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Stability analysis of a linear system coupled to a vectorial transport equation using integral inequalities^{*}

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Abstract

This paper deals with the stability analysis of a system of ordinary differential equations coupled to a vectorial transport equation. We develop here a new method to study the stability of this class of systems using linear matrix inequalities led by the choice of an appropriate Lyapunov functional. The main idea in our approach is to build a Lyapunov functional by enriching the basic energy of the coupled system under study with specific terms involving an approximation of the infinite dimensional state of the transport equation. To this end, we will exploit Legendre polynomials and their properties, and use a Bessel inequality to measure the contribution of our approximation. We will then give our exponential stability results and their proofs. Our approach will finally be tested on academic examples.

Key words: Transport equation, Lyapunov stability, integral inequalities.

1 Introduction

A systems coupling partial and ordinary differential equations present one kind of infinite dimensional systems. The robust control of what is also called *distributed parameter systems* has been a very active field for the last decades and has spawned many branches, such as for instance in stability analysis and stabilization design. This article is meant to perform a stability analysis of a system of linear ordinary differential equations (ODEs) coupled to a vectorial transport equation, which is an hyperbolic partial differential equation (PDE) of order one.

Analysing and controlling this type of system coupling ODEs and PDEs represent an attractive topic in applied mathematics and more recently in automatic control. A large number of papers already exists on stability of this class of systems: see for instance [16], [31], [23], [32], [21] among many others). Such systems now appear in the context of energy management and for instance in

modelling power converters connected to transmission lines (see, e.g. [33] and references therein).

The coupled system we will study in this article has the specificity to represent an alternative formulation of a time-delay system (TDS) with transport equations replacing delay terms, TDS also having a solution that evolves in an infinite dimensional space. In our opinion, this is an interesting manner to benefit of the different approaches for the stability and control in both domains (TDS and PDEs). On the one hand, we can refer to many documents devoted to stability study and control of PDEs such as the books [17], [6] and [2], and the references therein, and the articles [9], [28] or [29] for instance. On the other hand, TDS have been widely investigated in the literature (see e.g. [12], [13], [14], [26], [4], and [19]), and used in many areas, such as biological systems, where it was discovered for the first time, mechanical transmissions or networked control systems.

One should know that one of the most fruitful fields of research in stability of these TDS relies on the exhibition of Lyapunov-Krasovskii functional (LKF). The candidate Lyapunov functional, called complete LKF leads to a necessary and sufficient stability condition (see [13]). The parameters composing this complete LKF make this functional numerically difficult to handle, especially for high dimensional systems [10, 18]. Many investigations

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then turns to approximating this parameters, and more recently, approximation methods have been improved by considering polynomial like parameter of arbitrary degree [22].

The objective of this paper is to take advantage of some recent developments on the stability analysis of TDS in order to provide a new framework for the analysis of this coupled ODEs/PDE system. A preliminary study was given in [3]. The first difficulty is related to the infinite dimensional part of the system, which prevents from extending directly the existing methods of the finite dimension analysis. The second difficulty arises from the fact that stability of TDS can be assessed using the Lyapunov-Krasovskii Theorem and analysing the stability of our system cannot be performed using exactly the same theorem. Nevertheless, in order to provide efficient stability conditions, we employ a polynomial approximation of the infinite dimensional state expressed using Legendre polynomials, following the approach developed for TDS in [26].

Outline: In section 2 we detail the setting of the system we are going to study and give a general overview on Lyapunov stability. Section 3 is devoted to the presentation of the tools we use to lead this stability study. Our main results are given and proved in Section 4. Finally, section 5 shows the effectiveness of the presented results in numerical examples and concluding remarks end the paper in Section 6.

Notations: \mathbb{N} is the set of positive integer, \mathbb{R}^n is the n -dimensional Euclidean space with vector norm $|\cdot|_n$. I_n is the identity matrix in $\mathbb{R}^{n \times n}$, $0_{n,m}$ the null matrix in $\mathbb{R}^{n \times m}$, $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ replaces the symmetric matrix $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$. We denote $\mathbb{S}^n \subset \mathbb{R}^{n \times n}$ (respectively \mathbb{S}_+^n , and \mathbb{D}_+^n) the set of symmetric (resp. symmetric positive definite and diagonal positive definite) matrices and $\text{diag}(A, B)$ is a bloc diagonal matrix equal to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For any square matrix A , we define $\text{He}(A) = A + A^\top$. Finally, $L^2(0, 1; \mathbb{R}^n)$ represents the space of square integrable functions over the interval $[0, 1] \subset \mathbb{R}$ with values in \mathbb{R}^n and the partial derivative in time and space are denoted ∂_t and ∂_x , while the classical derivative are $\dot{X} = \frac{d}{dt}X$ and $\mathcal{L}' = \frac{d}{dx}\mathcal{L}$.

2 Formulation of the problem

2.1 Coupled ODE-PDE system

This article is devoted to the stability analysis of a system of ODEs coupled with a vectorial transport equation that takes the following shape:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1, t), & t > 0, \\ \partial_t z(x, t) + \Lambda \partial_x z(x, t) = 0, & x \in (0, 1), t > 0, \\ z(0, t) = C_1 X(t) + C_2 z(1, t), & t > 0. \end{cases} \quad (1)$$

The state of this coupled system is composed of $X(t) \in \mathbb{R}^n$ and $z(\cdot, t) \in L^2(0, 1; \mathbb{R}^m)$. A, B, C_1 et C_2 are constant matrices with appropriate dimensions. In system (1), $\Lambda \in \mathbb{D}_+^m$ is the matrix of propagation speed given by:

$$\Lambda = \text{diag}(\rho_i I_{m_i})_{\{i=1 \dots p\}}. \quad (2)$$

Thus, each velocity $\rho_i > 0$ is applied to m_i components of the state $z(x, t)$ such that $m = \sum_{i=1}^p m_i$. Moreover the matrix C_2 , who is responsible of a coupling between the transport PDEs, is so that its largest eigenvalue satisfies $\lambda_{\max}(C_2) < 1$.

The transport equation $\partial_t z + \Lambda \partial_x z = 0$ in (1) of unknown $z = z(x, t)$ is a simple vectorial PDE and if the initial data $z^0 \in L^2(0, 1; \mathbb{R}^m)$ and the lateral boundary data $z(0, \cdot) = u \in L^2(\mathbb{R}_+; \mathbb{R}^m)$ are given, it has a unique solution $z \in C(\mathbb{R}_+; L^2(0, 1; \mathbb{R}^m))$ such that (see e.g. [6]), for all $t > 0$:

$$\|z(t)\|_{L^2(0,1;\mathbb{R}^m)} \leq K \left(\|z^0\|_{L^2(0,1;\mathbb{R}^m)} + \|u\|_{L^2(\mathbb{R}_+;\mathbb{R}^m)} \right).$$

Considering now the finite dimensional system in $X(t)$ coupled to the transport equation in the variable $z(x, t)$, we can notice that the coupling is linear and the existence of solutions for this system can be proved by Galerkin's method (see [11]). Indeed, let us write the total energy of the system as follows:

$$E(X(t), z(t)) = |X(t)|_n^2 + \|z(t)\|_{L^2(0,1;\mathbb{R}^m)}^2,$$

and in the sequel, denote $E(t) = E(X(t), z(t))$ in order to simplify the notations. Easy formal calculations (using the equations of system (1)) give

$$\begin{aligned} \dot{E}(t) &= X^\top(t) (\text{He}(A) + C_1^\top \Lambda C_1) X(t) \\ &\quad + z^\top(1, t) (C_2^\top \Lambda C_2 - \Lambda) z(1, t) \\ &\quad + \text{He}(X^\top(t) (B + C_1^\top \Lambda C_2) z(1, t)). \end{aligned}$$

Since we assume $\Lambda \in \mathbb{D}_+^m$ and $\lambda_{\max}(C_2) < 1$, one has $C_2^\top \Lambda C_2 - \Lambda \prec 0$, so that there exists $\rho_0 > 0$ such that for all $\zeta \in \mathbb{R}^m$, $\zeta^\top (C_2^\top \Lambda C_2 - \Lambda) \zeta \leq -\rho_0 |\zeta|^2$. We refer to Remark 1 below for some details.

