Spatial quantile predictions for elliptical random fields

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Introduction

Kriging (see Krige (1951)) aims at predicting the conditional mean of a random field $(Z_t)_{t\in T}$ given the values $Z_{t_1},...,Z_{t_N}$ of the field at some points $t_1,...,t_N \in T$, where typically $T \subset \mathbb{R}^d$. It seems natural to predict, in the same spirit as Kriging, other functionals. In our study, we focus on quantiles for elliptical random fields.

Elliptical Distributions

Cambanis et al. (1981) give the representation : the random vector $X \in \mathbb{R}^d$ is elliptical with parameters $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$, if and only if

$$X = \mu + R\Lambda U^{(d)}, \qquad (1)$$

where $\Lambda\Lambda^T = \Sigma$, $U^{(d)}$ is a $d-$dimensional random vector uniformly distributed on $\mathcal{S}^{d-1}$ (the unit disk of dimension d), and R is a non-negative random variable independent of $U^{(d)}$. Furthermore, X is said consistent if :

$$R \overset{d}{=} \frac{\chi_d}{\epsilon} \qquad (2)$$

Distribution	ϵ
Gaussian	1
Student, $\nu > 0$	$\frac{\chi_\nu}{\sqrt{\nu}}$
Unimodal Gaussian Mixture	$\sum_{k=1}^{n} \pi_k \delta_{\theta_k}$
Laplace, $\lambda > 0$	$\frac{1}{\sqrt{\mathcal{E}(\frac{1}{\lambda})}}$
Uniform Gaussian Mixture	$\mathcal{U}(]0,1[)$

Table 1: Some consistent distributions

Now, we consider $X = (X_1, X_2)^T$ be a consistent $(R,d)-$elliptical random vector with with $X_1 \in \mathbb{R}^{d_1}$, $X_2 \in \mathbb{R}^{d_2}$, $d_1 + d_2 = d$ and parameters μ and Σ. Let us write:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}. \qquad (3)$$

The conditional distribution $X_2|(X_1 = x_1)$ has parameters:

$$\begin{cases} \mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \Sigma_{2|1} = \quad \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{cases} \qquad (4)$$

Furthermore, $X_2|(X_1 = x_1)$ is elliptical, with radius R^* given by :

$$R^* \overset{d}{=} R\sqrt{1-\beta} \,\Big|\, \left(R\sqrt{\beta}U^{(d)} = C_{11}^{-1}(x_1 - \mu_1)\right) \qquad (5)$$

where C_{11} is the Cholesky root of Σ_{11}, and $\beta \sim Beta(\frac{d_1}{2}, \frac{d_2}{2})$.

We can now define the notion of elliptical random fields. Indeed, a random field $(X(t))_{t\in T}$ is $R-$elliptical if $\forall n \in \mathbb{N}$, $\forall t_1,...,t_n \in T$, the vector $(X(t_1),...,X(t_n))$ is $(R,n)-$elliptical.

Conditional quantiles

From now, we consider the following context: $(X(t))_{t\in T}$ is an $R-$elliptical random field. We consider N observations at locations $t_1,...,t_n \in T$, called $(X(t_1),...,X(t_N))$. Our aim is to predict, at a site $t \in T$, the quantile of $X(t)$ given $X(t_1),...,X(t_N)$. Notice that the vector $(X(t), X(t_1),...,X(t_N))$ is $(R, N+1)-$elliptical. Thus, we can denote $X_2 = X(t) \in \mathbb{R}$ and $X_1 = (X(t_1),...,X(t_N)) \in \mathbb{R}^N$ and restrict ourselves to the study of the $q_\alpha(X_2|X_1)$.

General case

We denote :

$$\begin{cases} \Phi_R(x) = \mathbb{P}(RU^{(1)} \le x) \\ \Phi_{R^*}(x) = \mathbb{P}(R^*U^{(1)} \le x) \end{cases} \qquad (6)$$

Then the $\alpha-$quantile of $X_2|(X_1 = x_1)$ is given by :

$$\boxed{q_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_{R^*}^{-1}(\alpha)} \qquad (7)$$

Gaussian case

Since a conditional Gaussian distribution is still Gaussian, we have :

$$X_2|(X_1 = x_1) \sim \mathcal{N}(\mu_{2|1}, \Sigma_{2|1}) \qquad (8)$$

Then, the calculation of the conditional $\alpha-$quantile of $X_2|(X_1 = x_1)$ is immediate, and gives :

$$q_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi^{-1}(\alpha) \qquad (9)$$

Student case

Even if it is more calculative, we can also get theoretical formula. The conditional $\alpha-$quantile of $X_2|(X_1 = x_1)$ has the following expression

$$q_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\sqrt{\frac{\nu}{\nu+N}}\sqrt{1 + \frac{1}{\nu}q_1}\Phi_{\nu+N}^{-1}(\alpha). \qquad (10)$$

We did not obtain such simple results for other elliptical distributions. It is why we propose, in what follows, two approaches.

