Convergence to equilibrium for a second-order time semi-discretization of the Cahn-Hilliard equation

Paola Francesca Antonietti, Benoît Merlet, Morgan Pierre, Marco Verani

Convergence to equilibrium for a second-order time semi-discretization of the Cahn-Hilliard equation*

Paola F. Antonietti[♯], Benoît Merlet[†], Morgan Pierre[♭], Marco Verani[♯]

August 19, 2016

[♯] MOX– Laboratory for Modeling and Scientific Computing
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano, Italy
paola.antonietti@polimi.it, marco.verani@polimi.it

[†] Laboratoire Paul Painlevé, U.M.R. CNRS 8524
Université Lille 1, Cité Scientifique
F-59655 Villeneuve d'Ascq Cedex, France
and
Team RAPSODI
Inria Lille - Nord Europe, 40 av. Halley, F-59650 Villeneuve d'Ascq, France
merlet@cmap.polytechnique.fr

[♭] Laboratoire de Mathématiques et Applications
Université de Poitiers, CNRS
F-86962 Chasseneuil, France
Morgan.Pierre@math.univ-poitiers.fr

Keywords: Łojasiewicz-Simon inequality, gradient flow, Cahn-Hilliard, backward differentiation formula.

AMS Subject Classification: 65M10, 65P05, 37M.

Abstract

We consider a second-order two-step time semi-discretization of the Cahn-Hilliard equation with an analytic nonlinearity. The time-step is chosen small enough so that the pseudo-energy associated with the discretization is nonincreasing at every time iteration. We prove that the sequence generated by the scheme converges to a steady state as time tends to infinity. We also obtain convergence rates in the energy norm. The proof is based on the Łojasiewicz-Simon inequality.

*Paola F. Antonietti has been partially supported by SIR Project No. RBSI14VT0S *"PolyPDEs: Non-conforming polyhedral finite element methods for the approximation of partial differential equations"* funded by MIUR.

1 Introduction

In this paper, we consider a second-order time semi-discretization of the Cahn-Hilliard equation with an analytic nonlinearity, and we prove that any sequence generated by the scheme converges to a steady state as time goes to infinity, provided that the time-step is chosen small enough.

The Cahn-Hilliard equation [10] reads

$$\begin{cases} u_t = \Delta w \\ w = -\gamma \Delta u + f(u) \end{cases} \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^d$ $(1 \le d \le 3)$ with smooth boundary and $\gamma > 0$. A typical choice for the nonlinearity is

$$f(s) = c(s^3 - s) \tag{1.2}$$

with $c > 0$. More general conditions on f are given in Section 2, see (2.3)-(2.5). Equation (1.1) is completed with Neumann boundary conditions and an initial data.

The Cahn-Hilliard equation was analyzed by many authors and used in different contexts (see, e.g., [11, 37] and references therein). In particular, it is a H^{-1} gradient flow for the energy

$$E(u) = \int_\Omega \frac{\gamma}{2} |\nabla u|^2 + F(u)\, dx,$$

where F is an antiderivative of f. Convergence of single trajectories to equilibrium for (1.1)-(1.2) has been proved in [42]. The proof uses the gradient flow structure of the equation and a Łojasiewicz-Simon inequality [44].

In one space dimension, the set of steady states corresponding to (1.1)-(1.2) is finite [24, 32]. In this case, the use of a Łojasiewicz-Simon inequality can be avoided [51] but otherwise, the situation is highly complicated; if $d = 2$ or 3, there may even be a continuum of stationary solutions (see, e.g., [47] and references therein). The Łojasiewicz-Simon inequality allows to prove convergence to an equilibrium without any knowledge on the set of steady states. This celebrated inequality is based on the analyticity of f (see [27] for a recent overview). In contrast, for the related semilinear parabolic equation, convergence to equilibrium may fail for a nonlinearity of class C^∞ [39].

Using similar techniques, convergence to equilibrium for the non-autonomous Cahn-Hilliard equation was proved in [15], and the case of a logarithmic nonlinearity was considered in [1]. The Cahn-Hilliard equation endowed with dynamic or Wentzell boundary conditions was analyzed in [14, 40, 48, 49]. Coupled systems were also considered (see, e.g., [18, 30, 41]).

Since many space and/or time discretizations of the Cahn-Hilliard equation are available in the literature (see, e.g., [5, 17, 20, 21, 22, 26, 36, 43, 50]), it is natural to ask whether convergence to equilibrium also holds for these discretizations, by using similar techniques.

If we consider only a space semi-discretization of (1.1), and if this discretization can be shown to preserve the gradient flow structure, then convergence to equilibrium is a consequence of Łojasiewicz's classical convergence result [33] and its generalizations [8, 27]. Thanks to the finite dimension, the Łojasiewicz-Simon inequality reduces to the standard Łojasiewicz inequality. The latter is a direct consequence of analyticity of the discrete energy functional.

Thus, the situation regarding the space discretization is well understood, and we believe that the focus should be put on the time discretization, in the specific case where the time scheme preserves the gradient flow structure. In this regard, convergence to equilibrium for a fully discrete version of (1.1)-(1.2) was first proved in [34]: the time scheme was the backward Euler scheme and the space discretization was a finite element method. Fully discretized versions of Cahn-Hilliard type equations were considered in [12, 13, 29], where this nice feature of the backward Euler scheme was again demonstrated (see also [6, 25]). In [4], convergence to equilibrium was proved for several fully discretized versions of the closely related Allen-Cahn equation; the time scheme was either first order or second order, conditionnally or unconditionnally stable, and the time-step could possibly be variable. In addition, general conditions ensuring convergence to equilibrium for a time discretization were given (see also [7]).

Therefore, the fully discrete case is now also well understood. The last stage is to study the time semi-discrete case. This is all the more interesting since this approach is independent of a choice of a specific space discretization. Convergence to equilibrium was proved for the backward Euler time semi-discretization of the Allen-Cahn equation in [34] (see also [9]). A related damped wave equation was considered in [38].

For schemes different from the backward Euler method, the situation is not so clear, and this is well illustrated by the second order case. Indeed, there exist several second-order time semi-discretizations of (1.1)-(1.2) which preserve the gradient flow structure (see, e.g., [43, 50] and references therein). Most of these schemes are *one-step* methods, which can be seen as variants of the Crank-Nicolson scheme, such as the classical secant scheme [16, 17] or the more recent scheme of Gomez and Hughes [21], which is a Crank-Nicolson scheme with stabilization.

However, we have not been able to prove convergence to equilibrium for any of these second-order one-step schemes. One difficulty is that the gradient of E (cf. (3.2)) is treated in an implicit/explicit way, and another difficulty is that the discrete dynamical system associated with the scheme is defined on a space of infinite dimension. The first difficulty can be circumvented in finite dimension, as recently shown in [23], where convergence to equilibrium was proved for a fully discrete approximation of the modified phase-field crystal equation using the second-order time discretization of Gomez and Hughes. A related difficulty has been pointed out in [46] where the stability of the Crank-Nicolson scheme for the Navier-Stokes equation was proved in a finite dimensional setting only.

