



HAL
open science

Multipliers of uniform topological algebras

Mohammed El Azhari

► **To cite this version:**

| Mohammed El Azhari. Multipliers of uniform topological algebras. 2016. hal-01355345v1

HAL Id: hal-01355345

<https://hal.science/hal-01355345v1>

Preprint submitted on 23 Aug 2016 (v1), last revised 22 Jan 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multipliers of uniform topological algebras

M. El Azhari

Abstract. Let E be a complete uniform topological algebra with Arens-Michael normed factors $(E_\alpha)_{\alpha \in \Lambda}$. Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . If each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.

Mathematics Subject Classification 2010: 46H05, 46K05.

Keywords: Multiplier algebra, locally m-pseudoconvex algebra, uniform topological algebra.

1. Preliminaries

A topological algebra is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. For a topological algebra E , we denote by $\Delta(E)$ the set of all nonzero continuous multiplicative linear functionals on E . An approximate identity in a topological algebra E is a net $(e_\omega)_{\omega \in \Omega}$ such that for each $x \in E$ we have $xe_\omega \rightarrow_\omega x$ and $e_\omega x \rightarrow_\omega x$. Let E be an algebra, a function $p : E \rightarrow [0, \infty[$ is called a pseudo seminorm if there exists $0 < k \leq 1$ such that $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = |\lambda|^k p(x)$ and $p(xy) \leq p(x)p(y)$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$. k is called the homogeneity index of p . If $k = 1$, p is called a seminorm. A pseudo seminorm p is a pseudo norm if $p(x) = 0$ implies $x = 0$. A locally m-pseudoconvex algebra is a topological algebra E whose topology is determined by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of pseudo seminorms. For each $\alpha \in \Lambda$, let $\ker(p_\alpha) = \{x \in E : p_\alpha(x) = 0\}$, the quotient algebra $E_\alpha = E/\ker(p_\alpha)$ is a pseudo normed algebra in the pseudo norm $\bar{p}_\alpha(x_\alpha) = p_\alpha(x)$, $x_\alpha = x + \ker(p_\alpha)$. Let $f_\alpha : E \rightarrow E_\alpha$, $f_\alpha(x) = x + \ker(p_\alpha) = x_\alpha$, be the quotient map, f_α is a continuous homomorphism from E onto E_α . We endow the set Λ with the partial order: $\alpha \leq \beta$ if and only if $p_\alpha(x) \leq p_\beta(x)$ for all $x \in E$. Let $\alpha \leq \beta$ in Λ , since $\ker(p_\beta) \subset \ker(p_\alpha)$, we define the surjective continuous homomorphism $f_{\alpha\beta} : E_\beta \rightarrow E_\alpha$, $x_\beta = x + \ker(p_\beta) \rightarrow x_\alpha = x + \ker(p_\alpha)$. Thus $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is a projective system of pseudo normed algebras which is perfect in the sense of [2, Definition 4.1]. If E is complete, then $E \cong \varprojlim E_\alpha$ within a topological algebra isomorphism.

A locally m-convex algebra is a topological algebra E whose topology is defined by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms. For each $\alpha \in \Lambda$, let $\Delta_\alpha(E) = \{f \in \Delta(E) : |f(x)| \leq p_\alpha(x), x \in E\}$. Let E be an algebra with involution $*$, a seminorm on E is called a C^* -seminorm if $p(x^*x) = p(x)^2$ for all $x \in E$. A

complete locally m -convex $*$ -algebra $(E, (p_\alpha)_{\alpha \in \Lambda})$, for which each p_α is a C^* -seminorm, is called a locally C^* -algebra. A uniform seminorm on an algebra E is a seminorm p satisfying $p(x^2) = p(x)^2$ for all $x \in E$. A uniform topological algebra is a topological algebra whose topology is determined by a directed family of uniform seminorms. A uniform normed algebra is a normed algebra $(E, \|\cdot\|)$ such that $\|x^2\| = \|x\|^2$ for all $x \in E$.

