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# Hyperpfaffians

Ammar Aboud\*, Jean-Gabriel Luque†

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## Résumé

We define and investigate a generalization of the pfaffian for multiple array which interpolate between the hyperdeterminant and the hyperpfaffian.

## 1 Introduction

One of the simplest possible generalization of the determinant for higher-dimensional arrays is due to Cayley [5, 6] and consists in considering a multiple alternating sum. The pfaffian of a skew-symmetric matrix is defined as the square root of the determinant. In a more combinatorial way, it is also a signed sum but over perfect matchings instead of a signed sum over all the permutations. A rather natural way to define hyperpfaffian for  $k$ -tuple arrays consists in setting

$$\text{HPf}(M) := \frac{1}{n!} \sum \epsilon(\sigma) \prod_{i=1}^n M_{\sigma_1((i-1)k+1), \dots, \sigma_1(ik)}, \quad (1)$$

where the sum is over the permutations  $\sigma \in \mathfrak{S}_{nk}$  satisfying  $\sigma(1) < \dots < \sigma(k)$ ,  $\sigma(k+1) < \dots < \sigma(2k)$ ,  $\dots$ ,  $\sigma((n-1)k+1) < \dots < \sigma(nk)$ . This definition and a few variants are considered in [1, 3, 11, 12]. In this paper we propose a more general definition for hyperpfaffian which interpolate between the hyperdeterminant and the hyperpfaffian. We prove several formulas (generalization Laplace expansion, hyperpfaffian of a sum, composition formula). Our main tool is the Grassmann-Berezin calculus. We also consider a generalization of a formula due to Gherardeli [10] relying hyperdeterminants and the Alon-Tarsi constant [2].

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## 2 Hyperpfaffians and Grassmann variables

### 2.1 Combinatorial definition

Let  $M = (M_{i_1, \dots, i_{mk}})_{1 \leq i_1, \dots, i_{mk} \leq mn}$  be a tensor. We define the following polynomial which generalizes the combinatorial definition of the pfaffian of a matrix

$$\text{PF}^{(m)}(M) := \frac{1}{n!} \sum \epsilon(\sigma_1) \cdots \epsilon(\sigma_k) \prod_{i=1}^n M_{\sigma_1((i-1)m+1), \dots, \sigma_1(im), \dots, \sigma_k((i-1)m+1), \dots, \sigma_k(im)}, \quad (2)$$

where the sum runs over the  $k$ -tuples of permutations  $(\sigma_1, \dots, \sigma_k) \in \mathfrak{S}_{mn}^k$  satisfying  $\sigma_j((i-1)m+1) < \dots < \sigma_j(im)$  for any  $1 \leq j \leq k$  and  $1 \leq i \leq n$ .

For any  $M = (M_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k \leq n}$ , this definition allows to associate a polynomial  $\text{PF}^{(d)}(M)$  to  $M$  for any  $d$  which divides both  $n$  and  $k$ . Notice that if  $d = 1$  we recover the Cayley hyperdeterminant of  $M$

$$\text{PF}^{(1)}(M) = \text{Det}(M) = \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_1) \cdots \epsilon(\sigma_k) \prod_{i=1}^n M_{\sigma_1(i), \dots, \sigma_k(i)}. \quad (3)$$

In the other end, if  $k$  divides  $n$  and  $d = k$ , we recover the notion of hyperpfaffian as defined in [3, 11],

$$\text{PF}^{(k)}(M) = \text{HPf}(M) = \frac{1}{\left(\frac{n}{k}\right)!} \sum \epsilon(\sigma) \prod_{i=1}^{\frac{n}{k}} M_{\sigma((i-1)k+1), \dots, \sigma(ik)}, \quad (4)$$

where the sum runs over the permutations  $\sigma \in \mathfrak{S}_n$  satisfying  $\sigma((i-1)k+1) < \dots < \sigma(ik)$  for any  $1 \leq i \leq \frac{n}{k}$ . Remark that if  $d \neq n$  and  $k$  is odd then  $\text{PF}^{(d)}(M) = 0$ .