In this paper, instead of a Crank-Nicolson type method, we use a standard *two-step* scheme with fixed time-step, namely the backward differentiation formula of order two. It is well-known [43, 45] that this scheme enjoys a Lyapunov stability, namely, if the time-step is small enough, a so-called pseudo-energy (cf. (2.17)) is nonincreasing at every time iteration. Thanks to the implicit treatment of the gradient of E (cf. (2.13)), the proof of convergence is similar to the case of the backward Euler scheme in [34, 38]. Using the Lyapunov stability, we first prove Lasalle's invariance principle by a compactness argument (Proposition 3.1). Convergence to a steady state is then obtained as a consequence of an appropriate Łojasiewicz-Simon inequality (Lemma 3.2), which is the most technical point. In order to derive the convergence rate in H^1 norm, we also take advantage of the fact that the scheme is more dissipative than the original equation (see Remark 2.4).

It would be interesting to extend our convergence result to first-order or second-order schemes where the nonlinearity is treated explicitly. In order for such schemes to preserve the gradient structure, the standard approach is to truncate the cubic nonlinearity f (cf. (1.2)) at $\pm\infty$ so as to have a linear growth at most [43]. However, it is not known if the energy associated with such a nonlinearity satisfies a Łojasiewicz-Simon inequality, in contrast with the finite-dimensional case where it can be proved for certain space discretizations [4].

It could also be of interest to investigate whether a similar convergence result holds for the p-step backward differentiation formula (BDF), with $p \geq 3$. A favorable situation is the 3-step BDF method, which preserves the gradient flow structure, at least in finite dimension [45].

The paper is organized as follows. In Section 2, we introduce the scheme, we establish its well-posedness and we show that it is Lyapunov stable. In Section 3, we prove the convergence result.

2 The time semi-discrete scheme

2.1 Notation and assumptions

Let $H = L^2(\Omega)$ be equipped with the $L^2(\Omega)$ norm $|\cdot|_0$ and the $L^2(\Omega)$ scalar product $(\cdot,\cdot)$. We denote $V = H^1(\Omega)$ the standard Sobolev space based on the $L^2(\Omega)$ space. We use the hilbertian semi-norm $|\cdot|_1 = |\nabla\cdot|_0$ in V, and the norm in V is $\|v\|_1^2 = |v|_0^2 + |v|_1^2$. We denote $-\Delta : V \to V'$ the bounded operator associated with the inner product on V through

$$\langle -\Delta u, v\rangle_{V',V} = (\nabla u, \nabla v), \quad \forall u, v \in V,$$

where V' is the topological dual of V. As usual, we will denote $W^{k,p}(\Omega)$ the Sobolev spaces based on the $L^p(\Omega)$ space [19].

For a function $u \in L^2(\Omega)$, we denote

$$\langle u\rangle = \frac{1}{|\Omega|}\int_\Omega u \quad \text{and} \quad \dot{u} = u - \langle u\rangle,$$

where $|\Omega|$ is the Lebesgue measure of Ω. We also define

$$\dot{H} = \{u \in L^2(\Omega),\ \langle u\rangle = 0\}, \quad \dot{V} = V \cap \dot{H}.$$

We will use the continuous and dense injections

$$\dot{V} \subset \dot{H} = \dot{H}' \subset \dot{V}'.$$

As a consequence of the Poincaré-Wirtinger inequality, the norms $\|v\|_1$ and

$$v \mapsto (|v|_1^2 + \langle v \rangle^2)^{1/2} \tag{2.1}$$

are equivalent on V. The operator $-\dot{\Delta} : \dot{V} \to \dot{V}'$, that is the restriction of $-\Delta$, is an isomorphism. The scalar product in $\dot{V}'$ is given by

$$(\dot{u}, \dot{v})_{-1} = (\nabla(-\dot{\Delta})^{-1}\dot{u}, \nabla(-\dot{\Delta})^{-1}\dot{v}) = \langle \dot{u}, (-\dot{\Delta})^{-1}\dot{v} \rangle_{\dot{V}',\dot{V}}$$

and the norm is given by

$$|\dot{u}|_{-1}^2 = (\dot{u}, \dot{u})_{-1} = \langle \dot{u}, (-\dot{\Delta})^{-1}\dot{u} \rangle_{\dot{V}',\dot{V}}.$$

We recall the interpolation inequality

$$|\dot{u}|_0^2 \leq |\dot{u}|_{-1}|\dot{u}|_1, \quad \forall \dot{u} \in \dot{V}. \tag{2.2}$$

We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is *analytic* and if $d \geq 2$, we assume in addition that there exist a constant $C > 0$ and a real number $p \geq 0$ such that

$$|f'(s)| \leq C(1 + |s|^p), \quad \forall s \in \mathbb{R}, \tag{2.3}$$

with $p < 4$ if $d = 3$. No growth assumption is needed if $d = 1$. We also assume that

$$f'(s) \geq -c_f, \quad \forall s \in \mathbb{R}, \tag{2.4}$$

for some (optimal) nonnegative constant c_f, and that

$$\liminf_{|s| \to +\infty} \frac{f(s)}{s} > 0. \tag{2.5}$$

We define the energy functional

$$E(u) = \frac{\gamma}{2}|u|_1^2 + (F(u), 1), \tag{2.6}$$

where $F(s)$ is a given antiderivative of f. The Sobolev injection $V \subset L^{p+2}(\Omega)$ and the growth assumption (2.3) ensure that $E(u) < +\infty$ and $f(u) \in V'$, for all $u \in V$. In fact, by [31, Corollaire 17.8], the functional E is of class C^2 on V. For any $u, v, w \in V$, we have

$$\langle dE(u), v \rangle_{V',V} = \int_\Omega [\gamma \nabla u \cdot \nabla v + f(u)v] dx, \tag{2.7}$$

$$\langle d^2E(u)v, w \rangle_{V',V} = \int_\Omega [\gamma \nabla v \cdot \nabla w + f'(u)vw] dx, \tag{2.8}$$

where $dE(u) \in V'$ is the first differential of E at u and $d^2E(u) \in \mathcal{L}(V, V')$ is the differential of order two of E at u.

If u is a regular solution of (1.1), on computing we see that

$$\frac{d}{dt}E(u(t)) = -|w|_1^2 = -|u_t|_{-1}^2 \quad t \geq 0, \tag{2.9}$$

so that E is a Lyapunov functional associated with (1.1).

2.2 Existence, uniqueness and Lyapunov stability

Let $\tau > 0$ denote the time-step. The second-order backward differentiation scheme for (1.1) reads [43, 45]: let $(u_0, u_1) \in V \times V$ and for $n = 1, 2, \ldots$, let $(u_{n+1}, w_{n+1}) \in V \times V$ solve

$$\begin{cases} \frac{1}{2\tau}(3u_{n+1} - 4u_n + u_{n-1}, \varphi) + (\nabla w_{n+1}, \nabla\varphi) = 0 \\ (w_{n+1}, \psi) = \gamma(\nabla u_{n+1}, \nabla\psi) + (f(u_{n+1}), \psi), \end{cases} \tag{2.10}$$

for all $(\varphi, \psi) \in V \times V$. For simplicity, we assume that

$$\langle u_0 \rangle = \langle u_1 \rangle, \tag{2.11}$$

so that, by induction, any sequence (u_n) which complies with (2.10) satisfies $\langle u_n \rangle = \langle u_0 \rangle$ for all n (choose $\varphi = 1/|\Omega|$ in (2.10)). We note that w_0 and w_1 need not be defined.

