



HAL
open science

Optimal input experiment design and parameter estimation in core-scale pressure oscillation experiments

M.G. Potters, M. Mansoori, Xavier Bombois, J.D. Jansen, P.M.J. van den Hof

► To cite this version:

M.G. Potters, M. Mansoori, Xavier Bombois, J.D. Jansen, P.M.J. van den Hof. Optimal input experiment design and parameter estimation in core-scale pressure oscillation experiments. *Journal of Environmental Hydrology*, 2016, 534, pp.534-552. 10.1016/j.jhydrol.2016.01.043 . hal-01355013

HAL Id: hal-01355013

<https://hal.science/hal-01355013>

Submitted on 25 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Optimal Input Experiment Design and Parameter Estimation in Core-Scale Pressure Oscillation Experiments

M.G. Potters^{a,*}, M. Mansoori^b, X. Bombois^c, J.D. Jansen^d, P.M.J. Van den Hof^e

^a*Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands*

^b*Department of Chemical and Petroleum Engineering, Sharif University of Technology, Azadi Ave, Tehran, Iran*

^c*Laboratoire Ampère UMR CNRS 5005, Ecole Centrale de Lyon, 36 Avenue Guy de Collongue, 69134 Ecully Cedex, France*

^d*Faculty of Civil Engineering and Geosciences, Delft University of Technology, Stevinweg 1, 2600 GA Delft, The Netherlands*

^e*Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB, Eindhoven, The Netherlands*

Abstract

This paper considers Pressure Oscillation (PO) experiments for which we find the minimum experiment time that guarantees user-imposed parameter variance upper bounds and honours actuator limits. The parameters permeability and porosity are estimated with a classical least-squares estimation method for which an expression of the covariance matrix of the estimates is calculated. This expression is used to tackle the optimization problem. We study the Dynamic Darcy Cell experiment set-up [1] and focus on data generation using square wave actuator signals, which, as we shall prove, deliver shorter experiment times than sinusoidal ones. Parameter identification is achieved using either inlet pressure/outlet pressure measurements [1] or actuator position/outlet pressure measurements, where the latter is a novel approach. The solution to the optimization problem reveals that for both measurement methods an optimal excitation frequency, an optimal inlet volume, and an optimal outlet volume exist. We find that under the same parameter variance bounds and actuator

*Corresponding author, m.g.potters@tudelft.nl

constraints, actuator position/outlet pressure measurements result in required experiment times that are a factor fourteen smaller compared to inlet pressure/outlet pressure measurements. This result is analysed in detail and we find that the dominant effect driving this difference originates from an identifiability problem when using inlet-outlet pressure measurements for joint estimation of permeability and porosity. We illustrate our results with numerical simulations, and show excellent agreement with theoretical expectations.

Keywords: Experiment Design, Variance Constraints, Estimation, Porous Media

1. Introduction

Two key parameters influencing fluid flow in a porous medium are permeability (i.e. inverse resistance) and porosity (i.e. storage capacity). These parameters are important to characterise fluid flow in underground water resources [2], contaminated water disposal in underground storages [3], and subsurface hydrocarbon reservoirs [4]. Indeed, permeability and porosity estimates are used to initialise reservoir simulations, optimise the number of wells and their locations, and drilling and completion procedures.

At the core scale, estimation of both parameters *locally* may be carried out by performing an experiment on a cylindrically-shaped core sample of the porous medium, using either steady-state (SS), unsteady-state (USS), or pressure oscillation (PO) measurements. In an SS experiment a constant pressure difference is applied across the axis of the core sample and subsequently the flow rate is measured after the SS condition has been established. Permeability is then estimated based on the relationship between the flow rate and the pressure drop. In an USS experiment an impulse or step pressure change is applied at the upstream side of the sample while the pressure change is recorded downstream. The observed response is then analysed either graphically or numerically to estimate either permeability or porosity. Similarly, in a PO experiment, the recorded downstream pressure response is analyzed for parameter estimation -

21 the difference being that an oscillatory pressure signal is applied upstream. The
22 attenuation and phase shift between the up- and downstream signals are then
23 translated into parameter estimates [5, 1]. The oscillatory signal is usually a
24 single sinusoid with a frequency and amplitude specified by the experimenter.
25 The amplitude of the upstream signal is however bounded by the limits of the
26 actuator. In cases where a rather high actuator frequency is necessary to take
27 into account geometrical and physical properties of the sample, [6] suggested the
28 use of input signals with complex shapes including the required high frequencies.

29 The consensus in the literature is that a PO experiment has several advanta-
30 geous properties not shared by its SS and USS counterparts, e.g., less experiment
31 time, less stress on the core sample, and the possibility of simultaneously esti-
32 mating permeability and porosity [7, 3]. The effectiveness of PO experiments
33 for the estimation of permeability has been demonstrated in different set-ups
34 [1, 4, 8, 9, 6]. Despite its advantageous properties, however, it is observed that
35 measurements can result in large uncertainties in the estimates, particularly for
36 porosity [7, 3, 4]. Porosity estimates with an uncertainty exceeding an order of
37 magnitude, or that have negative values, have been reported [3, 7]. (Negative
38 values can however be easily circumvented by using log-transformed parame-
39 ters). One cause is measurement noise, but in this paper we show that other
40 ones also play an important role.

41 Furthermore, it is important to be able to reduce the experiment time with-
42 out loss of accuracy. In such a case, more core samples can be analysed in a given
43 time, which consequently reduces the experiment costs. Analogously, given a
44 maximum experiment time, it is important to get the best possible estimates.

45 Clearly, the challenge of estimating permeability and porosity with high ac-
46 curacy remains, especially in evaluating the production potential of tight forma-
47 tions in unconventional hydrocarbon reservoirs [4] or the sealing characteristics
48 of the cap rock in underground storage [3].

49

50 Motivated by the above problems we raise the question whether we can, for
51 a PO experiment, design the applied upstream pressure signal and utilise the

52 degrees of freedom (DOF) in the experiment set-up in order to increase parame-
53 ter accuracies. The dependence of the accuracy of the estimates on the selected
54 driving frequency has been first pointed out in [10], although no investigation
55 into this topic was pursued. From this question, we define the following op-
56 timization problem: find the minimal experiment time required to guarantee
57 user-imposed variance constraints on the estimates by utilising DOF in the ex-
58 periment set-up as well as designing the to-be-applied upstream pressure signal,
59 ensuring that this signal has an amplitude that honours the actuator limits.
60 Note that the solution can also be used to maximise the accuracy of the esti-
61 mates for a given experiment length. To address this optimization problem we
62 use techniques from Experiment Design.

63

64 Experiment Design addresses the long-standing issue of the lack of accurate
65 parameter estimates inferred from collected data, particularly at the catchment
66 scale. This issue is widely recognised; see for instance [11, 12, 13, 14, 15] and
67 the nice review of [16]. Some of the earliest works [17, 18, 19] in Experiment
68 Design (although not recognised under this name at that time) showed that
69 concepts such as *parameter correlation*, *identifiability*, *observability*, and exper-
70 iment length strongly affect the quality of the parameter estimates (i.e. their
71 variances). These works and those of [20, 21] were some of the first to quan-
72 titatively evaluate parameter uncertainty within a Bayesian framework. They
73 provided measures to find the best possible calibration data for computer mod-
74 els, using *a-posteriori* data, i.e. data from an experiment that had already taken
75 place. Other works [14, 22] analysed the role of tracer observations that influ-
76 ence parameter identifiability, and identifiability of unknown pollution sources.
77 The works [23, 24, 25] were the first to consider optimal experiment design for
78 groundwater hydrology *prior* to the actual inference experiment; they mainly
79 searched for optimal pumping and observation wells, keeping the pumping rates
80 constant, such that the experiment cost could be minimized subject to maxi-
81 mizing the overall accuracy in the parameters (using a D-optimality criterion).
82 More recently, a Bayesian methodology [26] was developed to find the optimal

83 investigation strategy, or sampling pattern, prior to the actual experimental
84 campaign.

85 We will take a non-Bayesian approach from linear systems theory [27] and
86 apply it to the core-scale PO experiment introduced above. The method is
87 different to the Bayesian methods in the sense that an optimal spectrum of
88 the input signal is calculated prior to the actual experiment, whereas in the
89 above methods the spectra of the inputs are not design variables. This optimal
90 spectrum reveals e.g. the time scales that are important for accurate parameter
91 estimation. We also consider variance constraints on the individual parameters,
92 which is particularly important to use for systems that have low sensitivities for
93 some parameters (in which case the D-optimality criterion, as used by e.g. [24],
94 can be ill-chosen).

95 In this paper, we tackle the experiment design problem as follows. We
96 perform parameter estimation using ordinary least squares using the measured
97 noise-corrupted downstream pressure signal [28, 29]. This signal is deduced from
98 the governing equations and boundary conditions, and depends on the applied
99 upstream signal. One benefit of this method is that it can deal in a rather
100 easy manner with (coloured) measurement noise; see [28] for details. A second
101 benefit is that a frequency-domain expression of the covariance matrix of the es-
102 timates can be formulated. This expression, which we introduce in Section 3, is
103 a function of the power spectrum of the applied signal and the DOF of the exper-
104 iment set-up. Consequently, we can formulate the above optimization problem
105 (of minimising the experiment time subject to parameter variance constraints
106 and actuator bounds by designing the optimal input signal and DOF of the
107 set-up) mathematically. We shall limit ourselves to sinusoidal and square-wave
108 actuator signals. The latter is easy to generate by rapidly switching between
109 two actuator levels, which can be done with current vibration exciters [1]. Other
110 reasons for this choice are explained in Section 5.

111

112 We apply our method to the Dynamic Darcy Cell experiment set-up, as de-
113 tailed in [1], but we stress that our methodology can be applied to many other

114 set-ups as well. The DOF in the Dynamic Darcy Cell set-up are the inlet volume
 115 and outlet volume. We introduce the Dynamic Darcy Cell in Section 2 and show
 116 how to apply sinusoidal and square wave signals to the set-up. Two types of
 117 measurements are then introduced: inlet pressure/outlet pressure measurements
 118 (Direct Method) and actuator position/outlet pressure measurements (Indirect
 119 Method). The former is one of the current ways to estimate parameters [1], in
 120 particular using sinusoidal signals. The latter has, to the authors' knowledge,
 121 not been investigated before. For both cases, we focus on square wave input
 122 signals, for which we prove that shorter experiment lengths than for sinusoidal
 123 ones can be obtained. We explain the data collection and estimation procedure
 124 in Section 3, and give an expression for the covariance matrix of the parameter
 125 estimates. In Sections 4 and 5 we use this expression to compute the opti-
 126 mal sinusoidal and square wave signals and DOF that minimize the experiment
 127 time for the estimation of permeability and porosity for the Direct and Indirect
 128 Methods. In the absence of a physical set-up, we illustrate the experiment de-
 129 sign results by simulating the noise-corrupted physical system and applying the
 130 optimal square wave signal in Section 6. Lastly, we draw conclusions in Section
 131 7.

132 2. The Dynamic Darcy Cell

To investigate the estimation problem the Dynamic Darcy Cell in [1] is con-
 sidered. Detailed and schematic layouts of this set-up are depicted in Figs. 1 and
 2. The porous core sample is mounted vertically in a core holder under a specific
 confined pressure p_{init} , with the inlet at the bottom (upstream side) and outlet
 at the top (downstream side). The sample with length L and cross-sectional
 area A is sealed at the top from the environment. The spatial coordinate axis x
 is oriented towards the outlet and is perpendicular to the cross-sectional plane.
 The bottom of the sample is located at $x = 0$. A vibration exciter with equi-
 librium position $x = -L_i$, also called the actuator, is moved according to the
 user-defined oscillatory signal $r(t)$. This signal is assumed to be continuous in

time until Section 3, where instead discrete-time signals for estimation of the parameters are used. The actuator position r as a function of time t is either a sinusoid, i.e.,

$$r(t) = C_\gamma \sin(\omega_{ex}t), \quad (1)$$

or a square wave,

$$r(t) = \frac{4}{\pi} C_\gamma \sum_{m=1}^{\infty} \frac{\sin([2m-1]\omega_c t)}{2m-1}. \quad (2)$$

In these expressions, ω_{ex} and ω_c in respectively (1), (2) represent the excitation frequency of the sinusoid and cycle frequency (slowest mode) of the square wave. Furthermore, C_γ is the amplitude of the actuator signal defined by

$$C_\gamma = \gamma L_i, \quad 0 < \gamma \leq \gamma_m. \quad (3)$$

133 Obviously, the actuator amplitude can not exceed the length L_i as can be seen
 134 from Fig. 2, so the user-chosen factor γ has a geometric constraint $\gamma \leq \gamma_m = 1$.
 135 However, the actuator amplitude may not necessarily be limited by the geometry
 136 of the setup but by its own movement restrictions, yielding a γ_m that is smaller
 137 than unity.

The square wave signal can be generated in practice by switching abruptly between $+C_\gamma$ and $-C_\gamma$. The actuator induces an upstream (inlet) pressure signal at position $x = 0$, i.e.,

$$p_i(t) = p(x = 0, t), \quad (4)$$

within the inlet volume V_i that is connected to the sample. This signal is measured by pressure transducer 1 (see Fig. 1) and assumed to be corrupted by sensor noise. At the sample outlet, which is connected to the end volume V_e , the output (downstream) pressure

$$p_e(t) = y_{nf}(t) \quad (5)$$

138 is measured by pressure transducer 2 at $x = L$, see Fig. 1. Here, $y_{nf}(t) =$
 139 $p_e(t) = p(x = L, t)$ is the noise-free pressure response of the core sample. The

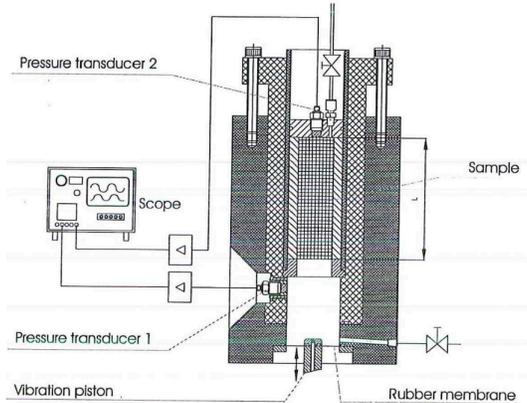


Figure 1: The Dynamic Darcy Cell. The core sample is mounted vertically in the holder. The actuator induces a pressure signal at inlet, measured by pressure transducer 1. The wave propagates through the porous core and is measured at the outlet by pressure transducer 2. Courtesy of [1].

140 input and output are continuous time signals. In Section 3 we will use sampled
 141 input and output data for parameter estimation. The measurements taken by
 142 this pressure transducer are also corrupted by sensor noise. We shall return to
 143 this point when we explain the estimation method in Section 3. The inlet and
 144 outlet volumes V_i and V_e can be set by the experimenter and are part of the
 145 DOF of the experiment set-up. (The set-up can for instance be designed to have
 146 easily-adaptable inlet and outlet volumes). Prior to the experiment the core,
 147 the inlet and outlet volume are filled with a gas (e.g., air) at pressure p_{init} .

148 2.1. System of Coupled Equations in the Time Domain

149 In the set-up we consider a porous cylindrical core that is homogeneous
 150 and isotropic with constant porosity ϕ and permeability k , and assume that the
 151 compressibility of the medium is small compared to the compressibility of the gas
 152 and therefore considered constant. Usually, both ϕ and k have to be identified.
 153 The core is defined in Table 1. Gravity forces and pressure dependencies of the
 154 viscosity μ are neglected and isothermal conditions are assumed.

155 We now derive the dynamics within the experiment set-up. We shall consider

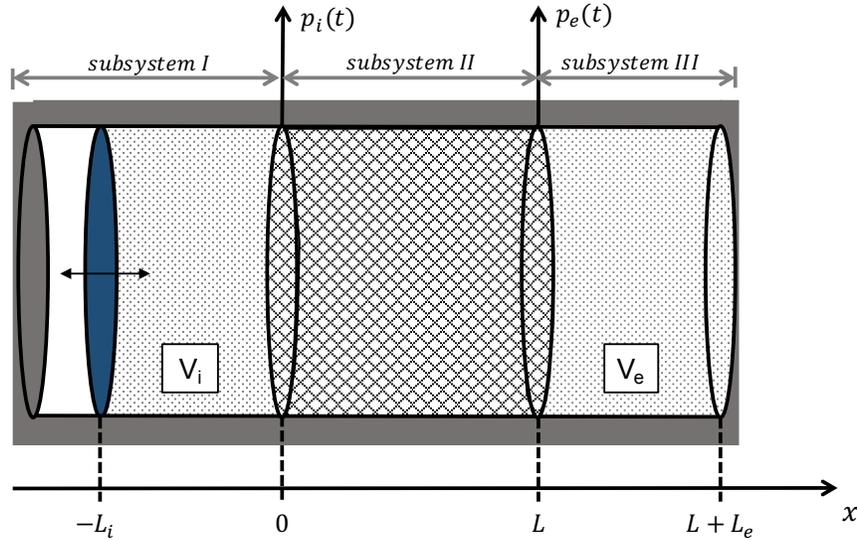


Figure 2: A sketch of the Dynamic Darcy Cell, rotated ninety degrees clock-wise. The cylindrical core sample is depicted in between the inlet volume V_i and outlet volume V_e . The sample has a cross-sectional area A and length L . The signals $p_i(t)$ and $p_e(t)$ are measured by pressure transducer 1 and 2 respectively, see Fig. 1. The actuator is visualised by the blue disk that can oscillate around its equilibrium position $x = -L_i$.

Quantity	Value
Length L	0.0512 m
Cross-sectional area A	$\pi 0.0375^2/4 \text{ m}^2$
Permeability k_0	$2.0 \times 10^{-13} \text{ m}^2$
Porosity ϕ_0	0.2
Fluid density ρ	1.225 kg m^3
Dynamic Viscosity μ	$1.84 \times 10^{-5} \text{ Pa s}$
Bulk modulus K	$1.0 \times 10^5 \text{ Pa}$

Table 1: Parameters defining the Dynamic Darcy Cell and the core sample. These values define a core sample used in an actual pressure oscillation experiment in ([1]). The parameters k_0 and ϕ_0 need to be estimated.

