\Omega(\prod_1^n X/G)) \longrightarrow \pi_{j-1}(I_{i_n(X/G)})$ given by the long exact sequence in homotopy of the homotopy fibration $I_{i_n(X/G)} \xrightarrow{p_{i_n(X/G)}} F_n(X/G) \xrightarrow{i_n(X/G)} \prod_1^n X/G$ of (2.7) coincides with the inclusion of $\pi_j(\prod_1^{n-1} X)$ in $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ via the usual identification of $\pi_j(X)$ with $\pi_{j-1}(\Omega(X))$.

Proof.

- (1) By hypothesis, $i''_{n-1}(X)$ is null homotopic. Let $H : F_{n-1}^G(X \setminus GQ_1) \times I \longrightarrow \prod_1^{n-1} X$ be a homotopy between $i''_{n-1}(X)$ and the constant map at the basepoint of $\prod_1^{n-1} X$, and let $\varphi(x) : I \longrightarrow (\prod_1^{n-1} X)^I$ be the path given by $\varphi(x)(t) = H(x, t)$. Applying Remark 1.2(e) to the homotopy fibration

$$(2.8) \quad I''_{n-1}(X) \xrightarrow{p_{i''_{n-1}(X)}} F_{n-1}^G(X \setminus GQ_1) \xrightarrow{i''_{n-1}(X)} \prod_1^{n-1} X,$$

we obtain a homotopy equivalence:

$$(2.9) \quad \mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X}) : I''_{n-1}(X) \longrightarrow F_{n-1}^G(X \setminus GQ_1) \times \Omega(\prod_1^{n-1} X)$$

defined by $\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})(x, \gamma) = (x, \varphi(x)^{-1} * \gamma)$ for all $(x, \gamma) \in I''_{n-1}(X)$. Since the map $\psi_{n-1}(X \setminus GQ_1) : F_{n-1}^G(X \setminus GQ_1) \longrightarrow F_{n-1}((X/G) \setminus \bar{Q}_1)$ is 1-connected, the map $\mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X)) : I''_{n-1}(X) \longrightarrow I'_{n-1}(X/G)$ is a weak homotopy equivalence by Corollary 1.8 and (2.3). Further, by (2.5), the map $\mathcal{I}(j_{n'}(X/G), j_n(X/G)) : I'_{n-1}(X/G) \longrightarrow I_{i_n(X/G)}$ is a homotopy equivalence, and this proves the first part of the statement.

Assume that G is a discrete group. Since X is a 1-connected manifold, the quotient map $\prod_1^{n-1} q(X) : \prod_1^{n-1} X \longrightarrow \prod_1^{n-1} X/G$ is the universal covering. Now $\dim(X/G) \geq 3$, and applying Corollary 2.4, we see that the homomorphism $\pi_1(i'_{n-1}(X/G)) : \pi_1(F_{n-1}((X/G) \setminus \bar{Q}_1)) \longrightarrow \pi_1(\prod_1^{n-1} X/G)$ is an isomorphism. It follows from diagram (2.2) and Lemma 1.1 that $\psi_{n-1}(X \setminus GQ_1) : F_{n-1}^G(X \setminus GQ_1) \longrightarrow F_{n-1}((X/G) \setminus \bar{Q}_1)$ is the universal covering.

- (2) Let $j \leq \min(\text{conn}(X), \dim(X/G) - 2)$. Applying the long exact homotopy sequence to the following commutative diagram whose rows are fibrations:

$$(2.10) \quad \begin{array}{ccccc} \prod_1^{n-1} G & \longrightarrow & F_{n-1}^G(X \setminus GQ_1) & \xrightarrow{\psi_{n-1}(X \setminus GQ_1)} & F_{n-1}((X/G) \setminus \bar{Q}_1) \\ & \parallel & \downarrow i''_{n-1}(X) & & \downarrow i'_{n-1}(X/G) \\ \prod_1^{n-1} G & \longrightarrow & \prod_1^{n-1} X & \xrightarrow{\prod_1^{n-1} q(X)} & \prod_1^{n-1} X/G \end{array}$$

yields the following commutative diagram whose rows are exact:

$$(2.11) \quad \begin{array}{ccccccc} \pi_j(\prod_1^{n-1} G) & \longrightarrow & \pi_j(F_{n-1}^G(X \setminus GQ_1)) & \xrightarrow{\pi_j(\psi_{n-1}(X \setminus GQ_1))} & \pi_j(F_{n-1}((X/G) \setminus \bar{Q}_1)) & \xrightarrow{\partial_j} & \pi_{j-1}(\prod_1^{n-1} G) \\ & \parallel & \downarrow \pi_j(i''_{n-1}(X)) & & \downarrow \pi_j(i'_{n-1}(X/G)) & & \parallel \\ \pi_j(\prod_1^{n-1} G) & \longrightarrow & \pi_j(\prod_1^{n-1} X) & \xrightarrow{\pi_j(\prod_1^{n-1} q(X))} & \pi_j(\prod_1^{n-1} X/G) & \xrightarrow{\bar{\partial}_j} & \pi_{j-1}(\prod_1^{n-1} G), \end{array}$$

where $\partial_j : \pi_j(F_{n-1}((X/G) \setminus \bar{Q}_1)) \rightarrow \pi_{j-1}(\prod_1^{n-1} G)$ (resp. $\bar{\partial}_j : \pi_j(\prod_1^{n-1} X/G) \rightarrow \pi_{j-1}(\prod_1^{n-1} G)$) is the boundary homomorphism corresponding to the first (resp. second) row of (2.10). Since $j \leq \dim(X/G) - 2$, by applying Corollary 2.4 to the manifold X/G , we see that the homomorphism $\pi_k(i'_{n-1}(X/G))$ is surjective for all $k \leq j + 1$ and is an isomorphism if $k \leq j$. The fact that $j \leq \text{conn}(X)$ implies that the homomorphism $\pi_k(\prod_1^{n-1} X/G) \rightarrow \pi_{k-1}(\prod_1^{n-1} G)$ is surjective for all $k \leq j + 1$ and is an isomorphism if $k \leq j$. By the commutativity of the diagram (2.11), it follows that the boundary homomorphism $\partial_k : \pi_k(F_{n-1}((X/G) \setminus \bar{Q}_1)) \rightarrow \pi_{k-1}(\prod_1^{n-1} G)$ is surjective for all $k \leq j + 1$ and is an isomorphism if $k \leq j$, and so by exactness of the uppermost row of (2.11), $\pi_j(F_{n-1}^G(X \setminus GQ_1)) = 0$ for all $j \leq \min(\text{conn}(X), \dim(X/G) - 2)$.