### 2.2 Grassmann-Berezin calculus

Let  $M = (M_{i_1, \dots, i_{mk}})_{1 \leq i_1, \dots, i_{mk} \leq mn}$  be a tensor where  $mk$  is even and consider  $k$  sets of formal variables  $\eta^{(i)} = \{\eta_{j_1}^{(i)}, \dots, \eta_{j_m}^{(i)}\}$  ( $i = 1 \dots k$ ) satisfying the commutations  $\eta_{j_1}^{(i)} \eta_{j_2}^{(i)} = -\eta_{j_2}^{(i)} \eta_{j_1}^{(i)}$  for any  $1 \leq i \leq k$ ,  $1 \leq j_1, j_2 \leq mn$  and  $\eta_{j_1}^{(i_1)} \eta_{j_2}^{(i_2)} = \eta_{j_2}^{(i_2)} \eta_{j_1}^{(i_1)}$  for any  $1 \leq i_1 \neq i_2 \leq k$ ,  $1 \leq j_1, j_2 \leq mn$ . Let us introduce the notation known as Berezin integrals. The Berezin integral is a convenient tool for computing in Grassmann algebra (i.e., with anticommutative variables). Let  $f$  be a polynomials in the variables  $\eta^{(1)}, \dots, \eta^{(k)}$ , we define  $\int d\eta_{j_1}^{(i_1)} \cdots d\eta_{j_m}^{(i_m)} f := \frac{\partial}{\partial \eta_{j_1}^{(i_1)}} \cdots \frac{\partial}{\partial \eta_{j_m}^{(i_m)}} f$ , where each  $\frac{\partial}{\partial \eta_j^{(i)}}$  acts on the Grassmann algebra as a left derivation ( $\eta_j^{(i)}$  is pushed to the left, with a sign, and hence erased). For simplicity we set also  $\eta_J^{(i)} = \eta_{j_1}^{(i)} \cdots \eta_{j_m}^{(i)}$  for  $J = \{j_1 \leq \dots \leq j_m\}$ . We define  $\Omega_m(M) := \sum M_{i_1 \dots i_{mk}} \eta_{\{i_1, \dots, i_m\}}^{(1)} \cdots \eta_{\{(i_{k-1})m+1, \dots, i_{km}\}}^{(k)}$ , where the sum

is over the  $km$ -tuples  $(i_1, \dots, i_{km})$  satisfying  $i_1 < \dots < i_m, \dots, i_{(k-1)m+1} < \dots < i_{km}$ . By reorganizing the monomials in the expansion of the polynomials one obtains the following result.

**Proposition 2.1** *Let  $\ell$  a divisor of  $n$  and  $I^{(1)}, \dots, I^{(k)}$  be  $k$  subsets of  $\{1, \dots, mn\}$  of cardinality  $\ell m$ . One has*

$$\frac{1}{\ell!} \int d\eta_{I^{(1)}}^{(1)} \cdots d\eta_{I^{(k)}}^{(k)} \Omega_m(M)^\ell = \text{PF}^{(m)} \left( M \left[ \overbrace{I^{(1)} | \dots | I^{(1)}}^{\times m} | \dots | \overbrace{I^{(k)} | \dots | I^{(k)}}^{\times m} \right] \right) \quad (5)$$

where  $M[I_1 | \dots | I_{mk}]$  is a hyperminor of  $M$  that is the tensor obtained by selecting the entries whose first index belongs in  $I_1$ , the second index belongs in  $I_2$  etc.

As a special case, one has

$$\frac{1}{n!} \int d\eta_{\{1, \dots, mn\}}^{(1)} \cdots d\eta_{\{1, \dots, mn\}}^{(k)} \Omega_m(M)^n = \int d\eta_{\{1, \dots, mn\}}^{(1)} \cdots d\eta_{\{1, \dots, mn\}}^{(k)} e^{\Omega_m(M)} = \text{PF}^{(m)}(M). \quad (6)$$

### 3 Some formulas

#### 3.1 Generalization of the Lapace formula

Let  $0 < n' < n$  be an integer. We split  $\Omega$  into two disjoint sums  $\Omega_m(M) = \Omega'_m(M) + \hat{\Omega}''_m(M)$  where

$$\Omega'_m(M) := \sum M_{i_1 \dots i_{mk}} \eta_{\{i_1, \dots, i_m\}}^{(1)} \cdots \eta_{\{i_{(k-1)m+1}, \dots, i_{km}\}}^{(k)} \quad (7)$$

where the sum runs over the  $km$ -tuples  $(i_1, \dots, i_{km})$  satisfying  $i_1 < \dots < i_m, \dots, i_{(k-1)m+1} < \dots < i_{km}$  and  $i_1 \in \{1, \dots, n'\}$ . The commutativity rules give

$$\Omega_m(M)^n = \binom{n}{n'} \Omega'_m(M)^{n'} \Omega''_m(M)^{n-n'} = \binom{n}{n'} \Omega'_m(M)^{n'} \hat{\Omega}''_m(M)^{n-n'}, \quad (8)$$