156 the inlet volume, the porous medium, and outlet volume as three subsystems
 157 with their own dynamics, input and output, and boundary conditions. The
 158 subsystems are interconnected at the inlet ($x = 0$) and outlet ($x = L$) locations.
 159 We shall use the dynamics to derive the relationship between the inlet pressure
 160 to outlet pressure signals, and the actuator position to outlet pressure signals
 161 in Section 2.3.

162 *Subsystem I: The Inlet Volume*

163 The actuator is a piston that vibrates in the x -direction within a confined gas
 164 in the inlet volume $V_i = AL_i$, where L_i is the length of the inlet. The dynamics
 165 of the gas volume as a consequence of the moving piston is governed by the
 166 momentum and mass conservation laws, which for this particular configuration
 167 read

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad (6)$$

$$\frac{\partial \rho}{\partial t} + w \frac{\partial \rho}{\partial x} + \rho \frac{\partial w}{\partial x} = 0, \quad (7)$$

168 where $w = w(x, t)$ is the velocity profile of the gas in the x -direction (we use
 169 air as gas, see Table 1), ρ the density of the gas, and $p = p(x, t)$ the pressure
 170 profile. We assume that the actuator has a small amplitude. In this case we
 171 may linearise (6)-(7) around the pressure p_{init} . The factor γ_m in (3) is thus
 172 either defined by limitations in the actuator movement or by the linearisation
 173 condition, whichever is most restrictive. Furthermore, we assume isothermal
 174 conditions and therefore replace $\frac{\partial \rho}{\partial t}$ with $\frac{1}{K} \frac{\partial p}{\partial t}$, where $K = \rho \frac{\partial p}{\partial \rho}$ the Bulk mod-
 175 ulus. Equations (6)-(7) then reduce to

$$\frac{\partial w}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad (8)$$

$$\frac{1}{K} \frac{\partial p}{\partial t} + \frac{\partial w}{\partial x} = 0, \quad (9)$$

176 where we furthermore neglected the nonlinear terms $w \frac{\partial w}{\partial x}$ and $w \frac{\partial \rho}{\partial x}$. Lastly, we
 177 take the derivative of (8) with respect to x and the derivative of (9) with respect

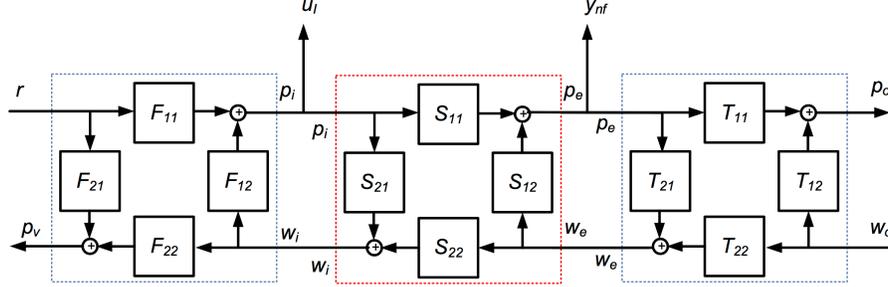


Figure 3: The network representation of the experimental setup. The dynamics inside the subsystems I and III are contained in respectively the left and right blue-dotted squares, whereas the dynamics in subsystem II is contained in the centre, red-dotted square.

178 to t and combine the two, resulting in

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho}{K} \frac{\partial^2 p}{\partial t^2}, \quad (10)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x}. \quad (11)$$

The boundary conditions, i.e. inputs, of this subsystem are given by

$$w(x = -L_i, t) = \frac{\partial r(t)}{\partial t}, \quad w(x = 0, t) = w_i(t), \quad (12)$$

where $r(t)$ is the movement of the actuator as a function of time, and $w_i(t)$ the inlet gas speed. We neglect friction effects between the wall of the inlet volume and the actuator. The outputs of this subsystem are

$$p(x = -L_i, t) = p_v(t), \quad p(x = 0, t) = p_i(t). \quad (13)$$

179 This subsystem is shown in Fig. 3 in the left blue-dotted square, which reveals
 180 the coupling of all dynamics in the Laplace domain. Notice that it is connected
 181 to subsystem II through the output $p_i(t)$, being the inlet pressure, and the inlet
 182 gas speed $w_i(t)$. We now continue with the dynamics in the porous medium.

183 *Subsystem II: The Porous Medium*

For the second system, i.e. the porous medium, mass conservation and Darcy's law show that the pressure change in the core is governed by the diffusion

equation

$$\frac{\partial p(x, t)}{\partial t} = \frac{kK}{\mu\phi} \frac{\partial^2 p(x, t)}{\partial x^2}. \quad (14)$$

The specific discharge, or Darcy velocity, is related to pressure by

$$w(x, t) = -\frac{k}{\mu} \frac{\partial p(x, t)}{\partial x}. \quad (15)$$

Note that this velocity is different to those in the inlet and outlet volumes (subsystems I and III). The boundary conditions, i.e. inputs, for this subsystem read

$$p(x = 0, t) = p_i(t), \quad w(x = L, t) = w_e(t), \quad (16)$$

in which $p_i(t)$ is the inlet pressure defined previously, and $w_e(t)$ the outlet gas speed. The outputs are defined by

$$w(x = 0, t) = w_i(t), \quad p(x = L, t) = p_e(t). \quad (17)$$

184 This subsystem is shown in Fig. 3 in the centre, red-dotted square. Note that
 185 the first and second subsystem are now connected. What remains is to connect
 186 subsystem II with subsystem III. We shall now derive the dynamics in the latter.

187 *Subsystem III: The Outlet Volume*

188 In the third subsystem, following the same reasoning as for the first subsys-
 189 tem, the dynamics are determined by the conservation of mass and momentum.
 190 The pressure profile follows the wave equation and reads (c.f. (10))

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho}{K} \frac{\partial^2 p}{\partial t^2}, \quad (18)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x}. \quad (19)$$

The boundary conditions are however different from subsystem I and are given by

$$p(x = L, t) = p_e(t), \quad w(x = L + L_e, t) = w_o(t), \quad (20)$$

in which $w_o(t)$ is an as-of-yet unspecified input for subsystem III. The outputs are defined by

$$w(x = L, t) = w_e(t), \quad p(x = L + L_e, t) = p_o(t), \quad (21)$$

191 where $w_e(t)$ is the gas speed at the outlet and $p_o(t)$ the pressure at the boundary
192 $x = L + L_e$.

193

194 At this point, we have defined all dynamics in the experiment set-up. The
195 three subsystems are now connected through their boundary conditions and
196 outputs. Before proceeding with introducing the scaled set-up, we reflect on the
197 assumptions made in the derivation of the governing equations in subsystems I
198 until III.

199 *Range of Applicability*

200 We have assumed that (a) the flow is essentially laminar and inviscid in the
201 inlet and outlet volumes, (b) that the flow in the core sample is laminar and
202 non-inertial (Darcy-dominated), and (c) that temperature changes (and the
203 associated changes in viscosity and density) of the working fluid are minimal,
204 so that isothermal conditions can be assumed.

205 One way to test whether or not the flow is laminar in the inlet and outlet
206 volume is to calculate the Reynolds number $\text{Re}_{I,O} = \frac{\rho v L}{\mu}$, in which v is the
207 maximum flow velocity with units [m/s], ρ the density of the fluid in [kg m⁻³],
208 L the characteristic travelled length of the fluid in [m], and μ the dynamic vis-
209 cosity in [Pa s]. The subscript I, O indicates that this definition of the Reynolds
210 number pertains to the inlet and outlet volumes. If $\text{Re}_{I,O} \leq 1$ the flow may be
211 considered laminar. To validate assumption (b), one can utilise the following
212 definition of the Reynolds number for porous media ([30]): $\text{Re} = \frac{\rho w d}{\mu}$, where ρ
213 is again the density of the fluid, w the specific discharge, and d the typical grain
214 diameter (which is roughly equivalent to the square root of permeability). If
215 $\text{Re} < 10$ then the flow in the core is Darcy-dominated. An alternative method is
216 to consider the Darcy-Forchheimer equation and compare the effect of the Darcy
217 term with the inertial term; see ([31, 32]). Anticipating low-frequency signals
218 and the fact that usually water or air is used as gas in a Darcy experiment, (c)
219 may be assumed to hold. For the core sample and excitation signals used in this
220 work, the Reynolds numbers are $\text{Re}_{I,O} \approx 0.8$ and $\text{Re} \approx 0.03$, and thus within

221 the accepted range.

222

223 We proceed by introducing the scaled set-up. The scaled equations are then
224 used to find the dynamic relationship between the actuator, inlet pressure, and
225 outlet pressure.

226 2.2. Scaling

We are interested to identify the porosity and the permeability of the porous medium. As shown in Table 1, the values of these parameters differ by twelve orders of magnitude. It is thus very important for the sake of numerical computations to scale the dynamical equations in such a way that the to-be-identified parameters are both of $O(1)$. To this end, we first choose k_s and ϕ_s equal to the order of magnitude of k and ϕ (which are usually known, either from prior experiment data, or for instance from the Rosetta Soil database from the US Salinity Lab [33]) and define the dimensionless parameters

$$\tilde{k} = \frac{k}{k_s}, \quad \tilde{\phi} = \frac{\phi}{\phi_s}. \quad (22)$$

227 Furthermore, we also define the following dimensionless variables

$$\tilde{x} = \frac{x}{L}, \quad \tilde{p}(\tilde{x}, \tilde{t}) = \frac{p(x/x_s, t/t_s)}{p_s}, \quad \tilde{t} = \frac{t}{t_s}, \quad \tilde{w}(\tilde{x}, \tilde{t}) = \frac{w(x/x_s, t/t_s)}{w_s}. \quad (23)$$

228 We note that the actuator signal $r(t)$ should be scaled in time and space accord-
229 ing to the above definitions, resulting in $\tilde{r}(\tilde{t}) = \frac{r(t/t_s)}{L}$. The scaled frequency
230 $\tilde{\omega}$ follows from the above-defined time scaling and is given by $\tilde{\omega} = \omega t_s$. Lastly,
231 choosing

$$p_s = \frac{K}{\phi_s}, \quad t_s = \frac{\phi_s \mu L^2}{k_s K}, \quad w_s = \frac{k_s p_s}{\mu L}, \quad (24)$$

232 leads to equations with $O(1)$ terms when substituting (22)-(24) into the equa-
 233 tions of subsystem I to III (10)-(18). The scaled system is thus given by

$$\frac{\partial^2 \tilde{p}(\tilde{x}, \tilde{t})}{\partial \tilde{x}^2} = \frac{\rho L^2}{K} t_s^{-2} \frac{\partial^2 \tilde{p}(\tilde{x}, \tilde{t})}{\partial \tilde{t}^2}, \quad (25)$$

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} = -\frac{t_s \rho_s}{\rho w_s L} \frac{\partial \tilde{p}}{\partial \tilde{x}}, \quad (26)$$

$$\frac{\partial \tilde{p}(\tilde{x}, \tilde{t})}{\partial \tilde{t}} = \frac{\tilde{k}}{\tilde{\phi}} \frac{\partial^2 \tilde{p}(\tilde{x}, \tilde{t})}{\partial \tilde{x}^2}, \quad (27)$$

$$\tilde{w}(\tilde{x}, \tilde{t}) = -\tilde{k} \frac{\partial \tilde{p}(\tilde{x}, \tilde{t})}{\partial \tilde{x}}, \quad (28)$$

234 with the scaled boundary conditions (12), (16), (20)

$$\tilde{w}(\tilde{x} = -\frac{L_i}{L}, \tilde{t}) = \frac{\partial \tilde{r}(\tilde{t})}{\partial \tilde{t}}, \quad \tilde{w}(\tilde{x} = 0, \tilde{t}) = \tilde{w}_i(\tilde{t}), \quad (29)$$

$$\tilde{p}(\tilde{x} = 0, \tilde{t}) = \tilde{p}_i(\tilde{t}), \quad \tilde{w}(\tilde{x} = 1, \tilde{t}) = \tilde{w}_e(\tilde{t}), \quad (30)$$

$$\tilde{p}(\tilde{x} = 1, \tilde{t}) = \tilde{p}_e(\tilde{t}), \quad \tilde{w}(\tilde{x} = 1 + \frac{L_e}{L}, \tilde{t}) = \tilde{w}_o(\tilde{t}), \quad (31)$$

235 and outputs (13), (17), (21)

$$\tilde{p}(\tilde{x} = -\frac{L_i}{L}, \tilde{t}) = \tilde{p}_v(\tilde{t}), \quad \tilde{p}(\tilde{x} = 0, \tilde{t}) = \tilde{p}_i(\tilde{t}) \quad (32)$$

$$\tilde{w}(\tilde{x} = 0, \tilde{t}) = \tilde{w}_i(\tilde{t}), \quad \tilde{p}(\tilde{x} = 1, \tilde{t}) = \tilde{p}_e(\tilde{t}), \quad (33)$$

$$\tilde{w}(\tilde{x} = 1, \tilde{t}) = \tilde{w}_e(\tilde{t}), \quad \tilde{p}(\tilde{x} = 1 + \frac{L_e}{L}, \tilde{t}) = \tilde{p}_o(\tilde{t}). \quad (34)$$

236 From now on we shall work with the scaled system and omit the tildes for the
 237 sake of simplicity. The unscaled results can easily be deduced using the scaling
 238 defined in this section. We proceed with the determination of the dynamic
 239 relationships between the actuator and the inlet and outlet pressures.

240 2.3. System of Coupled Equations in the Laplace Domain

241 For the estimation of the physical parameters we require explicit expressions
 242 of the transfer functions between the actuator and the outlet pressure, as well
 243 as the inlet to outlet pressure. For this purpose, we transform the equations for
 244 each system into the Laplace domain.

The dynamics in subsystem I are defined by (10)-(13). Their scaled equivalents are given by (25)-(26), (29), and (32). We take the Laplace transform of (25) and assume zero initial conditions. The resulting equation reads

$$\frac{\partial^2 P(x, s)}{\partial x^2} = \frac{\rho L^2 s^2}{K t_s^2} P(x, s), \quad (35)$$

where $P(x, s)$ is the Laplace transform of the scaled pressure profile $p(x, t)$. This equation has the general solution

$$P(x, s) = m_1 \sinh \left(x \frac{L}{t_s} \sqrt{\frac{\rho}{K}} s \right) + m_2 \cosh \left(x \frac{L}{t_s} \sqrt{\frac{\rho}{K}} s \right), \quad (36)$$

in which m_1 and m_2 are two unknown scalars. We solve the coefficients m_1, m_2 with the equation for the gas speed (26), which we transform into the Laplace domain (assuming zero initial conditions), giving

$$W(x, s) = -\frac{t_s p_s}{\rho w_s L s} \frac{\partial P(x, s)}{\partial x}. \quad (37)$$

Substitution of (36) into (37) then yields

$$W(x, s) = -\frac{p_s}{w_s \sqrt{\rho K}} \left\{ m_1 \cosh \left(x \frac{L}{t_s} \sqrt{\frac{\rho}{K}} s \right) + m_2 \sinh \left(x \frac{L}{t_s} \sqrt{\frac{\rho}{K}} s \right) \right\}. \quad (38)$$

Using the Laplace transformed boundary conditions (29) then leads to

$$\begin{aligned} m_1 &= -\frac{w_s}{p_s} \sqrt{\rho K} W_i(s), \\ m_2 &= -\frac{w_s}{p_s} \sqrt{\rho K} \left\{ \sinh^{-1} \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) s R(s) - \coth \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) W_i(s) \right\}. \end{aligned} \quad (39)$$

Substitution of these expressions into (36) and using the Laplace transformed outputs (32) allows us to determine the dynamical relationship between $R(s)$, $W_e(s)$ and $P_v(s), P_i(s)$:

$$\begin{bmatrix} P_i(s) \\ P_v(s) \end{bmatrix} = \frac{w_s}{p_s} \sqrt{\rho K} \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \begin{bmatrix} R(s) \\ W_i(s) \end{bmatrix}, \quad (41)$$

in which

$$\begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = \begin{bmatrix} -s \sinh^{-1} \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) & \coth \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) \\ -s \coth \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) & \sinh^{-1} \left(-s \frac{L_i}{t_s} \sqrt{\frac{\rho}{K}} \right) \end{bmatrix}. \quad (42)$$

The argument in each of the above hyperbolic functions contains the fraction $\frac{L_i\sqrt{\rho}}{t_s\sqrt{K}} = \frac{k_s L_i\sqrt{\rho K}}{L^2\mu\phi_s}$. Since this expression evaluates to a small number for typical values of each parameter, we approximate these hyperbolic functions with $\coth(z) \approx \frac{1}{z}$ and $\sinh(z) \approx z$ where $z \ll 1$. The transfer functions then reduce to the expressions

$$\begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = \frac{\phi_s L}{L_i s} \begin{bmatrix} s & -1 \\ s & -1 \end{bmatrix} = \frac{f_i}{s} \begin{bmatrix} s & -1 \\ s & -1 \end{bmatrix}, \quad (43)$$

where f_i is defined as

$$f_i = \frac{\phi_s AL}{V_i} = \frac{\phi_s L}{L_i}. \quad (44)$$

247 *Subsystem II*

We recall that the dynamics in subsystem II are defined by the equations (14)-(17). Their scaled equivalents are given by (27)-(28), (30), and (33). We apply the Laplace transform to the diffusion equation (27), again assuming zero initial conditions, and write it as

$$\left(\frac{\partial^2}{\partial x^2} - \frac{s}{k/\phi} \right) P(x, s) = 0, \quad (45)$$

where $P(x, s)$ is the Laplace transformed scaled pressure. The general solution to this equation reads

$$P(x, s) = c_1 e^{x\sqrt{\frac{s\phi}{k}}} + c_2 e^{-x\sqrt{\frac{s\phi}{k}}}, \quad (46)$$

where c_1 and c_2 are as-of-yet unknown scalar coefficients. Taking the Laplace transform of (28) and using the previous equation shows that the speed of gas in the porous medium is given by

$$W(x, s) = -kc_1 \sqrt{\frac{s\phi}{k}} e^{x\sqrt{\frac{s\phi}{k}}} + kc_2 \sqrt{\frac{s\phi}{k}} e^{-x\sqrt{\frac{s\phi}{k}}}. \quad (47)$$

The unknown coefficients c_1 and c_2 are then determined with the Laplace transformed boundary conditions (30). We then substitute these into (46) and use (33) to find the dynamical relationships between $P_e(s)$, $W_i(s)$ and $P_i(s)$, $W_e(s)$:

$$\begin{bmatrix} P_e(s) \\ W_i(s) \end{bmatrix} = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \begin{bmatrix} P_i(s) \\ W_e(s) \end{bmatrix}, \quad (48)$$

in which

$$\begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} = \begin{bmatrix} \cosh^{-1}\left(\sqrt{s\phi/k}\right) & \frac{-1}{\sqrt{s\phi k}} \tanh\left(\sqrt{s\phi/k}\right) \\ \sqrt{s\phi k} \tanh\left(\sqrt{s\phi/k}\right) & \cosh^{-1}\left(\sqrt{s\phi/k}\right) \end{bmatrix}. \quad (49)$$

248 Note that the argument in all hyperbolic functions are of $O(1)$ or higher. We
249 therefore do not approximate these functions as we did for subsystem I.

