



**HAL**  
open science

# On the Hartle-Hawking-Israel states for spacetimes with static bifurcate Killing horizons

Christian Gérard

► **To cite this version:**

Christian Gérard. On the Hartle-Hawking-Israel states for spacetimes with static bifurcate Killing horizons. 2016. hal-01355578v1

**HAL Id: hal-01355578**

**<https://hal.science/hal-01355578v1>**

Preprint submitted on 23 Aug 2016 (v1), last revised 19 Jan 2018 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the Hartle-Hawking-Israel states for spacetimes with static bifurcate Killing horizons

Christian GÉRARD

ABSTRACT. We revisit the construction by Sanders [S1] of the Hartle-Hawking-Israel state for a free quantum Klein-Gordon field on a spacetime with a static, bifurcate Killing horizon and a wedge reflection. Using the notion of the Calderón projector for elliptic boundary value problems and pseudodifferential calculus on manifolds, we give a short proof of its Hadamard property.

## 1. INTRODUCTION

Let  $(M, g)$  be a globally hyperbolic spacetime, with a *bifurcate Killing horizon*, see [KW], [S1] or Subsect. 2.1 for precise definition. The bifurcate Killing horizon  $\mathcal{H}$  is generated by the *bifurcation surface*  $\mathcal{B} = \{x \in M : V(x) = 0\}$ , where  $V$  is the Killing vector field. It allows to split  $(M, g)$  into four globally hyperbolic regions, the *right/left wedges*  $\mathcal{M}^+$ ,  $\mathcal{M}^-$  and the *future/past cones*  $\mathcal{F}$ ,  $\mathcal{P}$ , each invariant under the flow of  $V$ . An important object related with the Killing horizon  $\mathcal{H}$  is its *surface gravity*  $\kappa$ , which is a scalar, constant over all of  $\mathcal{H}$ .

Let us consider on  $(M, g)$  a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\square_g \phi(x) + m(x)\phi(x) = 0,$$

where  $m \in C^\infty(M, \mathbb{R})$ ,  $m(x) > 0$  is invariant under  $V$ , and its associated free field algebra.

If  $V$  is *time-like* in  $(\mathcal{M}^+, g)$ , ie if  $(\mathcal{M}^+, g, V)$  is a stationary spacetime, there exists (see [S2]) for any  $\beta > 0$  a *thermal state*  $\omega_\beta^+$  at temperature  $\beta^{-1}$  with respect to the group of Killing isometries of  $(\mathcal{M}^+, g)$  generated by  $V$ .

It was conjectured by Hartle and Hawking [HH] and Israel [I] that if  $\beta = \frac{2\pi}{\kappa}$  is the *inverse Hawking temperature*, denoted by  $\beta_H$  in the sequel, then  $\omega_\beta^+$  can be extended to the whole of  $M$  as a pure state, invariant under  $V$ , the *Hartle-Hawking-Israel state*, denoted in the sequel by  $\omega_{\text{HHI}}$ .

The rigorous construction of the HHI state was first addressed in an important paper by Kay and Wald [KW], who emphasized the importance of the *Hadamard condition*. They proved that a specific sub-algebra of the free field algebra has at most one state invariant under  $V$  and Hadamard. They also showed that if  $M$  admits a *wedge reflection* (see Subsect. 2.2) the restriction of such a state to  $\mathcal{M}^+$  will necessarily be a  $\beta_H$ -KMS state. These results were later improved in [K1].

The existence of such a state, ie of the HHI state, was however not proved in [HH]. The first proof of the existence of  $\omega_{\text{HHI}}$  was given by Sanders in the remarkable paper [S1], if the bifurcate Killing horizon is static, ie if  $V$  is static in  $\mathcal{M}^+$ , assuming also the existence of a wedge reflection. Sanders showed that there exists a unique Hadamard state  $\omega_{\text{HHI}}$  on  $M$  extending the *double  $\beta_H$ -KMS state*  $\omega_\beta$  on  $\mathcal{M}^+ \cup \mathcal{M}^-$ .

---

*Date:* August 2016.

*2010 Mathematics Subject Classification.* 81T20, 35S05, 35S15.

*Key words and phrases.* Hartle-Hawking state, Killing horizons, Hadamard states, pseudodifferential calculus, Calderón projector.

The double  $\beta_H$ -KMS state  $\omega_\beta$  is a pure state on  $\mathcal{M}^+ \cup \mathcal{M}^-$  which is the natural extension of  $\omega_\beta^+$  defined using the wedge reflection, see [K2, K3]. It is an exact geometrical analog of the Fock vacuum vector in the Araki-Woods representation of a thermal state.

**1.1. Content of the paper.** In this paper we revisit the construction in [S1] of the Hartle-Hawking-Israel state in a spacetime with a static bifurcate Killing horizon. Using the notion of the *Calderón projector* (see Sect. 5), which is a standard tool in elliptic boundary value problems, we significantly shorten the proof of the Hadamard property of  $\omega_{\text{HHI}}$ .

In [S1] the fact that  $\omega_{\text{HHI}}$  is Hadamard was proved by a careful comparison of the Hadamard parametrix construction for the D'Alembertian  $-\square_g + m$  associated to the Lorentzian metric  $g$  and for the Laplacian  $-\Delta_{\hat{g}} + m$  associated to the Riemannian metric  $\hat{g}$  obtained from  $g$  by Wick rotation in the Killing time coordinate.

In our paper we avoid working with the spacetime covariances of states and instead systematically work with the *Cauchy surface covariances* (see Subsect. 3.3) associated with a Cauchy surface  $\Sigma$  containing the bifurcation surface  $\mathcal{B}$ .

It turns out that the Cauchy surface covariances  $\lambda^\pm$  of the double  $\beta$ -KMS state  $\omega_\beta$  are related to a Calderón projector  $D$ .

Let us informally recall what is the Calderón projector associated to a elliptic boundary value problem, see Sect. 5 for more details:

let  $(N, \hat{g})$  be a complete Riemannian manifold and  $P = -\Delta_{\hat{g}} + m(x)$  for  $m \in C^\infty(N)$ ,  $m(x) > 0$  a Laplace-Beltrami operator. Let also  $\Omega \subset M$  a smooth open set. To  $\Omega$  is naturally associated the canonical surface density  $dS$ , defined by  $\langle dS|u \rangle = \int_{\partial\Omega} u d\sigma$ , for  $u \in C_c^\infty(M)$ , where  $d\sigma$  is the induced surface element on  $\partial\Omega$ .

If  $\partial_\nu$  is the external normal derivative to  $\partial\Omega$  and  $\gamma u = \begin{pmatrix} u|_{\partial\Omega} \\ \partial_\nu u|_{\partial\Omega} \end{pmatrix}$  for  $u \in C^\infty(\bar{\Omega})$  the Calderón projector  $D$  is a map from  $C_c^\infty(\partial\Omega) \otimes \mathbb{C}^2$  to  $C^\infty(\partial\Omega) \otimes \mathbb{C}^2$  defined by:

$$Df := \gamma \circ G(f_1(dVol_{\hat{g}})^{-1}dS - f_0(dVol_{\hat{g}}^{-1})\partial_\nu^*dS), \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C_c^\infty(\partial\Omega) \otimes \mathbb{C}^2,$$

where  $G = P^{-1}$ . It is easy to see that  $f \in C^\infty(\Sigma) \otimes \mathbb{C}^2$  equals  $\gamma u$  for some  $u \in C^\infty(\bar{\Omega})$  solution of  $Pu = 0$  in  $\Omega$  if and only if  $Df = f$ .

In our case we take  $N = \mathbb{S}_\beta \times \Sigma^+$ , where  $\mathbb{S}_\beta$  is the circle of length  $\beta$  and  $\Sigma^+ = \Sigma \cap \mathcal{M}^+$  is the right part of the Cauchy surface  $\Sigma$ . The Riemannian metric is  $\hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j$ , obtained by the *Wick rotation*  $t =: i\tau$  of the Lorentzian metric  $g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j$  on  $\mathcal{M}^+ \sim \mathbb{R} \times \Sigma^+$  where  $\mathcal{M}^+$  is identified to  $\mathbb{R} \times \Sigma^+$  using the Killing time coordinate  $t$ .

The existence of an extension of  $\omega_{\beta_H}$  to  $M$  is then an almost immediate consequence of the fact that  $(N, \hat{g})$  admits a smooth extension  $(N_{\text{ext}}, \hat{g}_{\text{ext}})$  if and only if  $\beta = \beta_H$ , a well-known result which plays also a crucial role in [S1].

In fact this geometrical fact implies that  $D$ , viewed as an operator defined on  $C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$  uniquely extends to a Calderon projector  $D_{\text{ext}}$ , defined on  $C_c^\infty(\Sigma) \otimes \mathbb{C}^2$ . From  $D_{\text{ext}}$  one can then easily obtain a pure quasi-free state  $\omega_{\text{HHI}}$  on the whole of  $M$ .

The Hadamard property of  $\omega_{\text{HHI}}$  follows then from the well-known fact that  $D_{\text{ext}}$ , being a Calderón projector, is a  $2 \times 2$  matrix of pseudodifferential operators on  $\Sigma$ , and of the Hadamard property of  $\omega_\beta$  in  $\mathcal{M}^+ \cup \mathcal{M}^-$ .

Beside shortening the proof of the Hadamard property of  $\omega_{\text{HHI}}$ , we think that our paper illustrates the usefulness of pseudodifferential calculus for the construction and study of Hadamard states, see also [GW1], [GW2], [GW3], [GOW] for other applications. We believe that Calderón projectors could also be used to construct

the Hartle-Hawking-Israel state in the still open case of spacetimes with a Killing horizon that is only stationary.

**1.2. Plan of the paper.** Let us now briefly give the plan of the paper. In Sect. 2 we recall the notion of a static bifurcate Killing horizon, following [S1] and introduce the associated Klein-Gordon equation.

Sect. 3 is devoted to background material on  $CCR^*$ -algebras, bosonic quasi-free states and their spacetime and Cauchy surface covariances in the case of quantum Klein-Gordon fields. We use the framework of *charged fields*, which is in our opinion more elegant, even when considering only neutral field equations. We also recall the notion of *pseudodifferential operators* on a manifold, which will be useful later on and formulate a consequence of [GW1] which states that the Cauchy surface covariances of *any* Hadamard state for Klein-Gordon fields is given by a matrix of pseudodifferential operators.

In Sect. 4 we define various 'Euclidean' Laplacians,  $K = -\Delta_{\hat{g}} + m$  acting on  $N = \mathbb{S}_\beta \times \Sigma^+$  and a related operator  $\tilde{K}$ , obtained from Wick rotation of the Lorentzian metric on  $M$  in the Killing time coordinate, which are considered in [S1]. It is sufficient for us to define these Laplacians by quadratic form techniques, which simplifies some arguments.

In Sect. 5 we recall the definition of the *Calderón projector*, which is a standard notion in elliptic boundary value problems. In Sect. 6, using the explicit expression for  $\tilde{K}^{-1}$ , we show that the projection associated to the double  $\beta$ -KMS state  $\omega_\beta$  equals to the Calderón projector  $D$  associated to  $K$  and the open set  $\Omega = ]0, \beta/2[ \times \Sigma^+$ .

