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# A Generalized Direction in Interior Point Method for Monotone Linear Complementarity Problems

Mounir Haddou · Tangi Migot · Jérémy Omer

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**Abstract** In this paper, we present a new interior point method with full Newton step for monotone linear complementarity problems. The specificity of our method is to compute the Newton step using a modified system similar to that introduced by Darvay in [10]. We prove that this new method possesses the best known upper bound complexity for these methods. Moreover, we extend results known in the literature since we consider a general family of smooth concave functions in the Newton system instead of the square root.

**Keywords** concave functions · interior-point methods · linear programming · linear complementarity problems · polynomial time complexity

**Mathematics Subject Classification (2000)** 49M05 · 90C33 · 90C51

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## 1 Introduction

In this paper we focus on the simplest interior point methods (IPMs) : the full Newton step interior point method [19], this method is one among the IPMs with the best worst-case complexity. Therefore, naturally any new try in the IPM framework must be validated on this method. In particular, we

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focus on this method applied to the monotone complementarity problem [15]. This problem is a generalization of the linear optimization problem and has been very popular in the literature due to its numerous applications [12]. In the literature of IPM, this problem has also been an active subject [15].

Darvay [10,11], introduces a modification in the interior point method for finding search directions for linear optimization problems, based on an algebraic transformation of the central path. In particular, he applied the square root function to both sides of the centering equation, and he used Newton's method to obtain the new direction. He proved that the corresponding full Newton step algorithm has  $O(\sqrt{n}L)$  iteration complexity. This new direction using the square root has become an active subject in the past few years [1–9,20–23]. Several authors generalized this approach to a wide class of optimization problems, for example for linear complementarity problems [1,3–6,8,23], convex quadratic programming [2], second-order cone optimization [21], semidefinite optimization [20] and symmetric cone optimization [7–9,22].

Inspired by Darvay's new approach we introduce here a new class of IPMs by considering a large family of smooth concave functions instead of the square root. This new class of methods generalizes the classical path-following IPMs, since we obtain them as a special case. The technique presented here does not include Darvay's algorithm, but we can consider a smoother version that belongs to our family of methods. Our main contribution is that we prove that the algorithm with the new directions converges to a solution with the best known complexity for this family of methods.

In Section 2, we introduce the problem and our new directions. In Section 3, we show the polynomial complexity of our new class of methods.

## Notations

Through this paper we will use the following notations:  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x \geq 0\}$ ,  $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n \mid x > 0\}$  and  $\mathbf{e}$  denotes the vector with all entries equal to one and whose dimension can be deduced from the context. Given two vectors  $z, s \in \mathbb{R}^n$ , we denote by  $z^T s$  the usual scalar product and by  $zs$  the Hadamard product of two vectors, that is  $zs = (z_i s_i)_{1 \leq i \leq n}$ . Moreover, we extend this component-wise operation to the division of two vectors and to the square root, that is  $\forall z \in \mathbb{R}^n, s \in (\mathbb{R} \setminus \{0\})^n$ ,  $z/s = (z_i/s_i)_{1 \leq i \leq n}$  and  $\forall z \in \mathbb{R}_+^n$ ,  $\sqrt{z} = (\sqrt{z_i})_{1 \leq i \leq n}$ .

## 2 Preliminaries and Problem Settings

An LCP consists in finding  $z, s \in \mathbb{R}^n$  such that for a square matrix  $M \in \mathbb{R}^{n \times n}$  and a vector  $q \in \mathbb{R}^n$

$$s = Mz + q, \quad z, s \geq 0, \quad zs = 0. \quad (\text{LCP})$$

A couple  $(z, s)$  such that  $s = Mz + q$  is said to be feasible for the LCP if we have  $z, s \geq 0$  and strictly feasible if  $z, s > 0$ . From now on, we use

standard notation  $\mathcal{F}_+ := \{(z, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n \mid s = Mz + q\}$  for the set of strictly feasible points of (LCP). In this paper we consider a monotone linear complementarity problem, i.e. an LCP where  $M$  is positive semi-definite. In this case the set of solutions of (LCP) is a convex set.

The main strategy of IPMs is to follow the central path  $(z(\mu), s(\mu))$  for  $\mu \in \mathbb{R}_{++}^n$ , defined by

$$s = Mz + q, \quad z, s \geq 0, \quad zs = \mu. \quad (\text{LCP}_\mu)$$

The couples  $(z(\mu), s(\mu))$  are also called  $\mu$ -centers and define an homotopic path. The limit when  $\mu \rightarrow 0$  satisfies the complementarity condition, and hence yields optimal solutions whenever the limit exists.

IPMs follow the central path approximately by solving an approached version of the non-linear system in  $(\text{LCP}_\mu)$  for several values of  $\mu$ . The main tool to solve such a system is the Newton method. A Newton step  $(\Delta z, \Delta s)$  is given as the solution of the following linear system

$$\begin{cases} M\Delta z = \Delta s \\ z\Delta s + s\Delta z = \mu - zs \end{cases} \quad (1)$$

There exists a wide variety of different IPMs that are based on this principle. In this paper we focus on the simplest IPM (see Algorithm 1): the full Newton step interior point method (FN-IPM).

**Input:** an accuracy parameter  $\epsilon > 0$  ;  
 a sequence of update parameters  $\{\theta^k\}$ ,  $0 < \theta^k < 1 \ \forall k \in \mathbb{N}$  ;  
 initial values  $(z^0, s^0) \in \mathcal{F}_+$ ,  $\mu^0 = z^0 s^0$  ;

```

1  $z := z^0, s := s^0, \mu := \mu^0, k := 0$  ;
2 while  $z^T s \geq n\epsilon$  do
3    $\mu := (1 - \theta^k)\mu$ ;
4   solve system (1) to find  $(\Delta z, \Delta s)$  ;
5    $(z, s) := (z, s) + (\Delta z, \Delta s)$ ;
6    $k := k + 1$ ;

```

**Algorithm 1:** Full Newton step IPM (FN-IPM)

In [11], Darvay introduces a modification in  $(\text{LCP}_\mu)$  by considering

$$s = Mz + q, \quad z, s \geq 0, \quad \varphi(zs) = \mu, \quad (2)$$

where  $\varphi : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  is assumed to be a smooth function such that  $\varphi(0) = 0$  and  $\varphi$  is defined by a component-wise extension of a real-valued function  $\varphi$  to  $\mathbb{R}^n$ , i.e. for  $t \in \mathbb{R}^n$   $\varphi(t) = (\varphi(t_i))_{1 \leq i \leq n}$ .

Darvay's method modifies the Newton steps. More precisely, the Newton step  $(\Delta z, \Delta s)$  is given by the linear system

$$\begin{cases} M\Delta z = \Delta s \\ \varphi'(zs)(z\Delta s + s\Delta z) = \mu - \varphi(zs) \end{cases} \quad (3)$$

In this paper we consider functions  $\varphi \in C^3(\mathbb{R}_+)$  such that  $\varphi(0) = 0$ ,  $\varphi$  is increasing and concave, and  $\varphi'''(t) \geq 0 \forall t \in \mathbb{R}_+$ . This class of functions allows for a generalization of the classical IPMs, since we obtain the classical central path system (LCP $_{\mu}$ ) for  $\varphi(t) = t$ . The square root function does not belong to this family since it is not differentiable at 0, but we can build a smooth version with  $\varphi_{\epsilon>0} : t \mapsto \sqrt{t+\epsilon} - \sqrt{\epsilon}$ . We modify Algorithm 1 to solve (3) instead of (1) at step 4, and call the resulting algorithm  $\varphi$ -FN-IPM. The main result of this article is that  $\varphi$ -FN-IPM, converges to an  $\epsilon$ -solution in at most  $\mathcal{O}(\sqrt{n} \log(\frac{n}{\epsilon}))$  iterations. This upper bound is the best one known for the FN-IPM.

