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On the Hartle-Hawking-Israel states for spacetimes with static bifurcate Killing horizons

Christian GÉRARD

ABSTRACT. We revisit the construction by Sanders [S1] of the Hartle-Hawking-Israel state for a free quantum Klein-Gordon field on a spacetime with a static, bifurcate Killing horizon and a wedge reflection. Using the notion of the Calderón projector for elliptic boundary value problems and pseudodifferential calculus on manifolds, we give a short proof of its Hadamard property.

1. INTRODUCTION

Let (M, g) be a globally hyperbolic spacetime, with a *bifurcate Killing horizon*, see [KW], [S1] or Subsect. 2.1 for precise definition. The bifurcate Killing horizon \mathcal{H} is generated by the *bifurcation surface* $\mathcal{B} = \{x \in M : V(x) = 0\}$, where V is the Killing vector field. It allows to split (M, g) into four globally hyperbolic regions, the *right/left wedges* \mathcal{M}^+ , \mathcal{M}^- and the *future/past cones* \mathcal{F} , \mathcal{P} , each invariant under the flow of V . An important object related with the Killing horizon \mathcal{H} is its *surface gravity* κ , which is a scalar, constant over all of \mathcal{H} .

Let us consider on (M, g) a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\square_g \phi(x) + m(x)\phi(x) = 0,$$

where $m \in C^\infty(M, \mathbb{R})$, $m(x) > 0$ is invariant under V , and its associated free field algebra.

If V is *time-like* in (\mathcal{M}^+, g) , ie if (\mathcal{M}^+, g, V) is a stationary spacetime, there exists (see [S2]) for any $\beta > 0$ a *thermal state* ω_β^+ at temperature β^{-1} with respect to the group of Killing isometries of (\mathcal{M}^+, g) generated by V .

It was conjectured by Hartle and Hawking [HH] and Israel [I] that if $\beta = \frac{2\pi}{\kappa}$ is the *inverse Hawking temperature*, denoted by β_H in the sequel, then ω_β^+ can be extended to the whole of M as a pure state, invariant under V , the *Hartle-Hawking-Israel state*, denoted in the sequel by ω_{HHI} .

The rigorous construction of the HHI state was first addressed by Kay in [K4], who constructed the HHI state in the Schwarzschild double wedge of the Kruskal spacetime. In such a double wedge, the HHI state is a *double KMS state*, see [K2, K3]. Later Kay and Wald [KW] considered the more general case of spacetimes with a bifurcate Killing horizon, and study general properties of stationary states on this class of spacetimes. They emphasized in particular the importance of the *Hadamard condition*. They proved that a specific sub-algebra of the free field algebra has at most one state invariant under V and Hadamard. They also showed that if M admits a *wedge reflection* (see Subsect. 2.2) the restriction of such a state to \mathcal{M}^+ will necessarily be a β_H -KMS state. These results were later improved in [K1].

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The existence of such a state, ie of the HHI state, was however not proved in [HH]. The first proof of the existence of ω_{HHI} was given by Sanders in the remarkable paper [S1], if the bifurcate Killing horizon is static, ie if V is static in \mathcal{M}^+ , assuming also the existence of a wedge reflection. Sanders showed that there exists a unique Hadamard state ω_{HHI} on M extending the *double β_{H} -KMS state* ω_{β} on $\mathcal{M}^+ \cup \mathcal{M}^-$. The double β_{H} -KMS state ω_{β} is a pure state on $\mathcal{M}^+ \cup \mathcal{M}^-$ which is the natural extension of ω_{β}^+ defined using the wedge reflection, see [K2, K3]. It is an exact geometrical analog of the Fock vacuum vector in the Araki-Woods representation of a thermal state.

1.1. Content of the paper. In this paper we revisit the construction in [S1] of the Hartle-Hawking-Israel state in a spacetime with a static bifurcate Killing horizon. Using the notion of the *Calderón projector* (see Sect. 5), which is a standard tool in elliptic boundary value problems, we significantly shorten the proof of the Hadamard property of ω_{HHI} .

In [S1] the fact that ω_{HHI} is Hadamard was proved by a careful comparison of the Hadamard parametrix construction for the D'Alembertian $-\square_g + m$ associated to the Lorentzian metric g and for the Laplacian $-\Delta_{\hat{g}} + m$ associated to the Riemannian metric \hat{g} obtained from g by Wick rotation in the Killing time coordinate.

In our paper we avoid working with the spacetime covariances of states and instead systematically work with the *Cauchy surface covariances* (see Subsect. 3.3) associated with a Cauchy surface Σ containing the bifurcation surface \mathcal{B} .

It turns out that the Cauchy surface covariances λ^{\pm} of the double β -KMS state ω_{β} are related to a Calderón projector D .

Let us informally recall what is the Calderón projector associated to an elliptic boundary value problem, see Sect. 5 for more details:

let (N, \hat{g}) be a complete Riemannian manifold and $P = -\Delta_{\hat{g}} + m(x)$ for $m \in C^{\infty}(N)$, $m(x) > 0$ a Laplace-Beltrami operator. Let also $\Omega \subset M$ a smooth open set. To Ω is naturally associated the canonical surface density dS , defined by $\langle dS|u \rangle = \int_{\partial\Omega} u d\sigma$, for $u \in C_c^{\infty}(M)$, where $d\sigma$ is the induced surface element on $\partial\Omega$.

If ∂_{ν} is the external normal derivative to $\partial\Omega$ and $\gamma u = \begin{pmatrix} u|_{\partial\Omega} \\ \partial_{\nu} u|_{\partial\Omega} \end{pmatrix}$ for $u \in C^{\infty}(\bar{\Omega})$ the Calderón projector D is a map from $C_c^{\infty}(\partial\Omega) \otimes \mathbb{C}^2$ to $C^{\infty}(\partial\Omega) \otimes \mathbb{C}^2$ defined by:

$$Df := \gamma \circ G(f_1(dVol_{\hat{g}})^{-1}dS - f_0(dVol_{\hat{g}}^{-1})\partial_{\nu}^*dS), \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C_c^{\infty}(\partial\Omega) \otimes \mathbb{C}^2,$$

where $G = P^{-1}$. It is easy to see that $f \in C^{\infty}(\Sigma) \otimes \mathbb{C}^2$ equals γu for some $u \in C^{\infty}(\bar{\Omega})$ solution of $Pu = 0$ in Ω if and only if $Df = f$.

In our case we take $N = \mathbb{S}_{\beta} \times \Sigma^+$, where \mathbb{S}_{β} is the circle of length β and $\Sigma^+ = \Sigma \cap \mathcal{M}^+$ is the right part of the Cauchy surface Σ . The Riemannian metric is $\hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j$, obtained by the *Wick rotation* $t =: i\tau$ of the Lorentzian metric $g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j$ on $\mathcal{M}^+ \sim \mathbb{R} \times \Sigma^+$ where \mathcal{M}^+ is identified to $\mathbb{R} \times \Sigma^+$ using the Killing time coordinate t .

The existence of an extension of $\omega_{\beta_{\text{H}}}$ to M is then an almost immediate consequence of the fact that (N, \hat{g}) admits a smooth extension $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ if and only if $\beta = \beta_{\text{H}}$, a well-known result which plays also a role in [S1].

In fact this geometrical fact implies that D , viewed as an operator defined on $C_c^{\infty}(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$ uniquely extends to a Calderon projector D_{ext} , defined on $C_c^{\infty}(\Sigma) \otimes \mathbb{C}^2$. From D_{ext} one can then easily obtain a pure quasi-free state ω_{HHI} on the whole of M .

The Hadamard property of ω_{HHI} follows then from the well-known fact that D_{ext} , being a Calderón projector, is a 2×2 matrix of pseudodifferential operators on Σ , and of the Hadamard property of ω_{β} in $\mathcal{M}^+ \cup \mathcal{M}^-$.

Beside shortening the proof of the Hadamard property of ω_{HHI} , we think that our paper illustrates the usefulness of pseudodifferential calculus for the construction and study of Hadamard states, see also [GW1], [GW2], [GW3], [GOW] for other applications. We believe that Calderón projectors could also be used to construct the Hartle-Hawking-Israel state in the still open case of spacetimes with a Killing horizon that is only stationary.

1.2. Plan of the paper. Let us now briefly give the plan of the paper. In Sect. 2 we recall the notion of a static bifurcate Killing horizon, following [S1] and introduce the associated Klein-Gordon equation.

Sect. 3 is devoted to background material on CCR^* -algebras, bosonic quasi-free states and their spacetime and Cauchy surface covariances in the case of quantum Klein-Gordon fields. We use the framework of *charged fields*, which is in our opinion more elegant, even when considering only neutral field equations. We also recall the notion of *pseudodifferential operators* on a manifold, which will be useful later on and formulate a consequence of [GW1] which states that the Cauchy surface covariances of *any* Hadamard state for Klein-Gordon fields is given by a matrix of pseudodifferential operators.

In Sect. 4 we define various 'Euclidean' Laplacians, $K = -\Delta_{\hat{g}} + m$ acting on $N = \mathbb{S}_\beta \times \Sigma^+$ and a related operator \tilde{K} , obtained from Wick rotation of the Lorentzian metric on M in the Killing time coordinate, which are considered in [S1]. It is sufficient for us to define these Laplacians by quadratic form techniques, which simplifies some arguments.

In Sect. 5 we recall the definition of the *Calderón projector*, which is a standard notion in elliptic boundary value problems. In Sect. 6, using the explicit expression for \tilde{K}^{-1} , we show that the projection associated to the double β -KMS state ω_β equals to the Calderón projector D associated to K and the open set $\Omega =]0, \beta/2[\times \Sigma^+$.