- (3) Let $\tilde{d} : \Omega(\prod_1^{n-1} X) \rightarrow I_{i''_{n-1}(X)}$ denote the boundary map of the homotopy fibration (2.8), and let $\tilde{\partial}_j : \pi_j(\prod_1^{n-1} X) \rightarrow \pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ denote the composition of the associated boundary homomorphism $\pi_{j-1}(\tilde{d})$ with the isomorphism $\pi_{j-1}(\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X}))$ induced by the homotopy equivalence given by (2.9). Identifying $\pi_j(X)$ with $\pi_{j-1}(\Omega(X))$, and applying Remark 1.2(e), it follows that $\tilde{\partial}_j$ coincides with the inclusion homomorphism

$$\pi_j \left(\prod_1^{n-1} X \right) \hookrightarrow \pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1} \left(\Omega \left(\prod_1^{n-1} X \right) \right).$$

The first and third rows of diagram (2.7) give rise to the following commutative diagram of homotopy fibrations:

$$(2.12) \quad \begin{array}{ccccc} I_{i''_{n-1}(X)} & \xrightarrow{p_{i''_{n-1}(X)}} & F_{n-1}^G(X \setminus GQ_1) & \xrightarrow{i''_{n-1}(X)} & \prod_1^{n-1} X \\ \downarrow k_n(X) & & \downarrow j'_n(X/G) \circ \psi_{n-1}(X \setminus GQ_1) & & \downarrow j_n(X/G) \circ \prod_1^{n-1} q(X) \\ I_{i_n(X/G)} & \xrightarrow{p_{i_n(X/G)}} & F_n(X/G) & \xrightarrow{i_n(X/G)} & \prod_1^n X/G, \end{array}$$

where the leftmost vertical map $k_n(X) : I_{i''_{n-1}(X)} \rightarrow I_{i_n(X/G)}$ is defined by $k_n(X) = \mathcal{I}(j'_n(X/G), j_n(X/G)) \circ \mathcal{I}(\psi_{n-1}(X \setminus GQ_1), \prod_1^{n-1} q(X))$, and is a weak homotopy equivalence by part (1). Now let $\hat{d} : \Omega(\prod_1^n X/G) \rightarrow I_{i_n(X/G)}$ be the boundary map of the lower homotopy fibration of (2.12), and let

$$\hat{\partial}_j : \pi_{j-1}(\Omega(\prod_1^n X/G)) \rightarrow \pi_{j-1}(I_{i_n(X/G)})$$

be the induced boundary homomorphism.

Consider the element $\omega = (\omega_1, \dots, \omega_{n-1}, c_{q_{n-1}}) \in \Omega(\prod_1^{n-1} X)$. The map $\prod_1^n q(X)$ induces a map $\Omega \prod_1^n q(X) : \Omega(\prod_1^n X) \rightarrow \Omega(\prod_1^n X/G)$, so

$$(2.13) \quad \begin{aligned} (\Omega \prod_1^n q(X))(\omega, c_{q_n}) &= (\bar{\omega}_1, \dots, \bar{\omega}_{n-1}, c_{\bar{q}_n}) \in \Omega(\prod_1^n X/G) \text{ and} \\ \hat{d} \circ (\Omega \prod_1^n q(X))(\omega) &= (\bar{q}_1, \dots, \bar{q}_n, \bar{\omega}_1, \dots, \bar{\omega}_{n-1}, c_{\bar{q}_n}) \in I_{i_n(X/G)} \end{aligned}$$

by Remark 1.2(e). On the other hand, $(q_1, \dots, q_{n-1}, \omega_1, \dots, \omega_{n-1}) \in I_{i''_{n-1}(X)}$, and by Remark 1.2(c), the definition of $k_n(X)$, (2.9) and (2.13), we have:

$$k_n(X)(q_1, \dots, q_{n-1}, \omega_1, \dots, \omega_{n-1}) = \hat{d} \circ (\Omega \prod_1^n q(X))(\omega) \text{ and}$$

$$\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})(q_1, \dots, q_{n-1}, \omega_1, \dots, \omega_{n-1}) = (q_1, \dots, q_{n-1}, \omega_1, \dots, \omega_{n-1}).$$

Thus the image of $([\omega], [c_{q_n}]) \in \pi_j(\prod_1^{n-1} X) \times \pi_j(X)$ under the composition

$$\pi_{j-1}(\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})) \circ (\pi_{j-1}(k_n(X)))^{-1} \circ \hat{\partial}_j \circ \pi_j(\prod_1^n q(X))$$

is equal to $([c_{q_1}], \dots, [c_{q_{n-1}}], [\omega])$. In particular, up to identification of $\pi_{j-1}(I_{i_n(X/G)})$ and $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ via the isomorphism

$$\pi_{j-1}(\mathcal{I}(\text{id}_{F_{n-1}^G(X \setminus GQ_1)}, \text{id}_{\prod_1^{n-1} X})) \circ (\pi_{j-1}(k_n(X)))^{-1},$$

the restriction of the homomorphism $\hat{\partial}_j : \pi_{j-1}(\Omega(\prod_1^n X/G)) \rightarrow \pi_{j-1}(I_{i_n(X/G)})$ to the subgroup $\pi_j(\prod_1^{n-1} X)$ of $\pi_j(\prod_1^n X/G)$ coincides with the inclusion of $\pi_j(\prod_1^{n-1} X)$ in $\pi_{j-1}(F_{n-1}^G(X \setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} X))$ which completes the proof. \square

Theorem 2.5 implies the following corollary.