with  $\hat{\Omega}''_m(M) := \sum M_{i_1 \dots i_{mk}} \eta_{\{i_1, \dots, i_m\}}^{(1)} \cdots \eta_{\{i_{(k-1)m+1}, \dots, i_{km}\}}^{(k)}$  where the sum runs over the  $km$ -tuples  $(i_1, \dots, i_{km})$  satisfying  $i_1 < \dots < i_m, \dots, i_{(k-1)m+1} < \dots < i_{km}$  and  $i_1, \dots, i_m \notin \{1, \dots, n'\}$ . But from Proposition 2.1, one has

$$\Omega'_m(M)^{n'} \hat{\Omega}''_m(M)^{n-n'} = n'!(n-n')! \sum^{\wedge} \text{PF}^{(m)}(M[I]) \text{PF}^{(m)}(M[J]) \eta_{I^{(1)}}^{(1)} \cdots \eta_{I^{(k)}}^{(k)} \eta_{J^{(1)}}^{(1)} \cdots \eta_{J^{(k)}}^{(k)}, \quad (9)$$

where  $\sum^{\wedge}$  means that the sum runs over the pairs of sequences

$$I = \left[ \overbrace{I^{(1)} | \dots | I^{(1)}}^{\times m} | \dots | \overbrace{I^{(k)} | \dots | I^{(k)}}^{\times m} \right] \text{ and } J = \left[ \overbrace{J^{(1)} | \dots | J^{(1)}}^{\times m} | \dots | \overbrace{J^{(k)} | \dots | J^{(k)}}^{\times m} \right]$$

satisfying  $\{1, \dots, n'\} \subset I^{(1)}$ ,  $\text{card}(I^{(s)}) = n'm$  and  $J^{(s)} = \{1, \dots, nm\} \setminus I^{(s)}$  for any  $1 \leq s \leq k$ . We deduce the following result which generalizes the Laplace expansion rule.

**Theorem 3.1** *One has*

$$\text{PF}^{(m)}(M) = \sum^{\wedge} (-1)^{\wr I} \text{PF}^{(m)}(M[I]) \text{PF}^{(m)}(M[J]) \quad (10)$$

where  $\wr I = k \binom{n'm+1}{2} + \sum_{i=1}^k \sum_{e \in I^{(i)}} e$ .

### 3.2 Hyperpfaffian of a sum

Let  $N = (N_{i_1, \dots, i_{mk}})_{1 \leq i_1, \dots, i_{mk} \leq mn}$  be another tensor. Since,  $\Omega_m(M + N) = \Omega_m(M) + \Omega_n(N)$  one obtains  $\Omega_m(M + N)^n = \sum_{\ell=0}^n \binom{n}{\ell} \Omega_m(M)^\ell \Omega_m(N)^{n-\ell}$ . Hence, proposition 2.1 implies the following result.

**Proposition 3.1** *One has*

$$\text{PF}^{(m)}(M + N) = \sum_{\ell=0}^n \sum (-1)^{\wr I} \text{PF}^{(m)}(M[I]) \text{PF}^{(m)}(N[J]), \quad (11)$$

where the second sum runs over the pairs of sequences

$$I = \left[ \overbrace{I^{(1)} | \dots | I^{(1)}}^{\times m} | \dots | \overbrace{I^{(k)} | \dots | I^{(k)}}^{\times m} \right] \text{ and } J = \left[ \overbrace{J^{(1)} | \dots | J^{(1)}}^{\times m} | \dots | \overbrace{J^{(k)} | \dots | J^{(k)}}^{\times m} \right]$$

satisfying  $\text{card}(I^{(s)}) = \ell m$  and  $J^{(s)} = \{1, \dots, nm\} \setminus I^{(s)}$  for any  $1 \leq s \leq k$ .