In Sect. 7, we recall the well-known fact that a smooth extension $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ of (N, \hat{g}) exists iff $\beta = \beta_{\text{H}}$. The extended Calderón projector D_{ext} generates a pure state on M , which is the Hartle-Hawking-Israel state ω_{HHI} . In Prop. 7.4, we show that such an extension is unique among quasi-free states whose spacetime covariances map $C_c^\infty(M)$ into $C^\infty(M)$ continuously. Finally we give in the proof of Thm. 7.5 a new and elementary proof of the Hadamard property of ω_{HHI} , using the pseudodifferential calculus on Σ .

2. SPACETIMES WITH A STATIC BIFURCATE KILLING HORIZON

2.1. Static bifurcate Killing horizons. We consider as in [S1] a globally hyperbolic spacetime (M, g) with a *static bifurcate Killing horizon*. We recall, see [S1, Def. 2.2], that this is a triple (M, g, V) , such that

- (1) the Lorentzian manifold (M, g) is globally hyperbolic,
- (2) V is a complete Killing vector field for (M, g) ,
- (3) $\mathcal{B} := \{x \in M : V(x) = 0\}$ is a compact, orientable submanifold of codimension 2,
- (4) there exists a Cauchy hypersurface Σ containing \mathcal{B} ,
- (5) V is g -orthogonal to Σ ,

see Figure 1 below where the vector field V is represented by arrows.

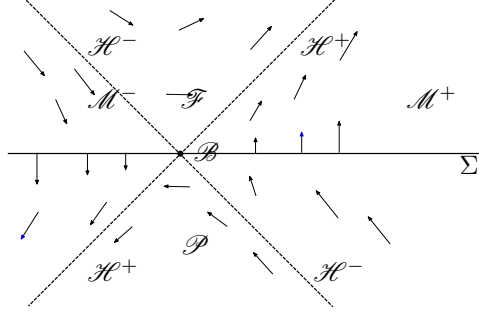


FIGURE 1.

For simplicity we will also assume that the bifurcation surface \mathcal{B} is connected. Denoting by n the future pointing normal vector field to Σ one introduces the *lapse function*:

$$(2.1) \quad v(x) := -n(x) \cdot g(x)V(x), \quad x \in \Sigma,$$

and Σ decomposes as

$$\Sigma = \Sigma^- \cup \mathcal{B} \cup \Sigma^+,$$

where $\Sigma^\pm = \{x \in \Sigma : \pm v(x) > 0\}$. The spacetime M splits as

$$M = \mathcal{M}^+ \cup \mathcal{M}^- \cup \overline{\mathcal{F}} \cup \overline{\mathcal{P}},$$

where the future cone $\mathcal{F} := I^+(\mathcal{B})$, the past cone $\mathcal{P} := I^-(\mathcal{B})$, the right/left wedges $\mathcal{M}^\pm := D(\Sigma^\pm)$, are all globally hyperbolic when equipped with g .

2.2. Wedge reflection. Additionally one has to assume the existence of a *wedge reflection*, see [S1, Def. 2.6], ie a diffeomorphism R of $\mathcal{M}^+ \cup \mathcal{M}^- \cup U$ onto itself, where U is an open neighborhood of \mathcal{B} such that:

- (1) $R \circ R = \text{Id}$,
- (2) R is an isometry of $(\mathcal{M}^+ \cup \mathcal{M}^-, g)$ onto itself, which reverses the time orientation,
- (3) $R = \text{Id}$ on \mathcal{B} ,
- (4) $R^*V = V$ on $\mathcal{M}^+ \cup \mathcal{M}^-$.

It follows that R preserves Σ , see [S1, Prop. 2.7], and we denote by r the restriction of R to Σ . Denoting by h the induced Riemannian metric on Σ one has:

$$(2.2) \quad r^*h = h, r^*v = -v.$$

2.3. Killing time coordinate. Denoting by $\Phi_s^V : M \rightarrow M$ the flow of the Killing vector field V , we obtain a diffeomorphism

$$\chi : \mathbb{R} \times (\Sigma \setminus \mathcal{B}) \ni (t, y) \mapsto \Phi_t^V(y) \in \mathcal{M}^+ \cup \mathcal{M}^-,$$

which defines the coordinate t on $\mathcal{M}^+ \cup \mathcal{M}^-$ called the *Killing time coordinate*. The metric g on $\mathcal{M}^+ \cup \mathcal{M}^-$ pulled back by χ takes the form (see [S1, Subsect. 2.1]):

$$(2.3) \quad g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j,$$

where the Riemannian metric $h_{ij}(y)dy^i dy^j$ is the restriction of g to Σ .

2.4. Klein-Gordon operator. We fix a real potential $m \in C^\infty(M)$. As in [S1] we assume that m is stationary w.r.t. the Killing vector field V and invariant under the wedge reflection, ie:

$$(2.4) \quad V^a \nabla_a m(x) = 0, \quad m \circ R(x) = m(x), \quad x \in \mathcal{M}^+ \cup \mathcal{M}^- \cup U.$$

For simplicity we also assume that

$$(2.5) \quad m(x) \geq m_0^2 > 0, \quad x \in M,$$

ie we consider only massive fields. Note that in [S1] the weaker condition $m(x) > 0$ was assumed. We consider the Klein-Gordon operator

$$(2.6) \quad P = -\square_g + m.$$

3. FREE KLEIN-GORDON FIELDS

In this section we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [GW1, Sect. 2] based on *charged fields*.

3.1. Charged CCR algebra.

3.1.1. Charged bosonic fields. Let \mathcal{Y} a complex vector space, \mathcal{Y}^* its anti-dual. Sesquilinear forms on \mathcal{Y} are identified with elements of $L(\mathcal{Y}, \mathcal{Y}^*)$ and the action of a sesquilinear form β is correspondingly denoted by $\bar{y}_1 \cdot \beta y_2$ for $y_1, y_2 \in \mathcal{Y}$. We fix $q \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ a non degenerate hermitian form on \mathcal{Y} , ie such that $\text{Ker } q = \{0\}$.

The *CCR *-algebra* $\text{CCR}(\mathcal{Y}, q)$ is the complex *-algebra generated by symbols $\mathbf{1}, \psi(y), \psi^*(y), y \in \mathcal{Y}$ and the relations:

$$\begin{aligned} \psi(y_1 + \lambda y_2) &= \psi(y_1) + \lambda \psi(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\ \psi^*(y_1 + \lambda y_2) &= \psi^*(y_1) + \lambda \psi^*(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\ [\psi(y_1), \psi(y_2)] &= [\psi^*(y_1), \psi^*(y_2)] = 0, \quad [\psi(y_1), \psi^*(y_2)] = \bar{y}_1 \cdot q y_2 \mathbf{1}, \quad y_1, y_2 \in \mathcal{Y}, \\ \psi(y)^* &= \psi^*(y), \quad y \in \mathcal{Y}. \end{aligned}$$

A state ω on $\text{CCR}(\mathcal{Y}, q)$ is (*gauge invariant*) *quasi-free* if

$$\omega\left(\prod_{i=1}^p \psi(y_i) \prod_{i=1}^q \psi^*(y_j)\right) = \begin{cases} 0 & \text{if } p \neq q, \\ \sum_{\sigma \in S_p} \prod_{i=1}^p \omega(\psi(y_i) \psi^*(y_{\sigma(i)})) & \text{if } p = q. \end{cases}$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2]. It is convenient to associate to ω its (*complex*) *covariances* $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ defined by:

$$\begin{aligned} \omega(\psi(y_1) \psi^*(y_2)) &=: \bar{y}_1 \cdot \lambda^+ y_2, \\ \omega(\psi^*(y_2) \psi(y_1)) &=: \bar{y}_1 \cdot \lambda^- y_2, \end{aligned} \quad y_1, y_2 \in \mathcal{Y}.$$

The following results are well-known, see eg [DG, Sect. 17.1] or [GW1, Sect. 2]:

– two hermitian forms $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ are the covariances of a quasi-free state ω iff

$$(3.1) \quad \lambda^\pm \geq 0, \quad \lambda^+ - \lambda^- = q.$$

– Let \mathcal{Y}_ω be the completion of \mathcal{Y} for the Hilbertian scalar product $\lambda^+ + \lambda^-$. If there exist linear operators $c^\pm \in L(\mathcal{Y}_\omega)$ such that

$$c^+ + c^- = \mathbf{1}, \quad (c^\pm)^2 = c^\pm,$$

(ie c^\pm is a pair of complementary projections) and $\lambda^\pm = \pm q \circ c^\pm$, then ω is a *pure state*.

3.1.2. *Neutral bosonic fields.* We complete this subsection by explaining the relationship with the formalism of neutral fields, see eg [GW1, Subsect. 2.5].

Let \mathcal{X} a real vector space, $\mathcal{X}^\#$ its dual, and $\sigma \in L_a(\mathcal{X}, \mathcal{X}^\#)$ a symplectic form on \mathcal{X} . The $*$ -algebra $\text{CCR}(\mathcal{X}, \sigma)$ is the complex $*$ -algebra generated by symbols $\mathbf{1}, \phi(x), x \in \mathcal{X}$ and relations:

$$\begin{aligned}\phi(x_1 + \lambda x_2) &= \phi(x_1) + \lambda \phi(x_2), \quad x_1, x_2 \in \mathcal{X}, \lambda \in \mathbb{R}, \\ [\phi(x_1), \phi(x_2)] &= ix_1 \cdot \sigma x_2 \mathbf{1}, \quad x_1, x_2 \in \mathcal{X}, \\ \phi(x)^* &= \phi(x), \quad x \in \mathcal{X}.\end{aligned}$$

To relate the neutral to the charged formalism one sets $\mathcal{Y} = \mathbb{C}\mathcal{X}$ and for $\beta \in L(\mathcal{X}, \mathcal{X}^\#)$ denote by $\beta_{\mathbb{C}} \in L(\mathcal{Y}, \mathcal{Y}^*)$ its sesquilinear extension. $\mathcal{Y}_{\mathbb{R}} \sim \mathcal{X} \oplus \mathcal{X}$ is the *real form* of \mathcal{Y} , ie $\mathcal{Y}_{\mathbb{R}} = \mathcal{Y}$ as a real vector space. Then $(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}}) \sim (\mathcal{X}, \sigma) \oplus (\mathcal{X}, \sigma)$ is a real symplectic space and we denote by $\phi(y), y \in \mathcal{Y}_{\mathbb{R}}$ the selfadjoint generators of $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$. Under the identification $\phi(y) \sim \phi(x) \otimes \mathbf{1} + \mathbf{1} \otimes \phi(x')$ for $y = x + ix'$ we can identify $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$ with $\text{CCR}(\mathcal{X}, \sigma) \otimes \text{CCR}(\mathcal{X}, \sigma)$ as $*$ -algebras.