For later purpose, we note that if $\langle u_n \rangle = \langle u_{n-1} \rangle$, then (2.10) is equivalent to

$$\begin{cases} \langle u_{n+1} \rangle = \langle u_n \rangle \\ (-\dot{\Delta})^{-1} \frac{(3u_{n+1} - 4u_n + u_{n-1})}{2\tau} + \dot{w}_{n+1} = 0 \\ \dot{w}_{n+1} = -\gamma\Delta u_{n+1} + f(u_{n+1}) - \langle f(u_{n+1}) \rangle \\ \langle w_{n+1} \rangle = \langle f(u_{n+1}) \rangle. \end{cases} \tag{2.12}$$

Eliminating w_{n+1} leads to

$$(-\dot{\Delta})^{-1} \frac{(3u_{n+1} - 4u_n + u_{n-1})}{2\tau} - \gamma\Delta u_{n+1} + f(u_{n+1}) - \langle f(u_{n+1}) \rangle = 0. \tag{2.13}$$

Proposition 2.1 (Existence for all τ)**.** *For all $(u_0, u_1) \in V \times V$ such that $\langle u_0 \rangle = \langle u_1 \rangle$, there exists at least one sequence $(u_n, w_n)_n$ which complies with* (2.10)*. Moreover, $\langle u_n \rangle = \langle u_0 \rangle$ for all n.*

Proof. Existence can be obtained by minimizing an appropriate functional. By induction, assume that for some $n \geq 1$, $(u_{n-1}, u_n) \in V \times V$ is defined, with $\langle u_n \rangle = \langle u_{n-1} \rangle = \langle u_0 \rangle$. Then, by (2.13), u_{n+1} can be obtained by solving

$$\min \{\mathcal{G}_n(v) \ : \ v \in V, \ \langle v \rangle = \langle u_0 \rangle\}, \tag{2.14}$$

where

$$\mathcal{G}_n(v) = \frac{3}{4\tau}|\dot{v}|_{-1}^2 + \frac{1}{2\tau}(-4\dot{u}_n + \dot{u}_{n-1}, \dot{v})_{-1} + E(v).$$

By (2.5), there exist $\kappa_1 > 0$ and $\kappa_2 \geq 0$ such that

$$F(s) \geq \kappa_1 s^2 - \kappa_2, \quad \forall s \in \mathbb{R}.$$

Thus, for all $v \in V$,

$$(F(v), 1) \geq \kappa_1 |v|_0^2 - \kappa_2 |\Omega|,$$

and so

$$E(v) \geq \kappa_3 \|v\|_1^2 - \kappa_2 |\Omega|, \tag{2.15}$$

with $\kappa_3 = \min\{\gamma/2, \kappa_1\} > 0$. Moreover, by the Cauchy-Schwarz inequality,

$$|(-4\dot{u}_n + \dot{u}_{n-1}, \dot{v})_{-1}| \leq |\dot{v}|_{-1} |-4\dot{u}_n + \dot{u}_{n-1}|_{-1} \leq \frac{3}{2}|\dot{v}|_{-1}^2 + C_n,$$

for some constant C_n which depends on $|\dot{u}_n|_{-1}$ and $|\dot{u}_{n-1}|_{-1}$. Summing up, we have proved that

$$\mathcal{G}_n(v) \geq \kappa_3 \|v\|_1^2 - \kappa_2|\Omega| - \frac{C_n}{2\tau}.$$

By considering a minimizing sequence (v_k) for problem (2.14), we obtain a minimizer, i.e. u_{n+1}. Then w_{n+1} can be recovered from u_{n+1} by (2.12). □

Proposition 2.2 (Uniqueness). *If $1/\tau > c_f^2/(6\gamma)$, then for every $(u_0, u_1) \in V \times V$ such that $\langle u_0 \rangle = \langle u_1 \rangle$, there exists at most one sequence $(u_n, w_n)_n$ which complies with* (2.10).

Proof. Assume that (u_{n+1}, w_{n+1}) and $(\tilde{u}_{n+1}, \tilde{w}_{n+1})$ are two solutions of (2.10), and denote $\delta u = u_{n+1} - \tilde{u}_{n+1}$, $\delta w = w_{n+1} - \tilde{w}_{n+1}$. On subtracting, we obtain

$$\begin{aligned} &3(\delta u, \varphi)/(2\tau) + (\nabla \delta w, \nabla \varphi) = 0, \\ &(\delta w, \psi) = \gamma(\nabla \delta u, \nabla \psi) + (f(u_{n+1}) - f(\tilde{u}_{n+1}), \psi), \end{aligned} \tag{2.16}$$

for all $(\varphi, \psi) \in V \times V$. Choosing $\varphi = \delta w$ and $\psi = \delta u$, yields

$$-(2\tau/3)|\delta w|_1^2 = \gamma|\delta u|_1^2 + (f(u_{n+1}) - f(\tilde{u}_{n+1}), \delta u).$$

Using the mean value inequality and (2.4) yields

$$(s-r)(f(s) - f(r)) = f'(\xi)(s-r)^2 \geq -c_f(s-r)^2,$$

for all $r, s \in \mathbb{R}$, for some $\xi \in \mathbb{R}$ depending on r, s. Thus,

$$c_f|\delta u|_0^2 \geq \gamma|\delta u|_1^2 + (2\tau/3)|\delta w|_1^2.$$

Using now (2.16) with $\varphi = \delta u$, we obtain

$$c_f|\delta u|_0^2 = -(2\tau c_f/3)(\nabla \delta w, \nabla \delta u) \leq \gamma|\nabla \delta u|_0^2 + \frac{\tau^2 c_f^2}{9\gamma}|\nabla \delta w|^2.$$

If $\tau c_f^2 < 6\gamma$, then $\delta \dot{w} = 0$, and by (2.16), $\delta u = 0$ also. Uniqueness follows. □

We define the following *pseudo-energy*

$$\mathcal{E}(u, v) = E(u) + \frac{1}{4\tau}|\dot{v}|_{-1}^2, \quad \forall (u, v) \in V \times V'. \tag{2.17}$$

For a sequence $(u_n)_n$, let also $\delta u_n = u_n - u_{n-1}$ denote the backard difference. The following relation will prove useful,

$$3u_{n+1} - 4u_n + u_{n-1} = 2\delta u_{n+1} + (\delta u_{n+1} - \delta u_n). \tag{2.18}$$

Proposition 2.3 (Lyapunov stability)**.** *Let* $\varepsilon \in [0,1)$. *If* $(u_n, w_n)_n$ *is a sequence which complies with* (2.10)-(2.11), *then for all* $n \geq 1$,

$$\mathcal{E}(u_{n+1}, \delta u_{n+1}) + \frac{\varepsilon\gamma}{2}|u_{n+1} - u_n|_1^2 + \left(\frac{1}{\tau} - \frac{c_f^2}{8\gamma(1-\varepsilon)}\right)|u_{n+1} - u_n|_{-1}^2$$
$$+\frac{1}{4\tau}|\delta u_{n+1} - \delta u_n|_{-1}^2 \leq \mathcal{E}(u_n, \delta u_n). \tag{2.19}$$