### 3 Polynomial Complexity

In this section, we consider the worst-case complexity of the  $\varphi$ -FN-IPM described in Algorithm 2 with  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  verifying

- (i)  $\varphi(0) = 0$ ;
- (ii)  $\varphi \in C^3([0, +\infty))$ ;
- (iii)  $\varphi'(t) > 0, \forall t \geq 0$ , i.e.  $\varphi$  is increasing;
- (iv)  $\varphi''(t) \leq 0, \forall t \geq 0$ , i.e.  $\varphi$  is concave;
- (v)  $\varphi'''(t) \geq 0, \forall t \geq 0$ .

These functions are invertible and can be extended in a smooth way for negative  $t$  by considering  $\varphi(t) = t\varphi'(0) + \frac{t^2}{2}\varphi''(0) + \frac{t^3}{6}\varphi'''(0)$ . Function  $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^n$  is then defined component-wisely:  $\forall \mathbf{t} \in \mathbb{R}^n, \varphi(\mathbf{t}) = (\varphi(t_i))_{1 \leq i \leq n}$ .

**Input:** an accuracy parameter  $\epsilon > 0$  ;  
 a sequence of update parameters  $\{\theta^k\}, 0 < \theta^k < 1 \forall k \in \mathbb{N}$  ;  
 initial values  $(z^0, s^0) \in \mathcal{F}_+, \mu^0 = z^0 s^0$ ;

```

1  $z := z^0, s := s^0, \mu := \mu^0, k := 0$  ;
2 while  $\sum_{i=1}^n \varphi(z_i s_i) \geq n\epsilon$  do
3    $\mu := (1 - \theta^k)\mu$ ;
4   solve system (3) to find  $(\Delta z, \Delta s)$  ;
5    $(z, s) := (z, s) + (\Delta z, \Delta s)$ ;
6    $k := k + 1$  ;
```

**Algorithm 2:**  $\varphi$ -Full Newton step IPM ( $\varphi$ -FN-IPM)

One important characteristic of  $\varphi$  is the existence of a constant  $T$  defined by

$$-\varphi''(0) = T(\varphi'(0))^2. \quad (4)$$

Noticing the conditions (iii) and (iv), we have  $T \geq 0$  and  $T = 0$  for  $\varphi(t) = t$ . It should be noted that arbitrary values of  $T > 0$  can be achieved by scaling  $\varphi$ .

Note that  $t \mapsto \sqrt{t}$ , in the same way as any function  $t \mapsto t^q, 0 < q < 1$ , does not satisfy these hypotheses since it is not differentiable in 0. However we can consider a smooth version for  $\epsilon > 0$  with  $t \mapsto (t + \epsilon)^q - (\epsilon)^q$ . As said in the

introduction the classical method is given by  $\varphi(t) = t$ . Other examples are  $\varphi : t \mapsto \log(1+t)$  and functions constructed as in [13, 16]: for instance  $\varphi : t \mapsto \frac{t}{t+1}$  and  $\varphi : t \mapsto 1 - \exp(-t)$ . It is interesting to note two latter functions  $\varphi$  are bounded. Moreover, notice that if a function  $\varphi$  satisfies all these hypotheses, then  $t \mapsto \alpha\varphi(Ct)$  with  $\alpha, C \in \mathbb{R}_{++}$  also satisfies these hypotheses.

The main result of this section (Theorem 3.5) states the polynomial worst-case complexity of the  $\varphi$ -FN-IPM described in Algorithm 2. In order to achieve this result we define a measure of the proximity to the central path in Section 3.1. Then, in Section 3.2, Theorem 3.1 estimates the error made at each Newton step. Sections 3.3 and 3.4 present conditions to ensure the correct behaviour of the algorithm: strict feasibility of the iterates (Theorem 3.2) and quadratic convergence of the Newton process (Theorem 3.3). Section 3.5 provides the sequence of update parameters (Theorem 3.4).

### 3.1 Proximity Measure

At each iteration, after updating parameter  $\mu$ , we compute the Newton direction  $(\Delta z, \Delta s)$  as a solution of system (3). Then we update the iterates with

$$z^+ = z + \alpha\Delta z \text{ and } s^+ = s + \alpha\Delta s. \quad (5)$$

Note that here we consider a damping factor  $\alpha \in [0, 1]$  to be more general. In this case we denote by  $(\alpha\Delta z, \alpha\Delta s)$  the Newton step with length  $\alpha$  and call it the  $\alpha$ -Newton step. Then the full Newton step is given for  $\alpha = 1$ .

In order to measure the distance to the target on the central path we consider a proximity measure  $\delta_\varphi(z, s, \mu)$  defined by

$$\delta_\varphi(z, s, \mu) := \frac{1}{2} \left\| \frac{\varphi'(0)}{\varphi'(zs)} \left( (v_\varphi(z, s, \mu)^{-1} - v_\varphi(z, s, \mu)) \right) \right\|_2,$$

with

$$v_\varphi(z, s, \mu) := \sqrt{\frac{\varphi(zs)}{\mu}} \text{ and } v_\varphi(z, s, \mu)^{-1} := \sqrt{\frac{\mu}{\varphi(zs)}}.$$

We may omit the arguments of  $v_\varphi(z, s, \mu)$  and  $\delta_\varphi(z, s, \mu)$ , when it is clear from the context. Notice that this proximity measure is a generalization of the one presented in [19], where the authors consider

$$\delta_{t \mapsto t} := \frac{1}{2} \left\| \sqrt{\frac{\mu}{zs}} - \sqrt{\frac{zs}{\mu}} \right\|_2.$$

Both proximity measures are equal for  $\varphi(t) = t$ . Moreover for any function  $\varphi$ , the two proximity measures are asymptotically similar (for  $zs \downarrow 0$ ).

The following two lemmas link the iterates and the proximity measure.