Note also that under the identification

$$\psi(y) \sim \frac{1}{\sqrt{2}}(\phi(y) + i\phi(iy)), \quad \psi^*(y) \sim \frac{1}{\sqrt{2}}(\phi(y) - i\phi(iy)), \quad y \in \mathcal{Y}$$

we can identify $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$ with $\text{CCR}(\mathcal{Y}, q)$ for $q = i\sigma_{\mathbb{C}}$.

A quasi-free state ω on $\text{CCR}(\mathcal{X}, \sigma)$ is determined by its *real covariance* $\eta \in L_s(\mathcal{X}, \mathcal{X}^\#)$ defined by:

$$\omega(\phi(x_1)\phi(x_2)) =: x_1 \cdot \eta x_2 + \frac{i}{2}x_1 \cdot \sigma x_2, \quad x_1, x_2 \in \mathcal{X}.$$

A symmetric form $\eta \in L_s(\mathcal{X}, \mathcal{X}^\#)$ is the covariance of a quasi-free state iff

$$\eta \geq 0, \quad |x_1 \cdot \sigma x_2| \leq 2(x_1 \cdot \eta x_1)^{\frac{1}{2}}(x_2 \cdot \eta x_2)^{\frac{1}{2}}, \quad x_1, x_2 \in \mathcal{X}.$$

To such a state ω we associate the quasi-free state $\tilde{\omega}$ on $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$ with real covariance $\text{Re}\eta_{\mathbb{C}}$. Then its complex covariances λ^\pm are given by (see [GW1, Subsect. 2.5]):

$$(3.2) \quad \lambda^\pm = \eta_{\mathbb{C}} \pm \frac{1}{2}i\sigma_{\mathbb{C}}.$$

Applying complex conjugation, we immediately see that in this case

$$(3.3) \quad \lambda^+ \geq 0 \Leftrightarrow \lambda^- \geq 0,$$

so it suffices to check for example that $\lambda^+ \geq 0$.

3.2. Free Klein-Gordon fields. Let $P = -\square_g + m(x)$, $m \in C^\infty(M, \mathbb{R})$ a Klein-Gordon operator on a globally hyperbolic spacetime (M, g) (we use the convention $(1, d)$ for the Lorentzian signature). Let E^\pm be the advanced/retarded inverses of P and $E := E^+ - E^-$. We apply the above framework to

$$\mathcal{Y} = \frac{C_c^\infty(M)}{PC_c^\infty(M)}, \quad [\bar{u}] \cdot q[u] = i(u|Eu)_M,$$

where $(u|v)_M = \int_M \bar{u}v d\text{Vol}_g$.

One restricts attention to quasi-free states on $\text{CCR}(\mathcal{Y}, q)$ whose covariances are given by distributions on $M \times M$, ie such that there exists $\Lambda^\pm \in \mathcal{D}'(M \times M)$ with

$$(3.4) \quad \begin{aligned}\omega(\psi([u_1])\psi^*([u_2])) &= (u_1|\Lambda^+u_2)_M, \\ \omega(\psi^*([u_2])\psi([u_1])) &= (u_1|\Lambda^-u_2)_M,\end{aligned} \quad u_1, u_2 \in C_c^\infty(M).$$

In the sequel the distributions $\Lambda^\pm \in \mathcal{D}'(M \times M)$ will be called the *spacetime covariances* of the state ω .

In (3.4) we identify distributions on M with distributional densities using the density $dVol_g$ and use the notation $(u|\varphi)_M$, $u \in C_c^\infty(M)$, $\varphi \in \mathcal{D}'(M)$ for the duality bracket. We have then

$$(3.5) \quad \begin{aligned} P(x, \partial_x)\Lambda^\pm(x, x') &= P(x', \partial_{x'})\Lambda^\pm(x, x') = 0, \\ \Lambda^+(x, x') - \Lambda^-(x, x') &= iE(x, x'). \end{aligned}$$

Such a state is called a *Hadamard state*, (see [R] for the neutral case and [GW1] for the complex case) if

$$(3.6) \quad \text{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm,$$

where $\text{WF}(\Lambda)'$ denotes the 'primed' wavefront set of Λ , ie $S' := \{((x, \xi), (x', -\xi')) : ((x, \xi), (x', \xi')) \in S\}$ for $S \subset T^*M \times T^*M$, and \mathcal{N}^\pm are the two connected components (positive/negative energy shell) of the characteristic manifold:

$$(3.7) \quad \mathcal{N} := \{(x, \xi) \in T^*M \setminus \{0\} : \xi_\mu g^{\mu\nu}(x) \xi_\nu = 0\}.$$

3.3. Cauchy surface covariances. Denoting by $Sol_{sc}(P)$ the space of smooth space-compact solutions of $P\phi = 0$, it is well known that

$$[E] : \frac{C_c^\infty(M)}{PC_c^\infty(M)} \ni [u] \mapsto Eu \in Sol_{sc}(P)$$

is bijective, with

$$i(u_1|Eu_2) = \overline{Eu_1} \cdot qEu_2, \quad u_i \in C_c^\infty(M),$$

for

$$(3.8) \quad \overline{\phi}_1 \cdot q\phi_2 := i \int_\Sigma (\nabla_\mu \overline{\phi}_1 \phi_2 - \overline{\phi}_1 \nabla_\mu \phi_2) n^\mu d\sigma,$$

where Σ is any spacelike Cauchy hypersurface, n^μ is the future directed unit normal vector field to Σ and $d\sigma$ the induced surface density. Setting

$$\rho : C_{sc}^\infty(M) \ni \phi \mapsto \left(\begin{array}{c} \phi|_\Sigma \\ i^{-1} \partial_\nu \phi|_\Sigma \end{array} \right) = f \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

Since the Cauchy problem

$$\begin{cases} P\phi = 0, \\ \rho u = f \end{cases}$$

as a unique solution $\phi \in Sol_{sc}(P)$ for $f \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$ the map

$$\frac{C_c^\infty(M)}{PC_c^\infty(M)} \ni [u] \mapsto \rho Eu \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

is bijective, and

$$i(u|Eu)_M = \overline{\rho Eu} \cdot q\rho Eu,$$

for

$$(3.9) \quad \overline{f} \cdot qf := \int_\Sigma \overline{f}_1 f_0 + \overline{f}_0 f_1 d\sigma_\Sigma, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

It follows that to a quasi-free state with spacetime covariances Λ^\pm one can associate its *Cauchy surface covariances* λ^\pm defined by:

$$(3.10) \quad \Lambda^\pm =: (\rho E)^* \lambda^\pm (\rho E).$$

Using the canonical scalar product $(f|f)_\Sigma := \int_\Sigma \overline{f}_1 f_1 + \overline{f}_0 f_0 d\sigma_\Sigma$ we identify λ^\pm with operators, still denoted by λ^\pm , belonging to $L(C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma), \mathcal{D}'(\Sigma) \oplus \mathcal{D}'(\Sigma))$.

A more explicit expression of λ^\pm in terms of Λ^\pm is as follows, see eg [GOW, Thm. 7.10]: let us introduce Gaussian normal coordinates to Σ

$$U \ni (t, y) \mapsto \chi(t, y) \in V,$$

where U is an open neighborhood of $\{0\} \times \Sigma$ in $\mathbb{R} \times \Sigma$ and V an open neighborhood of Σ in M , such that $\chi^*g = -dt^2 + h_{ij}(t, y)dy^i dy^j$. We denote by $\Lambda^\pm(t, y, t', y') \in \mathcal{D}'(U \times U)$ the restriction to $U \times U$ of the distributional kernel of Λ^\pm . By (3.5) and standard microlocal arguments, their restrictions to fixed times t, t' , denoted by $\Lambda^\pm(t, t') \in \mathcal{D}'(\Sigma \otimes \Sigma)$ are well defined.

We know also that $\partial_t^k \partial_{t'}^{k'} \Lambda^\pm(0, 0) \in \mathcal{D}'(\Sigma \times \Sigma)$ is well defined for $k, k' = 0, 1$. Then setting $\lambda^\pm =: \pm q \circ c^\pm$ we have:

$$(3.11) \quad c^\pm = \pm \begin{pmatrix} i\partial_{t'} \Lambda^\pm(0, 0) & \Lambda^\pm(0, 0) \\ \partial_t \partial_{t'} \Lambda^\pm(0, 0) & i^{-1} \partial_t \Lambda^\pm(0, 0) \end{pmatrix}.$$

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [GW1, GOW] using pseudodifferential calculus on Σ , see below for a short summary.

3.4. Pseudodifferential operators. We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [Sh, Sect. 4.3] for details.

For $m \in \mathbb{R}$ we denote by $\Psi^m(\mathbb{R}^d)$ the space of classical pseudodifferential operators of order m on \mathbb{R}^d , associated with poly-homogeneous symbols of order m see eg [Sh, Sect. 3.7].