Therefore, using Young's inequality ($2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$) on the last term, and choosing $\epsilon \leq \rho_0$, we get

$$\begin{aligned} \dot{E}(t) &\leq X^\top(t) (\text{He}(A) + C_1^\top \Lambda C_1) X(t) \\ &\quad + \frac{1}{\epsilon} |X^\top(t) (B + C_1^\top \Lambda C_2)|^2 + (\epsilon - \rho_0) |z(1, t)|^2 \\ &\leq K |X(t)|^2 \end{aligned}$$

where the constant $K > 0$ is depending on matrices A, B, C_1 and C_2 . Therefore, integrating in time, we get that for all $t > 0$, $E(t) \leq e^{Kt} E(0)$, leading to the existence of a unique solution (X, z) in the space $\mathbb{R}^n \times L^2(0, 1; \mathbb{R}^m)$.

2.2 Lyapunov stability

Inspired by the complete Lyapunov-Krasovskii functional which is a necessary and sufficient condition for stability of linear systems with constant delay (see [13]), we are looking for a candidate Lyapunov functional for system (1) of the form:

$$\begin{aligned} V(X(t), z(t)) &= X^\top(t)PX(t) + 2X^\top(t)\int_0^1 \mathcal{Q}(x)z(x, t)dx \\ &+ \int_0^1 \int_0^1 z^\top(x_1, t)\mathcal{T}(x_1, x_2)z(x_2, t)dx_1dx_2 \\ &+ \int_0^1 z^\top(x, t)Sz(x, t)dx + \int_0^1 \int_0^x z^\top(y, t)Rz(y, t)dydx, \end{aligned} \quad (3)$$

where the matrices $P \in \mathbb{S}_+^n$, $S, R \in \mathbb{S}_+^m$ and the functions $\mathcal{Q} \in L^2(0, 1; \mathbb{R}^{n \times m})$ and $\mathcal{T} \in L^\infty((0, 1)^2; \mathbb{S}^m)$ have to be determined. This functional is composed of five typical terms. The first quadratic term in $X(t)$ is dedicated to the state of the ODE, while the last four terms are concerned with the state of the PDE. It is worth mentioning that the last two terms can be interpreted as the weighted energy of the transport equation and have been widely used in the literature (see for instance [6]). The term depending on \mathcal{Q} is introduced in order to account for the coupling between the system of ODEs and the transport PDE.

While this class of functionals is already classical in the context of time delay systems, the interpretation of such functionals for PDEs is quite recent (see for instance [1] and [21]). The novelty of the present paper is closely related to these works. The difference of our approach relies on the use of efficient integral inequalities.

Our objectives in this paper is to provide an efficient numerical method to assess stability of system (1) by ensuring that the functional $V(X(t), z(t))$ in (3) is a Lyapunov functional for this system. As one will read in the proof of our stability result, it comes down to prove the existence of positive scalars ε_1 , ε_2 and ε_3 such that the following inequalities hold

$$\begin{aligned} \varepsilon_1 E(t) &\leq V(X(t), z(t)) \leq \varepsilon_2 E(t), \\ \dot{V}(X(t), z(t)) + 2\delta V(X(t), z(t)) &\leq -\varepsilon_3 E(t). \end{aligned} \quad (4)$$

The details will be developed in Section 4. Finally, we would like to highlight that thanks to the general formulation of the coupled system (1) we can relate our work to stability studies of many types of time-delay systems. Among them, we can mention:

- *Systems with single or multiple constant delays* which have been studied in many contribution on the subject (see [27], [15] and [5]).
- *Systems with Cross-Talking delays* (see [20]).

- *Systems with commensurate (or, rationally dependent) delays*, where a single delay and its multiples are involved (see e.g. [30]).
- *Delay Difference systems*, where the ODE is removed from the dynamics of (1) (see e.g. [8]).

These different cases will be illustrated thereafter in some numerical examples using our general setting.

Remark 1 *In the literature, the stability study of systems with cross-talking delays [20] and systems with commensurate delays [30], that our model can represent, is achieved in frequency domain only for small delays, and there are several methods to determine the maximal allowable delay for which the system remains stable (delay sweeping, pseudo-delay ...). Here, to avoid any limitation on the transport matrix Λ , we impose $\lambda_{\max}(C_2) < 1$, so that $C_2^\top \Lambda C_2 - \Lambda \prec 0$.*

3 Tools

The objective of this section is to present a simple methodology to construct Lyapunov functional like (3). The ultimate idea is to provide tractable stability conditions for the infinite dimensional system, conditions that can be evaluated on numerical examples. While the parameters P , S and R are matrices, which can be easily defined and evaluated on numerical simulations, the main difficulty in the selection of a candidate functional relies on the definition of the functions \mathcal{Q} and \mathcal{T} , that potentially vary with the integration parameters x , x_1 and x_2 . Among the possible choices that can be found in the literature, we will focus on polynomial functions of a given degree, expressed using a particular sequence of polynomials, e.g. the Legendre polynomials. This choice will be motivate in the sequel.

3.1 Construction of the Lyapunov functional

As mention above, the problem is to provide a simple methodology to design the functions \mathcal{Q} and \mathcal{T} that enter the functional (3). To do so, we propose the following construction as a truncated decomposition of \mathcal{Q} and \mathcal{T} over specific polynomials:

$$\begin{aligned} \mathcal{Q}(x) &= \sum_{k=0}^N Q(k)\mathcal{L}_k(x), \\ \mathcal{T}(x_1, x_2) &= \sum_{i=0}^N \sum_{j=0}^N T(i, j)\mathcal{L}_i(x_1)\mathcal{L}_j(x_2), \end{aligned} \quad (5)$$

where $N \in \mathbb{N}$, and where \mathcal{L}_k , for $k \in \mathbb{N}$, denote the shifted Legendre polynomials of degree k considered over the interval $[0, 1]$. More details about the definition of these polynomials and of their properties will be provided afterwards. The Lyapunov functional (3) becomes

easily

$$V_N(X(t), z(t)) = \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ Q_N^\top & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \quad (6)$$

$$+ \int_0^1 z^\top(x, t) S z(x, t) dx + \int_0^1 \int_0^x z^\top(y, t) R z(y, t) dy dx,$$

where

$$Z_N(t) = \begin{bmatrix} \int_0^1 z(x, t) \mathcal{L}_0(x) dx \\ \int_0^1 z(x, t) \mathcal{L}_1(x) dx \\ \vdots \\ \int_0^1 z(x, t) \mathcal{L}_N(x) dx \end{bmatrix} \in \mathbb{R}^{m(N+1)} \quad (7)$$

and

$$Q_N = [Q(0) \dots Q(N)] \text{ in } \mathbb{R}^{n, m(N+1)},$$

$$T_N = [T(i, j)]_{i, j=0..N} \text{ in } \mathbb{R}^{m(N+1), m(N+1)}.$$

The potential of using (6) over (3) relies on the fact that the parameters defining V_N are only matrices, and can be easily defined for numerical implementation. Nevertheless, this formulation requires an extensive studies of Legendre polynomials and the augmented vector $Z_N(t)$, given in the next subsections.

