Variance of the volume of random real algebraic submanifolds

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Abstract

Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and let M denote its real locus. We study the vanishing locus Z_{s_d} in M of a random real holomorphic section s_d of $\mathcal{E}\otimes\mathcal{L}^d$, where $\mathcal{L}\to\mathcal{X}$ is an ample line bundle and $\mathcal{E}\to\mathcal{X}$ is a rank r Hermitian bundle. When $r\in\{1,\ldots,n-1\}$, we obtain an asymptotic of order $d^{r-\frac{n}{2}}$, as d goes to infinity, for the variance of the linear statistics associated to Z_{s_d}, including its volume. Given an open set $U\subset M$, we show that the probability that Z_{s_d} does not intersect U is a O of $d^{-\frac{n}{2}}$ when d goes to infinity. When $n\geqslant 3$, we also prove almost sure convergence for the linear statistics associated to a random sequence of sections of increasing degree. Our framework contains the case of random real algebraic submanifolds of $\mathbb{RP}^n$ obtained as the common zero set of r independent Kostlan–Shub–Smale polynomials.

Keywords: Random submanifolds, Kac–Rice formula, Linear statistics, Kostlan–Shub–Smale polynomials, Bergman kernel, Real projective manifold.

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1 Introduction

Framework. Let us first describe our framework and state the main results of this article (see Section 2 for more details). Let $\mathcal{X}$ be a smooth complex projective manifold of positive complex dimension n. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ and let $\mathcal{E}$ be a rank r holomorphic vector bundle over $\mathcal{X}$, with $r\in\{1,\ldots,n\}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures and that the real locus M of $\mathcal{X}$ is not empty. Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on $\mathcal{E}$ and $\mathcal{L}$ respectively that are compatible with the real structures. We assume that $h_{\mathcal{L}}$ has positive curvature ω. Then ω is a Kähler form on $\mathcal{X}$ and its induces a Riemannian metric g on M.

For any $d\in\mathbb{N}$, the Kähler form ω, h_E and $h_{\mathcal{L}}$ induce a L^2-inner product on the space $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ of real holomorphic sections of $\mathcal{E}\otimes\mathcal{L}^d\to\mathcal{X}$ (see (2.1)). Let $d\in\mathbb{N}$ and $s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, we denote by Z_s the real zero set $s^{-1}(0)\cap M$ of s. For d large enough, for almost every $s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, Z_s is a codimension r smooth submanifold of M and we denote by $|\mathrm{d}V_s|$ the Riemannian measure on Z_s induced by g (see Sect. 2.2). In the sequel, we will consider $|\mathrm{d}V_s|$ as a positive Radon measure on M. Let us also denote by $|\mathrm{d}V_M|$ the Riemannian measure on M.

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Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then $|\mathrm{d}V_{s_d}|$ is a random positive Radon measure on M. We set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$ to avoid too many subscripts. In a previous paper [14, thm. 1.3], we computed the asymptotic of the expected Riemannian volume of Z_d as $d \to +\infty$. Namely, we proved that:

$$\mathbb{E}[\mathrm{Vol}\,(Z_d)] = d^{\frac{r}{2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O\big(d^{\frac{r}{2}-1}\big)\,, \tag{1.1}$$

where, $\mathrm{Vol}\,(M)$ is the volume of M for $|\mathrm{d}V_M|$ and the volumes of spheres are Euclidean volumes. Here and in throughout this paper, $\mathbb{E}[\cdot]$ denotes the expectation of the random variable between the brackets, and $\mathbb{S}^m$ stands for the unit Euclidean sphere of dimension m.

Let $\phi \in \mathcal{C}^0(M)$, we denote by $\|\phi\|_\infty = \max_{x\in M} |\phi(x)|$ its norm sup. Besides, we denote by $\langle \cdot\,, \cdot \rangle$ the duality pairing between $\big(\mathcal{C}^0(M), \|\cdot\|_\infty\big)$ and its topological dual. Then, (1.1) can be restated as:

$$\mathbb{E}[\langle |\mathrm{d}V_d|\,, \mathbf{1}\rangle] = d^{\frac{r}{2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O\big(d^{\frac{r}{2}-1}\big)\,,$$

where $\mathbf{1} \in \mathcal{C}^0(M)$ stands for the unit constant function on M. The same proof gives similar asymptotics for $\mathbb{E}[\langle |\mathrm{d}V_d|\,, \phi\rangle]$ for any continuous $\phi : M \to \mathbb{R}$ (see [14, section 5.3]).

Theorem 1.1. *Let $\mathcal{X}$ be a complex projective manifold of positive dimension n defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ Hermitian vector bundle and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as $d \to +\infty$:*

$$\forall \phi \in \mathcal{C}^0(M), \qquad \mathbb{E}[\langle |\mathrm{d}V_d|\,, \phi\rangle] = d^{\frac{r}{2}} \left(\int_M \phi\,|\mathrm{d}V_M|\right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_\infty\, O\big(d^{\frac{r}{2}-1}\big)\,. \tag{1.2}$$

Moreover the error term $O\big(d^{\frac{r}{2}-1}\big)$ does not depend on ϕ.

In particular, we can define a sequence of Radon measures $(\mathbb{E}[|\mathrm{d}V_d|])_{d\geqslant d_0}$ on M by: for every $d \geqslant d_0$ and every $\phi \in \mathcal{C}^0(M)$, $\langle \mathbb{E}[|\mathrm{d}V_d|]\,, \phi\rangle = \mathbb{E}[\langle |\mathrm{d}V_d|\,, \phi\rangle]$. Then Thm. 1.1 implies that:

$$\big(d^{-\frac{r}{2}}\big)\,\mathbb{E}[|\mathrm{d}V_d|] \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\,|\mathrm{d}V_M|\,, \tag{1.3}$$

as continuous linear functionals on $\big(\mathcal{C}^0(M), \|\cdot\|_\infty\big)$.

Statement of the results. The main result of this paper is an asymptotic for the covariances of the linear statistics $\big\{\langle |\mathrm{d}V_d|\,, \phi\rangle \;\big|\; \phi \in \mathcal{C}^0(M)\big\}$. Before we can state our theorem, we need to introduce some additional notations.

As usual, we denote by $\mathrm{Var}(X) = \mathbb{E}\Big[(X - \mathbb{E}[X])^2\Big]$ the variance of the real random variable X, and by $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])\,(Y - \mathbb{E}[Y])]$ the covariance of the real random variables X and Y. We call *variance* of $|\mathrm{d}V_d|$ and we denote by $\mathrm{Var}(|\mathrm{d}V_d|)$ the symmetric bilinear form on $\mathcal{C}^0(M)$ defined by:

$$\forall \phi_1, \phi_2 \in \mathcal{C}^0(M), \qquad \mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1, \phi_2) = \mathrm{Cov}(\langle |\mathrm{d}V_d|\,, \phi_1\rangle\,, \langle |\mathrm{d}V_d|\,, \phi_2\rangle)\,. \tag{1.4}$$

Definition 1.2. Let $\phi \in \mathcal{C}^0(M)$, we denote by ϖ_ϕ its *continuity modulus*, which is defined by:

$$\begin{array}{rccc} \varpi_\phi : & (0, +\infty) & \longrightarrow & [0, +\infty) \\ & \varepsilon & \longmapsto & \sup\big\{|\phi(x) - \phi(y)| \;\big|\; (x, y) \in M^2, \rho_g(x, y) \leqslant \varepsilon\big\}\,, \end{array}$$

where $\rho_g(\cdot, \cdot)$ stands for the geodesic distance on (M, g).

Since M is compact, ϖ_ϕ is well-defined for every $\phi \in \mathcal{C}^0(M)$. Moreover every $\phi \in \mathcal{C}^0(M)$ is uniformly continuous and we have:

$$\forall \phi \in \mathcal{C}^0(M), \qquad \varpi_\phi(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

Note that, if $\phi : M \to \mathbb{R}$ is Lipschitz continuous, then $\varpi_\phi(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.

Definition 1.3. Let $L : V \to V'$ be a linear map between two Euclidean spaces, we denote by $\left|\det{}^{\perp}(L)\right|$ the *Jacobian* of L:

$$\left|\det{}^{\perp}(L)\right| = \sqrt{\det(LL^*)},$$

where $L^* : V' \to V$ is the adjoint operator of L.

See Section 4.1 for a quick discussion of the properties of this Jacobian. If A is an element of $\mathcal{M}_{rn}(\mathbb{R})$, the space of matrices of size $r \times n$ with real coefficients, we denote by $\left|\det{}^{\perp}(A)\right|$ the Jacobian of the linear map from $\mathbb{R}^n$ to $\mathbb{R}^r$ associated to A in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^r$.

Definition 1.4. For every $t > 0$, we define $(X(t), Y(t))$ to be a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ with variance matrix:

$$\left(\begin{array}{ccccc|ccccc}
1-\frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}}-\frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & \vdots & 0 & e^{-\frac{t}{2}} & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & 0 & \vdots & & \ddots & e^{-\frac{t}{2}} & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} \\
\hline
e^{-\frac{t}{2}}-\frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & 1-\frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & e^{-\frac{t}{2}} & \ddots & & \vdots & 0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & e^{-\frac{t}{2}} & 0 & \vdots & & \ddots & 1 & 0 \\
0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) \otimes I_r,$$

where I_r is the identity matrix of size r. That is, if we denote by $X_{ij}(t)$ (resp. $Y_{ij}(t)$) the coefficients of $X(t)$ (resp. $Y(t)$), the couples $(X_{ij}(t), Y_{ij}(t))$ with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$ are independent from one another and the variance matrix of $(X_{ij}(t), Y_{ij}(t))$ is:

$$\begin{pmatrix} 1-\frac{te^{-t}}{1-e^{-t}} & e^{-\frac{t}{2}}\left(1-\frac{t}{1-e^{-t}}\right) \\ e^{-\frac{t}{2}}\left(1-\frac{t}{1-e^{-t}}\right) & 1-\frac{te^{-t}}{1-e^{-t}} \end{pmatrix} \quad \text{if } j = 1, \text{ and} \quad \begin{pmatrix} 1 & e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} & 1 \end{pmatrix} \quad \text{otherwise.}$$

Notation 1.5. We set $\alpha_0 = \dfrac{n-r}{7+(r+1)(n+1)}$.

We can now state our main result.

Theorem 1.6. *Let $\mathcal{X}$ be a complex projective manifold of dimension $n \geqslant 2$ defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \ldots, n-1\}$ Hermitian vector bundle and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.*

Let $\beta \in (0, \frac{1}{2})$, then there exists $C_\beta > 0$ such that, for all $\alpha \in (0, \alpha_0)$, for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:

$$\operatorname{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = d^{r-\frac{n}{2}} \left(\int_M \phi_1 \phi_2 \, |\mathrm{d}V_M| \right) \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} + \|\phi_1\|_\infty \|\phi_2\|_\infty O\left(d^{r-\frac{n}{2}-\alpha}\right) + \|\phi_1\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\left(d^{r-\frac{n}{2}}\right), \quad (1.5)$$

where

$$\mathcal{I}_{n,r} = \frac{1}{2} \int_0^{+\infty} \left(\frac{\mathbb{E}\left[\left|\det{}^\perp(X(t))\right| \left|\det{}^\perp(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r \left(\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r}\right)}{\operatorname{Vol}\left(\mathbb{S}^n\right)} \right)^2 \right) t^{\frac{n-2}{2}} \, \mathrm{d}t < +\infty. \quad (1.6)$$

Moreover the error terms $O\left(d^{r-\frac{n}{2}-\alpha}\right)$ and $O\left(d^{r-\frac{n}{2}}\right)$ in (1.5) do not depend on (ϕ_1, ϕ_2).

We obtain the variance of the volume of Z_d by applying Thm. 1.6 to $\phi_1 = \phi_2 = \mathbf{1}$. When $\phi_1 = \phi_2 = \phi$ we get the following.

Corollary 1.7 (Variance of the linear statistics)**.** *In the same setting as Thm. 1.6, let $\beta \in (0, \frac{1}{2})$, then there exists $C_\beta > 0$ such that, for all $\alpha \in (0, \alpha_0)$ and all $\phi \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:*

$$\operatorname{Var}(\langle |\mathrm{d}V_d| \, , \phi \rangle) = d^{r-\frac{n}{2}} \left(\int_M \phi^2 \, |\mathrm{d}V_M| \right) \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} + \|\phi\|_\infty^2 O\left(d^{r-\frac{n}{2}-\alpha}\right) + \|\phi\|_\infty \varpi_\phi\left(C_\beta d^{-\beta}\right) O\left(d^{r-\frac{n}{2}}\right). \quad (1.7)$$

Moreover, the error terms do not depend on ϕ.

Remarks 1.8. Some remarks are in order.

- The value of the constant α_0 should not be taken too seriously. This constant apppears for technical reasons and it is probably far from optimal.
- If ϕ_2 is Lipschitz continuous with Lipschitz constant K, then the error term in Eq. (1.5) can be replaced by:
$$\|\phi_1\|_\infty \left(\|\phi_2\|_\infty + K\right) O\left(d^{r-\frac{n}{2}-\alpha}\right)$$
by fixing $\beta > \alpha_0$, which is possible since $\frac{1}{2} > \alpha_0$.
- Thm. 1.6 shows that $\operatorname{Var}(|\mathrm{d}V_d|)$ is a continuous bilinear form on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ for d large enough. Moreover, denoting by $\langle \cdot \, , \cdot \rangle_M$ the L^2-inner product on $\mathcal{C}^0(M)$ defined by $\langle \phi_1 \, , \phi_2 \rangle_M = \displaystyle\int_M \phi_1 \phi_2 \, |\mathrm{d}V_M|$, we have:
$$d^{\frac{n}{2}-r} \operatorname{Var}(|\mathrm{d}V_d|) \xrightarrow[d\to+\infty]{} \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} \langle \cdot \, , \cdot \rangle_M$$
in the weak sense. A priori, there is no such convergence as continuous bilinear forms on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ since the estimate (1.5) involves the continuity modulus of ϕ_2.
- The fact that the constant $\mathcal{I}_{n,r}$ is finite is part of the statement and is proved below (Lemma 4.25). This constant is necessarily non-negative. Numerical evidence suggests that it is positive but we do not know how to prove it at this point.
- Thm. 1.6 does not apply in the case of maximal codimension ($r = n$). This case presents an additional singularity which causes our proof to fail. However, we believe a similar result to be true for $r = n$, at least in the case of the Kostlan–Shub–Smale polynomials described below (compare [6, 32]).

Corollary 1.9 (Concentration in probability)**.** *In the same setting as Thm. 1.6, let* $\alpha \geqslant \frac{r}{2} - \frac{n}{4}$ *and let* $\phi \in \mathcal{C}^0(M)$*. Then, for every* $\varepsilon > 0$*, we have:*

$$\mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \, , \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi\rangle]\right| > d^{\alpha}\varepsilon\right) = \frac{1}{\varepsilon^2} O\big(d^{r-\frac{n}{2}-2\alpha}\big) \, ,$$

where the error term is independent of ε*, but depends on* ϕ*.*

Corollary 1.10. *In the same setting as Thm. 1.6, let* $U \subset M$ *be an open subset, then as* $d \to +\infty$ *we have:*

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\big(d^{-\frac{n}{2}}\big) \, .$$

Our last corollary is concerned with the convergence of a random sequence of sections of increasing degree. Let us denote by $\mathrm{d}\nu_d$ the standard Gaussian measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (see (2.4)). Let $\mathrm{d}\nu$ denote the product measure $\bigotimes_{d\in\mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then we have the following.

Corollary 1.11 (Almost sure convergence)**.** *In the same setting as Thm. 1.6, let us assume that* $n \geqslant 3$*. Let* $(s_d)_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ *be a random sequence of sections. Then,* $\mathrm{d}\nu$*-almost surely, we have:*

$$\forall \phi \in \mathcal{C}^0(M), \qquad d^{-\frac{r}{2}} \langle |\mathrm{d}V_{s_d}| \, , \phi\rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} \left(\int_M \phi \, |\mathrm{d}V_M|\right) .$$

That is, $\mathrm{d}\nu$*-almost surely,*

$$d^{-\frac{r}{2}} |\mathrm{d}V_{s_d}| \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} |\mathrm{d}V_M| \, ,$$

in the sense of the weak convergence of measures.

Remark 1.12. We expect this result to hold for $n = 2$ as well, but our proof fails in this case.

The Kostlan–Shub–Smale polynomials Let us consider the simplest example of our framework. We choose $\mathcal{X}$ to be the complex projective space $\mathbb{CP}^n$, with the real structure defined by the usual conjugation in $\mathbb{C}^{n+1}$. Then M is the real projective space $\mathbb{RP}^n$. Let $\mathcal{L} = \mathcal{O}(1)$ be the hyperplane line bundle, equipped with its natural real structure and the metric dual to the standard metric on the tautological line bundle over $\mathbb{CP}^n$. Then the curvature form of $\mathcal{L}$ is the Fubini–Study form ω_{FS}, normalized so that the induced Riemannian metric is the quotient of the Euclidean metric on the unit sphere of $\mathbb{C}^{n+1}$. Let $\mathcal{E} = \mathbb{C}^r \times \mathbb{CP}^n \to \mathbb{CP}^n$ be the rank r trivial bundle with the trivial real structure and the trivial metric.

In this setting, the global holomorphic sections of $\mathcal{L}^d$ are the complex homogeneous polynomials of degree d in $n+1$ variables and those of $\mathcal{E} \otimes \mathcal{L}^d$ are r-tuples of such polynomials, since $\mathcal{E}$ is trivial. Finally, the real structures being just the usual conjugations, we have:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \mathbb{R}^d_{\mathrm{hom}}[X_0, \ldots, X_n]^r,$$

where $\mathbb{R}^d_{\mathrm{hom}}[X_0, \ldots, X_n]$ is the space of real homogeneous polynomials of degree d in $n+1$ variables. The r copies of this space in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ are pairwise orthogonal for the inner product (2.1). Hence a standard Gaussian in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is a r-tuple of independent standard Gaussian in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \ldots, X_n] = \mathbb{R}H^0\left(\mathcal{X}, \mathcal{L}^d\right)$.

It is well-known (cf. [3, 4, 11]) that the monomials are pairwise orthogonal for the L^2-inner product (2.1), but not orthonormal. Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, we denote its length by $|\alpha| = \alpha_0 + \cdots + \alpha_n$. We also define $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ and $\alpha! = (\alpha_0!) \cdots (\alpha_n!)$. Finally,

if $|\alpha| = d$, we denote by $\binom{d}{\alpha}$ the multinomial coefficient $\frac{d!}{\alpha!}$. Then, an orthonormal basis of $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ for the inner product (2.1) is given by the family:

$$\left(\sqrt{\frac{(d+n)!}{\pi^n d!}} \sqrt{\binom{d}{\alpha}} X^\alpha \right)_{|\alpha|=d}.$$

Thus a standard Gaussian vector in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ is a random polynomial:

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where the coefficients $(a_\alpha)_{|\alpha|=d}$ are independent real standard Gaussian variables. Since we are only concerned with the zero set of this random polynomial, we can drop the factor $\sqrt{\frac{(d+n)!}{\pi^n d!}}$.

Finally, in this setting, $|\mathrm{d}V_d|$ is the common zero set of r independent random polynomials in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ of the form:

$$\sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha, \tag{1.8}$$

with independent coefficients $(a_\alpha)_{|\alpha|=d}$ distributed according to the real standard Gaussian distribution. Such polynomials are known as the Kostlan–Shub–Smale polynomials. They were introduced in [11, 27] and were actively studied since (cf. [1, 5, 6, 21, 32]).

Related works. As we just said, zero sets of systems of independent random polynomials distributed as (1.8) were studied by Kostlan [11] and Shub and Smale [27]. The expected volume of these random algebraic manifolds was computed by Kostlan [11] and their expected Euler characteristic was computed by Podkorytov [21] in codimension 1, and by Bürgisser [5] in higher codimension. Both these results were extended to the setting of the present paper in [14].

In [32], Wschebor obtained an asymptotic bound, as the dimension n goes to infinity, for the variance of number of real roots of a system of n independent Kostlan–Shub–Smale polynomials. Recently, Dalmao [6] computed an asymptotic of order $\sqrt{d}$ for the variance of the number of real roots of one Kostlan–Shub–Smale polynomial in dimension $n = 1$. His result is very similar to (1.5), which leads us to think that such a result should hold for $r = n$. He also proved a central limit theorem for this number of real roots, using Wiener chaos methods.

In [12, thm. 3], Kratz and Leòn considered the level curves of a centered stationary Gaussian field with unit variance on the plane $\mathbb{R}^2$. More precisely, they considered the length of a level curve intersected with some large square $[-T, T] \times [-T, T]$. As $T \to +\infty$, they proved asymptotics of order T^2 for both the expectation and the variance of this length. They also proved that it satisfies a central limit theorem as $T \to +\infty$. In particular, their result applies to the centered Gaussian field on $\mathbb{R}^2$ with correlation function $(x, y) \mapsto \exp\left(-\frac{1}{2}\|x - y\|^2\right)$. This field can be seen as the scaling limit, in the sense of [19], of the centered Gaussian field $(s_d(x))_{x \in M}$ defined by our random sections, when $n = 2$ and $r = 1$.

The study of more general random algebraic submanifolds, obtained as the zero sets of random sections, was pioneered by Shiffman and Zelditch [23, 24, 25]. They considered the integration current over the common complex zero set Z_d of r independent random sections in $H^0(\mathcal{X}, \mathcal{L}^d)$, distributed as standard complex Gaussians. In [23], they computed

the asymptotic, as d goes to infinity, of the expectation of the associated smooth statistics when $r = 1$. They also provided an upper bound for the variance of these quantities and proved the equivalent of Cor. 1.11 in this complex algebraic setting. In [24], they gave an asymptotic of order $d^{2r-n-\frac{1}{2}}$ for the variance of the volume of $Z_d \cap U$, where $U \subset \mathcal{X}$ is a domain satisfying some regularity conditions. In [25], they proved a similar asymptotic for the variance of the smooth statistics associated to Z_d. When $r = 1$, they deduced a central limit theorem from these estimates and an asymptotic normality result of Sodin and Tsirelson [29]. Finally, in [26, thm. 1.4], Shiffman, Zelditch and Zrebiec proved that the probability that $Z_d \cap U = \emptyset$, where U is any open subset of $\mathcal{X}$, decreases exponentially fast as d goes to infinity.

Coming back to our real algebraic setting, one should be able to deduce from the general result of Nazarov and Sodin [19, thm. 3] that, given an open set $U \subset M$, the probability that $Z_d \cap U = \emptyset$ goes to 0 as d goes to infinity. Corollary 1.10 gives an upper bound for the convergence rate. In particular, this bounds the probability for Z_d to be empty. In the same spirit, Gayet and Welschinger [9] proved the following result. Let Σ be a fixed diffeomorphism type of codimension r submanifold of $\mathbb{R}^n$, let $x \in M$ and let $B_d(x)$ denote the geodesic ball of center x and radius $\frac{1}{\sqrt{d}}$. Then, the probability that $Z_d \cap B_d(x)$ contains a submanifold diffeomorphic to Σ is bounded from below. On the other hand, when $n = 2$ and $r = 1$, the Harnack–Klein inequality shows that the number of connected components of Z_d is bounded by a polynomial in d. In [7], Gayet and Welschinger proved that the probability for Z_d to have the maximal number of connected components decreases exponentially fast with d.

Another well-studied model of random submanifolds is that of Riemannian random waves, i.e. zero sets of random eigenfunctions of the Laplacian associated to some eigenvalue λ. In this setting, Rudnick and Wigman [22] computed an asymptotic bound, as $\lambda \to +\infty$, for the variance of the volume of a random hypersurface on the flat n-dimensional torus $\mathbb{T}^n$. On $\mathbb{T}^2$, this result was improved by Krishnapur, Kurlberg and Wigman [13] who computed the precise asymptotic of the variance of the lenght of a random curve. In [31], Wigman computed the asymptotic variance of the linear statistics associated to a random curve on the Euclidean sphere $\mathbb{S}^2$. His result holds for a large class of test-function that contains the characteristic functions of open sets satisfying some regularity assumption. In relation with Cor. 1.10, Nazarov and Sodin [18] proved that, on the Euclidean sphere $\mathbb{S}^2$, the number of connected components of a random curve times $\frac{1}{\lambda}$ converges exponentially fast in probability to a deterministic constant as $\lambda \to +\infty$.

About the proof. The idea of the proof is the following. The random section s_d defines a centered Gaussian field $(s_d(x))_{x\in\mathcal{X}}$. The correlation kernel of this field equals the Bergman kernel, that is the kernel of the orthogonal projection onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ for the inner product (2.1) (compare [3, 14, 23, 24, 25]).

In order to compute the covariance of the smooth statistics $\langle |\mathrm{d}V_s| \, , \phi_1 \rangle$ and $\langle |\mathrm{d}V_s| \, , \phi_2 \rangle$, we apply a Kac–Rice formula (cf. [2, 3, 6, 30, 31]). This allows us to write $\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1, \phi_2)$ as the integral over $M \times M$ of some function $\mathcal{D}_d(x, y)$, defined by (4.9). This density $\mathcal{D}_d(x, y)$ is the difference of two terms, coming respectively from

$$\mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_1 \rangle \, \langle |\mathrm{d}V_d| \, , \phi_2 \rangle] \qquad \text{and} \qquad \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_1 \rangle] \, \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_2 \rangle] \, .$$

Since the Bergman kernel decreases exponentially fast outside of the diagonal Δ in M^2 (see Section 3.4), the values of $s_d(x)$ and $s_d(y)$ are almost uncorrelated for (x, y) far from Δ. As a consequence, when the distance between x and y is much larger than $\frac{1}{\sqrt{d}}$, the above two terms in the expression of $\mathcal{D}_d(x, y)$ are equal, up to a small error (see Sect. 4.3.2 for a precise statement). Thus, $\mathcal{D}_d(x, y)$ is small far from Δ, and its integral over this domain only contributes a remainder term to $\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1, \phi_2)$.

The main contribution to the value of $\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2)$ comes from the integration of $\mathcal{D}_d(x, y)$ over a neighborhood of Δ of size about $\frac{1}{\sqrt{d}}$. We perform a change of variable in order to express this term as an integral over a domain of fixed size. This rescaling by $\frac{1}{\sqrt{d}}$ explains the factor $d^{-\frac{n}{2}}$ in (1.5). Besides, the order of growth of $\mathcal{D}_d(x, y)$ close to Δ is d^r, that is the order of growth of the square of $\mathbb{E}[|\mathrm{d}V_d|]$ (see Thm. 1.1). Finally, we get an order of growth of $d^{r-\frac{n}{2}}$ for $\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2)$. The constant in (1.5) appears as the scaling limit of the integral of $\mathcal{D}_d(x, y)$ over a neighborhood of Δ of typical size $\frac{1}{\sqrt{d}}$.

The difficulty in making this sketch of proof rigorous comes from the combination of two facts. First, we do not know exactly the value of the Bergman kernel (our correlation function) and its derivatives, but only asymptotics. In addition, the conditioning in the Kac–Rice formula is singular along Δ, and so is $\mathcal{D}_d$. Because of this, we lose all uniformity in the control of the error terms close to the diagonal. Nonetheless, by careful bookkeeping of the error terms, we can make the above heuristic precise.

Outline of the paper. In Section 2 we describe precisely our framework and the construction of the random measures $|\mathrm{d}V_{s_d}|$. We also introduce the Bergman kernel and explain how it is related to our random submanifolds.

In Section 3, we recall various estimates for the Bergman kernel that we use in the proof of our main theorem. These estimates were established by Ma and Marinescu [15, 16, 17] in a complex setting. Our main contribution in this section consists in checking that the preferred trivialization used by Ma and Marinescu to state their near-diagonal estimates is well-behaved with respect to the real structures on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ (see Section 3.1).

Section 4 is concerned with the proof of Thm. 1.6. In Sect. 4.1, we prove a Kac–Rice formula adapted to our problem, using Federer's coarea formula and Kodaira's embedding theorem. In Sect. 4.2 we prove an integral formula for the variance, using the Kac–Rice formula (Thm. 4.4). The core of the proof is contained in Sect. 4.3.

Finally, we prove Corollaries 1.9, 1.10 and 1.11 in Section 5.