250 *Subsystem III*

The dynamics in subsystem III are defined by the equations (18)-(21). Their scaled equivalents are given by (25)-(26), (31), and (34). We follow the exact same derivation as for subsystem I, albeit with the boundary conditions and outputs in (31) and (34). The result for subsystem III is that

$$\begin{bmatrix} P_o(s) \\ W_e(s) \end{bmatrix} = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} \begin{bmatrix} P_e(s) \\ W_o(s) \end{bmatrix} \quad (50)$$

in which

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} = \begin{bmatrix} \cosh^{-1}\left(\frac{L_e}{t_s} \sqrt{\frac{\rho}{K}} s\right) & -\frac{w_s}{p_s} \sqrt{\rho K} \tanh\left(\frac{L_e}{t_s} \sqrt{\frac{\rho}{K}} s\right) \\ \frac{p_s}{w_s \sqrt{\rho K}} \tanh\left(\frac{L_e}{t_s} \sqrt{\frac{\rho}{K}} s\right) & \cosh^{-1}\left(\frac{L_e}{t_s} \sqrt{\frac{\rho}{K}} s\right) \end{bmatrix}. \quad (51)$$

Following the derivation for subsystem I we remark that the arguments in all the hyperbolic functions are small. We approximate these functions around zero and find that we may write

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{k_s^2 K \rho}{\mu^2 L^2} \frac{s}{f_e} \\ \frac{s}{f_e} & 1 \end{bmatrix}, \quad (52)$$

where f_e is defined as

$$f_e = \frac{\phi_s AL}{V_e} = \frac{\phi_s L}{L_e}. \quad (53)$$

251 *The Experiment Set-Up System*

252 At this point we have coupled the different inputs and outputs of the total
253 system. In the Dynamic Darcy Cell experiment set-up, however, we do not have
254 gas entering the system at $x = 1 + \frac{L_e}{L}$ and thus $w_o(t) = 0$. Consequently, we

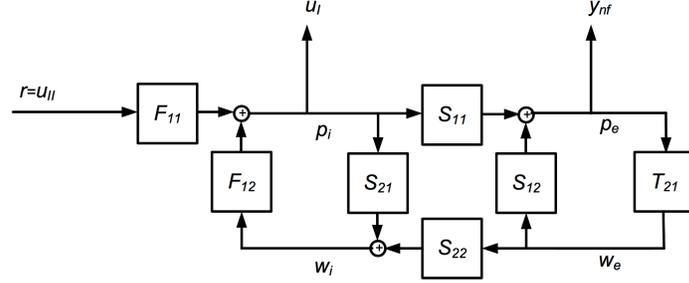


Figure 4: The network representation of the experimental setup after simplification.

255 can remove the transfer functions $T_{12}(s)$ and $T_{22}(s)$ from the network, see Fig.
 256 3. In addition, the transfer functions $F_{21}(s)$, $F_{22}(s)$ may be removed as they do
 257 not influence $p_i(t)$, $r(t)$, nor $p_e(t)$. Lastly, $T_{11}(s)$ can be discarded as it does not
 258 influence $p_e(t)$. The reduced network is depicted in Fig. 4. From this figure, we
 259 see the output signal $y_{nf}(t) = p_e(t)$ that we shall use to estimate permeability
 260 and porosity. For the two experiment types we consider (see Introduction),
 261 observe that $r(t) = u_{II}(t)$ is the input in the Indirect Method, and $u_I(t) = p_i(t)$
 262 the input in the Direct Method.

263 For the least-squares procedure that we shall introduce in Section 3 we
 264 require the transfer function between $r(t)$ and $y_{nf}(t) = p_e(t)$ and the transfer
 265 function between $p_i(t)$ and $y_{nf}(t) = p_e(t)$. We shall derive these transfer func-
 266 tions now.

267

We start with the relationship between P_i and P_e , which we shall use to find
 the time-domain connection between $p_e(t)$ and $p_i(t)$. We find from Fig. 4 that

$$P_e(s) = \frac{S_{11}(s)}{1 - S_{12}(s)T_{21}(s)} P_i(s). \quad (54)$$

268 This equation shows that the outlet pressure depends on both the inlet pressure
 269 P_i as well as itself through W_e . This feedback mechanism is clearly seen in Fig.
 270 4.

Next, the relationship between $R(s)$ and $P_i(s)$ can be deduced from (43),

(48)-(49), and (50) or Fig. 4 and reads

$$P_i(s) = F_{11}(s) \left[1 - F_{12}(s) \left(S_{21}(s) + S_{22}(s)T_{21}(s) \frac{S_{11}(s)}{1 - S_{12}(s)T_{21}(s)} \right) \right]^{-1} R(s). \quad (55)$$

271 Equations (54) and (55) reveal the transfer functions between the Laplace-
 272 transformed measurement variables $r(t)$, $p_i(t)$, and $p_e(t)$. In the next section we
 273 show how to calculate the time-domain response of $p_i(t)$ and $p_e(t)$ as a function
 274 of the actuator signal $r(t)$ using these transfer functions. These responses are a
 275 function of the physical parameters that we seek to estimate, and will be used
 276 as estimation functions in Section 3.

277 2.4. Input-Output Dynamics in the Time Domain

Suppose that an input signal $u(t)$ (which in our case can either be $p_i(t)$
 or $r(t)$) affects some noise-free output $y_{nf}(\boldsymbol{\theta}, t)$. In the Laplace domain, their
 relation is

$$Y_{nf}(\boldsymbol{\theta}, s) = G(s, \boldsymbol{\theta})U(s), \quad (56)$$

278 where $G(s, \boldsymbol{\theta})$ is a function of the physical parameters collected in the vector
 279 $\boldsymbol{\theta}$. This vector represents a family of physical systems, whereas $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ refers
 280 to a particular physical system characterised by its parameters $\boldsymbol{\theta}_0$. We shall
 281 refer to the system with $\boldsymbol{\theta}_0$ as the data-generating system, and $\boldsymbol{\theta}_0$ contains the
 282 physical parameters we need to estimate. As will become clear in Section 3,
 283 we shall use discrete-time input and output data to estimate the parameters
 284 using a least-squares method. To this end, we require a function that simulates
 285 the noise-free response $y_{nf}(\boldsymbol{\theta}, t)$ of the system. In this section, we derive this
 286 response for sinusoidal (1) and square wave (2) actuator signals.

287

Suppose we have a closed-form expression of $G(s, \boldsymbol{\theta})$ in (56). The time-
 domain signal $y_{nf}(\boldsymbol{\theta}, t)$ for an input signal of the form, assuming zero initial
 conditions, reads

$$u(t) = C_\gamma \sin(\omega t). \quad (57)$$

The corresponding noise-free output is then given by [28]

$$y_{nf}(\boldsymbol{\theta}, t) = C_\gamma |G(i\omega, \boldsymbol{\theta})| \sin(\omega t + \alpha), \quad (58)$$

where $G(i\omega, \boldsymbol{\theta})$ is defined in (56) and $\alpha = \angle G(i\omega, \boldsymbol{\theta})$. In other words, the output $y_{nf}(\boldsymbol{\theta}, t)$ is also a sinusoid with frequency ω but a different amplitude and phase. Using this insight, it is straightforward to write down an expression for the output when the input is a sum of sinusoids. Indeed, for a square wave input signal (2) with amplitude C and cycle frequency ω_c the output $y_{nf}(\boldsymbol{\theta}, t)$ reads

$$y_{nf}(\boldsymbol{\theta}, t) = \frac{4}{\pi} C_\gamma \sum_{m=1}^{\infty} \frac{|G(i[2m-1]\omega_c, \boldsymbol{\theta})|}{2m-1} \sin([2m-1]\omega_c t + \alpha_m), \quad (59)$$

288 where $\alpha_m = \angle G(i[2m-1]\omega_c, \boldsymbol{\theta})$. This is an elegant way to find the time-
 289 domain expression of the output of the system without requiring an inverse
 290 Laplace transform.

291

292 At this point a general expression of a linear system's response for sinusoidal
 293 and square wave input signals is defined. What remains is to find a closed-form
 294 expression for the transfer function $G(s, \boldsymbol{\theta})$ that connects the actuator signal $r(t)$
 295 with the inlet and outlet pressures $p_i(t)$ and $p_e(t)$. In the next two subsections
 296 we derive this transfer function for $p_i(t)$ to $p_e(t)$ and $r(t)$ to $p_e(t)$.

297 *Direct Method: Inlet-Outlet Pressure Measurements*

The first measurement method uses inlet and outlet pressure measurements. This is one way to collect data in pressure oscillation experiments, see e.g. [1]. The input signal $u(t) = u_I(t) = p_i(t)$ and the output is $y_{nf}(\boldsymbol{\theta}, t) = p_e(t)$, see Fig. 4. In the previous section we have derived the relationship between $P_i(s)$ and $Y_{nf}(s) = P_e(s)$ (c.f. (54)), being

$$P_e(s) = \frac{S_{11}}{1 - S_{12}T_{21}} P_i(s). \quad (60)$$

Substitution of S_{11} , S_{12} , and T_{21} defined in (49) and (50) results in

$$P_e(s) = G_{f_e}(s, \boldsymbol{\theta}) P_i(s), \quad (61)$$

in which

$$G_{f_e}(s, \boldsymbol{\theta}) = \frac{1}{\cosh \sqrt{\phi s/k} + f_e^{-1} \sqrt{s(\phi k)^{-1}} \sinh \sqrt{\phi s/k}}. \quad (62)$$

298 Note that $\boldsymbol{\theta} = [k, \phi]^T$ and that G_{f_e} also depends on the degree of freedom
 299 f_e . We have now obtained the closed-form expression of the transfer function
 300 between $p_i(t)$ and $p_e(t)$. Hence, we can easily calculate the output response in
 301 the time domain using (58). We remark that the dynamics between $p_i(t)$ and
 302 $p_e(t)$ have also been derived in [1] but that the concept of transfer function has
 303 not been used, see Appendix C for a discussion.

304 For future reference we also calculate the gradient of (62) with respect to
 305 $\boldsymbol{\theta} = (k, \phi)$:

$$\begin{aligned} \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial k} &= \frac{f_e \left\{ i\phi\omega \cosh \left(\sqrt{\frac{i\omega\phi}{k}} \right) + k [f_e\phi + 1] \sqrt{\frac{i\omega\phi}{k}} \sinh \left(\sqrt{\frac{i\omega\phi}{k}} \right) \right\}}{2k^2\phi \left\{ f_e \cosh \left(\sqrt{\frac{i\omega\phi}{k}} \right) + \sqrt{i\omega(k\phi)^{-1}} \sinh \left(\sqrt{\frac{i\omega\phi}{k}} \right) \right\}^2} \quad (63) \\ \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial \phi} &= - \frac{f_e \left\{ i\omega/k \cosh \left(\sqrt{\frac{i\omega\phi}{k}} \right) + \left[f_e \sqrt{\frac{i\omega\phi}{k}} - \sqrt{i\omega(k\phi)^{-1}} \right] \sinh \left(\sqrt{\frac{i\omega\phi}{k}} \right) \right\}}{2\phi \left\{ f_e \cosh \left(\sqrt{\frac{i\omega\phi}{k}} \right) + \sqrt{i\omega(\phi k)^{-1}} \sinh \left(\sqrt{\frac{i\omega\phi}{k}} \right) \right\}^2} \quad (64) \end{aligned}$$

306 We shall use these expressions in Section 4 to find the optimal input signals.
 307 The absolute values of the derivatives are shown in Figs. 5 and 6 as a function
 308 of ω for several values of f_e .

309 *Indirect Method: Actuator Position/Outlet Pressure Measurements*

Another possible method uses the actuator and outlet pressure signal for parameter estimation. These signals have so far not been considered in literature for identification, but have a major advantage with respect to the Direct Method. Indeed, since the actuator signal is applied and not measured, no sensor noise is present on the actuator signal. Therefore, it is not an Errors-in-Variables problem as the Direct Method is. For the Indirect Method, the input and output are respectively given by $u(t) = u_{II}(t) = r(t)$ and $y_{nf}(\boldsymbol{\theta}, t) = p_e(t)$, see

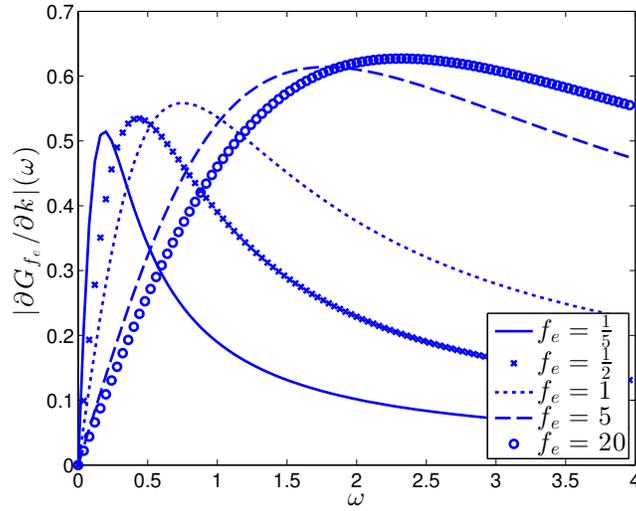


Figure 5: Absolute value of the derivative of the transfer function (62) with respect to permeability as a function of the scaled frequency ω , see (63). Observe that for increasing values of f_e , the maximum of the curve shifts to higher frequencies and higher values. A small (large) f_e value means that the outlet volume is large (small).

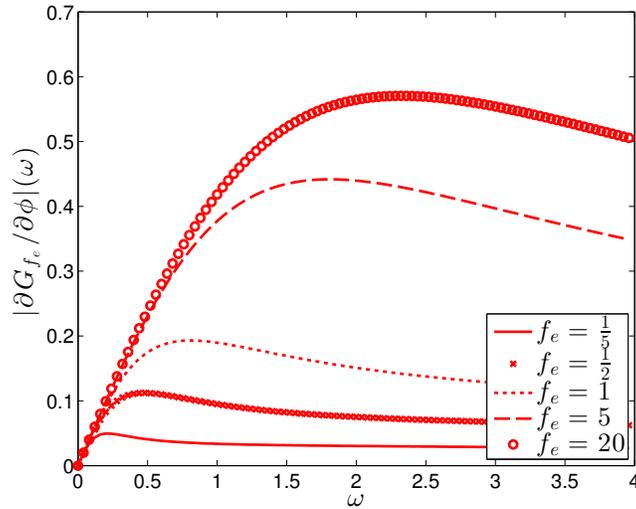


Figure 6: Absolute value of the derivative of the transfer function (62) with respect to porosity as a function of the scaled frequency ω , see (64). For a given ratio f_e , the absolute value of $\partial G_{f_e}(i\omega, \theta)/\partial\phi$ is smaller at almost all frequencies than the absolute value of $\partial G_{f_e}(i\omega, \theta)/\partial k$, see Figure 5.