In Sect. 7, we recall the well-known fact that a smooth extension  $(N_{\text{ext}}, \hat{g}_{\text{ext}})$  of  $(N, \hat{g})$  exists iff  $\beta = \beta_{\text{H}}$ . The extended Calderón projector  $D_{\text{ext}}$  generates a pure state on  $M$ , which is the Hartle-Hawking-Israel state  $\omega_{\text{HHI}}$ . In Prop. 7.4, we show that such an extension is unique among quasi-free states whose spacetime covariances map  $C_c^\infty(M)$  into  $C^\infty(M)$  continuously. Finally we give in the proof of Thm. 7.5 a new and elementary proof of the Hadamard property of  $\omega_{\text{HHI}}$ , using the pseudodifferential calculus on  $\Sigma$ .

## 2. SPACETIMES WITH A STATIC BIFURCATE KILLING HORIZON

**2.1. Static bifurcate Killing horizons.** We consider as in [S1] a globally hyperbolic spacetime  $(M, g)$  with a *static bifurcate Killing horizon*. We recall, see [S1, Def. 2.2], that this is a triple  $(M, g, V)$ , such that

- (1) the Lorentzian manifold  $(M, g)$  is globally hyperbolic,
- (2)  $V$  is a complete Killing vector field for  $(M, g)$ ,
- (3)  $\mathcal{B} := \{x \in M : V(x) = 0\}$  is a compact, orientable submanifold of codimension 2,
- (4) there exists a Cauchy hypersurface  $\Sigma$  containing  $\mathcal{B}$ ,
- (5)  $V$  is  $g$ -orthogonal to  $\Sigma$ ,

see Figure 1 below where the vector field  $V$  is represented by arrows.

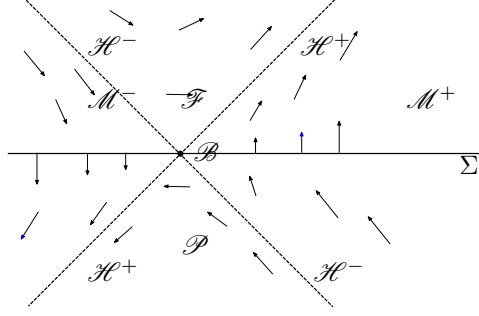


FIGURE 1.

For simplicity we will also assume that the bifurcation surface  $\mathcal{B}$  is connected. Denoting by  $n$  the future pointing normal vector field to  $\Sigma$  one introduces the *lapse function*:

$$(2.1) \quad v(x) := -n(x) \cdot g(x)V(x), \quad x \in \Sigma,$$

and  $\Sigma$  decomposes as

$$\Sigma = \Sigma^- \cup \mathcal{B} \cup \Sigma^+,$$

where  $\Sigma^\pm = \{x \in \Sigma : \pm v(x) > 0\}$ . The spacetime  $M$  splits as

$$M = \mathcal{M}^+ \cup \mathcal{M}^- \cup \overline{\mathcal{F}} \cup \overline{\mathcal{P}},$$

where the future cone  $\mathcal{F} := I^+(\mathcal{B})$ , the past cone  $\mathcal{P} := I^-(\mathcal{B})$ , the right/left wedges  $\mathcal{M}^\pm := D(\Sigma^\pm)$ , are all globally hyperbolic when equipped with  $g$ .

**2.2. Wedge reflection.** Additionally one has to assume the existence of a *wedge reflection*, see [S1, Def. 2.6], ie a diffeomorphism  $R$  of  $\mathcal{M}^+ \cup \mathcal{M}^- \cup U$  onto itself, where  $U$  is an open neighborhood of  $\mathcal{B}$  such that:

- (1)  $R \circ R = \text{Id}$ ,
- (2)  $R$  is an isometry of  $(\mathcal{M}^+ \cup \mathcal{M}^-, g)$  onto itself, which reverses the time orientation,
- (3)  $R = \text{Id}$  on  $\mathcal{B}$ ,
- (4)  $R^*V = V$  on  $\mathcal{M}^+ \cup \mathcal{M}^-$ .

It follows that  $R$  preserves  $\Sigma$ , see [S1, Prop. 2.7], and we denote by  $r$  the restriction of  $R$  to  $\Sigma$ . Denoting by  $h$  the induced Riemannian metric on  $\Sigma$  one has:

$$(2.2) \quad r^*h = h, r^*v = -v.$$

**2.3. Killing time coordinate.** Denoting by  $\Phi_s^V : M \rightarrow M$  the flow of the Killing vector field  $V$ , we obtain a diffeomorphism

$$\chi : \mathbb{R} \times (\Sigma \setminus \mathcal{B}) \ni (t, y) \mapsto \Phi_t^V(y) \in \mathcal{M}^+ \cup \mathcal{M}^-,$$

which defines the coordinate  $t$  on  $\mathcal{M}^+ \cup \mathcal{M}^-$  called the *Killing time coordinate*. The metric  $g$  on  $\mathcal{M}^+ \cup \mathcal{M}^-$  pulled back by  $\chi$  takes the form (see [S1, Subsect. 2.1]):

$$(2.3) \quad g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j,$$

where the Riemannian metric  $h_{ij}(y)dy^i dy^j$  is the restriction of  $g$  to  $\Sigma$ .

**2.4. Klein-Gordon operator.** We fix a real potential  $m \in C^\infty(M)$ . As in [S1] we assume that  $m$  is stationary w.r.t. the Killing vector field  $V$  and invariant under the wedge reflection, ie:

$$(2.4) \quad V^a \nabla_a m(x) = 0, \quad m \circ R(x) = m(x), \quad x \in \mathcal{M}^+ \cup \mathcal{M}^- \cup U.$$

For simplicity we also assume that

$$(2.5) \quad m(x) \geq m_0^2 > 0, \quad x \in M,$$

ie we consider only massive fields. Note that in [S1] the weaker condition  $m(x) > 0$  was assumed. We consider the Klein-Gordon operator

$$(2.6) \quad P = -\square_g + m.$$

### 3. FREE KLEIN-GORDON FIELDS

In this section we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [GW1, Sect. 2] based on *charged fields*.

#### 3.1. Charged CCR algebra.

**3.1.1. Charged bosonic fields.** Let  $\mathcal{Y}$  a complex vector space,  $\mathcal{Y}^*$  its anti-dual. Sesquilinear forms on  $\mathcal{Y}$  are identified with elements of  $L(\mathcal{Y}, \mathcal{Y}^*)$  and the action of a sesquilinear form  $\beta$  is correspondingly denoted by  $\bar{y}_1 \cdot \beta y_2$  for  $y_1, y_2 \in \mathcal{Y}$ . We fix  $q \in L_h(\mathcal{Y}, \mathcal{Y}^*)$  a non degenerate hermitian form on  $\mathcal{Y}$ , ie such that  $\text{Ker } q = \{0\}$ .

The *CCR \*-algebra*  $\text{CCR}(\mathcal{Y}, q)$  is the complex \*-algebra generated by symbols  $\mathbf{1}, \psi(y), \psi^*(y), y \in \mathcal{Y}$  and the relations:

$$\begin{aligned} \psi(y_1 + \lambda y_2) &= \psi(y_1) + \lambda \psi(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\ \psi^*(y_1 + \lambda y_2) &= \psi^*(y_1) + \lambda \psi^*(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\ [\psi(y_1), \psi(y_2)] &= [\psi^*(y_1), \psi^*(y_2)] = 0, \quad [\psi(y_1), \psi^*(y_2)] = \bar{y}_1 \cdot q y_2 \mathbf{1}, \quad y_1, y_2 \in \mathcal{Y}, \\ \psi(y)^* &= \psi^*(y), \quad y \in \mathcal{Y}. \end{aligned}$$

A state  $\omega$  on  $\text{CCR}(\mathcal{Y}, q)$  is (*gauge invariant*) *quasi-free* if

$$\omega\left(\prod_{i=1}^p \psi(y_i) \prod_{i=1}^q \psi^*(y_j)\right) = \begin{cases} 0 & \text{if } p \neq q, \\ \sum_{\sigma \in S_p} \prod_{i=1}^p \omega(\psi(y_i) \psi^*(y_{\sigma(i)})) & \text{if } p = q. \end{cases}$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2]. It is convenient to associate to  $\omega$  its (*complex*) *covariances*  $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$  defined by:

$$\begin{aligned} \omega(\psi(y_1) \psi^*(y_2)) &=: \bar{y}_1 \cdot \lambda^+ y_2, \\ \omega(\psi^*(y_2) \psi(y_1)) &=: \bar{y}_1 \cdot \lambda^- y_2, \end{aligned} \quad y_1, y_2 \in \mathcal{Y}.$$

The following results are well-known, see eg [DG, Sect. 17.1] or [GW1, Sect. 2]:

– two hermitian forms  $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$  are the covariances of a quasi-free state  $\omega$  iff

$$(3.1) \quad \lambda^\pm \geq 0, \quad \lambda^+ - \lambda^- = q.$$

– Let  $\mathcal{Y}_\omega$  be the completion of  $\mathcal{Y}$  for the Hilbertian scalar product  $\lambda^+ + \lambda^-$ . If there exist linear operators  $c^\pm \in L(\mathcal{Y}_\omega)$  such that

$$c^+ + c^- = \mathbf{1}, \quad (c^\pm)^2 = c^\pm,$$

(ie  $c^\pm$  is a pair of complementary projections) and  $\lambda^\pm = \pm q \circ c^\pm$ , then  $\omega$  is a *pure state*.

3.1.2. *Neutral bosonic fields.* We complete this subsection by explaining the relationship with the formalism of neutral fields, see eg [GW1, Subsect. 2.5].

Let  $\mathcal{X}$  a real vector space,  $\mathcal{X}^\#$  its dual, and  $\sigma \in L_a(\mathcal{X}, \mathcal{X}^\#)$  a symplectic form on  $\mathcal{X}$ . The  $*$ -algebra  $\text{CCR}(\mathcal{X}, \sigma)$  is the complex  $*$ -algebra generated by symbols  $\mathbf{1}, \phi(x), x \in \mathcal{X}$  and relations:

$$\begin{aligned}\phi(x_1 + \lambda x_2) &= \phi(x_1) + \lambda \phi(x_2), \quad x_1, x_2 \in \mathcal{X}, \lambda \in \mathbb{R}, \\ [\phi(x_1), \phi(x_2)] &= ix_1 \cdot \sigma x_2 \mathbf{1}, \quad x_1, x_2 \in \mathcal{X}, \\ \phi(x)^* &= \phi(x), \quad x \in \mathcal{X}.\end{aligned}$$

To relate the neutral to the charged formalism one sets  $\mathcal{Y} = \mathbb{C}\mathcal{X}$  and for  $\beta \in L(\mathcal{X}, \mathcal{X}^\#)$  denote by  $\beta_{\mathbb{C}} \in L(\mathcal{Y}, \mathcal{Y}^*)$  its sesquilinear extension.  $\mathcal{Y}_{\mathbb{R}} \sim \mathcal{X} \oplus \mathcal{X}$  is the *real form* of  $\mathcal{Y}$ , ie  $\mathcal{Y}_{\mathbb{R}} = \mathcal{Y}$  as a real vector space. Then  $(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}}) \sim (\mathcal{X}, \sigma) \oplus (\mathcal{X}, \sigma)$  is a real symplectic space and we denote by  $\phi(y), y \in \mathcal{Y}_{\mathbb{R}}$  the selfadjoint generators of  $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$ . Under the identification  $\phi(y) \sim \phi(x) \otimes \mathbf{1} + \mathbf{1} \otimes \phi(x')$  for  $y = x + ix'$  we can identify  $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$  with  $\text{CCR}(\mathcal{X}, \sigma) \otimes \text{CCR}(\mathcal{X}, \sigma)$  as  $*$ -algebras.