An algebra E is called proper if for any $x \in E$, $xE = Ex = \{0\}$ implies $x = 0$. If E has identity, then E is proper. Moreover, a topological algebra with approximate identity is proper. Also, a uniform topological algebra is proper. Let E be an algebra, a map $T : E \rightarrow E$ is called a multiplier if $T(x)y = xT(y)$ for all $x, y \in E$. We denote by $M(E)$ the set of all multipliers of E . It is known that if E is a proper algebra, then any multiplier T of E is linear with the property $T(xy) = T(x)y = xT(y)$ for all $x, y \in E$, and $M(E)$ is a commutative algebra with the identity map I of E as its identity. Let $(E, \|\cdot\|)$ be a uniform normed algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E with the operator norm $\|\cdot\|_{op}$. It is known that $\|\cdot\|_{op}$ has the square property and the map $l : (E, \|\cdot\|) \rightarrow (M_c(E), \|\cdot\|_{op})$, $l(x)(y) = xy$, is an isometric isomorphism (into).

2. Results

Proposition 2.1. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m -pseudoconvex algebra with proper normed factors $(E_\alpha)_{\alpha \in \Lambda}$. The following assertions are equivalent:

- (i) $T(\ker(f_\alpha)) \subset \ker(f_\alpha)$ for all $T \in M(E)$ and $\alpha \in \Lambda$;
- (ii) for each $T \in M(E)$, there exists a unique $(T_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} M(E_\alpha)$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ for all $\alpha \leq \beta$ in Λ . Furthermore, T is continuous if and only if T_α is continuous for all $\alpha \in \Lambda$.

Proof. (ii) \Rightarrow (i) : Let $T \in M(E)$ and $x \in \ker(f_\alpha)$, then $f_\alpha(T(x)) = T_\alpha(f_\alpha(x)) = 0$ and so $T(x) \in \ker(f_\alpha)$.

(i) \Rightarrow (ii) : Let $T \in M(E)$ and $\alpha \in \Lambda$, since $T(\ker(f_\alpha)) \subset \ker(f_\alpha)$ and by using a known theorem of algebra, there exists a unique linear map $T_\alpha : E_\alpha \rightarrow E_\alpha$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$. Let $\alpha \in \Lambda$ and $x, y \in E$, $T_\alpha(f_\alpha(x)f_\alpha(y)) = T_\alpha(f_\alpha(xy)) = f_\alpha(T(xy)) = f_\alpha(xT(y)) = f_\alpha(x)f_\alpha(T(y)) = f_\alpha(x)T_\alpha(f_\alpha(y))$ and similarly on the other side, so T_α is a multiplier of E_α . Let $\alpha \leq \beta$ in Λ , we have $T_\alpha \circ f_\alpha = f_\alpha \circ T$, then $T_\alpha \circ f_{\alpha\beta} \circ f_\beta = f_{\alpha\beta} \circ f_\beta \circ T = f_{\alpha\beta} \circ T_\beta \circ f_\beta$, hence $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ since f_β is surjective. Suppose that T is continuous. Let O_α be an open in E_α , we have $f_\alpha^{-1}(T_\alpha^{-1}(O_\alpha)) = (T_\alpha \circ f_\alpha)^{-1}(O_\alpha) = (f_\alpha \circ T)^{-1}(O_\alpha)$ which is open in E since $f_\alpha \circ T$ is continuous, then $T_\alpha^{-1}(O_\alpha)$ is open in E_α . Conversely, suppose that T_α is continuous for all $\alpha \in \Lambda$. Since E is topologically isomorphic to a subalgebra of $\varprojlim E_\alpha$, T is continuous if and only if $f_\alpha \circ T$ is continuous for all $\alpha \in \Lambda$. Since $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and T_α is continuous for all $\alpha \in \Lambda$, we deduce that T is continuous.

Proposition 2.2. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m -pseudoconvex algebra with

proper normed factors $(E_\alpha)_{\alpha \in \Lambda}$. The following assertions are equivalent:

- (j) $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ ;
- (jj) there exists a unique projective system $\{(M(E_\alpha), h_{\alpha\beta}), \alpha \leq \beta\}$ such that $h_{\alpha\beta}(U) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Furthermore, if E_α is complete for all $\alpha \in \Lambda$, then $h_{\alpha\beta}$ is continuous for all $\alpha \leq \beta$ in Λ .