Fig. 4. The relationship between $r(t)$ and $p_e(t)$ in the Laplace domain is found by combining (54), (55) and using (62), i.e.,

$$P_e(s) = [1 - F_{12}(S_{21} + S_{22}T_{21}G_{f_e}(s, \boldsymbol{\theta}))]^{-1} F_{11}(s)G_{f_e}(s, \boldsymbol{\theta})R(s) \quad (65)$$

Substitution of F_{11} , F_{12} (43), S_{11}, \dots, S_{22} (49), and T_{21} (50), into the previous expression results in

$$P_e(s) = L_{f_i, f_e}(s, \boldsymbol{\theta})G_{f_e}(s, \boldsymbol{\theta})R(s), \quad (66)$$

in which

$$L_{f_i, f_e}(s, \boldsymbol{\theta}) = \left[\frac{1}{f_i} + \frac{1}{s} \left(k\sqrt{\frac{s\phi}{k}} \tanh \sqrt{\frac{s\phi}{k}} + \frac{sG_{f_e}(s, \boldsymbol{\theta})}{f_e \cosh \sqrt{\frac{s\phi}{k}}} \right) \right]^{-1}. \quad (67)$$

This is an interesting result. It shows that the relationship between $R(s)$ and $P_e(s)$ is equal to the relationship between $P_i(s)$ and $P_e(s)$ multiplied by a complex filter $L_{f_i, f_e}(s, \boldsymbol{\theta})$. Thus, we have that $P_i(s) = L_{f_i, f_e}(s)R(s)$ and $P_e(s) = G_{f_e}(s)P_i(s)$. The input-output connection between $R(s)$ and $P_e(s)$ is thus $P_e(s) = L_{f_i, f_e}(s, \boldsymbol{\theta})G_{f_e}(s, \boldsymbol{\theta})R(s) = G_{f_i, f_e}(s, \boldsymbol{\theta})R(s)$, where

$$G_{f_i, f_e}(s, \boldsymbol{\theta}) = \left[\frac{1}{f_i} + \frac{1}{s} \left(k\sqrt{\frac{s\phi}{k}} \tanh \sqrt{\frac{s\phi}{k}} + \frac{sG_{f_e}(s, \boldsymbol{\theta})}{f_e \cosh \sqrt{\frac{s\phi}{k}}} \right) \right]^{-1} G_{f_e}(s, \boldsymbol{\theta}). \quad (68)$$

310 For future reference, we also calculate the derivatives of (68) with respect to k
311 and ϕ :

$$\begin{aligned} \frac{\partial G_{f_i, f_e}(i\omega, \boldsymbol{\theta})}{\partial k} &= \frac{k\phi G_{f_e}(i\omega, \boldsymbol{\theta}) \left[\sqrt{\frac{i\omega\phi}{k}} \operatorname{sech}^2 \sqrt{\frac{i\omega\phi}{k}} - \tanh \sqrt{\frac{i\omega\phi}{k}} \right] - if_e^{-1}\phi\omega G_{f_e}^2(i\omega, \boldsymbol{\theta})}{2k^2 \sqrt{\frac{i\omega\phi}{k}} \left(f_i^{-1} + f_e^{-1} G_{f_e}(i\omega, \boldsymbol{\theta}) \operatorname{sech} \sqrt{\frac{i\omega\phi}{k}} + \phi \sqrt{\frac{k}{i\omega\phi}} \tanh \sqrt{\frac{i\omega\phi}{k}} \right)^2} \times \operatorname{sech} \sqrt{\frac{i\omega\phi}{k}} \tanh \sqrt{\frac{i\omega\phi}{k}} + 2f_i^{-1}k^2 \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial k} \left[\sqrt{\frac{i\omega\phi}{k}} + f_i\phi \tanh \sqrt{\frac{i\omega\phi}{k}} \right] \quad (69) \\ \frac{\partial G_{f_i, f_e}(i\omega, \boldsymbol{\theta})}{\partial \phi} &= \frac{i\phi\omega \left(\sqrt{k} G_{f_e}(i\omega, \boldsymbol{\theta}) \operatorname{sech}^2 \sqrt{\frac{i\omega\phi}{k}} - 2f_i^{-1}\sqrt{k} \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial \phi} + \frac{\tanh \sqrt{\frac{i\omega\phi}{k}}}{\sqrt{i\omega\phi}} \right) \times \left[kG_{f_e}(i\omega, \boldsymbol{\theta}) - i\omega f_e^{-1} G_{f_e}^2(i\omega, \boldsymbol{\theta}) \operatorname{sech} \sqrt{\frac{i\omega\phi}{k}} - 2k\phi \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial \phi} \right]}{2k^{3/2} \left(f_e^{-1} \sqrt{\frac{i\omega\phi}{k}} G_{f_e}(i\omega, \boldsymbol{\theta}) \operatorname{sech} \sqrt{\frac{i\omega\phi}{k}} + f_i^{-1} \left[\sqrt{\frac{i\omega\phi}{k}} + f_i\phi \tanh \sqrt{\frac{i\omega\phi}{k}} \right] \right)} \quad (70) \end{aligned}$$

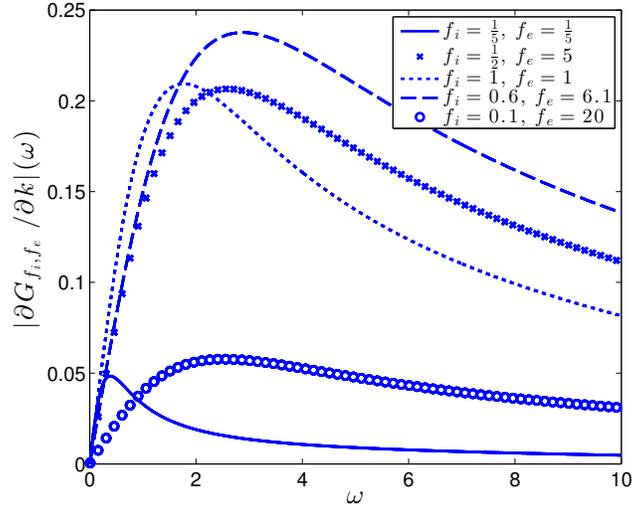


Figure 7: Absolute value of the derivative of the transfer function (68) with respect to permeability as a function of the scaled frequency ω , see (69). Observe that for increasing values of f_i and f_e , the maximum of the curve increases and shifts to higher frequencies. A small (large) f_e value means that the outlet volume is large (small). A small (large) f_i value means that the inlet volume is large (small).

312 In these expressions, the transfer function G_{f_e} and its derivatives are defined
 313 in (62)-(64). The absolute values of the above derivatives are shown in Figs. 7
 314 and 8 as a function of ω for several values of f_e and f_i .

315 3. Estimation Procedure

316 In this section we introduce the estimation procedure. Our goal is to esti-
 317 mate the physical parameters k and ϕ of the coupled system defined in Section
 318 2.1 for the Direct and Indirect Method.

319

We first rewrite the expression of the actuator amplitude C_γ (c.f. (3)). Since C_γ and the ratio f_i (44) are related to the inlet length L_i , we write the actuator amplitude as a function of f_i , i.e.

$$C_\gamma(f_i) = \gamma \frac{L_i}{L} = \gamma \frac{\phi_s}{f_i}, \quad (71)$$

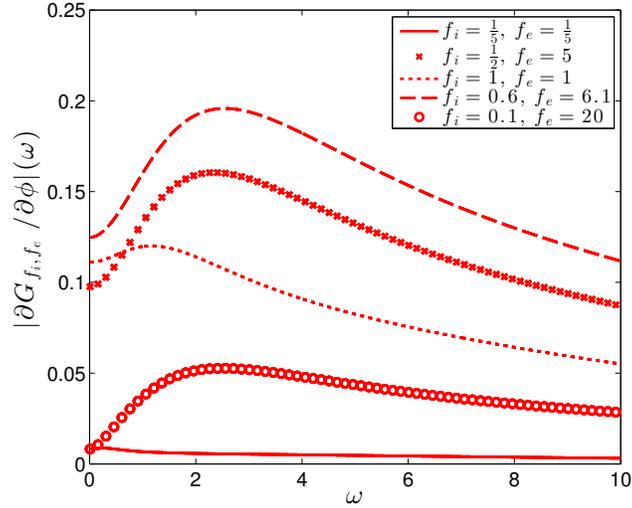


Figure 8: Absolute value of the derivative of the transfer function (68) with respect to porosity as a function of the scaled frequency ω , see (70).

320 where we still have that $0 < \gamma \leq \gamma_m$. It is already mentioned that $\gamma_m \leq 1$
 321 due to the movement limitations of the actuator. However, it is now also clear
 322 that γ_m can not be chosen large as otherwise the inlet volume is no longer
 323 approximately constant due to the actuator movement, and hence our above
 324 derivations no longer hold. It is important to use the this definition of the
 325 actuator amplitude as otherwise unphysical (wrong) results are obtained when
 326 using optimal experiment design. The value $\gamma_m \leq 1$ should be determined by
 327 the experimenter.

328

329 We continue with determining the data collection procedure and then define
 330 the least-squares method. In the remainder of this article we derive results
 331 based on a square wave actuator signal (with an as-of-yet undetermined am-
 332 plitude and cycle frequency), which we prove in Appendix D to deliver shorter
 333 experiment lengths than a sinusoidal one for the same values of f_i and f_e .

Let us for simplicity first explain the identification of the parameters for the Indirect Method. In system identification, parameters of a system are estimated based on input-output data collected on the system. For the Indirect Method we already mentioned that the input signal for the identification is a discrete-time version of the user-chosen actuator signal $r(t)$ and the output signal is a discrete-time version of the outlet pressure $p_e(t)$. An identification experiment can be performed as follows. We apply, for a certain duration, a continuous-time excitation signal $r(t)$ of the form (2) and we wait for the transients to die out. The corresponding steady-state output signal (i.e the outlet pressure) is then measured at a sampling rate T_s . We will suppose that an anti-aliasing filter is applied prior to the sampling. In this way, we are able to collect N output measurements $y_D[n]$ ($n = 1, \dots, N$) with NT_s the duration of the experiment. These measurements are of course corrupted by noise. In the sequel, we suppose that this noise is white and has a variance σ_e^2 . Coloured noise could also be taken into account, see e.g. [28]. The continuous-time input signal $r(t)$ also has to be transformed into a discrete-time signal that we will denote by $u_D[n]$. This discretization is done in the same way as for the output signal. First, the signal is filtered by an anti-aliasing filter and then sampled. Since we precisely know $r(t)$, we can exactly compute $u_D[n]$ ($n = 1, \dots, N$). This leads to the following input data ($n = 1, \dots, N$):

$$u_D[n] = \frac{4}{\pi} C_\gamma(f_i) \sum_{m=1}^{Q(T_s)} \frac{\sin([2m-1]\omega_c n T_s)}{2m-1}, \quad (72)$$

335 where we recall that ω_c is the cycle frequency of the square wave. Furthermore,
 336 observe that the sum no longer extends to infinity but to the finite integer
 337 $Q(T_s) = \left\lfloor \frac{1}{2} \left(\frac{\pi}{\omega_c T_s} + 1 \right) \right\rfloor$: it determines the highest mode that can be observed
 338 in the discrete-time data. Here, $\lfloor \cdot \rfloor$ represents the *floor* function. This integer
 339 can be easily deduced from the radial Nyquist rate $\frac{\pi}{T_s}$.

Using this procedure, we have obtained an input-output data set
 $Z_N = \{u_D[n], y_D[n]\}_{n=1, \dots, N}$ with which the unknown parameter vector $\theta_0 =$

(k_0, ϕ_0) will be estimated. In order to determine an optimal estimator for this purpose, we need to write an expression of the output data as a function of variables that represent these parameters. This can be easily done. Indeed, due to our data generating mechanism, in the frequency range $[0, \pi/T_s]$, the frequency response $G_d(e^{i\omega T_s}, \boldsymbol{\theta}_0)$ of the discrete-time transfer function $G_d(z, \boldsymbol{\theta}_0)$ between the input data $u_D[n]$ and the (noise-free) output data $y_{nf}[n]$ is perfectly equal to the frequency response $G_{f_i, f_e}(i\omega, \boldsymbol{\theta}_0)$ of the continuous-time transfer function $G_{f_i, f_e}(s, \boldsymbol{\theta}_0)$ between the continuous-time input and output. Consequently, we can write

$$y_D[n] = y_{nf}(\boldsymbol{\theta} = \boldsymbol{\theta}_0)[n] + e[n], \quad (73)$$

where $e[n]$ is a white noise realisation with variance σ_e^2 , and the definition of $y_{nf}(\boldsymbol{\theta})[n]$ for the square wave actuator signal (72) is given by

$$y_{nf}(\boldsymbol{\theta})[n] = \frac{4}{\pi} C_\gamma(f_i) \sum_{m=1}^{Q(T_s)} \frac{|G_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta})|}{2m-1} \sin([2m+1]\omega_c n T_s + \alpha_m), \quad (74)$$

in which $\alpha_m = \angle |G_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta})|$. Based on (73), an asymptotically efficient estimate for $\boldsymbol{\theta}_0$ is defined through [28]

$$\hat{\boldsymbol{\theta}}_N = \arg \min_N \frac{1}{N} \sum_{n=1}^N (y_D[n] - y_{nf}(\boldsymbol{\theta})[n])^2, \quad (75)$$

340 where $y_{nf}(\boldsymbol{\theta})$ is defined in (74). In Appendix A the expressions for u_D and
 341 $y_{nf}(\boldsymbol{\theta})[n]$ are given for a sinusoidal actuator signal. Replacing the expressions
 342 (72), (73) with the ones in Appendix A defines the estimate (75) for the sinu-
 343 soidal case.

344 We remark that the scaling introduced in the previous section should also
 345 be applied to the measured data set Z_N . For a detailed explanation on how
 346 identification can be used for scaled systems, we refer the reader to [34], pages
 347 6-7.

As shown in [28], $\hat{\boldsymbol{\theta}}_N$ is asymptotically (i.e. for large N) normally distributed with mean $\boldsymbol{\theta}_0$ and a covariance matrix \mathbf{P}_θ whose inverse is given by the Fisher

Information Matrix:

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1} = \frac{N}{\sigma_e^2} \bar{E} [\nabla_{\boldsymbol{\theta}} y_{nf}(\boldsymbol{\theta}) \nabla_{\boldsymbol{\theta}}^T y_{nf}(\boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}, \quad (76)$$

where $\bar{E}(\cdot) = \frac{1}{N} \sum_{n=1}^N E(\cdot)$ is a modified expectation operator and $\nabla_{\boldsymbol{\theta}}$ the gradient operator of $\boldsymbol{\theta}$. This matrix can be rewritten using Parseval's theorem as

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1}[\Phi_{u_D}] = \frac{NT_s}{2\pi\sigma_e^2} \int_{-\pi/T_s}^{\pi/T_s} [\nabla_{\boldsymbol{\theta}} G_{f_i, f_e}(i\omega, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} [\nabla_{\boldsymbol{\theta}} G_{f_i, f_e}(i\omega, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}^H \Phi_{u_D}(\omega) d\omega, \quad (77)$$

348 and shows that the covariance matrix is a functional in Φ_{u_D} , the power spectrum
 349 of $u_D[n]$. Superscript H denotes the Hermitian conjugate. Note that in (77) the
 350 discrete-time transfer function G_d is replaced by its continuous-time counterpart
 351 $G_{f_i, f_e}(i\omega)$ (68) since their frequency response are identical in the frequency range
 352 up to the Nyquist rate π/T_s .

353 The covariance matrix depends on the ratios f_i and f_e , the input signal u_D
 354 via its power spectrum, and on the number of data samples N . Using this expres-
 355 sion, we will be able to design the input spectrum that provides us the maximal
 356 information about the unknown parameter. This optimal spectrum can e.g. be
 357 defined as the spectrum that delivers parameter estimates with predetermined
 358 upper bounds on the parameter variances using the shortest experiment length
 359 (i.e. using the smallest N possible). This input spectrum selection is called
 360 optimal experiment design and will be the subject of the next sections. Note
 361 also the importance of the gradient of the transfer function, $\nabla_{\boldsymbol{\theta}} G_{f_i, f_e}(i\omega)$, in
 362 (77). This components of this gradient are depicted in Figs. 7 and 8.

363 Note that, from (77), we see that in order to calculate the covariance matrix,
 364 we need to know the unknown vector $\boldsymbol{\theta}_0$, which we aim to estimate! In practice,
 365 this vector is replaced by an initial guess that may or may not be derived from
 366 a prior experiment. Although this will change the optimal input signal, it has
 367 been shown in e.g. [34] that accurate estimates can nonetheless be obtained.

368 *Direct Method*

369 In the Direct Method we will also apply a square wave or sinusoidal excitation
 370 signal $r(t)$ to the system via the actuator. However, we will not use this signal
 371 as input for the identification, but we will use the measured inlet pressure $p_i(t)$
 372 instead. The output signal remains to be the outlet pressure and is measured
 373 in the same way as in the Indirect Method, yielding $y_D[n]$ ($n = 1, \dots, N$).
 374 This method is also followed in [1]. The input signal is transformed into a
 375 discrete-time signal $u_D[n]$ using the same measurement mechanism (anti-aliasing
 376 filter and sampling with T_s time steps). Since the signal $p_i(t)$ is measured
 377 (and not directly applied as $r(t)$), we are not able to retrieve precisely the
 378 inlet pressure (the measurements are indeed corrupted by noise). The signal
 379 entering the system is thus not known exactly, and consequently, we are facing
 380 an errors-in-variables (EIV) identification problem that may lead to serious
 381 problems in practice [35, 36]. As shown in [37] on another problem, a way
 382 to deal with this EIV problem is to apply a two-stage strategy [38]. First,
 383 identify a model \hat{L} of the relation L_{f_i, f_e} (c.f. 67) between the actuator signal
 384 and the noise-corrupted measurements of $p_i(t)$. Second, the actuator signal
 385 $r(t)$ is filtered by that model \hat{L} to lead to new input data. These new input
 386 data converge asymptotically to the noise-free inlet pressure if a good model
 387 structure is chosen for the identification. While this method alleviates the EIV
 388 issue, it will nevertheless increase the variance of the estimate $\hat{\theta}_N$. For the sake
 389 of comparison between the Direct and Indirect Method, we will disregard here
 390 this increase of variance and suppose that we can obtain the noise-free discrete-
 391 time inlet pressure using this procedure. This is the same as supposing that we
 392 know precisely the filter $L(s) = L_{f_i, f_e}(s, \theta_0)$.

For a square wave signal $r(t)$, the discrete-time signal $p_i[n]$ reads

$$u_D[n] = \frac{4}{\pi} C_\gamma(f_i) \sum_{m=1}^{Q(T_s)} |L_{f_i, f_e}(i[2m-1]\omega_c, \theta_0)| \frac{\sin([2m-1]\omega_c n T_s + \angle L_{f_i, f_e}(i[2m-1]\omega_c, \theta_0))}{2m-1}. \quad (78)$$

The output signal can therefore also be written as (73) with $y_{nf}(\theta)[n]$ now

defined as

$$y_{nf}(\boldsymbol{\theta})[n] = \frac{4}{\pi} C_\gamma(f_i) \sum_{m=1}^{Q(T_s)} |L_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta}_0)| \frac{|G_{f_e}(i[2m-1]\omega_c, \boldsymbol{\theta})|}{2m-1} \sin([2m+1]\omega_c n T_s + \alpha_m), \quad (79)$$

393 where $\alpha_m = \angle G_{f_e}(i[2m-1]\omega_c, \boldsymbol{\theta}) + \angle L_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta}_0)$. Based on the above
 394 expressions, we can use the data $Z_N = \{u_D[n], y_D[n]\}_{n=1, \dots, N}$ to estimate $\hat{\boldsymbol{\theta}}_N$
 395 using (75) but now with the new definitions of $y_{nf}(\boldsymbol{\theta})[n]$. Note that we now
 396 identify the parameter vector in the transfer function G_{f_e} while the considered
 397 transfer function was G_{f_i, f_e} for the Indirect Method. The covariance matrix for
 398 the estimate using the Direct Method is also given by (77) but with the same
 399 replacement and using the power spectrum corresponding to (78).

400 We remark that for the Direct Method we have assumed zero noise on the
 401 inlet pressure signal. This is never the case in practice and we therefore require
 402 a two-stage method. The variance in the parameters will consequently be larger
 403 than given by (77). These are strong arguments to prefer the Indirect Method
 404 which is the new approach we propose in this paper.

405 We furthermore point out that the zero noise assumption on the inlet pres-
 406 sure signal is alleviated in the literature [1] by averaging points of the signal
 407 $p_i(t)$ over a three-sample window to cancel out the measurement noise. This
 408 downsampling of data leads to information loss since high frequency dynamics
 409 is discarded, which leads to higher parameter variances.