Note also that under the identification

$$\psi(y) \sim \frac{1}{\sqrt{2}}(\phi(y) + i\phi(iy)), \quad \psi^*(y) \sim \frac{1}{\sqrt{2}}(\phi(y) - i\phi(iy)), \quad y \in \mathcal{Y}$$

we can identify  $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$  with  $\text{CCR}(\mathcal{Y}, q)$  for  $q = i\sigma_{\mathbb{C}}$ .

A quasi-free state  $\omega$  on  $\text{CCR}(\mathcal{X}, \sigma)$  is determined by its *real covariance*  $\eta \in L_s(\mathcal{X}, \mathcal{X}^\#)$  defined by:

$$\omega(\phi(x_1)\phi(x_2)) =: x_1 \cdot \eta x_2 + \frac{i}{2}x_1 \cdot \sigma x_2, \quad x_1, x_2 \in \mathcal{X}.$$

A symmetric form  $\eta \in L_s(\mathcal{X}, \mathcal{X}^\#)$  is the covariance of a quasi-free state iff

$$\eta \geq 0, \quad |x_1 \cdot \sigma x_2| \leq 2(x_1 \cdot \eta x_1)^{\frac{1}{2}}(x_2 \cdot \eta x_2)^{\frac{1}{2}}, \quad x_1, x_2 \in \mathcal{X}.$$

To such a state  $\omega$  we associate the quasi-free state  $\tilde{\omega}$  on  $\text{CCR}(\mathcal{Y}_{\mathbb{R}}, \text{Re}\sigma_{\mathbb{C}})$  with real covariance  $\text{Re}\eta_{\mathbb{C}}$ . Then its complex covariances  $\lambda^\pm$  are given by (see [GW1, Subsect. 2.5]):

$$(3.2) \quad \lambda^\pm = \eta_{\mathbb{C}} \pm \frac{1}{2}i\sigma_{\mathbb{C}}.$$

Applying complex conjugation, we immediately see that in this case

$$(3.3) \quad \lambda^+ \geq 0 \Leftrightarrow \lambda^- \geq 0,$$

so it suffices to check for example that  $\lambda^+ \geq 0$ .

**3.2. Free Klein-Gordon fields.** Let  $P = -\square_g + m(x)$ ,  $m \in C^\infty(M, \mathbb{R})$  a Klein-Gordon operator on a globally hyperbolic spacetime  $(M, g)$  (we use the convention  $(1, d)$  for the Lorentzian signature). Let  $E^\pm$  be the advanced/retarded inverses of  $P$  and  $E := E^+ - E^-$ . We apply the above framework to

$$\mathcal{Y} = \frac{C_c^\infty(M)}{PC_c^\infty(M)}, \quad \overline{[u]} \cdot q[u] = i^{-1}(u|Eu)_M,$$

where  $(u|v)_M = \int_M \bar{u}v dV_{ol_g}$ .

One restricts attention to quasi-free states on  $\text{CCR}(\mathcal{Y}, q)$  whose covariances are given by distributions on  $M \times M$ , ie such that there exists  $\Lambda^\pm \in \mathcal{D}'(M \times M)$  with

$$(3.4) \quad \begin{aligned}\omega(\psi([u_1])\psi^*([u_2])) &= (u_1|\Lambda^+u_2)_M, \\ \omega(\psi^*([u_2])\psi([u_1])) &= (u_1|\Lambda^-u_2)_M,\end{aligned} \quad u_1, u_2 \in C_c^\infty(M).$$

In the sequel the distributions  $\Lambda^\pm \in \mathcal{D}'(M \times M)$  will be called the *spacetime covariances* of the state  $\omega$ .

In (3.4) we identify distributions on  $M$  with distributional densities using the density  $dVol_g$  and use the notation  $(u|\varphi)_M$ ,  $u \in C_c^\infty(M)$ ,  $\varphi \in \mathcal{D}'(M)$  for the duality bracket. We have then

$$(3.5) \quad \begin{aligned} P(x, \partial_x)\Lambda^\pm(x, x') &= P(x', \partial_{x'})\Lambda^\pm(x, x') = 0, \\ \Lambda^+(x, x') - \Lambda^-(x, x') &= i^{-1}E(x, x'). \end{aligned}$$

Such a state is called a *Hadamard state*, (see [R] for the neutral case and [GW1] for the complex case) if

$$(3.6) \quad \text{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm,$$

where  $\text{WF}(\Lambda)'$  denotes the 'primed' wavefront set of  $\Lambda$ , ie  $S' := \{((x, \xi), (x', -\xi')) : ((x, \xi), (x', \xi')) \in S\}$  for  $S \subset T^*M \times T^*M$ , and  $\mathcal{N}^\pm$  are the two connected components (positive/negative energy shell) of the characteristic manifold:

$$(3.7) \quad \mathcal{N} := \{(x, \xi) \in T^*M \setminus \{0\} : \xi_\mu g^{\mu\nu}(x) \xi_\nu = 0\}.$$

**3.3. Cauchy surface covariances.** Denoting by  $Sol_{\text{sc}}(P)$  the space of smooth space-compact solutions of  $P\phi = 0$ , it is well known that

$$[E] : \frac{C_c^\infty(M)}{PC_c^\infty(M)} \ni [u] \mapsto Eu \in Sol_{\text{sc}}(P)$$

is bijective, with

$$i^{-1}(u_1|Eu_2) = \overline{Eu_1} \cdot qEu_2, \quad u_i \in C_c^\infty(M),$$

for

$$(3.8) \quad \overline{\phi}_1 \cdot q\phi_2 := i \int_\Sigma (\nabla_\mu \overline{\phi}_1 \phi_2 - \overline{\phi}_1 \nabla_\mu \phi_2) n^\mu d\sigma,$$

where  $\Sigma$  is any spacelike Cauchy hypersurface,  $n^\mu$  is the future directed unit normal vector field to  $\Sigma$  and  $d\sigma$  the induced surface density. Setting

$$\rho : C_{\text{sc}}^\infty(M) \ni \phi \mapsto \left( \begin{array}{c} \phi|_\Sigma \\ i^{-1} \partial_\nu \phi|_\Sigma \end{array} \right) = f \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

Since the Cauchy problem

$$\begin{cases} P\phi = 0, \\ \rho u = f \end{cases}$$

as a unique solution  $\phi \in Sol_{\text{sc}}(P)$  for  $f \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$  the map

$$\frac{C_c^\infty(M)}{PC_c^\infty(M)} \ni [u] \mapsto \rho Eu \in C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

is bijective, and

$$i^{-1}(u|Eu)_M = \overline{\rho Eu} \cdot q\rho Eu,$$

for

$$(3.9) \quad \overline{f} \cdot qf := \int_\Sigma \overline{f}_1 f_0 + \overline{f}_0 f_1 d\sigma_\Sigma, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

It follows that to a quasi-free state with spacetime covariances  $\Lambda^\pm$  one can associate its *Cauchy surface covariances*  $\lambda^\pm$  defined by:

$$(3.10) \quad \Lambda^\pm =: (\rho E)^* \lambda^\pm (\rho E).$$

Using the canonical scalar product  $(f|f)_\Sigma := \int_\Sigma \overline{f}_1 f_1 + \overline{f}_0 f_0 d\sigma_\Sigma$  we identify  $\lambda^\pm$  with operators, still denoted by  $\lambda^\pm$ , belonging to  $L(C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma), \mathcal{D}'(\Sigma) \oplus \mathcal{D}'(\Sigma))$ .

A more explicit expression of  $\lambda^\pm$  in terms of  $\Lambda^\pm$  is as follows, see eg [GOW, Thm. 7.10]: let us introduce Gaussian normal coordinates to  $\Sigma$

$$U \ni (t, y) \mapsto \chi(t, y) \in V,$$



where  $U$  is an open neighborhood of  $\{0\} \times \Sigma$  in  $\mathbb{R} \times \Sigma$  and  $V$  an open neighborhood of  $\Sigma$  in  $M$ , such that  $\chi^*g = -dt^2 + h_{ij}(t, y)dy^i dy^j$ . We denote by  $\Lambda^\pm(t, y, t', y') \in \mathcal{D}'(U \times U)$  the restriction to  $U \times U$  of the distributional kernel of  $\Lambda^\pm$ . By (3.5) and standard microlocal arguments, their restrictions to fixed times  $t, t'$ , denoted by  $\Lambda^\pm(t, t') \in \mathcal{D}'(\Sigma \otimes \Sigma)$  are well defined.

We know also that  $\partial_t^k \partial_{t'}^{k'} \Lambda^\pm(0, 0) \in \mathcal{D}'(\Sigma \times \Sigma)$  is well defined for  $k, k' = 0, 1$ . Then setting  $\lambda^\pm =: \pm q \circ c^\pm$  we have:

$$(3.11) \quad c^\pm = \pm \begin{pmatrix} i\partial_{t'} \Lambda^\pm(0, 0) & \Lambda^\pm(0, 0) \\ \partial_t \partial_{t'} \Lambda^\pm(0, 0) & i^{-1} \partial_t \Lambda^\pm(0, 0) \end{pmatrix}.$$

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [GW1, GOW] using pseudodifferential calculus on  $\Sigma$ , see below for a short summary.

**3.4. Pseudodifferential operators.** We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [Sh, Sect. 4.3] for details.

For  $m \in \mathbb{R}$  we denote by  $\Psi^m(\mathbb{R}^d)$  the space of classical pseudodifferential operators of order  $m$  on  $\mathbb{R}^d$ , associated with poly-homogeneous symbols of order  $m$  see eg [Sh, Sect. 3.7].

Let  $N$  be a smooth,  $d$ -dimensional manifold. Let  $U \subset N$  a precompact chart open set and  $\psi : U \rightarrow \tilde{U}$  a chart diffeomorphism, where  $\tilde{U} \subset \mathbb{R}^d$  is precompact, open. We denote by  $\psi^* : C_c^\infty(\tilde{U}) \rightarrow C_c^\infty(U)$  the map  $\psi^*u(x) := u \circ \psi(x)$ .