**Lemma 3.1** *If  $(z, s) \in \mathcal{F}_+$  and  $\delta_\varphi \leq 1$ , then  $\varphi(zs) \leq 6\mu$ .*

*Proof* Assume by contradiction that there exists  $i \in \{1, \dots, n\}$  such that  $\varphi(z_i s_i) > 6\mu_i$ . Since  $\varphi'$  is decreasing and  $zs > 0$ :

$$\begin{aligned} 2\delta_\varphi &\geq \|v_\varphi^{-1} - v_\varphi\|_2 = \left\| \frac{\sqrt{\mu}}{\sqrt{\varphi(zs)}} - \frac{\sqrt{\varphi(zs)}}{\sqrt{\mu}} \right\|_2 \geq \left| \frac{\sqrt{\mu_i}}{\sqrt{\varphi(z_i s_i)}} - \frac{\sqrt{\varphi(z_i s_i)}}{\sqrt{\mu_i}} \right| \\ &= \frac{\sqrt{\varphi(z_i s_i)}}{\sqrt{\mu_i}} - \frac{\sqrt{\mu_i}}{\sqrt{\varphi(z_i s_i)}} \\ &> \sqrt{6} - \frac{1}{\sqrt{6}} \approx 2.04, \end{aligned}$$

where the penultimate step comes from the increasing property of the function  $x \mapsto x - 1/x$  on  $\mathbb{R}_{++}$ . This is in contradiction with  $\delta_\varphi \leq 1$ .  $\square$

The following lemma gives bounds on  $\delta_\varphi$  that depend on some constant  $\Gamma(\|\mu\|_\infty)$  defined as

$$\Gamma(\|\mu\|_\infty) := \left( 1 - \frac{\varphi''(0)\varphi^{-1}(6\|\mu\|_\infty)}{\varphi'(\varphi^{-1}(6\|\mu\|_\infty))} \right).$$

**Lemma 3.2** *Let  $\delta = \frac{1}{2} \|v_\varphi^{-1} - v_\varphi\|_2$ , then*

$$\delta \leq \delta_\varphi \leq \left( 1 - \frac{\varphi''(0)\|zs\|_\infty}{\varphi'(\|zs\|_\infty)} \right) \delta.$$

*Furthermore in a close neighbourhood of the central path, i.e.  $\delta_\varphi \leq 1$ , we have*

$$\delta \leq \delta_\varphi \leq \Gamma(\|\mu\|_\infty) \delta. \quad (6)$$

*Proof* By convexity of function  $\varphi'$  for all  $i \in \{1, \dots, n\}$

$$\varphi'(z_i s_i) \geq \varphi'(0) + \varphi''(0) z_i s_i.$$

Then, for all  $i \in \{1, \dots, n\}$

$$1 \leq \frac{\varphi'(0)}{\varphi'(z_i s_i)} \leq 1 - \frac{\varphi''(0) z_i s_i}{\varphi'(z_i s_i)} \leq \max_i \left( 1 - \frac{\varphi''(0) z_i s_i}{\varphi'(z_i s_i)} \right).$$

Hence, by definition of  $\delta$  and  $\delta_\varphi$

$$\delta \leq \delta_\varphi \leq \left( 1 - \frac{\varphi''(0)\|zs\|_\infty}{\varphi'(\|zs\|_\infty)} \right) \delta.$$

The sharpest result when  $\delta_\varphi \leq 1$  is deduced from Lemma 3.1.  $\square$

In the previous lemma, equation (6), we define  $\Gamma$  as a function of  $\|\mu\|_\infty$  which depends on the choice of  $\varphi$ . For  $\varphi(t) = t$ , we get  $\Gamma(\|\mu\|_\infty) = 1$  for all  $\mu$ . Moreover, for any function  $\varphi$ ,  $\Gamma$  is increasing with respect to  $\|\mu\|_\infty$ , and converges to 1 as  $\|\mu\|_\infty \downarrow 0$ . Moreover, in the course of the proof we showed that if  $\delta_\varphi \leq 1$

$$\frac{\varphi'(0)}{\varphi'(z_i s_i)} \leq \Gamma(\|\mu\|_\infty), \quad \forall i \in \{1, \dots, n\}. \quad (7)$$

This result will be useful in a future proof.

### 3.2 Error Bound of the Newton Step

We use the first order Taylor-Lagrange formula applied to  $\varphi$  in  $zs$ . There exists  $\xi \in [z^+s^+, zs]$  (or  $\xi \in [zs, z^+s^+]$  if  $zs < z^+s^+$ ) such that

$$\begin{aligned} \varphi(z^+s^+) &= \varphi(zs) + \alpha\varphi'(zs)(z\Delta s + s\Delta z) \\ &\quad + \alpha^2\Delta z\Delta s\varphi'(zs) + \frac{\varphi''(\xi)}{2}(z^+s^+ - zs)^2, \end{aligned} \quad (8)$$

with  $(\Delta z, \Delta s)$  solution of (3). The update of  $\mu$  will be chosen such that  $\varphi(z^+s^+) < \varphi(zs)$  and thus  $0 < z^+s^+ < zs$ . Therefore the error we make when we say that  $\varphi(z^+s^+)$  is the  $\mu$ -center is

$$\eta(\alpha) := \alpha^2\Delta z\Delta s\varphi'(zs) + \frac{\varphi''(\xi)}{2}(z^+s^+ - zs)^2. \quad (9)$$

The following sequence of lemmas aims to bound this error in terms of the proximity measure. Before doing so, we recall a useful lemma from [14].

**Lemma 3.3 (Lemma 5.1, [14])** *Let  $(z, s) \in \mathcal{F}_+$  and  $a \in \mathbb{R}^n$ . Assume that matrix  $M$  is a positive semidefinite matrix. Let  $(\Delta z, \Delta s)$  be the solution of*

$$\begin{cases} -M\Delta z + \Delta s = 0 \\ s\Delta z + z\Delta s = a \end{cases}.$$

Then

$$\begin{aligned} \|\Delta z\Delta s\|_1 &\leq C_1 \left\| \frac{a}{\sqrt{zs}} \right\|_2^2, \\ \|\Delta z\Delta s\|_2 &\leq C_2 \left\| \frac{a}{\sqrt{zs}} \right\|_2^2, \\ \|\Delta z\Delta s\|_\infty &\leq C_\infty \left\| \frac{a}{\sqrt{zs}} \right\|_2^2, \end{aligned}$$

with  $C_1 = 1/2$ ,  $C_2 = 1/(2\sqrt{2})$  and  $C_\infty = 1/4$ .

Through the rest of this article we denote by  $C_p$  for  $p \in \{1, 2, \infty\}$  the constants defined as

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad C_\infty = \frac{1}{4}.$$

Straightforward application of this lemma for  $a = \frac{\mu - \varphi(zs)}{\varphi'(zs)}$  and  $v_\varphi = \sqrt{\frac{\varphi(zs)}{\mu}}$  gives the following lemma.

**Lemma 3.4** *Let  $(z, s) \in \mathcal{F}_+$ ,  $(\Delta z, \Delta s)$  be the solution of (3) and  $p \in \{1, 2, \infty\}$ , then*

$$\|\Delta z\Delta s\|_p \leq C_p \left\| \frac{\sqrt{\varphi(zs)}\sqrt{\mu}}{\sqrt{zs}\varphi'(zs)}(v_\varphi^{-1} - v_\varphi) \right\|_2^2.$$



The next lemma will bound  $\|\Delta z \Delta s \varphi'(zs)\|_p$  for  $p \in \{1, 2, \infty\}$ .