3.2 Shifted Legendre polynomials

The shifted Legendre polynomials we will use are denoted $\{\mathcal{L}_k\}_{k \in \mathbb{N}}$ and act over $[0, 1]$. It is crucial that the family $\{\mathcal{L}_k\}_{k \in \mathbb{N}}$ forms an orthogonal basis of $L^2(0, 1; \mathbb{R})$ and we have precisely

$$\langle \mathcal{L}_j, \mathcal{L}_k \rangle = \int_0^1 \mathcal{L}_j(x) \mathcal{L}_k(x) dx = \frac{1}{2k+1} \delta_{jk},$$

where δ_{jk} represents Kronecker's coefficient, equal to 1 if $j = k$ and 0 otherwise. We denote the corresponding norm of this inner scalar product $\|\mathcal{L}_k\| = \sqrt{\langle \mathcal{L}_k, \mathcal{L}_k \rangle} = 1/\sqrt{2k+1}$. The boundary values are given by :

$$\mathcal{L}_k(0) = (-1)^k, \quad \mathcal{L}_k(1) = 1. \quad (8)$$

Moreover, the derivative of those polynomials is given by

$$\mathcal{L}'_k(x) = \begin{cases} 0, & k = 0, \\ \sum_{j=0}^{k-1} \ell_{kj} \mathcal{L}_j(x), & k \geq 1. \end{cases} \quad (9)$$

with

$$\ell_{kj} = \begin{cases} (2j+1)(1 - (-1)^{k+j}), & \text{if } j \leq k-1, \\ 0, & \text{if } j \geq k. \end{cases} \quad (10)$$

The definition of the polynomials and the proof of these properties are omitted because they are not relevant to the paper. However one can find details about Legendre polynomials, for instance, in [7].

3.3 Truncated state

In this section, we provide an extensive study of the vector $Z_N(t)$ given in (7) and constructed for a prescribed integer N . We first note that the $m(N+1)$ components of $Z_N(t)$ are the projection of the m components of the state $z(x, t) = (z_i(x, t))_{i=1..m}$ of the transport PDE over the $N+1$ first Legendre polynomials $\{\mathcal{L}_k\}_{k \in \{0..N\}}$ with respect to the canonical inner product for any functions f, g in $L^2(0, 1; \mathbb{R})$.

In order to apply the Lyapunov theorem, a crucial technical step is the differentiation of $Z_N(t)$ with respect to the time. In order to complete this task, let us define the following notations related to the choice of the order N of the projection :

$$\mathbf{1}_N(\Lambda) = [\Lambda \dots \Lambda]^\top \in \mathbb{R}^{m(N+1), m},$$

$$\mathbf{1}_N^*(\Lambda) = [\Lambda \ -\Lambda \dots \ (-1)^N \Lambda]^\top \in \mathbb{R}^{m(N+1), m}, \quad (11)$$

$$L_N(\Lambda) = [\ell_{kj} \Lambda]_{j, k=0..N} \in \mathbb{R}^{m(N+1), m(N+1)},$$

where ℓ_{kj} are defined in (10).

The following lemma provides an expression of the time derivative of this vector $Z_N(t)$, which will be useful in the sequel.

Lemma 1 Consider $z \in C(\mathbb{R}_+; L^2(0, 1; \mathbb{R}^m))$ satisfying the transport equation in system (1). The time derivative of the vector $Z_N(t)$ is given by :

$$\dot{Z}_N(t) = (\mathbf{1}_N^*(\Lambda) C_2 - \mathbf{1}_N(\Lambda)) z(1, t) + \mathbf{1}_N^*(\Lambda) C_1 X(t) + L_N(\Lambda) Z_N(t), \quad (12)$$

using the notations defined in (10) and (11).

Proof : First, let us compute the time derivative of the projection of the infinite dimensional state $z(x, t)$ over the Legendre polynomial \mathcal{L}_k for any k in \mathbb{N} . Using the transport equation in (1), integration by parts and properties (8) and (9) of the Legendre polynomials, we obtain

the following expression:

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 z(x, t) \mathcal{L}_k(x) dx = \int_0^1 \partial_t z(x, t) \mathcal{L}_k(x) dx \\
& = - \int_0^1 \Lambda \partial_x z(x, t) \mathcal{L}_k(x) dx \\
& = - [\Lambda z(x, t) \mathcal{L}_k(x)]_0^1 + \int_0^1 \Lambda z(x, t) \mathcal{L}'_k(x) dx \\
& = -\Lambda z(1, t) + (-1)^k \Lambda z(0, t) + \int_0^1 \Lambda z(x, t) \mathcal{L}'_k(x) dx. \\
& = -\Lambda z(1, t) + (-1)^k \Lambda z(0, t) \\
& \quad + \sum_{j=0}^{\max[0, k-1]} \ell_{kj} \Lambda \int_0^1 z(x, t) \mathcal{L}_j(x) dx.
\end{aligned}$$

Consequently, using the notations recently introduced and omitting the time variable t , we have

$$\dot{Z}_N = -\mathbb{1}_N(\Lambda)z(1) + \mathbb{1}_N^*(\Lambda)z(0) + L_N(\Lambda)Z_N.$$

The proof is concluded by injecting the boundary condition $z(0, t) = C_1 X(t) + C_2 z(1, t)$ in the previous expression. \square

3.4 Bessel-Legendre inequality

The following lemma gives a Bessel-type inequality that compares an $L^2(0, 1)$ scalar product with the corresponding finite dimensional approximation product.

Lemma 2 *Let $z \in L^2(0, 1; \mathbb{R}^m)$ and $R \in \mathbb{S}_+^m$. The following integral inequality holds for all $N \in \mathbb{N}$:*

$$\int_0^1 z^\top(x) R z(x) dx \geq Z_N^\top R^N Z_N, \quad (13)$$

with

$$R^N = \text{diag}(R, 3R, \dots, (2N+1)R), \quad (14)$$

Proof: It relies on the orthogonality of the Legendre polynomials and on the Bessel inequality, see e.g. [26]. More precisely, the proof of this lemma results from the positive definiteness and the expansion of

$$\int_0^1 z_N^\top(x) R z_N(x) dx,$$

where

$$z_N(x) = z(x) - \sum_{k=0}^N \frac{\mathcal{L}_k(x)}{\|\mathcal{L}_k\|^2} \int_0^1 z(y) \mathcal{L}_k(y) dy$$

is the approximation error between the state z and its projection Z_N over the N first Legendre polynomials. \square

Remark 2 *Considering the Bessel-Legendre inequality with $N = 0$ and $N = 1$ leads to the particular cases of the Jensen Inequality and the Wirtinger-based inequality [24]. Moreover, when N tends to infinity, the inequality becomes an equality reflecting the well known Parseval Identity.*

It is worth noting that $\sum_{k=0}^N \frac{\mathcal{L}_k(x)}{\|\mathcal{L}_k\|^2} \int_0^1 z(y) \mathcal{L}_k(y) dy$

is the unique polynomial that minimizes the distance between z and the set of polynomials of degree less than N . This property results from the orthogonality of the Legendre polynomials. This also proves that inequality (13) is optimal.