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Contents

2 Random real algebraic submanifolds

2.1 General setting

In this section, we introduce our framework. It is the same as the algebraic setting of [14], see also [8, 9]. Classical references for the material of this section are [10, chap. 0] and [28, chap. 1].

Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension $n \geqslant 2$. We assume that $\mathcal{X}$ is defined over the reals, that is $\mathcal{X}$ is equipped with an anti-holomorphic involution $c_{\mathcal{X}}$. The real locus of $(\mathcal{X}, c_{\mathcal{X}})$ is the set of fixed points of $c_{\mathcal{X}}$. In the sequel, we assume that it is non-empty and we denote it by M. It is a classical fact that M is a smooth closed (i.e. compact without boundary) submanifold of $\mathcal{X}$ of real dimension n (see [28, chap. 1]).

Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle of rank $r \in \{1, \ldots, n-1\}$. Let $c_{\mathcal{E}}$ be a real structure on $\mathcal{E}$, compatible with $c_{\mathcal{X}}$ in the sense that the projection $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{X}$ satisfies $c_{\mathcal{X}} \circ \pi_{\mathcal{E}} = \pi_{\mathcal{E}} \circ c_{\mathcal{E}}$ and $c_{\mathcal{E}}$ is fiberwise $\mathbb{C}$-anti-linear. Let $h_{\mathcal{E}}$ be a real Hermitian metric on $\mathcal{E}$, that is $c_{\mathcal{E}}^{\star}(h_{\mathcal{E}}) = \overline{h_{\mathcal{E}}}$.

Similarly, let $\mathcal{L} \to \mathcal{X}$ be an ample holomorphic line bundle equipped with a compatible real structure $c_{\mathcal{L}}$ and a real Hermitian metric $h_{\mathcal{L}}$. Moreover, we assume that the curvature form ω of $h_{\mathcal{L}}$ is a Kähler form. Recall that if ζ is any non-vanishing holomorphic section on the open set $\Omega \subset \mathcal{X}$, then the restriction of ω to Ω is given by:

$$\omega_{/\Omega} = \frac{1}{2i} \partial\bar{\partial} \ln \left(h_{\mathcal{L}}(\zeta, \zeta) \right).$$

This Kähler form is associated to a Hermitian metric $g_{\mathbb{C}}$ on $\mathcal{X}$. The real part of $g_{\mathbb{C}}$ defines a Riemannian metric $g = \omega(\cdot, i\cdot)$ on $\mathcal{X}$, compatible with the complex structure. Note that, since $h_{\mathcal{L}}$ is compatible with the real structures on $\mathcal{X}$ and $\mathcal{L}$, we have $c_{\mathcal{L}}^{\star}(h_{\mathcal{L}}) = \overline{h_{\mathcal{L}}}$ and $c_{\mathcal{X}}^{\star}\omega = -\omega$. Then we have $c_{\mathcal{X}}^{\star} g_{\mathbb{C}} = \overline{g_{\mathbb{C}}}$, hence $c_{\mathcal{X}}^{\star} g = g$ and $c_{\mathcal{X}}$ is an isometry of $(\mathcal{X}, g)$.

Then g induces a Riemannian measure on every smooth submanifold of $\mathcal{X}$. In the case of $\mathcal{X}$, this measure is given by the volume form $\mathrm{d}V_{\mathcal{X}} = \frac{\omega^n}{n!}$. We denote by $|\mathrm{d}V_M|$ the Riemannian measure on (M, g).

Let $d \in \mathbb{N}$, then the rank r holomorphic vector bundle $\mathcal{E} \otimes \mathcal{L}^d$ can be endowed with a real structure $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d$, compatible with $c_{\mathcal{X}}$, and a real Hermitian metric $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$. If $x \in M$, then c_d induces a $\mathbb{C}$-anti-linear involution of the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the fixed points set of this involution, which is a dimension r real vector space.

Let $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. We can define a Hermitian inner product on $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ by:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \qquad \langle s_1 \, , s_2 \rangle = \int_{\mathcal{X}} h_d(s_1(x), s_2(x)) \, \mathrm{d}V_{\mathcal{X}}. \tag{2.1}$$

We say that a section $s \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is real if it is equivariant for the real structures, that is: $c_d \circ s = s \circ c_{\mathcal{X}}$. Let $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the real vector space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. The restriction of $\langle \cdot , \cdot \rangle$ to $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product.

Notation 2.1. In this paper, $\langle \cdot , \cdot \rangle$ will always denote either the inner product on the concerned Euclidean (or Hermitian) space or the duality pairing between a space and its topological dual. Which one will be clear from the context.

Let $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This space has finite complex dimension N_d by Hodge's theory (compare [15, thm. 1.4.1]). We denote by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \;\middle|\; c_d \circ s = s \circ c_{\mathcal{X}} \right\}. \tag{2.2}$$

The restriction of the inner product (2.1) to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ makes it into a Euclidean space of real dimension N_d.

Remark 2.2. Note that, even when we consider real sections restricted to M, the inner product is defined by integrating on the whole complex manifold $\mathcal{X}$.

2.2 Random submanifolds

This section is concerned with the definition of the random submanifolds we consider and the related random variables.

Let $d \in \mathbb{N}$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote the real zero set of s by $Z_s = s^{-1}(0) \cap M$. If the restriction of s to M vanishes transversally, then Z_s is a smooth submanifold of codimension r of M. In this case, we denote by $|\mathrm{d}V_s|$ the Riemannian measure on Z_s induced by g, seen as a Radon measure on M. Note that this includes the case where Z_s is empty.

Recall the following facts, that we already discussed in [14].

Definition 2.3 (see [20])**.** We say that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-*ample* if, for any $x \in M$, the evaluation map

$$\begin{array}{rccc} \mathrm{ev}_x^d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \\ & s & \longmapsto & s(x) \end{array} \tag{2.3}$$

is surjective.

Lemma 2.4 (see [14], cor. 3.10)**.** *There exists $d_1 \in \mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$, such that for all $d \geqslant d_1$, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample.*

Lemma 2.5 (see [14], section 2.6)**.** *If $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample, then for almost every section $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (for the Lebesgue measure), the restriction of s to M vanishes transversally.*

From now on, we only consider the case $d \geqslant d_1$, so that $|\mathrm{d}V_s|$ is a well-defined measure on M for almost every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, that is s_d is a random vector which distribution admits the density:

$$s \mapsto \frac{1}{\sqrt{2\pi}^{N_d}} \exp\left(-\frac{1}{2} \|s\|^2 \right) \tag{2.4}$$

with respect to the normalized Lebesgue measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Here $\|\cdot\|$ is the norm associated to the Euclidean inner product (2.1). Then Z_{s_d} is almost surely a submanifold of codimension r of M and $|\mathrm{d}V_{s_d}|$ is almost surely a random positive Radon measure on M.

To simplify notations, we set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$. For more details concerning Gaussian vectors, we refer to [14, appendix A] and the references therein.

Let $\phi \in \mathcal{C}^0(M)$, for every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ vanishing transversally, we set

$$\langle |\mathrm{d}V_s| \, , \phi \rangle = \int_{x \in Z_s} \phi(x) \, |\mathrm{d}V_s| \, . \tag{2.5}$$

Such a ϕ will be refered to as a *test-function*. Following [25], we call *linear statistic* of degree d associated to ϕ the real random variable $\langle |\mathrm{d}V_d| \, , \phi \rangle$.

2.3 The correlation kernel

Let $d \in \mathbb{N}$, then $(s_d(x))_{x \in \mathcal{X}}$ is a smooth centered Gaussian field on $\mathcal{X}$. As such, it is characterized by its correlation kernel. In this section, we recall that the correlation kernel of s_d equals the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. This is now a well-known fact (see [3, 8, 23, 25]) and was already used by the author in [14].

Let us first recall some facts about random vectors (see for example [14, appendix A]). In this paper, we only consider centered random vectors (that is their expectation vanishes), so we give the following definitions in this restricted setting. Let X_1 and X_2 be centered random vectors taking values in Euclidean (or Hermitian) vector spaces V_1 and V_2 respectively, then we define their *covariance operator* as:

$$\mathrm{Cov}(X_1, X_2) : v \longmapsto \mathbb{E}[X_1 \, \langle v \, , X_2 \rangle] \tag{2.6}$$

from V_2 to V_1. For every $v \in V_2$, we set $v^* = \langle \cdot \, , v \rangle \in V_2^*$. Then $\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 \otimes X_2^*]$ is an element of $V_1 \otimes V_2^*$. The *variance operator* of a centered random vector $X \in V$ is defined as $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathbb{E}[X \otimes X^*] \in V \otimes V^*$. We denote by $X \sim \mathcal{N}(\Lambda)$ the fact that X is a centered Gaussian vector with variance operator Λ. Finally, we say that $X \in V$ is a *standard* Gaussian vector if $X \sim \mathcal{N}(\mathrm{Id})$, where Id is the identity operator on V. A standard Gaussian vector admits the density (2.4) with respect to the normalized Lebesgue measure on V.

Recall that $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ stands for the bundle $P_1^* \left(\mathcal{E} \otimes \mathcal{L}^d \right) \otimes P_2^* \left(\left(\mathcal{E} \otimes \mathcal{L}^d \right)^* \right)$ over $\mathcal{X} \times \mathcal{X}$, where P_1 (resp. P_2) denotes the projection from $\mathcal{X} \times \mathcal{X}$ onto the first (resp. second) factor. The covariance kernel of $(s_d(x))_{x \in \mathcal{X}}$ is the section of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ defined by:

$$(x, y) \mapsto \mathrm{Cov}(s_d(x), s_d(y)) = \mathbb{E}[s_d(x) \otimes s_d(y)^*] \, . \tag{2.7}$$

The orthogonal projection from $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ admits a Schwartz kernel (see [15, thm. B.2.7]). That is, there exists a unique section E_d of $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$ such that, for any $s \in \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, the projection of s onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is given by:

$$x \longmapsto \int_{y \in \mathcal{X}} E_d(x, y) \, (s(y)) \, \mathrm{d}V_{\mathcal{X}} . \tag{2.8}$$

This section is called the *Bergman kernel* of $\mathcal{E} \otimes \mathcal{L}^d$. Note that E_d is also the Schwartz kernel of the orthogonal projection from $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, for the Hermitian inner product (2.1).

Proposition 2.6. *Let $d \in \mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then for all x and $y \in \mathcal{X}$, we have:*

$$\mathrm{Cov}(s_d(x), s_d(y)) = \mathbb{E}[s_d(x) \otimes s_d(y)^*] = E_d(x, y). \tag{2.9}$$

Proof. We will prove that $(x,y) \mapsto \mathbb{E}[s_d(x) \otimes s_d(y)^*]$ is the kernel of the orthogonal projection onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, i.e. satisfies (2.8). Let $s \in \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, then

$$\int_{y \in \mathcal{X}} \mathbb{E}[s_d(x) \otimes s_d(y)^*] \left(s(y)\right) \mathrm{d}V_{\mathcal{X}} = \mathbb{E}\left[s_d(x) \int_{y \in \mathcal{X}} s_d(y)^*(s(y)) \, \mathrm{d}V_{\mathcal{X}} \right] = \mathbb{E}[s_d(x) \left\langle s \, , s_d \right\rangle] \, .$$

If s is orthogonal to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ this quantity equals 0. If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then

$$\mathbb{E}[s_d(x) \left\langle s \, , s_d \right\rangle] = \mathbb{E}\left[\mathrm{ev}_x^d(s_d) s_d^*(s) \right] = \mathrm{ev}_x^d \left(\mathbb{E}[s_d \otimes s_d^*] \left(s\right) \right) = \mathrm{ev}_x^d(\mathrm{Var}(s_d) \, s) = \mathrm{ev}_x^d(s) = s(x)$$

since $\mathrm{Var}(s_d) = \mathrm{Id}$. Thus, for any $s \in \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, $\mathbb{E}[s_d(x) \left\langle s \, , s_d \right\rangle]$ is the value at x of the orthogonal projection of s on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Finally, the correlation kernel of $(s_d(x))_{x \in \mathcal{X}}$ satisfies (2.8) and equals E_d. □

Remark 2.7. If $(s_{1,d}, \ldots, s_{N_d,d})$ is any orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we have:

$$E_d : (x,y) \longmapsto \sum_{i=1}^{N_d} s_{i,d}(x) \otimes s_{i,d}(y)^* . \tag{2.10}$$

Remark 2.8. If $\mathcal{E}$ is the trivial bundle $\mathcal{X} \times \mathbb{C}^r \to \mathcal{X}$ then E_d splits as $E_d = \mathrm{Id} \otimes e_d$, where Id is the identity of $\mathbb{C}^r$ and e_d is the Bergman kernel of $\mathcal{L}^d$. There is no such splitting in general.

Remark 2.9. In a complex setting, E_d is also the covariance kernel of the centered Gaussian field associated with a standard complex Gaussian vector in $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

The Bergman kernel also describes the distribution of the derivatives of s_d. Let ∇^d denote any connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. Then ∇^d induces a connection $(\nabla^d)^*$ on $(\mathcal{E} \otimes \mathcal{L}^d)^* \to \mathcal{X}$, which is defined for all $\eta \in \Gamma\left(\left(\mathcal{E} \otimes \mathcal{L}^d \right)^* \right)$ by:

$$\forall s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right), \ \forall x \in \mathcal{X}, \qquad d_x \left\langle s \, , \eta \right\rangle = \left\langle \nabla_x^d s \, , \eta(x) \right\rangle + \left\langle s(x) \, , (\nabla^d)_x^* \eta \right\rangle , \tag{2.11}$$

where $\langle \cdot \, , \cdot \rangle$ is the duality pairing. Let $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right)$, then $s^\diamond : x \mapsto s(x)^* = \langle \cdot \, , s(x) \rangle$ defines a smooth section of $\left(\mathcal{E} \otimes \mathcal{L}^d \right)^*$. Note that we use the notation $s^\diamond$ because s^* already denotes $\langle \cdot \, , s \rangle$ which is a linear form on $\Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right)$. We want to understand the relation between $(\nabla^d)_x^* s^\diamond : T_x\mathcal{X} \to \left(\mathcal{E} \otimes \mathcal{L}^d \right)_x^*$ and $\left(\nabla_x^d s \right)^*$. Recall that $\left(\nabla_x^d s \right)^* = \left\langle \cdot \, , \nabla_x^d s \right\rangle$, where the inner product is the one on $\left(\mathcal{E} \otimes \mathcal{L}^d \right)_x \otimes T_x\mathcal{X}^*$ induced by h_d and $g_{\mathbb{C}}$. That is, $\left(\nabla_x^d s \right)^*$ is the adjoint operator of $\nabla_x^d s : T_x\mathcal{X} \to \left(\mathcal{E} \otimes \mathcal{L}^d \right)_x$. In order to get a nice relation, we have assume that ∇^d is a metric connection, i.e. that:

$$\forall s,t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right), \ \forall x \in \mathcal{X}, \qquad d_x \left\langle s \, , t \right\rangle = \left\langle \nabla_x^d s \, , t(x) \right\rangle + \left\langle s(x) \, , \nabla_x^d t \right\rangle . \tag{2.12}$$

Lemma 2.10. *Let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$, let $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right)$ and let $x \in \mathcal{X}$. Then for all $v \in T_x\mathcal{X}$,*

$$(\nabla^d)_x^* s^\diamond \cdot v = \left(\nabla_x^d s \cdot v \right)^* = v^* \circ \left(\nabla_x^d s \right)^* . \tag{2.13}$$

Proof. First, for all $s,t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right)$ and all $x \in \mathcal{X}$,

$$\langle t(x) \, , s(x) \rangle = \langle t(x) \, , s(x)^* \rangle = \langle t(x) \, , s^\diamond(x) \rangle \, . \tag{2.14}$$

Then, by taking the derivative of (2.14), we get that for all $s,t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d \right)$, for all $x \in \mathcal{X}$ and $v \in T_x\mathcal{X}$:

$$\left\langle t(x) \, , \nabla_x^d s \cdot v \right\rangle + \left\langle \nabla_x^d t \cdot v \, , s(x) \right\rangle = d_x \left(\langle t \, , s \rangle \right) \cdot v = \left\langle t(x) \, , (\nabla^d)_x^* s^\diamond \cdot v \right\rangle + \left\langle \nabla_x^d t \cdot v \, , s^\diamond(x) \right\rangle .$$

The first equality comes from the fact that ∇^d is metric (see (2.12)) and the second from the definition of the dual connection (2.11). Besides $\langle \nabla^d_x t \cdot v \, , s^\diamond(x) \rangle = \langle \nabla^d_x t \cdot v \, , s(x) \rangle$, hence for all $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ and all $x \in \mathcal{X}$ we have:

$$\forall v \in T_x\mathcal{X}, \qquad (\nabla^d)^*_x s^\diamond \cdot v = \left(\nabla^d_x s \cdot v\right)^* .$$

Recall that $\left(\nabla^d_x s\right)^*$ is the adjoint of $\nabla^d_x s$. Hence for all $v \in T_x\mathcal{X}$ and all $\zeta \in \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$,

$$\langle \zeta \, , \nabla^d_x s \cdot v \rangle = \left\langle \left(\nabla^d_x s\right)^* \zeta \, , v \right\rangle = v^* \circ \left(\nabla^d_x s\right)^* (\zeta),$$

which proves the second equality in (2.13). □

Remark 2.11. Conversely, one can show that a connection satisfying th first equality in (2.13) for every s, x and v is metric.

From now on, we assume that ∇^d is metric. Then ∇^d induces a natural connection ∇^d_1 on $P_1^*(\mathcal{E} \otimes \mathcal{L}^d) \to \mathcal{X} \times \mathcal{X}$ whose partial derivatives are: ∇^d with respect to the first variable, and the trivial connection with respect to the second. Similarly, $(\nabla^d)^*$ induces a connection ∇^d_2 on $P_2^*\left((\mathcal{E} \otimes \mathcal{L}^d)^*\right)$ and $\nabla^d_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^d_2$ is a connection on $(\mathcal{E} \otimes \mathcal{L}^d) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)^*$. We denote by ∂_x (resp. ∂_y) its partial derivative with respect to the first (resp. second) variable. By taking partial derivatives in (2.9), we get the following.

Corollary 2.12. *Let $d \in \mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then for all x and $y \in \mathcal{X}$, for all $(v, w) \in T_x\mathcal{X} \times T_y\mathcal{X}$,*

$$\mathrm{Cov}\left(\nabla^d_x s \cdot v, s(y)\right) = \mathbb{E}\left[\left(\nabla^d_x s \cdot v\right) \otimes s(y)^*\right] = \partial_x E_d(x, y) \cdot v, \tag{2.15}$$

$$\mathrm{Cov}\left(s(x), \nabla^d_y s \cdot w\right) = \mathbb{E}\left[s(x) \otimes \left(\nabla^d_y s \cdot w\right)^*\right] = \partial_y E_d(x, y) \cdot w, \tag{2.16}$$

$$\mathrm{Cov}\left(\nabla^d_x s \cdot v, \nabla^d_y s \cdot w\right) = \mathbb{E}\left[\left(\nabla^d_x s \cdot v\right) \otimes \left(\nabla^d_y s \cdot w\right)^*\right] = \partial_x \partial_y E_d(x, y) \cdot (v, w). \tag{2.17}$$

Proof. The first equality of each line is simply the definition of the covariance operator. By applying ∂_x to (2.9) we get:

$$\mathbb{E}\left[\left(\nabla^d_x s\right) \otimes s(y)^*\right] = \partial_x E_d(x, y),$$

which proves (2.15). We can rewrite (2.9) as: $\forall x, y \in \mathcal{X}$, $E_d(x, y) = \mathbb{E}[s(x) \otimes s^\diamond(y)]$. By applying ∂_y to this equality, we get:

$$\mathbb{E}\left[s(x) \otimes \left(\nabla^d\right)^*_y s^\diamond\right] = \partial_y E_d(x, y).$$

Then we apply this operator to $w \in T_y\mathcal{X}$, and we obtain (2.16) by Lemma 2.10. The proof of (2.17) is similar. □

We would like to write that $\partial_y E_d(x, y)$ is $\mathrm{Cov}\left(s(x), \nabla^d_y s\right) = \mathbb{E}\left[s(x) \otimes \left(\nabla^d_y s\right)^*\right]$. Unfortunately, this can not be true since

$$\partial_y E_d(x, y) \in T_y\mathcal{X}^* \otimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*_y$$

while

$$\mathbb{E}\left[s(x) \otimes \left(\nabla^d_y s\right)^*\right] \in T_y\mathcal{X} \otimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*_y .$$

Let $\partial_y^\sharp E_d(x,y) \in T_y\mathcal{X} \otimes \big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \otimes \big(\mathcal{E}\otimes\mathcal{L}^d\big)_y^*$ be defined by:

$$\forall w \in T_y\mathcal{X}, \qquad \partial_y^\sharp E_d(x,y)\cdot w^* = \partial_y E_d(x,y)\cdot w. \tag{2.18}$$

Similarly, let $\partial_x\partial_y^\sharp E_d(x,y) \in T_x\mathcal{X}^* \otimes T_y\mathcal{X} \otimes \big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \otimes \big(\mathcal{E}\otimes\mathcal{L}^d\big)_y^*$ be defined by:

$$\forall (v,w) \in T_x\mathcal{X}\times T_y\mathcal{X}, \qquad \partial_x\partial_y^\sharp E_d(x,y)\cdot(v,w^*) = \partial_x\partial_y E_d(x,y)\cdot(v,w). \tag{2.19}$$

Then by Lemma 2.10 and Corollary 2.12, we have the following.

Corollary 2.13. *Let $d\in\mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E}\otimes\mathcal{L}^d$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ then for all x and $y\in\mathcal{X}$, we have:*

$$\mathrm{Cov}\big(\nabla_x^d s, s(y)\big) = \mathbb{E}\big[\nabla_x^d s \otimes s(y)^*\big] = \partial_x E_d(x,y), \tag{2.20}$$

$$\mathrm{Cov}\big(s(x), \nabla_y^d s\big) = \mathbb{E}\Big[s(x)\otimes\big(\nabla_y^d s\big)^*\Big] = \partial_y^\sharp E_d(x,y), \tag{2.21}$$

$$\mathrm{Cov}\big(\nabla_x^d s, \nabla_y^d s\big) = \mathbb{E}\Big[\nabla_x^d s\otimes\big(\nabla_y^d s\big)^*\Big] = \partial_x\partial_y^\sharp E_d(x,y). \tag{2.22}$$

3 Estimates for the Bergman kernel

The goal of this section is to recall the estimates we need for the Bergman kernel. Most of what follows can be found in [15], with small additions from [16] and [17]. The first to use this kind of estimates in a random geometry context were Shiffman and Zelditch [23]. They used the estimates from [33] for the related Szegö kernel. See also [3, 24].

In order to state the near-diagonal estimates for the Bergman kernel, we first need to choose preferred charts on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ around any point in M. This is done in Section 3.1. Unlike our main reference [15], we are only concerned with a neighborhood of the real locus of $\mathcal{X}$, but we need to check that these charts are well-behaved with respect to the real structures. Sections 3.2, 3.3 and 3.4 state respectively near-diagonal, diagonal and far off-diagonal estimates for E_d.

3.1 Real normal trivialization

In this section, we define preferred local trivializations for $\mathcal{E}$ and $\mathcal{L}$ around any point in M. We also prove that these trivializations are compatible with the real and metric structures.

Let $R>0$ be such that the injectivity radius of $\mathcal{X}$ is larger than $2R$. Let $x_0\in M$, then the exponential map $\exp_{x_0} : T_{x_0}\mathcal{X}\to\mathcal{X}$ at x_0 is a diffeomorphism from the ball $B_{T_{x_0}\mathcal{X}}(0,2R)\subset T_{x_0}\mathcal{X}$ to the geodesic ball $B_{\mathcal{X}}(x_0,2R)\subset\mathcal{X}$. Note that this diffeomorphism is not biholomorphic in general.

Notation 3.1. Here and in the sequel, we always denote by $B_A(a,R)$ the open ball of center a and radius $R>0$ in the metric space A.

Since $c_{\mathcal{X}}$ is an isometry (see Sect. 2.1), we have that $c_{\mathcal{X}}\circ\exp_{x_0} = \exp_{x_0}\circ d_{x_0}c_{\mathcal{X}}$. Then $\exp_{x_0}$ sends $T_{x_0}M = \ker(d_{x_0}c_{\mathcal{X}} - \mathrm{Id})$ to M and agrees on $T_{x_0}M$ with the exponential map at x_0 in (M,g). By restriction, we get a diffeomorphism from $B_{T_{x_0}M}(0,2R)\subset T_{x_0}M$ to the geodesic ball $B_M(x_0,2R)\subset M$. Moreover, on $B_{T_{x_0}\mathcal{X}}(0,2R)$ we have:

$$d_{x_0}c_{\mathcal{X}} = (\exp_{x_0})^{-1}\circ c_{\mathcal{X}}\circ\exp_{x_0}. \tag{3.1}$$

We say that $\exp_{x_0}$ defines a *real normal chart* about x_0.

Since $i\cdot T_{x_0}M = \ker(d_{x_0}c_{\mathcal{X}} + \mathrm{Id})$, we have $T_{x_0}\mathcal{X} = T_{x_0}M\oplus i\cdot T_{x_0}M$. Note that $T_{x_0}M$ and $i\cdot T_{x_0}M$ are orthogonal for g_{x_0}, since these are distinct eigenspaces of an isometric

involution. Moreover, we know from Sect. 2.1 that $c_{\mathcal{X}}^{\star} g_{\mathbb{C}} = \overline{g_{\mathbb{C}}}$. This implies that $(g_{\mathbb{C}})_{x_0}$ takes real values on $T_{x_0}M \times T_{x_0}M$, i.e. the restrictions to $T_{x_0}M$ of $(g_{\mathbb{C}})_{x_0}$ and g_{x_0} are equal. Thus, $(g_{\mathbb{C}})_{x_0}$ is the sesquilinear extension of g_{x_0} restricted to $T_{x_0}M$. Let $\mathcal{I}$ be an isometry from $T_{x_0}M$ to $\mathbb{R}^n$ with its standard Euclidean structure, $\mathcal{I}$ extends as a $\mathbb{C}$-linear isometry $\mathcal{I}_{\mathbb{C}} : T_{x_0}\mathcal{X} \to \mathbb{C}^n$, such that $\mathcal{I}_{\mathbb{C}} \circ d_{x_0} c_{\mathcal{X}} \circ \mathcal{I}_{\mathbb{C}}^{-1}$ is the complex conjugation in $\mathbb{C}^n$. Thus, $\exp_{x_0} \circ \mathcal{I}_{\mathbb{C}}^{-1} : B_{\mathbb{C}^n}(0, 2R) \to B_{\mathcal{X}}(x_0, 2R)$ defines normal coordinates that induce normal coordinates $B_{\mathbb{R}^n}(0, 2R) \to B_M(x_0, 2R)$ and such that $\mathcal{I}_{\mathbb{C}} \circ (\exp_{x_0})^{-1} \circ c_{\mathcal{X}} \circ \exp_{x_0} \circ \mathcal{I}_{\mathbb{C}}^{-1}$ is the complex conjugation in $\mathbb{C}^n$. Such coordinates are called *real normal coordinates* about x_0.

We can now trivialize $\mathcal{E}$ over $B_{\mathcal{X}}(x_0, 2R)$. Let $\nabla^{\mathcal{E}}$ denote the Chern connection of $\mathcal{E}$. We identify the fiber at $\exp_{x_0}(z) \in B_{\mathcal{X}}(x_0, 2R)$ with $\mathcal{E}_{x_0}$, by parallel transport with respect to $\nabla^{\mathcal{E}}$ along the geodesic from x_0 to $\exp_{x_0}(z)$, defined by $t \mapsto \exp_{x_0}(tz)$ from $[0, 1]$ to $\mathcal{X}$ (cf. [15, sect. 1.6] and [16]). This defines a bundle map $\varphi_{x_0} : B_{T_{x_0}\mathcal{X}}(0, 2R) \times \mathcal{E}_{x_0} \to \mathcal{E}_{/B_{\mathcal{X}}(x_0,2R)}$ that covers $\exp_{x_0}$. We say that φ_{x_0} is the *real normal trivialization* of $\mathcal{E}$ over $B_{\mathcal{X}}(x_0, 2R)$.