Corollary 2.6. *Let X be a connected topological manifold without boundary such that $\dim(X) \geq 3$, let \tilde{X} be its universal covering, and let $Q_1 \in \tilde{X}$. Suppose that the map $i''_{n-1}(\tilde{X}) : F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1) \hookrightarrow \prod_1^{n-1} \tilde{X}$ is null homotopic.*

- (1) *The spaces $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)$ and $F_{n-1}(\widetilde{X \setminus \bar{Q}_1})$ are homeomorphic, and there exists a homotopy equivalence $I_{i''_{n-1}(\tilde{X})} \simeq F_{n-1}(\widetilde{X \setminus \bar{Q}_1}) \times \Omega(\prod_1^{n-1} \tilde{X})$, and a weak homotopy equivalence between $I_{i_n(X)}$ and $F_{n-1}(\widetilde{X \setminus \bar{Q}_1}) \times \Omega(\prod_1^{n-1} \tilde{X})$.*
- (2) *Let $j \geq 2$. Up to the identification of the groups $\pi_j(\prod_1^n \tilde{X})$ with $\pi_j(\prod_1^n X)$ via the isomorphism $\pi_j(q(\tilde{X})) : \pi_j(\tilde{X}) \rightarrow \pi_j(X)$ and the identification of the groups $\pi_{j-1}(I_{i_n(X)})$ and $\pi_{j-1}(F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} \tilde{X}))$ via the isomorphism given in the statement of Theorem 2.5(3), the restriction of the boundary homomorphism $\hat{\partial}_j : \pi_j(\prod_1^n \tilde{X}) \rightarrow \pi_{j-1}(I_{i_n(X)})$ of the homotopy long exact sequence of the homotopy fibration*

$$I_{i_n(X)} \xrightarrow{p_{i_n(X)}} F_n(X) \xrightarrow{i_n(X)} \prod_1^n X$$

to the subgroup $\pi_j(\prod_1^{n-1} \tilde{X})$ of $\pi_j(\prod_1^n \tilde{X})$ coincides with the inclusion of $\pi_j(\prod_1^{n-1} \tilde{X})$ in $\pi_{j-1}(F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} \tilde{X}))$ via the usual identification of $\pi_j(\tilde{X})$ with $\pi_{j-1}(\Omega(\tilde{X}))$.

Proof. In the whole of the proof, we shall replace the manifold X (resp. the group G) of Theorem 2.5 by \tilde{X} (resp. by $\pi_1(X)$). The group $\pi_1(X)$ is discrete, and since X is a manifold, $\pi_1(X)$ is countable and so is a 0-dimensional Lie group.

- (1) Since $\pi_1(X)$ is discrete, \tilde{X} is 1-connected and $\dim(X) \geq 3$, it follows from the second part of the statement of Theorem 2.5(1) that $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)$ is homeomorphic to $F_{n-1}(\widetilde{X \setminus \bar{Q}_1})$. Applying this to the first part of Theorem 2.5(1) yields the rest of the statement.
- (2) Since $q(\tilde{X}) : \tilde{X} \rightarrow X$ is a covering map, the induced homomorphism $\pi_j(q(\tilde{X})) : \pi_j(\tilde{X}) \rightarrow \pi_j(X)$ is injective for all $j \geq 1$ and is an isomorphism for all $j \geq 2$. Up to the identification of $\pi_j(\prod_1^n \tilde{X})$ with $\pi_j(\prod_1^n X)$ via this isomorphism, the result then follows directly from Theorem 2.5(3). \square

The following result brings together various descriptions for the homotopy type of the homotopy fibre $I_{i_n(X)}$, where X is a topological manifold without boundary whose universal covering is contractible.

Proposition 2.7. *Let X be a connected topological manifold without boundary such that $\dim(X) \geq 3$ and whose universal covering \tilde{X} is contractible, and let $Q_1 \in \tilde{X}$. Then the homotopy fibre $I_{i_n(X)}$ of the map $i_n(X) : F_n(X) \rightarrow \prod_1^n X$ is weakly*

homotopy equivalent to each of the following six spaces: $F_{n-1}(\widetilde{X \setminus \bar{Q}_1})$, $F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X}$, $\widetilde{F_n(X)}$, $F_n^{\pi_1(X)}(\tilde{X})$, $F_{n-1}(X \setminus \bar{Q}_1) \times_{\prod_1^{n-1} X} \prod_1^{n-1} \tilde{X}$ and $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)$.

Proof. Let $p(\tilde{X}) : \tilde{X} \rightarrow X$ denote the universal covering map.

- Since the space $\prod_1^n \tilde{X}$ is contractible, the map $i''_{n-1}(\tilde{X}) : F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1) \hookrightarrow \prod_1^{n-1} \tilde{X}$ is null homotopic. So by (1.7), $I''_{n-1}(\tilde{X}) \simeq F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)$. Further, $\pi_1(X)$ is discrete, \tilde{X} is 1-connected and $\dim(X) \geq 3$, it follows from Theorem 2.5(1) that $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1) \simeq F_{n-1}(\widetilde{X \setminus \bar{Q}_1})$.
- The strict pull-back (2.2) gives rise to a homotopy equivalence $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1) \simeq F_{n-1}(X \setminus \bar{Q}_1) \times_{\prod_1^{n-1} X} \prod_1^{n-1} \tilde{X}$.

So $I''_{n-1}(\tilde{X})$, $F_{n-1}^{\pi_1(X)}(\tilde{X} \setminus \pi_1(X)Q_1)$, $F_{n-1}(\widetilde{X \setminus \bar{Q}_1})$ and $F_{n-1}(X \setminus \bar{Q}_1) \times_{\prod_1^{n-1} X} \prod_1^{n-1} \tilde{X}$ are pairwise homotopy equivalent.

- Since $\dim(X) \geq 3$, the map $i_n(X) : F_n(X) \rightarrow \prod_1^n X$ is 2-connected by Theorem 2.2, and thus $\pi_1(i_n(X))$ is an isomorphism. Further, $\prod_1^n p(\tilde{X}) : \prod_1^n \tilde{X} \rightarrow \prod_1^n X$ is the universal covering, and so it follows from Lemma 1.1 that $\widetilde{F_n(X)} \simeq F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X}$.
- The pull-back (2.1) gives rise to a homotopy equivalence $F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X} \simeq F_n^{\pi_1(X)}(\tilde{X})$.
- Consider the (strict) pull-back:

$$\begin{array}{ccc} F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X} & \xrightarrow{\tilde{i}_n(X)} & \prod_1^n \tilde{X} \\ p'(X) \downarrow & & \downarrow \prod_1^n p(\tilde{X}) \\ F_n(X) & \xrightarrow{i_n(X)} & \prod_1^n X. \end{array}$$

Since the projection map $p'(X) : F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X} \rightarrow F_n(X)$ is 1-connected, the map $\mathcal{I}(p'(X), \prod_1^n p(\tilde{X})) : I_{\tilde{i}_n(X)} \rightarrow I_{i_n(X)}$ is a weak homotopy equivalence by Corollary 1.8 and (2.3). On the other hand, the map $\tilde{i}_n(X)$ is null homotopic because the space $\prod_1^n \tilde{X}$ is contractible, and it follows from (1.7) that $I_{\tilde{i}_n(X)} \simeq F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X}$.