Let N be a smooth, d -dimensional manifold. Let $U \subset N$ a precompact chart open set and $\psi : U \rightarrow \tilde{U}$ a chart diffeomorphism, where $\tilde{U} \subset \mathbb{R}^d$ is precompact, open. We denote by $\psi^* : C_c^\infty(\tilde{U}) \rightarrow C_c^\infty(U)$ the map $\psi^*u(x) := u \circ \psi(x)$.

Definition 3.1. A linear continuous map $A : C_c^\infty(N) \rightarrow C^\infty(N)$ belongs to $\Psi^m(N)$ if the following condition holds:

(C) Let $U \subset N$ be precompact open, $\psi : U \rightarrow \tilde{U}$ a chart diffeomorphism, $\chi_1, \chi_2 \in C_c^\infty(U)$ and $\tilde{\chi}_i = \chi_i \circ \psi^{-1}$. Then there exists $\tilde{A} \in \Psi^m(\mathbb{R}^d)$ such that

$$(3.12) \quad (\psi^*)^{-1} \chi_1 A \chi_2 \psi^* = \tilde{\chi}_1 \tilde{A} \tilde{\chi}_2.$$

Elements of $\Psi^m(N)$ are called (classical) pseudodifferential operators of order m on N .

The subspace of $\Psi^m(N)$ of pseudodifferential operators with properly supported kernels is denoted by $\Psi_c^m(N)$.

Note that if $\Psi_{(c)}^\infty(N) := \bigcup_{m \in \mathbb{R}} \Psi_{(c)}^m(N)$, then $\Psi_c^\infty(N)$ is an algebra, but $\Psi^\infty(N)$ is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

We denote by $T^*N \setminus \{0\}$ the cotangent bundle of N with the zero section removed.

To $A \in \Psi^m(N)$ one can associate its *principal symbol* $\sigma_{\text{pr}}(A) \in C^\infty(T^*N \setminus \{0\})$, which is homogeneous of degree m in the fiber variable ξ in T^*M , in $\{|\xi| \geq 1\}$. A is called *elliptic* in $\Psi^m(N)$ at $(x_0, \xi_0) \in T^*N \setminus \{0\}$ if $\sigma_{\text{pr}}(A)(x_0, \xi_0) \neq 0$.

If $A \in \Psi^m(N)$ there exists (many) $A_c \in \Psi_c^m(N)$ such that $A - A_c$ has a smooth kernel.

Finally one says that $(x_0, \xi_0) \notin \text{essupp}(A)$ for $A \in \Psi^\infty(N)$ if there exists $B \in \Psi_c^\infty(N)$ *elliptic* at (x_0, ξ_0) such that $A_c \circ B$ is smoothing, where $A_c \in \Psi_c^\infty(N)$ is as above, ie $A - A_c$ is smoothing.

3.5. The Cauchy surface covariances of Hadamard states. We now state a result which follows directly from a construction of Hadamard states in [GW1, Subsect. 8.2].

Theorem 3.2. Let ω be any Hadamard state for the free Klein-Gordon field on (M, g) and Σ a spacelike Cauchy hypersurface. Then its Cauchy surface covariances λ^\pm are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$.

Proof. It is well known (see eg [R]) that if ω_1, ω_2 are Hadamard states, then $\Lambda_1^\pm - \Lambda_2^\pm$ are smoothing operators on M . Using (3.10) this implies that $\lambda_1^\pm - \lambda_2^\pm$ are matrices of smoothing operators on Σ . From the definition of $\Psi^\infty(\Sigma)$ it hence suffices to construct *one* Hadamard state ω whose Cauchy surface covariances λ^\pm are matrices of pseudodifferential operators. The state constructed in [GW1, Subsect. 8.2] has this property, as can be seen from [GW1, Equ. (8.2)]. \square

4. EUCLIDEAN OPERATORS

The construction of the β -KMS state on \mathcal{M}^+ with respect to the Killing vector field V relies on the *Wick rotation*, where $(\mathbb{R} \times \Sigma^+, g)$ is replaced by $(\mathbb{S}_\beta \times \Sigma^+, \hat{g})$:

$$(4.1) \quad \hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j,$$

is the Riemannian metric obtained from (2.3) by setting $t = i\tau$ and $\mathbb{S}_\beta = [0, \beta[$ with endpoints identified is the circle of length β .

In this section we recall various 'Euclidean' operators related to \hat{g} appearing in [S1, S2]. It will be convenient to construct them by quadratic form techniques.

We set

$$N := \mathbb{S}_\beta \times \Sigma^+,$$

whose elements are denoted by (τ, y) . We equip N with the Riemannian metric \hat{g} in (4.1) and the associated density $dVol_{\hat{g}} = |v|(y)|h|^{\frac{1}{2}}(y)d\tau dy$. The hypersurface Σ^+ is equipped with the induced density $dVol_h = |h|^{\frac{1}{2}}(y)dy$.

4.1. Euclidean operator on N . We consider the operator

$$K := -\Delta_{\hat{g}} + m(y),$$

for m as in Subsect. 2.4. Note that m depends only on y since m is invariant under the Killing flow. We have

$$K = -v^{-2}(y)\bar{\partial}_\tau^2 - |v|^{-1}(y)|h|^{-\frac{1}{2}}(y)\bar{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\bar{\partial}_{y^j} + m(y).$$

K is well defined as a selfadjoint operator on $L^2(N, dVol_{\hat{g}})$ obtained from the quadratic form:

$$(4.2) \quad Q(u, u) := \int_N (|v|^{-2}|\partial_\tau u|^2 + \partial_i \bar{u} h^{ij} \partial_j u + m|u|^2) dVol_{\hat{g}},$$

which is closeable on $C_c^\infty(N)$, since K is symmetric and bounded from below on this domain. Denoting its closure again by Q and the domain of its closure by $\text{Dom } Q$, K is the selfadjoint operator associated to Q , ie the Friedrichs extension of its restriction to $C^\infty(\mathbb{S}_\beta) \otimes C_c^\infty(\Sigma^+)$. We know that $u \in \text{Dom } K$, $Ku = f$ iff

$$(4.3) \quad u \in \text{Dom } Q \text{ and } Q(u, u) = (w|f)_{L^2(N)}, \quad \forall w \in C_c^\infty(N).$$

From (2.5) we know that $K \geq m_0^2$ hence is boundedly invertible and we set

$$G := K^{-1}.$$

4.2. Change of volume form. Let us set $\hat{Q}(u, u) = Q(vu, vu)$, $\text{Dom } \hat{Q} = \{u \in L^2(N) : vu \in \text{Dom } Q\}$. By (2.5) we have $\hat{Q}(u, u) \geq m_0^2 \|vu\|^2$. If $u_n \in \text{Dom } \hat{Q}$, $u \in L^2(N)$ with $\|u_n - u\| \rightarrow 0$ and $\hat{Q}(u_n - u_m, u_n - u_m) \rightarrow 0$ then from the inequality above we obtain that $vu \in L^2(N)$ and $\|v(u_n - u)\| \rightarrow 0$. Since Q is closed we obtain that $u \in \text{Dom } \hat{Q}$ and $\hat{Q}(u_n - u, u_n - u) \rightarrow 0$, ie \hat{Q} is closed.

Let \hat{K} be the injective selfadjoint operator associated to \hat{Q} , (which is formally equal to vKv) and let $\hat{G} = \hat{K}^{-1}$. We claim that

$$(4.4) \quad G = v\hat{G}v, \text{ on } v^{-1}L^2(N).$$

This follows easily from the characterization (4.3) of G and similarly of \hat{G} .

Let now $U : L^2(N) \rightarrow L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)$ the unitary map given by $Uu = v^{\frac{1}{2}}u$. We set

$$\tilde{K} := UKU^*.$$

We have

$$\tilde{K} = -\bar{\partial}_\tau^2 + \epsilon^2(y, \bar{\partial}_y),$$

where:

$$\epsilon^2(y, \bar{\partial}_y) = -|v|^{\frac{1}{2}}(y)|h|^{-\frac{1}{2}}(y)\bar{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\bar{\partial}_{y^j}|v|^{\frac{1}{2}}(y) + v^2(y)m(y),$$

is obtained as above from the quadratic form

$$(4.5) \quad \int_{\Sigma^+} \left(\partial_i |v|^{\frac{1}{2}} \bar{u} |v| h^{ij} \partial_i |v|^{\frac{1}{2}} u + |v|^2 m |u|^2 \right) |h|^{\frac{1}{2}} dy.$$

If $\tilde{G} := \tilde{K}^{-1}$ we have by (4.4):

$$(4.6) \quad G = |v|^{1/2} \tilde{G} |v|^{3/2}, \text{ on } v^{-3/2} L^2(N).$$

We now recall a well known expression for \tilde{G} . Let

$$F(\tau) = \frac{e^{-\tau\epsilon} + e^{(\tau-\beta)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})}, \quad \tau \in [0, \beta],$$

extended to $\tau \in \mathbb{R}$ by β -periodicity. In particular we have:

$$(4.7) \quad F(\tau) = \frac{e^{-|\tau|\epsilon} + e^{(|\tau|-\beta)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})}, \quad \tau \in [-\beta, \beta]$$

The following expression for \tilde{G} is well-known (see eg [DG, Sect. 18.3.2]):

$$(4.8) \quad \tilde{G}\tilde{u}(\tau) = \int_{\mathbb{S}_\beta} F(\tau - \tau') \tilde{u}(\tau') d\tau', \quad \tilde{u} \in L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma \setminus \mathcal{B}).$$

Note that since $\epsilon^2 \geq mv^2$ by (4.5), we have also $\epsilon^{-2} \leq m^{-1}v^{-2}$ by Kato-Heinz theorem hence $C_c^\infty \Sigma^+ \subset \text{Dom } F(\tau)$.