Since $x_0 \in M$, $c_{\mathcal{E}}(\mathcal{E}_{x_0}) = \mathcal{E}_{x_0}$ and we denote by $c_{\mathcal{E},x_0}$ the restriction of $c_{\mathcal{E}}$ to $\mathcal{E}_{x_0}$. Then $(d_{x_0} c_{\mathcal{X}}, c_{\mathcal{E},x_0})$ is a real structure on $B_{T_{x_0}\mathcal{X}}(0, 2R) \times \mathcal{E}_{x_0}$ compatible with the real structure on $B_{T_{x_0}\mathcal{X}}(0, 2R)$. We want to check that φ_{x_0} is well-behaved with respect to the real structures, i.e. that for all $z \in B_{T_{x_0}\mathcal{X}}(0, 2R)$ and $\zeta^0 \in \mathcal{E}_{x_0}$,

$$c_{\mathcal{E}}(\varphi_{x_0}(z, \zeta^0)) = \varphi_{x_0}\left(d_{x_0} c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right). \tag{3.2}$$

This will be a consequence of Lemma 3.4 below.

Definition 3.2. Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle equipped with compatible real structures $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ and let ∇ be a connection on $\mathcal{E}$, we say that ∇ is a *real connection* if for every section $s \in \Gamma(\mathcal{E})$ we have:

$$\forall x \in \mathcal{X}, \qquad \nabla_x \left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) = c_{\mathcal{E}} \circ \nabla_{c_{\mathcal{X}}(x)} s \circ d_x c_{\mathcal{X}}.$$

Remark 3.3. Let $x \in M$, $v \in T_x M$ and $s \in \mathbb{R}\Gamma(\mathcal{E})$. If ∇ is a real connection on $\mathcal{E}$, then $\nabla_x s \cdot v \in \mathbb{R}\mathcal{E}_x$. Indeed,

$$\nabla_x s \cdot v = \nabla_{c_{\mathcal{X}}(x)} s \circ d_x c_{\mathcal{X}} \cdot v = c_{\mathcal{E}}\left(\nabla_x \left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \cdot v\right) = c_{\mathcal{E}}\left(\nabla_x s \cdot v\right).$$

Lemma 3.4. *Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle equipped with compatible real structures $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ and a real Hermitian metric $h_{\mathcal{E}}$. Then, the Chern connection $\nabla^{\mathcal{E}}$ of $\mathcal{E}$ is real.*

Proof. Since $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ are involutions and $(d_x c_{\mathcal{X}})^{-1} = d_{c_{\mathcal{X}}(x)} c_{\mathcal{X}}$, we need to check that

$$\forall s \in \Gamma(\mathcal{E}), \forall x \in \mathcal{X} \qquad \nabla^{\mathcal{E}}_x s = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)} \left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \circ d_x c_{\mathcal{X}}. \tag{3.3}$$

Let $\widetilde{\nabla}$ be defined by $\widetilde{\nabla}_x s = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)} \left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \circ d_x c_{\mathcal{X}}$, for all $s \in \Gamma(\mathcal{E})$ and $x \in \mathcal{X}$. Then $\widetilde{\nabla}$ is a connection on $\mathcal{E}$ and it is enough to check that it is compatible with both the metric and the complex structure. Indeed, in this case $\widetilde{\nabla} = \nabla^{\mathcal{E}}$ by unicity of the Chern connection, which proves (3.3).

Let us check that $\widetilde{\nabla}$ satisfies Leibniz' rule. Let $s \in \Gamma(\mathcal{E})$ and $f : \mathcal{X} \to \mathbb{C}$. We have:

$$\begin{aligned}
\widetilde{\nabla}_x (fs) &= c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)} \left((\overline{f \circ c_{\mathcal{X}}})(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= c_{\mathcal{E}} \circ \left(\overline{f(x)} \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)} \left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) + d_{c_{\mathcal{X}}(x)} (\overline{f \circ c_{\mathcal{X}}}) \otimes c_{\mathcal{E}}(s(x))\right) \circ d_x c_{\mathcal{X}} \\
&= f(x) \widetilde{\nabla}_x s + d_x f \otimes s(x).
\end{aligned}$$

Since $\nabla^{\mathcal{E}}$ is the Chern connection, its anti-holomorphic part is $\overline{\partial}^{\mathcal{E}}$. Then, $d_x c_{\mathcal{X}}$ and $c_{\mathcal{E}}$ being anti-linear (resp. fiberwise), the anti-linear part of $\widetilde{\nabla}_x s$ equals $c_{\mathcal{E}} \circ \overline{\partial}^{\mathcal{E}}_{c_{\mathcal{X}}(x)} (c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}) \circ d_x c_{\mathcal{X}}$. By computing in a local holomorphic frame, one can check that:

$$\forall s \in \Gamma(\mathcal{E}), \forall x \in \mathcal{X}, \qquad c_{\mathcal{E}} \circ \overline{\partial}^{\mathcal{E}}_{c_{\mathcal{X}}(x)} (c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}) \circ d_x c_{\mathcal{X}} = \overline{\partial}^{\mathcal{E}}_x s.$$

Thus, $\widetilde{\nabla}$ is compatible with the complex structure. Finally, we check the compatibility with the metric structure. Let $s, t \in \Gamma(\mathcal{E})$ and $x \in \mathcal{X}$, since $h_{\mathcal{E}} = c_{\mathcal{E}}^{\star}(\overline{h_{\mathcal{E}}})$ we have:

$$\begin{aligned}
d_x(h_{\mathcal{E}}(s,t)) &= d_x\left(\overline{h_{\mathcal{E}}}(c_{\mathcal{E}} \circ s, c_{\mathcal{E}} \circ t)\right) = d_{c_{\mathcal{X}}(x)}\left(\overline{h_{\mathcal{E}}}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}, c_{\mathcal{E}} \circ t \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= \overline{h_{\mathcal{E}}}\left(\nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}), c_{\mathcal{E}}(t(x))\right) \circ d_x c_{\mathcal{X}} \\
&\qquad + \overline{h_{\mathcal{E}}}\left(c_{\mathcal{E}}(s(x)), \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ t \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= h_{\mathcal{E}}\left(c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}), t(x)\right) \circ d_x c_{\mathcal{X}} \\
&\qquad + h_{\mathcal{E}}\left(s(x), c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= h_{\mathcal{E}}\left(\widetilde{\nabla}_x s, t(x)\right) + h_{\mathcal{E}}\left(s(x), \widetilde{\nabla}_x t\right).
\end{aligned}$$

□

Let us now prove (3.2). Let $z \in B_{T_{x_0}\mathcal{X}}(0, 2R)$, let $\zeta^0 \in \mathcal{E}_{x_0}$ and let $\zeta : B_{\mathcal{X}}(x_0, 2R) \to \mathcal{E}$ be the section defined by $\zeta : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta^0\right)$. We denote by $\gamma : [0,1] \mapsto \mathcal{X}$ the geodesic $t \mapsto \exp_{x_0}(tz)$. We have for all $t \in [0,1]$, $\zeta(\gamma(t)) = \varphi_{x_0}(tz, \zeta^0)$ and, by the definition of φ_{x_0}, we have:

$$\forall t \in [0,1], \qquad \nabla^{\mathcal{E}}_{\gamma(t)} \zeta \cdot \gamma'(t) = 0. \tag{3.4}$$

Let us denote $\widetilde{\zeta} = c_{\mathcal{E}} \circ \zeta \circ c_{\mathcal{X}}$ and $\overline{\gamma} = c_{\mathcal{X}} \circ \gamma$. Since $\nabla^{\mathcal{E}}$ is real, (3.4) implies that for all $t \in [0,1]$,

$$\nabla^{\mathcal{E}}_{\overline{\gamma}(t)} \widetilde{\zeta} \cdot \overline{\gamma}'(t) = \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(\gamma(t))} \widetilde{\zeta} \circ d_{\gamma(t)}(c_{\mathcal{X}}) \cdot \gamma'(t) = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{\gamma(t)} \zeta \cdot \gamma'(t) = 0. \tag{3.5}$$

Since $c_{\mathcal{X}}$ is an isometry, $\overline{\gamma}$ is a geodesic. More precisely, $\overline{\gamma} : t \mapsto \exp_{x_0}(t d_{x_0} c_{\mathcal{X}} \cdot z)$. Besides, $\widetilde{\zeta}(x_0) = c_{\mathcal{E}}(\zeta(x_0)) = c_{\mathcal{E},x_0}(\zeta^0)$. Then by (3.5), for all $t \in [0,1]$,

$$\varphi_{x_0}\left(t d_{x_0} c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right) = \varphi_{x_0}\left(t d_{x_0} c_{\mathcal{X}} \cdot z, \widetilde{\zeta}(x_0)\right) = \widetilde{\zeta}(\overline{\gamma}(t)).$$

Finally, we get (3.2) for $t = 1$:

$$\varphi_{x_0}\left(d_{x_0} c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right) = \widetilde{\zeta}(\overline{\gamma}(1)) = c_{\mathcal{E}}(\zeta(\gamma(1))) = c_{\mathcal{E}}\left(\varphi_{x_0}(z, \zeta^0)\right).$$

Recall that $\mathbb{R}\mathcal{E}$ is the set of fixed points of $c_{\mathcal{E}}$. Then $\mathbb{R}\mathcal{E}$ is naturally a rank r real vector bundle over M, as a subbundle of $\mathcal{E}_{/M}$. Let $\zeta^0 \in \mathbb{R}\mathcal{E}_{x_0}$, and $\zeta : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta^0\right)$ then, for all $x \in B_{\mathcal{X}}(x_0, 2R)$,

$$\begin{aligned}
c_{\mathcal{E}} \circ \zeta \circ c_{\mathcal{X}}(x) &= c_{\mathcal{E}} \circ \varphi_{x_0}\left((\exp_{x_0})^{-1}(c_{\mathcal{X}}(x)), \zeta^0\right) \\
&= c_{\mathcal{E}} \circ \varphi_{x_0}\left(d_{x_0} c_{\mathcal{X}} \circ (\exp_{x_0})^{-1}(x), \zeta^0\right) \\
&= \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), c_{\mathcal{E},x_0}(\zeta^0)\right) \\
&= \zeta(x).
\end{aligned}$$

Hence, ζ is a real local section of $\mathcal{E}$ and in particular, $\forall x \in M$, $\zeta(x) \in \mathbb{R}\mathcal{E}_x$. This shows that φ_{x_0} induces, by restriction, a bundle map $B_{T_{x_0}M}(0, 2R) \times \mathbb{R}\mathcal{E}_{x_0} \to \mathbb{R}\mathcal{E}_{/B_M(x_0, 2R)}$ that covers the restriction of $\exp_{x_0}$ to $B_{T_{x_0}M}(0, 2R)$.

Let $(\zeta_1^0, \dots, \zeta_r^0)$ be an orthonormal basis of $\mathbb{R}\mathcal{E}_{x_0}$. Since $\mathbb{R}\mathcal{E}_{x_0} = \ker(c_{\mathcal{E},x_0} - \mathrm{Id})$ and $c_{\mathcal{E},x_0}$ is $\mathbb{C}$-anti-linear, we have $\mathcal{E}_{x_0} = \mathbb{R}\mathcal{E}_{x_0} \oplus i \cdot \mathbb{R}\mathcal{E}_{x_0}$. Moreover, since $h_{\mathcal{E},x_0}$ and $c_{\mathcal{E},x_0}$ are compatible, $(\zeta_1^0, \dots, \zeta_r^0)$ is also an orthonormal basis of $\mathcal{E}_{x_0}$. Let $i \in \{1, \dots, r\}$, we denote by $\zeta_i : B_{\mathcal{X}}(x_0, 2R) \to \mathcal{E}$ the real local section defined by:

$$\zeta_i : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta_i^0\right).$$

Then, for every $x \in B_{\mathcal{X}}(0, 2R)$, $(\zeta_1(x), \dots, \zeta_r(x))$ is an orthonormal basis of $\mathcal{E}_x$. Indeed, the sections ζ_i are obtained by parallel transport for $\nabla^{\mathcal{E}}$ along geodesics starting at x_0, and $\nabla^{\mathcal{E}}$ is compatible with $h_{\mathcal{E}}$. Hence, for all i and $j \in \{1, \dots, r\}$, for all $z \in B_{\mathcal{X}}(x_0, 2R)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(h_{\mathcal{E}}(\zeta_i(\exp_{x_0}(tz)), \zeta_j(\exp_{x_0}(tz)))\right) &= h_{\mathcal{E}}\left(\nabla^{\mathcal{E}}_{\exp_{x_0}(tz)}\zeta_i \circ d_{tz}\exp_{x_0}\cdot z, \zeta_j(\exp_{x_0}(tz))\right)\\ &\quad + h_{\mathcal{E}}\left(\zeta_i(\exp_{x_0}(tz))), \nabla^{\mathcal{E}}_{\exp_{x_0}(tz)}\zeta_j \circ d_{tz}\exp_{x_0}\cdot z\right) = 0.\end{aligned}$$

The function $x \mapsto h_{\mathcal{E}}(\zeta_i(x), \zeta_j(x))$ is then constant along geodesics starting at x_0, hence on $B_{\mathcal{X}}(x_0, 2R)$. Since $(h_{\mathcal{E}}(\zeta_i(x), \zeta_j(x)))_{1 \leqslant i,j \leqslant r}$ is the identity matrix of size r at x_0, $(\zeta_1, \dots, \zeta_r)$ is a smooth unitary frame for $\mathcal{E}$ over $B_{\mathcal{X}}(0, 2R)$. In particular, this shows that the real normal trivialization φ_{x_0} is unitary. Since the ζ_i are real, $(\zeta_1(x), \dots, \zeta_r(x))$ is an orthonormal basis of $\mathbb{R}\mathcal{E}_x$ for all $x \in M$. Hence $(\zeta_1, \dots, \zeta_r)$ is also a smooth orthogonal frame for $\mathbb{R}\mathcal{E}$ over $B_M(0, 2R)$. We say that $(\zeta_1, \dots, \zeta_r)$ is a local *real unitary frame*.

Similarly, let φ'_{x_0} denote the real normal trivialization of $\mathcal{L}$ over $B_{\mathcal{X}}(x_0, 2R)$. Then any unit vector $\zeta_0^0 \in \mathbb{R}\mathcal{L}_{x_0}$ defines a local real unitary frame ζ_0 for $\mathcal{L}$:

$$\zeta_0 : x \mapsto \varphi'_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta_0^0\right).$$

Then, for any $d \in \mathbb{N}$, φ_{x_0} and φ'_{x_0} induce a trivialization $\varphi_{x_0} \otimes (\varphi'_{x_0})^d$ of $\mathcal{E} \otimes \mathcal{L}^d$. This trivialization is the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x_0, 2R)$, i.e. it is obtained by parallel transport along geodesics starting at x_0 for the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$. Moreover, a local real unitary frame for $\mathcal{E} \otimes \mathcal{L}^d$ is given by $(\zeta_1 \otimes \zeta_0^d, \dots, \zeta_r \otimes \zeta_0^d)$.

3.2 Near-diagonal estimates

We can now state the near-diagonal estimates of Ma and Marinescu for the Bergman kernel. In the sequel, we fix some $R > 0$ such that $2R$ is smaller than the injectivity radius of $\mathcal{X}$. Let $x \in M$, we have a natural real normal chart

$$\exp_x \times \exp_x : B_{T_x\mathcal{X}}(0, 2R) \times B_{T_x\mathcal{X}}(0, 2R) \to B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R).$$

Moreover, the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x, 2R)$ (see Section 3.1) induces a trivialization

$$B_{T_x\mathcal{X}}(0, 2R) \times B_{T_x\mathcal{X}}(0, 2R) \times \mathrm{End}\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right) \simeq \left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*_{/B_{\mathcal{X}}(x,2R) \times B_{\mathcal{X}}(x,2R)}$$

that covers $\exp_x \times \exp_x$. This trivialization coincides with the real normal trivialization of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ over $B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R)$.

Recall that $\mathrm{d}V_{\mathcal{X}}$ denotes the Riemannian measure on $\mathcal{X}$. When we read this measure in the real normal chart $\exp_x$, it admits a density $\kappa : B_{T_x\mathcal{X}}(0, 2R) \to \mathbb{R}_+$ with respect to the normalized Lebesgue measure of $(T_x\mathcal{X}, g_x)$. More precisely, we have $\kappa(z) = \sqrt{\det(g_{ij}(z))}$ where $(g_{ij}(z))$ is the matrix of $((\exp_x)^\star g)_z$, read in any real orthonormal basis of $(T_x\mathcal{X}, g_x)$. Since we use normal coordinates and $\mathcal{X}$ is compact, we have

$$\kappa(z) = 1 + O\left(\|z\|^2\right) \tag{3.6}$$

where $\|\cdot\|$ is induced by g_x and the estimate $O\Big(\|z\|^2\Big)$ does not depend on x.

Similarly, on the real locus (M,g), $|\mathrm{d}V_M|$ admits a density, in the real normal chart $\exp_x$, with respect to the normalized Lebesgue measure on (T_xM, g_x). This density is:

$$z \longmapsto \det\left(\left(\left(\exp_x^\star g\right)_z\right)_{/T_xM}\right)^{\frac{1}{2}}, \tag{3.7}$$

from $B_{T_xM}(0,2R)$ to $\mathbb{R}_+$. As we already explained in Sect. 3.1, on the real locus, $g_{\mathbb{C}}$ is the sesquilinear extension of the restriction of g to TM. Hence, for all $z \in B_{T_xM}(0,2R)$ we have:

$$\det\left(\left(\left(\exp_x^\star g\right)_z\right)_{/T_xM}\right)^2 = \det\left(\left(\exp_x^\star g\right)_z\right),$$

which means that the density of $|\mathrm{d}V_M|$ in the chart $\exp_x$ is $\sqrt{\kappa} : B_{T_xM}(0,2R) \to \mathbb{R}_+$.

The following result gives the asymptotic of the Bergman kernel E_d (see Sect. 2.3) and its derivatives, read in the real normal trivialization about x of $\left(\mathcal{E}\otimes\mathcal{L}^d\right)\boxtimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$.

Theorem 3.5 (Ma–Marinescu)**.** *There exists $C'>0$ such that, for any $p\in\mathbb{N}$, there exists C_p such that $\forall k\in\{0,\dots,p\}$, $\forall d\in\mathbb{N}$, $\forall z,w\in B_{T_x\mathcal{X}}(0,R)$,*

$$\left\| D^k_{(z,w)}\left(E_d(z,w) - \left(\frac{d}{\pi}\right)^n \frac{\exp\left(-\frac{d}{2}\left(\|z\|^2+\|w\|^2-2\langle z\,,w\rangle\right)\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}}\,\mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)\right\|$$
$$\leqslant C_p d^{n+\frac{p}{2}-1}\left(1+\sqrt{d}(\|z\|+\|w\|)\right)^{2n+6+p}\exp\left(-C'\sqrt{d}\,\|z-w\|\right)+O\big(d^{-\infty}\big),$$

where:

- *$D^k_{(z,w)}$ is the k-th differential at (z,w) for a map $T_x\mathcal{X}\times T_x\mathcal{X}\to\mathrm{End}\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$,*
- *the Hermitian inner product $\langle\cdot\,,\cdot\rangle$ comes from the Hermitian metric $(g_{\mathbb{C}})_x$,*
- *the norm $\|\cdot\|$ on $T_x\mathcal{X}$ is induced by g_x (or equivalently $\langle\cdot\,,\cdot\rangle$),*
- *the norm $\|\cdot\|$ on $\left(T_x\mathcal{X}^*\right)^{\otimes q}\otimes\mathrm{End}\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$ is induced by g_x and $(h_d)_x$.*

Moreover, the constants C_p and C' do not depend on x. The notation $O(d^{-\infty})$ means that, for any $l\in\mathbb{N}$, this term is $O\big(d^{-l}\big)$ with a constant that does not depend on x, z, w or d.

Proof. This is a weak version of [15, thm. 4.2.1], with $k=1$ and $m'=0$ in the notations of [15]. We used the fact that $\mathcal{F}_0$ in [15] is given by:

$$\mathcal{F}_0(z,w) = \frac{1}{\pi^n}\exp\left(-\frac{1}{2}\left(\|z\|^2+\|w\|^2-2\langle z\,,w\rangle\right)\right)\mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x},$$

(compare (4.1.84), (4.1.85) et (4.1.92) pp. 191–192 and (5.1.18) p. 46 in [15]) and $\mathcal{F}_1=0$. See [15, Rem. 1.4.26] and [16]. □

Remark 3.6. Note that our formula differs from the one in [15, 16] by a factor π in the exponential. This comes from different normalizations of the Kähler form ω.

We are only interested in the behavior of E_d at points of the real locus, hence we restrict our focus to points in M and derivatives in real directions. Similarly, for $x,y\in M$, $E_d(x,y)$ restricts to an element of $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\otimes\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y^*$, still denoted by $E_d(x,y)$. Note that we can recover the original $E_d(x,y):\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y\to\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x$ from its restriction by $\mathbb{C}$-linear extension.

First, we need to know the behavior of E_d and its derivatives up to order 1 in each variable in a neighborhood of the diagonal in $M\times M$.

Corollary 3.7. *There exist C and $C' > 0$, not depending on x, such that $\forall k \in \{0,1,2\}$, $\forall d \in \mathbb{N}$, $\forall z, w \in B_{T_xM}(0,R)$,*

$$\left\| D^k_{(z,w)} \left(E_d(z,w) - \left(\frac{d}{\pi}\right)^n \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right) \right\|$$
$$\leqslant C d^{n+\frac{k}{2}-1} \left(1 + \sqrt{d}(\|z\| + \|w\|)\right)^{2n+8} \exp\left(-C'\sqrt{d}\,\|z-w\|\right) + O\big(d^{-\infty}\big)\,,$$

where D^k is the k-th differential for a map from $T_xM \times T_xM$ to $\operatorname{End}\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$, the norm on T_xM is induced by g_x and the norm on $(T_xM^*)^{\otimes q} \otimes \operatorname{End}\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$ is induced by g_x and $(h_d)_x$.

Proof. We apply Theorem. 3.5 for $p = k \in \{0,1,2\}$ and set $C = \max(C_0, C_1, C_2)$. Then we restrict everything to the real locus. □

3.3 Diagonal estimates

In this section, we deduce diagonal estimates for E_d and its derivatives from Thm. 3.5. Let $x \in M$, then the usual differential for maps from $T_x\mathcal{X}$ to $(\mathcal{E}\otimes\mathcal{L}^d)_x$ defines a local trivial connection $\widetilde{\nabla}^d$ on $(\mathcal{E}\otimes\mathcal{L}^d)_{/B_{\mathcal{X}}(0,2R)}$, via the real normal trivialization. Since this trivialization is well-behaved with respect to both the metric and the real structure (cf. Sect. 3.1), $\widetilde{\nabla}^d$ is metric and real. By a partition of unity argument, there exists a real metric connection ∇^d on $\mathcal{E}\otimes\mathcal{L}^d$ such that ∇^d agrees with $\widetilde{\nabla}^d$ on $B_{\mathcal{X}}(0,R)$. In the remainder of this section, we use this connection ∇^d, and the induced connection on $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$, so that the connection is trivial in the real normal trivialization about x.

Recall that $\partial_y^\sharp E_d$ and $\partial_x\partial_y^\sharp E_d$ are defined by (2.18) and (2.19) respectively.

Corollary 3.8. *Let $x \in M$, let ∇^d be a real metric connection that is trivial over $B_{T_x\mathcal{X}}(0,R)$ in the real normal trivialization about x. Let $\partial_y^\sharp$ and ∂_x denote the associated partial derivatives for sections of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$, then we have the following estimates as $d \to +\infty$.*

$$E_d(x,x) = \frac{d^n}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\big(d^{n-1}\big)\,, \tag{3.8}$$
$$\partial_x E_d(x,x) = O\Big(d^{n-\frac{1}{2}}\Big)\,, \tag{3.9}$$
$$\partial_y^\sharp E_d(x,x) = O\Big(d^{n-\frac{1}{2}}\Big)\,, \tag{3.10}$$
$$\partial_x\partial_y^\sharp E_d(x,x) = \frac{d^{n+1}}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \otimes \operatorname{Id}_{T_xM^*} + O(d^n)\,. \tag{3.11}$$

Moreover the error terms do not depend on x.

Proof. Let $x \in M$ and let us choose an orthonormal basis of T_xM. We denote the corresponding coordinates on $T_xM \times T_xM$ by $(z_1,\dots,z_n,w_1,\dots,w_n)$ and by ∂_{z_i} and ∂_{w_j} the associated partial derivatives. Let us compute the partial derivatives of E_d read in the real normal trivialization of $\left(\mathcal{E}\otimes\mathcal{L}^d\right)\boxtimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$ about (x,x). By Cor. 3.7, we only need to compute the partial derivatives at $(0,0)$ of

$$\xi_d : (z,w) \mapsto \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \tag{3.12}$$

for any $d \in \mathbb{N}$. For all i and $j \in \{1, \dots, n\}$ and all $(z, w) \in B_{T_xM}(0, R)$ we have:

$$\partial_{z_i} \xi_d(z,w) = \left(-d(z_i - w_i) - \frac{1}{2} \frac{\partial_{z_i} \kappa(z)}{\kappa(z)} \right) \frac{\exp\left(-\frac{d}{2} \|z-w\|^2 \right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}}, \tag{3.13}$$

$$\partial_{w_j} \xi_d(z,w) = \left(d(z_j - w_j) - \frac{1}{2} \frac{\partial_{w_j} \kappa(w)}{\kappa(w)} \right) \frac{\exp\left(-\frac{d}{2} \|z-w\|^2 \right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \tag{3.14}$$

and

$$\begin{aligned} \partial_{z_i} \partial_{w_j} \xi_d(z,w) = {} & \frac{\exp\left(-\frac{d}{2} \|z-w\|^2 \right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \times \\ & \left(d\delta_{ij} - d^2 (z_i - w_i)(z_j - w_j) - \frac{d(z_j - w_j)}{2} \frac{\partial_{z_i} \kappa(z)}{\kappa(z)} + \frac{d(z_i - w_i)}{2} \frac{\partial_{w_j} \kappa(w)}{\kappa(w)} \right), \end{aligned} \tag{3.15}$$

where δ_{ij} equals 1 if $i = j$ and 0 otherwise. Recall that, by (3.6), $\kappa(0) = 1$ and the partial derivatives of κ vanish at the origin. Then evaluating the above expressions at $(0, 0)$ gives:

$$\xi_d(0,0) = 1, \qquad \partial_{z_i} \xi_d(0,0) = 0 = \partial_{w_j} \xi_d(0,0) \qquad \text{and} \qquad \partial_{z_i} \partial_{w_j} \xi_d(0,0) = \delta_{ij} d.$$

By Cor. 3.7, we have the following estimates for the partial derivatives of E_d read in the real normal trivialization about x: for all $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} E_d(0,0) &= \frac{d^n}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x} + O\left(d^{n-1}\right), \qquad \partial_{w_j} E_d(0,0) = O\left(d^{n-\frac{1}{2}}\right), \\ \partial_{z_i} \partial_{w_j} E_d(0,0) &= \delta_{ij} \frac{d^{n+1}}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x} + O(d^n), \quad \partial_{z_i} E_d(0,0) = O\left(d^{n-\frac{1}{2}}\right). \end{aligned} \tag{3.16}$$

Moreover these estimates are uniform in $x \in M$. Equations (3.8), (3.9), (3.10) and (3.11) are coordinate-free versions of these statements. □

3.4 Far off-diagonal estimates

Finally, we will use the fact that the Bergman kernel and its derivatives decrease fast enough outside of the diagonal. In this section we recall the far off-diagonal estimates of [17, thm. 5], see also [15, prop. 4.1.5].

Let $d \in \mathbb{N}$ and let S be a smooth section of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$. Let $x, y \in M$, we denote by $\|S(x,y)\|_{\mathcal{C}^k}$ the maximum of the norms of S and its derivatives of order at most k at the point (x, y). The derivatives of S are computed with respect to the connection induced by the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$ and the Levi–Civita connection on M. The norms of the derivatives are the one induced by h_d and g.

Theorem 3.9 (Ma–Marinescu)**.** *There exist $C' > 0$ and $d_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that $\forall d \geqslant d_0$, $\forall x, y \in M$*

$$\|E_d(x,y)\|_{\mathcal{C}^k} \leqslant C_k d^{n+\frac{k}{2}} \exp\left(-C' \sqrt{d}\, \rho_g(x,y) \right),$$

where $\rho_g(\cdot, \cdot)$ denotes the geodesic distance in (M, g).