**Definition 3.1.** A linear continuous map  $A : C_c^\infty(N) \rightarrow C^\infty(N)$  belongs to  $\Psi^m(N)$  if the following condition holds:

(C) Let  $U \subset N$  be precompact open,  $\psi : U \rightarrow \tilde{U}$  a chart diffeomorphism,  $\chi_1, \chi_2 \in C_c^\infty(U)$  and  $\tilde{\chi}_i = \chi_i \circ \psi^{-1}$ . Then there exists  $\tilde{A} \in \Psi^m(\mathbb{R}^d)$  such that

$$(3.12) \quad (\psi^*)^{-1} \chi_1 A \chi_2 \psi^* = \tilde{\chi}_1 \tilde{A} \tilde{\chi}_2.$$

Elements of  $\Psi^m(N)$  are called (classical) pseudodifferential operators of order  $m$  on  $N$ .

The subspace of  $\Psi^m(N)$  of pseudodifferential operators with properly supported kernels is denoted by  $\Psi_c^m(N)$ .

Note that if  $\Psi_{(c)}^\infty(N) := \bigcup_{m \in \mathbb{R}} \Psi_{(c)}^m(N)$ , then  $\Psi_c^\infty(N)$  is an algebra, but  $\Psi^\infty(N)$  is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

We denote by  $T^*N \setminus \{0\}$  the cotangent bundle of  $N$  with the zero section removed.

To  $A \in \Psi^m(N)$  one can associate its *principal symbol*  $\sigma_{\text{pr}}(A) \in C^\infty(T^*N \setminus \{0\})$ , which is homogeneous of degree  $m$  in the fiber variable  $\xi$  in  $T^*M$ , in  $\{|\xi| \geq 1\}$ .  $A$  is called *elliptic* in  $\Psi^m(N)$  at  $(x_0, \xi_0) \in T^*N \setminus \{0\}$  if  $\sigma_{\text{pr}}(A)(x_0, \xi_0) \neq 0$ .

If  $A \in \Psi^m(N)$  there exists (many)  $A_c \in \Psi_c^m(N)$  such that  $A - A_c$  has a smooth kernel.

Finally one says that  $(x_0, \xi_0) \notin \text{essupp}(A)$  for  $A \in \Psi^\infty(N)$  if there exists  $B \in \Psi_c^\infty(N)$  *elliptic* at  $(x_0, \xi_0)$  such that  $A_c \circ B$  is smoothing, where  $A_c \in \Psi_c^\infty(N)$  is as above, ie  $A - A_c$  is smoothing.

**3.5. The Cauchy surface covariances of Hadamard states.** We now state a result which follows directly from a construction of Hadamard states in [GW1, Subsect. 8.2].

**Theorem 3.2.** Let  $\omega$  be any Hadamard state for the free Klein-Gordon field on  $(M, g)$  and  $\Sigma$  a spacelike Cauchy hypersurface. Then its Cauchy surface covariances  $\lambda^\pm$  are  $2 \times 2$  matrices with entries in  $\Psi^\infty(\Sigma)$ .

**Proof.** It is well known (see eg [R]) that if  $\omega_1, \omega_2$  are Hadamard states, then  $\Lambda_1^\pm - \Lambda_2^\pm$  are smoothing operators on  $M$ . Using (3.10) this implies that  $\lambda_1^\pm - \lambda_2^\pm$  are matrices of smoothing operators on  $\Sigma$ . From the definition of  $\Psi^\infty(\Sigma)$  it hence suffices to construct *one* Hadamard state  $\omega$  whose Cauchy surface covariances  $\lambda^\pm$  are matrices of pseudodifferential operators. The state constructed in [GW1, Subsect. 8.2] has this property, as can be seen from [GW1, Equ. (8.2)].  $\square$

#### 4. EUCLIDEAN OPERATORS

The construction of the  $\beta$ -KMS state on  $\mathcal{M}^+$  with respect to the Killing vector field  $V$  relies on the *Wick rotation*, where  $(\mathbb{R} \times \Sigma^+, g)$  is replaced by  $(\mathbb{S}_\beta \times \Sigma^+, \hat{g})$ :

$$(4.1) \quad \hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j,$$

is the Riemannian metric obtained from (2.3) by setting  $t = i\tau$  and  $\mathbb{S}_\beta = [0, \beta[$  with endpoints identified is the circle of length  $\beta$ .

In this section we recall various 'Euclidean' operators related to  $\hat{g}$  appearing in [S1, S2]. It will be convenient to construct them by quadratic form techniques.

We set

$$N := \mathbb{S}_\beta \times (\Sigma^+ \setminus \mathcal{B}),$$

whose elements are denoted by  $(\tau, y)$ . We equip  $N$  with the Riemannian metric  $\hat{g}$  in (4.1) and the associated density  $dVol_{\hat{g}} = |v|(y)|h|^{\frac{1}{2}}(y)d\tau dy$ . The hypersurface  $\Sigma^+ \setminus \mathcal{B}$  is equipped with the induced density  $dVol_h = |h|^{\frac{1}{2}}(y)dy$ .

**4.1. Euclidean operator on  $N$ .** We consider the operator

$$K := -\Delta_{\hat{g}} + m(y),$$

for  $m$  as in Subsect. 2.4. Note that  $m$  depends only on  $y$  since  $m$  is invariant under the Killing flow. We have

$$K = -v^{-2}(y)\bar{\partial}_\tau^2 - |v|^{-1}(y)|h|^{-\frac{1}{2}}(y)\bar{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\bar{\partial}_{y^j} + m(y).$$

$K$  is well defined as a selfadjoint operator on  $L^2(N, dVol_{\hat{g}})$  obtained from the quadratic form:

$$(4.2) \quad Q(u, u) := \int_N (|v|^{-2}|\partial_\tau u|^2 + \partial_i \bar{u} h^{ij} \partial_j u + m|u|^2) dVol_{\hat{g}},$$

which is closeable on  $C_c^\infty(N)$ , since  $K$  is symmetric and bounded from below on this domain. Denoting its closure again by  $Q$  and the domain of its closure by  $\text{Dom } Q$ ,  $K$  is the selfadjoint operator associated to  $Q$ , ie the Friedrichs extension of its restriction to  $C^\infty(\mathbb{S}_\beta) \otimes C_c^\infty(\Sigma \setminus \mathcal{B})$ . We know that  $u \in \text{Dom } K$ ,  $Ku = f$  iff

$$(4.3) \quad Q(u, w) = (f|w)_{L^2(N)}, \quad \forall w \in C^\infty(\mathbb{S}_\beta) \otimes C_c^\infty(\Sigma \setminus \mathcal{B}).$$

From (2.5) we know that  $K \geq m_0^2$  hence is boundedly invertible and we set

$$G := K^{-1}.$$

**4.2. Change of volume form.** Let us define the quadratic form  $\tilde{Q}$  by  $\text{Dom } \tilde{Q} = |v|^{-\frac{1}{2}} \text{Dom } Q$  and

$$\tilde{Q}(\tilde{u}, \tilde{u}) := Q(u, u), \quad u = |v|^{\frac{1}{2}}\tilde{u}.$$

Since  $\tilde{u} \mapsto |v|^{\frac{1}{2}}\tilde{u}$  is unitary from  $L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+ \setminus \mathcal{B})$  to  $L^2(N)$ , we obtain that  $\tilde{Q}$  is closed. The operator  $\tilde{K}$  on  $L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma \setminus \mathcal{B})$  associated to  $\tilde{Q}$  equals

$$\tilde{K} := |v|^{3/2}K|v|^{1/2} =: -\bar{\partial}_\tau^2 + \epsilon^2(y, \bar{\partial}_y),$$

where:

$$\epsilon^2(y, \bar{\partial}_y) = -|v|^{\frac{1}{2}}(y)|h|^{-\frac{1}{2}}(y)\bar{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\bar{\partial}_{y^j}|v|^{\frac{1}{2}}(y) + v^2(y)m(y),$$

is obtain from the quadratic form

$$\int_{\Sigma^+ \setminus \mathcal{B}} \left( \partial_i |v|^{\frac{1}{2}} \bar{u} |v| h^{ij} \partial_i |v|^{\frac{1}{2}} u + |v|^2 m |u|^2 \right) |h|^{\frac{1}{2}} dy.$$

The operator  $\tilde{G} := \tilde{K}^{-1}$  is also well defined as a (possibly unbounded) operator on  $L^2(S_\beta) \otimes L^2(\Sigma^+)$ , since  $0 \notin \sigma_{\text{pp}}(\tilde{K})$ . We have the identity:

$$(4.4) \quad G = |v|^{1/2} \tilde{G} |v|^{3/2}.$$

We now recall a well known expression for  $\tilde{G}$ . Let

$$F(\tau) = \frac{e^{-\tau\epsilon} + e^{(\tau-\beta)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})}, \quad \tau \in [0, \beta[,$$

extended to  $\tau \in \mathbb{R}$  by  $\beta$ -periodicity. In particular we have:

$$(4.5) \quad F(\tau) = \frac{e^{-|\tau|\epsilon} + e^{(|\tau|-\beta)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})}, \quad \tau \in [-\beta, \beta]$$

The following expression for  $\tilde{G} = \tilde{K}^{-1}$  is well-known (see eg [DG, Sect. 18.3.2]):

$$(4.6) \quad \tilde{G}\tilde{u}(\tau) = \int_{\mathbb{S}_\beta} F(\tau - \tau') \tilde{u}(\tau') d\tau', \quad \tilde{u} \in L^2(S_\beta) \otimes L^2(\Sigma \setminus \mathcal{B}).$$

## 5. CALDERÓN PROJECTORS

In this section we recall some standard facts on *Calderón projectors*. We refer the reader to [CP, Sects. 5.1- 5.3] for details.

**5.1. The Calderón projector.** Let  $(N, h)$  a complete Riemannian manifold and  $P = -\Delta_h + m$ , where  $m \in C^\infty(N)$  is a real potential with  $m(x) \geq m_0^2 > 0$ . As in Sect. 4 we construct  $P$  as a selfadjoint operator on  $L^2(N, dVol_h)$  using the quadratic form

$$(5.1) \quad Q(u, u) = \int_N \partial_i \bar{u} h^{ij} \partial_j u + m(x) |u|^2(x) dVol_h.$$

We obtain that  $0 \in \rho(P)$ , hence  $G := P^{-1}$  is a bounded operator on  $L^2(N, dVol_h)$ , defined by

$$(5.2) \quad Q(Gv, w) = (v|w)_{L^2(N)}, \quad \forall w \in C_c^\infty(N).$$

Let  $\Omega \subset N$  an open set such that  $\partial\Omega = S = \bigcup_1^n S_i$ , where  $S_i$  are the connected components of  $S$  and are assumed to be smooth hypersurfaces. We denote by  $C^\infty(\bar{\Omega})$  the space of restrictions to  $\Omega$  of functions in  $C^\infty(N)$ .