In addition, inequality (13) can be interpreted as follows. The L^2 -norm of an element of $L^2(0, 1; \mathbb{R}^n)$ is greater than the sum of the norm of its projections over the normalized version of the orthogonal sequence of Legendre polynomial. In that sense, this inequality refers indeed to the Bessel inequality on Hilbert spaces.

To simplify the computation hereafter and the notations, we divide the Lyapunov functional (6) into three terms:

$$\begin{aligned}
V_{N,1}(t) &= \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ * & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}, \\
V_{N,2}(t) &= \int_0^1 z^\top(x, t) S z(x, t) dx, \\
V_{N,3}(t) &= \int_0^1 \int_0^x z^\top(y, t) R z(y, t) dy dx,
\end{aligned} \quad (15)$$

so that $V_N(X(t), z(t)) = V_{N,1}(t) + V_{N,2}(t) + V_{N,3}(t)$.

4 Main results and proofs

4.1 Exponential stability

We provide here a stability result for System (1), whose proof is based on the proposed Lyapunov functional (6) and the use of Property 1 and Lemma 2.

We first define the following set of matrices commuting with the transport speed matrix Λ in \mathbb{D}_+^m as follows

$$\mathcal{M}_\Lambda^m := \{M \in \mathbb{S}_+^m, M\Lambda = \Lambda M\} \quad (16)$$

Remark 3 *The matrix M belongs to the set \mathcal{M}_Λ^m if and only if $M \in \mathbb{S}_+^m$ is block diagonal and has the same Jordan structure as Λ in (2):*

$$M = \text{diag}(M_i)_{i=1\dots p} \text{ with } M_i \in \mathbb{S}_+^{m_i}.$$

We take $S, R \in \mathcal{M}_\Lambda^m$ and define the following $\mathbb{R}^{m(N+1), m(N+1)}$ matrices

$$\begin{aligned} R^N(\Lambda) &= \text{diag}(\Lambda R, 3\Lambda R, \dots, (2N+1)\Lambda R), \\ S^N &= \text{diag}(S, 3S, \dots, (2N+1)S), \\ \mathcal{I}^N &= \text{diag}(I_m, 3I_m, \dots, (2N+1)I_m). \end{aligned} \quad (17)$$

We recall that the matrices $L_N(\Lambda)$, $\mathbf{1}_N(\Lambda)$ and $\mathbf{1}_N^*(\Lambda)$ are defined in (11), and the matrix R^N is given by (14).

Theorem 1 *Consider System (1) with a given transport speed matrix $\Lambda \succ 0$. If there exists an integer $N > 0$ such that there exists*

- $\delta > 0$,
- $P \in \mathbb{S}_+^n$, $Q_N \in \mathbb{R}^{n, (N+1)m}$ and $T \in \mathbb{S}^{(N+1)m}$,
- S and $R \in \mathcal{M}_\Lambda^m$,

satisfying the following LMIs

$$\Phi_N = \begin{bmatrix} P & Q_N \\ * & T_N + S^N \end{bmatrix} \succ 0, \quad (18)$$

$$\Psi_N(\Lambda, \delta) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} \prec 0, \quad (19)$$

$$\Lambda R - 2\delta(S + R) \succ 0, \quad (20)$$

where

$$\begin{aligned} \Psi_{11} &= \text{He}(PA + Q_N \mathbf{1}_N^*(\Lambda) C_1) + C_1^\top \Lambda (R + S) C_1 + 2\delta P, \\ \Psi_{12} &= PB + Q_N (\mathbf{1}_N^*(\Lambda) C_2 - \mathbf{1}_N(\Lambda)) + C_1^\top \Lambda (R + S) C_2, \\ \Psi_{13} &= A^\top Q_N + C_1^\top \mathbf{1}_N^{*\top}(\Lambda) T_N + Q_N L_N(\Lambda) + 2\delta Q_N, \\ \Psi_{22} &= -\Lambda S + C_2^\top \Lambda (R + S) C_2, \\ \Psi_{23} &= B^\top Q_N + (\mathbf{1}_N^*(\Lambda) C_2 - \mathbf{1}_N(\Lambda))^\top T_N, \\ \Psi_{33} &= \text{He}(T_N L_N(\Lambda)) - R^N(\Lambda) + 2\delta(T_N + S^N + R^N), \end{aligned}$$

then, system (1) is exponentially stable.

Indeed, for a given transport speed matrix $\Lambda \in \mathbb{D}_+^m$, there exists a constant $K > 0$ and a guaranteed decay rate $\delta^* > \delta$ such that the energy of the system verifies, $\forall t > 0$,

$$E(t) \leq K e^{-2\delta^* t} \left(|z^0(0)|_m^2 + \|z^0\|_{L^2(0,1;\mathbb{R}^m)}^2 \right). \quad (21)$$

Proof : Our objective is to show that the Lyapunov functional V_N given in (6) verifies the inequalities

$$\varepsilon_1 E(t) \leq V_N(t) \leq \varepsilon_2 E(t), \quad (22)$$

$$\dot{V}_N(t) + 2\delta V_N(t) \leq -\varepsilon_3 E(t), \quad (23)$$

for some positive scalars $\varepsilon_1, \varepsilon_2$ and ε_3 . Therefore, the proof will successively relate the existence of each ε_i to one of the LMI given in Theorem 1.

Exponential stability: By proving that the Lyapunov functional verifies inequalities (22) and (23), we prove the exponential stability of system (1), since we get easily

$$\dot{V}_N(t) + (2\delta + \frac{\varepsilon_3}{\varepsilon_2}) V_N(t) \leq 0.$$

Thus, integrating on the interval $[0, t]$ and using $2\delta^* = 2\delta + \frac{\varepsilon_3}{\varepsilon_2}$, we obtain

$$V_N(t) \leq V_N(0) e^{-2\delta^* t} \quad \forall t > 0.$$

Using (22) once again, we get

$$\varepsilon_1 E(t) \leq V_N(t) \leq V_N(0) e^{-2\delta^* t} \leq \varepsilon_2 E(0) e^{-2\delta^* t},$$

which corresponds to (21) and ends the proof of Theorem 1, provided that inequalities (22) and (23) are satisfied.