Proof. We take the L^2 scalar product of equation (2.13) by δu_{n+1} and we use (2.18). We obtain

$$\frac{1}{\tau}|\delta u_{n+1}|_{-1}^2 + \frac{1}{2\tau}(\delta u_{n+1} - \delta u_n, \delta u_{n+1})_{-1} + \gamma(\nabla u_{n+1}, \nabla(u_{n+1} - u_n))$$
$$= (f(u_{n+1}), u_n - u_{n+1}).$$

By the Taylor-Lagrange formula, from (2.4), we deduce that

$$F(r) - F(s) \geq f(s)(r-s) - \frac{c_f}{2}|r-s|^2, \quad \forall r, s \in \mathbb{R}.$$

Thus,

$$(f(u_{n+1}), u_n - u_{n+1}) \leq (F(u_n), 1) - (F(u_{n+1}), 1) + \frac{c_f}{2}|u_{n+1} - u_n|_0^2.$$

Next, we use the well-known identity

$$(a, a-b)_m = \frac{1}{2}|a|_m^2 - \frac{1}{2}|b|_m^2 + \frac{1}{2}|a-b|_m^2,$$

for $m = -1$ and $m = 0$. We find

$$\frac{1}{\tau}|\delta u_{n+1}|_{-1}^2 + \frac{1}{4\tau}\left(|\delta u_{n+1}|_{-1}^2 - |\delta u_n|_{-1}^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right)$$
$$+\frac{\gamma}{2}\left(|u_{n+1}|_1^2 - |u_n|_1^2 + |u_{n+1} - u_n|_1^2\right)$$
$$\leq (F(u_n), 1) - (F(u_{n+1}), 1) + \frac{c_f}{2}|u_{n+1} - u_n|_0^2. \tag{2.20}$$

The interpolation inequality (2.2) and Young's inequality yield

$$\frac{c_f}{2}|u_{n+1} - u_n|_0^2 \leq \frac{\gamma(1-\varepsilon)}{2}|u_{n+1} - u_n|_1^2 + \frac{c_f^2}{8\gamma(1-\varepsilon)}|u_{n+1} - u_n|_{-1}^2.$$

Plugging this into (2.20) gives (2.19). □

Remark 2.4. If τ is small enough, then by choosing $\varepsilon \in (0,1)$, we see that the scheme (2.10) is more dissipative than the original equation (1.1), since the H^1 norm $|u_{n+1} - u_n|_1^2$ appears in (2.19); in contrast, only the H^{-1} norm $|u_t|_{-1}^2$ appears in (2.9).

3 Convergence to equilibrium

For a sequence $(u_n)_n$ in V, we define its omega-limit set by

$$\omega((u_n)_n) := \{u_\star \in V \ : \ \exists n_k \to \infty,\ u_{n_k} \to u_\star \text{ (strongly) in } V\}.$$

Let $M \in \mathbb{R}$ be given and consider the following affine subspace of V,

$$V_M = \{v \in V \ : \ \langle v \rangle = M\} = M + \dot{V}. \tag{3.1}$$

The set of critical points of E (see (2.6)) in V_M is

$$\mathcal{S}_M = \{u^\star \in V_M \ : \ -\gamma \Delta u^\star + f(u^\star) - \langle f(u^\star) \rangle = 0 \text{ in } \dot{V}'\}.$$

Indeed, we already know that $E \in C^2(V_M;\mathbb{R})$. Observe that, for any $u \in V_M$, $\dot{v} \in \dot{V}$, we have (see (2.7))

$$\begin{aligned}
\langle dE(u), v\rangle_{\dot{V}',\dot{V}} &= \int_\Omega [\gamma \nabla u \cdot \nabla v + f(u)v]dx \\
&= \int_\Omega [\gamma \nabla u \cdot \nabla v + (f(u) - \langle f(u)\rangle)v]dx \\
&= \langle -\gamma\Delta u + f(u) - \langle f(u)\rangle, v\rangle_{\dot{V}',\dot{V}}.
\end{aligned} \tag{3.2}$$

By definition, $u^\star$ is a critical point of E in V_M if $dE(u^\star) = 0$ in $\dot{V}'$. The definition of $\mathcal{S}_M$ follows.

Proposition 3.1. *Assume that $1/\tau > c_f^2/(8\gamma)$ and let $(u_n, w_n)_n$ be a sequence which complies with (2.10)-(2.11). Then $\delta u_n \to 0$ in V and $\omega((u_n)_n)$ is a nonempty compact and connected subset of V which is included in $\mathcal{S}_M$ with $M = \langle u_0 \rangle$. Moreover, E is constant on $\omega((u_n)_n)$.*

Proof. Using the assumption on τ, we may choose $\varepsilon \in (0,1)$ such that $1/\tau = c_f^2/(8\gamma(1-\varepsilon))$. Then (2.19) reads

$$\mathcal{E}(u_{n+1}, \delta u_{n+1}) + \frac{\varepsilon\gamma}{2}|u_{n+1} - u_n|_1^2 + \frac{1}{4\tau}|\delta u_{n+1} - \delta u_n|_{-1}^2 \le \mathcal{E}(u_n, \delta u_n), \tag{3.3}$$

for all $n \ge 1$. In particular, $(\mathcal{E}(u_n, \delta u_n))_n$ is non increasing. Moreover, by (2.15),

$$\mathcal{E}(u, v) \ge \kappa_3 \|u\|_1^2 + \frac{1}{4\tau}|\dot{v}|_{-1}^2 - \kappa_2|\Omega|, \quad \forall (u, v) \in V \times V'. \tag{3.4}$$

Since $\mathcal{E}(u_1, \delta u_1) < +\infty$, we deduce from (3.4) that $(u_n, \delta u_n)$ is bounded in $V \times V'$ and that $\mathcal{E}(u_n, \delta u_n)$ is bounded from below. Thus, $\mathcal{E}(u_n, \delta u_n)$ converges to some $\mathcal{E}_\star$ in $\mathbb{R}$. By induction, from (3.3)-(3.4) we also deduce that

$$\sum_{n=1}^{\infty} |u_{n+1} - u_n|_1^2 \le \frac{2}{\varepsilon\gamma}(\mathcal{E}(u_1, \delta u_1) + \kappa_2|\Omega|) < +\infty.$$

In particular, $\delta u_n \to 0$ in V. This implies that $E(u_n) \to \mathcal{E}_\star$, and so E is equal to $\mathcal{E}_\star$ on $\omega((u_n)_n)$.

Next, we claim that the sequence (u_n) is precompact in V. Let us first assume $d = 3$. We deduce from the Sobolev imbedding [19] that (u_n) is bounded in $L^6(\Omega)$. By the growth condition (2.3), there exists $2 \geq q > 6/5$ such that $\|f(u_{n+1})\|_{L^q(\Omega)} \leq M_1$, where M_1 is independent of n. By elliptic regularity [3], we deduce from (2.13) that $(u_{n+1})_{n\geq 1}$ is bounded in $W^{2,q}(\Omega)$. Finally, from the Sobolev imbedding [19], $W^{2,q}(\Omega)$ is compactly imbedded in V, and the claim is proved.