410 4. Experiment Design using the Direct Method

411 In the previous section we have defined the identification experiment and the
 412 identification criterion. Suppose now that our objective is to obtain estimates
 413 of both parameters with a variance that is smaller than a given threshold, i.e.,

$$\text{var}(\hat{k}_N) = \mathbf{e}_1^T \mathbf{P}_\theta \mathbf{e}_1 \leq c_k, \quad (80)$$

$$\text{var}(\hat{\phi}_N) = \mathbf{e}_2^T \mathbf{P}_\theta \mathbf{e}_2 \leq c_\phi, \quad (81)$$

414 where the covariance matrix is the inverse of (77). We would like to optimise
 415 the frequency content and the amplitude of the excitation signal $r(t)$ as well as

416 the DOF f_i and f_e in such a way that the above constraints are fulfilled with
 417 the shortest possible identification length N .

418

In this section we address this problem for the Direct Method. For the sake of brevity, we only derive the solution for the square wave actuator signal (2). The case of a sinusoidal excitation signal (1) can be treated in a similar fashion. Furthermore, we recall that, even though the excitation signal $r(t)$ is induced by the actuator, the signals that are used for the identification are the reconstructed inlet pressure $p_i(t)$ and the measured outlet pressure $p_e(t)$. The transfer function that is to be identified is thus $G_{f_e}(s, \boldsymbol{\theta}_0)$.

The spectrum of the sampled inlet pressure signal with amplitude $C_\gamma(f_i)$ and cycle frequency ω_c is given by

$$\Phi_{u_D}(\omega) = \frac{16}{\pi^2} \frac{\pi C_\gamma^2(f_i)}{2T_s} \sum_{m=1}^{Q(T_s)} \frac{|L_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta}_0)|^2}{(2m-1)^2} \sum_{l=\{-1,1\}} \delta(\omega - l[2m-1]\omega_c). \quad (82)$$

419 Since the filter $L_{f_i, f_e}(s, \boldsymbol{\theta}_0)$ is not identified but only $G_{f_e}(s, \boldsymbol{\theta})$, L_{f_i, f_e} for
 420 the Direct Method is not a function of $\boldsymbol{\theta}$ anymore, but assumed known and
 421 therefore always evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Substitution of (82) in the expression
 422 of the covariance matrix (77) and replacing G_{f_i, f_e} with G_{f_e} (62) result in the
 423 inverse of the covariance matrix for the Direct Method:

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1}[\omega_c, C_\gamma(f_i), f_i, f_e] = \frac{16}{\pi^2} \frac{N C_\gamma^2(f_i)}{2\sigma_e^2} \sum_{m=1}^{Q(T_s)} \frac{|L_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta}_0)|^2}{(2m-1)^2} \times \text{Re} \left\{ [\nabla_{\boldsymbol{\theta}} G_{f_e}(i[2m-1]\omega_c, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \cdot [\text{C.C.}] \right\}, \quad (83)$$

424 where C.C. stands for complex conjugate.

425 4.1. Optimization Problem and Its Solution

Due to the amplitude limitation of the actuator, the optimal experiment design problem for the Direct Method is defined mathematically as

$$\min_{\omega_c, C_\gamma(f_i), f_i, f_e} \text{Experiment length } N \quad (84)$$

426 subject to

$$\text{var}(\hat{k}_N) = \mathbf{e}_1^T \mathbf{P}_\theta[\omega_c, C_\gamma(f_i), f_i, f_e] \mathbf{e}_1 \leq c_k, \quad (85)$$

$$\text{var}(\hat{\phi}_N) = \mathbf{e}_2^T \mathbf{P}_\theta[\omega_c, C_\gamma(f_i), f_i, f_e] \mathbf{e}_2 \leq c_\phi \quad (86)$$

427 in which c_k and c_ϕ are the scaled user-imposed constraints on respectively the
 428 variance of the estimates \hat{k}_N and $\hat{\phi}_N$, \mathbf{e}_i the i -th unit vector, and \mathbf{P}_θ the in-
 429 verse of matrix (83). The optimization problem for a sinusoidal actuator signal
 430 is defined and solved in Appendix B.

431

432 We wish to ensure that the variance of the estimates in $\hat{\boldsymbol{\theta}}_N = (\hat{k}_N, \hat{\phi}_N)^T$
 433 do not exceed their respective bounds c_k and c_ϕ using the smallest experiment
 434 length. The solution is found as follows:

435 1. Set the factor $\gamma = \gamma_m$ in the actuator amplitude $C_\gamma(f_i)$.

436 2. Define the functions

$$N_k(f_i, f_e, \omega_c) = \frac{1}{c_k} \mathbf{e}_1^T \left(\frac{\mathbf{P}_\theta^{-1}[\omega_c, C_{\gamma_m}(f_i), f_i, f_e]}{N} \right)^{-1} \mathbf{e}_1, \quad (87)$$

$$N_\phi(f_i, f_e, \omega_c) = \frac{1}{c_\phi} \mathbf{e}_2^T \left(\frac{\mathbf{P}_\theta^{-1}[\omega_c, C_{\gamma_m}(f_i), f_i, f_e]}{N} \right)^{-1} \mathbf{e}_2, \quad (88)$$

437 where \mathbf{P}_θ^{-1} is given by (83) and we note that γ has been replaced by γ_m .

3. Using the functions from the previous item, define

$$N_{min}(f_i, f_e) = \min_{\omega_c} \{ \max [N_k(f_i, f_e, \omega_c), N_\phi(f_i, f_e, \omega_c)] \}. \quad (89)$$

438 4. The optimal experiment length is given by

$$N_{opt} = N_{min}(f_{i,opt}, f_{e,opt}), \quad \text{where } \{f_{i,opt}, f_{e,opt}\} = \arg \min_{f_i, f_e} N_{min}(f_i, f_e). \quad (90)$$

439 The solution to the problem is given by the quartet $N_{opt}, f_{i,opt}, f_{e,opt},$

440 and $\omega_{c,min}(f_{i,opt}, f_{e,opt}) = \arg N_{min}(f_{i,opt}, f_{e,opt})$.

441 In step 1 the coefficient γ in the actuator amplitude $C_\gamma(f_i)$ is set to $\gamma = \gamma_m$.

442 Equation (83) shows that the parameter variances scale inversely proportional
 443 to $C_\gamma(f_i)$. Thus, selecting the largest possible amplitude of the actuator is a

444 requirement to find the minimal experiment time. The actuator amplitude is
 445 now only a function of f_i . In step 2 the functions N_k and N_ϕ respectively
 446 define the required experiment length to ensure that (i) $\text{var}(\hat{k}_N) = c_k$ and (ii)
 447 $\text{var}(\hat{\phi}_N) = c_\phi$. Smaller values than N_k and N_ϕ violate the respective constraints
 448 in (86), whereas larger ones lead to longer experiment lengths than required.
 449 Both constraints are honoured simultaneously, i.e. satisfying situation (i) and
 450 (ii), with the function $N_{min}(f_i, f_e)$ calculated in step 3. It returns the minimal
 451 required experiment length as a function of f_i and f_e by finding the optimal
 452 cycle frequency. The last step then finds the optimal values of f_i and f_e such
 453 that the global minimum of $N_{min}(f_i, f_e)$, i.e. N_{opt} , is obtained.

454

455 Previously it was mentioned that in order to compute the optimal input sig-
 456 nal the true parameter vector θ_0 should be used. However, it is this parameter
 457 that we want to identify. Unfortunately, this so-called chicken-and-egg problem
 458 is unavoidable in (optimal) Experiment Design. In order to design an optimal
 459 signal, some knowledge of the system (in this case the true parameter vector)
 460 is required. Consequently, to solve the optimization problem (84)-(86), we re-
 461 quire a prior estimate or an initial estimate θ_{init} to evaluate the inverse of the
 462 covariance matrix (83). This substitution inevitably leads to suboptimal exper-
 463 iment lengths. However, a recent study [34] showed that much better estimates
 464 can be obtained using Experiment Design compared to an arbitrary selection
 465 of signals and frequencies, and that in many cases the variance constraints are
 466 still honoured. In [39, 40, 41] advanced techniques to circumvent this issue are
 467 discussed.

468 4.2. Numerical Results

469 In this section we follow the four steps defined in the previous section to
 470 numerically find the solution (90) that solves (84)-(86). To this end, we grid
 471 the frequency $\omega_c \in [0, \pi/T_s]$ and ratios $f_i \in [0.05, 20]$, and $f_e \in [0.05, 20]$.
 472 The grid resolutions are respectively $\Delta\omega_c = 0.05$, $\Delta f_i = \Delta f_e = 0.05$. The
 473 minimum (maximum) values in the interval of f_i and f_e correspond to unscaled

474 maximum (minimum) inlet and outlet lengths of respectively $L_i = L_e = 0.2$
 475 meters ($L_i = L_e = 0.5$ mm). Values outside these intervals are assumed to be
 476 practically infeasible. The physical system is defined through the values in Table
 477 1; the experiment conditions and experiment design values in Table 2. We wish
 478 to ensure that the standard deviation of the estimate \hat{k}_N and $\hat{\phi}_N$ are respectively
 479 less than or equal to 5% of their true values k_0 and ϕ_0 , defined in Table 1, using
 480 minimal experiment time. We remark that the optimal experiment lengths
 481 found in this section depend strongly on the choice of noise variance σ_e^2 and
 482 actuator amplitude bound γ_m . Thus, comparison with results in the existing
 483 literature is only fair under the same experiment conditions.

484 We first discuss separate parameter estimation results, followed by joint
 485 parameter estimation results.

486 4.2.1. *Separate Parameter Estimation*

487 Separate parameter estimation refers to the situation where either k_0 or ϕ_0
 488 is known, and respectively ϕ_0 and k_0 is unknown and needs to be estimated.
 489 We thus wish to ensure that the standard deviation of the estimate \hat{k}_N or
 490 $\hat{\phi}_N$ is less than or equal to 5% of their true value k_0 or ϕ_0 in minimal time.
 491 Separate estimation is a special case of the situation considered in the previous
 492 section where both parameters are identified together (joint estimation), and
 493 the optimal experiment design problem in this case can thus be solved following
 494 a very similar procedure of the one presented in the previous section. This
 495 procedure is given in Appendix B for the case of the sinusoidal excitation signal.
 496 The problem and solution for a square wave excitation signal can **also** be trivially
 497 formulated using that appendix.

498 The different results for separate estimation of k and ϕ with sinusoidal or
 499 square wave excitation are summarized in Table 3. In order to interpret these
 500 results, it makes sense to start discussing the result corresponding to the si-
 501 nusoidal excitation and in particular the function $N_{min}(f_i, f_e)$ defined for this
 502 signal in (B.4) in Appendix B. This function is represented in Fig. B.13 of
 503 Appendix B for the case where the parameter k is identified. We observe the

504 following:

505 • For any value of f_i the required experiment length reduces with increas-
 506 ing f_e . Inspection of $N_{min}(f_i, f_e)$ (c.f. (B.4)) shows that increasing
 507 $\partial G_{f_e}(i\omega_{ex}, \theta)/\partial \theta$ decreases N_{min} . From Fig. 5 we see that increasing
 508 f_e results in a larger absolute value of the derivative. At $f_e = 20$ we find
 509 a maximum at $\omega_{ex} \approx 2.4$. Similar behaviour is shown for the case $\theta = \phi$.
 510 The optimal value is $f_{e,opt} = 20$.

• For any value of f_e the required experiment length reduces with decreasing
 f_i . Let us analyse this result. Note first that f_i is only present in the
 term $C_{\gamma_m}(f_i)|L_{f_i, f_e}(i\omega, \boldsymbol{\theta}_0)|$, i.e. the amplitude of the inlet pressure $p_i(t)$.
 Obviously, by decreasing f_i , we increase the inlet volume and thus also
 the maximal amplitude of the actuator $C_{\gamma_m}(f_i)$ (see remark below (3) and
 (71)), which in turn decreases the required experiment time (c.f. (B.4)).
 However, the inlet amplitude attenuation due to the filter $|L_{f_i, f_e}|$ decreases
 with decreasing f_i , yet increases with frequency. It is easy to show that

$$\forall f_i, f_e : \max_{\omega} C_{\gamma_m}(f_i)|L_{f_i, f_e}(i\omega, \boldsymbol{\theta}_0)| = \gamma_m \phi_s.$$

511 Thus, the maximal amplitude of the inlet pressure is equal to $\gamma_m \phi_s$ and
 512 independent of f_i and f_e . However, for small f_i this asymptotic value is
 513 reached quicker at lower frequencies than for larger f_i , see Fig. 10. It
 514 is also reached quicker for larger f_e . This explains why one should opt
 515 for taking $f_{i,opt} = 0.1$ for any f_e . It is also at low frequencies where
 516 the derivative of $G_{f_e}(i\omega, \boldsymbol{\theta})$ with respect to k is large, as explained in the
 517 previous item.

518 • The optimal ratios are $f_{i,opt} = 0.1$ and $f_{e,opt} = 20$. The corresponding
 519 optimal experiment length $N_{opt} = 2.86 \times 10^5$ at optimal frequency $\omega_{opt} =$
 520 2.45 , see Table 3. The value of $f_{e,opt}$ corresponds with a choice made in
 521 literature [1]: minimising the outlet volume. The optimal values occur at
 522 the boundary of the considered intervals of f_i and f_e . The aim in practice
 523 is thus to maximise the inlet volume and minimise the outlet volume.

Quantity	Scaled Value	Real Value
Sampling time	$\tilde{T}_s = 0.021$	$T_s = 0.01$ s
Sensor noise variance	$\tilde{\sigma}_e^2 = 0.05$	$\sigma_e^2 = 1.25 \times 10^9$ Pa ²
Permeability variance constraint	$\tilde{c}_k = 2.5 \times 10^{-3}$	$c_k = 2.5 \times 10^{-3} k_0^2$ m ⁴
Porosity variance constraint	$\tilde{c}_\phi = 2.5 \times 10^{-3}$	$c_\phi = 2.5 \times 10^{-3} \phi_0^2$
Actuator amplitude constraint	$\gamma_m = 0.1$	$\gamma_m = 0.1$

Table 2: The experiment is designed using the above quantities. The scaled system is defined through the choices $L = 0.05$, $k_s = k_0$, $\phi_s = \phi_0$. These scalars define the scaled system in Section 2.2.

- 524 • Quantitatively the same results and analysis holds for the case where $\theta =$
525 ϕ . The experiment lengths in Table 3 show that porosity is more difficult
526 to estimate than permeability, as $\partial G_{f_e} / \partial \phi$ is smaller than $\partial G_{f_e} / \partial k$ for all
527 values of f_e depicted in Fig. 6.
- 528 • Square wave actuator signals result in shorter experiment lengths than
529 sinusoidal ones, in agreement with our result in Appendix E.

530 4.2.2. Joint Parameter Estimation

531 We follow the procedure outlined at the start of this section, using the same
532 experiment conditions, to analyse the joint parameter estimation experiment
533 design solutions. The optimization problem is given by (84)-(86). The results
534 are also given in Table 4 for the case of the sinusoidal and square wave signals.
535 The logarithm of the $N_{min}(f_i, f_e)$ (c.f. 89)) is shown in Fig. 9 as a function
536 of f_i and f_e . Starting once again the discussion with the sinusoidal signal, we
537 observe the following:

- 538 • The optimal input signal for joint estimation, defined through (90), is
539 found for $\omega_{opt} = \omega_{c,min}(f_{i,opt}, f_{e,opt}) = 2.35$ at optimal ratios $f_{i,opt} = 0.1$
540 and $f_{e,opt} = 1.0$, see Table 3 and Fig. 9. The minimal experiment length
541 for this combination is $N_{opt} = 4.47 \times 10^7$, and corresponds to an unscaled
542 experiment time of about five days ($T_s = 0.01$ seconds). The experiment

Direct Method	k		ϕ		$\{k, \phi\}$	
	Sine	SW	Sine	SW	Sine	SW
N_{opt}	2.86×10^5	1.66×10^5	3.46×10^5	2.0×10^5	4.47×10^7	2.51×10^7
ω_{opt}	2.45	2.3	2.45	2.3	2.35	2.1
$f_{i,opt}$	0.1	0.1	0.1	0.1	0.1	0.1
$f_{e,opt}$	20	20	20	20	1.0	1.0

Table 3: Optimal experiment lengths, scaled frequencies, and optimal ratios $f_{i,opt}$, $f_{e,opt}$ of input using the Direct Method for sinusoidal (Sine) and square wave (SW) actuator signal. Three cases are shown: estimation of only k or ϕ , and the joint estimation of the parameters $\{k, \phi\}$.

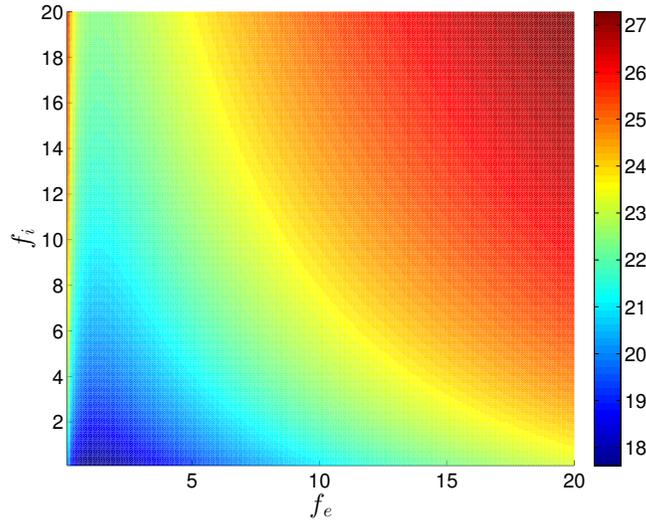


Figure 9: The logarithm minimal experiment length $N_{min}(f_i, f_e)$ (90) for the simultaneous estimation of k and ϕ using a sinusoidal excitation signal required to honour their respective variance constraint and the amplitude constraint $\gamma_m = 0.1$, is plotted against the ratios f_i and f_e . In sharp contrast to single parameter estimation experiments, the optimal ratio $f_{e,opt} = 1.0$. The optimal inlet ratio $f_{i,opt} = 0.1$.