Proof. This is the first part of [17, thm. 5], where we only considered the restriction of E_d and its derivatives to M. Note that the Levi–Civita connection on M is the restriction of the Levi–Civita connection on $\mathcal{X}$. Hence the norm $\|\cdot\|_{\mathcal{C}^k}$, such as we defined it, is smaller than the one used in [17]. □

4 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. Recall that $\mathcal{X}$ is a compact Kähler manifold of dimension $n \geqslant 2$ defined over the reals and that M denotes its real locus, assumed to be non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n-1\}$ real Hermitian vector bundle and $\mathcal{L} \to \mathcal{X}$ be real a Hermitian line bundle whose curvature form is ω, the Kähler form of $\mathcal{X}$. We assume that $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. For all $d \in \mathbb{N}$, E_d denotes the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Finally, s_d denotes a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, which real zero set is denoted by Z_d, and $|\mathrm{d}V_d|$ is the measure of integration over Z_d.

4.1 The Kac–Rice formula

The first step in our proof of Thm. 1.6 is to prove a version of the Kac–Rice formula adapted to our problem. This is the goal of this section. First, we recall the Kac–Rice formula we used in [14] to compute the expectation of $\mathrm{Vol}\,(Z_d)$ (Thm. 4.1). Then we prove a Kac–Rice formula adapted to the computation of the covariance (Thm. 4.4), compare [2, thm. 6.3] and [30, chap. 11.5].

Let $L : V \to V'$ be a linear map between two Euclidean spaces, recall that we denote by $\left|\det^{\perp}(L)\right|$ its Jacobian (cf. Def. 1.3). Since LL^* is a semi-positive symmetric endomorphism of V', $\det(LL^*) \geqslant 0$ and $\left|\det^{\perp}(L)\right|$ is well-defined. The range of L^* is $\ker(L)^{\perp}$, hence $\ker(LL^*) = \ker(L^*) = L(V)^{\perp}$. Thus $\left|\det^{\perp}(L)\right| > 0$ if and only if LL^* is injective, that is if and only if L is surjective. In fact, if L is surjective, let A be the matrix of the restriction of L to $\ker(L)^{\perp}$ in any orthonormal basis of $\ker(L)^{\perp}$ and V', then we have:

$$\left|\det^{\perp}(L)\right| = \sqrt{\det\left(AA^{\mathrm{t}}\right)} = \left|\det(A)\right|.$$

Theorem 4.1 (Kac–Rice formula)**.** *Let $d \geqslant d_1$, where d_1 is defined by Lem. 2.4 and let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then for any Borel measurable function $\phi : M \to \mathbb{R}$ we have:*

$$\mathbb{E}\left[\int_{x \in Z_d} \phi(x)\,|\mathrm{d}V_d|\right] = (2\pi)^{-\frac{r}{2}} \int_{x \in M} \frac{\phi(x)}{\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|} \mathbb{E}\Big[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right| \,\Big|\, s_d(x) = 0\Big]\,|\mathrm{d}V_M| \quad (4.1)$$

whenever one of these integrals is well-defined.

The expectation on the right-hand side of (4.1) is to be understood as the conditional expectation of $\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|$ given that $s_d(x) = 0$. This result is a consequence of [14, thm. 5.3]. See also Section 5.3 of [14], where we applied this result with $\phi = \mathbf{1}$, in order to compute the expected volume of Z_d.

Let us denote by $\Delta = \{(x, y) \in M^2 \mid x = y\}$ the diagonal in M^2. Let $d \in \mathbb{N}$ and let $(x, y) \in M^2 \setminus \Delta$ we denote by $\mathrm{ev}_{x,y}^d$ the evaluation map:

$$\begin{array}{rccc} \mathrm{ev}_{x,y}^d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) & \longrightarrow & \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_y. \\ & s & \longmapsto & (s(x), s(y)) \end{array} \quad (4.2)$$

The following proposition is the equivalent of Lemma 2.4 for two points $(x, y) \notin \Delta$. One could prove this result using only the estimates of Section 3. We give instead a less technical proof, using the Kodaira embedding theorem. See [15, sect. 5.1] for a discussion of the relations between these approaches.

Proposition 4.2. *There exists $d_2 \in \mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$, such that for every $d \geqslant d_2$ and every $(x, y) \in M^2 \setminus \Delta$, the evaluation map $\mathrm{ev}_{x,y}^d$ is surjective.*

Proof. Recall that there exists $d_1 \in \mathbb{N}$ such that, for all $d \geqslant d_1$, the map ev_x^d defined by (2.3) is surjective for any $x \in M$ (see Lem. 2.4). Then, for all $d \geqslant d_1$ and all $x \in M$, the complexified map $\widetilde{\mathrm{ev}}_x^d : H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ defined by $\widetilde{\mathrm{ev}}_x^d(s) = s(x)$ is also surjective.

For any $l \in \mathbb{N}$, we denote by $\Psi_l : \mathcal{X} \to \mathbb{P}\left(H^0(\mathcal{X}, \mathcal{L}^l)^*\right)$ the Kodaira map, defined by $\Psi_l(x) = \left\{s \in H^0(\mathcal{X}, \mathcal{L}^l) \;\middle|\; s(x) = 0\right\}$. By the Kodaira embedding theorem (see [10, chap. 1.4]), there exists $l_0 \in \mathbb{N}$ such that Ψ_{l_0} is well-defined and is an embedding.

We set $d_2 = l_0 + d_1$. Let $d \geqslant d_2$ and let $(x, y) \in M^2 \setminus \Delta$. Since $\Psi_{l_0}(x)$ and $\Psi_{l_0}(y)$ are distincts hyperplanes in $H^0(\mathcal{X}, \mathcal{L}^{l_0})$, there exist σ_x and $\sigma_y \in H^0(\mathcal{X}, \mathcal{L}^{l_0})$ such that:

$$\begin{cases} \sigma_x(x) \neq 0, \\ \sigma_x(y) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \sigma_y(x) = 0, \\ \sigma_y(y) \neq 0. \end{cases}$$

Since $d - l_0 \geqslant d_1$, $\widetilde{\mathrm{ev}}_x^d$ is onto and there exist $\sigma_{1,x}, \ldots, \sigma_{r,x} \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^{d-l_0})$ such that $(\sigma_{1,x}(x), \ldots, \sigma_{r,x}(x))$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^{d-l_0}\right)_x$. Similarly there exist $\sigma_{1,y}, \ldots, \sigma_{r,y}$ such that $(\sigma_{1,y}(y), \ldots, \sigma_{r,y}(y))$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^{d-l_0}\right)_y$. We define global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$ by $s_{k,x} = \sigma_{k,x} \otimes \sigma_x$ and $s_{k,y} = \sigma_{k,y} \otimes \sigma_y$ for all $k \in \{1, \ldots, r\}$. These sections are such that $(s_{k,x}(x))_{1 \leqslant k \leqslant r}$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$, $(s_{k,y}(y))_{1 \leqslant k \leqslant r}$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y$ and for all $k \in \{1, \ldots, r\}$, $s_{k,x}(y) = 0 = s_{k,y}(x)$. This proves that the map

$$\begin{array}{rccc} \widetilde{\mathrm{ev}}_{x,y}^d : & H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) & \longrightarrow & \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \oplus \left(\mathcal{E} \otimes \mathcal{L}^d\right)_y . \\ & s & \longmapsto & (s(x), s(y)) \end{array}$$

has rank at least $2r$ (as a $\mathbb{C}$-linear map). Since $\widetilde{\mathrm{ev}}_{x,y}^d$ is the complexified map of $\mathrm{ev}_{x,y}^d$, the latter must have rank at least $2r$ (as a $\mathbb{R}$-linear map), hence it is onto. □

Remark 4.3. In Prop. 4.2, $\mathrm{ev}_{x,y}^d$ is surjective if and only if $\left|\det{}^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right| > 0$, that is if and only if $\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*$ is non-singular. Since the latter is the variance operator of $\mathrm{ev}_{x,y}^d(s_d)$, where $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we see that the surjectivity of $\mathrm{ev}_{x,y}^d$ is equivalent to the non-degeneracy of the distribution of $(s_d(x), s_d(y))$.

We can now deduce a Kac–Rice type formula from Prop. 4.2. For any $d \in \mathbb{N}$, we define F_d to be the following bundle map over M^2:

$$\begin{array}{rccc} F_d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2 & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) \times \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) . \\ & (s, x, y) & \longmapsto & (s(x), s(y)) \end{array}$$

Let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ (see Def. 3.2). Then by Rem. 3.3, the restriction of ∇^d defines a connection on $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$. Let $\nabla^d F_d$ denote the vertical component of the diffential of F_d. Then, for all $(s_0, x, y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$, we have:

$$\begin{array}{rccc} \nabla^d_{(s_0,x,y)} F_d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times T_xM \times T_yM & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y . \\ & (s, v, w) & \longmapsto & \left(s(x) + \nabla^d_x s_0 \cdot v, s(y) + \nabla^d_y s_0 \cdot w\right) \end{array}$$

We denote by $\partial_1^d F_d$ the partial derivative of F_d with respect to the first variable (meaning s), and by $\partial_2^d F_d$ its partial derivative with respect to the second variable (meaning (x, y)). Then for all $(s_0, x, y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$ we have:

$$\partial_1^d F_d(s_0, x, y) = \mathrm{ev}_{x,y}^d \qquad \text{and} \qquad \partial_2^d F_d(s_0, x, y) : (v, w) \mapsto \left(\nabla^d_x s_0 \cdot v, \nabla^d_y s_0 \cdot w\right) . \tag{4.3}$$

From now on, we assume that $d \geqslant d_2$, where d_2 is given by Prop. 4.2. We define an incidence manifold Σ_d by:

$$\Sigma_d = \left(F_d\right)^{-1}(0) \cap \left(\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \left(M^2 \setminus \Delta\right)\right) .$$

By Prop. 4.2 and Eq. 4.3, for all $(s,x,y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) \times \big(M^2 \setminus \Delta\big)$, $\partial_1^d F_{d,p}(s,x,y)$ is surjective. Thus, the restriction of F_d to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) \times \big(M^2 \setminus \Delta\big)$ is a submersion and Σ_d is a submanifold of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) \times M^2$ of codimension $2r$. Note that we are only concerned with the zero set of F_d, hence none of this depends on the choice of ∇^d. We can now state the Kac–Rice formula in this context.

Theorem 4.4 (Kac–Rice formula)**.** *Let $d \geqslant d_2$, where d_2 is given by Prop. 4.2, and let ∇^d be any real connection on $\mathcal{E}\otimes\mathcal{L}^d$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then for any Borel measurable function $\Phi : \Sigma_d \to \mathbb{R}$ we have:*

$$\mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta} \Phi(s_d,x,y)\,|\mathrm{d}V_d|^2\right] = \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta} \frac{1}{\left|\det^\perp\left(\mathrm{ev}^d_{x,y}\right)\right|}\times \mathbb{E}\Big[\Phi(s_d,x,y)\left|\det^\perp\left(\nabla^d_x s_d\right)\right|\left|\det^\perp\left(\nabla^d_y s_d\right)\right| \,\Big|\, s_d(x)=0=s_d(y)\Big]\,|\mathrm{d}V_M|^2 \quad (4.4)$$

whenever one of these integrals is well-defined. Here, $|\mathrm{d}V_M|^2$ stands for the product measure on M^2 induced by $|\mathrm{d}V_M|$. Similarly, $|\mathrm{d}V_d|^2$ is the product measure on $(Z_d)^2$.

The expectation on the right-hand side of (4.4) is to be understood as the conditional expectation of $\Phi(s_d,x,y)\left|\det^\perp\left(\nabla^d_x s_d\right)\right|\left|\det^\perp\left(\nabla^d_y s_d\right)\right|$ given that $s_d(x)=0=s_d(y)$.

Proof. The proof of Thm. 4.4 uses the double fibration trick, that is apply Federer's coarea formula twice. See for example [14, App. C] and the reference therein.

The Euclidean inner product on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ defined by Eq. (2.1) and the Riemannian metric g induce a Riemannian metric on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)\times M^2$, and on Σ_d by restriction. Let $\pi_1 : \Sigma_d \to \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ and $\pi_2 : \Sigma_d \to M^2\setminus\Delta$ denote the projections from Σ_d to the first and second factor, respectively. For all $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, $\pi_1^{-1}(s)$ is isometric to Z_s and we identify these spaces. Similarly, for all $(x,y)\in M^2\setminus\Delta$ we identify $\pi_2^{-1}(x,y)$ with the isometric space $\ker(\mathrm{ev}^d_{x,y})$.

We denote by $\mathrm{d}s$ the Lebesgue measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ or any of its subspaces, normalized so that a unit cube has volume 1. Let $\Phi : \Sigma_d \to \mathbb{R}$ be a Borel measurable function. Then

$$\mathbb{E}\left[\int_{(Z_d)^2\setminus\Delta} \Phi\,|\mathrm{d}V_d|^2\right] = \int_{s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)} \left(\int_{(x,y)\in\pi_1^{-1}(s)} \Phi(s,x,y)\frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\,|\mathrm{d}V_d|^2\right)\mathrm{d}s,$$

where N_d is the dimension of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then, by the double fibration trick [14, Prop. C.3] this quantity equals:

$$\int_{(x,y)\in M^2\setminus\Delta} \left(\int_{s\in\ker(\mathrm{ev}^d_{x,y})} \Phi(s,x,y)\frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\frac{\left|\det^\perp\left(\partial_2^d F_d(s,x,y)\right)\right|}{\left|\det^\perp\left(\partial_1^d F_d(s,x,y)\right)\right|}\,\mathrm{d}s\right)|\mathrm{d}V_M|^2\,. \quad (4.5)$$

Then Eq. (4.3) shows that $\partial_2^d F_{d,p}(s,x,y) = \nabla^d_x s \oplus \nabla^d_y s$. Moreover, by definition of the metrics, T_xM is orthogonal to T_yM and $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ is orthogonal to $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_y$. Thus

$$\begin{aligned}
\left|\det^\perp\left(\partial_2^d F_d(s,x,y)\right)\right| &= \det\left(\partial_2^d F_d(s,x,y)\left(\partial_2^d F_d(s,x,y)\right)^*\right)^{\frac{1}{2}}\\
&= \det\left(\begin{pmatrix}\nabla^d_x s & 0\\ 0 & \nabla^d_y s\end{pmatrix}\begin{pmatrix}(\nabla^d_x s)^* & 0\\ 0 & (\nabla^d_y s)^*\end{pmatrix}\right)^{\frac{1}{2}}\\
&= \det\begin{pmatrix}\nabla^d_x s(\nabla^d_x s)^* & 0\\ 0 & \nabla^d_y s(\nabla^d_y s)^*\end{pmatrix}^{\frac{1}{2}}\\
&= \left|\det^\perp\left(\nabla^d_x s\right)\right|\left|\det^\perp\left(\nabla^d_y s\right)\right|.
\end{aligned}$$

Besides, Eq.(4.3) also shows that $\left|\det^{\perp}\left(\partial_1^d F_d(s,x,y)\right)\right| = \left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|$, which does not depend on s, so that (4.5) equals:

$$\int_{(x,y)\in M^2\setminus\Delta} \frac{1}{\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|}\left(\int_{s\in\ker\left(\mathrm{ev}_{x,y}^d\right)} \Phi\left|\det^{\perp}\left(\nabla_x^d s\right)\right|\left|\det^{\perp}\left(\nabla_y^d s\right)\right| \frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\,\mathrm{d}s\right)|\mathrm{d}V_M|^2 .$$

Finally, by Prop. 4.2, $\ker(\mathrm{ev}_x^d)$ is a subspace of codimension $2r$ of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Hence, the inner integral in (4.5) can be expressed as a conditional expectation given that $\mathrm{ev}_{x,y}^d(s_d)=0$, up to a factor $(2\pi)^r$. This concludes the proof of Thm. 4.4. □

4.2 An integral formula for the variance

In this section, we fix some $d \geqslant \max(d_0,d_1,d_2)$ where d_0, d_1 and d_2 are defined by Thm. 3.9, Lem. 2.4 and Prop. 4.2 respectively. We denote by ∇^d a real connection on $\mathcal{E}\otimes\mathcal{L}^d$. Let $\phi_1,\phi_2\in\mathcal{C}^0(M)$, we want to compute:

$$\begin{aligned}\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) &= \mathrm{Cov}(\langle|\mathrm{d}V_d|\,,\phi_1\rangle,\langle|\mathrm{d}V_d|\,,\phi_2\rangle)\\ &= \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle\langle|\mathrm{d}V_d|\,,\phi_2\rangle] - \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_2\rangle].\end{aligned} \tag{4.6}$$

First, by Thm. 4.1, we have:

$$\begin{aligned}&\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_2\rangle] = \frac{1}{(2\pi)^r}\times\\ &\int_{M^2}\phi_1(x)\phi_2(y)\frac{\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|\,\middle|\,s_d(x)=0\right]}{\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|}\frac{\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_y^d s_d\right)\right|\,\middle|\,s_d(y)=0\right]}{\left|\det^{\perp}\left(\mathrm{ev}_y^d\right)\right|}|\mathrm{d}V_M|^2 .\end{aligned} \tag{4.7}$$

On the other hand,

$$\begin{aligned}\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle\langle|\mathrm{d}V_d|\,,\phi_2\rangle] &= \mathbb{E}\left[\left(\int_{x\in Z_d}\phi_1(x)\,|\mathrm{d}V_d|\right)\left(\int_{y\in Z_d}\phi_2(y)\,|\mathrm{d}V_d|\right)\right]\\ &= \mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right].\end{aligned}$$

Indeed, Z_d is almost surely of dimension $n-r>0$, so that $(Z_d)^2\cap\Delta$ (that is the diagonal in $(Z_d)^2$) has measure 0 for $|\mathrm{d}V_d|^2$. We compute this integral by Thm. 4.4:

$$\begin{aligned}\mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right] &= \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta}\frac{\phi_1(x)\phi_2(y)}{\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|}\times\\ &\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|\left|\det^{\perp}\left(\nabla_y^d s_d\right)\right|\,\middle|\,s_d(x)=0=s_d(y)\right]|\mathrm{d}V_M|^2 .\end{aligned} \tag{4.8}$$

Let $\mathcal{D}_d$ be the function defined by: $\forall(x,y)\in M^2\setminus\Delta$,

$$\begin{aligned}\mathcal{D}_d(x,y) = \Bigg(&\frac{\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|\left|\det^{\perp}\left(\nabla_y^d s_d\right)\right|\,\middle|\,\mathrm{ev}_{x,y}^d(s_d)=0\right]}{\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|}\\ &-\frac{\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|\,\middle|\,s_d(x)=0\right]\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_y^d s_d\right)\right|\,\middle|\,s_d(y)=0\right]}{\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|\left|\det^{\perp}\left(\mathrm{ev}_y^d\right)\right|}\Bigg)\end{aligned} \tag{4.9}$$

Since $\dim M = n > 0$, Δ has measure 0 in M^2 for $|\mathrm{d}V_M|^2$. Thus, by (4.6), (4.7), (4.8) and (4.9), we have:

$$\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) = \frac{1}{(2\pi)^r}\int_{M^2}\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\,. \tag{4.10}$$

Remark 4.5. At this stage, it is worth noticing that the values of the conditional expectations appearing in the definition of $\mathcal{D}_d$ (see Eq. (4.9)) do not depend on the choice of ∇^d. In fact, the whole conditional distribution of $\nabla^d_x s_d$ given that $s_d(x) = 0$ (resp. of $\nabla^d_y s_d$ given that $s_d(y) = 0$, resp. of $(\nabla^d_x s_d, \nabla^d_y s_d)$ given that $s_d(x) = 0 = s_d(y)$) is independent of the choice of ∇^d. Indeed, if $s_d(x) = 0$ then $\nabla^d_x s_d$ does not depend on ∇^d, and we conditioned on the vanishing of $s_d(x)$ (resp. $s_d(y)$, resp. $s_d(x)$ and $s_d(y)$). Thus, in the sequel, we can use any real connection we like, even one that depends on $(x,y) \in M^2 \setminus \Delta$.

4.3 Asymptotic for the variance

In this section we compute the asymptotic of the integral in Eq. (4.10). The main point is to write M^2 as the disjoint union of a neighborhood of Δ, of size about $\frac{\ln d}{\sqrt{d}}$, and its complement. In (4.10), the set of points that are far from the diagonal will contribute a term of smaller order than the neighborhood of Δ. This is a consequence of the fast decrease of the Bergman kernel outside of the diagonal. The values of s_d at x and y are not correlated, up to some small error, outside of a neighborhood of Δ.

We still assume that $d \geqslant \max(d_0, d_1, d_2)$ and we denote by s_d a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$.

4.3.1 Asymptotics for the uncorrelated terms

Let us first compute asymptotics for the terms in the expression of $\mathcal{D}_d$ (see Eq. (4.9)) that only depend on one point, say $x \in M$. For all $x \in M$, ev^d_x is linear. Hence $s_d(x) = \mathrm{ev}^d_x(s_d)$ is a centered Gaussian vector in $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$ with variance operator:

$$\mathrm{ev}^d_x\left(\mathrm{ev}^d_x\right)^* = \mathbb{E}\left[s_d(x)\otimes(s_d(x))^*\right] = E_d(x,x), \tag{4.11}$$

where E_d is the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$ and the last equality is given by Prop. 2.6.

Lemma 4.6. *For every $x \in M$, we have:*

$$\left(\frac{\pi}{d}\right)^{\frac{nr}{2}}\left|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)\right| = 1 + O\left(d^{-1}\right),$$

where the error term $O\left(d^{-1}\right)$ does not depend on x.

Proof. Let $x \in M$, then $\left|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)\right|^2 = \det E_d(x,x)$ by (4.11). By (3.8), we have:

$$\left(\frac{\pi}{d}\right)^{nr}\left|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)\right|^2 = \det\left(\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{-1}\right)\right) = 1 + O\left(d^{-1}\right).$$

The error term in (3.8) is independent of x, hence the same is true here. □

Let ∇^d be a real connection on $\mathcal{E}\otimes\mathcal{L}^d$. We assume that ∇^d is a metric connection, so that Lem. 2.10 and Cor. 2.12 are valid in this context. Recall that the Chern connection is an example of real metric connection.

For all $x \in M$, let $j^d_x : s \mapsto \left(s(x), \nabla^d_x s\right)$ denote the evaluation of the 1-jet at x, from $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ to $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes (\mathbb{R}\oplus T_xM^*)$. Since j^d_x is linear, $\left(s_d(x), \nabla^d_x s_d\right)$ is a centered

Gaussian vector with variance operator $j_x^d \left(j_x^d\right)^*$. This operator splits according to the direct sum $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \otimes T_xM^*$:

$$\begin{aligned} j_x^d \left(j_x^d\right)^* &= \mathbb{E}\left[j_x^d(s_d)\otimes\left(j_x^d(s_d)\right)^*\right] \\ &= \begin{pmatrix} \mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}\left[s_d(x)\otimes(\nabla_x^d s_d)^*\right] \\ \mathbb{E}\left[\nabla_x^d s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla_x^d s_d\otimes(\nabla_x^d s_d)^*\right] \end{pmatrix} \\ &= \begin{pmatrix} E_d(x,x) & \partial_y^\sharp E_d(x,x) \\ \partial_x E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,x) \end{pmatrix}, \end{aligned} \tag{4.12}$$

where the last equality comes from Cor. 2.13. We chose $d \geqslant d_1$, so that ev_x^d is surjective (see Lem. 2.4), i.e. $\det\left(\mathrm{ev}_x^d\left(\mathrm{ev}_x^d\right)^*\right) > 0$. Hence, the distribution of $s_d(x)$ is non-degenerate. Then (see [2, prop. 1.2]), the distribution of $\nabla_x^d s_d$ given that $s_d(x)=0$ is a centered Gaussian in $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\otimes T_xM^*$ with variance operator:

$$\partial_x\partial_y^\sharp E_d(x,x) - \partial_x E_d(x,x)\left(E_d(x,x)\right)^{-1}\partial_y^\sharp E_d(x,x). \tag{4.13}$$

Lemma 4.7. *For every $x\in M$, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{r}{2}} \mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x)\right] = (2\pi)^{\frac{r}{2}}\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\left(1+O\big(d^{-1}\big)\right),$$

where the error term is independent of x.

Proof. Let $x\in M$, and let $L_d(x)$ be a centered Gaussian vector in $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\otimes T_xM^*$ with variance operator:

$$\Lambda_d(x) = \frac{\pi^n}{d^{n+1}}\left(\partial_x\partial_y^\sharp E_d(x,x) - \partial_x E_d(x,x)\left(E_d(x,x)\right)^{-1}\partial_y^\sharp E_d(x,x)\right). \tag{4.14}$$

By (4.13) and the above discussion, the distribution of $\nabla_x^d s_d$ given that $s_d(x)=0$ equals that of $\left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{1}{2}} L_d(x)$. Then,

$$\mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x)\right] = \mathbb{E}\left[\left|\det{}^\perp\left(\left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{1}{2}} L_d(x)\right)\right|\right] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}}\mathbb{E}\left[\left|\det{}^\perp\left(L_d(x)\right)\right|\right]. \tag{4.15}$$

Recall that the distribution of $\nabla_x^d s_d$ given that $s_d(x)=0$ does not depend on the choice of ∇^d (Rem. 4.5). Hence $\Lambda_d(x)$ does not depend on the choice of ∇^d. For the following computation, we choose ∇^d to be trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Then we can use the diagonal estimates of Cor. 3.8 for the Bergman kernel and its derivatives. We have: $\Lambda_d(x) = \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\otimes\mathrm{Id}_{T_xM^*} + O\big(d^{-1}\big)$, where the error does not depend on x. Hence,

$$\det\left(\Lambda_d(x)\right) = 1 + O\big(d^{-1}\big). \tag{4.16}$$

Besides, there exists some $K>0$ such that $\left\|\Lambda_d(x)^{-1}-\mathrm{Id}\right\| \leqslant Kd^{-1}$ for all d large enough. Then, by the mean value inequality, for all $L\in\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes T_xM^*$

$$\left(\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x)^{-1}-\mathrm{Id}\right)L, L\right\rangle\right)-1\right) \leqslant \frac{K}{2d}\|L\|^2\exp\left(\frac{K}{2d}\|L\|^2\right).$$

Let $L_d^0(x) \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_xM^*$ and let $\mathrm{d}L$ denote the normalized Lebesgue measure on $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_xM^*$. Then we have:

$$\begin{aligned}(2\pi)^{\frac{nr}{2}} &\left|\det\left(\Lambda_d(x)\right)^{\frac{1}{2}}\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d(x)\right)\right|\right] - \mathbb{E}\left[\left|\det{}^{\perp}\left(L_d^0(x)\right)\right|\right]\right| \\ &\leqslant \int \left|\det{}^{\perp}\left(L\right)\right| e^{-\frac{1}{2}\|L\|^2}\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x)^{-1}-\mathrm{Id}\right)L\,,L\right\rangle\right)-1\right|\mathrm{d}L \\ &\leqslant \frac{K}{2d}\int\left|\det{}^{\perp}\left(L\right)\right|\exp\left(-\frac{1}{2}\left(1-\frac{K}{d}\right)\|L\|^2\right)\mathrm{d}L.\end{aligned}$$

The integral on the last line converges to some finite limit as $d\to+\infty$. Thus, by (4.16),

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d(x)\right)\right|\right] &= \det\left(\Lambda_d(x)\right)^{-\frac{1}{2}}\left(\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d^0(x)\right)\right|\right]+O\left(d^{-1}\right)\right) \\ &= \mathbb{E}\left[\left|\det{}^{\perp}\left(L_d^0(x)\right)\right|\right]+O\left(d^{-1}\right),\end{aligned} \tag{4.17}$$

uniformly in $x\in M$. Lemma 4.7 follows from (4.15), (4.17) and the following equality, that was proved in [14, lem. A.14]:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d^0(x)\right)\right|\right] = (2\pi)^{\frac{r}{2}}\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}. \tag{4.18}$$

□

4.3.2 Far off-diagonal asymptotics for the correlated terms

We can now focus on computing terms in the expression of $\mathcal{D}_d$ that depend on both x and y. For all $(x,y)\in M^2\setminus\Delta$, $\mathrm{ev}_{x,y}^d(s_d) = (s_d(x), s_d(y))$ is a centered Gaussian vector with variance operator:

$$\begin{aligned}\mathrm{ev}_{x,y}^d(\mathrm{ev}_{x,y}^d)^* &= \mathbb{E}\left[\mathrm{ev}_{x,y}^d(s_d)\otimes\mathrm{ev}_{x,y}^d(s_d)^*\right] \\ &= \begin{pmatrix}\mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}[s_d(x)\otimes s_d(y)^*] \\ \mathbb{E}[s_d(y)\otimes s_d(x)^*] & \mathbb{E}[s_d(y)\otimes s_d(y)^*]\end{pmatrix} \\ &= \begin{pmatrix}E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix},\end{aligned} \tag{4.19}$$

where we decomposed this operator according to the direct sum $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y$. Since we assumed $d\geqslant d_2$, $\left|\det{}^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right| > 0$ (see Prop. 4.2) and the distribution of $(s_d(x), s_d(y))$ is non-degenerate.