5. CALDERÓN PROJECTORS

In this section we recall some standard facts on *Calderón projectors*. We refer the reader to [CP, Sects. 5.1- 5.3] for details.

5.1. The Calderón projector. Let (N, h) a complete Riemannian manifold and $P = -\Delta_h + m$, where $m \in C^\infty(N)$ is a real potential with $m(x) \geq m_0^2 > 0$. As in Sect. 4 we construct P as a selfadjoint operator on $L^2(N, dVol_h)$ using the quadratic form

$$(5.1) \quad Q(u, u) = \int_N \partial_i \bar{u} h^{ij} \partial_j u + m(x) |u|^2(x) dVol_h.$$

We obtain that $0 \in \rho(P)$, hence $G := P^{-1}$ is a bounded operator on $L^2(N, dVol_h)$, defined by

$$(5.2) \quad Q(Gv, w) = (v|w)_{L^2(N)}, \quad \forall w \in C_c^\infty(N).$$

Let $\Omega \subset N$ an open set such that $\partial\Omega = S = \bigcup_1^n S_i$, where S_i are the connected components of S and are assumed to be smooth hypersurfaces. We denote by $C^\infty(\bar{\Omega})$ the space of restrictions to Ω of functions in $C^\infty(N)$.

We associate to S_i the distribution density dS_i defined by:

$$\langle dS_i | u \rangle := \int_{S_i} u d\sigma_h^{(i)}, \quad u \in C_c^\infty(N),$$

where $d\sigma_h^{(i)}$ is the induced Riemannian density on S_i and we set

$$dS = \sum_{i=1}^n dS_i.$$

We denote by ∂_ν the unit exterior normal vector field to S and set

$$\langle \partial_\nu^* dS | u \rangle := \langle dS | \partial_\nu u \rangle, \quad u \in C_c^\infty(N).$$

For $u \in C^\infty(\bar{\Omega})$ we set

$$\gamma u := \begin{pmatrix} u|_S \\ \partial_\nu u|_S \end{pmatrix} =: \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}.$$

For $v \in C_c^\infty(S)$ we denote by $\tilde{v} \in C_c^\infty(N)$ an extension of v to N such that $\tilde{v}|_S = v$, $\partial_\nu \tilde{v}|_S = 0$.

Definition 5.1. Let $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C_c^\infty(S) \oplus C_c^\infty(S)$. We set:

$$Df := \gamma \circ G(\tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h^{-1})\partial_\nu^*dS).$$

– The operator $D : C_c^\infty(S) \oplus C_c^\infty(S) \rightarrow C^\infty(S) \oplus C^\infty(S)$ is continuous and is called the Calderón projector associated to (P, S) .

– The operator D is a 2×2 matrix of pseudodifferential operators on S .

Note that dS and ∂_ν^*dS are distributional densities, hence $(dVol_h)^{-1}dS$ and $(dVol_h)^{-1}\partial_\nu^*dS$ are distributions on N , supported on S .

Note also that the Calderón projector is obviously covariant under diffeomorphisms: if $\chi : (N, h) \rightarrow (N', h')$ is an isometric diffeomorphism with $S' = \chi(S)$, $P = \chi^*P'$, then

$$D = \psi^*D',$$

where $\psi : S \rightarrow S'$ is the restriction of χ to S .

5.1.1. *Expression in Gaussian normal coordinates.* Let U_i be a neighborhood of $\{0\} \times S_i$ in $\mathbb{R} \times S_i$ and V_i a neighborhood of S_i in N such that Gaussian normal coordinates to S_i induce a diffeomorphism:

$$\chi_i : U_i \ni x \mapsto (s, y) \in V_i$$

from U_i to V_i , and $ds^2 + k_s(y)dy^2 = \chi_i^*h$ on U_i . Then for $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$ we have

$$(5.3) \quad \begin{aligned} & \chi_i^* \left(\tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h)^{-1}\partial_\nu^*dS \right) \\ &= \delta_0(s) \otimes (f_1(y) - r_0(y)f_0(y)) - \delta_0'(s) \otimes f_0(y), \end{aligned}$$

where $r_s(y) = |k_s|^{-\frac{1}{2}}(y)\partial_s|k_s|^{\frac{1}{2}}$.

If $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$, $\int \varphi(s)ds = 1$, setting $\varphi_n(s) = n\varphi(ns)$, we can compute Df for $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$ as

$$(5.4) \quad Df = \lim_{n \rightarrow +\infty} \gamma \circ G(\varphi_n(s) \otimes (f_1(y) - r_0(y)f_0(y)) - \varphi_n'(s) \otimes f_1(y)),$$

where the limit takes place in $C^\infty(S) \oplus C^\infty(S)$.

Note that it is not obvious that $Df \in C^\infty(S) \oplus C^\infty(S)$. To prove it one can first replace G by a properly supported pseudodifferential parametrix $P^{(-1)} \in \Psi_c^{-2}(N)$. Using then Gaussian normal coordinates near a point $x^0 \in S$, one is reduced locally to $N = \mathbb{R}^d$, $S = \{x_1 = 0\}$. The details can be found for example in [CP, Sects. 5.1- 5.3].

Another useful identity is the following: for $u \in C^\infty(\bar{\Omega})$ let Iu be the extension of u by 0 in $N \setminus \bar{\Omega}$. Then

$$(5.5) \quad PIu = \tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h)^{-1}\partial_\nu^*dS + IPu, \quad \text{for } f = \gamma u.$$

6. THE DOUBLE β -KMS STATE

In this section we consider the *double β -KMS state* ω_β in $\mathcal{M}^+ \cup \mathcal{M}^-$. It is obtained as the natural extension to $\mathcal{M}^+ \cup \mathcal{M}^-$ of the state ω_β^+ in \mathcal{M}^+ , which is a β -KMS state in \mathcal{M}^+ with respect to the Killing flow. Its construction, for the more general stationary case is given in [S1, Thm. 3.5].

Since $\Sigma \setminus \mathcal{B}$ is a Cauchy surface for $\mathcal{M}^+ \cup \mathcal{M}^-$, we associate to ω_β its (complex) Cauchy surface covariances on $\Sigma \setminus \mathcal{B}$ λ^\pm , and (since ω_β is a pure state), the pair of complementary projections $c^\pm = \pm q^{-1} \circ \lambda^\pm$, see Subsect. 3.1. We will study in details the projection c^+ .

We identify $C_c^\infty(\Sigma \setminus \mathcal{B})$ with $C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2$ using the map

$$(6.1) \quad \begin{aligned} \hat{R}: C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2 &\rightarrow C_c^\infty(\Sigma^+) \oplus C_c^\infty(\Sigma^-) \\ g = g^{(0)} \oplus g^{(\beta/2)} &\mapsto f = g^{(0)} \oplus r^* g^{(\beta/2)}, \end{aligned}$$

where $r: \Sigma \rightarrow \Sigma$ is the restriction to Σ of the wedge reflection R , see Subsect. 2.2.

We will show that

$$C := \hat{R}^{-1} \circ c^+ \circ \hat{R}$$

is exactly the Calderón projector for the Euclidean operator K_+ acting on (N, \hat{g}) , see Subsect. 4.1, and the open set

$$\Omega := \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

6.1. The double β -KMS state. We recall now the expression of ω_β given by Sanders, see [S1, Sect. 3.3].

There are some differences in signs and factors of i with the expression given by Sanders in [S1, Sect. 3.3]. They come from two differences between our convention for quantized Klein-Gordon fields and the one of Sanders:

- our convention for Cauchy data of a solution of $Pu = 0$ is given the map $\rho u = \begin{pmatrix} u|_\Sigma \\ i^{-1} \partial_\nu u|_\Sigma \end{pmatrix} =: f$, which is more natural for complex fields and leads to a more symmetric formulation of the Hadamard condition, while Sanders uses $\rho u = \begin{pmatrix} u|_\Sigma \\ \partial_\nu u|_\Sigma \end{pmatrix} = g$, so $f = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} g$.

- we use as complex symplectic form $\bar{f} \cdot \sigma f' = (f_1|f'_0) - (f_0|f'_1)$, while Sanders uses $g \cdot \sigma g' = (g_0|g'_1) - (g_1|g'_0)$. In terms of spacetime fields, we use $i^{-1}E$, Sanders uses iE .

Let us unitarily identify $L^2(\Sigma, |h|^{1/2} dy)$ with $L^2(\Sigma^+, |h|^{1/2} dy) \oplus L^2(\Sigma^-, |h|^{1/2} dy)$, by

$$u \mapsto u_+ \oplus u_-, \quad u_\pm = u|_{\Sigma^\pm}.$$

Under this identification the action of the wedge reflection $r^*u = u \circ r$ will be denoted by T , with:

$$(6.2) \quad T(u_+ \oplus u_-) := r^*u_- \oplus r^*u_+.$$

A direct comparison with the formulas in [S1, Sect. 3.3], using the identity (3.2) gives the following proposition.