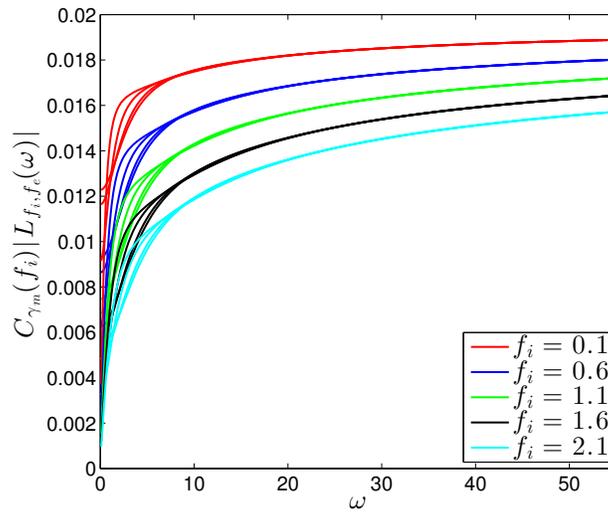


Figure 10: The inlet pressure amplitude $C_{\gamma_m}(f_i)|L_{f_i, f_e}(i\omega, \theta_0)|$ as a function of frequency ω for various values of f_i . For each ratio f_i the inlet amplitude is shown for various values of f_e in the same color. For the same f_i value, smaller values of f_e correspond to smaller amplitudes. Observe that a smaller f_i leads to a higher amplitude at any given frequency, although all ratios of f_i lead to the same asymptotic value of $\gamma_m \phi_s = 0.02$. Notice that f_e only affects the amplitude at low frequencies and after $\omega = 7.0$ all ratios f_e yield the same amplitude at their respective value of f_i .

543 length when using a square wave actuator signal is shorter than when using
 544 a sinusoidal actuator signal, as expected from the result in Appendix D.

- 545 • For any value of f_e it is clear from the figure that by lowering f_i shorter
 546 experiment times are obtained. This effect has already been explained in
 547 Section 4.2.1.
- 548 • In sharp contrast to the single parameter experiments (see Table 3), we
 549 observe that now, instead, f_e should be chosen equal to $f_e = f_{e,opt} = 1.0$ as
 550 opposed to $f_e \gg 1$. The experiment length N is affected by f_e through the
 551 gradient of G_{f_e} , see (83). In Figs. 5 and 6 we see that both components of
 552 this gradient increase with increasing f_e , which in principle should shorten
 553 N . This is not the result we obtain from the optimal experiment design
 554 results.

In order to understand the curiosity mentioned in the last item, i.e. why the optimal ratio is now $f_{e,opt} = 1.0$ instead of $f_e = 20$, we need to analyse the physical system further. In fact, we will now show that the parameters k and ϕ are not identifiable in the limit $f_e \rightarrow \infty$. To this end, we introduce the general condition for identifiability ([42, 28])

$$G(i\omega, \boldsymbol{\theta}_1) = G(i\omega, \boldsymbol{\theta}_2) \forall \omega \Rightarrow \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2. \quad (91)$$

Taking the limit $f_e \rightarrow \infty$ of the frequency response $G_{f_e}(i\omega, \boldsymbol{\theta})$ in (62) yields

$$\lim_{f_e \rightarrow \infty} G_{f_e}(i\omega, \boldsymbol{\theta}) = \frac{1}{\cosh \sqrt{i\omega\phi/k}}. \quad (92)$$

555 Suppose that $\boldsymbol{\theta}_1 = [k_1, \phi_1]^T$ and $\boldsymbol{\theta}_2 = [\kappa k_1, \kappa \phi_1]^T$, where $\kappa \in \mathbb{R}$. Substitution
 556 in (92) then shows that $G_\infty(i\omega, \boldsymbol{\theta}_1) = G_\infty(i\omega, \boldsymbol{\theta}_2)$ for any frequency ω , whereas
 557 $\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$.

558 We have thus shown that in the limit of $f_e \rightarrow \infty$ the transfer function G_{f_e}
 559 contains the quotient $\frac{\phi}{k}$. As a consequence, infinitely many equal quotients exist
 560 by multiplying k and ϕ with the same scalar κ . This problem does not exist for
 561 the estimation of only one parameter, as the other one is known. How is this
 562 effect then reflected in (90)?

It turns out that the components of the gradient $\nabla_{\boldsymbol{\theta}} G_{f_e}(i\omega, \boldsymbol{\theta})$ are linearly dependent for all frequencies in the limit $f_e \rightarrow \infty$. Consequently, the inverse of the covariance matrix $\mathbf{P}_{\boldsymbol{\theta}}^{-1}$ is rank deficient for any input spectrum $\Phi_{u_D}(\omega)$. To see this, consider the derivative of the transfer function (62) with respect to the parameters k and ϕ ; see (63) and (64). Dividing the numerator and denominator of each derivative by f_e^2 and taking the limit $f_e \rightarrow \infty$ shows that

$$\frac{\partial G_{f_e}(i\omega, \theta)}{\partial k} = -\frac{k}{\phi} \frac{\partial G_{f_e}(i\omega, \theta)}{\partial \phi}. \quad (93)$$

This shows that the elements in $\nabla_{\boldsymbol{\theta}} G_{f_e}(i\omega, \boldsymbol{\theta})$ are linearly dependent at all frequencies. Consequently, for any input spectrum Φ_{u_D} , the inverse of the covariance matrix for the Direct Method in this limit, i.e.

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1} = \frac{NT_s}{2\pi\sigma_e^2} \begin{pmatrix} k^2/\phi^2 & -k/\phi \\ -k/\phi & 1 \end{pmatrix} \int_{-\pi/T_s}^{\pi/T_s} \left| \frac{\partial G_{f_e}(i\omega, \boldsymbol{\theta})}{\partial \phi} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}^2 |L_{f_i, f_e}(i\omega, \boldsymbol{\theta}_0)|^2 \Phi_{u_D}(\omega) d\omega \quad (94)$$

563 is rank deficient for all ω . Indeed, we find that $\det(\mathbf{P}_{\boldsymbol{\theta}}^{-1}) = 0$. It is clear that
 564 the variances of the estimates \hat{k}_N and $\hat{\phi}_N$ approach infinity when increasing f_e
 565 towards the limit $f_e \rightarrow \infty$.

566 4.2.3. Discussion

567 The above calculations show that when identifying the two parameters jointly,
 568 high values of f_e should be avoided. We see now that there are two compet-
 569 ing mechanisms. One the one hand, the sensitivity of the physical system with
 570 respect to the parameters increases with increasing f_e , which is beneficial to re-
 571 duce the individual variances of the parameters (and thus shorten the required
 572 experiment time). On the other hand, joint estimation of the two parameters
 573 becomes more difficult as the derivatives become more and more similar for
 574 large f_e . Hence, the parameters become increasingly correlated for increasing
 575 f_e for any excitation frequency (or, in fact, any input spectrum), making the
 576 covariance matrix singular in the limit $f_e \rightarrow \infty$. This effect already plays a
 577 dominant role at $f_e \approx 20$, as can be observed from Figs. 5 and 6. Consequently,
 578 a trade-off between these two mechanisms has to be made. Optimal experiment

579 design shows that the best trade-off is reached by choosing $f_e = 1.0$ and using
580 an optimal frequency of $\omega_{c,opt} = 2.1$. Due to this trade-off, the minimal experi-
581 ment time increases to $N = 2.51 \times 10^7$ (approximately five days), which is two
582 orders of magnitude larger than the experiment lengths obtained for separate
583 parameter estimation.

584 5. Experiment Design using the Indirect Method

585 In this section we consider experiment design for the Indirect Method. We
586 recall that we estimate the parameters in the transfer function $G_{f_i, f_e}(s, \boldsymbol{\theta}_0) =$
587 $L_{f_i, f_e}(s, \boldsymbol{\theta}_0)G_{f_e}(s, \boldsymbol{\theta}_0)$. The input in this case is the actuator signal $r(t)$ and the
588 output is the outlet pressure $p_e(t)$, see Fig. 4. We use the same core sample
589 values and scaling as in the previous section, see Tables 1 and 2. As in Section
590 4, we only derive the results for the square wave signal. The result for the
591 sinusoidal excitation can be treated in a similar fashion. The experiment design
592 theory on single parameter estimation can be found in Appendix C.

The power spectrum of the square wave actuator signal, corresponding to
the input signal (72), is defined by

$$\Phi_{u_D}(\omega) = \frac{16}{\pi^2} \frac{\pi C_\gamma^2(f_i)}{2T_s} \sum_{m=1}^{Q(T_s)} \frac{1}{(2m-1)^2} \sum_{l=\{-1,1\}} \delta(\omega - l[2m-1]\omega_c). \quad (95)$$

593 Substitution of this spectrum in the covariance matrix (77) yields

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1}[\omega_c, C_\gamma(f_i), f_i, f_e] = \frac{16}{\pi^2} \frac{NC_\gamma^2(f_i)}{2\sigma_e^2} \times \sum_{m=1}^{Q(T_s)} \frac{1}{(2m-1)^2} \text{Re} \left\{ [\nabla_{\boldsymbol{\theta}} G_{f_i, f_e}(i[2m-1]\omega_c, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \cdot [\text{C.C.}] \right\}, \quad (96)$$

594 where C.C. stands for complex conjugate and the transfer function G_{f_i, f_e} is given
595 by (68). As in the previous section, we search for the minimal identification
596 length N that nonetheless fulfils variance constraints on the estimates, and
597 respects the actuator amplitude limitations, by finding optimal values for γ , ω_c ,
598 f_i , and f_e .

599 *5.1. Optimization Problem and Its Solution*

The optimization problem in the Indirect Method is given by

$$\min_{\omega_c, f_i, f_e, C_\gamma(f_i)} \text{Experiment length } N \quad (97)$$

600 subject to

$$\text{var}(\hat{k}_N) = \mathbf{e}_1^T \mathbf{P}_\theta[\omega_c, C_\gamma(f_i), f_i, f_e] \mathbf{e}_1 \leq c_k, \quad (98)$$

$$\text{var}(\hat{\phi}_N) = \mathbf{e}_2^T \mathbf{P}_\theta[\omega_c, C_\gamma(f_i), f_i, f_e] \mathbf{e}_2 \leq c_\phi, \quad (99)$$

601 in which the inverse of the covariance matrix for two parameters is given by
 602 (96) for the transfer function G_{f_i, f_e} (68). Notice that the optimization problem
 603 is similar to (84)-(86), although we now consider the transfer function is G_{f_i, f_e}
 604 and the amplitude of the input is different.

605

606 The solution is found by following the same reasoning as in Section 4.1. We
 607 follow the steps 1-4 in Section 4.1, where we instead use (96) for \mathbf{P}_θ^{-1} .

608 *5.2. Numerical Results*

609 Let us now investigate the experiment design solutions, where we use the
 610 same physical set-up, experiment parameters, and gridding method as defined
 611 in Section 4.2. For convenience we recall that we wish to ensure that the variance
 612 of the estimate \hat{k}_N and $\hat{\phi}_N$ are respectively less than or equal to 5% of their true
 613 values k_0 and ϕ_0 , defined in Table 1. All parameters defining the porous medium
 614 are also mentioned in this table. Other parameters are defined in Table 2.
 615 We will first consider separate parameter estimation experiment design results,
 616 followed by the joint parameter results.

617 *5.2.1. Separate parameter estimation*

618 In Appendix C the optimization problem (B.1)-(B.2) and solution are given
 619 for the separate parameter estimation problem in case of a sinusoidal signal.
 620 The case of a square wave input signal is trivially formulated and solved with
 621 the use of (96).

622 The optimal experiment times are tabulated in Table 4. The function
 623 $N_{min}(f_i, f_e)$ (c.f. (C.2)) plotted against f_i and f_e is qualitatively similar to
 624 the Direct Method case (see Fig. B.13) for both $\theta = k$ and $\theta = \phi$. Furthermore,
 625 we observe the following:

- 626 • For both $\theta = k$ and $\theta = \phi$ the optimal ratios are $f_{i,opt} = 0.1$ and $f_{e,opt} =$
 627 20. Apparently, it is best practice to maximize the inlet volume and
 628 minimize the outlet volume.
- 629 • From (C.2) we find that f_e influences the gradient $\partial G_{f_i, f_e}(i\omega_{ex}, \theta)/\partial \theta$.
 630 The larger the gradient, the smaller $N_{min}(f_i, f_e)$ for any value of f_i . Fig-
 631 ures 7 and 8 show that increasing f_e increases the gradient of both k and
 632 ϕ . This explains why one should opt for maximising f_e .
- 633 • The same figures also show that decreasing f_i decreases the gradient, which
 634 increases the required experiment length. Yet, we see from Table 4 that
 635 f_i should be minimised. The actuator amplitude $C_{\gamma_m}(f_i)$ scales inversely
 636 proportional to f_i , and so minimising f_i increases the amplitude and re-
 637 duces the required experiment length. The two effects are thus competing,
 638 but the latter one is dominant. Hence, f_i should be minimised.
- 639 • The optimal excitation frequencies lie close to those values where the gra-
 640 dients are maximal, see Figs. 7 and 8.

641 5.2.2. Joint parameter estimation

642 The solution for the joint estimation of permeability and porosity is also
 643 presented in Table 4 and is discussed now. The numerical results are obtained
 644 through the same procedure as in Section 4, but using (96) to define \mathbf{P}_θ . We
 645 discuss here directly the case of the square wave signal:

- 646 • The function $N_{min}(f_i, f_e)$ attains its minimum for $f_{i,opt} = 0.6$ and $f_{e,opt} =$
 647 6.1, giving $N_{opt} = N_{min}(0.6, 6.1) = 1.8 \times 10^6$ (5 hours) and an optimal
 648 cycle frequency of $\omega_{c,opt} = 1.0$, see Fig. 11.

Indirect Method	k		ϕ		$\{k, \phi\}$	
	Sine	SW	Sine	SW	Sine	SW
	N_{opt}	3.00×10^5	1.74×10^5	3.6×10^5	2.1×10^5	3.3×10^6
ω_{opt}	2.55	2.4	2.5	2.4	1.2	1.0
$f_{i,opt}$	0.1	0.1	0.1	0.1	0.6	0.6
$f_{e,opt}$	20	20	20	20	6.4	6.1

Table 4: Scaled optimal experiment lengths, frequencies, and ratios f_i , f_e of input for the Indirect Method. The scaled system is defined through Table 1. These scalars define the scaled system in Section 2.2. For simultaneous estimation in case of a SW actuator signal, note that $f_{i,opt}$ and $f_{e,opt}$ translate into an inlet length of $L_i = L/3 \approx 1.7$ cm and an outlet length of $L_e = L/30 \approx 1.7$ mm, both of which are feasible in practice.

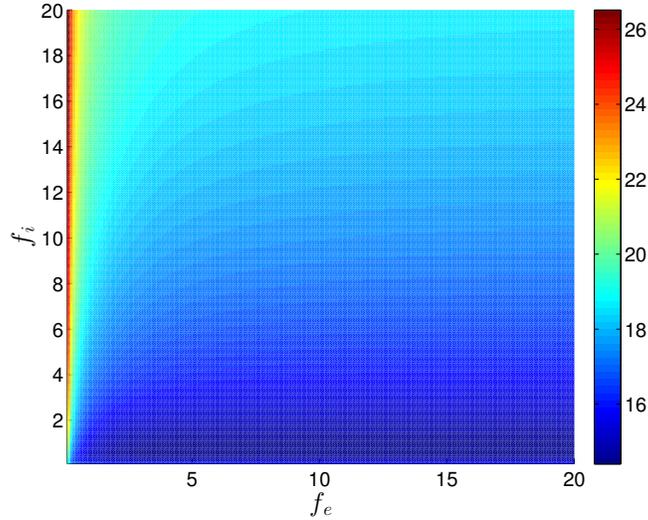


Figure 11: The logarithm of the minimal experiment length $N_{min}(f_i, f_e)$ as a function of the ratios f_i and f_e . Note that the shortest experiment length is obtained for $f_{i,opt} = 0.6$, $f_{e,opt} = 6.1$.

- 649 • These optimal values differ significantly from the separate estimation re-
650 sults, see Table 4. From Figs. 7 and 8 we see that for $f_i = 0.1$, $f_e = 20$ the
651 gradients of k and ϕ are very similar, resulting in high correlation between
652 the two parameters in a joint estimation. This increases the required ex-
653 periment length. However, we have here no identifiability problem as for
654 the Direct Method.
- 655 • In Section 5.2.1 it is explained that the actuator amplitude is maximised
656 by minimising f_i and therefore for those cases $f_{i,opt} = 0.1$. By virtue of the
657 previous item, a trade-off between parameter correlation and amplitude
658 maximisation has to be made. We see from Figs. 7 and 8 that the gradients
659 are large and not identical at the optimal ratios of f_i and f_e , so the increase
660 in f_i (reducing the amplitude and thus increasing the required experiment
661 length) is compensated by larger gradients.

662 5.2.3. Discussion

663 We now discuss some differences between the Indirect and Direct Method
664 results.

665 Let us first discuss the case of separate estimation. For this specific case,
666 observe from Tables 3 and 4 that permeability or porosity is estimated with
667 slightly shorter experiment times when using the Direct Method. However, we
668 point out that we have assumed the input signal $p_i(t)$ is noise free, corresponding
669 to the assumption that we know the filter $L_{f_i, f_e}(i\omega, \theta_0)$. In reality, this is not
670 the case, and will increase the variance of the estimates. Probably the Indirect
671 Method is then preferred.

672 Let us now go to the more interesting case of joint estimation of the two
673 parameters. Experiment Design has revealed that the optimal experiment length
674 strongly depends on the selected input signals. Indeed, the experiment time for
675 the Indirect Method is a factor fourteen smaller for joint estimation compared
676 to the Direct Method. We have seen for the latter method that f_e had to
677 be reduced due to singularity issues at $f_e \gg 1$. This back-off reduces both
678 the inlet pressure amplitude and the gradients, hereby increasing the required

679 experiment length enormously compared to the separate estimations. For the
680 Indirect Method, the combination ($f_i = 0.1, f_e = 20$) corresponds to highly
681 correlated parameters and thus a long experiment time. However, we found
682 that for $f_{i,opt} = 0.6$ and $f_{e,opt} = 6.1$ the gradients can be increased significantly
683 even though the actuator amplitude is less. The difference in the required
684 experiment length for joint estimation compared to the separate estimations is
685 therefore less drastic than in the Direct Method.