Quantile Regression

Quantile regression, introduced by Koenker and Bassett (1978), approximates the conditional quantile as follows :

$$\hat{q}_\alpha(X_2|X_1 = x_1) = \beta^{*T}x_1 + \beta_0^*, \qquad (11)$$

where β^* and β_0^* are solutions of the following minimization problem

$$(\beta^*, \beta_0^*) = \underset{\beta\in\mathbb{R}^N, \beta_0\in\mathbb{R}}{\arg\min} \; \mathbb{E}[\phi_\alpha(X_2 - \beta^T X_1 - \beta_0)]. \qquad (12)$$

and where the scoring function ϕ_α is

$$\phi_\alpha(x) = (\alpha - 1)x\mathbb{1}_{\{x<0\}} + \alpha x\mathbb{1}_{\{x>0\}} = \alpha x - x\mathbb{1}_{\{x<0\}}. \qquad (13)$$

In our context of elliptical random fields, we are able to solve this minimization problem, and then define the Quantile Regression Predictor :

$$\boxed{\hat{q}_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_R^{-1}(\alpha)} \qquad (14)$$

Furthermore, its distribution is

$$\hat{q}_\alpha(X_2|X_1) \sim \mathcal{E}_1\left(\mu_2 + \sqrt{\Sigma_{2|1}}\Phi_R^{-1}(\alpha), \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, R\right) \qquad (15)$$

Extremal quantiles

In this section, the aim is to establish a relation between Φ_R^{-1} and $\Phi_{R^*}^{-1}$ for extremal values of α. For that, we do an assumption : Their exist $0 < \ell < +\infty$ and $\gamma \in \mathbb{R}$ such as :

$$\lim_{x\to+\infty} \frac{1 - \Phi_{R^*}(x)}{1 - \Phi_R(x^\gamma)} = \ell \qquad (16)$$

Under this assumption, we can define Extreme Conditional Quantiles Predictors :

$$\boxed{\begin{cases} \hat{\hat{q}}_{\alpha\uparrow}(X_2|X_1 = x_1) = \mu_{2|1} + \sigma_{2|1}\left[\Phi_R^{-1}\left(1 - \frac{1}{\frac{\ell}{1-\alpha} + 2(1-\ell)}\right)\right]^{\frac{1}{\gamma}} \\ \hat{\hat{q}}_{\alpha\downarrow}(X_2|X_1 = x_1) = \; \mu_{2|1} - \sigma_{2|1}\left[\Phi_R^{-1}\left(1 - \frac{1}{\frac{\ell}{\alpha} + 2(1-\ell)}\right)\right]^{\frac{1}{\gamma}} \end{cases}} \qquad (17)$$

Distribution	γ	ℓ
Gaussian	1	1
Student, $\nu > 0$	$\frac{N+\nu}{\nu}$	$\frac{\Gamma\left(\frac{\nu+N+1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+N}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{q_1}{\nu}\right)^{\frac{N+\nu}{2}}\frac{\nu^{\frac{N}{2}+1}}{\nu+N}$
Unimodal GM	1	$\frac{\min(\theta_1,...,\theta_n)^N \exp\left(-\frac{\min(\theta_1,...,\theta_n)^2}{2}q_1\right)}{\sum_{k=1}^{n}\pi_k\theta_k^N \exp\left(-\frac{\theta_k^2}{2}q_1\right)}$
Uniform GM	$N+1$	$\frac{\Gamma\left(\frac{N+2}{2}\right)q_1^{\frac{N+1}{2}}\sqrt{2}}{\Gamma\left(\frac{N+1}{2}\right)(N+1)\chi_{N+1}^2(q_1)}$

Table 2: Some examples

Thanks to the paper of Djurčić and Torgašev (2001), we are able to prove that these predictors $\hat{\hat{q}}_{\alpha\uparrow}$ and $\hat{\hat{q}}_{\alpha\downarrow}$ are asymptoticaly equivalent to the theoretical quantiles respectively when $\alpha \to 1$ and $\alpha \to 0$.