Proposition 6.1. *The double β -KMS state on $\mathcal{M}^+ \cup \mathcal{M}^-$ is given by the Cauchy surface covariance $\lambda^+ = \begin{pmatrix} \lambda_{00}^+ & \lambda_{01}^+ \\ \lambda_{10}^+ & \lambda_{11}^+ \end{pmatrix}$ where:*

$$(6.3) \quad \begin{aligned} \lambda_{00}^+ &= \frac{1}{2} |v|^{1/2} \left(\epsilon^{-1} \coth(\frac{\beta}{2} \epsilon) + \epsilon^{-1} T \text{sh}^{-1}(\frac{\beta}{2} \epsilon) \right) |v|^{1/2}, \\ \lambda_{11}^+ &= \frac{1}{2} |v|^{-1/2} \left(\epsilon \coth(\frac{\beta}{2} \epsilon) - \epsilon T \text{sh}^{-1}(\frac{\beta}{2} \epsilon) \right) |v|^{-1/2}, \\ \lambda_{01}^+ &= \lambda_{10}^+ = \frac{1}{2} \mathbf{1}. \end{aligned}$$

As in Subsect. 3.1 we have $\lambda^- = \lambda^+ - q$, where the charge $q = i\sigma$ is given by the matrix $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We introduce the operators $c^\pm := \pm q^{-1} \lambda^\pm$ and obtain

$$(6.4) \quad c^+ = \begin{pmatrix} \frac{1}{2} & \lambda_{00}^+ \\ \lambda_{11}^+ & \frac{1}{2} \end{pmatrix}.$$

Note that if

$$b_0 = \epsilon^{-1} \coth\left(\frac{\beta}{2}\epsilon\right) + \epsilon^{-1} T \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon\right), \quad b_1 = \epsilon \coth\left(\frac{\beta}{2}\epsilon\right) - \epsilon T \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon\right),$$

then using that $[T, \epsilon] = 0$ we obtain that

$$b_0 b_1 = b_1 b_0 = \coth\left(\frac{\beta}{2}\epsilon\right)^2 - \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon\right)^2 = 1,$$

from which it follows easily that c^\pm are (formally) projections. This is expected since the double β -KMS state ω_β is a pure state in $\mathcal{M}^+ \cup \mathcal{M}^-$.

6.2. Conjugation by \hat{R} . The map \hat{R} defined in (6.1) allows to unitarily identify $L^2(\Sigma^+) \otimes \mathbb{C}^2$ with $L^2(\Sigma^+) \oplus L^2(\Sigma^-)$. We have:

$$(6.5) \quad \hat{R}^{-1} \epsilon \hat{R} = \epsilon_+ \oplus \epsilon_+, \quad \hat{R}^{-1} T \hat{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denoting by c_{ij}^+ for $i, j \in \{0, 1\}$ the entries of the matrix c^+ and setting

$$C_{ij} := \hat{R}^{-1} \circ c_{ij}^+ \circ \hat{R},$$

we obtain after an easy computation using (6.3), (6.1):

$$(6.6) \quad \begin{aligned} C_{00}g_0 &= \frac{1}{2}g_0^{(0)} \oplus \frac{1}{2}g_0^{(\beta/2)}, \\ C_{11}g_1 &= \frac{1}{2}g_1^{(0)} \oplus \frac{1}{2}g_1^{(\beta/2)}, \\ C_{01}g_1 &= \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(0)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(\beta/2)} \\ &\quad \oplus \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(\beta/2)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(0)}, \\ C_{10}g_0 &= \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(0)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(\beta/2)} \\ &\quad \oplus \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(\beta/2)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(0)}. \end{aligned}$$

In (6.6) the upper indices (0), $(\beta/2)$ refer to the two connected components $\{\tau = 0\}$ and $\{\tau = \beta/2\}$ of $\partial\Omega$, while the lower indices 0, 1 refer to the two components of g .

6.3. The Calderón projector. We now compute the Calderón projector for K_+ , associated to the Riemannian manifold (N, \hat{g}) . We choose

$$\Omega = \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

We have $S = \partial\Omega = S_0 \cup S_{\beta/2}$ and we write $f \in C_c^\infty(S) \oplus C_c^\infty(S)$ as $f = f^{(0)} \oplus f^{(\beta/2)}$ for $f^{(i)} \in C_c^\infty(S_i) \oplus C_c^\infty(S_i)$.

We denote by $\gamma^{(i)}$, $i = 0, \beta/2$ the trace operators on S_i defined by $\gamma u = \gamma^{(0)}u \oplus \gamma^{(\beta/2)}u$ for $u \in C^\infty(\bar{\Omega})$. We have:

$$(6.7) \quad \begin{aligned} \gamma^{(0)}u &= \lim_{\tau \rightarrow 0^+} \begin{pmatrix} u(\tau, y) \\ -|v(y)|^{-1} \partial_\tau u(\tau, y) \end{pmatrix}, \\ \gamma^{(\beta/2)}u &= \lim_{\tau \rightarrow (\beta/2)^-} \begin{pmatrix} u(\tau, y) \\ |v(y)|^{-1} \partial_\tau u(\tau, y) \end{pmatrix}. \end{aligned}$$

We denote similarly by $\partial_\nu^{(i)}$ the exterior normal derivatives on S_i .

We compute the Calderón projector D defined in Subsect. 5.1 using the coordinates (τ, y) . Since $dS_i = |h|^{\frac{1}{2}}(y)dy$ and $dVol_{\hat{g}} = |v|^{\frac{1}{2}}(y)|h|^{\frac{1}{2}}(y)dy$, we obtain:

$$(6.8) \quad Df = D^{(0)}f \oplus D^{(\beta/2)}f,$$

for

$$(6.9) \quad \begin{aligned} D^{(i)}f &= \gamma^{(i)} \circ G \circ |v|^{-1} \left(\partial_\nu^{(0)} \delta_0(\tau) \otimes f_0^{(0)}(y) + \delta_0(\tau) \otimes f_1^{(0)}(y) \right. \\ &\quad \left. + \partial_\nu^{(\beta/2)} \delta_{\beta/2}(\tau) \otimes f_0^{(\beta/2)}(y) + \delta_{\beta/2}(\tau) \otimes f_1^{(\beta/2)}(y) \right). \end{aligned}$$

Since $G = |v|^{\frac{1}{2}} \tilde{G} |v|^{3/2}$ we have $G \circ |v|^{-1} = |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}}$. Denoting by $D_{kl}^{(i)(j)}$ for $i, j \in \{0, \beta/2\}$ and $k, l \in \{0, 1\}$ the various entries of D , we obtain:

$$(6.10) \quad D_{kl}^{(i)(j)} v = \begin{cases} \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\partial_\nu^{(j)} \delta_j(\tau) \otimes v(y)), & l = 0, \\ \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\delta_j(\tau) \otimes v(y)), & l = 1. \end{cases}$$

We also set

$$\partial_\tau^{(i)} = \mp \partial_\tau, \text{ for } i = 0, \beta/2,$$

so that $\partial_\nu^{(i)} = |v|^{-1}(y) \partial_\tau^{(i)}$.

Proposition 6.2. *We have $D = \hat{R}^{-1} \circ c^+ \circ \hat{R}$.*

Proof. We recall that C_{ij} are the entries of $\hat{R}^{-1} \circ c^+ \circ \hat{R}$. We compute $D_{kl}^{(i)(j)}$ using (6.10) and the explicit formulas (4.7), (4.8) for the kernel $\tilde{G}(\tau, \tau')$ of \tilde{G} .
Computation of D_{00} :

$$\begin{aligned} D_{00}^{(0)(0)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \lim_{\tau \rightarrow 0^+} \partial_{\tau'} \tilde{G}(\tau, 0) |v|^{-\frac{1}{2}} u = \frac{1}{2} u, \\ D_{00}^{(0)(\beta/2)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \lim_{\tau \rightarrow 0^+} \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = 0, \\ D_{00}^{(\beta/2)(0)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \lim_{\tau \rightarrow \beta/2^-} \partial_{\tau'} \tilde{G}(\tau, 0) u = 0, \\ D_{00}^{(\beta/2)(\beta/2)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \lim_{\tau \rightarrow \beta/2^-} \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = \frac{1}{2} u. \end{aligned}$$

Hence

$$D_{00} g_0 = C_{00} g_0.$$

Computation of D_{11} :

$$\begin{aligned} D_{11}^{(0)(0)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_0 \otimes u \\ &= -|v|^{-\frac{1}{2}} \lim_{\tau \rightarrow 0^+} \partial_\tau \tilde{G}(\tau, 0) |v|^{\frac{1}{2}} u = \frac{1}{2} u, \\ D_{11}^{(0)(\beta/2)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_{\beta/2} \otimes u \\ &= -|v|^{-\frac{1}{2}} \lim_{\tau \rightarrow 0^+} \partial_\tau \tilde{G}(\tau, \beta/2) |v|^{\frac{1}{2}} u = 0, \\ D_{11}^{(\beta/2)(0)} u &= \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_0 \otimes u \\ &= |v|^{-\frac{1}{2}} \lim_{\tau \rightarrow \beta/2^-} \partial_\tau \tilde{G}(\tau, 0) |v|^{\frac{1}{2}} u = 0, \\ D_{11}^{(\beta/2)(\beta/2)} u &= \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_{\beta/2} \otimes u \\ &= |v|^{-\frac{1}{2}} \lim_{\tau \rightarrow \beta/2^-} \partial_\tau \tilde{G}(\tau, \beta/2) |v|^{\frac{1}{2}} u = \frac{1}{2} u. \end{aligned}$$

Hence

$$D_{11}g_0 = C_{11}g_0.$$

Computation of D_{01} :

$$\begin{aligned} D_{01}^{(0)(0)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0 \otimes u \\ &= |v|^{\frac{1}{2}}\lim_{\tau \rightarrow 0^+}\tilde{G}(\tau, 0)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\ D_{01}^{(0)(\beta/2)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2} \otimes u \\ &= |v|^{\frac{1}{2}}\lim_{\tau \rightarrow 0^+}\tilde{G}(\tau, \beta/2)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\ D_{01}^{(\beta/2)(0)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0 \otimes u \\ &= |v|^{\frac{1}{2}}\lim_{\tau \rightarrow \beta/2^-}\tilde{G}(\tau, 0)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\ D_{01}^{(\beta/2)(\beta/2)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2} \otimes u \\ &= |v|^{\frac{1}{2}}\lim_{\tau \rightarrow \beta/2^-}\tilde{G}(\tau, \beta/2)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u. \end{aligned}$$