We denote by $j_{x,y}^d : s\mapsto\left(s(x), s(y), \nabla_x^d s, \nabla_y^d s\right)$ the evaluation of the 1-jet at (x,y). Then $j_{x,y}^d$ is a linear map from $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ to

$$\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \oplus \left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_xM^*\right) \oplus \left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \otimes T_yM^*\right), \tag{4.20}$$

and $j_{x,y}^d(s_d)$ is a centered Gaussian vector, with variance operator $j_{x,y}^d\left(j_{x,y}^d\right)^*$. We can split this variance operator according to the direct sum (4.20). Then by Cor. 2.13, we have:

$$\begin{aligned}j_{x,y}^d\left(j_{x,y}^d\right)^* &= \mathbb{E}\left[j_{x,y}^d(s_d)\otimes\left(j_{x,y}^d(s_d)\right)^*\right] \\ &= \begin{pmatrix}\mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}[s_d(x)\otimes s_d(y)^*] & \mathbb{E}\left[s_d(x)\otimes(\nabla_x^d s_d)^*\right] & \mathbb{E}\left[s_d(x)\otimes(\nabla_y^d s_d)^*\right] \\ \mathbb{E}[s_d(y)\otimes s_d(x)^*] & \mathbb{E}[s_d(y)\otimes s_d(y)^*] & \mathbb{E}\left[s_d(y)\otimes(\nabla_x^d s_d)^*\right] & \mathbb{E}\left[s_d(y)\otimes(\nabla_y^d s_d)^*\right] \\ \mathbb{E}\left[\nabla_x^d s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla_x^d s_d\otimes s_d(y)^*\right] & \mathbb{E}\left[\nabla_x^d s_d\otimes(\nabla_x^d s_d)^*\right] & \mathbb{E}\left[\nabla_x^d s_d\otimes(\nabla_y^d s_d)^*\right] \\ \mathbb{E}\left[\nabla_y^d s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla_y^d s_d\otimes s_d(y)^*\right] & \mathbb{E}\left[\nabla_y^d s_d\otimes(\nabla_x^d s_d)^*\right] & \mathbb{E}\left[\nabla_y^d s_d\otimes(\nabla_y^d s_d)^*\right]\end{pmatrix} \\ &= \begin{pmatrix}E_d(x,x) & E_d(x,y) & \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ E_d(y,x) & E_d(y,y) & \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \\ \partial_x E_d(x,x) & \partial_x E_d(x,y) & \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) & \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y)\end{pmatrix}.\end{aligned} \tag{4.21}$$

Since the distribution of $(s_d(x), s_d(y))$ is non-degenerate, the distribution of $\left(\nabla^d_x s, \nabla^d_y s\right)$ given that $\mathrm{ev}_{x,y}(s_d) = 0$ is a centered Gaussian with variance operator:

$$\begin{pmatrix} \partial_x \partial_y^\sharp E_d(x,x) & \partial_x \partial_y^\sharp E_d(x,y) \\ \partial_x \partial_y^\sharp E_d(y,x) & \partial_x \partial_y^\sharp E_d(y,y) \end{pmatrix} - \\ \begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix}. \tag{4.22}$$

Definition 4.8. For every $(x,y) \in M^2 \setminus \Delta$ and every d large enough, we define $\Lambda_d(x,y)$ to be the operator such that $\frac{d^{n+1}}{\pi^n}\Lambda_d(x,y)$ equals (4.22). That is, $\Lambda_d(x,y)$ is the conditional variance of $\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \left(\nabla^d_x s, \nabla^d_y s\right)$ given that $\mathrm{ev}_{x,y}(s_d) = 0$.

Let $C' > 0$ be the constant appearing in the exponential in Thm. 3.9. We denote

$$b_n = \frac{1}{C'}\left(\frac{n}{2} + 1\right) \tag{4.23}$$

and

$$\Delta_d = \left\{ (x,y) \in M^2 \;\middle|\; \rho_g(x,y) < b_n \frac{\ln d}{\sqrt{d}} \right\}, \tag{4.24}$$

where, as before, ρ_g is the geodesic distance in (M,g).

Lemma 4.9. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right| = \left|\det{}^\perp\left(\mathrm{ev}^d_x\right)\right| \left|\det{}^\perp\left(\mathrm{ev}^d_y\right)\right| \left(1 + O\big(d^{-\frac{n}{2}-1}\big)\right),$$

where the error term is uniform in $(x,y) \in M^2 \setminus \Delta_d$

Proof. For all $(x,y) \in M^2 \setminus \Delta_d$, we have $\rho_g(x,y) \geqslant b_n \frac{\ln d}{\sqrt{d}}$. Then, by Thm. 3.9,

$$\|E_d(x,y)\| \leqslant C_0 d^n \exp\left(-C' b_n \ln d\right) \leqslant C_0 d^{\frac{n}{2}-1}.$$

Then, by (4.19) we have:

$$\begin{aligned} \mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^* &= \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix} \\ &= \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} + O\big(d^{\frac{n}{2}-1}\big). \end{aligned}$$

Besides, by (3.8),

$$\begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix}^{-1} = \left(\frac{\pi}{d}\right)^n \left(\mathrm{Id} + O\big(d^{-1}\big)\right) = O\big(d^{-n}\big), \tag{4.25}$$

so that

$$\begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix} = \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} \left(\mathrm{Id} + O\big(d^{-\frac{n}{2}-1}\big)\right). \tag{4.26}$$

We conclude the proof by taking the square root of the determinant of this last equality (recall (4.11)). ∎

Lemma 4.10. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big] = \\ \mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\Big|\,s_d(x)=0\Big]\,\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,s_d(y)=0\Big]\left(1+O\big(d^{-\frac{n}{2}-1}\big)\right),$$

where the error term is uniform in $(x,y) \in M^2 \setminus \Delta_d$

This lemma is a consequence of the following technical result.

Lemma 4.11. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}\left(\mathrm{Id} + O\big(d^{-\frac{n}{2}-1}\big)\right),$$

uniformly in $(x,y) \in M^2 \setminus \Delta_d$.

Proof of Lemma 4.10. Let $(L_d(x), L_d(y))$ and $(L'_d(x), L'_d(y))$ be centered Gaussian vectors in

$$\big(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \otimes T_xM^*\big) \oplus \Big(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y \otimes T_yM^*\Big)$$

such that: the variance of $(L'_d(x), L'_d(y))$ is $\Lambda_d(x,y)$ (recall Def. 4.8), and $L_d(x)$ and $L_d(y)$ are independent with variances $\Lambda_d(x)$ and $\Lambda_d(y)$ respectively (see (4.14)). Then, the distribution of $(L_d(x), L_d(y))$ is a centered Gaussian with variance $\left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)$. From the definitions of $\Lambda_d(x)$, $\Lambda_d(y)$ and $\Lambda_d(x,y)$, we have:

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\Big|\,s_d(x)=0\Big] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}}\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\big],$$

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,s_d(y)=0\Big] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}}\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big],$$

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big] = \left(\frac{d^{n+1}}{\pi^n}\right)^{r}\mathbb{E}\big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|\big|\det{}^{\perp}\left(L'_d(y)\right)\big|\big].$$

Since $L_d(x)$ and $L_d(y)$ are independent, we only need to prove that:

$$\mathbb{E}\big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L'_d(y)\right)\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big]\left(1+O\big(d^{-\frac{n}{2}-1}\big)\right). \tag{4.27}$$

By Lemma 4.11,

$$\begin{aligned}\det\left(\Lambda_d(x,y)\right) &= \det\left(\begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}\left(\mathrm{Id}+O\big(d^{-\frac{n}{2}-1}\big)\right)\right) \\ &= \det\left(\Lambda_d(x)\right)\det\left(\Lambda_d(x)\right)\left(1+O\big(d^{-\frac{n}{2}-1}\big)\right).\end{aligned} \tag{4.28}$$

Besides Lem. 4.11 shows that:

$$\Lambda_d(x,y)^{-1} = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}^{-1}\left(\mathrm{Id}+O\big(d^{-\frac{n}{2}-1}\big)\right).$$

By Cor. 3.8 and Eq. (4.14), we have: $\left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right) = \mathrm{Id}+O\big(d^{-1}\big)$. Hence,

$$\begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}^{-1} = \mathrm{Id}+O\big(d^{-1}\big) \tag{4.29}$$

uniformly in (x,y), and

$$\Lambda_d(x,y)^{-1} - \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}^{-1} = O\big(d^{-\frac{n}{2}-1}\big)\,.$$

Thus there exists $K>0$ such that, for all d large enough,

$$\left\| \Lambda_d(x,y)^{-1} - \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}^{-1} \right\| \leqslant \frac{K}{d^{\frac{n}{2}+1}}.$$

Then, for every $L=(L_1,L_2) \in \big(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \otimes T_xM^*\big) \oplus \Big(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y \otimes T_yM^*\Big)$, we have:

$$\left|\exp\left(-\frac{1}{2}\left\langle \left(\Lambda_d(x,y)^{-1} - \left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)^{-1}\right) L\,,L\right\rangle\right) - 1\right| \leqslant \frac{K\,\|L\|^2}{2d^{\frac{n}{2}+1}} \exp\left(\frac{K\,\|L\|^2}{2d^{\frac{n}{2}+1}}\right).$$

Let $\mathrm{d}L$ denote the normalized Lebesgue measure on this vector space. We have:

$$\begin{aligned}
&(2\pi)^r \Big|\det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L'_d(y)\right)\big|\big] \\
&\qquad\qquad - \det\left(\Lambda_d(x)\right)^{\frac{1}{2}} \det\left(\Lambda_d(y)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big]\Big| \\
&\leqslant \int \big|\det{}^{\perp}\left(L_1\right)\big|\,\big|\det{}^{\perp}\left(L_2\right)\big| \exp\left(-\frac{1}{2}\left\langle \left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)^{-1} L\,,L\right\rangle\right) \times \\
&\qquad\qquad \left|\exp\left(-\frac{1}{2}\left\langle \left(\Lambda_d(x,y)^{-1} - \left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)^{-1}\right) L\,,L\right\rangle\right) - 1\right| \mathrm{d}L \\
&\leqslant \frac{K}{2d^{\frac{n}{2}+1}} \int \big|\det{}^{\perp}\left(L_1\right)\big|\,\big|\det{}^{\perp}\left(L_2\right)\big|\,\|L\|^2 \times \\
&\qquad\qquad \exp\left(-\frac{1}{2}\left\langle \left(\left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)^{-1} - \frac{K}{2d^{\frac{n}{2}+1}}\,\mathrm{Id}\right) L\,,L\right\rangle\right) \mathrm{d}L \\
&= O\big(d^{-\frac{n}{2}-1}\big)\,.
\end{aligned}$$

Let us prove the last equality. By Eq. (4.29), for every d large enough (uniform in (x,y)),

$$\left\| \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix}^{-1} - \left(1 + \frac{K}{2d^{\frac{n}{2}+1}}\right) \mathrm{Id} \right\| \leqslant \frac{1}{2},$$

so that:

$$\begin{aligned}
&\int \big|\det{}^{\perp}\left(L_1\right)\big|\,\big|\det{}^{\perp}\left(L_2\right)\big|\,\|L\|^2 \exp\left(-\frac{1}{2}\left\langle \left(\left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)^{-1} - \frac{K}{2d^{\frac{n}{2}+1}}\,\mathrm{Id}\right) L\,,L\right\rangle\right) \mathrm{d}L \\
&\qquad\qquad \leqslant \int \big|\det{}^{\perp}\left(L_1\right)\big|\,\big|\det{}^{\perp}\left(L_2\right)\big|\,\|L\|^2 \exp\left(-\frac{1}{4}\,\|L\|^2\right) \mathrm{d}L.
\end{aligned}$$

And the last integral is finite since $\big|\det{}^{\perp}\left(L_1\right)\big|\,\big|\det{}^{\perp}\left(L_2\right)\big|\,\|L\|^2$ is a polynomial in L.

Eq. (4.16) and (4.28) show that $\det(\Lambda_d(x,y)) = 1 + O\big(d^{-1}\big)$. Then, by the previous computations and Eq. (4.28), we have:

$$\begin{aligned}
&\mathbb{E}\big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L'_d(y)\right)\big|\big] \\
&\qquad = \left(\frac{\det\left(\Lambda_d(x)\right)\det\left(\Lambda_d(y)\right)}{\det\left(\Lambda_d(x,y)\right)}\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big] + O\big(d^{-\frac{n}{2}-1}\big) \\
&\qquad = \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big]\,\big(1 + O\big(d^{-\frac{n}{2}-1}\big)\big) + O\big(d^{-\frac{n}{2}-1}\big)\,.
\end{aligned}$$

Equations (4.17) and (4.18) proves that

$$\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(x)\right)\big|\big]\,\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d(y)\right)\big|\big]$$

converges to some positive constant. This proves (4.27) and establishes Lemma 4.10. □

Proof of Lemma 4.11. First, recall that $\Lambda_d(x,y)$, $\Lambda_d(x)$ and $\Lambda_d(y)$ do not depend on the choice of ∇^d (see Rem. 4.5). In this proof, we use the Chern connection which is both real and metric. Let $(x,y) \in M^2 \setminus \Delta_d$, then $\rho_g(x,y) \geqslant b_n \frac{\ln d}{\sqrt{d}}$. By Thm. 3.9, we have:

$$\|\partial_x E_d(x,y)\| \leqslant C_1 d^{n+\frac{1}{2}} \exp\left(-C' b_n \ln d\right) \leqslant C_1 d^{\frac{n}{2}-\frac{1}{2}}.$$

Similarly, $\|\partial_x E_d(y,x)\|$, $\left\|\partial_y^\sharp E_d(x,y)\right\|$ and $\left\|\partial_y^\sharp E_d(y,x)\right\|$ are smaller than $C_1 d^{\frac{n-1}{2}}$. Then

$$\begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} = \begin{pmatrix} \partial_x E_d(x,x) & 0 \\ 0 & \partial_x E_d(y,y) \end{pmatrix} + O\Big(d^{\frac{n-1}{2}}\Big) \tag{4.30}$$

$$\begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix} = \begin{pmatrix} \partial_y^\sharp E_d(x,x) & 0 \\ 0 & \partial_y^\sharp E_d(y,y) \end{pmatrix} + O\Big(d^{\frac{n-1}{2}}\Big) \tag{4.31}$$

and, by Eq. (4.26),

$$\begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} = \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix}^{-1} \left(\mathrm{Id} + O\big(d^{-\frac{n}{2}-1}\big)\right). \tag{4.32}$$

Using Eq. (3.9), (3.10) and (4.25), we get:

$$\begin{aligned} &\begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix} \\ &\quad = \begin{pmatrix} \partial_x E_d(x,x) & 0 \\ 0 & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & 0 \\ 0 & \partial_y^\sharp E_d(y,y) \end{pmatrix} \\ &\qquad\qquad + O\big(d^{\frac{n}{2}-1}\big) \end{aligned} \tag{4.33}$$

Using Thm. 3.9 once more, we know that $\left\|\partial_x \partial_y^\sharp E_d(x,y)\right\|$ and $\left\|\partial_x \partial_y^\sharp E_d(y,x)\right\|$ are smaller than $C_2 d^{\frac{n}{2}}$. Then we have:

$$\begin{pmatrix} \partial_x \partial_y^\sharp E_d(x,x) & \partial_x \partial_y^\sharp E_d(x,y) \\ \partial_x \partial_y^\sharp E_d(y,x) & \partial_x \partial_y^\sharp E_d(y,y) \end{pmatrix} = \begin{pmatrix} \partial_x \partial_y^\sharp E_d(x,x) & 0 \\ 0 & \partial_x \partial_y^\sharp E_d(y,y) \end{pmatrix} + O\big(d^{\frac{n}{2}}\big). \tag{4.34}$$

We substract Eq. (4.33) to Eq. (4.34) and divide by $\frac{d^{n+1}}{\pi^n}$. By definition of $\Lambda_d(x,y)$, $\Lambda_d(x)$ and $\Lambda_d(y)$ (see Def. 4.8 and Eq. (4.14)),

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix} + O\big(d^{-\frac{n}{2}-1}\big) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix} \left(\mathrm{Id} + O\big(d^{-\frac{n}{2}-1}\big)\right),$$

where we used the fact that $\Lambda_d(x) = \mathrm{Id} + O\big(d^{-1}\big) = \Lambda_d(y)$ to obtain the last equality. □

Proposition 4.12. *Let $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, then we have the following as $d \to +\infty$:*

$$\int_{M^2 \setminus \Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\left|\mathrm{d}V_M\right|^2 = \|\phi_1\|_\infty \|\phi_2\|_\infty O\big(d^{r-\frac{n}{2}-1}\big),$$

where the error term is independent of (ϕ_1, ϕ_2).

Proof. We combine Lemmas 4.9 and 4.10, which gives:

$$\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]}{\big|\det{}^{\perp}\left(\mathrm{ev}^d_{x,y}\right)\big|} =$$
$$\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\Big|\,s_d(x)=0\Big]\,\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,s_d(y)=0\Big]}{|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)|\,\big|\det{}^{\perp}\left(\mathrm{ev}^d_y\right)\big|}\left(1+O\big(d^{-\frac{n}{2}-1}\big)\right).$$

for all $(x,y)\in M^2\setminus\Delta_d$. Besides, by Lemmas 4.6 and 4.7,

$$\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\Big|\,s_d(x)=0\Big]}{|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)|}=O\big(d^{\frac{r}{2}}\big)=\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,s_d(y)=0\Big]}{\big|\det{}^{\perp}\left(\mathrm{ev}^d_y\right)\big|}.$$

Recalling the definition of $\mathcal{D}_d$ (Eq. (4.9)), we obtain that:

$$\forall (x,y)\in M^2\setminus\Delta_d,\qquad \mathcal{D}_d(x,y)=O\big(d^{r-\frac{n}{2}-1}\big),$$

uniformly in $(x,y)\notin\Delta_d$. Then, for any continuous ϕ_1 and $\phi_2\in\mathcal{C}^0(M)$, we have:

$$\left|\int_{M^2\setminus\Delta_d}\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\right|\leqslant\|\phi_1\|_\infty\|\phi_2\|_\infty\,\mathrm{Vol}\left(M^2\right)\left(\sup_{M^2\setminus\Delta_d}|\mathcal{D}_d|\right)$$
$$=\|\phi_1\|_\infty\|\phi_2\|_\infty\,O\big(d^{r-\frac{n}{2}-1}\big).$$

and the error term does not depend on (ϕ_1,ϕ_2). □

4.3.3 Properties of the limit distribution

Before we tackle the computation of the dominant term in (4.10), that is the integral over Δ_d, we introduce the random variables that will turn out to be the scaling limits of $\left(\nabla^d_x s_d,\nabla^d_y s_d\right)$ given that $\mathrm{ev}^d_{x,y}(s_d)=0$. We also establish some of their properties.

Notation 4.13. Let $x\in M$ and $z\in T_xM$, we denote by $z^*\otimes z\in T_xM^*\otimes T_xM$ the linear map:

$$z^*\otimes z:\begin{array}{ccc} T_xM^* & \longrightarrow & T_xM^*.\\ \eta & \longmapsto & \eta(z)z^*\end{array}$$

Let $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$ be an orthonormal basis of T_xM and let $(dx_1,\dots,dx_n)$ denote its dual basis. If $z=\sum z_i\frac{\partial}{\partial x_i}$ then $z^*\otimes z=\sum z_iz_j dx_i\otimes\frac{\partial}{\partial x_j}$, i.e. the matrix of $z^*\otimes z$ in $(dx_1,\dots,dx_n)$ is $(z_iz_j)_{1\leqslant i,j\leqslant n}$.

Definition 4.14. For all $x\in M$ and $z\in T_xM\setminus\{0\}$, we define

$$\Lambda_x(z)\in\mathrm{End}\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes T_xM^*\right)$$

by:

$$\Lambda_x(z)=\begin{pmatrix}\mathrm{Id}_{T_xM^*}-\frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}}z^*\otimes z & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*}-\frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right)\\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*}-\frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right) & \mathrm{Id}_{T_xM^*}-\frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}}z^*\otimes z\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}.$$

We need information about $\Lambda_x(z)$, especially concerning the vanishing of its eigenvalues. This will be useful in the estimates involving $\Lambda_x(z)$ below.

Lemma 4.15. *For all $x \in M$ and $z \in T_xM \setminus \{0\}$, the eigenvalues of $\Lambda_x(z)$ are:*

- $1 - e^{-\frac{1}{2}\|z\|^2}$ *and* $1 + e^{-\frac{1}{2}\|z\|^2}$, *each with multiplicity* $(n-1)r$,
- $\dfrac{1 - e^{-\|z\|^2} + \|z\|^2 e^{-\frac{1}{2}\|z\|^2}}{1 + e^{-\frac{1}{2}\|z\|^2}}$ *and* $\dfrac{1 - e^{-\|z\|^2} - \|z\|^2 e^{-\frac{1}{2}\|z\|^2}}{1 - e^{-\frac{1}{2}\|z\|^2}}$, *each with multiplicity* r.

Proof. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. By definition of $\Lambda_x(z)$, its eigenvalues are the same as that of

$$\begin{pmatrix} \mathrm{Id}_{T_xM^*} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - \frac{z^* \otimes z}{1-e^{-\|z\|^2}}\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - \frac{z^* \otimes z}{1-e^{-\|z\|^2}}\right) & \mathrm{Id}_{T_xM^*} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix}, \tag{4.35}$$

with multiplicities multiplied by r. Hence, it is enough to compute the eigenvalues of the operator (4.35).

Let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ of T_xM such that $z = \|z\| \frac{\partial}{\partial x_1}$, and let us denote by $(dx_1, \dots, dx_n)$ the dual basis. Then, $z^* \otimes z = \|z\|^2 dx_1 \otimes \frac{\partial}{\partial x_1}$. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$, then the matrix of

$$\begin{pmatrix} \mathrm{Id}_{T_xM^*} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - \frac{z^* \otimes z}{1-e^{-\|z\|^2}}\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - \frac{z^* \otimes z}{1-e^{-\|z\|^2}}\right) & \mathrm{Id}_{T_xM^*} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix}$$

in the orthonormal basis $(e_1 \otimes dx_1, \dots, e_1 \otimes dx_n, e_2 \otimes dx_1, \dots, e_2 \otimes dx_n)$ is:

$$\left(\begin{array}{cc|cc} 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 & e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0 \\ 0 & I_{n-1} & 0 & e^{-\frac{1}{2}\|z\|^2} I_{n-1} \\ \hline e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0 & 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 \\ 0 & e^{-\frac{1}{2}\|z\|^2} I_{n-1} & 0 & I_{n-1} \end{array}\right), \tag{4.36}$$

where I_{n-1} stands for the identity matrix of size $n-1$. If we re-order the basis, the matrix of this operator in $(e_1 \otimes dx_1, e_2 \otimes dx_1, \dots, e_1 \otimes dx_n, \dots, e_2 \otimes dx_n)$ is:

$$\left(\begin{array}{cc|c} 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0 \\ e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 \\ \hline 0 & 0 & \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right). \tag{4.37}$$

The bottom-right block has eigenvalues $1 - e^{-\frac{1}{2}\|z\|^2}$ and $1 + e^{-\frac{1}{2}\|z\|^2}$, each with multiplicity $n-1$. To conclude the proof of Lemma 4.15, we only need to observe that, for all $t > 0$, the eigenvalues of

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) \\ e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) & 1 - \frac{te^{-t}}{1-e^{-t}} \end{pmatrix}$$

are:

$$1 - \frac{te^{-t}}{1-e^{-t}} + e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) = \frac{1 - e^{-t} - te^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}}$$

and

$$1-\frac{te^{-t}}{1-e^{-t}}-e^{-\frac{1}{2}t}\left(1-\frac{t}{1-e^{-t}}\right)=\frac{1-e^{-t}+te^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}.$$

Note that the latter one is the largest. □

Definition 4.16. We define the function $f:(0,+\infty)\to\mathbb{R}$ by:

$$\forall t>0,\qquad f(t)=\frac{1-e^{-\frac{1}{2}t}}{1-e^{-t}-te^{-\frac{1}{2}t}}.$$

Corollary 4.17. *Let $x\in M$ and $z\in T_xM\setminus\{0\}$, then we have:*

$$\det\left(\Lambda_x(z)\right)=\left(1-e^{-\|z\|^2}\right)^{r(n-2)}\left(1-e^{-\|z\|^2}+\|z\|^2e^{-\frac{1}{2}\|z\|^2}\right)^r\left(1-e^{-\|z\|^2}-\|z\|^2e^{-\frac{1}{2}\|z\|^2}\right)^r>0. \tag{4.38}$$

Moreover,

$$\|\Lambda_x(z)\|<2\qquad\text{and}\qquad\left\|\Lambda_x(z)^{-1}\right\|=f\left(\|z\|^2\right), \tag{4.39}$$

where $\|\cdot\|$ denote the operator norm on $\mathrm{End}\left(\mathbb{R}^2\otimes\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes T_xM^*\right)$.

Proof. First, the formula for $\det\left(\Lambda_x(z)\right)$ is a direct consequence of Lem. 4.15, and we only need to check that the eigenvalues of $\Lambda_x(z)$ are positive. Clearly, $1\pm e^{-\frac{1}{2}t}>0$ when $t>0$.

Then, for all positive t, we have:

$$\frac{1-e^{-t}-te^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}=\frac{e^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}\left(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}-t\right)=\frac{e^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}\left(2\sinh\left(\frac{t}{2}\right)-t\right),$$

and $2\sinh\left(\frac{t}{2}\right)>t$. Besides,

$$\frac{1-e^{-t}+te^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}=\frac{e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\left(2\sinh\left(\frac{t}{2}\right)+t\right)>0.$$

Recall that $\|\Lambda_x(z)\|$ is the larger eigenvalue of $\Lambda_x(z)$, and $\left\|\Lambda_x(z)^{-1}\right\|$ is the inverse of the smallest eigenvalue of $\Lambda_x(z)$. For all $t>0$ we have

$$0<1-e^{-\frac{t}{2}}<1+e^{-\frac{t}{2}}<2.$$

Besides,

$$\frac{1-e^{-t}-te^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}+\frac{1-e^{-t}+te^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}=2\left(1-\frac{te^{-t}}{1-e^{-t}}\right)<2,$$

and we just proved that both these terms are positive. Hence, each of them is smaller than 2. Thus, all the eigenvalues of $\Lambda_x(z)$ are smaller than 2 and $\|\Lambda_x(z)\|<2$.

For all $t>0$,

$$\frac{1-e^{-t}-te^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}<1-e^{-\frac{t}{2}}\iff1-e^{-t}-te^{-\frac{1}{2}t}<1-2e^{-\frac{t}{2}}+e^{-t}\iff1-\frac{t}{2}<e^{-\frac{t}{2}},$$

and this is always true by convexity of the exponential. Thus, the smallest eigenvalue of $\Lambda_x(z)$ is $\dfrac{1-e^{-\|z\|^2}-\|z\|^2e^{-\frac{1}{2}\|z\|^2}}{1-e^{-\frac{1}{2}\|z\|^2}}=\dfrac{1}{f(\|z\|^2)}$, which proves our last claim. □

Remark 4.18. To better understand the estimate (4.39), note that f is a decreasing function on $(0, +\infty)$. Moreover,

$$f(t) \xrightarrow[t\to+\infty]{} 1, \qquad \text{and} \qquad f(t) \sim \frac{12}{t^2}$$

when t goes to 0.