$$\begin{cases} \hat{\hat{q}}_{\alpha\uparrow}(X_2|X_1 = x_1) \underset{\alpha\to1}{\sim} q_\alpha(X_2|X_1 = x_1) \\ \hat{\hat{q}}_{\alpha\downarrow}(X_2|X_1 = x_1) \underset{\alpha\to0}{\sim} q_\alpha(X_2|X_1 = x_1) \end{cases} \qquad (18)$$

Numerical study

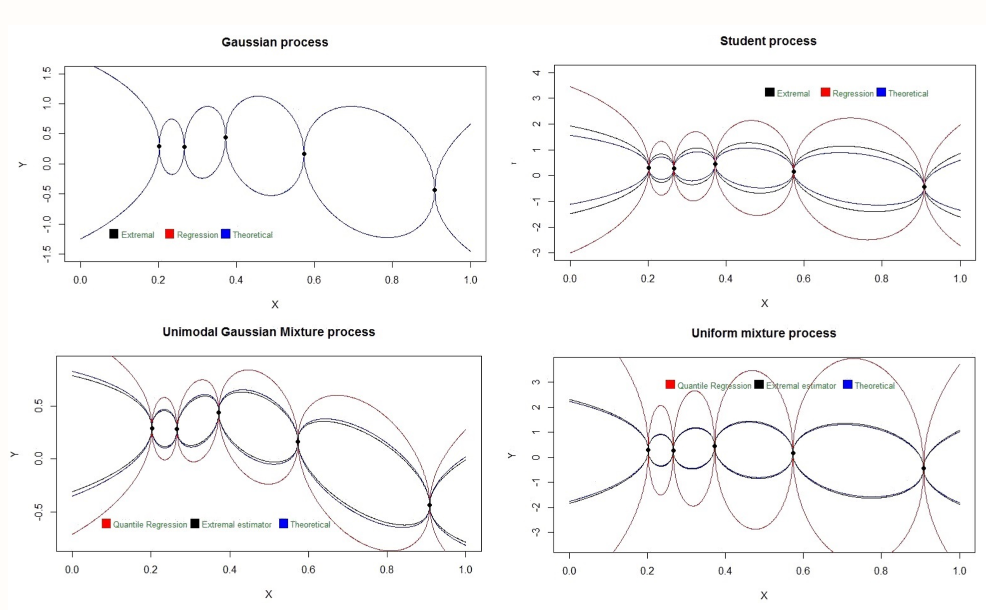

Figure 1: Levels of quantile $\alpha = 0.995$ and $\alpha = 0.005$

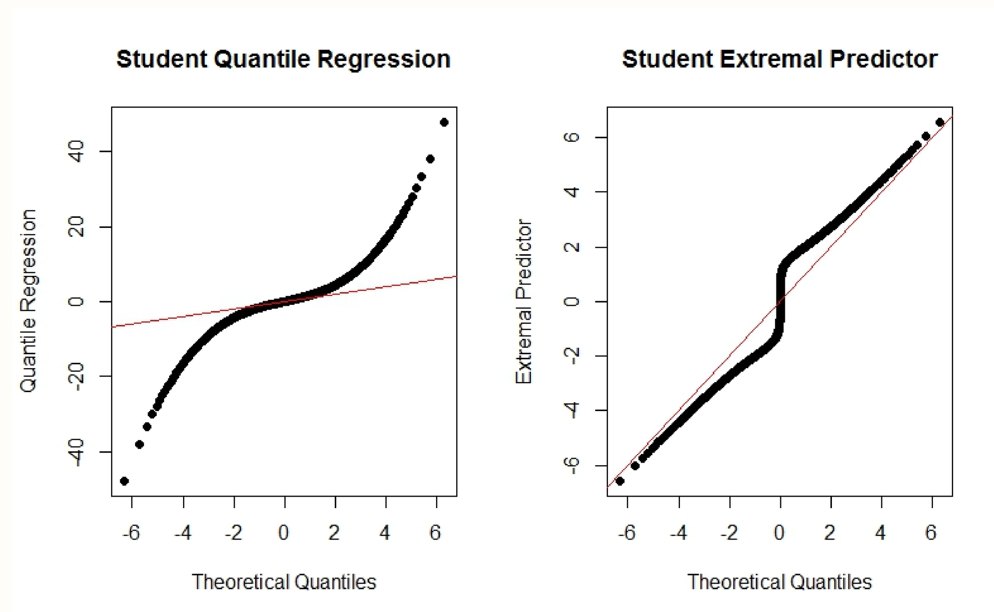

Figure 2: Q-Q plots for Student example

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