Existence of ε_1 : One the one hand, since $S \succ 0$ and $\Phi_N \succ 0$, there exists a sufficiently small $\varepsilon_1 > 0$ such that

$$S \succ \varepsilon_1 I_m, \quad \Phi_N = \begin{bmatrix} P & Q_N \\ * & T_N + S^N \end{bmatrix} \succ \varepsilon_1 \begin{bmatrix} I_n & 0 \\ * & \mathcal{I}^N \end{bmatrix}.$$

On the other hand, the matrix R being positive definite, V_N satisfies, $\forall t \geq 0$,

$$\begin{aligned} V_N(t) &\geq \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \Phi_N \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} - Z_N^\top(t) S^N Z_N(t) \\ &\quad + \int_0^1 z^\top(x, t) S z(x, t) dx. \end{aligned}$$

Replacing Φ_N by its lower bound depending on ε_1 and introducing ε_1 in the last integral term, we have

$$\begin{aligned} V_N(t) &\geq \varepsilon_1 |X(t)|_n^2 - Z_N^\top(t) (S^N - \varepsilon_1 \mathcal{I}^N) Z_N(t) \\ &\quad + \int_0^1 z^\top(x, t) (S - \varepsilon_1 I_m) z(x, t) dx \\ &\quad + \varepsilon_1 \int_0^1 z^\top(x, t) z(x, t) dx. \end{aligned}$$

By noting that $S - \varepsilon_1 I_m \succ 0$, Lemma 2 ensures

$$\begin{aligned} \int_0^1 z^\top(x, t) (S - \varepsilon_1 I_m) z(x, t) dx \\ \geq Z_N^\top(t) (S^N - \varepsilon_1 \mathcal{I}^N) Z_N(t). \end{aligned}$$

We thus obtain a lower bound of $V_N(t)$ depending on the energy function $E(t)$:

$$V_N(t) \geq \varepsilon_1 |X(t)|_n^2 + \varepsilon_1 \|z(t)\|_{L^2(0,1;\mathbb{R}^m)}^2 = \varepsilon_1 E(t).$$

Existence of ε_2 : There exists a sufficiently large scalar $\beta > 0$ such that

$$\begin{bmatrix} P & Q_N \\ Q_N^\top & T_N \end{bmatrix} \preceq \beta \begin{bmatrix} I_n & 0 \\ * & \mathcal{I}^N \end{bmatrix},$$

yielding, under the assumptions $S \succ 0$ and $R \succ 0$, and after an integration by parts, that

$$\begin{aligned} V_N(t) &\leq \beta |X(t)|_n^2 + \beta Z_N^\top(t) \mathcal{I}^N Z_N(t) \\ &\quad + \int_0^1 z^\top(x, t) (S + (1-x)R) z(x, t) dx \\ &\leq \beta |X(t)|_n^2 + \beta Z_N^\top(t) \mathcal{I}^N Z_N(t) \\ &\quad + \int_0^1 z^\top(x, t) (S + R) z(x, t) dx. \end{aligned}$$

Applying Lemma 2 to the second term of the right-hand side gives

$$\begin{aligned} V_N(t) &\leq \beta |X(t)|_n^2 + \int_0^1 z^\top(x, t) (\beta I_m + S + R) z(x, t) dx \\ &\leq \beta |X(t)|_n^2 + \varepsilon_2 \|z\|_{L^2(0,1;\mathbb{R}^m)}^2 \leq \varepsilon_2 E(t), \end{aligned}$$

where $\varepsilon_2 = \beta + \lambda_{\max}(S) + \lambda_{\max}(R)$. Therefore, the proof of (22) is complete.

Existence of ε_3 : We define here an augmented approximate state vector, of size $n + (N + 2)m$ given by

$$\xi_N(t) = \begin{bmatrix} X^\top(t) & z^\top(1, t) & Z_N^\top(t) \end{bmatrix}^\top.$$

We can compute the derivative of $(X(t), Z_N(t))$, using the first equation in system (1) and the new formulation of Lemma 1, as follows

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} &= \begin{bmatrix} A \\ \mathbf{1}_N^*(\Lambda) C_1 \end{bmatrix} X(t) \\ &\quad + \begin{bmatrix} B \\ \mathbf{1}_N^*(\Lambda) C_2 - \mathbf{1}_N(\Lambda) \end{bmatrix} z(1, t) + \begin{bmatrix} 0 \\ L_N(\Lambda) \end{bmatrix} Z_N(t). \end{aligned}$$

Then, using notations (15), we can calculate $\dot{V}_{N,1}$:

$$\begin{aligned} \dot{V}_{N,1}(t) &= \frac{d}{dt} \left(\begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q_N \\ Q_N^\top & T_N \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \right) \\ &= \xi_N^\top(t) \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \\ * & 0 & \Psi_{23} \\ * & * & \text{He}(T_N L_N(\Lambda)) \end{bmatrix} \xi_N(t), \end{aligned} \quad (24)$$

$$\begin{aligned} \psi_1 &= \text{He}(PA + Q_N \mathbf{1}_N^*(\Lambda) C_1), \\ \text{with } \psi_2 &= PB + Q_N (\mathbf{1}_N^*(\Lambda) C_2 - \mathbf{1}_N(\Lambda)), \\ \psi_3 &= A^\top Q_N + C_1^\top \mathbf{1}_N^*(\Lambda) T + Q_N L_N(\Lambda). \end{aligned}$$

Now, using the transport equation in (1) and since $S \in \mathcal{M}_\Lambda^m$, we have

$$\begin{aligned} \dot{V}_{N,2}(t) &= \int_0^1 \partial_t z^\top(x, t) S z(x, t) + z^\top(x, t) S \partial_t z(x, t) dx \\ &= - \int_0^1 \partial_x z^\top(x, t) \Lambda S z(x, t) + z^\top(x, t) S \Lambda \partial_x z(x, t) dx. \\ &= - \int_0^1 \partial_x (z^\top(x, t) \Lambda S z(x, t)) dx \\ &= (C_1 X(t) + C_2 z(1, t))^\top \Lambda S (C_1 X(t) + C_2 z(1, t)) \\ &\quad - z(1, t)^\top \Lambda S z(1, t) \\ &= \xi_N^\top(t) \begin{bmatrix} C_1^\top \Lambda S C_1 & C_1^\top \Lambda S C_2 & 0 \\ * & -\Lambda S + C_2^\top \Lambda S C_2 & 0 \\ * & * & 0 \end{bmatrix} \xi_N(t). \end{aligned}$$

Note that the assumption $S \in \mathcal{M}_\Lambda^m$ implies $\Lambda S = S \Lambda \in \mathbb{S}_+^m$ which is crucial to derive this expression. Using the same arguments, since $R \in \mathcal{M}_\Lambda^m$, we have

$$\begin{aligned} \dot{V}_{N,3}(t) &= \int_0^1 \int_0^x \partial_t (z^\top(y, t) R z(y, t)) dy dx \\ &= \int_0^1 \partial_t z^\top(x, t) R z(x, t) + z^\top(x, t) R \partial_t z(x, t) dx \\ &= - \int_0^1 \partial_x z^\top(x, t) \Lambda R z(x, t) + z^\top(x, t) R \Lambda \partial_x z(x, t) dx, \\ &= - \int_0^1 \int_0^x \partial_y (z^\top(y, t) \Lambda R z(y, t)) dy dx \\ &= (C_1 X(t) + C_2 z(1, t))^\top \Lambda R (C_1 X(t) + C_2 z(1, t)) \\ &\quad - \int_0^1 z^\top(x, t) \Lambda R z(x, t) dx. \end{aligned}$$