Proof. (jj) \Rightarrow (j) : Let $U \in M(E_\beta)$ and $x_\beta \in \ker(f_{\alpha\beta})$, then $f_{\alpha\beta}(U(x_\beta)) = h_{\alpha\beta}(U)(f_{\alpha\beta}(x_\beta)) = 0$ and so $U(x_\beta) \in \ker(f_{\alpha\beta})$.

(j) \Rightarrow (jj) : Let $\alpha \leq \beta$ in Λ and $U \in M(E_\beta)$, since $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ and by using a known theorem of algebra, there exists a unique linear map $V : E_\alpha \rightarrow E_\alpha$ such that $V \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$. Let $x_\beta, y_\beta \in E_\beta$, $V(f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(y_\beta)) = V(f_{\alpha\beta}(x_\beta y_\beta)) = f_{\alpha\beta}(U(x_\beta y_\beta)) = f_{\alpha\beta}(x_\beta U(y_\beta)) = f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(U(y_\beta)) = f_{\alpha\beta}(x_\beta) V(f_{\alpha\beta}(y_\beta))$ and similarly on the other side, so V is a multiplier of E_α .

This shows the existence of the map $h_{\alpha\beta} : M(E_\beta) \rightarrow M(E_\alpha)$ such that $h_{\alpha\beta}(U) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Let $\alpha \leq \beta$ in Λ , $U_1, U_2 \in M(E_\beta)$ and $\lambda \in \mathbb{C}$, $h_{\alpha\beta}(U_1 + \lambda U_2) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ (U_1 + \lambda U_2) = (f_{\alpha\beta} \circ U_1) + \lambda (f_{\alpha\beta} \circ U_2) = h_{\alpha\beta}(U_1) \circ f_{\alpha\beta} + \lambda h_{\alpha\beta}(U_2) \circ f_{\alpha\beta} = (h_{\alpha\beta}(U_1) + \lambda h_{\alpha\beta}(U_2)) \circ f_{\alpha\beta}$, hence $h_{\alpha\beta}(U_1 + \lambda U_2) = h_{\alpha\beta}(U_1) + \lambda h_{\alpha\beta}(U_2)$ since $f_{\alpha\beta}$ is surjective. Also, $h_{\alpha\beta}(U_1 \circ U_2) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U_1 \circ U_2 = h_{\alpha\beta}(U_1) \circ f_{\alpha\beta} \circ U_2 = h_{\alpha\beta}(U_1) \circ h_{\alpha\beta}(U_2) \circ f_{\alpha\beta}$, then $h_{\alpha\beta}(U_1 \circ U_2) = h_{\alpha\beta}(U_1) \circ h_{\alpha\beta}(U_2)$ since $f_{\alpha\beta}$ is surjective. Let $\alpha \leq \beta \leq \gamma$ in Λ and $W \in M(E_\gamma)$, $(h_{\alpha\beta} \circ h_{\beta\gamma})(W) \circ f_{\alpha\beta} = h_{\alpha\beta}(h_{\beta\gamma}(W)) \circ f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\beta} \circ h_{\beta\gamma}(W) \circ f_{\beta\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma} \circ W = f_{\alpha\gamma} \circ W = h_{\alpha\gamma}(W) \circ f_{\alpha\gamma}$, consequently $(h_{\alpha\beta} \circ h_{\beta\gamma})(W) = h_{\alpha\gamma}(W)$ since $f_{\alpha\gamma}$ is surjective. Thus $h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}$. Let $\alpha \in \Lambda$, if E_α is complete, then every multiplier of E_α is continuous. Now by assuming that E_α is complete for all $\alpha \in \Lambda$, we will show that $h_{\alpha\beta}$ is continuous for all $\alpha \leq \beta$ in Λ . For $\alpha \in \Lambda$ and $r \geq 0$, let $B_\alpha(0, r) = \{x_\alpha \in E_\alpha : \bar{p}_\alpha(x_\alpha) \leq r\}$.