In the case $d = 1$ or 2, we obtain directly from the Sobolev imbedding that $f(u_{n+1})$ is bounded in $L^q(\Omega)$, for any $q < +\infty$, and we conclude similarly.

As a consequence, $\omega((u_n)_n)$ is a nonempty compact subset of V. Since $|u_{n+1} - u_n|_1 \to 0$, $\omega((u_n)_n)$ is also connected. Let finally $u_\star$ belong to $\omega((u_n)_n)$, with $n_k \to \infty$ such that $u_{n_k} \to u_\star$ in V. We let n_k tend to ∞ in (2.13). Thanks to (2.11), the whole sequence (u_n) belongs to V_M and $u_\star$ as well, where $M = \langle u_0 \rangle$. By (2.18), the term corresponding to the discrete time derivative tends to 0 in V, and we obtain that $u_\star$ belongs to $\mathcal{S}_M$. □

If the critical points of E are isolated, i.e. $\mathcal{S}_M$ is discrete, then Proposition 3.1 ensures that the sequence $(u_n)_n$ converges in V. However, as pointed out in the introduction, the structure of $\mathcal{S}_M$ is generally not known, and there may even be a continuum of steady states. In such cases, the Łojasiewicz-Simon inequality which follows is needed to ensure convergence of the whole sequence (u_n).

Lemma 3.2. *Let $u^\star \in \mathcal{S}_M$. Then there exist constants $\theta \in (0, 1/2)$ and $\delta > 0$ depending on $u^\star$ such that, for any $u \in V_M$ satisfying $|u - u^\star|_1 < \delta$, there holds*

$$|E(u) - E(u^\star)|^{1-\theta} \leq | - \gamma\Delta u + f(u) - \langle f(u)\rangle|_{-1}. \tag{3.5}$$

Proof. We will apply the abstract result of Theorem 11.2.7 in [27]. We introduce the auxiliary functional $E_M(v) = E(M + v)$ on $\dot{V}$. We will also use the auxiliary functions

$$f_M(s) = f(M + s) \quad \text{and} \quad F_M(s) = F(M + s).$$

It is obvious that

$$E_M(v) = \int_\Omega \frac{\gamma}{2}|\nabla v|^2 + F_M(v)\, dx.$$

The function E_M is of class C^2 on $\dot{V}$ and by (3.2), for any $v \in \dot{V}$, we have

$$dE_M(v) = -\gamma\Delta v + f_M(v) - \langle f_M(v)\rangle \text{ in } \dot{V}'.$$

Similarly, by (2.8), for any $v, \varphi \in \dot{V}$, we have

$$d^2E_M(v)\varphi = -\gamma\Delta\varphi + f'_M(v)\varphi - \langle f'_M(v)\varphi\rangle \text{ in } \dot{V}'. \tag{3.6}$$

Let $v^\star \in \dot{V}$ be a critical point of E_M, i.e. a solution of $dE_M(v^\star) = 0$ in $\dot{V}'$. Using (2.3) and elliptic regularity, we obtain that $v^\star \in C^0(\overline{\Omega}) \subset L^\infty(\Omega)$. In particular, $f'_M(v) \in L^\infty(\Omega)$. The operator $A = d^2E_M(v^\star) \in \mathcal{L}(\dot{V}, \dot{V}')$ (cf. (3.6)) can be written

$$A = -\gamma\Delta + P_0(f'_M(v^\star)Id),$$

where $-\gamma\Delta : \dot{V} \to \dot{V}'$ is an isomorphism, $P_0 : H \to \dot{H}$ is the L^2-projection operator, and $f'_M(v^\star)Id : \dot{V} \to H$ is a multiplication operator. Since $\dot{V}$ is compactly imbedded in $\dot{H}$ [19], $f_M'(v^\star)Id : \dot{V} \to H$ is compact, and $P_0(f_M'(v^\star)Id)$ as well. Using [27, Theorem 2.2.5], we obtain that A is a semi-Fredholm operator.

Next, let $N(A)$ denote the kernel of A, and $\Pi : \dot{V} \to N(A)$ the L^2 projection. By [27, Corollary 2.2.6], $\mathcal{L} := A + \Pi : \dot{V} \to \dot{V}'$ is an isomorphism. We choose $Z = \dot{H}$ and denote $W = \mathcal{L}^{-1}(Z)$; W is a Banach space for the norm $\|w\|_W = |\mathcal{L}(w)|_0$. We claim that W is continuously imbedded in $W^{2,2}(\Omega)$. Indeed, by definition, $w \in W$ if and only if $w \in \dot{V}$ and $\mathcal{L}(w) = g$ for some $g \in Z$, i.e.

$$w \in \dot{V} \text{ and } -\gamma\Delta w + f_M'(v^\star)w - \langle f_M'(v^\star)w\rangle + \Pi w = g.$$

Thus, $-\Delta w \in \dot{H}$. By elliptic regularity [3], $w \in W^{2,2}(\Omega)$. Moreover, by the triangle inequality,

$$\gamma|-\Delta w|_0 \leq C\|f_M'(v^\star)\|_{L^\infty}|w|_0 + |\Pi w|_0 + |\mathcal{L}(w)|_0 \leq C\|w\|_W,$$

where C is a constant independent of w. But, by elliptic regularity [3], we also know that $\|w\|_{W^{2,2}} \leq C|-\Delta w|_0$ for all $w \in \dot{V}$. This proves the claim.

The Nemytskii operator $f_M : v \mapsto f_M(v)$ is analytic from $L^\infty(\Omega)$ into $L^\infty(\Omega)$ (see [27, Example 2.3]). Using [27, Proposition 2.3.4], we find that the functional $v \mapsto \int_\Omega F_M(v)$ is real analytic from $L^\infty(\Omega)$ into $\mathbb{R}$. Thus, E_M, which is the sum of a continuous quadratic functional and of a functional which is real analytic on $W \subset W^{2,2}(\Omega) \subset L^\infty(\Omega)$, is real analytic on W. We also obtain that $dE_M : W \to Z$ is real analytic.

We are therefore in position to apply the abstract Theorem 11.2.7 in [27], which shows that there exist $\theta \in (0, 1/2)$ and $\delta > 0$ such that for all $v \in \dot{V}$,

$$|v - v^\star|_1 < \delta \Rightarrow |E_M(v) - E_M(v^\star)|^{1-\theta} \leq |dE_M(v)|_{-1}. \tag{3.7}$$

Finally, we note that any $u^\star \in \mathcal{S}_M$ can be written $u^\star = M + v^\star$, where $v^\star$ is a critical point of E_M; by definition of V_M, any $u \in V_M$ can be written $u = M + v$ with $v \in \dot{V}$. The expected Łojasiewicz-Simon inequality (3.5) is exactly (3.7) written in terms of $u^\star$, u, E and f. □

Theorem 3.3. *Assume that $1/\tau > c_f^2/(8\gamma)$ and let $(u_n, w_n)_n$ be a sequence which complies with* (2.10)-(2.11). *Then the whole sequence converges to (u_∞, w_∞) in $V \times V$, with $u_\infty \in \mathcal{S}_M$, $M = \langle u_0\rangle$, and w_∞ constant. Moreover, the following convergence rate holds*

$$\|u_n - u_\infty\|_1 + \|w_n - w_\infty\|_1 \leq Cn^{-\frac{\theta}{1-\theta}}, \tag{3.8}$$

for all $n \geq 2$, where C is a constant depending on $\|u_0\|_1$, $\|u_1\|_1$, f, γ, τ, and θ, while $\theta \in (0, 1/2)$ may depend on u_∞.