We associate to  $S_i$  the distribution density  $dS_i$  defined by:

$$\langle dS_i | u \rangle := \int_{S_i} u d\sigma_h^{(i)}, \quad u \in C_c^\infty(N),$$

where  $d\sigma_h^{(i)}$  is the induced Riemannian density on  $S_i$  and we set

$$dS = \sum_{i=1}^n dS_i.$$

We denote by  $\partial_\nu$  the unit exterior normal vector field to  $S$  and set

$$\langle \partial_\nu^* dS | u \rangle := \langle dS | \partial_\nu u \rangle, \quad u \in C_c^\infty(N).$$

For  $u \in C^\infty(\bar{\Omega})$  we set

$$\gamma u := \begin{pmatrix} u|_S \\ \partial_\nu u|_S \end{pmatrix} =: \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}.$$

For  $v \in C_c^\infty(S)$  we denote by  $\tilde{v} \in C_c^\infty(N)$  an extension of  $v$  to  $N$  such that  $\tilde{u}|_S = u, \partial_\nu \tilde{u}|_S = 0$ .

**Definition 5.1.** Let  $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C_c^\infty(S) \oplus C_c^\infty(S)$ . We set:

$$Df := \gamma \circ G(\tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h^{-1})\partial_\nu^*dS).$$

– The operator  $D : C_c^\infty(S) \oplus C_c^\infty(S) \rightarrow C^\infty(S) \oplus C^\infty(S)$  is continuous and is called the Calderón projector associated to  $(P, S)$ .

– The operator  $D$  is a  $2 \times 2$  matrix of pseudodifferential operators on  $S$ .

Note that  $dS$  and  $\partial_\nu^*dS$  are distributional densities, hence  $(dVol_h)^{-1}dS$  and  $(dVol_h)^{-1}\partial_\nu^*dS$  are distributions on  $N$ , supported on  $S$ .

Note also that the Calderón projector is obviously covariant under diffeomorphisms: if  $\chi : (N, h) \rightarrow (N', h')$  is an isometric diffeomorphism with  $S' = \chi(S)$ ,  $P = \chi^*P'$ , then

$$D = \psi^*D',$$

where  $\psi : S \rightarrow S'$  is the restriction of  $\chi$  to  $S$ .

5.1.1. *Expression in Gaussian normal coordinates.* Let  $U_i$  be a neighborhood of  $\{0\} \times S_i$  in  $\mathbb{R} \times S_i$  and  $V_i$  a neighborhood of  $S_i$  in  $N$  such that Gaussian normal coordinates to  $S_i$  induce a diffeomorphism:

$$\chi_i : U_i \ni x \mapsto (s, y) \in V_i$$

from  $U_i$  to  $V_i$ , and  $ds^2 + k_s(y)dy^2 = \chi_i^*h$  on  $U_i$ . Then for  $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$  we have

$$(5.3) \quad \begin{aligned} & \chi_i^* \left( \tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h)^{-1}\partial_\nu^*dS \right) \\ &= \delta_0(s) \otimes (f_1(y) - r_0(y)f_0(y)) - \delta_0'(s) \otimes f_0(y), \end{aligned}$$

where  $r_s(y) = |k_s|^{-\frac{1}{2}}(y)\partial_s|k_s|^{\frac{1}{2}}$ .

If  $\varphi \in C_c^\infty(\mathbb{R})$  with  $\varphi \geq 0, \int \varphi(s)ds = 1$ , setting  $\varphi_n(s) = n\varphi(ns)$ , we can compute  $Df$  for  $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$  as

$$(5.4) \quad Df = \lim_{n \rightarrow +\infty} \gamma \circ G(\varphi_n(s) \otimes (f_1(y) - r_0(y)f_0(y)) - \varphi_n'(s) \otimes f_1(y)),$$

where the limit takes place in  $C^\infty(S) \oplus C^\infty(S)$ .

Note that it is not obvious that  $Df \in C^\infty(S) \oplus C^\infty(S)$ . To prove it one can first replace  $G$  by a properly supported pseudodifferential parametrix  $P^{(-1)} \in \Psi_c^{-2}(N)$ . Using then Gaussian normal coordinates near a point  $x^0 \in S$ , one is reduced locally to  $N = \mathbb{R}^d, S = \{x_1 = 0\}$ . The details can be found for example in [CP, Sects. 5.1- 5.3].

Another useful identity is the following: for  $u \in C^\infty(\bar{\Omega})$  let  $Iu$  be the extension of  $u$  by 0 in  $N \setminus \bar{\Omega}$ . Then

$$(5.5) \quad PIu = \tilde{f}_1(dVol_h)^{-1}dS - \tilde{f}_0(dVol_h^{-1})\partial_\nu^*dS + IPu, \text{ for } f = \gamma u.$$

## 6. THE DOUBLE $\beta$ -KMS STATE

In this section we consider the *double  $\beta$ -KMS state*  $\omega_\beta$  in  $\mathcal{M}^+ \cup \mathcal{M}^-$ . It is obtained as the natural extension to  $\mathcal{M}^+ \cup \mathcal{M}^-$  of the state  $\omega_\beta^+$  in  $\mathcal{M}^+$ , which is a  $\beta$ -KMS state in  $\mathcal{M}^+$  with respect to the Killing flow. Its construction, for the more general stationary case is given in [S1, Thm. 3.5].

Since  $\Sigma \setminus \mathcal{B}$  is a Cauchy surface for  $\mathcal{M}^+ \cup \mathcal{M}^-$ , we associate to  $\omega_\beta$  its (complex) Cauchy surface covariances on  $\Sigma \setminus \mathcal{B}$   $\lambda^\pm$ , and (since  $\omega_\beta$  is a pure state), the pair of complementary projections  $c^\pm = \pm q^{-1} \circ \lambda^\pm$ , see Subsect. 3.1. We will study in details the projection  $c^+$ .

We identify  $C_c^\infty(\Sigma \setminus \mathcal{B})$  with  $C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2$  using the map

$$(6.1) \quad \begin{aligned} \hat{R} : C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2 &\rightarrow C_c^\infty(\Sigma^+) \oplus C_c^\infty(\Sigma^-) \\ g = g^{(0)} \oplus g^{(\beta/2)} &\mapsto f = g^{(0)} \oplus r^*g^{(\beta/2)}, \end{aligned}$$

where  $r : \Sigma \rightarrow \Sigma$  is the restriction to  $\Sigma$  of the wedge reflection  $R$ , see Subsect. 2.2.

We will show that

$$C := \hat{R}^{-1} \circ c^+ \circ \hat{R}$$

is exactly the Calderón projector for the Euclidean operator  $K_+$  acting on  $(N, \hat{g})$ , see Subsect. 4.1, and the open set

$$\Omega := \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

**6.1. The double  $\beta$ -KMS state.** We recall now the expression of  $\omega_\beta$  given by Sanders, see [S1, Sect. 3.3].

There are some differences in signs and factors of  $i$  with the expression given by Sanders in [S1, Sect. 3.3]. They come from two differences between our convention for quantized Klein-Gordon fields and the one of Sanders:

- our convention for Cauchy data of a solution of  $Pu = 0$  is given the map  $\rho u = \begin{pmatrix} u|_\Sigma \\ i^{-1}\partial_\nu u|_\Sigma \end{pmatrix} =: f$ , which is more natural for complex fields and leads to a more symmetric formulation of the Hadamard condition, while Sanders uses  $\rho u = \begin{pmatrix} u|_\Sigma \\ \partial_\nu u|_\Sigma \end{pmatrix} = g$ , so  $f = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} g$ .

- we use as complex symplectic form  $\bar{f} \cdot \sigma f' = (f_1|f'_0) - (f_0|f'_1)$ , while Sanders uses  $g \cdot \sigma g' = (g_0|g'_1) - (g_1|g'_0)$ . In terms of spacetime fields, we use  $i^{-1}E$ , Sanders uses  $iE$ .

Let us unitarily identify  $L^2(\Sigma, |h|^{1/2} dy)$  with  $L^2(\Sigma^+, |h|^{1/2} dy) \oplus L^2(\Sigma^-, |h|^{1/2} dy)$ , by

$$u \mapsto u_+ \oplus u_-, \quad u_\pm = u|_{\Sigma^\pm}.$$

Under this identification the action of the wedge reflection  $r^*u = u \circ r$  will be denoted by  $T$ , with:

$$(6.2) \quad T(u_+ \oplus u_-) := r^*u_- \oplus r^*u_+.$$

A direct comparison with the formulas in [S1, Sect. 3.3], using the identity (3.2) gives the following proposition.

**Proposition 6.1.** *The double  $\beta$ -KMS state on  $\mathcal{M}^+ \cup \mathcal{M}^-$  is given by the Cauchy surface covariance  $\lambda^+ = \begin{pmatrix} \lambda_{00}^+ & \lambda_{01}^+ \\ \lambda_{10}^+ & \lambda_{11}^+ \end{pmatrix}$  where:*

$$(6.3) \quad \begin{aligned} \lambda_{00}^+ &= \frac{1}{2}|v|^{1/2} \left( \epsilon^{-1} \coth(\frac{\beta}{2}\epsilon) + \epsilon^{-1} T \text{sh}^{-1}(\frac{\beta}{2}\epsilon) \right) |v|^{1/2}, \\ \lambda_{11}^+ &= \frac{1}{2}|v|^{-1/2} \left( \epsilon \coth(\frac{\beta}{2}\epsilon) - \epsilon T \text{sh}^{-1}(\frac{\beta}{2}\epsilon) \right) |v|^{-1/2}, \\ \lambda_{01}^+ &= \lambda_{10}^+ = \frac{1}{2} \mathbf{1}. \end{aligned}$$

As in Subsect. 3.1 we have  $\lambda^- = \lambda^+ - q$ , where the charge  $q = i\sigma$  is given by the matrix  $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . We introduce the operators  $c^\pm := \pm q^{-1} \lambda^\pm$  and obtain

$$(6.4) \quad c^+ = \begin{pmatrix} \frac{1}{2} & \lambda_{00}^+ \\ \lambda_{11}^+ & \frac{1}{2} \end{pmatrix}.$$

Note that if

$$b_0 = \epsilon^{-1} \coth(\frac{\beta}{2}\epsilon) + \epsilon^{-1} T \text{sh}^{-1}(\frac{\beta}{2}\epsilon), \quad b_1 = \epsilon \coth(\frac{\beta}{2}\epsilon) - \epsilon T \text{sh}^{-1}(\frac{\beta}{2}\epsilon),$$

then using that  $[T, \epsilon] = 0$  we obtain that

$$b_0 b_1 = b_1 b_0 = \coth\left(\frac{\beta}{2}\epsilon\right)^2 - \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon\right)^2 = 1,$$

from which it follows easily that  $c^\pm$  are (formally) projections. This is expected since the double  $\beta$ -KMS state  $\omega_\beta$  is a pure state in  $\mathcal{M}^+ \cup \mathcal{M}^-$ .