Definition 4.19. For every $x \in M$, and $z \in T_xM \setminus \{0\}$, let $(L_x(0), L_x(z))$ be a centered Gaussian vector in $\mathbb{R}^2 \otimes \mathbb{R} \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T_xM^*$ with variance operator $\Lambda_x(z)$.

Recall that we defined the random vector $(X(t), Y(t))$ for all $t > 0$ in the introduction (see Def. 1.4). Then $(X(t), Y(t))$ and $(L_x(0), L_x(z))$ are related as follows.

Lemma 4.20. *Let $x \in M$ and $z \in T_xM \setminus \{0\}$, then there exists an orthonormal basis of T_xM such that, for every orthonormal basis of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$, the couple of $r \times n$ matrices associated to $(L_x(0), L_x(z))$ in these bases is distributed as $(X(\|z\|^2), Y(\|z\|^2))$.*

Proof. As in the proof of Lem. 4.15, let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ of T_xM such that $z = \|z\| \frac{\partial}{\partial x_1}$. Let $(dx_1, \dots, dx_n)$ denote its dual basis. Let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$, and let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$.

Then $z^* \otimes z = \|z\|^2 dx_1 \otimes \frac{\partial}{\partial x_1}$ and the matrix of the operator (4.35) in the orthonormal basis $(e_1 \otimes dx_1, \dots, e_1 \otimes dx_n, e_2 \otimes dx_1, \dots, e_2 \otimes dx_n)$ is given by (4.36). Since $\Lambda_x(z)$ is exactly this operator tensor $\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$, the matrix of $\Lambda_x(z)$ in the orthonormal basis:

$$\begin{aligned}(&e_1 \otimes dx_1 \otimes \zeta_1, \dots, e_1 \otimes dx_n \otimes \zeta_1, e_2 \otimes dx_1 \otimes \zeta_1, \dots, e_2 \otimes dx_n \otimes \zeta_1,\\ &\qquad e_1 \otimes dx_1 \otimes \zeta_2, \dots, e_2 \otimes dx_n \otimes \zeta_2, \dots e_1 \otimes dx_1 \otimes \zeta_r, \dots, e_2 \otimes dx_n \otimes \zeta_r)\end{aligned}$$

is exactly the variance matrix of $(X(\|z\|^2), Y(\|z\|^2))$ (cf. Def. 1.4).

Let $M_x(0)$ and $M_x(z)$ denote the matrices of $L_x(0)$ and $L_x(d)$ in the bases $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and $(\zeta_1, \dots, \zeta_r)$. Then $(M_x(0), M_x(z))$ is a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R})^2$. Moreover, we have just seen that the variance matrix of this random vector is the same as that of $(X(\|z\|^2), Y(\|z\|^2))$. This concludes the proof. □

Corollary 4.21. *Let $x \in M$ and $z \in T_xM \setminus \{0\}$, then we have:*

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_x(0)\right)\right| \left|\det{}^{\perp}\left(L_x(z)\right)\right|\right] = \mathbb{E}\left[\left|\det{}^{\perp}\left(X(\|z\|^2)\right)\right| \left|\det{}^{\perp}\left(Y(\|z\|^2)\right)\right|\right].$$

Proof. With the same notations as in the proof of Lemma 4.20 above, we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(M_x(0)\right)\right| \left|\det{}^{\perp}\left(M_x(z)\right)\right|\right] = \mathbb{E}\left[\left|\det{}^{\perp}\left(X(\|z\|^2)\right)\right| \left|\det{}^{\perp}\left(Y(\|z\|^2)\right)\right|\right],$$

since $(M_x(0), M_x(z))$ and $(X(\|z\|^2), Y(\|z\|^2))$ have the same distribution. Besides,

$$\left|\det{}^{\perp}\left(L_x(0)\right)\right| = \left|\det{}^{\perp}\left(M_x(0)\right)\right| \qquad \text{and} \qquad \left|\det{}^{\perp}\left(L_x(z)\right)\right| = \left|\det{}^{\perp}\left(M_x(0)\right)\right|.$$

□

Let us now establish some facts about the distribution of $(X(t), Y(t))$ for $t > 0$.

Lemma 4.22. *For all $t > 0$, we have:*

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(X(t)\right)\right| \left|\det{}^{\perp}\left(Y(t)\right)\right|\right] \leqslant \left(n - \frac{te^{-t}}{1-e^{-t}}\right)^r \leqslant n^r.$$

Proof. First, by the Cauchy–Schwarz inequality, we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(X(t)\right)\right|\left|\det{}^{\perp}\left(Y(t)\right)\right|\right] \leqslant \mathbb{E}\left[\left|\det{}^{\perp}\left(X(t)\right)\right|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^{\perp}\left(Y(t)\right)\right|^2\right]^{\frac{1}{2}}.$$

Then, the definition of $(X(t), Y(t))$ (Def. 1.4) shows that both $X(t)$ and $Y(t)$ are centered Gaussian vectors in $\mathcal{M}_{rn}(\mathbb{R})$ with variance matrix:

$$\begin{pmatrix} 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 \\ 0 & I_{n-1} \end{pmatrix} \otimes I_r. \tag{4.40}$$

in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^r$. Here I_r and I_{n-1} stand for the identity matrices of size r and $n-1$ respectively. Hence,

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(X(t)\right)\right|\left|\det{}^{\perp}\left(Y(t)\right)\right|\right] \leqslant \mathbb{E}\left[\left|\det{}^{\perp}\left(X(t)\right)\right|^2\right] = \mathbb{E}\left[\det\left(X(t)X(t)^{\mathrm{t}}\right)\right].$$

We denote by $X_1(t), \ldots, X_r(t)$ the rows of $X(t)$. Then

$$X(t)X(t)^{\mathrm{t}} = \left(\left\langle X_i(t), X_j(t)\right\rangle\right)_{1 \leqslant i,j \leqslant r},$$

where we see $X_i(t)$ as an element of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. Hence, $\det\left(X(t)X(t)^{\mathrm{t}}\right)$ is the Gram determinant of the family $(X_1(t), \ldots, X_r(t))$, which is known to be the square of the r-dimensional volume of the parallelepiped spanned by these vectors. In particular,

$$\det\left(X(t)X(t)^{\mathrm{t}}\right) \leqslant \|X_1(t)\|^2 \cdots \|X_r(t)\|^2.$$

By (4.40), the $X_i(t)$ are independent identically distributed centered Gaussian vectors with variance matrix:

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

so that:

$$\det\left(X(t)X(t)^{\mathrm{t}}\right) \leqslant \mathbb{E}\left[\|X_1(t)\|^2 \cdots \|X_r(t)\|^2\right] \leqslant \mathbb{E}\left[\|X_1(t)\|^2\right]^r = \left(n - \frac{te^{-t}}{1-e^{-t}}\right)^r. \qquad \square$$

Lemma 4.23. *We have the following estimate as $t \to +\infty$:*

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(X\left(t\right)\right)\right|\left|\det{}^{\perp}\left(Y\left(t\right)\right)\right|\right] = (2\pi)^r \left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\right)^2 + O\left(te^{-\frac{t}{2}}\right).$$

Proof. Let $(X(\infty), Y(\infty))$ be a standard Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R})^2 \simeq \mathbb{R}^{2nr}$, i.e. $X(\infty)$ and $Y(\infty)$ are independent standard Gaussian vectors in $\mathcal{M}_{rn}(\mathbb{R})$. Then,

$$\begin{aligned} \mathbb{E}\left[\left|\det{}^{\perp}\left(X(\infty)\right)\right|\left|\det{}^{\perp}\left(Y(\infty)\right)\right|\right] &= \mathbb{E}\left[\left|\det{}^{\perp}\left(X(\infty)\right)\right|\right]\mathbb{E}\left[\left|\det{}^{\perp}\left(Y(\infty)\right)\right|\right] \\ &= \mathbb{E}\left[\left|\det{}^{\perp}\left(X(\infty)\right)\right|\right]^2 \\ &= (2\pi)^r \left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\right)^2, \end{aligned}$$

where we used (4.18) to get the last equality.

Then the proof is basically the same as that of Lemma 4.7. From Definition 1.4, we see that the variance operator $\Lambda(t)$ of $(X(t), Y(t))$ equals $\mathrm{Id} + O\left(te^{-\frac{t}{2}}\right)$. Hence:

$$\det\left(\Lambda(t)\right) = 1 + O\left(te^{-\frac{t}{2}}\right) \qquad \text{and} \qquad \Lambda(t)^{-1} = \mathrm{Id} + O\left(te^{-\frac{t}{2}}\right).$$

Let $C > 0$ such that $\left\|\Lambda(t)^{-1} - \mathrm{Id}\right\| \leqslant Cte^{-\frac{t}{2}}$. We denote by $L = (L_1, L_2)$ a generic element of $\mathcal{M}_{rn}(\mathbb{R})^2$ and by $\mathrm{d}L$ the normalized Lebesgue measure on this space. Then,

$$\begin{aligned}
&(2\pi)^{rn}\left|\det(\Lambda(t))^{\frac{1}{2}}\,\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\right] - \mathbb{E}\left[\left|\det{}^{\perp}(X(\infty))\right|\left|\det{}^{\perp}(Y(\infty))\right|\right]\right| \\
&\quad = \int \left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\left(\exp\left(-\frac{1}{2}\left\langle\left(\Lambda(t)^{-1} - \mathrm{Id}\right)L, L\right\rangle\right) - 1\right)e^{-\frac{1}{2}\|L\|^2}\,\mathrm{d}L \\
&\quad \leqslant \frac{C}{2}te^{-\frac{t}{2}}\int \left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\|L\|^2\exp\left(-\frac{1}{2}\left(1 - \frac{C}{2}te^{-\frac{t}{2}}\right)\|L\|^2\right)\mathrm{d}L \\
&\quad = O\left(te^{-\frac{t}{2}}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\right] \\
&\qquad = \det(\Lambda(t))^{-\frac{1}{2}}\left(\mathbb{E}\left[\left|\det{}^{\perp}(X(\infty))\right|\left|\det{}^{\perp}(Y(\infty))\right|\right] + O\left(te^{-\frac{t}{2}}\right)\right) \\
&\qquad = (2\pi)^r\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2 + O\left(te^{-\frac{t}{2}}\right).
\end{aligned}$$

□

Definition 4.24. Let $D_{n,r} : (0, +\infty) \to \mathbb{R}$ be the function defined by:

$$\forall t \in (0, +\infty), \qquad D_{n,r}(t) = \frac{\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\right]}{(1 - e^{-t})^{\frac{r}{2}}} - (2\pi)^r\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2.$$

Lemma 4.25. *We have:*

$$\int_0^{+\infty} |D_{n,r}(t)|\, t^{\frac{n-2}{2}}\,\mathrm{d}t < +\infty.$$

Proof. We first check the integrability of $|D_{n,r}(t)|\, t^{\frac{n-2}{2}}$ at $t = 0$. By Lemma 4.22, about $t = 0$ we have:

$$\begin{aligned}
|D_{n,r}(t)|\, t^{\frac{n-2}{2}} &\leqslant t^{\frac{n-2}{2}}\frac{\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\right]}{(1 - e^{-t})^{\frac{r}{2}}} + t^{\frac{n-2}{2}}(2\pi)^r\left(\frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)}\right)^2 \\
&\leqslant t^{\frac{n-2}{2}}\frac{n^r}{(1 - e^{-t})^{\frac{r}{2}}} + O(t^{\frac{n-2}{2}}) = O\left(t^{\frac{n-2-r}{2}}\right).
\end{aligned}$$

And this is integrable at $t = 0$ since $n - r \leqslant 1$.

Then, by Lemma 4.23, we have: $|D_{n,r}(t)|\, t^{\frac{n-2}{2}} = O\left(t^{\frac{n}{2}}e^{-\frac{t}{2}}\right)$ when t goes to infinity. This proves the integrability at infinity. □

4.3.4 Near-diagonal asymptotics for the correlated terms

The next step of the proof is the compute the contribution of the integral (4.10) on Δ_d. Let $R > 0$ be such that $2R$ is smaller than the injectivity radius of $\mathcal{X}$, as in Section 3. Let $d_3 \in \mathbb{N}$ be such that $\forall d \geqslant d_3$, $b_n\frac{\ln d}{\sqrt{d}} \leqslant R$. In the sequel we consider $d \geqslant \max(d_0, d_1, d_2, d_3)$.

Since we chose d large enough that $b_n\frac{\ln d}{\sqrt{d}} \leqslant R$ we can compute everything in the expo-

nential chart about x. Let $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, we have:

$$\begin{aligned}
&\int_{\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\left|\mathrm{d}V_M\right|^2 \\
&\quad= \int_{x\in M}\left(\int_{y\in B_M\left(x,b_n\frac{\ln d}{\sqrt{d}}\right)} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\left|\mathrm{d}V_M\right|\right)\left|\mathrm{d}V_M\right| \\
&\quad= \int_{x\in M}\left(\int_{z\in B_{T_xM}\left(0,b_n\frac{\ln d}{\sqrt{d}}\right)} \phi_1(x)\phi_2(\exp_x(z))\mathcal{D}_d(x,\exp_x(z))\sqrt{\kappa(z)}\,\mathrm{d}z\right)\left|\mathrm{d}V_M\right|,
\end{aligned} \tag{4.41}$$

where $\sqrt{\kappa}$ is the density of $\left|\mathrm{d}V_M\right|$ with respect to the normalized Lebesgue measure on T_xM (see Sect. 3.2). Let $x \in M$, for all $z \in B_{T_xM}\left(0, b_n \ln d\right)$ we define

$$D_d(x,z) = \mathcal{D}_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right), \tag{4.42}$$

where $\mathcal{D}_d$ is defined by (4.9). Then, by a change of variable in (4.41),

$$\begin{aligned}
&\int_{\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\left|\mathrm{d}V_M\right|^2 = \\
&d^{-\frac{n}{2}}\int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_d(x,z)\left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{1}{2}}\mathrm{d}z\right)\left|\mathrm{d}V_M\right|,
\end{aligned} \tag{4.43}$$

and we need to compute the asymptotic of $D_d(x,z)$ as d goes to infinity. We start by computing $\left|\det{}^{\perp}\left(\mathrm{ev}^d_{x,y}\right)\right|$ when $(x,y) \in \Delta_d$.

Proposition 4.26. *Let $\alpha \in \left(0, \frac{1}{2r+1}\right)$, let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Then we have:*

$$\left(\frac{\pi}{d}\right)^{2nr}\det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) = \left(1-e^{-\|z\|^2}\right)^r\left(1+O\big(d^{-\alpha}\big)\right), \tag{4.44}$$

where the error term does not depend on (x,z).

We will deduce Proposition 4.26 from the following two lemmas.

Lemma 4.27. *Let $\beta \in (0,1)$ and $d \geqslant d_3$, then for every $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$, we have:*

$$\left(\frac{\pi}{d}\right)^{2nr}\det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)\right) = \left(1-e^{-\|z\|^2}\right)^r + O\big(d^{\beta-1}\big),$$

where y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$. Moreover the error term depends on β but not on (x,z).

Lemma 4.28. *There exists $\widetilde{C} > 0$ such that, for all $\beta \in (0,1)$, there exists $d_\beta \in \mathbb{N}$ such that: $\forall d \geqslant d_\beta$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,*

$$\left|\left(\frac{\pi}{d}\right)^{2nr}\det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right)\left(1-e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{C}d^{\beta-1},$$

where y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$.

Let us assume Lemmas 4.27 and 4.28 for now, and prove Prop. 4.26.

Proof of Proposition 4.26. First, note that if (4.44) holds for $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, then the same estimate holds for $z \in B_{T_xM}(0, b_n \ln d)$ since both sides of the equality vanish when $z = 0$. In the sequel we assume that $z \neq 0$.

Let $\alpha \in \left(0, \frac{1}{2r+1}\right)$, let $d \geqslant d_3$ and let $x \in M$. Then for any $z \in T_xM$ such that $\|z\| \geqslant d^{-\alpha}$, we have:

$$\left(1 - e^{-\|z\|^2}\right)^{-r} \leqslant \left(1 - \exp\left(-d^{-2\alpha}\right)\right)^{-r}. \tag{4.45}$$

Since $1 - e^{-t} = t\left(1 - \frac{t}{2} + O(t^2)\right)$ as $t \to 0$, there exists $\widetilde{C}_0$ such that for all $t \in (0,1)$,

$$\left|\left(1 - e^{-t}\right)^{-r} - t^{-r}\right| \leqslant \widetilde{C}_0 t^{1-r}. \tag{4.46}$$

Hence, by (4.45), for any $d \geqslant d_3$, for any $x \in M$ and any $z \in T_xM$ such that $\|z\| \geqslant d^{-\alpha}$, we have:

$$\left(1 - e^{-\|z\|^2}\right)^{-r} \leqslant \left(d^{2r\alpha} + \widetilde{C}_0 d^{(2r-2)\alpha}\right) \leqslant d^{2r\alpha}\left(1 + \widetilde{C}_0\right). \tag{4.47}$$

Let $\beta = 1 - (2r+1)\alpha$ and $\beta' = 1 - \alpha$, then β and $\beta' \in (0,1)$. By Lemma 4.27, there exists $\widetilde{K}_\beta > 0$ such that: for all $d \geqslant d_3$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d)$,

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right) - \left(1 - e^{-\|z\|^2}\right)^r\right| \leqslant \widetilde{K}_\beta d^{\beta-1} = \widetilde{K}_\beta d^{-(2r+1)\alpha},$$

where $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Then, by (4.47), we have: $\forall d \geqslant d_3$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant d^{-\alpha} = d^{\beta'-1}$,

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{K}_\beta d^{-\alpha}\left(1 + \widetilde{C}_0\right),$$

Besides, let $d \geqslant d_{\beta'}$ and $x \in M$, then for all $z \in B_{T_xM}\left(0, d^{-\alpha}\right) \setminus \{0\}$ we have:

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{C} d^{-\alpha},$$

by Lemma 4.28. Finally, for all $d \geqslant \max(d_{\beta'}, d_3)$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, b_n \ln d\right) \setminus \{0\}$, we have:

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant d^{-\alpha} \max\left(\widetilde{C}, 2\widetilde{K}_\beta\left(1 + \widetilde{C}_0\right)\right). \qquad \square$$

Proof of Lemma 4.27. Let $d \geqslant d_3$, let $x \in M$ and let $z \in B_{T_xM}(0, b_n \ln d)$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Since $\frac{\|z\|}{\sqrt{d}} < R$, let us write Eq. (4.19) in the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ about x (see Sect. 3.1). We have:

$$\left(\frac{\pi}{d}\right)^n \mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)^* = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}.$$

Then, by the near-diagonal estimates of Cor. 3.7, we have:

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} & e^{-\frac{1}{2}\|z\|^2}\left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-\frac{1}{2}} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} \\ e^{-\frac{1}{2}\|z\|^2}\left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-\frac{1}{2}} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} & \left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-1} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} \end{pmatrix} + O\left(\frac{(\ln d)^{2n+8}}{d}\right),$$

where the error term does not depend on (x,z). Recall that κ satisfies (3.6). Hence for all $z \in B(0, b_n \ln d)$,

$$\kappa\left(\frac{z}{\sqrt{d}}\right) = 1 + O\left(\frac{(\ln d)^2}{d}\right),$$

uniformly in x and z. Let $\beta \in (0,1)$, then we have:

$$\left(\frac{\pi}{d}\right)^n \mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^* = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right), \tag{4.48}$$

and the constant in the term $O\big(d^{\beta-1}\big)$ does not depend on (x,z). Since the dominant term on the right-hand side of (4.48) has bounded coefficients, we get the result by taking the determinant of (4.48). □

Proof of Lemma 4.28. Let $d \geqslant \max(d_0, d_3)$ and let $x \in M$. Recall that $D^k_{(z,w)}$ denotes the k-th differential at (z,w) of a map from $T_x\mathcal{X} \times T_x\mathcal{X}$ to $\mathrm{End}\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$.

The Chern connection reads $D + \mu_x$ in the real normal trivialization about x, where μ_x is a 1-form on $B_{T_x\mathcal{X}}(0,2R)$. By definition of the real normal trivialization, $\mu_x(0) = 0$. Besides $\mu_x(z)$ is a smooth function of (x,z). Then, by compacity of M, there exists $K > 0$ such that $\|\mu_x(z)\| \leqslant K$ for all $x \in M$ and all $z \in B_{T_x\mathcal{X}}(0,R)$. Hence, there exists $K' > 0$ independent of x such that, for any smooth section S of $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$ over $B_{T_xM}(0,R) \times B_{T_xM}(0,R)$, we have:

$$\forall z, w \in B_{T_xM}(0,R), \qquad \left\|D_{(z,w)}S\right\| \leqslant K' \left\|S\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^1},$$

where $\|\cdot\|_{\mathcal{C}^1}$ was defined in Section 3.4. Since we use the exponential chart, we can argue similarly for the Levi–Civita connection. This gives a similar result for the higher derivatives of S. For all $k \in \mathbb{N}$, there exists $K_k > 0$ independent of x such that, for any smooth section S of $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$ over $B_{T_xM}(0,R) \times B_{T_xM}(0,R)$, we have:

$$\forall z, w \in B_{T_xM}(0,R), \qquad \left\|D^k_{(z,w)}S\right\| \leqslant K_k \left\|S\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^k}. \tag{4.49}$$

Since $d \geqslant d_0$, by Eq. (4.49) and Thm. 3.9 we have: $\forall z, w \in B_{T_xM}(0,R)$,

$$\left\|D^2_{(z,w)}E_d\right\| \leqslant K_2 \left\|E_d\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^2} \leqslant C_2 K_2 d^{n+1}. \tag{4.50}$$

Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let us write Eq. (4.19), in the real normal trivialization of $\mathcal{E}\otimes\mathcal{L}^d$ about x, as in the proof of Lemma 4.27. We have:

$$\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^* = \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}.$$

Then, by elementary operations on rows and columns,

$$\frac{1}{\|z\|^{2r}} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) = \frac{1}{\|z\|^{2r}} \det\begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} =$$

$$\det\begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \end{pmatrix} \end{pmatrix}. \tag{4.51}$$

By Taylor's formula, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\left\| E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \right\| \leqslant \frac{\|z\|^2}{2d} \left(\sup_{w \in [0, \frac{z}{\sqrt{d}}]} \left\| D^2_{(0,w)} E_d \right\| \right). \quad (4.52)$$

Then, by (4.50), we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} \left\| E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \right\| \leqslant \|z\|\, C_2 K_2 \pi^n. \quad (4.53)$$

Similarly, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} \left\| E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) \right\| \leqslant \|z\|\, C_2 K_2 \pi^n. \quad (4.54)$$

A second order Taylor's formula gives:

$$\left\| \left(E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \right) - D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right) \left(\frac{z}{\sqrt{d}}, 0\right) \right) \right\| \leqslant \left(\frac{\|z\|}{\sqrt{d}}\right)^3 \left(\sup_{[0, \frac{z}{\sqrt{d}}]^2} \left\| D^3 E_d \right\| \right),$$

and since $d \geqslant d_0$, by Thm. 3.9 and Eq. (4.49) we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|^2} \left\| \left(E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \right) - D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right) \left(\frac{z}{\sqrt{d}}, 0\right) \right) \right\| \leqslant \|z\|\, C_3 K_3 \pi^n. \quad (4.55)$$

Finally, by Equations (4.53), (4.54) and (4.55),

$$\begin{aligned} &\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \end{pmatrix} \end{pmatrix} \\ &= \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \\ \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) & \frac{1}{\|z\|^2} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right) \left(\frac{z}{\sqrt{d}}, 0\right) \right) \end{pmatrix} + O(\|z\|), \end{aligned} \quad (4.56)$$

where the error term is uniform in x and d.

On the other hand, for every $x \in M$ and every $z \in T_xM \setminus \{0\}$, the diagonal estimates of Sect. 3.3 give (see (3.16)):

$$\begin{aligned} \left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|^2} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right) \left(\frac{z}{\sqrt{d}}, 0\right) \right) &= \frac{\pi^n}{d^{n+1}} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\|z\|}\right) \left(\frac{z}{\|z\|}, 0\right) \right) \\ &= \mathrm{Id}_{\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x} + O(d^{-1}), \end{aligned}$$

where the error term is independent of x and z. Similarly,

$$\begin{aligned} \left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) &= \left(\frac{\pi}{d}\right)^n \frac{1}{\sqrt{d}} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\|z\|}\right) = O(d^{-1}), \\ \left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) &= \left(\frac{\pi}{d}\right)^n \frac{1}{\sqrt{d}} D_{(0,0)}E_d \cdot \left(\frac{z}{\|z\|}, 0\right) = O(d^{-1}), \end{aligned}$$

and

$$\left(\frac{\pi}{d}\right)^n E_d(0,0) = \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O(d^{-1}).$$

Thus

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|} D_{(0,0)} E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \\ \frac{1}{\|z\|} D_{(0,0)} E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) & \frac{1}{\|z\|^2} D^2_{(0,0)} E_d \left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right) \end{pmatrix} = \mathrm{Id} + O(d^{-1}), \quad (4.57)$$

where the error term is uniform in (x, z). By (4.56) and (4.57), there exist $\widetilde{C}_1$ and $\widetilde{C}_2 > 0$ such that we have: $\forall d \geqslant \max(d_0, d_3)$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$,

$$\left\| \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \end{pmatrix} \end{pmatrix} - \mathrm{Id} \right\| \leqslant \widetilde{C}_1 \|z\| + \widetilde{C}_2 \frac{1}{d}. \quad (4.58)$$

Let $\beta \in (0,1)$, then for all $d \geqslant \max(d_0, d_3)$, for all $x \in M$ and all $z \in B_{T_xM}\left(0, d^{\beta-1}\right)$, we have: $\widetilde{C}_1 \|z\| + \widetilde{C}_2 d^{-1} \leqslant d^{\beta-1}\left(\widetilde{C}_1 + \widetilde{C}_2\right)$. Let $d_\beta \in \mathbb{N}$ be such that $(d_\beta)^{\beta-1}\left(\widetilde{C}_1 + \widetilde{C}_2\right) \leqslant \frac{1}{2}$. Since the determinant is a smooth function, there exists $\widetilde{C}_3 > 0$ such that, for every operator Λ, if $\|\Lambda\| \leqslant \frac{1}{2}$, then $|\det(\mathrm{Id} + \Lambda) - 1| \leqslant \widetilde{C}_3 \|\Lambda\|$. Hence, by Eq. (4.51) and (4.58), we have: for all $d \geqslant d_\beta$, for all $x \in M$, for all $z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,

$$\left| \frac{1}{\|z\|^{2r}} \left(\frac{\pi}{d}\right)^{2rn} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) - 1 \right| \leqslant \left(\widetilde{C}_1 + \widetilde{C}_2\right) \widetilde{C}_3 d^{\beta-1}. \quad (4.59)$$

Recall that $\widetilde{C}_0$ was defined in the proof of Prop. 4.26 (see Eq. (4.46)) and that, for all $x \in M$, for all $z \in B_{T_xM}(0,1) \setminus \{0\}$, we have:

$$\left| \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \leqslant \widetilde{C}_0 \|z\|^2.$$

Then we have: $\forall d \geqslant d_\beta$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,

$$\begin{aligned}
&\left| \left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) \left(1 - e^{-\|z\|^2}\right)^{-r} - 1 \right| \\
&\quad = \left| \left(\frac{\pi}{d}\right)^{2nr} \frac{1}{\|z\|^{2r}} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \\
&\quad \leqslant \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} \left| \frac{1}{\|z\|^{2r}} \left(\frac{\pi}{d}\right)^{2rn} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) - 1 \right| + \left| \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \\
&\quad \leqslant \left(1 + \widetilde{C}_0 d^{2\beta-2}\right)\left(\widetilde{C}_1 + \widetilde{C}_2\right)\widetilde{C}_3 d^{\beta-1} + \widetilde{C}_0 d^{2\beta-2} \\
&\quad \leqslant d^{\beta-1}\left(\left(\widetilde{C}_1 + \widetilde{C}_2\right)\widetilde{C}_3\left(1 + \widetilde{C}_0\right) + \widetilde{C}_0\right) = d^{\beta-1}\widetilde{C},
\end{aligned}$$

where we define $\widetilde{C} > 0$ by the equality on the last line. □

We now want to compute the limit of the conditional distribution of $\frac{\pi^n}{d^{n+1}}\left(\nabla^d_x s_d, \nabla^d_y s_d\right)$ given that $s_d(x) = 0 = s_d(y)$ for $(x,y) \in \Delta_d$. It is enough to compute the limit of $\Lambda_d(x,y)$ as $d \to +\infty$. Recall that Λ_d is defined by Def. 4.8. Since we work near the diagonal, we can write everything in the real normal trivialization centered at x (see Sect. 3.1).