Thus $I_{\tilde{i}_n(X)}$, $F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X}$, $F_n(X) \times_{\prod_1^n X} \prod_1^n \tilde{X}$ and $F_n^{\pi_1(X)}(\tilde{X})$ are pairwise homotopy equivalent, and are weakly homotopy equivalent to $I_{i_n(X)}$. Finally, (2.5) implies that $I_{i'_{n-1}(X)} \simeq I_{i_n(X)}$, and by Corollary 1.8 and (2.3), there is a weak homotopy equivalence $I_{i''_{n-1}(\tilde{X})} \rightarrow I_{i'_{n-1}(X)}$, and the result follows. \square

The family of manifolds covered by Euclidean spaces is properly contained in the family of manifolds whose universal covering is contractible. The fact that the

inclusion is strict may be seen by considering for example the Whitehead open 3-manifold that is contractible but not homeomorphic to \mathbb{R}^3 .

REMARK 2.8. By [5, Theorem 1.6, Chapter V, p. 229] and [21], the universal covering of any paracompact aspherical manifold is contractible.

3. The homotopy fibre of the inclusion map $F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$ and the associated long exact sequence in homotopy

In this section, we start by studying the homotopy fibre of the inclusion $i_n(\mathbb{S}^k/G) : F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$, where \mathbb{S}^k/G is the orbit manifold associated with a free tame action of a Lie group G on the k -sphere \mathbb{S}^k . One of our aims is to describe completely the long exact sequence in homotopy of the homotopy fibration

$$I_{i_n(\mathbb{S}^k/G)} \longrightarrow F_n(\mathbb{S}^k/G) \longrightarrow \prod_1^n \mathbb{S}^k/G$$

in the case that G is discrete (so finite). Let $k \geq 3$, and consider a free tame action $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ of a compact Lie group G on \mathbb{S}^k . If $\dim(G) = 0$ then G is finite, and comparing Euler characteristics, we obtain $\chi(\mathbb{S}^k) = |G| \cdot \chi(\mathbb{S}^k/G)$, so G is trivial or \mathbb{Z}_2 if k is even, and $|G| \geq 2$ if k is odd. In particular, for $G = \mathbb{Z}_2$ we get $\mathbb{S}^k/\mathbb{Z}_2 = \mathbb{R}P^k$, the k -dimensional real projective space. If $\dim(G) > 0$ then we have the following result.

Theorem 3.1 ([5, 8.5. Theorem]). *Suppose that G is a compact Lie group such that $\dim(G) > 0$ and that acts freely on \mathbb{S}^k . Then G is isomorphic to a subgroup of \mathbb{S}^3 . More precisely, up to isomorphism, G is one of the following groups: \mathbb{S}^1 ; $N(\mathbb{S}^1)$, the normaliser of \mathbb{S}^1 in the group \mathbb{S}^3 ; or \mathbb{S}^3 .*

Using Theorem 3.1, we obtain the following proposition.

Proposition 3.2. *Let G be a compact Lie group G , and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of G on the sphere \mathbb{S}^k .*

- (1) *If G is finite with $|G| > 2$ or $G = \mathbb{S}^1$ then k is odd.*
- (2) *If $G = N(\mathbb{S}^1)$ or $G = \mathbb{S}^3$ then $k = 4m + 3$ for some $m \geq 0$.*

Proof.

- (1) First, suppose that G is finite with $|G| > 2$. Given a free action $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$, we see that $\chi(\mathbb{S}^k) = |G| \cdot \chi(\mathbb{S}^k/G)$, so k is odd because $|G| > 2$. Now assume that $G = \mathbb{S}^1$. A free action $\mathbb{S}^1 \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ of \mathbb{S}^1 on \mathbb{S}^k induces a free action of the cyclic group \mathbb{Z}_l on \mathbb{S}^k for all $l \geq 1$. Comparing Euler characteristics once more, we obtain $\chi(\mathbb{S}^k) = l \cdot \chi(\mathbb{S}^k/\mathbb{Z}_l)$, and taking $l \geq 3$, we deduce once more that k is odd.

(2) Suppose that $G = N(\mathbb{S}^1)$ or \mathbb{S}^3 . A free action $\mathbb{S}^3 \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ of \mathbb{S}^3 on \mathbb{S}^k induces a free action $N(\mathbb{S}^1) \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ of $N(\mathbb{S}^1)$ on \mathbb{S}^k , and a free action $N(\mathbb{S}^1) \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ of G on \mathbb{S}^k induces a free action $\mathbb{S}^1 \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ of \mathbb{S}^1 on \mathbb{S}^k . By (1), it follows that $k = 2m' + 1$ for some $m' \geq 0$, from which we obtain an induced free action $\mathbb{Z}_2 \times \mathbb{S}^{2m'+1}/\mathbb{S}^1 \rightarrow \mathbb{S}^{2m'+1}/\mathbb{S}^1$ of \mathbb{Z}_2 on $\mathbb{S}^{2m'+1}/\mathbb{S}^1$. Using the Euler characteristic relation $\chi(\mathbb{S}^{2m'+1}/\mathbb{S}^1) + \chi((\mathbb{S}^{2m'+1}/\mathbb{S}^1)^{\mathbb{Z}_2}) = 2\tilde{\chi}((\mathbb{S}^{2m'+1}/\mathbb{S}^1)/\mathbb{Z}_2)$ given in [12, p. 146] for the reduced Euler characteristic $\tilde{\chi}$, we see that m' is also odd, and so $k = 4m + 3$ for some $m \geq 0$. \square

If $k \geq 3$, let $G \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ be a tame, free action of a compact Lie group G on \mathbb{S}^k , and consider the topological manifold without boundary \mathbb{S}^k/G that is the orbit space for this action. If $Q_1 \in \mathbb{S}^k$, the map $\mathbb{S}^k \setminus GQ_1 \hookrightarrow \mathbb{S}^k$ factors through the contractible space $\mathbb{S}^k \setminus Q_1$, and so is null homotopic. If $1 \leq i \leq n$, the fact that the following diagram:

$$\begin{array}{ccc} F_n^G(\mathbb{S}^k \setminus GQ_1) & \xhookrightarrow{i''_n(\mathbb{S}^k)} & \prod_1^n \mathbb{S}^k \\ \downarrow J_i(\mathbb{S}^k)|_{F_n^G(\mathbb{S}^k/GQ_1)} & & \downarrow J_i(\mathbb{S}^k) \\ \mathbb{S}^k \setminus GQ_1 & \xhookrightarrow{\quad} & \mathbb{S}^k \end{array}$$

is commutative implies that the map $i''_n(\mathbb{S}^k) : F_n^G(\mathbb{S}^k \setminus GQ_1) \hookrightarrow \prod_1^n \mathbb{S}^k$ is also null homotopic. We obtain the following corollary by applying parts (1) and (2) of Theorem 2.5 to the map $i''_n(\mathbb{S}^k)$.