**6.2. Conjugation by  $\hat{R}$ .** The map  $\hat{R}$  defined in (6.1) allows to unitarily identify  $L^2(\Sigma^+) \otimes \mathbb{C}^2$  with  $L^2(\Sigma^+) \oplus L^2(\Sigma^-)$ . We have:

$$(6.5) \quad \hat{R}^{-1} \epsilon \hat{R} = \epsilon_+ \oplus \epsilon_+, \quad \hat{R}^{-1} T \hat{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denoting by  $c_{ij}^+$  for  $i, j \in \{0, 1\}$  the entries of the matrix  $c^+$  and setting

$$C_{ij} := \hat{R}^{-1} \circ c_{ij}^+ \circ \hat{R},$$

we obtain after an easy computation using (6.3), (6.1):

$$(6.6) \quad \begin{aligned} C_{00}g_0 &= \frac{1}{2}g_0^{(0)} \oplus \frac{1}{2}g_0^{(\beta/2)}, \\ C_{11}g_1 &= \frac{1}{2}g_1^{(0)} \oplus \frac{1}{2}g_1^{(\beta/2)}, \\ C_{01}g_1 &= \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(0)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(\beta/2)} \\ &\quad \oplus \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(\beta/2)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1} \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{\frac{1}{2}}g_1^{(0)}, \\ C_{10}g_0 &= \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(0)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(\beta/2)} \\ &\quad \oplus \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \coth\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(\beta/2)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+ \operatorname{sh}^{-1}\left(\frac{\beta}{2}\epsilon_+\right)|v|^{-\frac{1}{2}}g_0^{(0)}. \end{aligned}$$

In (6.6) the upper indices (0),  $(\beta/2)$  refer to the two connected components  $\{\tau = 0\}$  and  $\{\tau = \beta/2\}$  of  $\partial\Omega$ , while the lower indices 0, 1 refer to the two components of  $g$ .

**6.3. The Calderón projector.** We now compute the Calderón projector for  $K_+$ , associated to the Riemannian manifold  $(N, \hat{g})$ . We choose

$$\Omega = \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

We have  $S = \partial\Omega = S_0 \cup S_{\beta/2}$  and we write  $f \in C_c^\infty(S) \oplus C_c^\infty(S)$  as  $f = f^{(0)} \oplus f^{(\beta/2)}$  for  $f^{(i)} \in C_c^\infty(S_i) \oplus C_c^\infty(S_i)$ .

We denote by  $\gamma^{(i)}$ ,  $i = 0, \beta/2$  the trace operators on  $S_i$  defined by  $\gamma u = \gamma^{(0)}u \oplus \gamma^{(\beta/2)}u$  for  $u \in C^\infty(\bar{\Omega})$ . We have:

$$(6.7) \quad \begin{aligned} \gamma^{(0)}u &= \lim_{\tau \rightarrow 0^+} \begin{pmatrix} u(\tau, y) \\ -|v(y)|^{-1} \partial_\tau u(\tau, y) \end{pmatrix}, \\ \gamma^{(\beta/2)}u &= \lim_{\tau \rightarrow (\beta/2)^-} \begin{pmatrix} u(\tau, y) \\ |v(y)|^{-1} \partial_\tau u(\tau, y) \end{pmatrix}. \end{aligned}$$

We denote similarly by  $\partial_\nu^{(i)}$  the exterior normal derivatives on  $S_i$ .

We compute the Calderón projector  $D$  defined in Subsect. 5.1 using the coordinates  $(\tau, y)$ . Since  $dS_i = |h|^{\frac{1}{2}}(y)dy$  and  $dVol_{\hat{g}} = |v|^{\frac{1}{2}}(y)|h|^{\frac{1}{2}}(y)dy$ , we obtain:

$$(6.8) \quad Df = D^{(0)}f \oplus D^{(\beta/2)}f,$$

for

$$(6.9) \quad \begin{aligned} D^{(i)}f &= \gamma^{(i)} \circ G \circ |v|^{-1} \left( \partial_\nu^{(0)} \delta_0(\tau) \otimes f_0^{(0)}(y) + \delta_0(\tau) \otimes f_1^{(0)}(y) \right. \\ &\quad \left. + \partial_\nu^{(\beta/2)} \delta_{\beta/2}(\tau) \otimes f_0^{(\beta/2)}(y) + \delta_{\beta/2}(\tau) \otimes f_1^{(\beta/2)}(y) \right). \end{aligned}$$

Since  $G = |v|^{\frac{1}{2}} \tilde{G} |v|^{3/2}$  we have  $G \circ |v|^{-1} = |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}}$ . Denoting by  $D_{kl}^{(i)(j)}$  for  $i, j \in \{0, \beta/2\}$  and  $k, l \in \{0, 1\}$  the various entries of  $D$ , we obtain:

$$(6.10) \quad D_{kl}^{(i)(j)} v = \begin{cases} \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\partial_\nu^{(j)} \delta_j(\tau) \otimes v(y)), & l = 0, \\ \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\delta_j(\tau) \otimes v(y)), & l = 1. \end{cases}$$

We also set

$$\partial_\tau^{(i)} = \mp \partial_\tau, \text{ for } i = 0, \beta/2,$$

so that  $\partial_\nu^{(i)} = |v|^{-1}(y) \partial_\tau^{(i)}$ .

**Proposition 6.2.** *We have  $D = \hat{R}^{-1} \circ c^+ \circ \hat{R}$ .*

**Proof.** We recall that  $C_{ij}$  are the entries of  $\hat{R}^{-1} \circ c^+ \circ \hat{R}$ . We compute  $D_{kl}^{(i)(j)}$  using (6.10) and the explicit formulas (4.5), (4.6) for the kernel  $\tilde{G}(\tau, \tau')$  of  $\tilde{G}$ . *Computation of  $D_{00}$ :*

$$\begin{aligned} D_{00}^{(0)(0)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \partial_\tau \tilde{G}(0^+, 0^+) |v|^{-\frac{1}{2}} u = \frac{1}{2} u, \\ D_{00}^{(0)(\beta/2)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \partial_\tau \tilde{G}(0^+, \beta/2^-) |v|^{-\frac{1}{2}} u = 0, \\ D_{00}^{(\beta/2)(0)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \partial_\tau \tilde{G}(\beta/2^-, 0^+) u = 0, \\ D_{00}^{(\beta/2)(\beta/2)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \partial_\tau \tilde{G}(\beta/2^-, \beta/2^-) |v|^{-\frac{1}{2}} u = \frac{1}{2} u. \end{aligned}$$

Hence

$$D_{00} g_0 = C_{00} g_0.$$

*Computation of  $D_{11}$ :*

$$\begin{aligned} D_{11}^{(0)(0)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_0 \otimes u \\ &= -|v|^{-\frac{1}{2}} \partial_\tau \tilde{G}(0^+, 0^+) |v|^{\frac{1}{2}} u = \frac{1}{2} u, \\ D_{11}^{(0)(\beta/2)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_{\beta/2} \otimes u \\ &= -|v|^{-\frac{1}{2}} \partial_\tau \tilde{G}(0^+, \beta/2^-) |v|^{\frac{1}{2}} u = 0, \\ D_{11}^{(\beta/2)(0)} u &= \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_0 \otimes u \\ &= |v|^{-\frac{1}{2}} \partial_\tau \tilde{G}(\beta/2^-, 0^+) |v|^{\frac{1}{2}} u = 0, \\ D_{11}^{(\beta/2)(\beta/2)} u &= \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_{\beta/2} \otimes u \\ &= |v|^{-\frac{1}{2}} \partial_\tau \tilde{G}(\beta/2^-, \beta/2^-) |v|^{\frac{1}{2}} u = \frac{1}{2} u. \end{aligned}$$

Hence

$$D_{11} g_0 = C_{00} g_0.$$

Computation of  $D_{01}$ :

$$\begin{aligned}
D_{01}^{(0)(0)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0 \otimes u \\
&= |v|^{\frac{1}{2}}\tilde{G}(0^+, 0^+)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\
D_{01}^{(0)(\beta/2)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2} \otimes u \\
&= |v|^{\frac{1}{2}}\tilde{G}(0^+, \beta/2^-)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\
D_{01}^{(\beta/2)(0)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0 \otimes u \\
&= |v|^{\frac{1}{2}}\tilde{G}(\beta/2^-, 0^+)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u, \\
D_{01}^{(\beta/2)(\beta/2)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2} \otimes u \\
&= |v|^{\frac{1}{2}}\tilde{G}(\beta/2^-, \beta/2^-)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u.
\end{aligned}$$

Hence

$$D_{01}g_1 = C_{01}g_1.$$

Computation of  $D_{10}$ :

$$\begin{aligned}
D_{10}^{(0)(0)}u &= \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(0)}\delta_0 \otimes u \\
&= -|v|^{-\frac{1}{2}}\partial_\tau\partial_{\tau'}\tilde{G}(0^+, 0^+)|v|^{-\frac{1}{2}}u = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\
D_{10}^{(0)(\beta/2)}u &= \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(\beta/2)}\delta_{\beta/2}u \\
&= |v|^{-\frac{1}{2}}\partial_\tau\partial_{\tau'}\tilde{G}(0^+, \beta/2^-)u = -\frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\
D_{10}^{(\beta/2)(0)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\partial_\nu^{(0)}\delta_0 \otimes u \\
&= -|v|^{-\frac{1}{2}}\partial_\tau\partial_{\tau'}G(\beta/2^-, 0^+)|v|^{-\frac{1}{2}}u = -\frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\text{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u, \\
D_{10}^{(\beta/2)(\beta/2)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\partial_\nu^{(\beta/2)}\delta_{\beta/2} \otimes u \\
&= -|v|^{-\frac{1}{2}}\partial_\tau\partial_{\tau'}\tilde{G}(\beta/2, \beta/2)|v|^{-\frac{1}{2}}u = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u.
\end{aligned}$$

Hence

$$D_{10}g_0 = C_{10}g_0.$$

This completes the proof of the proposition.  $\square$

## 7. THE HARTLE-HAWKING-ISRAEL STATE AND ITS PROPERTIES

**7.1. The smooth extension of  $(N, \hat{g})$  and the Hawking temperature.** The existence of the Hartle-Hawking-Israel state and the definition of the Hawking temperature  $T_H = \kappa(2\pi)^{-1}$  (where  $\kappa$  is the surface gravity) rely on the existence of an extension  $(N_{\text{ext}}, \hat{g}_{\text{ext}})$  of  $(N, \hat{g})$  such that the two components  $S_0, S_{\beta/2} \sim \Sigma^+$  of  $\partial\Omega$  are smoothly glued together into  $\Sigma \subset N_{\text{ext}}$ .

The extended Riemannian metric  $\hat{g}_{\text{ext}}$  is smooth iff  $\beta = (2\pi)\kappa^{-1}$  (for other values of  $\beta$   $(N_{\text{ext}}, \hat{g}_{\text{ext}})$  has a conic singularity on  $\mathcal{B}$ ).

Let us embed  $\Sigma \setminus \mathcal{B}$  into  $N$  by:

$$\hat{r} : \begin{cases} x \mapsto (0, x) \text{ for } x \in \Sigma^+, \\ x \mapsto (\beta/2, r(x)) \text{ for } x \in \Sigma^-, \end{cases}$$

Note that for  $\hat{R}$  defined in (6.1) we have

$$(7.1) \quad \hat{R} = (\hat{r})^*.$$

We recall that the function  $m : \Sigma \rightarrow \mathbb{R}^+$  was introduced in Subsect. 2.4.