Lemma 4.29. *Let $x \in M$ and let ∇^d be a real metric connection which is trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Let $\beta \in (0,1)$, then, in the real normal trivialization about x, we have: $\forall z \in B_{T_xM}(0, b_n \ln d)$,*

$$\frac{\pi^n}{d^{n+1}} \begin{pmatrix} \partial_x \partial_y^\sharp E_d(0,0) & \partial_x \partial_y^\sharp E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} = \\ \begin{pmatrix} \mathrm{Id}_{T_xM^*} & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - z^* \otimes z\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_xM^*} - z^* \otimes z\right) & \mathrm{Id}_{T_xM^*} \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} + O\left(d^{\beta-1}\right),$$

where the error term does not depend on (x,z).

Proof. Let $x \in M$ and let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ of T_xM. We denote the corresponding coordinates on $T_xM \times T_xM$ by $(z_1, \ldots, z_n, w_1, \ldots, w_n)$ and by ∂_{z_i} and ∂_{w_j} the associated partial derivatives. Let $(dx_1, \ldots, dx_n)$ denote the dual basis of $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. By definition of ∇^d and $\partial_x \partial_y^\sharp E_d$ (see Eq. (2.19)), for all $z, w \in B_{T_xM}(0,R)$, the matrix of $\partial_x \partial_y^\sharp E_d(z,w)$ in the orthonormal basis $(dx_1, \ldots, dx_n)$ is:

$$\left(\partial_{z_i}\partial_{w_j} E_d(z,w)\right)_{1 \leqslant i,j \leqslant n}.$$

Note that this is a matrix with values in $\mathrm{End}\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$. Recall that we defined the function ξ_d by (3.12). Then, by Cor. 3.7, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\partial_{z_i}\partial_{w_j} E_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) = \left(\frac{d}{\pi}\right)^n \partial_{z_i}\partial_{w_j}\xi_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} + O\left((\ln d)^{2n+8}\right).$$

Then, Eq. (3.15) shows that:

$$\partial_{z_i}\partial_{w_j}\xi_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) = \exp\left(-\frac{1}{2}\|z-w\|^2\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{-\frac{1}{2}}\kappa\left(\frac{w}{\sqrt{d}}\right)^{-\frac{1}{2}} \times \\ \left(d\delta_{ij} - d(z_i - w_i)(z_j - w_j) - \frac{\sqrt{d}(z_j - w_j)\partial_{z_i}\kappa\left(\frac{z}{\sqrt{d}}\right)}{2\kappa\left(\frac{z}{\sqrt{d}}\right)} + \frac{\sqrt{d}(z_i - w_i)\partial_{w_j}\kappa\left(\frac{w}{\sqrt{d}}\right)}{2\kappa\left(\frac{w}{\sqrt{d}}\right)}\right) \\ = d\exp\left(-\frac{1}{2}\|z-w\|^2\right)\left(\delta_{ij} - (z_i - w_i)(z_j - w_j)\right) + O\left((\ln d)^4\right),$$

where we used the fact that, uniformly in $z \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\kappa\left(\frac{z}{\sqrt{d}}\right) = 1 + O\left(\frac{(\ln d)^2}{d}\right)$$

$$\text{and } \forall i \in \{1, \ldots, n\}, \qquad \partial_{z_i}\kappa\left(\frac{z}{\sqrt{d}}\right) = O\left(\frac{\ln d}{\sqrt{d}}\right).$$

Hence, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\frac{\pi^n}{d^{n+1}}\partial_{z_i}\partial_{w_j} E_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) = \\ \exp\left(-\frac{1}{2}\|z-w\|^2\right)\left(\delta_{ij} - (z_i - w_i)(z_j - w_j)\right)\mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} + O\left(\frac{(\ln d)^{2n+8}}{d}\right),$$

where the error term is independent of x, z and w. Furthermore, for any $\beta \in (0,1)$, the term $O\Big(\frac{(\ln d)^{2n+8}}{d}\Big)$ can be replaced by $O\left(d^{\beta-1}\right)$. Finally, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\frac{\pi^n}{d^{n+1}} \partial_x \partial_y^\sharp E_d(z,w) = \exp\left(-\frac{1}{2}\|z-w\|^2\right) \left(\mathrm{Id}_{T_xM^*} - (z-w)^* \otimes (z-w)\right) \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right),$$

which yields the result. □

A similar proof, using Cor. 3.7 and the expressions (3.13) and (3.14) for the partial derivatives of ξ_d yields the following.

Lemma 4.30. *Let $x \in M$ and let ∇^d be a real metric connection which is trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Let $\beta \in (0,1)$, then, in the real normal trivialization about x, we have: $\forall z \in B_{T_xM}(0, b_n \ln d)$,*

$$\frac{\pi^n}{d^{n+\frac{1}{2}}} \begin{pmatrix} \partial_x E_d(0,0) & \partial_x E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} = e^{-\frac{1}{2}\|z\|^2} \begin{pmatrix} 0 & z^* \\ -z^* & 0 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\big(d^{\beta-1}\big)$$

$$\frac{\pi^n}{d^{n+\frac{1}{2}}} \begin{pmatrix} \partial_y^\sharp E_d(0,0) & \partial_y^\sharp E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} = e^{-\frac{1}{2}\|z\|^2} \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\big(d^{\beta-1}\big),$$

where $z^* \in T_xM^*$ is to be understood as the constant map $t \mapsto z^*$ from $\mathbb{R}$ to T_xM^* and $z \in T_xM$ is to be understood as the evaluation on z from T_xM^* to $\mathbb{R}$. Moreover, the error terms does not depend on (x,z).

We would like to get a similar asymptotic for the last term in the conditional variance operator (4.22), namely:

$$\begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}^{-1}.$$

Unfortunately, this term is singular on Δ, and this kills all hope to get a uniform estimate on $B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Instead, we obtain a uniform estimate on $B_{T_xM}(0, b_n \ln d) \setminus B_{T_xM}(0,\rho)$ for some $\rho > 0$. We need to carefully check how this estimate depends on ρ.

Lemma 4.31. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. Then, in the real normal trivialization about x, we have:*

$$\left(\frac{d}{\pi}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}^{-1} = \frac{1}{1-e^{-\|z\|^2}} \begin{pmatrix} 1 & -e^{-\frac{1}{2}\|z\|^2} \\ -e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)\right).$$

Here, the notation $O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)$ means a quantity such that there exists $C > 0$ and $\varepsilon > 0$, independent of x, z, d and ρ, such that whenever $\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}} \leqslant \varepsilon$, the norm of this quantity is smaller than $C\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}$.

Proof. By Eq. (4.19) and (4.48), we have:

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0,\frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}},0\right) & E_d\left(\frac{z}{\sqrt{d}},\frac{z}{\sqrt{d}}\right) \end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right),$$

where the error term is independent of (x,z). Besides,

$$\left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1} = \frac{1}{1-e^{-\|z\|^2}} \begin{pmatrix} 1 & -e^{-\frac{1}{2}\|z\|^2} \\ -e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}, \tag{4.60}$$

and the eigenvalues of

$$\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$$

are $1-e^{-\frac{1}{2}\|z\|^2}$ and $1+e^{-\frac{1}{2}\|z\|^2}$, which shows that:

$$\left\|\left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1}\right\| \leqslant \frac{1}{1-e^{-\frac{1}{2}\|z\|^2}},$$

where $\|\cdot\|$ is the operator norm on $\mathrm{End}\left(\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$. Then, if $\|z\| \geqslant \rho$, we have:

$$\frac{1}{1-e^{-\frac{1}{2}\|z\|^2}} \leqslant \frac{1}{1-e^{-\frac{1}{2}\rho^2}}.$$

Thus,

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0,\frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}},0\right) & E_d\left(\frac{z}{\sqrt{d}},\frac{z}{\sqrt{d}}\right) \end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)\right). \tag{4.61}$$

Taking the inverse of Eq. (4.61), we get:

$$\left(\frac{d}{\pi}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0,\frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}},0\right) & E_d\left(\frac{z}{\sqrt{d}},\frac{z}{\sqrt{d}}\right) \end{pmatrix}^{-1} = \left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1} \left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)\right),$$

where we used the mean value inequality and the fact that the differential of $\Lambda \mapsto \Lambda^{-1}$ is bounded from above on the closed ball of center Id and radius $\frac{1}{2}$. Finally, Eq. (4.60) gives the result. □

Recall that $\Lambda_x(z)$ is defined for $x \in M$ and $z \in T_xM \setminus \{0\}$ by Def. 4.42. Recall also that $\Lambda_d(x,y)$ is defined by Def. 4.8.

Lemma 4.32. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then, in the real normal trivialization about x, we have:*

$$\Lambda_d(x,y) = \Lambda_x(z) + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right),$$

where the constant in the error term does not depend on (x,z), d or ρ.

Proof. We know that $\Lambda_d(x,y)$ does not depend on the choice of ∇^d (see Rem.4.5). Hence, we can compute $\Lambda_d(x,y)$ with ∇^d trivial over $B_{T_xM}(0,R)$ in the real normal trivialization of $\mathcal{E}\otimes\mathcal{L}^d$ about x.

Let $\beta \in (0,1)$ and $\rho \in (0,1)$, we apply Lemmas 4.30 and 4.31 for $\frac{\beta}{2}$. Then, in the real normal trivialization about x, we have:

$$\begin{aligned}
&\frac{\pi^n}{d^{n+1}}\begin{pmatrix}\partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix}\begin{pmatrix}E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}^{-1}\begin{pmatrix}\partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y)\\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y)\end{pmatrix}\\
&= \left(\begin{pmatrix}0 & z^*\\ -z^* & 0\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\frac{\beta}{2}-1}\right)\right)\left(\begin{pmatrix}1 & e^{-\frac{1}{2}\|z\|^2}\\ e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1}\times\\
&\qquad\left(\mathrm{Id} + O\left(\frac{d^{\frac{\beta}{2}-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)\right)\left(\begin{pmatrix}0 & -z\\ z & 0\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\frac{\beta}{2}-1}\right)\right). \quad (4.62)
\end{aligned}$$

Since, $\rho \leqslant |z| < b_n \ln d$, the norm of

$$\left(\begin{pmatrix}1 & e^{-\frac{1}{2}\|z\|^2}\\ e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1}$$

is smaller than $\left(1-e^{-\frac{1}{2}\|z\|^2}\right)^{-1} \leqslant \left(1-e^{-\frac{1}{2}\rho^2}\right)^{-1}$, and the norms of the other matrices appearing in (4.62) are $O(\ln d)$. Hence, the expression (4.62) equals:

$$\begin{aligned}
&\frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}}\begin{pmatrix}0 & z^*\\ -z^* & 0\end{pmatrix}\begin{pmatrix}1 & -e^{-\frac{1}{2}\|z\|^2}\\ -e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix}\begin{pmatrix}0 & -z\\ z & 0\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right)\\
&\qquad = \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}}\begin{pmatrix}z^*\otimes z & e^{-\frac{1}{2}\|z\|^2}z^*\otimes z\\ e^{-\frac{1}{2}\|z\|^2}z^*\otimes z & z^*\otimes z\end{pmatrix}\otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right), \quad (4.63)
\end{aligned}$$

where the error term is independent of (x,z). Finally, Eq. (4.63) and Lemma 4.29 yield the result. □

Lemma 4.33. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then,*

$$\begin{aligned}
&\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d) = 0\right] =\\
&\qquad\qquad \mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right| \left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right] + O\left(f(\rho)^{\frac{r(n+1)}{2}+4} d^{\beta-1}\right),
\end{aligned}$$

where constant in the error term does not depend on (x,z), d or ρ.

Proof. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, let $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$ then we have:

$$\begin{aligned}\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\Big[\left|\det{}^{\perp}\left(\nabla^d_x s_d\right)\right|\left|\det{}^{\perp}\left(\nabla^d_y s_d\right)\right| \Big| \mathrm{ev}^d_{x,y}(s_d) = 0\Big] \\ = \mathbb{E}\left[\left|\det{}^{\perp}\left(\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \nabla^d_x s_d\right)\right|\left|\det{}^{\perp}\left(\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \nabla^d_y s_d\right)\right| \Big| \mathrm{ev}^d_{x,y}(s_d) = 0\right] \\ = \mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(x)\right)\right|\left|\det{}^{\perp}\left(L'_d(y)\right)\right|\right],\end{aligned}$$

where $(L'_d(x), L'_d(y))$ is a centered Gaussian vector in

$$\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_xM^* \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \otimes T_yM^*$$

with variance operator $\Lambda_d(x,y)$. We can consider $(L'_d(x), L'_d(y))$ as a random vector in $\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_xM^*$, via the real normal trivialization about x. From now on, we work in this trivialization. Let $\rho \in (0,1)$ and $\beta \in (0,1)$, we assume that $\rho \leqslant \|z\| < b_n \ln d$. Then, by Lemma 4.32, we have:

$$\Lambda_d(x,y) = \Lambda_x(z) + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right).$$

Moreover, by Cor. 4.17, $\|\Lambda_x(z)\|^{-1} \leqslant f\left(\|z\|^2\right) \leqslant f(\rho)$. Hence, we have:

$$\Lambda_d(x,y) = \Lambda_x(z)\left(\mathrm{Id} + O\left(f(\rho)\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right)\right) = \Lambda_x(z)\left(\mathrm{Id} + O\big(f(\rho)^3 d^{\beta-1}\big)\right),$$

where we used the fact that $\frac{1}{1-e^{-\frac{1}{2}\rho^2}} \leqslant f(\rho)$ (see the proof of Cor. 4.17). Then, we get:

$$\det\left(\Lambda_d(x,y)\right) = \det\left(\Lambda_x(z)\right)\left(1 + O\big(f(\rho)^3 d^{\beta-1}\big)\right) \tag{4.64}$$

and

$$\Lambda_d(x,y)^{-1} = \Lambda_x(z)^{-1}\left(\mathrm{Id} + O\big(f(\rho)^3 d^{\beta-1}\big)\right) = \Lambda_x(z)^{-1} + O\big(f(\rho)^4 d^{\beta-1}\big).$$

Thus there exists $K > 0$ and $\varepsilon > 0$ such that, whenever $f(\rho)^4 d^{\beta-1} \leqslant \varepsilon$,

$$\left\|\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right\| \leqslant K f(\rho)^4 d^{\beta-1}.$$

By the mean value inequality, for every $L = (L_1, L_2) \in \mathbb{R}^2 \otimes T_xM^* \otimes \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x$ we have:

$$\begin{aligned}\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right)L, L\right\rangle\right) - 1\right| \\ \leqslant \frac{K}{2}\|L\|^2 f(\rho)^4 d^{\beta-1} \exp\left(\frac{K}{2}\|L\|^2 f(\rho)^4 d^{\beta-1}\right),\end{aligned}$$

whenever $f(\rho)^4 d^{\beta-1} \leqslant \varepsilon$. Let $\mathrm{d}L$ denote the normalized Lebesgue measure on this vector

space, and recall that we defined $(L_x(0), L_x(z))$ above (Def. 4.19). Then, we have:

$$\begin{aligned}
&(2\pi)^{nr} \Big| \det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d'(x)\right)\big| \big|\det{}^{\perp}\left(L_d'(y)\right)\big|\big] \\
&\qquad\qquad - \det\left(\Lambda_x(z)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_x(0)\right)\big| \big|\det{}^{\perp}\left(L_x(z)\right)\big|\big] \Big| \\
&\leqslant \int \big|\det{}^{\perp}\left(L_1\right)\big| \big|\det{}^{\perp}\left(L_2\right)\big| \exp\left(-\frac{1}{2}\left\langle \Lambda_x(z)^{-1} L\,, L\right\rangle\right) \times \\
&\qquad\qquad \left|\exp\left(-\frac{1}{2}\left\langle \left(\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right) L\,, L\right\rangle\right) - 1\right| \mathrm{d}L \\
&\leqslant \frac{K}{2} f(\rho)^4 d^{\beta-1} \int \big|\det{}^{\perp}\left(L_1\right)\big| \big|\det{}^{\perp}\left(L_2\right)\big| \left\|L\right\|^2 \times \\
&\qquad\qquad \exp\left(-\frac{1}{2}\left\langle \left(\Lambda_x(z)^{-1} - \frac{K}{2} f(\rho)^4 d^{\beta-1}\,\mathrm{Id}\right) L\,, L\right\rangle\right) \mathrm{d}L,
\end{aligned}$$

whenever $f(\rho)^4 d^{\beta-1} \leqslant \varepsilon$. Since $\Lambda_x(d) < 2$ by Cor. 4.17, the smallest eigenvalue of $\Lambda_x(z)^{-1}$ is larger than $\frac{1}{2}$. Thus, if $f(\rho)^4 d^{\beta-1} \leqslant \frac{1}{2K}$, for every L we have:

$$\left\langle \left(\Lambda_x(z)^{-1} - \frac{K}{2} f(\rho)^4 d^{\beta-1}\,\mathrm{Id}\right) L\,, L\right\rangle \geqslant \frac{1}{4}\left\|L\right\|^2.$$

Hence, the last integral above is bounded by:

$$\int \big|\det{}^{\perp}\left(L_1\right)\big| \big|\det{}^{\perp}\left(L_2\right)\big| \left\|L\right\|^2 \exp\left(-\frac{1}{8}\left\|L\right\|^2\right) \mathrm{d}L < +\infty.$$

Then, we have:

$$\begin{aligned}
&\det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_d'(x)\right)\big| \big|\det{}^{\perp}\left(L_d'(y)\right)\big|\big] = \\
&\qquad\qquad \det\left(\Lambda_x(z)\right)^{\frac{1}{2}} \mathbb{E}\big[\big|\det{}^{\perp}\left(L_x(0)\right)\big| \big|\det{}^{\perp}\left(L_x(z)\right)\big|\big] + O\big(f(\rho)^4 d^{\beta-1}\big),
\end{aligned}$$

and by (4.64), we obtain:

$$\begin{aligned}
\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d'(x)\right)\big| \big|\det{}^{\perp}\left(L_d'(y)\right)\big|\big] &= \mathbb{E}\big[\big|\det{}^{\perp}\left(L_x(0)\right)\big| \big|\det{}^{\perp}\left(L_x(z)\right)\big|\big] \left(1 + O\big(f(\rho)^3 d^{\beta-1}\big)\right) \\
&\quad + \det\left(\Lambda_x(z)\right)^{-\frac{1}{2}} O\big(f(\rho)^4 d^{\beta-1}\big) \left(1 + O\big(f(\rho)^3 d^{\beta-1}\big)\right).
\end{aligned}$$

Since, for all $t > 0$ we have (see Lem. 4.15):

$$\frac{1}{1 + e^{-\frac{1}{2}t}} \leqslant 1, \qquad \frac{1}{1 - e^{-\frac{1}{2}t}} \leqslant f(t) \qquad \text{and} \qquad \frac{1 + e^{-\frac{1}{2}t}}{1 - e^{-t} + t e^{-\frac{1}{2}t}} \leqslant f(t),$$

by Cor. 4.17 we have: $\det\left(\Lambda_x(z)\right)^{-\frac{1}{2}} \leqslant f(\rho)^{\frac{r(n+1)}{2}}$. Besides, by Cor. 4.21, we have:

$$\mathbb{E}\big[\big|\det{}^{\perp}\left(L_x(0)\right)\big| \big|\det{}^{\perp}\left(L_x(z)\right)\big|\big] = \mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2\Big)\Big| \Big|\det{}^{\perp}\Big(Y(\|z\|^2\Big)\Big|\Big],$$

and by Lemma 4.22 this quantity is bounded from above by n^r. Finally, we have:

$$\begin{aligned}
\mathbb{E}\big[\big|\det{}^{\perp}\left(L_d'(x)\right)\big| \big|\det{}^{\perp}\left(L_d'(y)\right)\big|\big] =& \mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2\Big)\Big| \Big|\det{}^{\perp}\Big(Y(\|z\|^2\Big)\Big|\Big] \\
&+ O\Big(f(\rho)^{4+\frac{r(n+1)}{2}} d^{\beta-1}\Big).
\end{aligned}$$

□

The following corollary is not necessary to the proof of Thm. 1.6 but is worth mentioning.

Corollary 4.34. *Let $\beta \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big| \,\Big|\, \mathrm{ev}^d_{x,y}(s_d) = 0\Big] = \\ \mathbb{E}\Big[\Big|\det{}^{\perp}\left(X(\|z\|^2\right)\Big| \Big|\det{}^{\perp}\left(Y(\|z\|^2\right)\Big|\Big] + O\big(d^{\beta-1}\big)\,,$$

where the error term depends on z but not on x.

Proof. Let us fix, β, x and z, then we set $\rho = \|z\|$ and we apply Lemma 4.33. □

Before we can conclude the proof of Thm. 1.6, we need one last lemma.

Lemma 4.35. *Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let $\beta \in (0,1)$ and let ∇^d be any real metric connection. Then, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big| \,\Big|\, \mathrm{ev}^d_{x,y}(s_d) = 0\Big] \leqslant \frac{(2r)!}{r!} n^r + O\big(d^{\beta-1}\big)\,,$$

where the error term is independent of (x,z).

Proof. Let $x \in M$, let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ and let $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. As in the proof of Lem. 4.33, let $(L'_d(x), L'_d(y))$ be a centered Gaussian vector in $\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T_xM^*$ which variance operator is $\Lambda_d(x,y)$, read in the real normal trivialization about x. In the sequel, we work in this trivialization. We have:

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big| \,\Big|\, \mathrm{ev}^d_{x,y}(s_d) = 0\Big] = \\ \mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(x)\right)\right| \left|\det{}^{\perp}\left(L'_d(y)\right)\right|\right].$$

The proof follows the same lines as that of Lem. 4.22, the main difference being that the variance operator is not explicit. An additional difficulty comes from the fact that the estimate for $\Lambda_d(x,y)$ given by Lemma 4.32 is not uniform in $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, hence it is useless here. Fortunately, we only need to bound its trace, which is bounded from above by that of the unconditional variance operator:

$$\frac{\pi^n}{d^{n+1}} \begin{pmatrix} \partial_x \partial^\sharp_y E_d(0,0) & \partial_x \partial^\sharp_y E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x \partial^\sharp_y E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x \partial^\sharp_y E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix},$$

and Lemma 4.29 allows us to bound the latter.

By the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(x)\right)\right| \left|\det{}^{\perp}\left(L'_d(y)\right)\right|\right] \leqslant \mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(x)\right)\right|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(y)\right)\right|^2\right]^{\frac{1}{2}}. \tag{4.65}$$

Let $\Lambda_{d,1}(x,y)$ and $\Lambda_{d,2}(x,y)$ denote the variance operators of $L'_d(x)$ and $L'_d(y)$ respectively, so that:

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_{d,1}(x,y) & * \\ * & \Lambda_{d,2}(x,y) \end{pmatrix}. \tag{4.66}$$

Let us choose orthonormal bases of T_xM and $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$. We denote by $(L'_d(x)_{ij})_{\substack{1\leqslant i\leqslant r\\1\leqslant j\leqslant n}}$ the coefficients of the matrix of $L'_d(x)$ in these bases, and by $(L'_d(x)_i)_{1\leqslant i\leqslant r}$ its rows. As in the proof of Lem. 4.22, we have:

$$\begin{aligned}\left|\det{}^{\perp}\left(L'_d(x)\right)\right|^2 &= \det\left(L'_d(x)\left(L'_d(x)\right)^*\right) = \det\left(\left\langle L'_d(x)_i\,,L'_d(x)_j\right\rangle\right)\\ &\leqslant \left\|L'_d(x)_1\right\|^2\cdots\left\|L'_d(x)_r\right\|^2.\end{aligned} \tag{4.67}$$

Then, we have:

$$\begin{aligned}\mathbb{E}\left[\left\|L'_d(x)_1\right\|^2\cdots\left\|L'_d(x)_r\right\|^2\right] &= \mathbb{E}\left[\prod_{i=1}^r\left(\sum_{j=1}^n\left(L'_d(x)_{ij}\right)^2\right)\right]\\ &= \sum_{1\leqslant j_1,\ldots,j_r\leqslant n}\mathbb{E}\left[\prod_{i=1}^r\left(L'_d(x)_{i(j_i)}\right)^2\right].\end{aligned} \tag{4.68}$$

Let $j_1,\ldots,j_r\in\{1,\ldots,r\}$, we denote $X_i = L'_d(x)_{i(j_i)}$. Then, by Wick's formula (see [30, lem. 11.6.1]), we have:

$$\mathbb{E}\left[\prod_{i=1}^r\left(L'_d(x)_{i(j_i)}\right)^2\right] = \mathbb{E}\left[\prod_{i=1}^r(X_i)^2\right] = \sum_{(\{a_i,b_i\})}\prod_{i=1}^r\mathbb{E}\left[X_{\lfloor\frac{a_i}{2}\rfloor}X_{\lfloor\frac{b_i}{2}\rfloor}\right],$$

where we sum over all the partitions into pairs $(\{a_i,b_i\})_{1\leqslant i\leqslant r}$ of $\{1,\ldots,2r\}$. Hence, by Cauchy-Schwarz inequality again, we get:

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^r\left(L'_d(x)_{i(j_i)}\right)^2\right] &\leqslant \sum_{(\{a_i,b_i\})}\prod_{i=1}^r\mathbb{E}\left[\left(X_{\lfloor\frac{a_i}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(X_{\lfloor\frac{b_i}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\\ &\leqslant \sum_{(\{a_i,b_i\})}\prod_{k=1}^{2r}\mathbb{E}\left[\left(X_{\lfloor\frac{k}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\\ &\leqslant \sum_{(\{a_i,b_i\})}\prod_{l=1}^r\mathbb{E}\left[(X_l)^2\right]\\ &\leqslant \frac{(2r)!}{2^r r!}\prod_{i=1}^r\mathbb{E}\left[\left(L'_d(x)_{i(j_i)}\right)^2\right].\end{aligned}$$

Thus, we have:

$$\begin{aligned}\sum_{1\leqslant j_1,\ldots,j_r\leqslant n}\mathbb{E}\left[\prod_{i=1}^r\left(L'_d(x)_{i(j_i)}\right)^2\right] &\leqslant \frac{(2r)!}{2^r r!}\sum_{1\leqslant j_1,\ldots,j_r\leqslant n}\prod_{i=1}^r\mathbb{E}\left[\left(L'_d(x)_{i(j_i)}\right)^2\right]\\ &\leqslant \frac{(2r)!}{2^r r!}\prod_{i=1}^r\left(\sum_{j=1}^n\mathbb{E}\left[\left(L'_d(x)_{ij}\right)^2\right]\right)\\ &\leqslant \frac{(2r)!}{2^r r!}\left(\sum_{i=1}^r\sum_{j=1}^n\mathbb{E}\left[\left(L'_d(x)_{ij}\right)^2\right]\right)^r\\ &\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^r,\end{aligned} \tag{4.69}$$

where Tr stands for the trace operator. Finally, by (4.67), (4.68) and (4.69), we have:

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|^2\Big] \leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^r,$$

and similarly,

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(L'_d(y)\right)\big|^2\Big] \leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,2}(x,y)\right)^r.$$

Thus, by (4.65), we get:

$$\begin{aligned}\mathbb{E}\big[\big|\det{}^{\perp}\left(L'_d(x)\right)\big|\,\big|\det{}^{\perp}\left(L'_d(y)\right)\big|\big] &\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^{\frac{r}{2}}\operatorname{Tr}\left(\Lambda_{d,2}(x,y)\right)^{\frac{r}{2}}\\ &\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_d(x,y)\right)^r.\end{aligned} \tag{4.70}$$

Let $\beta \in (0,1)$, by Eq. (4.70), we only need to prove that $\operatorname{Tr}\left(\Lambda_d(x,y)\right) \leqslant 2n + O\big(d^{\beta-1}\big)$ to complete the proof. By Eq. (4.19),

$$\begin{pmatrix} E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}$$

is a variance operator. Hence it is a positive symmetric operator and so is its inverse. Besides, by (4.21), we know that:

$$\begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y)\\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y)\end{pmatrix} = \begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix}^*.$$

Then, the diagonal coefficients of:

$$\begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix}\begin{pmatrix} E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}^{-1}\begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y)\\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y)\end{pmatrix}$$

are non-negative, and so is its trace. Finally, by the definition of $\Lambda_d(x,y)$ (Def. 4.8), we have:

$$\operatorname{Tr}\left(\Lambda_d(x,y)\right) \leqslant \frac{\pi^n}{d^{n+1}}\operatorname{Tr}\begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y)\\ \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y)\end{pmatrix}. \tag{4.71}$$

Note that what we have done so far works for any choice of connection since $\Lambda_d(x,y)$ is independent of this choice. However, the right-hand side of Eq. (4.71) depends on the choice ∇^d. We use a real metric connection that is trivial on $B_{T_xM}(0,R)$ in the real normal trivialization about x. Then, by Lemma 4.29, we have:

$$\operatorname{Tr}\left(\Lambda_d(x,y)\right) \leqslant 2n + O\big(d^{\beta-1}\big). \qquad \square$$

4.3.5 Conclusion of the proof

We can now prove Thm. 1.6. Recall that we defined $\alpha_0 = \dfrac{n-r}{7+(r+1)(n+1)}$ (see Ntn. 1.5). Let us denote $\alpha_1 = \dfrac{\alpha_0}{n-r} = \dfrac{1}{7+(r+1)(n+1)}$.