Proof. Let $M = \langle u_0\rangle$. For every $u_\star \in \omega((u_n)_n)$, there exist $\theta \in (0, 1)$ and $\delta > 0$ which may depend on $u_\star$ such that the inequality (3.5) holds for every $u \in B_\delta(u_\star) = \{u \in V_M \;:\; |u - u_\star| < \delta\}$. The union of balls $\{B_\delta(u_\star) \;:\; u_\star \in \omega((u_n)_n)\}$ forms an open covering of $\omega((u_n)_n)$ in V_M. Due to the compactness of $\omega((u_n)_n)$ in V, we can find a finite subcovering $\{B_{\delta^i}(u_\star^i)\}_{i=1}^m$ such that the constants δ^i and θ^i corresponding to $u_\star^i$ in Lemma 3.2 are indexed by i.

From the definition of $\omega((u_n)_n)$, we know that there exists a sufficiently large n_0 such that $u_n \in \mathcal{U} = \cup_{i=1}^m B_{\delta^i}(u_\star^i)$ for all $n \geq n_0$. Taking $\theta = \min_{i=1}^m\{\theta^i\}$, we deduce from Lemma 3.2 and Proposition 3.1 that for all $n \geq n_0$,

$$|E(u_n) - \mathcal{E}_\star|^{1-\theta} \leq |-\gamma\Delta u_n + f(u_n) - \langle f(u_n)\rangle|_{-1}, \tag{3.9}$$

where $\mathcal{E}_\star$ is the value of E on $\omega((u_n)_n)$. We may also assume (by taking a larger n_0 if necessary) that for all $n \geq n_0$, $|\delta u_n|_{-1} \leq 1$.

We denote $\Phi_n = \mathcal{E}(u_n, \delta u_n) - \mathcal{E}_\star$, so that $\Phi_n \geq 0$ and Φ_n is nonincreasing. Let $n \geq n_0$. Using the inequality $(a+b)^{1-\theta} \leq a^{1-\theta} + b^{1-\theta}$, valid for all $a, b \geq 0$, together with (3.9), we obtain

$$\begin{aligned}
\Phi_{n+1}^{1-\theta} &\leq |E(u_{n+1}) - \mathcal{E}_\star|^{1-\theta} + (4\tau)^{\theta-1}|\delta u_{n+1}|_{-1}^{2(1-\theta)} \\
&\leq |-\gamma\Delta u_{n+1} + f(u_{n+1}) - \langle f(u_{n+1})\rangle|_{-1} + (4\tau)^{\theta-1}|\delta u_{n+1}|_{-1} \\
&\leq C\left(|u_{n+1} - u_n|_{-1} + |\delta u_{n+1} - \delta u_n|_{-1}\right), \\
&\leq C\left(|u_{n+1} - u_n|_1^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right)^{1/2}
\end{aligned} \tag{3.10}$$

where $C = C(\tau, \theta, \|(-\dot{\Delta})^{-1}\|_{\mathcal{L}(\dot{V}', \dot{V}')}, \|(-\dot{\Delta})^{-1}\|_{\mathcal{L}(\dot{V}, \dot{V})})$ (here and in the following, C denotes a generic positive constant independent of n). For the third inequality, we have used (2.13) and (2.18). Next, we choose $\varepsilon \in (0,1)$ such that $1/\tau = c_f^2/(8\gamma(1-\varepsilon))$. Then (3.3) holds, and it can be written

$$\Phi_n - \Phi_{n+1} \geq C\left(|u_{n+1} - u_n|_1^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right), \tag{3.11}$$

with $C = C(\tau, \gamma, \varepsilon) > 0$.

Assume first that $\Phi_{n+1} > \Phi_n/2$. Then

$$\Phi_n^\theta - \Phi_{n+1}^\theta = \theta\int_{\Phi_{n+1}}^{\Phi_n} x^{\theta-1}dx \geq \theta\frac{\Phi_n - \Phi_{n+1}}{\Phi_n} \geq 2^{\theta-1}\theta\frac{\Phi_n - \Phi_{n+1}}{\Phi_{n+1}^{1-\theta}}.$$

Using (3.10) and (3.11), we find

$$\Phi_n^\theta - \Phi_{n+1}^\theta \geq C\left(|u_{n+1} - u_n|_1^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right)^{1/2},$$

where $C = C(\tau, \theta, \gamma, \varepsilon, \|(-\dot{\Delta})^{-1}\|_{\mathcal{L}(\dot{V}', \dot{V}')}, \|(-\dot{\Delta})^{-1}\|_{\mathcal{L}(\dot{V}, \dot{V})})$.

Now, if $\Phi_{n+1} \leq \Phi_n/2$, then

$$\Phi_n^{1/2} - \Phi_{n+1}^{1/2} \geq (1 - 1/\sqrt{2})\Phi_n^{1/2} \geq (1 - 1/\sqrt{2})(\Phi_n - \Phi_{n+1})^{1/2}$$

and using (3.11) again, we find

$$\Phi_n^{1/2} - \Phi_{n+1}^{1/2} \geq C\left(|u_{n+1} - u_n|_1^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right)^{1/2}.$$

Thus, in both cases, we have

$$|u_{n+1} - u_n|_1 \leq C(\Phi_n^\theta - \Phi_{n+1}^\theta) + C(\Phi_n^{1/2} - \Phi_{n+1}^{1/2}), \tag{3.12}$$

for all $n \geq n_0$. Summing on $n \geq n_0$, we obtain

$$\sum_{n=n_0}^{\infty} |u_{n+1} - u_n|_1 \leq C\Phi_{n_0}^{\theta} + C\Phi_{n_0}^{1/2} < +\infty. \tag{3.13}$$

Using the Cauchy criterion, we find that the whole sequence (u_n) converges to some u_∞ in V. By Proposition 3.1, u_∞ belongs to $\mathcal{S}_M$. Using the second equation in (2.12), we see that $\dot{w}_n \to 0$. For the term $\langle w_n \rangle$, we write

$$\int_\Omega |f(u_n) - f(u_\infty)|dx = \int_\Omega \left| \int_0^1 f'((1-s)u_n + su_\infty)(u_n - u_\infty)ds \right| dx.$$

Using assumption (2.3), Hölder's inequality and Sobolev imbeddings, we find

$$\int_\Omega |f(u_n) - f(u_\infty)|dx \leq C(\|u_n\|_1, \|u_\infty\|_1)\|u_n - u_\infty\|_1.$$

Since (u_n) is bounded in V, this yields, for all $n \geq 2$,

$$|\langle w_n \rangle - w_\infty| = |\langle f(u_n)\rangle - \langle f(u_\infty)|\rangle \leq \langle |f(u_n) - f(u_\infty)|\rangle \leq C\|u_n - u_\infty\|_1, \tag{3.14}$$

where we have used the last equation in (2.12) and where $w_\infty = \langle f(u_\infty)\rangle$. This implies that $w_n \to w_\infty$ in V (see (2.1)), and it concludes the proof of convergence.