**Lemma 3.5** *Let  $(z, s) \in \mathcal{F}_+$ ,  $(\Delta z, \Delta s)$  be the solution of (3) and  $p \in \{1, 2, \infty\}$ , then*

$$\|\Delta z \Delta s \varphi'(zs)\|_p \leq \|\Delta z \Delta s \varphi'(0)\|_p \leq 4C_p \|\mu\|_\infty \delta_\varphi^2.$$

*Proof* By concavity of  $\varphi$  we have that  $\varphi(z_i s_i) \leq \varphi'(0) z_i s_i$ ,  $\forall i$ , so

$$\frac{\varphi(zs) \varphi'(0)}{zs (\varphi'(zs))^2} \leq \frac{(\varphi'(0))^2}{(\varphi'(zs))^2}. \quad (10)$$

Furthermore for  $p \in \{1, 2, \infty\}$  and using Lemma 3.4, followed by (10)

$$\begin{aligned} \|\Delta z \Delta s \varphi'(zs)\|_p &\leq \|\Delta z \Delta s\|_p \varphi'(0) \\ &\leq C_p \left\| \frac{\sqrt{\varphi(zs)} \sqrt{\mu}}{\sqrt{zs} \varphi'(zs)} ((v_\varphi^{-1} - v_\varphi)) \right\|_2^2 \varphi'(0) \\ &\leq \left( \sum_{i=1}^n C_p \frac{\varphi(z_i s_i) \mu_i}{z_i s_i (\varphi'(z_i s_i))^2} (v_{\varphi_i}^{-1} - v_{\varphi_i})^2 \right) \varphi'(0) \\ &\leq \sum_{i=1}^n C_p \mu_i \left( \frac{\varphi'(0)}{\varphi'(z_i s_i)} \right)^2 (v_{\varphi_i}^{-1} - v_{\varphi_i})^2 \\ &\leq 4C_p \|\mu\|_\infty \delta_\varphi^2. \end{aligned}$$

□

Now we move to the main result which gives a bound for the complete error.

**Lemma 3.6** *Let  $(z, s) \in \mathcal{F}_+$  and  $(\Delta z, \Delta s)$  be the solution of (3). For  $p \in \{2, \infty\}$  we have*

$$\begin{aligned} \|\eta\|_p &\leq (4C_p + 2T\varphi(\|zs\|_\infty)) \alpha^2 \|\mu\|_\infty \delta_\varphi^2 \\ &\quad + \left( C_\infty \sqrt{\varphi(\|zs\|_\infty)} + \alpha C_p^2 \sqrt{\|\mu\|_\infty} \delta_\varphi \right) 8\alpha^3 T \|\mu\|_\infty^{3/2} \delta_\varphi^3 \end{aligned}$$

*Proof* By equation (9) we have

$$\begin{aligned} \eta &= \alpha^2 \Delta z \Delta s \varphi'(zs) + \frac{\alpha^2 \varphi''(\xi)}{2} (z \Delta s + s \Delta z)^2 + \frac{\alpha^4 \varphi''(\xi)}{2} (\Delta s \Delta z)^2 \\ &\quad + \alpha^3 \varphi''(\xi) (\Delta s \Delta z) (z \Delta s + s \Delta z) \end{aligned}$$

Taking the p-norm for  $p \in \{2, \infty\}$  and using triangle inequalities

$$\begin{aligned} \|\eta\|_p &\leq \alpha^2 \|\Delta z \Delta s \varphi'(zs)\|_p + \alpha^2 \left\| \frac{\varphi''(\xi)}{2} (z \Delta s + s \Delta z)^2 \right\|_p \\ &\quad + \alpha^4 \left\| \frac{\varphi''(\xi)}{2} (\Delta s \Delta z)^2 \right\|_p + \alpha^3 \|\varphi''(\xi) (\Delta s \Delta z) (z \Delta s + s \Delta z)\|_p. \end{aligned}$$

We now bound each term of the above right-hand side. First, Lemma 3.5 gives

$$\|\Delta z \Delta s \varphi'(zs)\|_p \leq 4C_p \|\mu\|_\infty \delta_\varphi^2.$$

Using successively (4),  $(\mu - \varphi(zs))^2 = (\mu v_\varphi(v_\varphi^{-1} - v_\varphi))^2 = \mu \varphi(zs)(v_\varphi^{-1} - v_\varphi)^2$  and  $\varphi(z_i s_i) \leq \varphi(\|zs\|_\infty)$ ,  $\forall i \in \{1, \dots, n\}$ , we obtain step by step

$$\begin{aligned} \left\| \frac{\varphi''(\xi)}{2} (z\Delta s + s\Delta z)^2 \right\|_p &\leq \left\| \frac{\varphi''(0)}{2} (z\Delta s + s\Delta z)^2 \right\|_p \\ &= \frac{T}{2} \left\| \left( \frac{\varphi'(0)}{\varphi'(zs)} \right)^2 (\mu - \varphi(zs))^2 \right\|_p \\ &\leq \frac{T\varphi(\|zs\|_\infty) \|\mu\|_\infty}{2} \left\| \left( \frac{\varphi'(0)}{\varphi'(zs)} (v_\varphi^{-1} - v_\varphi) \right)^2 \right\|_p \\ &\leq 2T\varphi(\|zs\|_\infty) \|\mu\|_\infty \delta_\varphi^2. \end{aligned}$$

To bound the third term, we use Lemma 3.5, equality (4) and the inequality  $\|u^2\|_p \leq \|u\|_p^2$  for all  $u \in \mathbb{R}^n$ , and we obtain

$$\begin{aligned} \left\| \frac{\varphi''(\xi)}{2} (\Delta s \Delta z)^2 \right\|_p &\leq \frac{T}{2} \|(\varphi'(0) \Delta s \Delta z)^2\|_p \leq \frac{T}{2} \|\varphi'(0) \Delta s \Delta z\|_p^2 \\ &\leq \frac{T}{2} (\|\mu\|_\infty 4C_p \delta_\varphi^2)^2 = 8TC_p^2 \|\mu\|_\infty^2 \delta_\varphi^4. \end{aligned}$$

Finally, the definition of  $v_\varphi$  implies that

$$\begin{aligned} \|\varphi'(0)(z\Delta s + s\Delta z)\|_p &= \left\| \frac{\varphi'(0)}{\varphi'(zs)} \mu v_\varphi(v_\varphi^{-1} - v_\varphi) \right\|_p \\ &\leq \left\| \frac{\varphi'(0)}{\varphi'(zs)} \sqrt{\|\mu\|_\infty} \sqrt{\varphi(\|zs\|_\infty)} (v_\varphi^{-1} - v_\varphi) \right\|_p \end{aligned}$$

Using the above inequality, as well as  $\|uw\|_p \leq \|u\|_\infty \|w\|_p$ ,  $\forall (u, w) \in \mathbb{R}^n \times \mathbb{R}^n$ , and (4), we get

$$\begin{aligned} \|\varphi''(\xi)(\Delta s \Delta z)(z\Delta s + s\Delta z)\|_p &\leq T \|\varphi'(0)(\Delta s \Delta z)\|_\infty \|\varphi'(0)(z\Delta s + s\Delta z)\|_p \\ &\leq 8TC_\infty \sqrt{\varphi(\|zs\|_\infty)} \|\mu\|_\infty^{3/2} \delta_\varphi^3, \end{aligned}$$

which completes the proof.  $\square$

In the special case where we are in a close neighbourhood of the central path we get an improved version of the result:

**Theorem 3.1** *Let  $(z, s) \in \mathcal{F}_+$ ,  $\delta_\varphi \leq 1$  and  $(\Delta z, \Delta s)$  be the solution of (3) and  $p \in \{2, \infty\}$ , we have*

$$\|\eta\|_p \leq (4C_p + 12T\|\mu\|_\infty) \alpha^2 \|\mu\|_\infty \delta_\varphi^2 + (\sqrt{6}C_\infty + \alpha C_p^2 \delta_\varphi) \alpha^3 8T \|\mu\|_\infty^2 \delta_\varphi^3 \quad (11)$$