We denote by $\|\cdot\|_\alpha$ the operator pseudo norm on $M(E_\alpha)$. Let $\alpha \leq \beta$ in Λ , $f_{\alpha\beta}$ is open by the open mapping theorem, so there is $\lambda \geq 0$ such that $\lambda B_\alpha(0, 1) \subset f_{\alpha\beta}(B_\beta(0, 1))$ i.e. $B_\alpha(0, 1) \subset f_{\alpha\beta}(B_\beta(0, r))$ where $r = \lambda^{-k_\beta}$ and k_β is the homogeneity index of \bar{p}_β . Let $U \in M(E_\beta)$,

$$\begin{aligned} \|h_{\alpha\beta}(U)\|_\alpha &= \sup \{\bar{p}_\alpha(h_{\alpha\beta}(U)(f_\alpha(x))) : f_\alpha(x) \in B_\alpha(0, 1)\} \\ &\leq \sup \{\bar{p}_\alpha(h_{\alpha\beta}(U)(f_{\alpha\beta}(f_\beta(x)))) : f_\beta(x) \in B_\beta(0, r)\} \\ &= \sup \{\bar{p}_\alpha(f_{\alpha\beta}(U(f_\beta(x)))) : f_\beta(x) \in B_\beta(0, r)\} \\ &\leq \sup \{\bar{p}_\beta(U(f_\beta(x))) : f_\beta(x) \in B_\beta(0, r)\} \\ &\leq \sup \{\|U\|_\beta \bar{p}_\beta(f_\beta(x)) : f_\beta(x) \in B_\beta(0, r)\} \\ &= r \|U\|_\beta. \end{aligned}$$

Therefore $h_{\alpha\beta}$ is continuous.

Theorem 2.3. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally m-pseudoconvex algebra with proper normed factors $(E_\alpha)_{\alpha \in \Lambda}$. Assume that E satisfies conditions (i) and (j). Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its pseudo seminorm topology.

Proof. By Propositions 2.1 and 2.2, we define the map $\varphi : M(E) \rightarrow \varprojlim M(E_\alpha)$, $T \rightarrow (T_\alpha)_{\alpha \in \Lambda}$. We will show that φ is an algebra isomorphism. Let $T, S \in M(E)$ and $\lambda \in \mathbb{C}$, $T_\alpha \circ f_\alpha = f_\alpha \circ T$ and $S_\alpha \circ f_\alpha = f_\alpha \circ S$, then $(T_\alpha + \lambda S_\alpha) \circ f_\alpha = f_\alpha \circ (T + \lambda S)$, so $(T + \lambda S)_\alpha = T_\alpha + \lambda S_\alpha$ by Proposition 2.1. Also, $T_\alpha \circ S_\alpha \circ f_\alpha = T_\alpha \circ f_\alpha \circ S = f_\alpha \circ T \circ S$, hence $(T \circ S)_\alpha = T_\alpha \circ S_\alpha$ by Proposition 2.1. Let $T \in M(E)$, if $T_\alpha = 0$ for all $\alpha \in \Lambda$, then $f_\alpha \circ T = T_\alpha \circ f_\alpha = 0$ for all $\alpha \in \Lambda$ and consequently $T = 0$. Let $(U_\alpha)_{\alpha \in \Lambda} \in \varprojlim M(E_\alpha)$ and define the map $T = \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi : E \rightarrow E$ where $\varprojlim U_\alpha$ is the multiplier of $\varprojlim E_\alpha$ defined by $(\varprojlim U_\alpha)(x_\alpha)_\alpha = (U_\alpha(x_\alpha))_\alpha$ and $\Phi : E \rightarrow \varprojlim E_\alpha$ is the topological algebra isomorphism given by $\Phi(x) = (f_\alpha(x))_\alpha$. Clearly T is a multiplier of E , also $f_\alpha \circ T = f_\alpha \circ \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi = U_\alpha \circ f_\alpha$ for all $\alpha \in \Lambda$, so $\varphi(T) = (U_\alpha)_\alpha$. If E_α is complete for all $\alpha \in \Lambda$, then every multiplier of E_α is continuous, hence every multiplier of E is continuous by Proposition 2.1. The pseudo seminorm topology on $M(E)$ is the topology defined by the family of pseudo seminorms $q_\alpha(T) = \|T_\alpha\|_\alpha$, $\alpha \in \Lambda$, so φ is a topological algebra isomorphism.