Merging the expressions of $\dot{V}_{N,1}$, $\dot{V}_{N,2}$ and $\dot{V}_{N,3}$, we obtain

$$\begin{aligned} \dot{V}_N(t) &= \xi_N^\top \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \\ * & 0 & \Psi_{23} \\ * & * & \text{He}(T_N L_N(\Lambda)) \end{bmatrix} \xi_N \\ &+ \xi_N^\top \begin{bmatrix} C_1^\top \Lambda(S+R)C_1 & C_1^\top \Lambda(S+R)C_2 & 0 \\ * & -\Lambda S + C_2^\top \Lambda(S+R)C_2 & 0 \\ * & * & 0 \end{bmatrix} \xi_N \\ &\quad - \int_0^1 z^\top(x,t) \Lambda R z(x,t) dx. \end{aligned}$$

Using the definition of the matrix $\Psi_N(\Lambda, \delta)$ in (19), the following estimate of $\dot{V}_N + 2\delta V_N$ can be obtained:

$$\begin{aligned} \dot{V}_N(t) + 2\delta V_N(t) &\leq \xi_N^\top(t) \Psi_N(\Lambda, \delta) \xi_N(t) \\ &\quad - \int_0^1 z^\top(x,t) (\Lambda R - 2\delta(S+R)) z(x,t) dx \\ &\quad + Z_N^\top(t) [R^N(\Lambda) - 2\delta(S^N + R^N)] Z_N(t), \end{aligned} \quad (25)$$

where we have also used the fact that

$$\begin{aligned} \int_0^1 \int_0^x z^\top(y,t) R z(y,t) dy dx &= \int_0^1 (1-x) z^\top(x,t) R z(x,t) dx \\ &\leq \int_0^1 z^\top(x,t) R z(x,t) dx. \end{aligned}$$

Following the same procedure as for the existence of ε_1 , the LMIs (19) and (20) ensure that there exists a sufficiently small $\varepsilon_3 > 0$ such that

$$\Lambda R - 2\delta(S+R) \succ \varepsilon_3 I_m,$$

and $\Psi_N(\Lambda, \delta) \prec -\varepsilon_3 \begin{bmatrix} I_n & 0 & 0 \\ * & 0 & 0 \\ * & * & \mathcal{I}^N \end{bmatrix}$. Hence, using these

two LMIs in estimate (25), and using also $R \succ 0$ to get rid of the last term, one can obtain

$$\begin{aligned} \dot{V}_N(t) + 2\delta V_N(t) &\leq -\varepsilon_3 |X(t)|_n^2 - \varepsilon_3 \int_0^1 |z(x,t)|^2 dx \\ &\quad - \int_0^1 z^\top(x,t) (\Lambda R - 2\delta(S+R) - \varepsilon_3 I_m) z(x,t) dx \\ &\quad + Z_N^\top(t) [R^N(\Lambda) - 2\delta(S^N + R^N) - \varepsilon_3 \mathcal{I}^N] Z_N(t). \end{aligned}$$

Since $\Lambda R - 2\delta(S+R) - \varepsilon_3 I_m \in \mathbb{S}_+^m$, Lemma 2 can be applied and gives

$$\begin{aligned} & - \int_0^1 z^\top(x,t) (\Lambda R - 2\delta(S+R) - \varepsilon_3 I_m) z(x,t) dx \\ & \leq -Z_N^\top(t) [R^N(\Lambda) - 2\delta(S^N + R^N) - \varepsilon_3 \mathcal{I}^N] Z_N(t), \end{aligned}$$

so that the Lyapunov functional V_N satisfies, for all $t > 0$:

$$\dot{V}_N(t) + 2\delta V_N(t) \leq -\varepsilon_3 E(t).$$

One can therefore conclude on the exponential stability of system (1) with respect to the norm $E(t)$. \square

Remark 4 *The general results proved here include the particular case presented in [3] that involves only one transport speed ($\Lambda = \rho I_m$) for all components of the infinite dimensional transport variable $z(x,t)$. These general results allow the stability study, not only for systems with one constant delay as in [3] and [25] leading the same stability study, but also for many other types of time-delay system, as systems with Cross-Talking delays [20] or systems with commensurate delays [30].*

4.2 Hierarchy of LMI conditions

Following the previous studies on delay systems with Bessel-Legendre inequality [26], the stability conditions of Theorem 1 form a hierarchy of LMI conditions. This is formulated in the following theorem.

Theorem 2 *Considering the coupled system (1), and using the notations introduced for Theorem 1, define, for a given $\delta > 0$, the set $\mathcal{P}_N(\delta) \subset \mathbb{D}_+^m$ by*

$$\mathcal{P}_N(\delta) := \left\{ \begin{array}{l} \Lambda \in \mathbb{D}_+^m, \text{ such that } \Phi_N \succ 0, \\ \Psi_N(\Lambda, \delta) \prec 0, \Lambda R - 2\delta(S+R) \succ 0 \\ \text{for } P \in \mathbb{S}_+^n, S, R \in \mathcal{M}_\Lambda^m \\ T_N \in \mathbb{S}^{(N+1)m}, Q_N \in \mathbb{R}^{n, (N+1)m} \end{array} \right\}.$$

Therefore, the following inclusions hold,

- (i) $\forall (N, N') \in \mathbb{N}^2, \forall \delta \in \mathbb{R},$
 $N < N' \Rightarrow \mathcal{P}_N(\delta) \subset \mathcal{P}_{N'}(\delta)$
- (ii) $\forall N \in \mathbb{N}, \forall (\delta, \delta') \in \mathbb{R}^2,$
 $\delta' < \delta \Rightarrow \mathcal{P}_N(\delta) \subset \mathcal{P}_N(\delta').$

The set $\mathcal{P}_N(\delta)$, for a given positive scalar δ , represent the set of transport speed matrices Λ that are proven to give an exponentially stable system (1) with decay rate δ , according to the conditions of Theorem 1 at the order N . The two inclusions stated here means that (i) increasing N , or (ii) decreasing δ in the conditions of Theorem 1 can only enlarge the set of allowable transport speed matrices Λ .

Proof : Let us start with assertion (i). Let us consider a given $\delta > 0$ and two integers N and N' such that $N < N'$. Without loss of generality, assume that $N' = N + 1$. If $\mathcal{P}_N(\delta)$, denoted here \mathcal{P}_N , is empty, the inclusion is easily obtained. If \mathcal{P}_N is not empty, then for a

given transport speed matrix $\Lambda \in \mathcal{P}_N$, we have from the definition of \mathcal{P}_N the LMIs $\Phi_N \succ 0$, $\Psi_N(\Lambda, \delta) \prec 0$ and $\Lambda R - 2\delta(S + R) \succ 0$ satisfied for $\delta \geq 0$, $P \in \mathbb{S}_+^n$, $T_N \in \mathbb{S}_+^{(N+1)m}$, $Q_N \in \mathbb{R}_+^{n, (N+1)m}$, S and $R \in \mathcal{M}_\Lambda^m$.