*Proof* The proof is similar to the proof of Lemma 3.6, but we use  $\varphi(zs) \leq 6\mu$  from Lemma 3.1 instead of  $\varphi(z_i s_i) \leq \varphi(\|zs\|_\infty)$ ,  $\forall i$ .  $\square$

For instance, with  $\alpha = 1$ , using  $\delta_\varphi^q \leq \delta_\varphi$  for  $q \geq 1$ , (11) becomes

$$\|\eta\|_\infty \leq \left( \|\mu\|_\infty + \left( \frac{25 + 4\sqrt{6}}{2} \right) T \|\mu\|_\infty^2 \right) \delta_\varphi^2.$$

For  $\varphi(t) = t$  we get the same result as in [17]:  $\|\eta\|_\infty \leq \|\mu\|_\infty \delta_\varphi^2$ .

### 3.3 Feasibility of the Newton Step

A Newton step is feasible (strictly feasible) if the couple  $(z^+, s^+)$  defined by (5) is feasible (strictly feasible).

**Theorem 3.2** *Let  $\alpha$  be in  $[0, 1]$  and  $\delta_\varphi \leq 1$ . The  $\alpha$ -Newton step is strictly feasible for  $(z, s)$  if*

$$\alpha \delta_\varphi^2 < \frac{1}{1 + \left( \frac{25 + 4\sqrt{6}}{2} \right) T \|\mu\|_\infty}. \quad (12)$$

*Note that for  $\alpha = 1$ , the above condition makes the full Newton step be strictly feasible.*

For  $\alpha = 1$  condition (12) implies that the proximity measure,  $\delta_\varphi$ , must be less than 1, which will not be a restrictive assumption.

*Proof* For  $\alpha = 0$  the result trivially holds. Therefore, we can assume without loss of generality that  $\alpha \in ]0, 1]$ . Let  $\beta \in ]0, \alpha]$ , and define  $z^\beta := z + \beta \Delta z$  and  $s^\beta := s + \beta \Delta s$ , where  $(\Delta z, \Delta s)$  is the solution of (3). The proof rests upon a continuity argument. Indeed, we assume  $z, s > 0$  so  $\varphi(zs) > 0$  and, using equations (3) and (8):

$$\begin{aligned} \varphi(z^\beta s^\beta) &= \varphi(zs) + \beta \varphi'(zs)(z \Delta s + s \Delta z) + \eta(\beta) \\ &= \varphi(zs)(1 - \beta) + \beta \left( \mu + \frac{\eta(\beta)}{\beta} \right) \\ &\geq \varphi(zs)(1 - \beta) + \beta \left( \mu - e \frac{\|\eta(\beta)\|_\infty}{\beta} \right) \end{aligned}$$

which is positive for all  $z, s > 0$  if  $\|\mu\|_\infty > \|\eta(\beta)\|_\infty / \beta$ . Using Lemma 3.6 this condition holds if

$$\begin{aligned} \|\mu\|_\infty &> (4C_\infty + 2T\varphi(\|zs\|_\infty))\beta \|\mu\|_\infty \delta_\varphi^2 \\ &\quad + \left( C_\infty \sqrt{\varphi(\|zs\|_\infty)} + \beta C_\infty^2 \sqrt{\|\mu\|_\infty} \delta_\varphi \right) 8\beta^2 T \|\mu\|_\infty^{3/2} \delta_\varphi^3. \end{aligned}$$

The right-hand side is increasing with respect to  $\beta$ , so it is sufficient to verify

$$\begin{aligned} \|\boldsymbol{\mu}\|_\infty &> (4C_\infty + 2T\varphi(\|zs\|_\infty))\alpha\|\boldsymbol{\mu}\|_\infty\delta_\varphi^2 \\ &\quad + \left(C_\infty\sqrt{\varphi(\|zs\|_\infty)} + \alpha C_\infty^2\sqrt{\|\boldsymbol{\mu}\|_\infty}\delta_\varphi\right)8\alpha^2T\|\boldsymbol{\mu}\|_\infty^{3/2}\delta_\varphi^3. \end{aligned}$$

Therefore, since  $\delta_\varphi \leq 1$ , using Lemma 3.1, it suffices to have

$$1 - \delta_\varphi^2\alpha \left(4C_\infty + \left(12 + 8\sqrt{6}C_\infty + 8C_\infty^2\right)T\|\boldsymbol{\mu}\|_\infty\right) > 0,$$

which corresponds to our assumption. It follows that for all  $\beta \in [0, \alpha]$ ,  $\varphi(z^\beta s^\beta) > 0$ . By continuity of  $\varphi$ , this implies that none of  $z^\beta$  or  $s^\beta$  vanish for  $\beta \in [0, \alpha]$ , so the result follows.  $\square$

### 3.4 Quadratic Decrease of the Proximity Measure

The Newton method is known to behave well in a close neighbourhood of the solution. The following theorem states a condition on the proximity measure,  $\delta_\varphi := \delta_\varphi(z, s, \boldsymbol{\mu})$ , that ensures a quadratic convergence of the Newton step. We denote by  $\bar{\delta}_\varphi := \delta_\varphi(z^+, s^+, \boldsymbol{\mu})$  and  $\bar{v}_\varphi := v_\varphi(z^+, s^+, \boldsymbol{\mu})$  the proximity measure and the function  $v_\varphi$  after the Newton step.

**Theorem 3.3** *Let  $(z, s) \in \mathcal{F}_+$  and  $(z^+, s^+)$  be defined as in (5) for  $\alpha = 1$ . Let  $\Gamma(\|\boldsymbol{\mu}\|_\infty)$  defined in (6) and*

$$Q(\|\boldsymbol{\mu}\|_\infty) := \frac{1 - (\Gamma(\|\boldsymbol{\mu}\|_\infty)(\sqrt{2} + (13 + 2\sqrt{6})T\|\boldsymbol{\mu}\|_\infty))^2/4}{1 + \left(\frac{25+4\sqrt{6}}{2}\right)T\|\boldsymbol{\mu}\|_\infty}. \quad (13)$$

*If  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)$ , then  $\bar{\delta}_\varphi \leq \delta_\varphi^2$ .*

*Proof* Let  $\Gamma(\|\boldsymbol{\mu}\|_\infty)$  be defined as in Section 3.1:

$$\bar{\delta}_\varphi \leq \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)}{2} \|(\bar{v}_\varphi)^{-1} - (\bar{v}_\varphi)\|_2 = \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)}{2} \|(\bar{v}_\varphi)^{-1}(\mathbf{e} - (\bar{v}_\varphi)^2)\|_2 \quad (14)$$

$$\leq \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)}{2} \left\| \frac{\frac{\eta(\alpha)}{\boldsymbol{\mu}}}{\sqrt{\mathbf{e} + \frac{\eta(\alpha)}{\boldsymbol{\mu}}}} \right\|_2 \quad (15)$$

$$\leq \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty) \left\| \frac{\eta(\alpha)}{\boldsymbol{\mu}} \right\|_2}{2\sqrt{1 - \left\| \frac{\eta(\alpha)}{\boldsymbol{\mu}} \right\|_\infty}}. \quad (16)$$

Let  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty) \leq 1$ , using Theorem 3.1 with a full step, i.e.  $\alpha = 1$ , it follows

$$\bar{\delta}_\varphi \leq \delta_\varphi^2 \Gamma(\|\boldsymbol{\mu}\|_\infty) \frac{4C_2 + (12 + 8\sqrt{6}C_\infty + 8C_2^2)T\|\boldsymbol{\mu}\|_\infty}{2\sqrt{1 - \delta_\varphi^2(4C_\infty + (12 + 8\sqrt{6}C_\infty + 8C_\infty^2)T\|\boldsymbol{\mu}\|_\infty)}}.$$

So,  $\bar{\delta}_\varphi \leq \delta_\varphi^2$  if  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)$ .  $\square$

Considering  $\varphi(t) = t$ , the condition of Theorem 3.3 becomes the same as in [17]:  $\delta_\varphi^2 \leq 1/2$ .