Corollary 3.3. *Let $G \times \mathbb{S}^k \rightarrow \mathbb{S}^k$ be a free, tame action of a compact Lie group G on \mathbb{S}^k .*

- (1) *There is a homotopy equivalence $I_{i''_{n-1}(\mathbb{S}^k)} \simeq F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \times \Omega(\prod_1^{n-1} \mathbb{S}^k)$. If the group G is finite and $\dim(\mathbb{S}^k/G) \geq 3$ then $F_{n-1}(\widehat{\mathbb{S}^k \setminus GQ_1}) \simeq F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)$.*
- (2) *If $j \leq k - \dim G - 2$ then $\pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)) = 0$.*

Let $k \geq 3$. We now proceed to study the long exact sequence in homotopy of the homotopy fibration

$$I_{i_n(\mathbb{S}^k/G)} \xrightarrow{p_{i_n(\mathbb{S}^k/G)}} F_n(\mathbb{S}^k/G) \xrightarrow{i_n(\mathbb{S}^k/G)} \prod_1^n \mathbb{S}^k/G.$$

In this case, there is a homotopy equivalence $I_{i''_{n-1}(\mathbb{S}^k)} \xrightarrow{\simeq} F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \times \Omega(\prod_1^{n-1} \mathbb{S}^k)$ by Theorem 2.5(1). We restrict our attention to the case where k is odd and $\dim(G) = 0$ i.e., G is finite. Up to the identification of the group $\pi_j(I_{i_n(\mathbb{S}^k/G)})$ with $\pi_{j-1}(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \times \Omega(\prod_1^{n-1} \mathbb{S}^k))$ described in Theorem 2.5(3), our aim is to

describe the boundary homomorphism:

$$\hat{\partial}_j : \pi_j \left(\prod_1^n \mathbb{S}^k \right) \longrightarrow \pi_{j-1} \left(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1) \times \Omega \left(\prod_1^{n-1} \mathbb{S}^k \right) \right),$$

as well as other homomorphisms of that exact sequence. Since $k \geq 3$, Corollary 3.3(2) implies that $\pi_1(I_{i_n}(\mathbb{S}^k/G)) = 0$. So we may assume that $j > 2$. In what follows, we shall need the following result (cf. [10, Theorem 3.1]).

Lemma 3.4. *Let $k \geq 3$ be odd, and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of a finite group G on \mathbb{S}^k . Then for all $1 \leq i, l \leq n$, there exists a map $\lambda_i(\mathbb{S}^k) : \mathbb{S}^k \longrightarrow F_n^G(\mathbb{S}^k)$ such that the composition $\mathbb{S}^k \xrightarrow{\lambda_i(\mathbb{S}^k)} F_n^G(\mathbb{S}^k) \xrightarrow{\psi_n(\mathbb{S}^k)} F_n(\mathbb{S}^k/G) \xrightarrow{p_{J_l}(\mathbb{S}^k/G)|_{F_n(\mathbb{S}^k/G)}} \mathbb{S}^k/G$ is homotopic to the quotient map $q(\mathbb{S}^k) : \mathbb{S}^k \longrightarrow \mathbb{S}^k/G$ for $1 \leq i \leq n$. If $l = i$ then the composition coincides with $q(\mathbb{S}^k)$.*

Proof. To construct the continuous map $\lambda_i(\mathbb{S}^k) : \mathbb{S}^k \longrightarrow F_n^G(\mathbb{S}^k)$, consider a non-vanishing vector field on \mathbb{S}^k . Such a vector field exists because k is odd. Since the given action $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ is free, there exists $\epsilon > 0$ such that the distance between any two distinct points belonging to the same G -orbit is greater than ϵ . Given $x \in \mathbb{S}^k$, consider the n -tuple (x, x_2, \dots, x_n) , where for $m = 2, \dots, n$, x_m is the point on the integral curve of the vector field passing through x and corresponding to the parameter value $t = (m-1)/R$ for $R > 0$ sufficiently large. Thus the distance between any two points of this n -tuple is less than ϵ , and so the points x, x_2, \dots, x_{n-1} and x_n belong to distinct orbits. Given $1 \leq i \leq n$, set $\lambda_i(x) = (x_2, \dots, x_i, \underbrace{x}_{i^{\text{th}} \text{ position}}, x_{i+1}, \dots, x_n)$.

By construction, $\lambda_i(x)$ belongs to the orbit configuration space $F_n^G(\mathbb{S}^k)$ and λ_i is continuous. Further,

$$p_{J_l}(\mathbb{S}^k/G) |_{F_n(\mathbb{S}^k/G)} \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(x) = \begin{cases} q(x_{i+1}) & \text{if } l < i, \\ q(x) & \text{if } l = i, \\ q(x_i) & \text{if } l > i. \end{cases}$$

In each case, one may use the integral curve of the vector field to show that the composition $p_{J_l}(\mathbb{S}^k/G) |_{F_n(\mathbb{S}^k/G)} \circ \psi_n(\mathbb{S}^k) \circ \lambda_i : \mathbb{S}^k \longrightarrow \mathbb{S}^k/G$ is homotopic to $q(\mathbb{S}^k)$ as required. \square