Hence

$$D_{01}g_1 = C_{01}g_1.$$

Computation of D_{10} :

$$\begin{aligned} D_{10}^{(0)(0)}u &= \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(0)}\delta_0 \otimes u \\ &= -|v|^{-\frac{1}{2}}\lim_{\tau \rightarrow 0^+}\partial_\tau\partial_{\tau'}\tilde{G}(\tau, 0)|v|^{-\frac{1}{2}}u = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\ D_{10}^{(0)(\beta/2)}u &= \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(\beta/2)}\delta_{\beta/2} \otimes u \\ &= |v|^{-\frac{1}{2}}\lim_{\tau \rightarrow 0^+}\partial_\tau\partial_{\tau'}\tilde{G}(\tau, \beta/2)|v|^{-\frac{1}{2}}u = -\frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\ D_{10}^{(\beta/2)(0)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(0)}\delta_0 \otimes u \\ &= |v|^{-\frac{1}{2}}\lim_{\tau \rightarrow \beta/2^-}\partial_\tau\partial_{\tau'}\tilde{G}(\tau, 0)|v|^{-\frac{1}{2}}u = -\frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\ D_{10}^{(\beta/2)(\beta/2)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(\beta/2)}\delta_{\beta/2} \otimes u \\ &= -|v|^{-\frac{1}{2}}\lim_{\tau \rightarrow \beta/2^-}\partial_\tau\partial_{\tau'}\tilde{G}(\tau, \beta/2)|v|^{-\frac{1}{2}}u = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u. \end{aligned}$$

Hence

$$D_{10}g_0 = C_{10}g_0.$$

This completes the proof of the proposition. \square

7. THE HARTLE-HAWKING-ISRAEL STATE AND ITS PROPERTIES

7.1. The smooth extension of (N, \hat{g}) and the Hawking temperature. The existence of the Hartle-Hawking-Israel state and the definition of the Hawking temperature $T_H = \kappa(2\pi)^{-1}$ (where κ is the surface gravity) rely on the existence of an extension $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ of (N, \hat{g}) such that the two components $S_0, S_{\beta/2} \sim \Sigma^+$ of $\partial\Omega$ are smoothly glued together into $\Sigma \subset N_{\text{ext}}$.

The extended Riemannian metric \hat{g}_{ext} is smooth iff $\beta = (2\pi)\kappa^{-1}$ (for other values of β $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ has a conic singularity on \mathcal{B}).

Let us embed $\Sigma \setminus \mathcal{B}$ into N by:

$$\hat{r} : \begin{cases} x \mapsto (0, x) \text{ for } x \in \Sigma^+, \\ x \mapsto (\beta/2, r(x)) \text{ for } x \in \Sigma^-, \end{cases}$$

Note that for \hat{R} defined in (6.1) we have

$$(7.1) \quad \hat{R} = (\hat{r})^*.$$

We recall that the function $m : \Sigma \rightarrow \mathbb{R}^+$ was introduced in Subsect. 2.4.

Proposition 7.1. [S1, Subsect. 2.2] *Assume that $\beta = (2\pi)\kappa^{-1}$. Then there exists a smooth complete Riemannian manifold $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ and*

- (1) *a smooth isometric embedding $\psi : \Sigma \rightarrow N_{\text{ext}}$,*
 - (2) *a smooth isometric embedding $\chi : (N, \hat{g}) \rightarrow (N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}}, \hat{g}_{\text{ext}})$ for $\mathcal{B}_{\text{ext}} = \psi(\mathcal{B})$,*
 - (3) *a smooth function $m_{\text{ext}} : N_{\text{ext}} \rightarrow \mathbb{R}$ with $m_{\text{ext}} \geq m_0^2 > 0$*
- such that*

$$\psi|_{\Sigma \setminus \mathcal{B}} = \chi \circ \hat{r}, \quad \psi^* m_{\text{ext}} = m|_N .$$

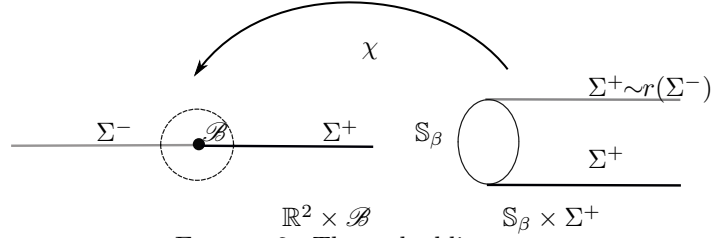


FIGURE 2. The embedding χ

This fundamental fact is well explained in [S1, Subsect. 2.2]. Let us briefly recall the construction of χ following [S1]: one introduces Gaussian normal coordinates to \mathcal{B} in (Σ, h) , where h is the Riemannian metric induced by g on Σ . We choose the unit normal vector field to \mathcal{B} pointing towards Σ^+ . Using these coordinates we can, since \mathcal{B} is compact, identify a small neighborhood U of \mathcal{B} in Σ with $] - \delta, \delta[\times \mathcal{B}$. Denoting by ω local coordinates on \mathcal{B} we have a map

$$\begin{aligned} \phi :] - \delta, \delta[\times \mathcal{B} \ni (s, \omega) &\mapsto y = \exp_{\omega}^h(s) \in U, \\ U^+ &= \phi(]0, \delta[\times \mathcal{B}), \\ \phi^* h &= ds^2 + k_{\alpha\beta}(s, \omega) d\omega^\alpha d\omega^\beta, \end{aligned}$$

for $U^+ = \Sigma^+ \cap U$. In the local coordinates (τ, s, ω) on $\mathcal{S}_\beta \times U^+$ the embedding χ takes the form:

$$(7.2) \quad \chi : \begin{aligned} &\mathcal{S}_\beta \times]0, \delta[\times \mathcal{B} \rightarrow B_2(0, \delta) \times \mathcal{B} \\ &(\tau, s, \omega) \mapsto (s \cos(\beta(2\pi)^{-1}\tau), s \sin(\beta(2\pi)^{-1}\tau), \omega) =: (X, Y, \omega), \end{aligned}$$

where $B_2(0, \delta) = \{(X, Y) \in \mathbb{R}^2 : 0 < X^2 + Y^2 < \delta^2\}$. A straightforward computation performed in [S1, Subsect. 2.2] shows that \hat{g} admits a smooth extension \hat{g}_{ext} to N_{ext} iff $\beta = (2\pi)\kappa^{-1}$.

7.2. The extension of ω_β to M . We recall from Subsect. 4.1 that K is defined from the closure \bar{Q} of the quadratic form Q on $C_c^\infty(N)$.

Similarly $K_{\text{ext}} = -\Delta_{\hat{g}_{\text{ext}}} + m_{\text{ext}}$, acting on N_{ext} is defined using the corresponding quadratic form Q_{ext} .

The following lemma is equivalent to [S1, Prop. 5.2], for completeness we give a short proof using quadratic form arguments (note that we assume the stronger condition that $\inf m(x) > 0$).

Lemma 7.2. *Let $U : C_c^\infty(N) \rightarrow C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$ defined by*

$$Uu = u \circ \chi^{-1}.$$

Then U extends as a unitary operator $U : L^2(N) \rightarrow L^2(N_{\text{ext}})$ with $K_{\text{ext}} = UKU^$.*

Proof. U clearly extends as a unitary operator. To check the second statement it suffices, taking into account the way K and K_{ext} are defined, to prove that $C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$ is a form core for Q_{ext} . The domain of $\overline{Q}_{\text{ext}}$ is the Sobolev space $H^1(N_{\text{ext}})$ associated to \hat{g}_{ext} , so we need to show that $C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$ is dense in $H^1(N_{\text{ext}})$. Using the coordinates (X, Y, ω) near $\mathcal{B}_{\text{ext}} \sim \{0\} \times \mathcal{B}$, this follows from the fact that $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $H^1(\mathbb{R}^2)$, see eg [A, Thm. 3.23]. \square

We recall that the projection c^\pm associated to the double β -KMS state ω_β was defined in (6.4). Let us identify in the sequel Σ with $\Sigma_{\text{ext}} = \psi(\Sigma) \subset N_{\text{ext}}$.

Theorem 7.3. *Let D_{ext} the Calderón projector for (K_{ext}, Σ) . Then for $f, g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$ we have:*

$$(g|c^+g)_{L^2(\Sigma)} = (g_{\text{ext}}|D_{\text{ext}}f_{\text{ext}})_{L^2(\Sigma)},$$

where $f_{\text{ext}} = (\psi^*)^{-1}f$, $g_{\text{ext}} = (\psi^*)^{-1}g$.

Proof. This follows from Prop. 6.2, the fact that \hat{R} is implemented by the embedding \hat{r} of $\Sigma \setminus \mathcal{B}$ into N , (see (7.1)) and Lemma 7.2. \square

7.3. Uniqueness of the extension. We discuss now the uniqueness of extensions of ω_β to M . Other types of uniqueness results were obtained before in [KW] and [K1].