Proposition 2.4. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m-pseudoconvex algebra with approximate identity $(e_\omega)_{\omega \in \Omega}$. Then E satisfies conditions (i) and (j).

Proof. Let $T \in M(E)$, $x \in \ker(f_\alpha)$ and $\omega \in \Omega$,
 $f_\alpha(T(x)) = f_\alpha(T(x - xe_\omega + xe_\omega))$
 $= f_\alpha(T(x) - T(xe_\omega)) + f_\alpha(T(xe_\omega)) = f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(xT(e_\omega))$
 $= f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(x)f_\alpha(T(e_\omega)) = f_\alpha(T(x) - T(x)e_\omega)$.
Since $T(x)e_\omega \rightarrow_\omega T(x)$ and f_α is continuous, we deduce that $f_\alpha(T(x)) = 0$. Now we will show that $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Since $(e_\omega)_{\omega \in \Omega}$ is an approximate identity in E and $f_\beta : E \rightarrow E_\beta$ is a surjective continuous homomorphism, it follows that $(f_\beta(e_\omega))_{\omega \in \Omega}$ is an approximate identity in E_β . Let $U \in M(E_\beta)$, $x_\beta \in \ker(f_{\alpha\beta})$ and $\omega \in \Omega$,
 $f_{\alpha\beta}(U(x_\beta)) = f_{\alpha\beta}(U(x_\beta - x_\beta f_\beta(e_\omega) + x_\beta f_\beta(e_\omega)))$
 $= f_{\alpha\beta}(U(x_\beta) - U(x_\beta f_\beta(e_\omega))) + f_{\alpha\beta}(U(x_\beta f_\beta(e_\omega)))$
 $= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta U(f_\beta(e_\omega)))$
 $= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(U(f_\beta(e_\omega)))$
 $= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega))$.
Since $U(x_\beta) f_\beta(e_\omega) \rightarrow_\omega U(x_\beta)$ and $f_{\alpha\beta}$ is continuous, we deduce that $f_{\alpha\beta}(U(x_\beta)) = 0$.

Corollary 2.5. [4, Theorems 2.6 and 2.12] Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally m-pseudoconvex algebra with approximate identity. Suppose that each factor $E_\alpha = E/\ker(p_\alpha)$ in the generalized Arens-Michael decomposition of E is complete. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its pseudo seminorm topology.

Proof. It follows from Theorem 2.3 and Proposition 2.4.

Corollary 2.6. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally C^* -algebra. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.

Proof. By [6, Theorem 2.6] and [8, Corollary 1.12], E has an approximate identity and each factor E_α is complete.

Now we will describe multipliers algebras of complete uniform topological algebras.

Proposition 2.7. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra. Then $\ker(f_\alpha) = \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$ and $\ker(f_{\alpha\beta}) = \cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$ for all $\alpha \leq \beta$ in Λ .

Proof. By [1, Theorem 6], $p_\alpha(x) = \sup\{|\chi(x)| : \chi \in \Delta_\alpha(E)\}$ for all $x \in E$ and $\alpha \in \Lambda$, then $\ker(f_\alpha) = \ker(p_\alpha) = \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. Let $\alpha \leq \beta$ in Λ and $x_\beta \in E_\beta$,
 $x_\beta \in \ker(f_{\alpha\beta}) \Leftrightarrow f_{\alpha\beta}(x_\beta) = 0$
 $\Leftrightarrow \mu(f_{\alpha\beta}(x_\beta)) = 0$ for all $\mu \in \Delta(E_\alpha)$
 $\Leftrightarrow x_\beta \in \cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$.