*Remark 1* The condition in Theorem 3.3 implies the condition in Theorem 3.2. So, if the iterates locate in the neighbourhood of quadratic convergence, the full Newton step will provide strictly feasible iterates.

*Remark 2* Notice that since the proximity measure is always non-negative, the condition from Theorem 3.3 can hold only when  $\boldsymbol{\mu}$  is sufficiently small, i.e. when  $\Gamma(\|\boldsymbol{\mu}\|_\infty) (\sqrt{2} + (13 + 2\sqrt{6})T\|\boldsymbol{\mu}\|_\infty) \leq 2$ . This is not a restrictive assumption, because we can always scale a given initial point so that it satisfies this condition.

### 3.5 Updating Parameter Strategy

The sequence of parameters  $\{\theta^k\}$  must be chosen such that the iterates remain strictly feasible and satisfy the condition of Theorem 3.3. In this section, Proposition 3.1 gives an upper bound on the proximity measure after an update on  $\boldsymbol{\mu}$ , that is  $\boldsymbol{\mu}^+ = \boldsymbol{\mu}(1 - \theta^k)$ , and then Theorem 3.4 describes how to build the sequence  $\{\theta^k\}$ .

First, we provide an upper bound of the proximity measure after an update of the parameter, denoted  $\delta_\varphi^+ := \delta_\varphi(z^+, s^+, \boldsymbol{\mu}(1 - \theta^k))$ , in terms of the update  $\theta^k$ , the proximity measure before this update, denoted  $\bar{\delta}_\varphi := \delta_\varphi(z^+, s^+, \boldsymbol{\mu})$ , and the proximity measure before the Newton step, denoted  $\delta_\varphi := \delta_\varphi(z, s, \boldsymbol{\mu})$ . The computation of this upper bound is based on the following lemma.

**Lemma 3.7** *Let  $(z, s) \in \mathcal{F}_+$  and  $(\Delta z, \Delta s)$  be the solution of (3). Then,*

$$\sum_{i=1}^n \frac{\varphi(z_i^+ s_i^+)}{\boldsymbol{\mu}_i} \leq n + 2\delta_\varphi^2.$$

*Proof* We first show that

$$\left\| \frac{\Delta z \Delta s \boldsymbol{\varphi}'(zs)}{\boldsymbol{\mu}} \right\|_1 \leq 2\delta_\varphi^2. \quad (17)$$

For this, we apply Lemma 3.3 with  $\Delta z/\sqrt{\boldsymbol{\mu}}$ ,  $\Delta s/\sqrt{\boldsymbol{\mu}}$  and  $\mathbf{a} = \frac{\boldsymbol{\mu} - \varphi(zs)}{\sqrt{\boldsymbol{\mu}} \boldsymbol{\varphi}'(zs)}$  to obtain

$$\left\| \frac{\Delta z \Delta s}{\boldsymbol{\mu}} \right\|_1 \leq \frac{1}{2} \left\| \frac{\boldsymbol{\mu} - \varphi(zs)}{\sqrt{\boldsymbol{\mu}} \boldsymbol{\varphi}'(zs) \sqrt{zs}} \right\|_2^2.$$

Then, we get (17) by following the same steps as in the proof of Lemma 3.5.

From equation (8) and by concavity of  $\varphi$ ,  $\varphi(z^+ s^+)/\boldsymbol{\mu} \leq \mathbf{e} + \Delta z \Delta s \boldsymbol{\varphi}'(zs)/\boldsymbol{\mu}$ . So,  $\sum_{i=1}^n \varphi(z_i^+ s_i^+)/\boldsymbol{\mu}_i \leq \sum_{i=1}^n 1 + |\Delta z_i \Delta s_i \boldsymbol{\varphi}'(z_i s_i)/\boldsymbol{\mu}_i| \leq n + 2\delta_\varphi^2$ .  $\square$

**Proposition 3.1** Let  $\overline{v}_\varphi := v_\varphi(z^+, s^+, \boldsymbol{\mu})$  and  $v_\varphi^+ := (z^+, s^+, \boldsymbol{\mu}^+)$ , where  $\boldsymbol{\mu}^+ := (1 - \theta^k)\boldsymbol{\mu}$ . Then,

$$(\delta_\varphi^+)^2 \leq (1 - \theta^k)(\overline{\delta}_\varphi)^2 + \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)^2}{4(1 - \theta^k)} (n(\theta^k)^2 + (4\theta^k - 2(\theta^k)^2)\delta_\varphi^2) .$$

Furthermore, assuming that  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)$  yields

$$(\delta_\varphi^+)^2 \leq (1 - \theta^k)Q(\|\boldsymbol{\mu}\|_\infty)^2 + \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)^2}{4(1 - \theta^k)} (n(\theta^k)^2 + (4\theta^k - 2(\theta^k)^2)Q(\|\boldsymbol{\mu}\|_\infty)) .$$

*Proof* Noticing that  $v_\varphi^+ = \overline{v}_\varphi / \sqrt{1 - \theta^k}$ , it follows that

$$\begin{aligned} (\delta_\varphi^+)^2 &= \frac{1}{4} \sum_{i=1}^n \left( \frac{\varphi'(0)}{\varphi'(z_i^+ s_i^+)} \right)^2 ((v_{\varphi_i}^+)^{-2} + (v_{\varphi_i}^+)^2 - 2) \\ &= \frac{1}{4} \sum_{i=1}^n \left( \frac{\varphi'(0)}{\varphi'(z_i^+ s_i^+)} \right)^2 \left( (1 - \theta^k)(\overline{v}_{\varphi_i})^{-2} + \frac{\overline{v}_{\varphi_i}^2}{(1 - \theta^k)} - 2 \right) \\ &= (1 - \theta^k) \left( \frac{1}{4} \sum_{i=1}^n \left( \frac{\varphi'(0)}{\varphi'(z_i^+ s_i^+)} \right)^2 ((\overline{v}_{\varphi_i})^{-2} + \overline{v}_{\varphi_i}^2 - 2) \right) \\ &\quad + \frac{1}{4} \sum_{i=1}^n \left( \frac{\varphi'(0)}{\varphi'(z_i^+ s_i^+)} \right)^2 \left( -2\theta^k + \frac{2\theta^k - (\theta^k)^2}{1 - \theta^k} \overline{v}_{\varphi_i}^2 \right) \\ &= (1 - \theta^k)(\overline{\delta}_\varphi)^2 + \frac{1}{4} \sum_{i=1}^n \left( \frac{\varphi'(0)}{\varphi'(z_i^+ s_i^+)} \right)^2 \left( -2\theta^k + \frac{2\theta^k - (\theta^k)^2}{1 - \theta^k} \overline{v}_{\varphi_i}^2 \right) \end{aligned}$$