Lemma 4.36. *Let $\alpha \in (0,\alpha_1)$, let $\phi \in \mathcal{C}^0(M)$ and $x \in M$, then we have:*

$$\left|\int_{B_{T_xM}(0,d^{-\alpha})}\phi\left(\frac{z}{\sqrt{d}}\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\left(\frac{1}{d^r}D_d(x,z) - D_{n,r}(\|z\|^2)\right)\mathrm{d}z\right| = \|\phi\|_\infty\, O\Big(d^{(r-n)\alpha}\Big),$$

where the error term does not depend on x or ϕ.

Proof. We have:

$$\left|\int_{B_{T_xM}(0,d^{-\alpha})}\phi\left(\frac{z}{\sqrt{d}}\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\left(\frac{1}{d^r}D_d(x,z)-D_{n,r}(\|z\|^2)\right)\mathrm{d}z\right|$$
$$\leqslant\|\phi\|_\infty\left(\sup_{B_{T_xM}\left(0,b_n\frac{\ln d}{\sqrt{d}}\right)}|\kappa|^{\frac{1}{2}}\right)\int_{B_{T_xM}(0,d^{-\alpha})}\left(\left|\frac{1}{d^r}D_d(x,z)\right|+\left|D_{n,r}(\|z\|^2)\right|\right)\mathrm{d}z.$$

Since $\kappa(z)=1+O\left(\|z\|^2\right)$ uniformly in x (see (3.6)), we have:

$$\sup_{B_{T_xM}\left(0,b_n\frac{\ln d}{\sqrt{d}}\right)}|\kappa|^{\frac{1}{2}}=1+O\Big(\frac{(\ln d)^2}{d}\Big),$$

and this term is bounded. Thus, we only need to consider the integral of

$$\left|\frac{1}{d^r}D_d(x,z)-D_{n,r}(\|z\|^2)\right|.$$

By Lemma 4.22, we have:

$$\begin{aligned}\int_{B(0,d^{-\alpha})}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z&\leqslant\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{\rho=0}^{d^{-\alpha}}\frac{\mathbb{E}\left[\left|\det{}^\perp\left(X(\rho^2)\right)\right|\left|\det{}^\perp\left(Y(\rho^2)\right)\right|\right]}{\left(1-e^{-\rho^2}\right)^{\frac{r}{2}}}\rho^{n-1}\,\mathrm{d}\rho\\&\quad+(2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\right)^2\mathrm{Vol}\left(B_{T_xM}\left(0,d^{-\alpha}\right)\right)\\&\leqslant\frac{n^r}{2}\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{d^{-2\alpha}}\frac{t^{\frac{n-2}{2}}}{\left(1-e^{-t}\right)^{\frac{r}{2}}}\,\mathrm{d}t+O\big(d^{-n\alpha}\big).\end{aligned}\tag{4.72}$$

Then, since there exists $C>0$ such that $\frac{t}{1-e^{-t}}\leqslant C$ for all $t\in(0,1]$, we get:

$$\int_{t=0}^{d^{-2\alpha}}\frac{t^{\frac{n-2}{2}}}{\left(1-e^{-t}\right)^{\frac{r}{2}}}\,\mathrm{d}t\leqslant C\int_{t=0}^{d^{-2\alpha}}t^{\frac{n-2-r}{2}}\,\mathrm{d}t=O\Big(d^{(r-n)\alpha}\Big).\tag{4.73}$$

Hence, $\displaystyle\int_{B(0,d^{-\alpha})}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z=O\Big(d^{(r-n)\alpha}\Big)$. By the definition of $D_d(x,z)$ (cf. (4.42)), we have:

$$\begin{aligned}\frac{1}{d^r}\left|D_d(x,z)\right|&\leqslant\frac{1}{d^r}\frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_xs_d\right)\right|\left|\det{}^\perp\left(\nabla^d_ys_d\right)\right|\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]}{\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right|}\\&\quad+\frac{1}{d^r}\frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_xs_d\right)\right|\Big|\,s_d(x)=0\Big]}{\left|\det{}^\perp\left(\mathrm{ev}^d_x\right)\right|}\frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_ys_d\right)\right|\Big|\,s_d(y)=0\Big]}{\left|\det{}^\perp\left(\mathrm{ev}^d_y\right)\right|}.\end{aligned}$$

Then, let $\beta\in(0,1)$ and $\beta'\in\left(0,\frac{1}{2r+1}\right)$, by Prop. 4.26 and Lem. 4.35 we have:

$$\begin{aligned}\frac{1}{d^r}\frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_xs_d\right)\right|\left|\det{}^\perp\left(\nabla^d_ys_d\right)\right|\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]}{\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right|}&\leqslant\frac{\frac{(2r)!}{r!}n^r+O\big(d^{\beta-1}\big)}{\left(1-e^{-\|z\|^2}\right)^{\frac{r}{2}}}\left(1+O\Big(d^{-\beta'}\Big)\right)\\&\leqslant C\left(\frac{1}{1-e^{-\|z\|^2}}\right)^{\frac{r}{2}},\end{aligned}$$

for some large C. By a polar change of coordinates similar to (4.72) and (4.73), we show that the integral of this term over $B_{T_xM}(0, d^{-\alpha})$ is a $O\big(d^{(r-n)\alpha}\big)$. Finally, by Lem. 4.6 and 4.7 we have:

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\,\Big|\,s_d(x)=0\Big]}{\big|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)\big|}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,s_d(y)=0\Big]}{\big|\det{}^{\perp}\left(\mathrm{ev}^d_y\right)\big|}=O(1)\,.$$

Hence the integral of this term over $B_{T_xM}(0, d^{-\alpha})$ is a $O(d^{-n\alpha})$. □

Lemma 4.37. *Let $\alpha \in (0, \alpha_1)$, let $\phi \in \mathcal{C}^0(M)$ and $x \in M$, then we have:*

$$\left|\int_{d^{-\alpha}\leqslant\|z\|<b_n\ln d}\phi\left(\frac{z}{\sqrt{d}}\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\left(\frac{1}{d^r}D_d(x,z)-D_{n,r}(\|z\|^2)\right)\mathrm{d}z\right|=\|\phi\|_\infty\, O\Big(d^{(r-n)\alpha}\Big)\,,$$

where the error term does not depend on x or ϕ.

Proof. As in the proof of Lemma 4.36, since $\kappa^{\frac{1}{2}}$ is bounded on $B_{T_xM}\left(0, b_n\frac{\ln d}{\sqrt{d}}\right)$, we only need to prove that:

$$\left|\frac{1}{d^r}D_d(x,z)-D_{n,r}(\|z\|^2)\right|=O\Big(d^{(r-n)\alpha-\beta}\Big)$$

for some $\beta > 0$. Then, since $\mathrm{Vol}\left(B_{T_xM}(0, b_n\ln d)\right) = O((\ln d)^n) = O\left(d^{\beta}\right)$, we get the result by integrating over $B_{T_xM}(0, b_n\ln d)\setminus B_{T_xM}(0, d^{-\alpha})$.

Let us fix some $\beta \in (0,1)$ which value will be chosen below. By Lem. 4.33, for every $x \in M$ and $z \in B_{T_xM}(0, b_n\ln d)$ such that $\|z\| \geqslant d^{-\alpha}$ we have:

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]=$$
$$\mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\Big|\,\Big|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\Big|\Big]+O\Big(f(d^{-\alpha})^{\frac{r(n+1)}{2}+4}d^{\beta-1}\Big)\,,\quad(4.74)$$

where, as usual, y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$. Recall that we have: $f(t)\sim\frac{12}{t^2}$ as $t\to 0$ (cf. Rem. 4.18). Then, we get:

$$f(d^{-\alpha})^{\frac{r(n+1)}{2}+4}=O\Big(d^{\alpha(8+r(n+1))}\Big)\,,$$

so that the error term in (4.74) is a $O\big(d^{\alpha(8+r(n+1))-1+\beta}\big)$. Let us now choose β to be:

$$\beta=\frac{1}{2}\left(1-\frac{\alpha}{\alpha_1}\right)\in\left(0,\frac{1}{2}\right).$$

Then we have:

$$\alpha\left(8+r(n+1)\right)-1+\beta=-\alpha n+\frac{\alpha}{\alpha_1}-1+\beta=-\alpha n-\beta,\quad(4.75)$$

and the error term in (4.74) is a $O\big(d^{-n\alpha-\beta}\big)$, with $\beta>0$. Now, let $\beta'\in\left(0,\frac{1}{2r+1}\right)$, by Prop. 4.26 and Eq. (4.74) we have:

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s_d\right)\big|\big|\det{}^{\perp}\left(\nabla^d_y s_d\right)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]}{\big|\det{}^{\perp}\left(\mathrm{ev}^d_{x,y}\right)\big|}=$$
$$\frac{\mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\Big|\,\Big|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\Big|\Big]+O\big(d^{-n\alpha-\beta}\big)}{\left(1-e^{-\|z\|^2}\right)^{\frac{r}{2}}}\left(1+O\Big(d^{\beta'-1}\Big)\right),\quad(4.76)$$

for all $x \in M$ and $z \in T_xM$ such that $d^{-\alpha} \leqslant \|z\| < b_n \ln d$. Since $\left(1 - e^{-d^{-2\alpha}}\right)^{-\frac{r}{2}} = O(d^{r\alpha})$, and the numerator of (4.76) is bounded (cf. Lem. 4.22), the right-hand side of Eq. (4.76) equals:

$$\frac{\mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\Big|\,\Big|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\Big|\Big]}{\big(1-e^{-\|z\|^2}\big)^{\frac{r}{2}}} + O\Big(d^{(r-n)\alpha-\beta}\Big) + O\Big(d^{r\alpha+\beta'-1}\Big)\,.$$

Moreover, $1 - n\alpha - \beta = \alpha\,(8 + r(n+1)) + \beta > 0$ (see Eq. (4.75)), so that we can assume that $\beta' < 1 - n\alpha - \beta$. Then we have: $r\alpha + \beta' - 1 < (r-n)\alpha - \beta$, and

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_x s_d\big)\big|\big|\det{}^{\perp}\big(\nabla^d_y s_d\big)\big|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]}{\big|\det{}^{\perp}\big(\mathrm{ev}^d_{x,y}\big)\big|} = \frac{\mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\Big|\,\Big|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\Big|\Big]}{\big(1-e^{-\|z\|^2}\big)^{\frac{r}{2}}} + O\Big(d^{(r-n)\alpha-\beta}\Big)\,.$$

On the other hand, by Lemmas 4.6 and 4.7, we have:

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_x s_d\big)\big|\,\Big|\,s_d(x)=0\Big]}{\big|\det{}^{\perp}\big(\mathrm{ev}^d_x\big)\big|}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_y s_d\big)\big|\,\Big|\,s_d(y)=0\Big]}{\big|\det{}^{\perp}\big(\mathrm{ev}^d_y\big)\big|} = (2\pi)^r\left(\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\right)^2 + O\big(d^{-1}\big)\,.$$

Once again, Eq. (4.75) shows that $-n\alpha - \beta > -1$, a fortiori $(r-n)\alpha - \beta > -1$. Thus, for all $x \in M$ and $z \in T_xM$ such that $d^{-\alpha} \leqslant \|z\| < b_n \ln d$, we have:

$$\left|\frac{1}{d^r}D_d(x,z) - D_{n,r}(\|z\|^2)\right| = O\Big(d^{(r-n)\alpha-\beta}\Big)\,,$$

where $\beta > 0$ and the error term is independent of (x,z). □

Proposition 4.38. *Let $\alpha \in (0,\alpha_0)$, let ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, we have the following asymptotic as $d \to +\infty$:*

$$\begin{aligned}\frac{1}{d^r}\int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z\right)|\mathrm{d}V_M| \\ = \int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_{n,r}(\|z\|^2)\,\mathrm{d}z\right)|\mathrm{d}V_M| \\ + \|\phi_1\|_\infty\,\|\phi_2\|_\infty\,O\big(d^{-\alpha}\big)\,,\end{aligned}$$

where the error term does not depend on (ϕ_1,ϕ_2).

Proof. Let $\alpha \in (0,\alpha_0)$, we set $\alpha' = \frac{\alpha}{n-r} \in (0,\alpha_1)$. Let $\phi_1,\phi_2 \in \mathcal{C}^0(M)$ and let $x \in M$, we apply Lemmas 4.36 and 4.37 for α' and ϕ_2. Then, we have:

$$\begin{aligned}\frac{1}{d^r}\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z \\ = \int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_{n,r}(\|z\|^2)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z \\ + |\phi_1(x)|\,\|\phi_2\|_\infty\,O\Big(d^{(r-n)\alpha'}\Big)\,, \quad (4.77)\end{aligned}$$

and the error term can be rewritten as $O(d^{-\alpha})$.

Since $\kappa(z)^{\frac{1}{2}} = 1 + O\left(\|z\|^2\right)$ (cf. (3.6)), there exists $C > 0$ independent of x such that for all $z \in B_{T_xM}(0,R)$, $\left|\kappa(z)^{\frac{1}{2}} - 1\right| \leqslant C\|z\|^2$. Then, we get:

$$\begin{aligned}\left|\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\left(\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}} - 1\right)\mathrm{d}z\right| \\ \leqslant |\phi_1(x)|\,\|\phi_2\|_\infty\, C\frac{(b_n\ln d)^2}{d}\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z \\ \leqslant |\phi_1(x)|\,\|\phi_2\|_\infty\,\frac{C}{2}\frac{(b_n\ln d)^2}{d}\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{(b_n\ln d)^2}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t.\end{aligned}$$

Since $|D_{n,r}(t)|\,t^{\frac{n-2}{2}}$ is integrable on $(0,+\infty)$ (Lem. 4.25) and $\alpha < 1$, we have:

$$\begin{aligned}&\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z = \\ &\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z + |\phi_1(x)|\,\|\phi_2\|_\infty\, O\big(d^{-\alpha}\big), \quad (4.78)\end{aligned}$$

where the error term in independent of x. By (4.77) and (4.78), we have:

$$\begin{aligned}&\frac{1}{d^r}\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z = \\ &\qquad\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z + |\phi_1(x)|\,\|\phi_2\|_\infty\, O\big(d^{-\alpha}\big),\end{aligned}$$

uniformly in $x \in M$. Integrating this relation over M yields the result. □

Now, let $\alpha \in (0,\alpha_0)$, let ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, then by Eq. (4.10), Prop. 4.12, Eq. (4.43) and Prop. 4.38 we have:

$$\begin{aligned}&\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) = \\ &\quad\frac{d^{r-\frac{n}{2}}}{(2\pi)^r}\int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z\right)|\mathrm{d}V_M| \\ &\qquad\qquad + \|\phi_1\|_\infty\,\|\phi_2\|_\infty\, O\big(d^{r-\frac{n}{2}-\alpha}\big), \quad (4.79)\end{aligned}$$

where the error term is independent of (ϕ_1,ϕ_2). Then, we have:

$$\begin{aligned}\left|\int_{z\in B_{T_xM}(0,b_n\ln d)}\left(\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) - \phi_1(x)\phi_2(x)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z\right| \\ \leqslant \|\phi_1\|_\infty\,\varpi_{\phi_2}\left(\frac{b_n\ln d}{\sqrt{d}}\right)\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z,\end{aligned}$$

where ϖ_{ϕ_2} is the continuity modulus of ϕ_2 (see Def. 1.2). Besides, by a polar change of coordinates, we have:

$$\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z = \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{(b_n\ln d)^2}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t, \quad (4.80)$$

and this quantity is bounded, by Lemma 4.25. Then,

$$\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z = \phi_1(x)\phi_2(x)\int_{z\in B_{T_xM}(0,b_n\ln d)} D_{n,r}(\|z\|^2)\,\mathrm{d}z + \|\phi_1\|_\infty\,\varpi_{\phi_2}\left(\frac{b_n\ln d}{\sqrt{d}}\right)O(1), \quad (4.81)$$

where the error term is independent of (ϕ_1,ϕ_2).

Let $\beta\in\left(0,\frac{1}{2}\right)$, then there exists $C_\beta>0$ such that for all $d\in\mathbb{N}^*$, $b_n\frac{\ln d}{\sqrt{d}}\leqslant C_\beta d^{-\beta}$. Since ϖ_{ϕ_2} is a non-decreasing function, we have $\varpi_{\phi_2}\left(b_n\frac{\ln d}{\sqrt{d}}\right)\leqslant\varpi_{\phi_2}\left(C_\beta d^{-\beta}\right)$. By (4.79), (4.80) and (4.81), we obtain:

$$\begin{aligned}\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) = {}& d^{r-\frac{n}{2}}\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r}\left(\int_M\phi_1\phi_2\,|\mathrm{d}V_M|\right)\left(\frac{1}{2}\int_{t=0}^{(b_n\ln d)^2} D_{n,r}(t)t^{\frac{n-2}{2}}\,\mathrm{d}t\right)\\ &+\|\phi_1\|_\infty\|\phi_2\|_\infty O\left(d^{r-\frac{n}{2}-\alpha}\right)+\|\phi_1\|_\infty\varpi_{\phi_2}\left(C_\beta d^{-\beta}\right)O\left(d^{r-\frac{n}{2}}\right). \quad (4.82)\end{aligned}$$

By Lemma 4.23, we have: $|D_{n,r}(t)| = O\left(te^{-\frac{t}{2}}\right)$. Then there exists some $C>0$ such that, for all t large enough,

$$|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\leqslant Ce^{-\frac{t}{4}}.$$

Then, for d large enough we have:

$$\left|\int_{t=(b_n\ln d)^2}^{+\infty} D_{n,r}(t)t^{\frac{n-2}{2}}\,\mathrm{d}t\right|\leqslant C\int_{t=(b_n\ln d)^2}^{+\infty} e^{-\frac{t}{4}}\,\mathrm{d}t\leqslant 4C\exp\left(-\frac{1}{4}b_n^2(\ln d)^2\right)=O\left(d^{-1}\right). \quad (4.83)$$

By Equations (4.82) and (4.83), we get:

$$\begin{aligned}\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) = {}& d^{r-\frac{n}{2}}\left(\int_M\phi_1\phi_2\,|\mathrm{d}V_M|\right)\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r}\left(\frac{1}{2}\int_0^{+\infty}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t\right)\\ &+\|\phi_1\|_\infty\|\phi_2\|_\infty O\left(d^{r-\frac{n}{2}-\alpha}\right)+\|\phi_1\|_\infty\varpi_{\phi_2}\left(C_\beta d^{-\beta}\right)O\left(d^{r-\frac{n}{2}}\right). \quad (4.84)\end{aligned}$$

Finally, recall that we defined $I_{n,r}$ by Eq. (1.6) and $D_{n,r}$ by Def. 4.24. Hence, we have:

$$\mathcal{I}_{n,r}=\frac{1}{2}\int_0^{+\infty}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t,$$

and this quantity is finite by Lemma 4.25. This concludes the proof of Theorem 1.6.

5 Proofs of the corollaries

5.1 Proof of Corollary 1.9

Corollary 1.9 is a direct of Thm. 1.6 and the Markov inequality. Let $\phi\in\mathcal{C}^0(M)$, then, by (1.7) we have:

$$\mathrm{Var}(\langle|\mathrm{d}V_d|\,,\phi\rangle)=O\left(d^{r-\frac{n}{2}}\right),$$

where the error term depends on ϕ. Now, let $\alpha \geqslant \frac{r}{2} - \frac{n}{4}$ and $\varepsilon > 0$. We have:

$$\begin{aligned}\mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \, , \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi\rangle]\right| > d^{\alpha}\varepsilon\right) &= \mathbb{P}\left(d^{-\alpha}\left|\langle |\mathrm{d}V_d| \, , \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi\rangle]\right| > \varepsilon\right) \\ &\leqslant \frac{1}{\varepsilon^2}\operatorname{Var}\left(d^{-\alpha}\langle |\mathrm{d}V_d| \, , \phi\rangle\right) \\ &\leqslant \frac{1}{\varepsilon^2} d^{-2\alpha}\operatorname{Var}(\langle |\mathrm{d}V_d| \, , \phi\rangle)\,.\end{aligned}$$

5.2 Proof of Corollary 1.10

We obtain Cor. 1.10 as a consequence of Cor. 1.9. Let $U \subset M$ be an open subset. We denote by $\phi_U \in \mathcal{C}^0(M)$ the function such that $\phi_U(x)$ is the geodesic distance from x to the complement of U in (M, g). Then we have:

$$U = \{x \in M \mid \phi_U(x) > 0\}\,,$$

and ϕ_U is non-negative. Hence, $Z_d \cap U = \emptyset$ if and only if $\langle |\mathrm{d}V_d| \, , \phi_U\rangle = 0$. Let $\varepsilon > 0$ such that:

$$\varepsilon < \frac{1}{2}\left(\int_M \phi_U \, |\mathrm{d}V_M|\right)\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}.$$

Then, by Thm. 1.1, for d large enough we have:

$$d^{-\frac{r}{2}}\mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_U\rangle] - \varepsilon \geqslant \frac{1}{2}\left(\int_M \phi_U \, |\mathrm{d}V_M|\right)\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)} > 0.$$

Thus, for d large enough, we have:

$$\begin{aligned}\mathbb{P}(Z_d \cap U = \emptyset) &= \mathbb{P}(\langle |\mathrm{d}V_d| \, , \phi_U\rangle = 0) \\ &\leqslant \mathbb{P}\left(\langle |\mathrm{d}V_d| \, , \phi_U\rangle < \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_U\rangle] - d^{\frac{r}{2}}\varepsilon\right) \\ &\leqslant \mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \, , \phi_U\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi_U\rangle]\right| > d^{\frac{r}{2}}\varepsilon\right).\end{aligned}$$

And by Cor. 1.9, this is a $O\left(d^{-\frac{n}{2}}\right)$.

5.3 Proof of Corollary 1.11

In this section we assume that $n \geqslant 3$. We consider a random sequence $(s_d)_{d\in\mathbb{N}}$ of sections of increasing degree, distributed according to the probability measure $\mathrm{d}\nu = \bigotimes_{d\in\mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Strictly speaking, $|\mathrm{d}V_{s_d}|$ is not defined for small d. However, $\mathrm{d}\nu$-almost surely, $|\mathrm{d}V_{s_d}|$ is well-defined for all $d \geqslant d_1$, so the statement of Cor. 1.11 makes sense.

Our proof follows the lines of the proof of Shiffman and Zelditch [23, sect. 3.3] in the complex case. First, we prove that for every fixed $\phi \in \mathcal{C}^0(M)$ we have:

$$d^{-\frac{r}{2}}\langle |\mathrm{d}V_{s_d}| \, , \phi\rangle \xrightarrow[d\to+\infty]{} \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M \phi \, |\mathrm{d}V_M|\right). \tag{5.1}$$

Then we use a separability argument to get the result. In the complex algebraic setting of [23], the scaled volume of $s_d^{-1}\{0\} \subset \mathcal{X}$ is a deterministic constant, independent of d. In our real algebraic setting this is not the case.

Let $\phi \in \mathcal{C}^0(M)$, then we have:

$$\mathbb{E}\left[\sum_{d\in\mathbb{N}}\left(d^{-\frac{r}{2}}\left(\langle |\mathrm{d}V_{s_d}| \, , \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi\rangle]\right)\right)^2\right] = \sum_{d\in\mathbb{N}} d^{-r}\operatorname{Var}(\langle |\mathrm{d}V_d| \, , \phi\rangle) < +\infty,$$

since $d^{-r}\operatorname{Var}(\langle|\mathrm{d}V_d|\,,\phi\rangle) = O\big(d^{-\frac{n}{2}}\big)$ by Cor. 1.7. Hence, $\mathrm{d}\nu$-almost surely, we have:

$$\sum_{d\in\mathbb{N}}\Big(d^{-\frac{r}{2}}\Big(\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle - \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi\rangle]\Big)\Big)^2 < +\infty,$$

and

$$\Big(d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle - d^{-\frac{r}{2}}\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi\rangle]\Big) \xrightarrow[d\to+\infty]{} 0.$$

Then, by Thm. 1.1, $\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle$ satisfies (5.1) $\mathrm{d}\nu$-almost surely.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a dense sequence in the separable space $\big(\mathcal{C}^0(M),\|\cdot\|_\infty\big)$. Without loss of generality, we can assume that $\phi_0 = \mathbf{1}$, the unit constant function on M. Then, $\mathrm{d}\nu$-almost surely, we have:

$$\forall k\in\mathbb{N},\qquad d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi_k\rangle \xrightarrow[d\to+\infty]{} \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M \phi_k\,|\mathrm{d}V_M|\right). \tag{5.2}$$

Let $\underline{s} = (s_d)_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}}\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ be a fixed sequence such that (5.2) holds. For every $\phi\in\mathcal{C}^0(M)$ and $k\in\mathbb{N}$ we have:

$$\begin{aligned}
\left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle - \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M\phi\,|\mathrm{d}V_M|\right)\right| &\\
&\leqslant \left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle - d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi_k\rangle\right| \\
&\quad + \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left|\int_M\phi_k\,|\mathrm{d}V_M| - \int_M\phi\,|\mathrm{d}V_M|\right| \\
&\quad + \left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi_k\rangle - \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M\phi_k\,|\mathrm{d}V_M|\right)\right| \\
&\leqslant \|\phi-\phi_k\|_\infty\left(d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\mathbf{1}\rangle + \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\operatorname{Vol}(M)\right) \\
&\quad + \left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi_k\rangle - \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M\phi_k\,|\mathrm{d}V_M|\right)\right|.
\end{aligned}$$

Recall that $\phi_0 = \mathbf{1}$. Then, by (5.2), the sequence $(d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\mathbf{1}\rangle)_{d\in\mathbb{N}}$ converges. Hence it is bounded by some positive constant $K_{\underline{s}}$. Let $\phi\in\mathcal{C}^0(M)$ and let $\varepsilon>0$. Let $k\in\mathbb{N}$ be such that:

$$\|\phi-\phi_k\|_\infty \leqslant \varepsilon\left(K_{\underline{s}} + \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\operatorname{Vol}(M)\right)^{-1}.$$

Then, for every d large enough we have:

$$\left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi_k\rangle - \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M\phi_k\,|\mathrm{d}V_M|\right)\right| \leqslant \varepsilon,$$

and

$$\left|d^{-\frac{r}{2}}\langle|\mathrm{d}V_{s_d}|\,,\phi\rangle - \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)}\left(\int_M\phi\,|\mathrm{d}V_M|\right)\right| \leqslant 2\varepsilon.$$

Thus, ϕ satisfies (5.1).

Finally, whenever (5.2) is satisfied we have: for every $\phi\in\mathcal{C}^0(M)$, ϕ satisfies (5.1). Since the condition (5.2) is satisfied $\mathrm{d}\nu$-almost surely, this proves Cor. 1.11.

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