Proposition 2.8. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra. Then E satisfies conditions (i) and (j).

Proof. By Proposition 2.7, $\ker(f_\alpha) = \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. If T is a multiplier of E , then $T(\ker(\chi)) \subset \ker(\chi)$ for all $\chi \in \Delta_\alpha(E)$ by [5, Theorem 2.9], so

$$\begin{aligned} T(\ker(f_\alpha)) &= T(\cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}) \\ &\subset \cap \{T(\ker(\chi)) : \chi \in \Delta_\alpha(E)\} \\ &\subset \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\} = \ker(f_\alpha). \end{aligned}$$

By Proposition 2.7, $\ker(f_{\alpha\beta}) = \cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$ for all $\alpha \leq \beta$ in Λ . If U is a multiplier of E_β , then $U(\ker(\delta)) \subset \ker(\delta)$ for all $\delta \in \Delta(E_\beta)$ by [5, Theorem 2.9], so $U(\ker(\mu \circ f_{\alpha\beta})) \subset \ker(\mu \circ f_{\alpha\beta})$ for all $\mu \in \Delta(E_\alpha)$, and consequently

$$\begin{aligned} U(\ker(f_{\alpha\beta})) &= U(\cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}) \\ &\subset \cap \{U(\ker(\mu \circ f_{\alpha\beta})) : \mu \in \Delta(E_\alpha)\} \\ &\subset \cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\} = \ker(f_{\alpha\beta}). \end{aligned}$$

Theorem 2.9. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete uniform topological algebra. Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.

Proof. It follows from Theorem 2.3 and Proposition 2.8.

Remark. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete uniform topological algebra which is also a symmetric $*$ -algebra, then $(E, (p_\alpha)_{\alpha \in \Lambda})$ is a locally C^* -algebra, and so each factor E_α is complete.

As an application of previous results, we deduce the Arhipainen unitization theorem [1, Theorem 4] on uniform topological algebras.

Proposition 2.10. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E . Then there is a family of seminorms $(q_\alpha)_{\alpha \in \Lambda}$ on $M_c(E)$ such that

1. $(M_c(E), (q_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra;
2. the map $L : E \rightarrow M_c(E)$, $L(x)(y) = xy$, is an algebra isomorphism (into) and $q_\alpha(L(x)) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in \Lambda$.

Proof. 1. By Propositions 2.1, 2.2 and 2.8, we define the map $\psi : M_c(E) \rightarrow \varinjlim M_c(E_\alpha), T \rightarrow (T_\alpha)_{\alpha \in \Lambda}$. As in the proof of Theorem 2.3, ψ is an injective homomorphism. We endow $M_c(E)$ with the topology defined by the family of seminorms $q_\alpha(T) = \|T_\alpha\|_\alpha, \alpha \in \Lambda$, where $\|\cdot\|_\alpha$ is the operator norm on $M_c(E_\alpha)$. Let $T \in M_c(E), q_\alpha(T^2) = \|(T^2)_\alpha\|_\alpha = \|(T_\alpha)^2\|_\alpha = \|T_\alpha\|_\alpha^2 = q_\alpha(T)^2$ since $\|\cdot\|_\alpha$ has the square property. Let $T \in M_c(E)$ with $q_\alpha(T) = 0$ for all $\alpha \in \Lambda$, then $T_\alpha = 0$ for all $\alpha \in \Lambda$, so $T = 0$ since ψ is injective.

2. Since E is proper, L is an algebra isomorphism (into). Let $x \in E$ and $\alpha \in \Lambda, (L(x))_\alpha \circ f_\alpha = f_\alpha \circ L(x)$, then $(L(x))_\alpha(f_\alpha(y)) = (f_\alpha \circ L(x))(y) = f_\alpha(xy) = f_\alpha(x)f_\alpha(y)$ for all $y \in E$. Since the map $l : (E_\alpha, \bar{p}_\alpha) \rightarrow (M_c(E_\alpha), \|\cdot\|_\alpha), l(x_\alpha)(y_\alpha) = x_\alpha y_\alpha$, is an isometric isomorphism (into), it follows that $\|(L(x))_\alpha\|_\alpha = \bar{p}_\alpha(f_\alpha(x)) = p_\alpha(x)$, so $q_\alpha(L(x)) = p_\alpha(x)$.