Using successively equation (7), Lemma 3.7 and  $\Gamma(\|\boldsymbol{\mu}^+\|_\infty) \leq \Gamma(\|\boldsymbol{\mu}\|_\infty)$ , we obtain

$$\begin{aligned} (\delta_\varphi^+)^2 &\leq (1 - \theta^k)(\overline{\delta}_\varphi)^2 + \Gamma(\|\boldsymbol{\mu}^+\|_\infty)^2 \left( -\frac{2\theta^k}{4}n + \frac{2\theta^k - (\theta^k)^2}{4(1 - \theta^k)}(n + 2\delta_\varphi^2) \right) \\ &\leq (1 - \theta^k)(\overline{\delta}_\varphi)^2 + \Gamma(\|\boldsymbol{\mu}\|_\infty)^2 \left( \frac{n(\theta^k)^2}{4(1 - \theta^k)} + \frac{2\theta^k - (\theta^k)^2}{2(1 - \theta^k)}\delta_\varphi^2 \right) . \end{aligned}$$

This proves the first part of the proposition.

Now, assuming that  $\delta_\varphi \leq Q(\|\boldsymbol{\mu}\|_\infty)$  allows us to use Theorem 3.3 and so  $\overline{\delta}_\varphi \leq \delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)^2$  gives the result.  $\square$

We conclude this section by a description of the choice of the update parameters  $\theta^k$ .

**Theorem 3.4** Let  $\boldsymbol{\mu}$  be such that  $Q(\|\boldsymbol{\mu}\|_\infty) > 0$  and  $(z, s) \in \mathcal{F}_+$  such that  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)$ . Define  $\theta^k$  as

$$\theta^k = \frac{-b + \sqrt{b^2 - 4ac}}{2a} ,$$

with

$$\begin{aligned} a &= \Gamma(\|\boldsymbol{\mu}\|_\infty)^2 n - 2\Gamma(\|\boldsymbol{\mu}\|_\infty)^2 Q(\|\boldsymbol{\mu}\|_\infty) + 4Q(\|\boldsymbol{\mu}\|_\infty)^2, \\ b &= 4\Gamma(\|\boldsymbol{\mu}\|_\infty)^2 Q(\|\boldsymbol{\mu}\|_\infty) - 8Q(\|\boldsymbol{\mu}\|_\infty)^2 + 4Q(\|\boldsymbol{\mu}\|_\infty), \\ c &= 4Q(\|\boldsymbol{\mu}\|_\infty)^2 - 4Q(\|\boldsymbol{\mu}\|_\infty). \end{aligned}$$

The proximity measure  $\delta_\varphi^+ := \delta_\varphi(z^+, s^+, \boldsymbol{\mu}(1 - \theta^k))$  satisfies the conditions of feasibility in Theorem 3.2 and quadratic convergence of the Newton step in Theorem 3.3.

By definition  $Q(\|\boldsymbol{\mu}\|_\infty) < 1$ , thus  $c$  is negative. Furthermore, for  $n$  sufficiently large  $a$  is positive and so  $b^2 - 4ac$  and  $\theta^k$  are positive.

We would also like to point out that the value of  $\theta^k$  is of order  $1/\sqrt{n}$  for  $n$  large. This observation is fundamental considering the complexity of the algorithm.

*Proof* As pointed out earlier in Remark 1 the condition of Theorem 3.2 is weaker than the condition of Theorem 3.3. Thus, it is sufficient to satisfy the latter condition to ensure strict feasibility of the iterates.

According to the condition of Theorem 3.3, after an update of  $\boldsymbol{\mu}$ , i.e.  $\boldsymbol{\mu}^+ = (1 - \theta^k)\boldsymbol{\mu}$ , the proximity measure  $\delta_\varphi^+$  must satisfy

$$(\delta_\varphi^+)^2 \leq Q(\|\boldsymbol{\mu}^+\|_\infty).$$

As  $Q$  is decreasing with respect to  $\|\boldsymbol{\mu}\|_\infty$ , it is sufficient to ensure that

$$(\delta_\varphi^+)^2 \leq Q(\|\boldsymbol{\mu}\|_\infty).$$

By Proposition 3.1 in the case  $\delta_\varphi^2 \leq Q(\|\boldsymbol{\mu}\|_\infty)$ , we can choose any  $\theta^k$  satisfying

$$(1 - \theta^k)Q(\|\boldsymbol{\mu}\|_\infty)^2 + \frac{\Gamma(\|\boldsymbol{\mu}\|_\infty)^2}{4(1 - \theta^k)} (n(\theta^k)^2 + (4\theta^k - 2(\theta^k)^2)Q(\|\boldsymbol{\mu}\|_\infty)) \leq Q(\|\boldsymbol{\mu}\|_\infty).$$

Therefore, it is sufficient to choose  $\theta^k > 0$  such that

$$\theta^k \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

with  $a, b$  and  $c$  defined as in the statement of the theorem.  $\square$

*Remark 3* For a more explicit characterization of  $\theta^k$ , we can study its asymptotical behaviour. By definition of  $\Gamma$  and  $Q$ ,

$$\lim_{\boldsymbol{\mu} \rightarrow 0} Q(\|\boldsymbol{\mu}\|_\infty) = 1/2 \text{ and } \lim_{\boldsymbol{\mu} \rightarrow 0} \Gamma(\|\boldsymbol{\mu}\|_\infty) = 1,$$

$$\text{so } \lim_{\boldsymbol{\mu} \rightarrow 0} \theta^k = \frac{-1 + \sqrt{1 + 4n}}{2n} \leq \frac{1}{\sqrt{n}}.$$

### 3.6 Complexity Analysis of the Full Newton Step IPM

The complexity of this algorithm is obtained by the extension of a classical lemma, whose proof can be found for instance in [19].

**Lemma 3.8** *Let  $\bar{\theta}$  be such that  $0 < \bar{\theta} \leq \theta^k \forall k \in \mathbb{N}$ . The  $\varphi$ -FN-IPM for monotone LCP described in Algorithm 2 provides an  $\epsilon$ -solution  $(z, s)$ , which satisfies  $\sum_{i=1}^n \varphi(z_i s_i) \leq n\epsilon$  after at most  $\log(\sum_{i=1}^n \varphi(z_i^0 s_i^0)/\epsilon) / \bar{\theta}$  iterations.*

The sequence  $\{\theta^k\}$  is given by Theorem 3.4. As already stated,  $\theta^k$  is of order  $1/\sqrt{n}$  for  $n$  large, which justify the existence of  $\bar{\theta}$ . Moreover, we can choose  $\bar{\theta}$  of order  $1/\sqrt{n}$  without loss of generality. As a result, we can now state our main theorem.