686 6. Simulation of the Experiment Set-Up: Estimation Results

687 In this last section we apply the optimal square wave input signal of the In-
688 direct Method detailed in Section 5, Table 4, to the core sample. All parameters
689 are defined in Tables 1 and 2. In the absence of a physical set-up, we instead
690 generate numerical experiment data $Z_N = \{u_D[n], y_D[n]\}_{n=1,\dots,N}$ by applying
691 our optimal input signal u_D (using the values in Table 4 and (72)) and gener-
692 ating noise-corrupted output sequences y_D using (74) and (73). We then follow
693 the estimation procedure for the Indirect Method detailed in Section 3.

694
695 We consider the simultaneous estimation of permeability and porosity from a
696 single experiment. We performed five thousand Monte Carlo simulations yield-
697 ing the estimated pairs $\{\hat{k}_N, \hat{\phi}_N\}$. The optimal square wave signal has a cycle
698 frequency of $\omega_{c,opt} = 1.0$, the experiment length is $N_{opt} = 1.8 \times 10^6$, and the
699 optimal ratio $f_{i,opt} = 0.6$, $f_{e,opt} = 6.1$, see Section 5. A scatter plot of the esti-
700 mates is shown in Fig. 12. The mean of all points is $(1.0, 1.0)$, corresponding to
701 the true scaled parameter values $k_0 = \phi_0 = 1.0$. The variance of the estimates
702 \hat{k}_N and $\hat{\phi}_N$ are respectively $\sigma_k^2 = 2.49 \times 10^{-3}$ and $\sigma_\phi^2 = 2.5 \times 10^{-3}$. The variance
703 constraints defined in Table 2 are clearly respected.

704 In conclusion, we see the optimal input signal and set-up conditions indeed
705 generate estimates that respect the variance constraints that we set prior to the
706 experiment.

707 We note that in the experiment design sections we have made use of the fact

708 that we know θ_0 in order to find the optimal input spectra, since the inverse
709 of the covariance matrix depends on θ_0 . Unfortunately, this dependence is uni-
710 versal in experiment design: no optimal experiment can be designed without
711 prior knowledge of the system. This work considers a non-Bayesian Experiment
712 Design approach; the unknown true parameter vector θ_0 in the inverse covari-
713 ance matrix should therefore be replaced by an initial estimate θ_{init} . We have
714 shown in [34] that by replacing the true vector by an initial guess nonetheless
715 delivers better results than an arbitrary input signal or degrees-of-freedom in
716 the set-up. Other solutions to tackle this problem are reported in [39, 41]. It
717 is worthwhile to remark that this problem has a close analog in the Bayesian
718 Experiment Design approaches. Indeed, in these approaches a *prior* has to be
719 defined for each parameter through e.g. assuming a uniform distribution [13].
720 Another method, known as "preposterior data analysis" [26], uses a collection
721 of simulated data sets that account for all possible experiment outcomes. This
722 set is used to find the best experiment set-up given that you do not know the
723 actual data prior to the experiment.

724 We furthermore note that our results are based on the values k_0 and ϕ_0
725 shown in Table 1. Different values will lead to different optimal frequencies
726 and optimal ratios $f_{i,opt}$ and $f_{e,opt}$. However, the Indirect Method will remain
727 a better estimation method than the Direct Method, regardless of the actual
728 values of permeability and porosity.

729 7. Discussion and Conclusions

730 In this paper we have introduced a novel estimation method that allows find-
731 ing the minimal experiment time that is required to estimate permeability and
732 porosity under user-specified parameter variance constraints and actuator limits.
733 We illustrated our methodology on the Dynamic Darcy Cell. We approached
734 the problem by introducing a classical least-squares estimation procedure, from
735 which we derived an expression for the covariance matrix of the estimates. This
736 expression allowed computation of the optimal input signal (either a sinusoid

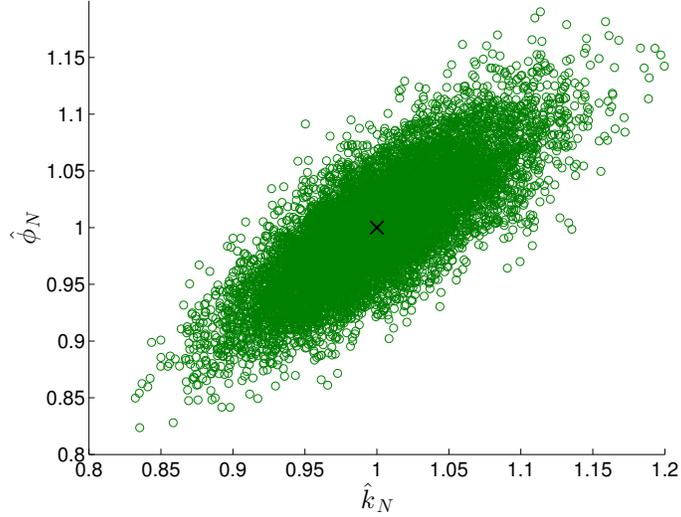


Figure 12: Scatter plot of the couples $\{\hat{k}_N, \hat{\phi}_N\}$ resulting from 5×10^3 experiments. The variance of permeability is $\sigma_k^2 = 9.9183 \times 10^{-5}$ and of porosity $\sigma_\phi^2 = 5.1211 \times 10^{-5}$. The black cross indicates the coordinate of the mean values of permeability and porosity, which were $\langle \hat{k}_N \rangle = 1.0$ and $\langle \hat{\phi}_N \rangle = 1.0$ and hence respectively equal to $k_0 = 1.0$ and $\phi_0 = 1.0$.

737 or square wave) and optimal experiment set-up degrees of freedom, being the
 738 ratios of the pore volume to inlet volume, f_i , and pore volume to outlet volume,
 739 f_e using Experiment Design techniques. We considered sinusoidal and square
 740 wave actuator signals and two measurement types: actuator position/outlet
 741 pressure measurements (the Indirect Method, novel in this article) and inlet
 742 pressure/outlet pressure measurements (Direct Method).

743

744 We have proven that square wave signals deliver shorter experiment lengths
 745 than sinusoidal ones under the same variance constraints and actuator limits.

746 The experiment design results for the Direct Method were as follows. For the
 747 joint estimation of permeability and porosity, we found that the optimal ratios
 748 $f_{e,opt} = 1.0$ and $f_{i,opt} = 0.1$, in contrast to separate parameter experiments for
 749 which $f_{e,opt} = 20$. This curiosity originates from a trade-off between variance
 750 reduction of permeability and porosity (which is obtained for high f_e values, and

751 reduces the variance of each separate parameter) and their correlation (which
752 increases with increasing f_e , and drives the inverse of the covariance matrix
753 to singularity). As a result, the minimum experiment time in this case is two
754 orders of magnitude larger than for the separate parameter experiments.

755 We also investigated the Indirect Method, novel in this article. We found
756 it superior to the Direct Method for joint parameter estimation: experiment
757 lengths of a factor fourteen less are found. This difference originates mainly
758 from the fact that, compared to the Direct Method, the sensitivity of the system
759 with respect to changes in permeability and porosity for the Indirect Method is
760 large over a broader range of f_i and f_e values.

761 Our analytical results are verified by simulating the Dynamic Darcy cell nu-
762 merically, and we found excellent agreement between the numerical results and
763 theoretical predictions.

764

765 Experiment design, and the introduction of the novel estimation method, clearly
766 have enormous potential in practice. This theoretical work is therefore also an
767 invitation to experimentalists to apply our technique to real cores in a labora-
768 tory.

769

770 Future work could consist of applying our methodology to a system at the catch-
771 ment scale, e.g. an aquifer. On this scale, multiple input and output signals
772 are typically available, which can be used to identify the spatially-dependent
773 permeability and porosity values of an inhomogeneous reservoir. This is for
774 instance the case in the Hydrogeological Experiment Site of Poitiers, France,
775 where pumping at several wells of the aquifer is possible ([43, 44]).

776 For applicability of our methodology to catchment-scale problems several
777 hurdles are to be overcome. First, the experiment design method has to be
778 generalised to multiple-input, multiple-output systems (e.g. an aquifer). Sec-
779 ond, our methodology is only suitable for linear models. Consequently, lin-
780 earisation techniques have to be applied to the governing nonlinear equations
781 of the catchment-scale problem, and possibly a dimensional analysis has to be

782 performed to validate whether or not the linear dynamics are sufficient to accu-
783 rately model the process. A third hurdle is the parameter identifiability issue
784 which, on the catchment scale, is likely much larger than on the core scale. User
785 choices such as the spatial resolution of the problem (determining the number
786 of porosity and permeability values that have to be estimated), and the number
787 of input and output locations will play an important role to address this issue.
788 An interesting option that could aid identifiability and input design is the pos-
789 sibility to gather additional information about the aquifer or reservoir with a
790 method known as Fiber-Optic Distributed Acoustic Sensing ([45, 46]).

791 **References**

- 792 [1] K. Heller, H. Bruining, D. Smeulders, Permeability obtained from pressure
793 oscillation experiments, part i, one phase flow, *Computational Methods in*
794 *Water Resources* 47 (2002) 201–207.
- 795 [2] M. Cardiff, T. Bakhos, P. Kitanidis, W. Barrash, Aquifer heterogeneity
796 characterisation with oscillatory pumping: Sensitivity analysis and imaging
797 potential, *Water Resources Research* 49 (2013) 5395–5410.
- 798 [3] I. Song, J. Renner, Analysis of oscillatory fluid flow through rock samples,
799 *International Journal of Geophysics* 170 (2007) 195–204.
- 800 [4] Y. Wang, R. Knabe, Permeability characterisation on tight gas samples
801 using pore pressure oscillation method, *Petrophysics* 52 (2011) 437–433.
- 802 [5] G. Fischer, The determination of permeability and storage capacity: Pore
803 pressure oscillation method, *International Geophysics Series* 51 (1992) 187–
804 211.
- 805 [6] G. Boitnott, Use of complex pore transients to measure permeability of
806 rocks, in: *SPE Annual Technical Conference and Exhibition*, 1997.

- 807 [7] M. Bernabé, B. Evans, A note on the oscillating flow method for measuring
808 rock permeability, *International Journal of Rock Mechanics and Mining*
809 *Sciences* 43 (2006) 311–316.
- 810 [8] P. Suri, M. Azeemuddin, M. Zaman, A. Kukreti, J.-C. Roegiers, Stress-
811 dependent permeability measurements using the oscillating pulse technique,
812 *Journal of Petroleum Science and Engineering* 17 (1997) 247–264.
- 813 [9] A. Hasanov, M. Batzle, Pore pressure pulsing effects on reservoir transport
814 properties, in: *Second International Workshop on Rock Physics*, 2013.
- 815 [10] R. Kranz, J. Saltzman, J. Blacic, Hydraulic diffusivity measurements on
816 laboratory rock samples using an oscillating pore pressure method, *Inter-
817 national Journal of Rock Mechanics and Mining Sciences & Geomechanics*
818 *Abstracts* 27 (1990) 345–352.
- 819 [11] V. Gupta, S. Sorooshian, The relationship between data and the precision of
820 parameter estimates of hydrologic models, *Journal of Hydrology* 81 (1985)
821 57–77.
- 822 [12] F. Kleissen, M. Beck, H. Weather, The identifiability of conceptual hy-
823 drochemical models, *Water Resources Research* 26 (1990) 2979–2992.
- 824 [13] K. Beven, A. Binley, The future of distributed models: Model calibration
825 and uncertainty prediction, *Hydrological Processes* 6 (2012) 279–298.
- 826 [14] B. Wagner, Simultaneous parameter estimation and contaminant source
827 characterisation for coupled groundwater flow and contaminant transport
828 modelling, *Journal of Hydrology* 135 (1992) 275–303.
- 829 [15] P. Yapo, H. Gupta, S. Sorooshian, Automatic calibration of conceptual
830 rainfall-runoff models: sensitivity to calibration data, *Journal of Hydrology*
831 181 (1996) 23–48.
- 832 [16] J. Kool, J. Parker, M. Van Genuchten, Parameter estimation for unsatu-
833 rated flow and transport models – a review, *Journal of Hydrology* 91 (1987)
834 255–293.

- 835 [17] S. Sorooshian, V. Gupta, J. Fulton, Evaluation of maximum likelihood pa-
836 rameter estimation techniques for conceptual rainfall-runoff models: Influ-
837 ence of calibration data variability and length on model credibility, *Water*
838 *Resources Research* 19 (1983) 251–259.
- 839 [18] S. Sorooshian, V. Gupta, Automatic calibration of conceptual rainfall-
840 runoff models: The question of parameter observability and uniqueness,
841 *Water Resources Research* 19 (1983) 260–268.
- 842 [19] S. Sorooshian, V. Gupta, The analysis of structural identifiability: The-
843 ory and application to conceptual rainfall-runoff models, *Water Resources*
844 *Research* 21 (1985) 487–495.
- 845 [20] G. Kuczera, Improved parameter inference in catchment models 1. evaluat-
846 ing parameter uncertainty, *Water Resources Research* 19 (1983) 1151–1162.
- 847 [21] G. Kuczera, Improved parameter inference in catchment models 2. com-
848 bining different kinds of hydrological data and testing their compatability,
849 *Water Resources Research* 19 (1983) 1163–1172.
- 850 [22] P. Mahar, B. Datta, Optimal identification of ground-water pollution
851 sources and parameter estimation, *Journal of Water Resources Planning*
852 *and Management* 127 (2001) 20–29.
- 853 [23] N.-S. Hsu, W.-G. Yeh, Optimum experimental design for parameter iden-
854 tification in groundwater hydrology, *Water Resources Research* 25 (1989)
855 1025–1040.
- 856 [24] T. Nishikawa, W.-G. Yeh, Optimum pumping test design for the parameter
857 identification of groundwater systems, *Water Resources Research* 25 (1989)
858 1737–1747.
- 859 [25] J. McCarthy, W.-G. Yeh, Optimal pumping test design for parameter es-
860 timation and prediction in groundwater hydrology, *Water Resources Re-*
861 *search* 26 (1990) 779–791.

- 862 [26] P. Leube, A. Geiges, W. Nowak, Bayesian assessment of the expected data
863 impact on prediction confidence in optimal sampling design, *Water Re-*
864 *sources Research* 48.
- 865 [27] X. Bombois, G. Scorletti, P. Van den Hof, R. Hilderbrand, Least-costly
866 identification experiment for control, *Automatica* 42 (2006) 1651–1662.
- 867 [28] L. Ljung, *System Identification: Theory for the User*, Prentice Hall, 1999.
- 868 [29] R. Aster, B. Borchers, C. Thurber, *Parameter Estimation and Inverse Prob-*
869 *lems*, Academic Press, 2005.
- 870 [30] J. Bear, *Dynamics of Fluids in Porous Media*, Dover Publications, 1972.
- 871 [31] P. Forchheimer, *Wasserbewegung durch boden*, *Zeit. Ver. Deutsch. Ing.* 45
872 (1901) 1781–1788.
- 873 [32] L. Dake, *Developments in Petroleum Science 8: Fundamentals of Reservoir*
874 *Engineering*, Elsevier, 1978.
- 875 [33] U. S. Lab, Rosetta soil database.
876 URL <http://www.ars.usda.gov/News/docs.htm?docid=8953>
- 877 [34] M. Potters, X. Bombois, M. Mansoori, P. Van den Hof, Estimat-
878 ing parameters with pre-specified accuracies in distributed parame-
879 ter systems, Accepted in the *International Journal of Control*, DOI:
880 10.1080/00207179.2016.1138143.
- 881 [35] T. Söderström, Errors-in-variables methods in system identification, *Auto-*
882 *matica* 43 (939–958).
- 883 [36] T. Söderström, L. Wang, R. Pintelon, J. Schoukens, Can errors-in-variables
884 systems be identified from closed-loop experiments, *Automatica* 49 (2013)
885 681–684.
- 886 [37] M. Mansoori, P. Van den Hof, J. Jansen, D. Rashtchian, Pressure tran-
887 sient analysis of bottomhole pressure and rate measurements using system
888 identification techniques, *SPE Journal* 20 (2015) 1005–1027.

- 889 [38] P. Van den Hof, R. Schrama, Indirect method for transfer function estima-
890 tion from closed loop data, *Automatica* 29 (6) (1993) 1523–1527.
- 891 [39] L. Gerencsèr, H. Hjalmarsson, J. Mårtensson, Identification of ARX sys-
892 tems with non-stationary inputs - asymptotic analysis with applications to
893 adaptive input design, *Automatica* 45 (2009) 623–633.
- 894 [40] C. Larsson, M. Annergren, H. Hjalmarsson, C. Rojas, X. Bombois, A. Mes-
895 bah, P. Modén, Model predictive control with integrated experiment design
896 for OE systems, in: *Proceedings of the 2013 European Control Conference*,
897 2013.
- 898 [41] M. Forgiione, X. Bombois, P. Van den Hof, Data-driven model improvement
899 for model-based control, *Automatica* 52 (2015) 118–124.
- 900 [42] H. Pohjanpalo, System identifiability based on the power series expansion
901 of the solution, *Mathematical Biosciences* 41 (1978) 21–33.
- 902 [43] A. Chamroo, R. Ouvrard, T. Poinot, G. Porel, B. Nauleau, J. Bodin,
903 Continuous-time model identification of wells interaction on the hydroge-
904 ological experimental site of poitiers, in: *European Control Conference*,
905 2014.
- 906 [44] A. Chamroo, R. Ouvrard, T. Poinot, J. Bodin, B. Nauleau, G. Porel, Hy-
907 drogeological experimental site of poitiers (france), in: *17th IFAC Sympo-*
908 *sium on System Identification*, 2015.
- 909 [45] J. van der Horst, et. al., Fiber optic sensing for improved wellbore pro-
910 duction surveillance, in: *International Petroleum Technology Conference*,
911 2014.
- 912 [46] J. Koelman, J. Lopez, J. Potters, Optical fibers: The neurons for future
913 intelligent wells, in: *SPE Intelligent Energy International*, 2012.