For the convergence rate, we will first show that

$$0 \leq \Phi_n \leq Cn^{-\frac{1}{1-2\theta}}, \tag{3.15}$$

for all $n \geq n_1$, for some $n_1 > n_0$ large enough. The exponent θ is the same as above. If $\Phi_{n_1} = 0$ for some $n_1 \geq n_0$, then $\Phi_n = 0$ for all $n \geq n_1$, and estimate (3.15) is obvious. So we may assume that $\Phi_n > 0$ for all n. Let $n \geq n_0$ and denote $G(s) = \dfrac{1}{s^{1-2\theta}}$. The sequence $G(\Phi_n)$ is nondecreasing and tends to $+\infty$.

If $\Phi_{n+1} > \Phi_n/2$, then

$$\begin{aligned} G(\Phi_{n+1}) - G(\Phi_n) &= \int_{\Phi_{n+1}}^{\Phi_n} \frac{1-2\theta}{s^{2-2\theta}} ds \\ &\geq (1-2\theta)2^{2\theta-2}\Phi_{n+1}^{2\theta-2}[\Phi_n - \Phi_{n+1}] \\ &\overset{(3.11)}{\geq} C\Phi_{n+1}^{2\theta-2}\left(|u_{n+1} - u_n|_1^2 + |\delta u_{n+1} - \delta u_n|_{-1}^2\right) \\ &\overset{(3.10)}{\geq} C_1, \end{aligned}$$

where C_1 is a positive constant independent of n.

If $\Phi_{n+1} \leq \Phi_n/2$ and $\Phi_n \leq 1$, then

$$G(\Phi_{n+1}) - G(\Phi_n) \geq \frac{2^{1-2\theta} - 1}{\Phi_n^{1-2\theta}} \geq 2^{1-2\theta} - 1.$$

Let $n_0' \geq n_0$ be large enough so that $\Phi_{n_0'} \leq 1$. Then, in both cases, for all $n \geq n_0'$, we have

$$G(\Phi_{n+1}) - G(\Phi_n) \geq C_2,$$

where $C_2 = \min\{C_1, 2^{1-2\theta} - 1\} > 0$. By summation on n, we obtain

$$G(\Phi_n) - G(\Phi_{n_0'}) \geq C_2(n - n_0'),$$

for all $n \geq n_0'$. Thus, by choosing $n_1 > n_0'$ large enough, we have

$$G(\Phi_n) \geq \frac{C_2}{2} n,$$

for all $n \geq n_1$, which yields (3.15).

Now, by summing estimate (3.12) on n, we find

$$|u_n - u_\infty|_1 \leq \sum_{k=n}^{\infty} |u_{k+1} - u_k|_1 \leq C\Phi_n^\theta + C\Phi_n^{1/2} \leq C\Phi_n^\theta,$$

for all $n \geq n_1$. Using (3.15) yields

$$\|u_n - u_\infty\|_1 \leq Cn^{-\frac{\theta}{1-2\theta}}, \tag{3.16}$$

for all $n \geq n_1$. We may change the constant C in order for the estimate to hold for all $n \geq 2$. From (3.16) and the second equation in (2.12), we obtain the convergence rate for $(\dot{w}_n)$. The convergence rate for $\langle w_n \rangle$ is a consequence of (3.16) and (3.14). This concludes the proof. $\square$

Remark 3.4. It is possible to show that a local minimizer of E in V_M is stable uniformly with respect to τ. More precisely, let $(u_n^\tau)_n$ denote a sequence which complies with (2.13) and corresponding to a time-step τ. We assume $\tau \in (0, \tau^\star]$ where $\tau^\star > 0$ is such that $1/\tau^\star > c_f^2/(8\gamma)$. If $u^\star \in V_M$ is a local minimizer of E in V_M, and if $u_0^\tau = u_1^\tau$ is close enough to $u^\star$ in V_M, then the whole sequence $(u_n^\tau)_n$ remains close to $u^\star$, uniformly with respect to $\tau \in (0, \tau^\star]$. The proof of this stability result is based on the Łojasiewicz-Simon inequality (it may be false for a C^∞ nonlinearity, see [2]). It is proved in [4] for several fully discrete approximations of the Allen-Cahn equation. The case of the semi-discrete scheme (2.13) is more involved. Indeed, dissipative estimates (uniform in τ) are needed to obtain pre-compactness of the set $\{u_n^\tau \ : \ \tau \in (0, \tau^\star],\ n \in \mathbb{N}\}$ in V_M. Moreover, as $\tau \to 0^+$, the dissipation due to the scheme vanishes (cf. Remark 2.4). Thus, instead of the series $\sum_n |u_{n+1}^\tau - u_n^\tau|_1$ (cf. (3.13)), we have to deal with the series $\sum_n |u_{n+1}^\tau - u_n^\tau|_{-1}$. We refer the interested reader to [28, 35] for the proof of stability of a local minimizer in an infinite dimensional setting (for continuous dynamical systems).

References

[1] H. Abels and M. Wilke. Convergence to equilibrium for the Cahn-Hilliard equation with a logarithmic free energy. *Nonlinear Anal.*, 67(11):3176–3193, 2007.

[2] P.-A. Absil and K. Kurdyka. On the stable equilibrium points of gradient systems. *Systems Control Lett.*, 55(7):573–577, 2006.

[3] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[4] N. E. Alaa and M. Pierre. Convergence to equilibrium for discretized gradient-like systems with analytic features. *IMA J. Numer. Anal.*, 33(4):1291–1321, 2013.

[5] P. F. Antonietti, L. Beirão da Veiga, S. Scacchi, and M. Verani. A C^1 virtual element method for the Cahn-Hilliard equation with polygonal meshes. *SIAM J. Numer. Anal.*, 54(1):34–56, 2016.

[6] H. Attouch and J. Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. *Math. Program.*, 116(1-2, Ser. B):5–16, 2009.

[7] H. Attouch, J. Bolte, and B. F. Svaiter. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward-backward splitting, and regularized Gauss-Seidel methods. *Math. Program.*, 137(1-2, Ser. A):91–129, 2013.

[8] T. Bárta, R. Chill, and E. Fašangová. Every ordinary differential equation with a strict Lyapunov function is a gradient system. *Monatsh. Math.*, 166(1):57–72, 2012.

[9] J. Bolte, A. Daniilidis, O. Ley, and L. Mazet. Characterizations of Łojasiewicz inequalities: subgradient flows, talweg, convexity. *Trans. Amer. Math. Soc.*, 362(6):3319–3363, 2010.

[10] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Phys.*, 28(2):258–267, 1958.

[11] L. Cherfils, A. Miranville, and S. Zelik. The Cahn-Hilliard equation with logarithmic potentials. *Milan J. Math.*, 79(2):561–596, 2011.

[12] L. Cherfils and M. Petcu. A numerical analysis of the Cahn-Hilliard equation with non-permeable walls. *Numer. Math.*, 128(3):517–549, 2014.