Proposition 7.4. *There exists at most one quasi-free state ω for the Klein-Gordon field on M such that:*

- (1) *the restriction of ω to $\mathcal{M}^+ \cup \mathcal{M}^-$ equals ω_β ,*
- (2) *the spacetime covariances Λ^\pm of ω map $C_c^\infty(M)$ into $C^\infty(M)$.*

Proof. Let ω a quasi-free state for the Klein-Gordon operator P in M , with spacetime covariances Λ^\pm . We assume that $\Lambda^\pm : C_c^\infty(M) \rightarrow C^\infty(M)$. Denoting by $\Lambda^\pm(x, x')$ their Schwartz kernels, we have $P(x, \partial_x)\Lambda^\pm(x, x') = P(x', \partial_{x'})\Lambda^\pm(x, x') = 0$, which implies that

$$(7.3) \quad \text{WF}(\Lambda^\pm)' \subset \mathcal{N} \times \mathcal{N},$$

where \mathcal{N} is defined in (3.7). We claim that the entries $c_{k,k'}^\pm$, $k, k' = 0, 1$ of c^\pm defined in (3.11) map $C_c^\infty(\Sigma)$ into $C^\infty(\Sigma)$. In fact by (7.3) we have $\Lambda^\pm = \Lambda^\pm \circ A$ modulo smoothing, where $A \in \Psi^0(M)$ is a pseudodifferential operator with $\text{essupp}(A)$ included in an arbitrary small conical neighborhood of \mathcal{N} . For $u \in C_c^\infty(\Sigma)$ we have, modulo factors of i :

$$c_{k,k'}^\pm u = \partial_t^k \Lambda^\pm \circ A(-\partial_t^{k'} \delta_0 \otimes u)|_{t=0},$$

see (3.11). Since $\text{WF}((-\partial_t^{k'} \delta_0) \otimes u) \subset N^*\Sigma$, where $N^*\Sigma \subset T^*M$ is the conormal bundle to Σ and Σ is spacelike, we have $N^*\Sigma \cap \mathcal{N} = \emptyset$, hence $A(-\partial_t^{k'} \delta_0 \otimes u) \in C^\infty(M)$, which proves our claim.

Let now ω_i , $i = 1, 2$ be two quasi-free states as in the proposition. Since $(u|(\Lambda_1^+ - \Lambda_2^+)v)_{L^2(M)} = 0$ for $u, v \in C_c^\infty(\mathcal{M}^+ \cup \mathcal{M}^-)$ we obtain that $(f|(\lambda_1^+ - \lambda_2^+)g)_{L^2(\Sigma) \otimes \mathbb{C}^2} = 0$ for $f, g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$ hence $\text{supp}(\lambda_1^+ - \lambda_2^+)g \subset \mathcal{B}$ for $g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$. Since we have seen that $\lambda_i^+ : C_c^\infty(\Sigma) \otimes \mathbb{C}^2 \rightarrow C^\infty(\Sigma) \otimes \mathbb{C}^2$ this implies that $(\lambda_1^+ - \lambda_2^+)f = 0$ for $f \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$. Since λ_i^+ are selfadjoint for $L^2(\Sigma, dV ol_h) \otimes \mathbb{C}^2$ this implies that $\text{supp}(\lambda_1^+ - \lambda_2^+)f \subset \mathcal{B}$ for $f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2$, hence $(\lambda_1^+ - \lambda_2^+)f = 0$ using again that $\lambda_i^+ : C_c^\infty(\Sigma) \otimes \mathbb{C}^2 \rightarrow C^\infty(\Sigma) \otimes \mathbb{C}^2$. \square

7.4. The Hartle-Hawking-Israel state.

Theorem 7.5 ([S1]). *Let us set*

$$\lambda_{\text{HHI}}^+ := q \circ D_{\text{ext}}, \quad \lambda_{\text{HHI}}^- := \lambda_{\text{HHI}}^+ - q,$$

where D_{ext} is the Calderón projector for (K_{ext}, Σ) and the charge quadratic form q is defined in (3.8). Then:

- (1) λ_{HHI}^\pm are the Cauchy surface covariances for the Cauchy surface Σ of a quasi-free state ω_{HHI} for the free Klein-Gordon field on M .
- (2) the Hartle-Hawking-Israel state ω_{HHI} is a pure Hadamard state and is the unique extension to M of the double β -KMS state ω_β with the property that its spacetime covariances Λ_{HHI}^\pm map continuously $C_c^\infty(M)$ into $C^\infty(M)$.

Proof. Let us first prove (1). By (3.3) it suffices to check the positivity of λ_{HHI}^+ . This was shown in [S1, Thm. 5.3] using reflection positivity. For the reader's convenience, let us briefly repeat the argument:

for $u \in L^2(N)$ we set $Ru(\tau, y) = u(-\tau, y)$, for $\tau \in [-\beta/2, \beta/2] \sim \mathbb{S}_\beta$. The operator $G = K^{-1}$ is reflection positive, i.e.

$$(7.4) \quad (Ru|Gu)_{L^2(N)} \geq 0, \quad \forall u \in L^2(N), \quad \text{supp}u \subset [0, \beta/2] \times \Sigma^+.$$

In fact setting $\tilde{u} = |v|^{3/2}u$, (7.4) is equivalent to

$$(7.5) \quad (R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)} \geq 0, \quad \forall \tilde{u} \in L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+), \quad \text{supp}\tilde{u} \subset [0, \beta/2] \times \Sigma^+.$$

Using (4.7) we obtain

$$\begin{aligned} & (R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)} \\ &= (u_0 | \frac{1}{2\epsilon(1-e^{-\beta\epsilon})} u_0)_{L^2(\Sigma^+)} + (u_\beta | \frac{1}{2\epsilon(1-e^{-\beta\epsilon})} u_\beta)_{L^2(\Sigma^+)}, \end{aligned}$$

for

$$u_0 = \int_{\mathbb{S}_\beta} e^{-\tau\epsilon} \tilde{u}(\tau) d\tau, \quad u_\beta = \int_{\mathbb{S}_\beta} e^{(\tau-\beta/2)\epsilon} \tilde{u}(\tau) d\tau$$

where \tilde{u} is identified with the map $\mathbb{S}_\beta \ni \tau \mapsto \tilde{u}(\tau) \in L^2(\mathbb{S}_\beta; L^2(\Sigma^+))$. This proves (7.4).

By Lemma 7.2 and using that $G_{\text{ext}} = K_{\text{ext}}^{-1}$ is bounded on $L^2(N_{\text{ext}})$, we deduce from (7.4) that G_{ext} is also reflection positive, ie

$$(7.6) \quad (R_{\text{ext}}u | G_{\text{ext}}u_{\text{ext}})_{L^2(N_{\text{ext}})} \geq 0, \quad u \in L^2(N_{\text{ext}}), \quad \text{supp}u \subset N_{\text{ext}}^+ = \chi([0, \beta/2] \times \Sigma^+),$$

for $R_{\text{ext}} = URU^*$. By the remark before [S1, Thm. 5.3], if (s, y) are Gaussian normal coordinates to Σ in N_{ext} we have $R_{\text{ext}}u(s, y) = u(-s, y)$, ie R_{ext} is given by the reflection in Gaussian normal coordinates. This map is a isometry of $(N_{\text{ext}}, \hat{g}_{\text{ext}})$, which implies that if $\hat{g}_{\text{ext}} = ds^2 + h_{\text{ext}}(s, y)dy^2$ near Σ , we have $h_{\text{ext}}(s, y) = h_{\text{ext}}(-s, y)$ hence if $r_s(y) = |h_{\text{ext}}(s, y)|^{-\frac{1}{2}} \partial_s |h_{\text{ext}}(s, y)|^{\frac{1}{2}}$ we have $r_0(y) \equiv 0$.

If $f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2$ it follows from (5.3) that $D_{\text{ext}}f = \gamma G_{\text{ext}}Tf$ for

$$Tf = \delta_0(s) \otimes f_1 - \delta'_0(s) \otimes f_0.$$

We have $R_{\text{ext}}Tf = \delta_0(s) \otimes f_1 + \delta'_0(s) \otimes f_0$. Applying the reflection positivity (7.6) to $u = Tf$ we obtain that:

$$(R_{\text{ext}}Tf | G_{\text{ext}}Tf)_{L^2(N_{\text{ext}})} = (f | q D_{\text{ext}}f)_{L^2(\Sigma)} \geq 0,$$

which proves the positivity of λ_{HHI}^+ . To make the argument rigorous is suffices to approximate δ_0 by a sequence φ_n as in (5.4). This completes the proof of (1).

Let us now prove (2). The fact that ω_{HHI} is the unique extension of ω_β to M with the stated properties has been proved in Prop. 7.4. It remains to prove that ω_{HHI} is a pure Hadamard state in M .

The fact that ω_{HHI} is pure follows from the fact that D_{ext} is a projection. To prove the Hadamard property let us fix a reference Hadamard state ω_{ref} for the Klein-Gordon field in M . By Thm. 3.2 its Cauchy surface covariances on Σ $\lambda_{\text{ref}}^{\pm}$ are matrices of pseudodifferential operators on Σ . The same is true of $c_{\text{ref}}^{\pm} = \pm q^{-1} \circ \lambda_{\text{ref}}^{\pm}$ and of c_{HHI}^{\pm} , since Calderón projectors are given by matrices of pseudodifferential operators on Σ .

Moreover we know that the restriction of ω_{HHI} to $\mathcal{M}^+ \cup \mathcal{M}^-$ is a Hadamard state. The same is obviously true of the restriction of ω_{ref} to $\mathcal{M}^+ \cup \mathcal{M}^-$. Going to Cauchy surface covariances, this implies that if $\chi \in C_c^{\infty}(\Sigma^{\pm})$ then

$$\chi \circ (c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}) \circ \chi \text{ is a smoothing operator on } \Sigma.$$

We claim that this implies that $c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}$ is smoothing, which will imply that ω_{HHI} is a Hadamard state.

If fact let a be one of the entries of $c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}$, which is a scalar pseudodifferential operator belonging to $\Psi^m(\Sigma)$ for some $m \in \mathbb{R}$. We know that $\chi \circ a \circ \chi$ is smoothing for any $\chi \in C_c^{\infty}(\Sigma \setminus \mathcal{B})$. Then its principal symbol $\sigma_{\text{pr}}(a)$ vanishes on $T^*(\Sigma \setminus \mathcal{B})$ hence on $T^*\Sigma$ by continuity, so $a \in \Psi^{m-1}(\Sigma)$. Iterating this argument we obtain that a is smoothing, which proves our claim and completes the proof of the theorem. \square

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