If the group G is trivial, Lemma 3.4 implies that the fibration $F_n(\mathbb{S}^k) \longrightarrow \mathbb{S}^k$ admits a section (cf. [10, Theorem 3.1]). The lemma also yields some information about the homomorphism $\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$ induced by the inclusion map $i_n(\mathbb{S}^k/G) : F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$, and which is relevant to understanding the long exact homotopy sequence in homotopy of the homotopy fibration $I_{i_n}(\mathbb{S}^k/G) \longrightarrow F_n(\mathbb{S}^k/G) \hookrightarrow \prod_1^n \mathbb{S}^k/G$. In order to study the homomorphism

$\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$, consider a basepoint $(x_1, \dots, x_n) \in F_n(\mathbb{S}^k/G)$, where we assume that the set $\{x_1, \dots, x_n\}$ is contained in a small disc whose centre is a point $x_0 \in \mathbb{S}^k/G$. If $1 \leq i \leq n$, choosing a path from x_i to x_0 inside this disc allows us to identify $\pi_j(\mathbb{S}^k/G, x_i)$ with $\pi_j(\mathbb{S}^k/G, x_0)$, and in this way, the group $\pi_j(\prod_1^n \mathbb{S}^k/G, (x_1, \dots, x_n))$ may be identified isomorphically with the group $\prod_1^n \pi_j(\mathbb{S}^k/G, x_0)$. In what follows, we shall consider the composition of the homomorphism $\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G, (x_1, \dots, x_n))$ by this isomorphism, which by abuse of notation, we shall also denote by $\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$.

Lemma 3.5. *Let $k \geq 3$ be odd, and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of a finite group G on \mathbb{S}^k . Then $\pi_0(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) = 0$, $\pi_1(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) = \prod_1^n G$, and the homomorphism $\pi_j(i'_n(\mathbb{S}^k/G)) : \pi_j(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$ induced by the inclusion map $i'_n(\mathbb{S}^k/G) : F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1) \hookrightarrow \prod_1^n \mathbb{S}^k/G$ is trivial for all $j \geq 2$.*

Proof. By Corollary 2.4, the homomorphism $\pi_j(i'_n(\mathbb{S}^k/G)) : \pi_j(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$ is an isomorphism for $j = 0, 1$, whence $\pi_0(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) = 0$ and $\pi_1(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) = \prod_1^n G$. So assume that $j \geq 2$. Taking $X = \mathbb{S}^k$ and $r = n = 1$ in (2.2), we obtain the following pull-back diagram:

$$(3.1) \quad \begin{array}{ccc} \mathbb{S}^k \setminus GQ_1 & \xhookrightarrow{i''_1(\mathbb{S}^k)} & \mathbb{S}^k \\ \psi_1(\mathbb{S}^k \setminus GQ_1) \downarrow & & \downarrow q(\mathbb{S}^k) \\ (\mathbb{S}^k/G) \setminus \bar{Q}_1 & \xhookrightarrow{i'_1(\mathbb{S}^k/G)} & \mathbb{S}^k/G. \end{array}$$

Now $\pi_1(i'_1(\mathbb{S}^k/G))$ is an isomorphism, and so $\psi_1(\mathbb{S}^k \setminus GQ_1) : \mathbb{S}^k \setminus GQ_1 \longrightarrow (\mathbb{S}^k/G) \setminus \bar{Q}_1$ is the universal covering by Lemma 1.1. In particular, $\pi_j(\psi_1(\mathbb{S}^k \setminus GQ_1))$ is an isomorphism. Further, as we saw just before Corollary 3.3, the map $i''_1(\mathbb{S}^k) : \mathbb{S}^k \setminus GQ_1 \hookrightarrow \mathbb{S}^k$ is null homotopic. We conclude from (3.1) that the homomorphism $\pi_j(i'_1(\mathbb{S}^k/G)) : \pi_j((\mathbb{S}^k/G) \setminus \bar{Q}_1) \longrightarrow \pi_j(\mathbb{S}^k/G)$ is trivial. But if $1 \leq i \leq n$, the composition of the map $i'_n(\mathbb{S}^k/G) : F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1) \hookrightarrow \prod_1^n \mathbb{S}^k/G$ with the projection $p_{J_i(\mathbb{S}^k/G)} : \prod_1^n \mathbb{S}^k/G \longrightarrow \mathbb{S}^k/G$ factors through the map $i'_1(\mathbb{S}^k/G) : (\mathbb{S}^k/G) \setminus \bar{Q}_1 \hookrightarrow \mathbb{S}^k/G$, and so the homomorphism $\pi_j(i'_n(\mathbb{S}^k/G)) : \pi_j(F_n((\mathbb{S}^k/G) \setminus \bar{Q}_1)) \longrightarrow \pi_j(\prod_1^n \mathbb{S}^k/G)$ is trivial. \square

Let $\Delta_j^n(\mathbb{S}^k/G)$ denote the diagonal subgroup of $\prod_1^n \pi_j(\mathbb{S}^k/G)$. We end this paper with the following proposition.

Proposition 3.6. *Let $k \geq 3$ be odd, let $j \geq 2$, and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of a finite group G on \mathbb{S}^k . Then:*

- (1) *the image of the homomorphism $\pi_j(i_n(\mathbb{S}^k/G)) : \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \prod_1^n \pi_j(\mathbb{S}^k/G)$ is the diagonal subgroup $\Delta_j^n(\mathbb{S}^k/G)$.*

(2) there are split short exact sequences:

$$(3.2) \quad 0 \longrightarrow \Delta_j^n(\mathbb{S}^k/G) \longrightarrow \pi_j\left(\prod_1^n \mathbb{S}^k/G\right) \longrightarrow \pi_{j-1}\left(\prod_1^{n-1} \Omega(\mathbb{S}^k/G)\right) \longrightarrow 0 \quad \text{and}$$

$$(3.3) \quad 0 \longrightarrow \pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ_1)) \longrightarrow \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \Delta_j^n(\mathbb{S}^k/G) \longrightarrow 0.$$

In particular, if G is the trivial group then there is a split short exact sequence:

$$0 \longrightarrow \pi_j(F_{n-1}(\mathbb{R}^k)) \longrightarrow \pi_j(F_n(\mathbb{S}^k)) \longrightarrow \Delta_j^n(\mathbb{S}^k) \longrightarrow 0.$$

Proof. Let $j \geq 2$.