914 **Appendix A. Definitions using Sinusoidal Excitation Signals**

915 In this appendix we define the sinusoidal actuator signal $r[n]$ for the Direct
 916 and Indirect Method that can be used to estimate the parameters permeability
 917 and porosity in Section 3.

918 *Sinusoidal Input in the Indirect Method*

The power spectrum of a sinusoidal actuator signal, i.e. the input, with amplitude $C_\gamma(f_i)$ and frequency ω_{ex} is defined by

$$\Phi_{u_D}(\omega) = \frac{\pi C_\gamma^2(f_i)}{2T_s} \sum_{l=\{-1,1\}} \delta(\omega - l\omega_{ex}). \quad (\text{A.1})$$

This leads to the discrete-time actuator signal $r[n]$, which is the input $u_D[n]$, given by

$$u_D[n] = C_\gamma(f_i) \sin(\omega_{ex}nT_s) \quad (\text{A.2})$$

In this case the noise-free outlet pressure is given by

$$y_{nf}(\boldsymbol{\theta})[n] = C_\gamma(f_i) |G_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta})| \sin(\omega_{ex}nT_s + \alpha), \quad (\text{A.3})$$

919 where $\alpha = \angle |G_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta})|$. Replacing (72) and (73) by respectively (A.2)
 920 and (A.3) defines the estimation problem (75) for the Indirect Method using a
 921 sinusoidal actuator signal.

922 *Sinusoidal Input in the Direct Method*

The power spectrum the discrete-time inlet pressure signal, defined as the input signal in the Direct Method, as a consequence of a sinusoidal actuator signal with amplitude $C_\gamma(f_i)$ and frequency ω_{ex} is defined by

$$\Phi_{u_D}(\omega) = \frac{\pi C_\gamma^2(f_i)}{2T_s} |L_{f_i, f_e}(\omega_{ex}, \boldsymbol{\theta}_0)| \sum_{l=\{-1,1\}} \delta(\omega - l\omega_{ex}). \quad (\text{A.4})$$

The discrete-time pressure signal $p_i[n]$ is thus given by $u_D[n]$ and reads

$$u_D[n] = C_\gamma(f_i) |L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0)| \sin(\omega_{ex}nT_s + \angle L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0)), \quad (\text{A.5})$$

where the filter $L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0)$ is defined in (67). The noise-free outlet pressure is then given by

$$y_{nf}(\boldsymbol{\theta})[n] = C_\gamma(f_i) |L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0)| |G_{f_e}(i\omega_{ex}, \boldsymbol{\theta})| \sin(\omega_{ex}nT_s + \angle L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0) + \angle G_{f_e}(i\omega_{ex}, \boldsymbol{\theta})). \quad (\text{A.6})$$

923 Replacing (72) and (73) by respectively (A.5) and (A.6) defines the estimation
924 problem (75) for the Direct Method.

925 Appendix B. Single Parameter Experiment Design for Direct Method

In this appendix we consider the estimation of either permeability of porosity using a sinusoidal actuator signal. The power spectrum of the sinusoidal inlet pressure signal (c.f. (A.5)) is given by (A.4). The true parameter vector is denoted by $\boldsymbol{\theta}_0$ and is thus either equal to k_0 or ϕ_0 . To find the minimal experiment time that honours the variance constraint on one of the estimates given the amplitude constraints on the inlet pressure signal reads:

$$\min_{\omega_{ex}, C_\gamma(f_i), f_i, f_e} \text{Experiment length } N \quad (\text{B.1})$$

926 subject to

$$\text{var}(\hat{\theta}_N) = P_\theta[\omega_{ex}, C_\gamma(f_i), f_i, f_e] \leq c_\theta, \quad (\text{B.2})$$

where c_θ is either c_k (constraint value for permeability) or c_ϕ (constraint value for porosity), G_{f_e} is given by (62), and the inverse of the covariance matrix (??) is given by the scalar

$$P_\theta^{-1} = \frac{NC_\gamma^2(f_i)}{2\sigma_e^2} |L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta})|^2 \left| \frac{\partial G_{f_e}(i\omega_{ex}, \theta)}{\partial \theta} \right|_{\theta=\theta_0}^2. \quad (\text{B.3})$$

927 From this expression it is clear that a requirement to find the optimal solution
928 is that the condition $\text{var}(\hat{\theta}_N) \equiv P_\theta = c_\theta$ is met, and that furthermore that we
929 should select $\gamma = \gamma_m$, see previous sections for motivation. The solution to
930 (B.1)-(B.2) for given ratios of (f_i, f_e) is then trivially found to be

$$N_{min}(f_i, f_e) = \frac{2\sigma_e^2}{c_\theta C_{\gamma_m}^2(f_i)} \min_{\omega_{ex}} \left[|L_{f_i, f_e}(i\omega_{ex}, \boldsymbol{\theta}_0)|^{-2} \left| \frac{\partial G_{f_e}(i\omega_{ex}, \theta)}{\partial \theta} \right|_{\theta=\theta_0}^{-2} \right] \quad (\text{B.4})$$

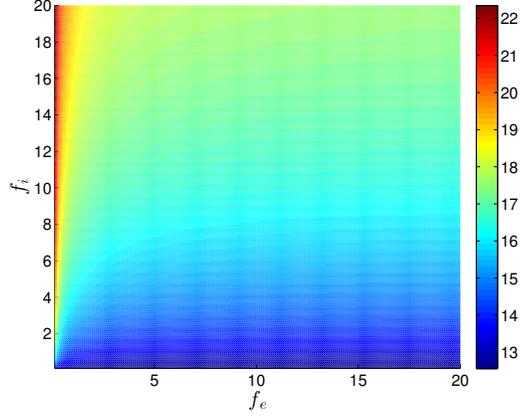


Figure B.13: The logarithm of $N_{min}(f_i, f_e)$ as a function of f_i and f_e for $\theta = k$. The optimal ratios are $f_{i,opt} = 0.1$ and $f_{e,opt} = 20$.

where the optimal excitation frequency $\omega_{ex,min}(f_i, f_e)$ is a function of f_i and f_e and is the frequency at which the minimum in (B.4) is obtained. The shortest possible experiment length is then found to be:

$$N_{opt} = N_{min}(f_{i,opt}, f_{e,opt}), \quad (f_{i,opt}, f_{e,opt}) = \arg \min_{f_i, f_e} N_{min}(f_i, f_e). \quad (\text{B.5})$$

931 The logarithm of the function $N_{min}(f_i, f_e)$ (c.f. (B.4)) is plotted for $\theta = k$ in
 932 Figure B.13 as a function of f_i and f_e . The result for porosity is qualitatively
 933 the same. All optimal values are reported in Table 3. All other parameters
 934 are defined in Tables 1 and 2. The constraint value is respectively $c_\theta = c_k$ and
 935 $c_\theta = c_\phi$ for these cases, see Table 2.

936 Appendix C. Single Parameter Experiment Design for Indirect Method

In this appendix we follow the exact same procedure as in Appendix B, but now for actuator to outlet measurements. The actuator limit is given by $\gamma_m = 0.1$. All parameters are defined in Tables 1 and 2. The optimization problem for the Indirect Method is given by (B.1)-(B.2), where P_θ should be replaced with

$$P_\theta^{-1}[\omega_c, C_\gamma(f_i), f_i, f_e] = \frac{NC_\gamma^2(f_i)}{2\sigma_e^2} \left| \frac{\partial G_{f_i, f_e}(i\omega_{ex}, \theta)}{\partial \theta} \right|_{\theta=\theta_0}^2. \quad (\text{C.1})$$

Following the same arguments as in the previous appendix we find that the minimal experiment time for a given set of $\{f_i, f_e\}$ is given by

$$N_{min}(f_i, f_e) = \frac{2\sigma_e^2}{c_\theta C_{\gamma_m}^2(f_i)} \min_{\omega_{ex}} \left| \frac{\partial G_{f_i, f_e}(i\omega_{ex})}{\partial \theta} \right|_{\theta=\theta_0}^{-2}, \quad (\text{C.2})$$

937 where G_{f_i, f_e} is defined in (68). The optimal experiment length N_{opt} is then
938 found as in the previous appendix.

939 **Appendix D. Dynamical Relationship between Inlet Pressure and** 940 **Outlet Pressure**

An alternative way to find the dynamical relationship between the inlet pressure $p_i(t)$ and outlet pressure $p_e(t)$ has been considered in [1]. In that paper, the expression for the system output when applying an input signal $u(t) = C \cos(\omega t)$ was found to be

$$y_{nf}(t) = p(x = 1, t) = C \frac{\Psi_1(\omega) \cos(\omega t) + \Psi_2(\omega) \sin(\omega t)}{\Psi_1^2(\omega) + \Psi_2^2(\omega)} \quad (\text{D.1})$$

941 in which (after some additional algebraic manipulation to their expressions)

$$\begin{aligned} \Psi_1(\omega) &= \cosh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \cos\left(\sqrt{\frac{\omega\phi}{2k}}\right) + \sqrt{\frac{\omega\phi}{2k}} f \times \\ &\quad \left[\sinh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \cos\left(\sqrt{\frac{\omega\phi}{2k}}\right) - \cosh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \sin\left(\sqrt{\frac{\omega\phi}{2k}}\right) \right], \\ \Psi_2(\omega) &= \sinh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \sin\left(\sqrt{\frac{\omega\phi}{2k}}\right) + \sqrt{\frac{\omega\phi}{2k}} f \times \\ &\quad \left[\sinh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \cos\left(\sqrt{\frac{\omega\phi}{2k}}\right) + \cosh\left(\sqrt{\frac{\omega\phi}{2k}}\right) \sin\left(\sqrt{\frac{\omega\phi}{2k}}\right) \right]. \end{aligned}$$

It is clear that (58) is more compact and easier to use, especially for more complex input signals. A simple time plot of these two signals shows that the expressions (58) and (D.1) are indeed equivalent for $u(t) = C \cos(\omega t)$ (not shown here). Note that for this comparison, $\sin(\cdot)$ in (58) needs to be replaced by $\cos(\cdot)$. In fact, one can show with trigonometric identities that their amplitude ratio between the outlet and inlet pressure

$$R = \frac{2}{\sqrt{\Psi_1^2 + \Psi_2^2}} \quad (\text{D.2})$$

is equal to $|G_{f_e}|$ in (62), and that their phase shift Θ between outlet and inlet pressure

$$\Theta = \arctan \frac{-\Psi_2}{\Psi_1} \quad (\text{D.3})$$

942 is equal to $\alpha = \angle G_{f_e}$.

943 Importantly, regardless of the different notations, the way of deriving the
944 expression of the ratio R and phase shift Θ in [1] is more involved than when
945 considering the coupled subsystems in Section 2. This becomes especially true
946 when considering the full inlet to outlet dynamics, i.e. subsystems I until III.

947 Appendix E. Square Wave versus Sinusoid

948 In this appendix we prove that a square wave input signal can deliver shorter
949 experiment times under the same variance constraints compared to a sinusoidal
950 input signal, and give a lower bound on how much shorter the experiment time
951 can be. This result is valid for the estimation of one or two parameters simulta-
952 neously. Indeed, one cannot identify more than two parameters with one single
953 sinusoid. The result derived here does not only pertain to the Direct and Indi-
954 rect Methods, but holds in general. The dimension of θ must however be equal
955 to or less than two.

We introduce some notations. The spectrum for a sinusoidal input signal with frequency ω_{ex} and amplitude C reads

$$\Phi_u(\omega) = \frac{\pi C^2}{2T_s} \sum_{l=\{-1,1\}} \delta(\omega - l\omega_{ex}), \quad (\text{E.1})$$

in which T_s is the sample time. The spectrum for a square wave with cycle frequency ω_c and amplitude C reads

$$\Phi_u(\omega) = \frac{8C^2}{\pi T_s} \sum_{m=1}^{Q(T_s)} \frac{1}{(2m-1)^2} \sum_{l=\{-1,1\}} \delta(\omega - l[2m-1]\omega_c), \quad (\text{E.2})$$

956 where $Q(T_s) = \left\lfloor \frac{1}{2} \left(\frac{\pi}{\omega_c T_s} + 1 \right) \right\rfloor$.

Lastly, for an input $u(t)$ that is connected to output $y(t)$ through a transfer function $G_{\Xi}(i\omega, \theta_0)$, where Ξ contains all degrees-of-freedom of the set-up, assuming white measurement noise and no feedback, the inverse of the covariance

matrix of the estimates $\hat{\boldsymbol{\theta}}_N$ is given by

$$\mathbf{P}_{\boldsymbol{\theta}}^{-1}[\Phi_u(\omega)] = \frac{NT_s}{2\pi\sigma_e^2} \int_{-\pi/T_s}^{\pi/T_s} [\nabla_{\boldsymbol{\theta}} G_{\Xi}(i\omega, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} [\nabla_{\boldsymbol{\theta}}^* G_{\Xi}(i\omega, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \Phi_u(\omega) d\omega, \quad (\text{E.3})$$

957 where σ_e^2 is the variance of the white noise, N the experiment length, and the
 958 asterisk denotes complex conjugation.

959

960 We are now ready to prove the following lemma.

Lemma 1. *Let N_1 , $\omega_{ex} = \omega_{opt}$, Ξ_{opt} , C be respectively the minimal experiment time, optimal excitation frequency, optimal set of degrees-of-freedom in the set-up, and amplitude C of the sinusoid $u(t) = C \sin(\omega_{ex}t)$ that solve the optimization problem*

$$\min_{\omega_{ex}, \Xi} N \quad (\text{E.4})$$

subject to

$$\forall \theta_i \in \boldsymbol{\theta} : \text{var}(\hat{\theta}_{i,N}) \leq c_{\theta_i}, \quad (\text{E.5})$$

961 where c_{θ_i} is the variance constraint for estimate $\hat{\theta}_{i,N}$. Then, using a square wave
 962 signal with cycle frequency $\omega_c = \omega_{opt}$ and amplitude C , the minimal experiment
 963 time N_{SW} that fulfil constraint (E.5) can be guaranteed to be at least a factor
 964 $\pi^2/16$ times smaller than N_1 .

Proof. The expression of the covariance matrix (E.3) for a single sinusoid with spectrum (E.1) using the optimal parameters $N = N_1$, $\omega_{ex} = \omega_{opt}$, $\Xi = \Xi_{opt}$, and amplitude C reads

$$\mathbf{P}_{\boldsymbol{\theta}, opt, 1}^{-1} = \frac{N_1 C^2}{2\sigma_e^2} \text{Re} \left\{ [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}^* \right\}. \quad (\text{E.6})$$

965 By definition this covariance matrix honours the variance constraints. The
 966 substitution of (E.2), $\omega_c = \omega_{opt}$, and $N = N_{SW}$ into the covariance matrix
 967 (E.3) results in

$$\mathbf{P}_{\boldsymbol{\theta}, SW}^{-1} = \frac{16 N_{SW} C^2}{\pi^2 2\sigma_e^2} \sum_{m=1}^{Q(T_s)} \frac{1}{(2m-1)^2} \text{Re} \left\{ [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i[2m-1]\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \times \right. \\ \left. [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i[2m-1]\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}^* \right\} \quad (\text{E.7})$$

968 Separating the expression $\mathbf{P}_{\boldsymbol{\theta},SW}^{-1}$ into the sum of modes $m = 1$ and $m > 1$ we
 969 find

$$\mathbf{P}_{\boldsymbol{\theta},SW}^{-1} = \frac{16 N_{SW} C^2}{\pi^2 2\sigma_e^2} \text{Re} \left\{ [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} [\nabla_{\boldsymbol{\theta}} G_{\Xi_{opt}}(i\omega_{opt}, \boldsymbol{\theta})]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}^* \right\} \\ + \text{extra positive definite terms.}$$

Clearly, by choosing $N_{SW} = \frac{\pi^2}{16} N_1 < N_1$ and using (E.6), (E.7), the above equation may be rewritten as

$$\mathbf{P}_{\boldsymbol{\theta},SW}^{-1} = \mathbf{P}_{\boldsymbol{\theta},opt,1}^{-1} + \text{extra positive definite terms.} \quad (\text{E.8})$$

What remains to show is that the variance(s) of $\hat{\theta}_{N,i}$ has (have) not increased. To this end, using Schur's complement, we rewrite the constraints in (E.5) for covariance matrix $\mathbf{P}_{\boldsymbol{\theta},SW}^{-1}$ as:

$$\forall \theta_i \in \boldsymbol{\theta} : \begin{pmatrix} c\theta_i & \mathbf{e}_i \\ \mathbf{e}_i^T & \mathbf{P}_{\boldsymbol{\theta},SW}^{-1} \end{pmatrix} \succeq \mathbf{0}. \quad (\text{E.9})$$

Denote $\boldsymbol{\Omega} \succeq \mathbf{0}$ the extra positive definite terms in (E.8). For the constraint on $\hat{\theta}_{N,i}$, we find by substitution of (E.8) that we require

$$\forall \theta_i \in \boldsymbol{\theta} : \begin{pmatrix} c\theta_i & \mathbf{e}_i \\ \mathbf{e}_i^T & \mathbf{P}_{\boldsymbol{\theta},opt,1}^{-1} + \boldsymbol{\Omega} \end{pmatrix} = \begin{pmatrix} c\theta_i & \mathbf{e}_i \\ \mathbf{e}_i^T & \mathbf{P}_{\boldsymbol{\theta},opt,1}^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\Omega} \end{pmatrix} \succeq \mathbf{0}. \quad (\text{E.10})$$

970 The first terms on the r.h.s. of this equation is positive definite by virtue of
 971 the fact that the constraint is satisfied for $\mathbf{P}_{\boldsymbol{\theta},opt,1}^{-1}$. Multiplying this equation
 972 on the left by an arbitrary vector $[\boldsymbol{\xi}, \boldsymbol{\theta}]^T$ and $[\boldsymbol{\xi}, \boldsymbol{\theta}]$ on the right, with $\boldsymbol{\xi}$ a scalar
 973 entry, shows that we require that $\boldsymbol{\theta}^T \boldsymbol{\Omega} \boldsymbol{\theta} > 0$ for all $\boldsymbol{\theta}$. This condition is clearly
 974 honoured since $\boldsymbol{\Omega} \succeq \mathbf{0}$. Hence, by selecting $N_{SW} = \pi^2/16N_1 < N_1$ we can
 975 honour the constraints with a shorter experiment time when using a square
 976 wave signal.

977

□