### 3.3 Composition of Hyperpfaffians

Suppose now  $m = pm'$ , in this case  $\Omega_{m'}(M)^{pn} = (\Omega_{m'}(M)^p)^n$ . But from Proposition 2.1, one has

$$\Omega_{m'}(M)^p = p! \sum \text{PF}^{(m)} M \left[ \overbrace{I^{(1)} | \dots | I^{(1)}}^{\times m'} | \dots | \overbrace{I^{(k)} | \dots | I^{(k)}}^{\times m'} \right] \eta_{I^{(1)}}^{(1)} \dots \eta_{I^{(k)}}^{(k)}, \quad (12)$$

where the sum is over the sets  $I^{(1)}, \dots, I^{(k)} \subset \{1, \dots, nm\}$  of cardinality  $m$ . So  $\Omega_{m'}(M)^p$  is written as  $\Omega_m(M')$  where  $M'$  is a  $mn^{\otimes m'k}$  tensor. More explicitly, applying again Proposition 2.1, we obtain

**Proposition 3.2** *One has*

$$\text{PF}^{(m)} \left( \text{PF}^{(m')} (M \langle i_1, \dots, i_{mk} \rangle) \right)_{1 \leq i_1, \dots, i_{mk} \leq mn} = \frac{\binom{np}{p, \dots, p}}{n!} \text{PF}^{(m')} (M) \quad (13)$$

with

$$M \langle i_1, \dots, i_{mk} \rangle = M \left[ \overbrace{\{i_1, \dots, i_m\} | \dots | \{i_1, \dots, i_m\}}^{\times m'} | \dots | \overbrace{\{i_{m(k-1)+1}, \dots, i_{mk}\} | \dots | \{i_{m(k-1)+1}, \dots, i_{mk}\}}^{\times m'} \right].$$

## 4 Hyperpfaffians and generalized latin squares

A  $(m, k)$ -latin quasisquare is a  $m \times mk$  matrix

$$\begin{bmatrix} \sigma_1(1) & \cdots & \sigma_1(m) & \cdots & \sigma_k(1) & \cdots & \sigma_k(m) \\ \vdots & & \vdots & \cdots & \vdots & & \vdots \\ \sigma_1((k-1)m+1) & \cdots & \sigma_1(km) & \cdots & \sigma_k((k-1)m+1) & \cdots & \sigma_k(km) \end{bmatrix}$$

where each  $\sigma_i$  is a permutation and each line is a permutation  $\tau \in \mathfrak{S}_{km}$  satisfying  $\tau((\ell-1)m+1) < \cdots < \tau(\ell m)$  for any  $1 \leq \ell \leq k$ . We denote by  $LQ(m, k)$  the set of the  $(m, k)$ -latin quasisquares. To each  $c \in LQ(m, k)$  we associate a sign  $\varepsilon(c)$  which is the product of the signs of the permutations  $\sigma_i$ 's and the signs of the lines.

Let  $\mathcal{A}^n$  be the unique  $n^{\otimes n}$  antisymmetric tensor such that  $\mathcal{A}_{1, \dots, n}^n = 1$ .

**Proposition 4.1** *One has*

$$\text{PF}^{(m)}(\mathcal{A}^{mk}) = \frac{1}{k!} \sum_{c \in LQ(m, k)} \varepsilon(c) \quad (14)$$

*Proof* – Observe that

$$\Omega_m(\mathcal{A}^{mk}) = \sum_{\substack{\tau \in \mathfrak{S}_{mk} \\ \forall \ell \in \{1, \dots, k\}, \tau((\ell-1)m+1) < \cdots < \tau(\ell m)}} \varepsilon(\tau) \eta_{\{\tau(1), \dots, \tau(m)\}}^{(1)} \cdots \eta_{\{\tau((k-1)m+1), \dots, \tau(km)\}}^{(k)}$$

And so  $(\Omega_m(\mathcal{A}^{mk}))^k = \sum_{c \in LQ(m, k)} \varepsilon(c) \eta_{\{1, \dots, mk\}}^{(1)} \cdots \eta_{\{1, \dots, mk\}}^{(k)}$ . Proposition 2.1 allows us to conclude.  $\square$

This proposition generalizes a result due to Gherardelli [10] relying the hyperdeterminant of an antisymmetric tensor and the Alon-Tarsi constant [2]. More precisely, since  $LQ(1, k)$  is the set of  $k \times k$ -latin squares, we recover it for  $m = 1$ ,  $\text{Det}(\mathcal{A}^k) = \frac{1}{k!} \sum_{c \in LQ(1, k)} \varepsilon(c)$ . The following table contains the first values of  $\text{PF}^{(m)}(\mathcal{A}^{mk})$ .

$m \setminus k$	1	2	3	4	5	6	7	8
1	1	1	0	4	0	2304	0	6210846720
2	1	3	90	204120	?	?	?	?
3	0	10	0	?	0	?	0	?
4	1	35	519750	?	?	?	?	?
5	0	126	0	?	0	?	0	?
6	1	462	?	?	?	?	?	?