(1) First, note that the quotient map $q(\mathbb{S}^k) : \mathbb{S}^k \longrightarrow \mathbb{S}^k/G$ induces an isomorphism $\pi_j(q(\mathbb{S}^k)) : \pi_j(\mathbb{S}^k) \longrightarrow \pi_j(\mathbb{S}^k/G)$. Let $s_j : \pi_j(\mathbb{S}^k/G) \longrightarrow \pi_j(F_n(\mathbb{S}^k/G))$ be defined by $s_j = \pi_j(\psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k)) \circ (\pi_j(q(\mathbb{S}^k)))^{-1}$. It follows from Lemma 3.4 that:

$$\pi_j(p_{J_i}(\mathbb{S}^k/G)) \big|_{\pi_j(F_n(\mathbb{S}^k/G))} \circ s_j = \text{id}_{\pi_j(\mathbb{S}^k/G)}$$

for all $1 \leq i \leq n$. In particular, the homomorphism $\pi_j(p_{J_i}(\mathbb{S}^k/G)) \big|_{\pi_j(F_n(\mathbb{S}^k/G))}$ is surjective and admits a section s_j . Taking the long exact sequence in homotopy of the fibration

$$F_{n-1}(\mathbb{S}^k/G \setminus \bar{Q}_1) \xrightarrow{j'_n(\mathbb{S}^k/G)} F_n(\mathbb{S}^k/G) \xrightarrow{p_{J_i}(\mathbb{S}^k/G) \big|_{F_n(\mathbb{S}^k/G)}} \mathbb{S}^k/G,$$

we obtain the following split short exact sequence:

$$0 \longrightarrow \pi_j(F_{n-1}(\mathbb{S}^k/G \setminus \bar{Q}_1)) \xrightarrow{\pi_j(j'_n(\mathbb{S}^k/G))} \pi_j(F_n(\mathbb{S}^k/G)) \xrightarrow{\pi_j(p_{J_i}(\mathbb{S}^k/G) \big|_{F_n(\mathbb{S}^k/G)})} \pi_j(\mathbb{S}^k/G) \longrightarrow 0.$$

In particular,

$$(3.4) \quad \pi_j(F_n(\mathbb{S}^k/G)) = \text{Im}(\pi_j(j'_n(\mathbb{S}^k/G))) \oplus \text{Im}(s_j).$$

Now let $1 \leq i, l \leq n$. By Lemma 3.4, the composition $p_{J_l}(\mathbb{S}^k/G) \big|_{F_n(\mathbb{S}^k/G)} \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k)$ is homotopic to $q(\mathbb{S}^k)$, from which we deduce that:

$$\pi_j(i_n(\mathbb{S}^k/G) \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k))(\pi_j(\mathbb{S}^k)) \subseteq \Delta_j^n(\mathbb{S}^k/G).$$

Conversely, if $\alpha \in \pi_j(\mathbb{S}^k/G)$ then:

$$(3.5) \quad \pi_j(i_n(\mathbb{S}^k/G) \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k))((\pi_j(q(\mathbb{S}^k)))^{-1}(\alpha)) = (\alpha, \dots, \alpha),$$

and we conclude that:

$$\begin{aligned} \Delta_j^n(\mathbb{S}^k/G) &= \pi_j(i_n(\mathbb{S}^k/G) \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k))(\pi_j(\mathbb{S}^k)) \\ &= \pi_j(i_n(\mathbb{S}^k/G) \circ \psi_n(\mathbb{S}^k) \circ \lambda_i(\mathbb{S}^k)) \circ (\pi_j(q(\mathbb{S}^k)))^{-1}(\pi_j(\mathbb{S}^k/G)) \\ &= \pi_j(i_n(\mathbb{S}^k/G)) \circ s_j(\pi_j(\mathbb{S}^k/G)), \end{aligned}$$

so

$$(3.6) \quad \pi_j(i_n(\mathbb{S}^k/G))(\text{Im}(s_j)) = \Delta_j^n(\mathbb{S}^k/G).$$

In the commutative diagram of fibrations (2.4), replace X by \mathbb{S}^k/G , and Q_1 by \bar{Q}_1 . By Lemma 3.5, the homomorphism

$$\pi_j(i'_{n-1}(\mathbb{S}^k/G)) : \pi_j(F_{n-1}((\mathbb{S}^k/G)\setminus\bar{Q}_1)) \longrightarrow \pi_j\left(\prod_1^{n-1} \mathbb{S}^k/G\right)$$

is trivial. Taking the long exact sequence in homotopy of (2.4), and using the commutativity of the resulting diagram, we deduce that $\text{Im}(\pi_j(j'_n(\mathbb{S}^k/G))) \subset \ker \pi_j(i_n(\mathbb{S}^k/G))$. By taking the image of (3.4), it follows from this inclusion and (3.6) that $\text{Im}(\pi_j(i_n(\mathbb{S}^k/G))) = \Delta_j^n(\mathbb{S}^k/G)$ as required.

(2) Consider the homotopy fibration

$$(3.7) \quad I_{i_n(\mathbb{S}^k/G)} \xrightarrow{p_{i_n(\mathbb{S}^k/G)}} F_n(\mathbb{S}^k/G) \xrightarrow{i_n(\mathbb{S}^k/G)} \prod_1^n \mathbb{S}^k/G.$$

Taking the long exact sequence in homotopy and using part (1) and the isomorphism between $\pi_j(I_{i_n(\mathbb{S}^k/G)})$ and $\pi_j(F_{n-1}^G(\mathbb{S}^k\setminus GQ_1))$ given by Proposition 2.7, we obtain the short exact sequence (3.3). Let $s'_j : \Delta_j^n(\mathbb{S}^k/G) \longrightarrow \pi_j(F_n(\mathbb{S}^k/G))$ be defined by $s'_j = s_j \circ \pi_j(p_{J_i}(\mathbb{S}^k/G)) \Big|_{\Delta_j^n(\mathbb{S}^k/G)}$. If $(\alpha, \dots, \alpha) \in \Delta_j^n(\mathbb{S}^k/G)$ then $\pi_j(i_n(\mathbb{S}^k/G)) \circ s'_j(\alpha, \dots, \alpha) = \pi_j(i_n(\mathbb{S}^k/G)) \circ s_j(\alpha) = (\alpha, \dots, \alpha)$ by (3.5), which shows that (3.3) splits.