[13] L. Cherfils, M. Petcu, and M. Pierre. A numerical analysis of the Cahn-Hilliard equation with dynamic boundary conditions. *Discrete Contin. Dyn. Syst.*, 27(4):1511–1533, 2010.

[14] R. Chill, E. Fašangová, and J. Prüss. Convergence to steady state of solutions of the Cahn-Hilliard and Caginalp equations with dynamic boundary conditions. *Math. Nachr.*, 279(13-14):1448–1462, 2006.

[15] R. Chill and M. A. Jendoubi. Convergence to steady states in asymptotically autonomous semilinear evolution equations. *Nonlinear Anal.*, 53(7-8):1017–1039, 2003.

[16] Q. Du and R. A. Nicolaides. Numerical analysis of a continuum model of phase transition. *SIAM J. Numer. Anal.*, 28(5):1310–1322, 1991.

[17] C. M. Elliott. The Cahn-Hilliard model for the kinetics of phase separation. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 35–73. Birkhäuser, Basel, 1989.

[18] C. G. Gal and A. Miranville. Robust exponential attractors and convergence to equilibria for non-isothermal Cahn-Hilliard equations with dynamic boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S*, 2(1):113–147, 2009.

[19] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[20] H. Gomez, V. M. Calo, Y. Bazilevs, and T. J. R. Hughes. Isogeometric analysis of the Cahn-Hilliard phase-field model. *Comput. Methods Appl. Mech. Engrg.*, 197(49-50):4333–4352, 2008.

[21] H. Gomez and T. J. R. Hughes. Provably unconditionally stable, second-order time-accurate, mixed variational methods for phase-field models. *J. Comput. Phys.*, 230(13):5310–5327, 2011.

[22] L. Goudenège, D. Martin, and G. Vial. High order finite element calculations for the Cahn-Hilliard equation. *J. Sci. Comput.*, 52(2):294–321, 2012.

[23] M. Grasselli and M. Pierre. Energy stable and convergent finite element schemes for the modified phase field crystal equation. *ESAIM Math. Model. Numer. Anal.*, to appear.

[24] M. Grinfeld and A. Novick-Cohen. Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments. *Proc. Roy. Soc. Edinburgh Sect. A*, 125(2):351–370, 1995.

[25] F. Guillén-González and M. Samsidy Goudiaby. Stability and convergence at infinite time of several fully discrete schemes for a Ginzburg-Landau model for nematic liquid crystal flows. *Discrete Contin. Dyn. Syst.*, 32(12):4229–4246, 2012.

[26] J. Guo, C. Wang, S. M. Wise, and X. Yue. An H^2 convergence of a second-order convex-splitting, finite difference scheme for the three-dimensional Cahn-Hilliard equation. *Commun. Math. Sci.*, 14(2):489–515, 2016.

[27] A. Haraux and M. A. Jendoubi. *The convergence problem for dissipative autonomous systems*. SpringerBriefs in Mathematics. Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2015. Classical methods and recent advances, BCAM SpringerBriefs.

[28] S.-Z. Huang. *Gradient inequalities*, volume 126 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006. With applications to asymptotic behavior and stability of gradient-like systems.

[29] S. Injrou and M. Pierre. Error estimates for a finite element discretization of the Cahn-Hilliard-Gurtin equations. *Adv. Differential Equations*, 15(11-12):1161–1192, 2010.

[30] J. Jiang, H. Wu, and S. Zheng. Well-posedness and long-time behavior of a non-autonomous Cahn-Hilliard-Darcy system with mass source modeling tumor growth. *J. Differential Equations*, 259(7):3032–3077, 2015.

[31] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1993.

[32] S. Kosugi, Y. Morita, and S. Yotsutani. Stationary solutions to the one-dimensional Cahn-Hilliard equation: proof by the complete elliptic integrals. *Discrete Contin. Dyn. Syst.*, 19(4):609–629, 2007.

[33] S. Łojasiewicz. Une propriété topologique des sous-ensembles analytiques réels. In *Les Équations aux Dérivées Partielles (Paris, 1962)*, pages 87–89. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

[34] B. Merlet and M. Pierre. Convergence to equilibrium for the backward Euler scheme and applications. *Commun. Pure Appl. Anal.*, 9(3):685–702, 2010.

[35] A. Miranville and A. Rougirel. Local and asymptotic analysis of the flow generated by the Cahn-Hilliard-Gurtin equations. *Z. Angew. Math. Phys.*, 57(2):244–268, 2006.

[36] F. Nabet. Convergence of a finite-volume scheme for the Cahn-Hilliard equation with dynamic boundary conditions. *IMA J. Numer. Anal.*, to appear.

[37] A. Novick-Cohen. The Cahn-Hilliard equation. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 201–228. Elsevier/North-Holland, Amsterdam, 2008.

[38] M. Pierre and P. Rogeon. Convergence to equilibrium for a time semi-discrete damped wave equation. *J. Appl. Anal. Comput.*, 6(4), 2016.

[39] P. Poláčik and F. Simondon. Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains. *J. Differential Equations*, 186(2):586–610, 2002.

[40] J. Prüss, R. Racke, and S. Zheng. Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions. *Ann. Mat. Pura Appl. (4)*, 185(4):627–648, 2006.

[41] J. Prüss and M. Wilke. Maximal L_p-regularity and long-time behaviour of the non-isothermal Cahn-Hilliard equation with dynamic boundary conditions. In *Partial differential equations and functional analysis*, volume 168 of *Oper. Theory Adv. Appl.*, pages 209–236. Birkhäuser, Basel, 2006.

[42] P. Rybka and K.-H. Hoffmann. Convergence of solutions to Cahn-Hilliard equation. *Comm. Partial Differential Equations*, 24(5-6):1055–1077, 1999.

[43] J. Shen and X. Yang. Numerical approximations of Allen-Cahn and Cahn-Hilliard equations. *Discrete Contin. Dyn. Syst.*, 28(4):1669–1691, 2010.

[44] L. Simon. Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. *Ann. of Math. (2)*, 118(3):525–571, 1983.

[45] A. M. Stuart and A. R. Humphries. *Dynamical systems and numerical analysis*, volume 2 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 1996.

[46] F. Tone. On the long-time stability of the Crank-Nicolson scheme for the 2D Navier-Stokes equations. *Numer. Methods Partial Differential Equations*, 23(5):1235–1248, 2007.

[47] J. Wei and M. Winter. On the stationary Cahn-Hilliard equation: interior spike solutions. *J. Differential Equations*, 148(2):231–267, 1998.

[48] H. Wu. Convergence to equilibrium for a Cahn-Hilliard model with the Wentzell boundary condition. *Asymptot. Anal.*, 54(1-2):71–92, 2007.

[49] H. Wu and S. Zheng. Convergence to equilibrium for the Cahn-Hilliard equation with dynamic boundary conditions. *J. Differential Equations*, 204(2):511–531, 2004.

[50] X. Wu, G. J. van Zwieten, and K. G. van der Zee. Stabilized second-order convex splitting schemes for Cahn-Hilliard models with application to diffuse-interface tumor-growth models. *Int. J. Numer. Methods Biomed. Eng.*, 30(2):180–203, 2014.

[51] S. Zheng. Asymptotic behaviour to the solution of the Cahn-Hilliard equation. *Applic. Anal.*, 23:165–184, 1986.