**Proposition 7.1.** [S1, Subsect. 2.2] *Assume that  $\beta = (2\pi)\kappa^{-1}$ . Then there exists a smooth complete Riemannian manifold  $(N_{\text{ext}}, \hat{g}_{\text{ext}})$  and*

- (1) *a smooth isometric embedding  $\psi : \Sigma \rightarrow N_{\text{ext}}$ ,*
  - (2) *a smooth isometric embedding  $\chi : (N, \hat{g}) \rightarrow (N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}}, \hat{g}_{\text{ext}})$  for  $\mathcal{B}_{\text{ext}} = \psi(\mathcal{B})$ ,*
  - (3) *a smooth function  $m_{\text{ext}} : N_{\text{ext}} \rightarrow \mathbb{R}$  with  $m_{\text{ext}} \geq m_0^2 > 0$*
- such that*

$$\psi|_{\Sigma \setminus \mathcal{B}} = \chi \circ \hat{r}, \quad \psi^* m_{\text{ext}} = m|_N .$$

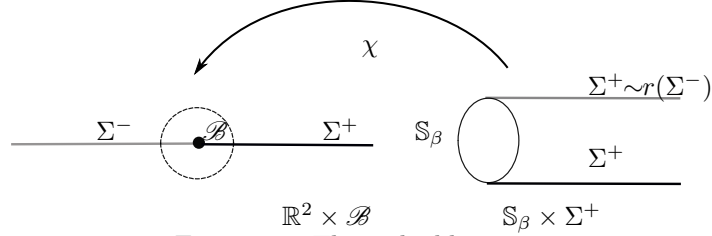


FIGURE 2. The embedding  $\chi$

This fundamental fact is well explained in [S1, Subsect. 2.2]. Let us briefly recall the construction of  $\chi$  following [S1]: one introduces Gaussian normal coordinates to  $\mathcal{B}$  in  $(\Sigma, h)$ , where  $h$  is the Riemannian metric induced by  $g$  on  $\Sigma$ . We choose the unit normal vector field to  $\mathcal{B}$  pointing towards  $\Sigma^+$ . Using these coordinates we can, since  $\mathcal{B}$  is compact, identify a small neighborhood  $U$  of  $\mathcal{B}$  in  $\Sigma$  with  $] - \delta, \delta[ \times \mathcal{B}$ . Denoting by  $\omega$  local coordinates on  $\mathcal{B}$  we have a map

$$\begin{aligned} \phi : ] - \delta, \delta[ \times \mathcal{B} \ni (s, \omega) &\mapsto y = \exp_{\omega}^h(s) \in U, \\ U^+ &= \phi(]0, \delta[ \times \mathcal{B}), \\ \phi^* h &= ds^2 + k_{\alpha\beta}(s, \omega) d\omega^\alpha d\omega^\beta, \end{aligned}$$

for  $U^+ = (\Sigma^+ \setminus \mathcal{B}) \cap U$ . In the local coordinates  $(\tau, s, \omega)$  on  $\mathcal{S}_\beta \times U^+$  the embedding  $\chi$  takes the form:

$$(7.2) \quad \chi : \begin{aligned} &\mathcal{S}_\beta \times ]0, \delta[ \times \mathcal{B} \rightarrow B_2(0, \delta) \times \mathcal{B} \\ &(\tau, s, \omega) \mapsto (s \cos(\beta(2\pi)^{-1}\tau), s \sin(\beta(2\pi)^{-1}\tau), \omega) =: (X, Y, \omega), \end{aligned}$$

where  $B_2(0, \delta) = \{(X, Y) \in \mathbb{R}^2 : 0 < X^2 + Y^2 < \delta^2\}$ . A straightforward computation performed in [S1, Subsect. 2.2] shows that  $\hat{g}$  admits a smooth extension  $\hat{g}_{\text{ext}}$  to  $N_{\text{ext}}$  iff  $\beta = (2\pi)\kappa^{-1}$ .

**7.2. The extension of  $\omega_\beta$  to  $M$ .** We recall from Subsect. 4.1 that  $K$  is defined from the closure  $\bar{Q}$  of the quadratic form  $Q$  on  $C_c^\infty(N)$ .

Similarly  $K_{\text{ext}} = -\Delta_{\hat{g}_{\text{ext}}} + m_{\text{ext}}$ , acting on  $N_{\text{ext}}$  is defined using the corresponding quadratic form  $Q_{\text{ext}}$ .

The following lemma is equivalent to [S1, Prop. 5.2], for completeness we give a short proof using quadratic form arguments (note that we assume the stronger condition that  $\inf m(x) > 0$ ).

**Lemma 7.2.** *Let  $U : C_c^\infty(N) \rightarrow C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$  defined by*

$$Uu = u \circ \chi^{-1}.$$

*Then  $U$  extends as a unitary operator  $U : L^2(N) \rightarrow L^2(N_{\text{ext}})$  with  $K_{\text{ext}} = UKU^*$ .*

**Proof.**  $U$  clearly extends as a unitary operator. To check the second statement it suffices, taking into account the way  $K$  and  $K_{\text{ext}}$  are defined, to prove that  $C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$  is a form core for  $Q_{\text{ext}}$ . The domain of  $\bar{Q}_{\text{ext}}$  is the Sobolev space  $H^1(N_{\text{ext}})$  associated to  $\hat{g}_{\text{ext}}$ , so we need to show that  $C_c^\infty(N_{\text{ext}} \setminus \mathcal{B}_{\text{ext}})$  is dense in  $H^1(N_{\text{ext}})$ . Using the coordinates  $(X, Y, \omega)$  near  $\mathcal{B}_{\text{ext}} \sim \{0\} \times \mathcal{B}$ , this follows from the fact that  $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$  is dense in  $H^1(\mathbb{R}^2)$ , see eg [A, Thm. 3.23].  $\square$

We recall that the projection  $c^\pm$  associated to the double  $\beta$ -KMS state  $\omega_\beta$  was defined in (6.4). Let us identify in the sequel  $\Sigma$  with  $\Sigma_{\text{ext}} = \psi(\Sigma) \subset N_{\text{ext}}$ .

**Theorem 7.3.** *Let  $D_{\text{ext}}$  the Calderón projector for  $(K_{\text{ext}}, \Sigma)$ . Then for  $f, g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$  we have:*

$$(g|c^+g)_{L^2(\Sigma)} = (g_{\text{ext}}|D_{\text{ext}}f_{\text{ext}})_{L^2(\Sigma)},$$

where  $f_{\text{ext}} = (\psi^*)^{-1}f$ ,  $g_{\text{ext}} = (\psi^*)^{-1}g$ .

**Proof.** This follows from Prop. 6.2, the fact that  $\hat{R}$  is implemented by the embedding  $\hat{r}$  of  $\Sigma \setminus \mathcal{B}$  into  $N$ , (see (7.1)) and Lemma 7.2.  $\square$

**7.3. Uniqueness of the extension.** We discuss now the uniqueness of extensions of  $\omega_\beta$  to  $M$ . Other types of uniqueness results were obtained before in [KW] and [K1].

**Proposition 7.4.** *There exists at most one quasi-free state  $\omega$  for the Klein-Gordon field on  $M$  such that:*

- (1) *the restriction of  $\omega$  to  $\mathcal{M}^+ \cup \mathcal{M}^-$  equals  $\omega_\beta$ ,*
- (2) *the spacetime covariances  $\Lambda^\pm$  of  $\omega$  map  $C_c^\infty(M)$  into  $C^\infty(M)$ .*

**Proof.** Let  $\omega$  a quasi-free state for the Klein-Gordon operator  $P$  in  $M$ , with spacetime covariances  $\Lambda^\pm$ . We assume that  $\Lambda^\pm : C_c^\infty(M) \rightarrow C^\infty(M)$ . Denoting by  $\Lambda^\pm(x, x')$  their Schwartz kernels, we have  $P(x, \partial_x)\Lambda^\pm(x, x') = P(x', \partial_{x'})\Lambda^\pm(x, x') = 0$ , which implies that

$$(7.3) \quad \text{WF}(\Lambda^\pm)' \subset \mathcal{N} \times \mathcal{N},$$

where  $\mathcal{N}$  is defined in (3.7). We claim that the entries  $c_{k,k'}^\pm$ ,  $k, k' = 0, 1$  of  $c^\pm$  defined in (3.11) map  $C_c^\infty(\Sigma)$  into  $C^\infty(\Sigma)$ . In fact by (7.3) we have  $\Lambda^\pm = \Lambda^\pm \circ A$  modulo smoothing, where  $A \in \Psi^0(M)$  is a pseudodifferential operator with  $\text{essupp}(A)$  included in an arbitrary small conical neighborhood of  $\mathcal{N}$ . For  $u \in C_c^\infty(\Sigma)$  we have, modulo factors of  $i$ :

$$c_{k,k'}^\pm u = \partial_t^k \Lambda^\pm \circ A(-\partial_t^{k'} \delta_0 \otimes u)|_{t=0},$$

see (3.11). Since  $\text{WF}((-\partial_t^{k'} \delta_0) \otimes u) \subset N^*\Sigma$ , where  $N^*\Sigma \subset T^*M$  is the conormal bundle to  $\Sigma$  and  $\Sigma$  is spacelike, we have  $N^*\Sigma \cap \mathcal{N} = \emptyset$ , hence  $A(-\partial_t^{k'} \delta_0 \otimes u) \in C^\infty(M)$ , which proves our claim.

Let now  $\omega_i$ ,  $i = 1, 2$  be two quasi-free states as in the proposition. Since  $(u|(\Lambda_1^+ - \Lambda_2^+)v)_{L^2(M)} = 0$  for  $u, v \in C_c^\infty(\mathcal{M}^+ \cup \mathcal{M}^-)$  we obtain that  $(f|(\lambda_1^+ - \lambda_2^+)g)_{L^2(\Sigma) \otimes \mathbb{C}^2} = 0$  for  $f, g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$  hence  $\text{supp}(\lambda_1^+ - \lambda_2^+)g \subset \mathcal{B}$  for  $g \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$ . Since we have seen that  $\lambda_i^+ : C_c^\infty(\Sigma) \otimes \mathbb{C}^2 \rightarrow C^\infty(\Sigma) \otimes \mathbb{C}^2$  this implies that  $(\lambda_1^+ - \lambda_2^+)f = 0$  for  $f \in C_c^\infty(\Sigma \setminus \mathcal{B}) \otimes \mathbb{C}^2$ . Since  $\lambda_i^+$  are selfadjoint for  $L^2(\Sigma, dV ol_h) \otimes \mathbb{C}^2$  this implies that  $\text{supp}(\lambda_1^+ - \lambda_2^+)f \subset \mathcal{B}$  for  $f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2$ , hence  $(\lambda_1^+ - \lambda_2^+)f = 0$  using again that  $\lambda_i^+ : C_c^\infty(\Sigma) \otimes \mathbb{C}^2 \rightarrow C^\infty(\Sigma) \otimes \mathbb{C}^2$ .  $\square$

#### 7.4. The Hartle-Hawking-Israel state.

**Theorem 7.5** ([S1]). *Let us set*

$$\lambda_{\text{HHI}}^+ := q \circ D_{\text{ext}}, \quad \lambda_{\text{HHI}}^- := \lambda_{\text{HHI}}^+ - q,$$

where  $D_{\text{ext}}$  is the Calderón projector for  $(K_{\text{ext}}, \Sigma)$  and the charge quadratic form  $q$  is defined in (3.8). Then:

- (1)  $\lambda_{\text{HHI}}^\pm$  are the Cauchy surface covariances for the Cauchy surface  $\Sigma$  of a quasi-free state  $\omega_{\text{HHI}}$  for the free Klein-Gordon field on  $M$ .
- (2) the Hartle-Hawking-Israel state  $\omega_{\text{HHI}}$  is a pure Hadamard state and is the unique extension to  $M$  of the double  $\beta$ -KMS state  $\omega_\beta$  with the property that its spacetime covariances  $\Lambda_{\text{HHI}}^\pm$  map continuously  $C_c^\infty(M)$  into  $C^\infty(M)$ .