First of all, we select the following matrices

$$Q_{N+1} = \begin{bmatrix} Q_N & 0_{n,m} \end{bmatrix} \in \mathbb{R}^{n, m(N+2)},$$

$$T_{N+1} = \begin{bmatrix} T_N & 0_{m, m(N+1)} \\ * & 0_{m,m} \end{bmatrix} \in \mathbb{R}^{m(N+2), m(N+2)},$$

and we keep the same matrices S and R . Obviously, the linear matrix inequality $\Lambda R - 2\delta(S + R) \succ 0$ still hold since it is required by the definition of $\mathcal{P}_N(\delta)$.

Let us now express the matrices Φ_{N+1} and $\Psi_{N+1}(\Lambda, \delta)$ for this selection of Q_{N+1} and T_{N+1} . Now, we note that the matrix Φ_{N+1} can be written as

$$\Phi_{N+1} = \begin{bmatrix} \Phi_N & 0_{n+m(N+1), m} \\ * & (2N+3)S \end{bmatrix}.$$

Since $S \in \mathcal{M}_\Lambda^m \subset \mathbb{S}_+^m$ and $\Phi_N \succ 0$ by assumption, then $\Phi_{N+1} \succ 0$ also holds. For the remaining LMI to be proven, we first note that the matrices $L_{N+1}(\Lambda)$, S_{N+1} and R_{N+1} can be written as

$$L_{N+1}(\Lambda) = \begin{bmatrix} L_N(\Lambda) & 0_{m(N+1), m} \\ [\ell_{N+1,0}\Lambda, \dots, \ell_{N+1,N}\Lambda] & 0_{m,m} \end{bmatrix},$$

$$S^{N+1} = \begin{bmatrix} S^N & 0_{m(N+1), m} \\ * & (2N+3)S \end{bmatrix}, R^{N+1} = \begin{bmatrix} R^N & 0_{m(N+1), m} \\ * & (2N+3)R \end{bmatrix}.$$

From these expressions, the matrix $\Psi_{N+1}(\Lambda, \delta)$ that depends on $L_{N+1}(\Lambda)$, S^{N+1} and R^{N+1} can also be written as

$$\Psi_{N+1}(\Lambda, \delta) = \begin{bmatrix} \Psi_N(\Lambda, \delta) & 0_{n+m(N+2), m} \\ * & -(2N+3)(\Lambda R - 2\delta(S + R)) \end{bmatrix}.$$

Since $\Psi_N(\Lambda, \delta) \prec 0$ and $\Lambda R - 2\delta(S + R) \succ 0$, we get $\Psi_{N+1}(\Lambda, \delta) \prec 0$, and we conclude that $\mathcal{P}_N \subset \mathcal{P}_{N+1}$. Finally, for any $N' > N$, the inclusion $\mathcal{P}_N \subset \mathcal{P}_{N'}$ is obtained by a recursive reasoning.

Let us now prove assertion (ii). Assume that for a given $N \in \mathbb{N}$ and two decay rates $0 < \delta' < \delta$, the set $\mathcal{P}_N(\delta)$ is not empty. For any transport speed matrix $\Lambda \in \mathcal{P}_N(\delta)$ the LMIs $\Phi_N \succ 0$, $\Psi_N(\Lambda, \delta) \prec 0$ and $\Lambda R - 2\delta(S + R) \succ 0$ hold. The first LMI is independent of δ and the third one, since $\delta' < \delta$ and $S, R \in \mathbb{S}_+^m$, implies easily

$$\Lambda R - 2\delta'(S + R) \succ 0$$

Finally from the definition of $\Psi_N(\Lambda, \delta)$ we have

$$\Psi_N(\Lambda, \delta') = \Psi_N(\Lambda, \delta) - 2(\delta - \delta') \begin{bmatrix} P & 0_{n,m} & Q_N \\ * & 0_{m,m} & 0_{m, (N+1)m} \\ * & * & T_N + S^N + R^N \end{bmatrix}.$$

Then, $\Psi_N(\Lambda, \delta) \prec 0$ and $S, R \in \mathbb{S}_+^m$ together with $\delta' < \delta$ allow to conclude $\Psi_N(\Lambda, \delta') \prec 0$, that ends the proof. \square

5 Numerical examples

Since there is a wide literature in TDS, we will take advantage of the numerous example it offers and we will present here some of them. In the sequel, Systems (1) will be translated as a TDS model each time. We will make very clear what are the parameters (Λ , A , B , C_1 and C_2) we use in our representation but switch to the TDS formulation at once.

5.1 Example 1

In this example, we will consider the first particular case of a single transport speed in system (1), *i.e.* $\Lambda = \rho I_m$, and a matrix $C_2 = 0_{m,m}$. The translation of this situation in a TDS gives a constant delay $h = 1/\rho$ in equation

$$\dot{X}(t) = AX(t) + BC_1X(t-h)$$

and simple examples of this type of TDS were studied in [3] using the same approach based on the coupled transport-ODEs system we have here. Now, we are willing to consider the following controlled system which is extracted from the dynamics of machining chatter [27, 34] and given by

$$\begin{cases} \dot{X}(t) = A'X(t) + B'u(t) \\ y(t) = C'x(t). \end{cases} \quad \text{with}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, B' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top.$$

A delayed static output feedback controller is proposed:

$$u(t) = -Ky(t) + Ky(t-h),$$

where K is the gain of the controller and ($h = 1/\rho$) is an unknown constant delay. The resulting dynamics is

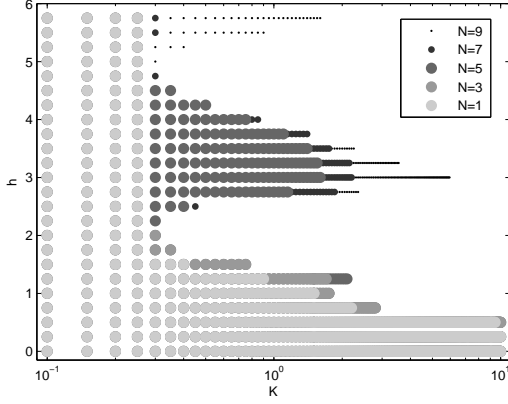


Fig. 1. Stability regions in the plane (K, h) obtained by Theorem 1 for several values of N with $\delta = 0$, in example 1.

thus modeled by our system (1) with $A = A' - B'KC'$, $B = B'KC'$, $C_1 = I_4$ and $C_2 = 0_{4,4}$.

This example illustrates Theorem 2 which proves that the stability conditions form a double hierarchy: with respect to the order N of the approximation of the infinite dimensional state and also to the decay rate δ of the energy.

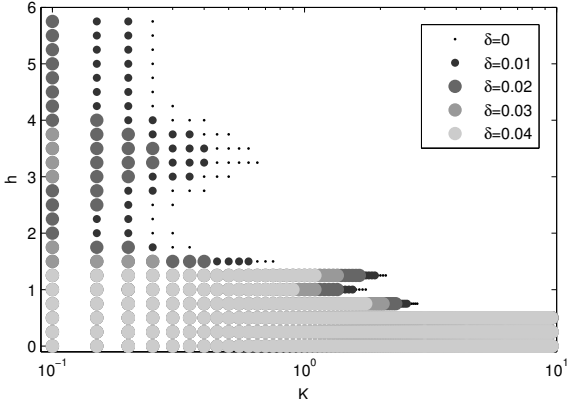


Fig. 2. Stability regions in the plane (K, h) obtained by Theorem 1 for several values of δ with $N = 4$, in example 1.