Observe that the first line is the Alon-Tarsi constant for the even values of  $k$ . The second column is  $\binom{2m-1}{m}$ . This can be easily shown by remarking that any quasisquare in  $LQ(m, 2)$  is on the form

$$\begin{bmatrix} \sigma_1(1) & \cdots & \sigma_1(m) & \sigma_2(1) & \cdots & \sigma_2(m) \\ \sigma_1(m+1) & \cdots & \sigma_1(2m) & \sigma_2(m+1) & \cdots & \sigma_2(2m) \end{bmatrix}.$$

We deduce that  $\text{card}(LQ(m, 2)) = \binom{2m}{m}$  and a straightforward examination shows that any quasisquare has a positive sign.

When  $m$  is even, Theorem 3.1 allows us to write

$$\text{PF}^{(m)}(\mathcal{A}^{mk}) = \sum_{\wedge} (-1)^{\ell I} \text{PF}^{(m)}(\mathcal{A}^{mk}[I]) \text{PF}^{(m)}(\mathcal{A}^{mk}[J]), \quad (15)$$

where  $\text{card}(I^{(s)}) = \text{card}(J^{(s)}) = \frac{m}{2}$  for any  $1 \leq s \leq k$ . Numerical evidences suggest that each term having a non-zero contribution in the sum satisfies  $(-1)^{\ell I} \text{PF}^{(m)}(\mathcal{A}^{mk}[I]) \text{PF}^{(m)}(\mathcal{A}^{mk}[J]) = \text{PF}^{(m)}(\mathcal{A}^{mk}[I])^2$ . If we assume this conjecture, then we show that  $\frac{1}{k!} \sum_{c \in LQ(m, k)} \varepsilon(c) \geq 0$ . This is still an open problem. For  $m = 1$ , we recover a weak version of the Alon-Tarsi conjecture as stated in [13].

## 5 Concluding remarks

The construction proposed in this paper allows us to place the Alon-Tarsi conjecture in a broader context. Indeed, a sound knowledge of the algebraic dependences of the different hyperpfaffians for antisymmetric tensors could help us to understand the combinatoric of the Alon-Tarsi sum. The first (and well known) example is given by  $\text{Pf} = \det^2$  for antisymmetric matrices. This is no longer the case for higher tensors and the complete picture remains to be discovered.

We notice also that there are unsigned version of most of the equalities stated in the paper. These equalities involve hyperhafnians, unsigned analogues of hyperpfaffians obtained by replacing the Grassmann variables by commuting nilpotent (i.e.  $x_i^2 = 0$ ) variables.

## References

- [1] A. Abdelessam, *The Grassmann-Berezin calculus and theorems of the matrix-tree type*, Advances in Applied Mathematics 33 1 (2004), 51-70.
- [2] N. Alon, M. Tarsi, *Coloring and orientations of graphs*, Combinatoria 12 (1992), 125-143.
- [3] A.I. Barvinok, *New algorithm for linear  $k$ -matroid intersection and matroid  $k$ -parity problems*, Mathematical Programming, 69 (1995), 449-470.
- [4] F. A. Berezin, *The method of second quantization*, Academic Press (1966)
- [5] A. Cayley, *On the theory of determinants*, Trans. Cambridge Phil. Soc. VIII (1843), 1-16.
- [6] A Cayley, *Mémoire sur les hyperdéterminants*, J. reine und angew. Math. 30 (1846), 1-37
- [7] A.A. Drisko, *On the number of even and odd latin squares of order  $p + 1$*
- [8] A.A. Drisko, *Proof of the Alon-Tarsi Conjecture for  $n = 2^r p$* .

- [9] L. Gegenbauer *Über Determinanten hheren Ranges*, Denkschriften der Kais. Akademie der Wissenschaften in Wien, Vienne, **43**, 17-32, 1882.
- [10] F. Gherardelli, *Osservazioni sugli iperdeterminanti*, Istit. Lombardo Acad. Sci. Let. Rend., A 126 (1993), 107-113.
- [11] J.-G. Luque, J.-Y. Thibon, *Pfaffian and Hafnian identities in shuffle algebras*, Advances in Applied Mathematics 29 **4** (2002), 620-646.
- [12] D. Redelmeier, *Hyperpfaffians in algebraic combinatorics*, PhD University of Waterloo (2006).
- [13] P. Zappa, *The Cayley determinant of the determinant tensor and the Alon-Tarsi conjecture*, Adv. Appl. Math., 19 (1997), 31-44.