**Proof.** Let us first prove (1). By (3.3) it suffices to check the positivity of  $\lambda_{\text{HHI}}^+$ . This was shown in [S1, Thm. 5.3] using reflection positivity. For the reader's convenience, let us briefly repeat the argument:

for  $u \in L^2(N)$  we set  $Ru(\tau, y) = u(-\tau, y)$ , for  $\tau \in [-\beta/2, \beta/2] \sim \mathbb{S}_\beta$ . The operator  $G = \kappa^{-1}$  is reflection positive, i.e.

$$(7.4) \quad (Ru|Gu)_{L^2(N)} \geq 0, \quad \forall u \in L^2(N), \quad \text{supp} u \subset [0, \beta/2] \times \Sigma^+.$$

In fact setting  $\tilde{u} = |v|^{3/2}u$ , (7.4) is equivalent to

$$(7.5) \quad (R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)} \geq 0, \quad \forall \tilde{u} \in L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+), \quad \text{supp} \tilde{u} \subset [0, \beta/2] \times \Sigma^+.$$

Using (4.5) we obtain

$$\begin{aligned} & (R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)} \\ &= (u_0 | \frac{1}{2\epsilon(1-e^{-\beta\epsilon})} u_0)_{L^2(\Sigma^+)} + (u_\beta | \frac{1}{2\epsilon(1-e^{-\beta\epsilon})} u_\beta)_{L^2(\Sigma^+)}, \end{aligned}$$

for

$$u_0 = \int_{\mathbb{S}_\beta} e^{-\tau\epsilon} \tilde{u}(\tau) d\tau, \quad u_\beta = \int_{\mathbb{S}_\beta} e^{(\tau-\beta/2)\epsilon} \tilde{u}(\tau) d\tau$$

where  $\tilde{u}$  is identified with the map  $\mathbb{S}_\beta \ni \tau \mapsto \tilde{u}(\tau) \in L^2(\mathbb{S}_\beta; L^2(\Sigma^+))$ . This proves (7.4).

By Lemma 7.2 and using that  $G_{\text{ext}} = K_{\text{ext}}^{-1}$  is bounded on  $L^2(N_{\text{ext}})$ , we deduce from (7.4) that  $G_{\text{ext}}$  is also reflection positive, ie

$$(7.6) \quad (R_{\text{ext}}u | G_{\text{ext}}u_{\text{ext}})_{L^2(N_{\text{ext}})} \geq 0, \quad u \in L^2(N_{\text{ext}}), \quad \text{supp} u \subset N_{\text{ext}}^+ = \chi([0, \beta/2] \times \Sigma^+),$$

for  $R_{\text{ext}} = URU^*$ . By the remark before [S1, Thm. 5.3], if  $(s, y)$  are Gaussian normal coordinates to  $\Sigma$  in  $N_{\text{ext}}$  we have  $R_{\text{ext}}u(s, y) = u(-s, y)$ , ie  $R_{\text{ext}}$  is given by the reflection in Gaussian normal coordinates. This map is a isometry of  $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ , which implies that if  $\hat{g}_{\text{ext}} = ds^2 + h_{\text{ext}}(s, y)dy^2$  near  $\Sigma$ , we have  $h_{\text{ext}}(s, y) = h_{\text{ext}}(-s, y)$  hence if  $r_s(y) = |h_{\text{ext}}(s, y)|^{-\frac{1}{2}} \partial_s |h_{\text{ext}}(s, y)|^{\frac{1}{2}}$  we have  $r_0(y) \equiv 0$ .

If  $f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2$  it follows from (5.3) that  $D_{\text{ext}}f = \gamma G_{\text{ext}}Tf$  for

$$Tf = \delta_0(s) \otimes f_1 - \delta'_0(s) \otimes f_0.$$

We have  $R_{\text{ext}}Tf = \delta_0(s) \otimes f_1 + \delta'_0(s) \otimes f_0$ . Applying the reflection positivity (7.6) to  $u = Tf$  we obtain that:

$$(R_{\text{ext}}Tf | G_{\text{ext}}Tf)_{L^2(N_{\text{ext}})} = (f | q D_{\text{ext}}f)_{L^2(\Sigma)} \geq 0,$$

which proves the positivity of  $\lambda_{\text{HHI}}^+$ . To make the argument rigorous is suffices to approximate  $\delta_0$  by a sequence  $\varphi_n$  as in (5.4). This completes the proof of (1).

Let us now prove (2). The fact that  $\omega_{\text{HHI}}$  is the unique extension of  $\omega_\beta$  to  $M$  with the stated properties has been proved in Prop. 7.4. It remains to prove that  $\omega_{\text{HHI}}$  is a pure Hadamard state in  $M$ .

The fact that  $\omega_{\text{HHI}}$  is pure follows from the fact that  $D_{\text{ext}}$  is a projection. To prove the Hadamard property let us fix a reference Hadamard state  $\omega_{\text{ref}}$  for the Klein-Gordon field in  $M$ . By Thm. 3.2 its Cauchy surface covariances on  $\Sigma$   $\lambda_{\text{ref}}^{\pm}$  are matrices of pseudodifferential operators on  $\Sigma$ . The same is true of  $c_{\text{ref}}^{\pm} = \pm q^{-1} \circ \lambda_{\text{ref}}^{\pm}$  and of  $c_{\text{HHI}}^{\pm}$ , since Calderón projectors are given by matrices of pseudodifferential operators on  $\Sigma$ .

Moreover we know that the restriction of  $\omega_{\text{HHI}}$  to  $\mathcal{M}^+ \cup \mathcal{M}^-$  is a Hadamard state. The same is obviously true of the restriction of  $\omega_{\text{ref}}$  to  $\mathcal{M}^+ \cup \mathcal{M}^-$ . Going to Cauchy surface covariances, this implies that if  $\chi \in C_c^{\infty}(\Sigma^{\pm} \setminus \mathcal{B})$  then

$$\chi \circ (c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}) \circ \chi \text{ is a smoothing operator on } \Sigma.$$

We claim that this implies that  $c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}$  is smoothing, which will imply that  $\omega_{\text{HHI}}$  is a Hadamard state.

If fact let  $a$  be one of the entries of  $c_{\text{HHI}}^{\pm} - c_{\text{ref}}^{\pm}$ , which is a scalar pseudodifferential operator belonging to  $\Psi^m(\Sigma)$  for some  $m \in \mathbb{R}$ . We know that  $\chi \circ a \circ \chi$  is smoothing for any  $\chi \in C_c^{\infty}(\Sigma \setminus \mathcal{B})$ . Then its principal symbol  $\sigma_{\text{pr}}(a)$  vanishes on  $T^*(\Sigma \setminus \mathcal{B})$  hence on  $T^*\Sigma$  by continuity, so  $a \in \Psi^{m-1}(\Sigma)$ . Iterating this argument we obtain that  $a$  is smoothing, which proves our claim and completes the proof of the theorem.  $\square$

**Acknowledgments.** It is a pleasure to thank Michał Wrochna, Ko Sanders and Francis Nier for useful discussions.

## REFERENCES

- [A] Adams, R.: *Sobolev Spaces* Academic Press, (1975).
- [CP] Chazarain, J., Piriou, A.: *Introduction to the Theory of Linear Partial Differential Operators*, North-Holland, (1982).
- [DG] Dereziński, J., Gérard, C.: *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs in Mathematical Physics, Cambridge University Press, 2013.
- [HH] Hartle, J. B. , Hawking, S. W.: Path-integral derivation of black-hole radiance, Phys. Rev. D **13** (1976), 2188–2203.
- [I] Israel, W.: Thermo-field dynamics of black holes, Phys. Lett. **57A** (1976), 107–110.
- [K1] Kay, B. S.: Sufficient conditions for quasifree states and an improved uniqueness theorem for quantum fields on spacetimes with horizons, J. Math. Phys. **34** (1993), 4519–4539.
- [K2] Kay, B. S.: A uniqueness result for quasifree KMS states, Helv. Phys. Acta **58** (1985), 1017–1029.
- [K3] Kay, B. S.: Purification of KMS states, Helv. Phys. Acta **58** (1985), 1030–1040.
- [KW] Kay, B. S., Wald, R. M.: Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon, Phys. Rep. **207** (1991), 49–136.
- [GOW] C. Gérard, O. Oulhazi, M. Wrochna: Hadamard states for the Klein-Gordon equation on Lorentzian manifolds of bounded geometry, preprint [arXiv:1602.00930](https://arxiv.org/abs/1602.00930) (2016).
- [GW1] C. Gérard, M. Wrochna: Construction of Hadamard states by pseudo-differential calculus, Comm. Math. Phys. **325**, (2014), 713–755.
- [GW2] Gérard, C., Wrochna, M.: Hadamard states for the linearized Yang-Mills equation on curved spacetime, Comm. Math. Phys. **337** (2015), 253–320.
- [GW3] Gérard, C., Wrochna, M.: Hadamard property of the *in* and *out* states for Klein-Gordon fields on asymptotically static spacetimes, preprint [hal-01337432](https://arxiv.org/abs/1601.01337) (2016).
- [R] Radzikowski, M.: Micro-local approach to the Hadamard condition in quantum field theory on curved spacetime, Comm. Math. Phys. **179** (1996), 529–553.
- [S1] Sanders, K.: On the construction of Hartle-Hawking-Israel states across a static bifurcate Killing horizon, Letters in Math. Phys. **105** (2015), 575–640.
- [S2] Sanders, K.: Thermal equilibrium states of a linear scalar quantum field in stationary spacetimes, **28** (2013) 1330010.
- [Sh] Shubin, M.: *Pseudodifferential Operators and Spectral Theory*, Springer 2001.

UNIVERSITÉ PARIS-SUD XI, DÉPARTEMENT DE MATHÉMATIQUES, 91405 ORSAY CEDEX,  
FRANCE

*E-mail address:* `christian.gerard@math.u-psud.fr`