Proposition 2.11. Let E be a uniform topological algebra without unit, and let E_e be the algebra obtained from E by adjoining the unit. Then the map $g : E_e \rightarrow M_c(E), g((x, \lambda)) = L(x) + \lambda I$ is an algebra isomorphism (into).

Proof. It is easy to show that g is an algebra homomorphism. Let $(x, \lambda) \in E_e$ with $g((x, \lambda)) = 0$, then $L(x) = -\lambda I$. Suppose $\lambda \neq 0, I = -\lambda^{-1}L(x) = L(-\lambda^{-1}x)$, so $-\lambda^{-1}x$ is a left unit in E . Since E is commutative, $-\lambda^{-1}x$ is a unit in E , a contradiction. Thus $L(x) = 0$ and consequently $x = 0$ since E is proper.

Corollary 2.12. [1, Theorem 4] Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra without unit. Then there is a family of seminorms $(s_\alpha)_{\alpha \in \Lambda}$ on E_e such that $(E_e, (s_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra and $s_\alpha((x, 0)) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in \Lambda$.

Proof. For each $\alpha \in \Lambda$, we define a seminorm on E_e by $s_\alpha((x, \lambda)) = q_\alpha(L(x) + \lambda I)$ for all $x \in E$ and $\lambda \in \mathbb{C}$. By Propositions 2.10 and 2.11, $(E_e, (s_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra

and $s_\alpha((x, 0)) = q_\alpha(L(x)) = p_\alpha(x)$ for all $x \in E$.

Remark. We have $s_\alpha((x, \lambda)) = q_\alpha(L(x) + \lambda I) \leq q_\alpha(L(x)) + |\lambda|q_\alpha(I) = p_\alpha(x) + |\lambda|$ for all $x \in E$ and $\lambda \in \mathbb{C}$. This shows that the topology on E_e defined by the family of seminorms $(s_\alpha)_{\alpha \in \Lambda}$ is weaker than the usual topology on E_e defined by the family of seminorms $(\tilde{p}_\alpha)_{\alpha \in \Lambda}$ where $\tilde{p}_\alpha((x, \lambda)) = p_\alpha(x) + |\lambda|$.

References

- [1] J. Arhippainen, On locally convex square algebras, *Funct. Approx.* 22 (1993), 57-63.
- [2] M. Haralampidou, L. Palacios, C. Signoret, Multipliers in locally convex *-algebras, *Rocky Mountain J. Math.*, 43 (2013), 1931-1940.
- [3] M. Haralampidou, L. Palacios, C. Signoret, Multipliers in perfect locally m-convex algebras, *Banach J. Math. Anal.*, 9 (2015), 137-143.
- [4] M. Haralampidou, L. Palacios, C. Signoret, Multipliers in some perfect locally m-pseudoconvex algebras, *Proceedings of the 8th International Conference on Topological Algebras and their Applications*, 2014, Ed. by A. Katz, Series: De Gruyter Proceedings in Mathematics (to appear).
- [5] T. Husain, Multipliers of topological algebras, *Dissertationes Math. (Rozprawy Mat.)*, 285 (1989), 36 pp.
- [6] A. Inoue, Locally C*-algebras, *Mem. Faculty Sci. Kyushu Univ. (Ser. A)*, 25 (1971), 197-235.
- [7] E. A. Michael, Locally multiplicatively τ convex topological algebras, *Mem. Amer. Math. Soc.*, 11 (1952).
- [8] N. C. Phillips, Inverse limits of C*-algebras, *J. Operator Theory*, 19 (1988), 159-195.

Ecole Normale Supérieure
Avenue Oued Akreuch
Takaddoum, BP 5118, Rabat
Morocco

E-mail: mohammed.elazhari@yahoo.fr