We now derive the short exact sequence (3.2). We take $X = \mathbb{S}^k$ in Theorem (2.5)(3). The hypotheses of the theorem are satisfied because the inclusion $\mathbb{S}^k\setminus GQ_1 \hookrightarrow \mathbb{S}^k$ is null homotopic, and the homomorphism $\pi_m(q(\mathbb{S}^k)) : \pi_m(\mathbb{S}^k) \longrightarrow \pi_m(\mathbb{S}^k/G)$ is an isomorphism for all $m \geq 2$ and is injective if $m = 1$. As in the statement of Theorem (2.5)(3), we consider the homotopy fibration (3.7), and we identify $\pi_{j-1}(I_{i_n(\mathbb{S}^k/G)})$ with $\pi_j(F_{n-1}^G(\mathbb{S}^k\setminus GQ_1)) \times \pi_{j-1}(\prod_1^{n-1} \Omega(\mathbb{S}^k/G))$, and $\pi_{j-1}(\Omega(\mathbb{S}^k/G))$ with $\pi_j(\mathbb{S}^k/G)$. It then follows that the restriction to the subgroup $\pi_j(\prod_1^{n-1} \mathbb{S}^k/G)$ of $\pi_j(\prod_1^n \mathbb{S}^k/G)$ of the boundary homomorphism $\hat{\partial}_j : \pi_{j-1}(\Omega(\prod_1^n \mathbb{S}^k/G)) \longrightarrow \pi_{j-1}(I_{i_n(\mathbb{S}^k/G)})$ coincides with the inclusion of $\pi_j(\prod_1^{n-1} \mathbb{S}^k/G)$ in $\pi_{j-1}(F_{n-1}^G(\mathbb{S}^k\setminus GQ_1)) \times \pi_{j-1}(\Omega(\prod_1^{n-1} \mathbb{S}^k/G))$. On the other hand, if $\alpha \in \pi_j(\mathbb{S}^k/G)$ then by part (1), $(\alpha, \dots, \alpha) \in \text{Im}(\pi_j(i_n(\mathbb{S}^k/G)))$, and so $\hat{\partial}_j(\alpha, \dots, \alpha) = 0$ by exactness. Applying Theorem (2.5)(3) once more, we see that $\hat{\partial}_j(0, \dots, 0, \alpha) = \hat{\partial}_j(-\alpha, \dots, -\alpha, 0) = (0, -\alpha, \dots, -\alpha)$, where in the final expression, 0 (resp. $(-\alpha, \dots, -\alpha)$) corresponds to the $\pi_j(F_{n-1}^G(\mathbb{S}^k\setminus GQ_1))$ -factor (resp. to the $\pi_{j-1}(\prod_1^{n-1} \Omega(\mathbb{S}^k/G))$ -factor). It follows from these considerations that $\text{Im}(\hat{\partial}_j) = \pi_{j-1}(\prod_1^{n-1} \Omega(\mathbb{S}^k/G))$. We then obtain the short exact sequence (3.2) by noting that $\ker(\hat{\partial}_j) = \Delta_j^n(\mathbb{S}^k/G)$ by exactness and part (1). The fact that (3.2) splits is a consequence of Theorem (2.5)(3). \square

Note that Proposition 3.6 gives a complete description of the long exact homotopy sequence in homotopy of the homotopy fibration (3.7) in the case where G is finite and k is odd. The study of the configuration spaces $F_n(\mathbb{R}P^{2k})$ constitutes work in progress.

REFERENCES

- [1] G. Allaud, Delooping homotopy equivalences, *Arch. Math. Basel* **23** (1972), 167–169.
- [2] M. Arkowitz, Introduction to homotopy theory, Universitext, Springer, New York (2011).
- [3] J. S. Birman, On braid groups, *Comm. Pure and Appl. Math.* **22** (1969), 41–72.
- [4] J. S. Birman, Braids, links and mapping class groups, *Ann. Math. Stud.* **82**, Princeton University Press (1974).
- [5] G. E. Bredon, Introduction to compact transformation groups, Academic Press, New York and London (1972).
- [6] F. R. Cohen, Introduction to configuration spaces and their applications. In Braids, volume 19, *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, World Sci. Publ., Hackensack, NJ (2010), 183–261.
- [7] F. R. Cohen and M. A. Xicoténcatl, On orbit configuration spaces associated to the Gaussian integers: homotopy and homology groups, in Arrangements in Boston: a Conference on Hyperplane Arrangements (1999), *Topol. Appl.* **118** (2002), 17–29.
- [8] A. L. Edmonds, Taming free circle actions, *Proc. Amer. Math. Soc.* **62** (1977), 337–343.
- [9] E. Fadell and S. Y. Husseini, Fixed point theory for non-simply-connected manifolds, *Topology* **20** (1981), 53–92.
- [10] E. Fadell and S. Y. Husseini, Geometry and topology of configuration spaces, Springer Monographs in Mathematics, Springer-Verlag, Berlin (2001).
- [11] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962), 111–118.
- [12] E. E. Floyd, On periodic maps and the Euler characteristics of associated spaces, *Trans. Amer. Math. Soc.* **72** (1952), 138–147.
- [13] C. H. Goldberg, An exact sequence of braid groups, *Math. Scand.* **33** (1973), 69–82.
- [14] D. L. Gonçalves and J. Guaschi, Inclusion of configuration spaces in Cartesian products, and the virtual cohomological dimension of the braid groups of \mathbb{S}^2 and $\mathbb{R}P^2$, *Pacific J. Math.*, to appear.
- [15] D. L. Gonçalves and J. Guaschi, The homotopy fibre of the inclusion $F_n(M) \hookrightarrow \prod_1^n M$ for M either \mathbb{S}^2 or $\mathbb{R}P^2$ and orbit configuration spaces, in preparation.
- [16] P. Hilton, Homotopy Theory and Duality, Gordon and Breach Science Publishers, New York-London-Paris (1965).
- [17] J. M. Lee, Introduction to Smooth Manifolds, Graduate Texts in Mathematics **218**, Second edition, Springer, New York, 2013.
- [18] W. S. Massey, Algebraic topology: an introduction, reprint of the 1967 edition. Graduate Texts in Mathematics **56**. Springer-Verlag, New York-Heidelberg, 1977.
- [19] M. Mather, Pull-backs in homotopy theory, *Canad. J. Math.* **XXVIII**, No. 2, 1976, 225–263.
- [20] C. Westerland, Configuration spaces in topology and geometry, *Austral. Math. Soc. Gaz.* **38** (2011), 279–283.
- [21] A. Zabrodsky, Covering spaces of paracompact spaces, *Pacific J. Math.* **14** (1964), 1489–1503.

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