**Theorem 3.5** *Let  $\mu^0 = z^0 s^0$ . Algorithm 2, with the sequence of update parameters  $\theta^k$  described above, guarantees strict feasibility of the iterates and quadratic convergence of the proximity measure. Moreover, it provides an  $\epsilon$ -solution  $(z, s)$ , which satisfies  $\sum_{i=1}^n \varphi(z_i s_i) \leq n\epsilon$  after at most  $\mathcal{O}(\sqrt{n} \log(\frac{n}{\epsilon}))$  iterations.*

Preliminary computational experiments give similar results to the classical method on a small selection of LPs from the NETLIB repository<sup>1</sup>. This confirms the validity of our approach. Some informations regarding these results are presented on Appendix A. Further investigations on more sophisticated methods may get the best out of this new direction.

## 4 Conclusions

The method presented in this article shows a generalization of the FN-IPM with polynomial upper bound complexity for monotone LCPs considering a general family of smooth increasing concave functions. The main contributions of this article are that we extend the classical path-following method and Darvay's method with  $\varphi(t) = \sqrt{t}$  and prove that these new methods have the best known worst-case complexity.

Further research may extend this result to a more general family of LCPs such that  $P_*(\kappa)$ -LCPs or  $P_0$ -LCPs with bounded level sets as in [4, 5, 23].

Recent developments by Roos in [18] consider an infeasible IPM with full Newton step using only one feasibility step. More investigations regarding the method presented in this paper could extend the results in [18].

Despite having the best worst case upper-bound for IPMs the full Newton step is not the most used approach for numerics, since this upper-bound is attained in general [19]. We believe that the philosophy applied in this paper can be generalized to other IPMs approaches. In particular, we are planning to study a predictor-corrector implementation of this approach.

<sup>1</sup> <http://www.netlib.org/lp/>



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## A Numerics

To validate the theoretical results, we first implemented the  $\varphi$ -FN-IPM described in Algorithm 2 with the sequence of update parameters given by Theorem 3.4 and with the functions  $\varphi$  given in Table 1. The datasets and the residuals after convergence are detailed

Name	Id.	$\theta^1$	log	$( )^\alpha$
Fct.	t	$\frac{t}{t+1}$	$\log(1+t)$	$(t+10^{-3})^\alpha - (10^{-3})^\alpha$ with $\alpha \in (0, 1)$

**Table 1**  $\varphi$ -functions used in the computational tests

in Table 2. The stopping criterion is  $z^T s \leq n\epsilon$ , where  $\epsilon = 10^{-6}$ , and we compute the residuals as  $\|zs\|_\infty \times 10^6$ . For every  $\varphi$  function that appears in Table 1, the method converged in the same number of iterations, so we only display the number of iterations once. This phenomenon is not surprising, since the  $\varphi$ -FN-IPM used here stays very close to the central path, as shown by Theorem 3.3. Therefore, this implementation does not exploit fully the new directions. Nonetheless, some small differences remain in the residuals as illustrated in Table 2.

Instance	Iter.	Id.	$\theta^1$	log	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
ADLITTLE	268	0.995583	0.995853	0.995856	0.995851	0.995855	0.995854
AFIRO	170	0.943919	0.944204	0.944207	0.944200	0.944205	0.944205
BEACONFD	418	0.995479	0.995728	0.995730	0.995724	0.995727	0.995728
BOEING2	420	0.974064	0.974307	0.974309	0.974309	0.974309	0.974307
BLEND	264	0.984332	0.984599	0.984602	0.984600	0.984599	0.984599
GROW7	406	0.983538	0.983785	0.983788	0.983780	0.983787	0.983785
ISRAEL	428	0.988911	0.989157	0.989155	0.989148	0.989154	0.989156
KB2	203	0.944257	0.944530	0.944532	0.944530	0.944531	0.944530
RECIPELP	332	0.960634	0.960884	0.960887	0.960884	0.960885	0.960885
SC50A	218	0.951084	0.951353	0.951356	0.951350	0.951354	0.951354
SC50B	218	0.951084	0.951353	0.951356	0.951350	0.951354	0.951354
SC105	316	0.973217	0.973472	0.973475	0.973473	0.973474	0.973473
SCAGR7	342	0.985376	0.985631	0.985633	0.985632	0.985632	0.985632
SHARE1B	372	0.968387	0.968633	0.968635	0.968634	0.968632	0.968634
SHARE2B	310	0.972849	0.973105	0.973107	0.973102	0.973107	0.973106
STOCFOR1	324	0.982774	0.983031	0.983033	0.983033	0.983031	0.983031

**Table 2** Value of  $\text{Res.} = \|zs\|_\infty \times 10^6$ , after the algorithm reaches  $z^T s \leq n\epsilon = 10^{-6}n$

To illustrate the possible differences between the functions  $\varphi$ , we run another experiment that considers a fixed value for  $\theta^k = \frac{1}{\sqrt{n}}$  for all  $k \in \mathbb{N}$ . Table 3 illustrates the different behaviours observed for different choices of  $\varphi$ . We notice that a smaller number of iterations seems to be required when the derivative in zero is larger for the methods with  $\alpha = 0.25, 0.5$  and  $0.75$ . Thus, further research exploring interior-point methods in large neighbourhood of the central path may get the best out of these differences.

Instance	Id.		$\theta^1$		log		$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
	Res.	Iter.	Res.	Iter.	Res.	Iter.	Res.	Iter.	Res.	Iter.	Res.	Iter.
ADLITTLE	0.995	268	0.975	255	0.993	263	0.967	212	0.983	226	0.966	245
AFIRO	0.943	170	0.988	161	0.930	167	0.970	134	0.972	143	0.954	155
BEACONFD	0.995	418	0.998	397	1.000	410	0.984	330	0.996	352	0.968	382
BOEING2	0.974	420	0.974	399	0.977	412	0.986	331	0.967	354	0.975	383
BLEND	0.984	264	0.975	251	0.986	259	0.950	209	0.954	223	0.973	241
GROW7	0.983	406	0.973	386	0.996	398	0.993	320	0.982	342	0.992	370
ISRAEL	0.988	428	0.976	407	0.987	420	0.976	338	0.972	361	0.998	390
KB2	0.944	203	0.938	193	0.957	199	0.968	160	0.954	171	0.959	185
RECIPELP	0.960	332	0.978	315	0.992	325	0.963	262	0.991	279	0.995	302
SC50A	0.951	218	0.960	207	0.946	214	0.962	172	0.999	183	0.944	199
SC50B	0.951	218	0.960	207	0.946	214	0.962	172	0.999	183	0.944	199
SC105	0.973	316	0.982	300	0.976	310	0.989	249	0.983	266	0.982	288
SCAGR7	0.985	342	0.981	325	0.968	336	0.978	270	0.988	288	0.981	312
SHARE1B	0.968	372	0.984	353	0.969	365	0.989	293	0.983	313	0.979	339
SHARE2B	0.972	310	0.996	294	0.981	304	0.957	245	0.981	261	0.962	283
STOCFOR1	0.982	324	0.975	308	0.979	318	0.967	256	0.980	273	0.961	296

**Table 3** Value of  $\text{Res.} = \|zs\|_\infty \times 10^6$  after the algorithm reaches  $z^T s \leq n\epsilon = 10^{-6}n$  and number of iterations (Iter.).