Figure 1 and Figure 2 show the stability regions in the (K, h) plane. On the one hand, we note that the stability of the system is independent of the delay term (for $K \leq 0.3$) at the order $N = 0$ which corresponds to the use of Jensen's inequality for the stability conditions. We find the same results as [24] at the order $N = 1$ which uses Wirtinger's inequality. Furthermore, we remark in Figure 1 that the system remains stable, with a given transport speed ρ , for all the orders greater than N if it is for N , for the same transport speed ρ . Thus, we have $\mathcal{P}_1 \subset \mathcal{P}_3 \subset \dots \subset \mathcal{P}_9$. On the other hand, Figure 2 shows the hierarchy of the stability conditions compared to the decay rate δ of the energy, i.e at the order N and for a given transport speed ρ , if the system is stable for the

maximal allowable decay rate δ_{max} , it remains stable for all $\delta < \delta_{max}$ at the same order N and for the same transport speed ρ , and we have $\mathcal{P}_N(\delta_{0.04}) \subset \mathcal{P}_N(\delta_{0.03}) \subset \dots \subset \mathcal{P}_N(\delta_0)$.

The stability results in Figure 1 and Figure 2 were found using the steps $\Delta_h = 0.25$ for the delay h and $\Delta_K = 0.05$ for the gain K of the controller.

5.2 Example 2

In this example, we deal with a more general case, studying the stability of a system with multiple transport speeds, specifically with $\Lambda = \text{diag}(\rho_1, \rho_2)$. We will take one of the most usual examples in TDS for the corresponding class of systems with multiple constant delays, which was studied in [27]. It uses the following matrices in system (1):

$$A = \begin{bmatrix} -1.3 \end{bmatrix}, B = \begin{bmatrix} -1 & -0.5 \end{bmatrix}, C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

thus, the corresponding TDS state equation is given by

$$\dot{X}(t) = -1.3X(t) - X(t - h_1) - 0.5X(t - h_2),$$

where the delays h_i are given by $1/\rho_i$, for $i = 1, 2$.

For this example, Figure 3 gives the stability regions for different values of h_1 and h_2 . We can remark that increasing N allows us to broaden the stability region of the coupled system.

Figure 3 also illustrates the principles presented in Theorem 2 since we can see that $\mathcal{P}_1 \subset \mathcal{P}_3 \subset \dots \subset \mathcal{P}_9$. Indeed by increasing N , the stability regions obtained through the conditions of Theorem 1 become larger and larger. Finally, knowing that the stability region can be exactly calculated with a frequency domain approach as in [27], one should know that the stability region we derive here for $N = 9$ provides a very accurate approximation. This demonstrates the potential of the methodology proposed in this paper.

5.3 Example 3

Now, we consider a more general case for the coupled system (1) taking an example for which the matrix C_2 is neither null nor diagonal. We have thus an interconnection between the components of the transport state z and we manage to model a situation with cross talking delays in the TDS framework. System (1) is presented by the following matrices :

$$A = \begin{bmatrix} 0 & 1 \\ -20 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & -4 & -1 & -1 & 0 \end{bmatrix},$$

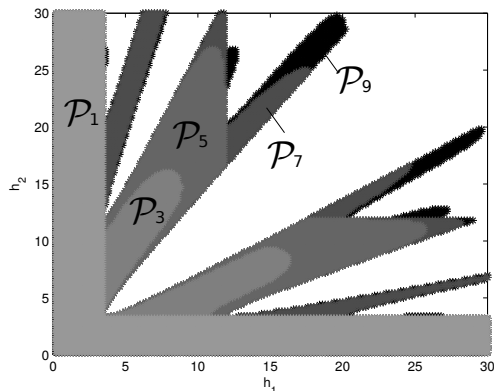


Fig. 3. Stability regions in the plane (h_1, h_2) obtained by Theorem 1 for several values of N , in example 2.

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \Lambda = \begin{bmatrix} \rho_1 I_2 & 0 \\ 0 & \rho_2 I_4 \end{bmatrix}.$$

This system refers to the example studied by [27]. The state equation of this system, under the form of cross-talking delay systems, is given by :

$$\begin{aligned} \dot{X}(t) = & \begin{bmatrix} 0 & 1 \\ -20 & -1 \end{bmatrix} X(t) + \begin{bmatrix} 0 & 0 \\ -3 & -2 \end{bmatrix} X(t - h_1) \\ & + \begin{bmatrix} 0 & 0 \\ -4 & -1 \end{bmatrix} X(t - h_2) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} X(t - h_1 - h_2), \end{aligned}$$

where again the delay h_i is given by $1/\rho_i$, for $i = 1, 2$.

Figure 4 depicts the stability regions obtained by the conditions of Theorem 1 in the plane (h_1, h_2) with $N = 10$. Again, one can check that Theorem 1 is able to provide a good inner approximation of the stability region, which matches with the region obtained through the frequency domain analysis provided in [27]. Moreover, it is worth noting that usual Lyapunov-Krasovskii analysis of such systems with commensurate delays would require to consider three independent delays, i.e. h_1 , h_2 and also $h_1 + h_2$, which misses the link between these three delays. The methodology provided in this paper allows dealing with this class of systems in a direct and generic manner.

6 Conclusion

In this article we give a general presentation of a coupled ODEs-transport PDE systems which allows to study

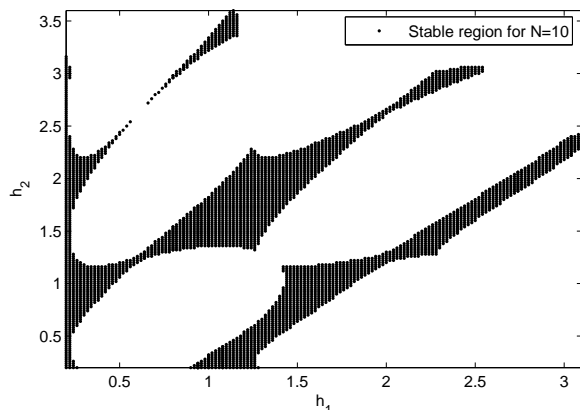


Fig. 4. Stability regions in the plane (h_1, h_2) obtained by Theorem 1 for $N = 10$, in example 3.

many types of TDS, and we provide a new approach for the stability analysis of this kind of systems. The approach consists in a Lyapunov method that gives LMI conditions depending on the transport speed matrix, on the degree of the polynomial approximation which is based on Legendre polynomials and on the guaranteed decay rate δ of the energy of the system. This work give a more general setting for the analysis than the one provided in [3] which takes into account a single transport speed and no cross talking transport states. This paper also generalizes the work proposed in [26], on single constant delay, to the case of multiple and commensurate delays. In addition, we prove that the set of stability conditions forms a hierarchy of LMI indexed by the polynomial degree N , in the sense that increasing N reduces the conservatism of the proposed method.

This stability study can be seen as a milestone for future research on infinite dimensional systems. A first direction of research would be to extend this stability study to uncertain or time-varying speed matrices Λ and give a new approach to evaluate stability of the corresponding kind of time-delay systems. Second, we aim at extending such an analysis to a wider class of PDEs (with or without a coupling to ODEs), including for instance heat equation, wave equation among many other.

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