On the Distinguishing Number of Cyclic Tournaments: Towards Albertson-Collins Conjecture

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August 13, 2016

Abstract

A distinguishing r-labeling of a digraph G is a mapping λ from the set of vertices of G to the set of labels $\{1,\dots,r\}$ such that no nontrivial automorphism of G preserves all the labels. The distinguishing number $D(G)$ of G is then the smallest r for which G admits a distinguishing r-labeling. Albertson and Collins conjectured in 1999 that $D(T) \leq 2$ for every cyclic tournament T of (odd) order $2p+1 \geq 3$, with $V(T) = \{0,\dots,2p\}$, and, more precisely, that the canonical 2-labeling λ^* given by $\lambda^*(i) = 1$ if and only if $i \leq p$ is distinguishing.

We prove that whenever one of the subtournaments of T induced by vertices $\{0,\dots,p\}$ or $\{p+1,\dots,2p\}$ is rigid, T satisfies Albertson-Collins Conjecture. Using this property, we prove that several classes of cyclic tournaments satisfy Albertson-Collins Conjecture. Moreover, we also prove that every Paley tournament satisfies Albertson-Collins Conjecture.

Keywords: Distinguishing number; Automorphism group; Cyclic tournament; Albertson-Collins Conjecture.
MSC 2010: 05C20, 20B25.

1 Introduction

An *r-labeling* of a graph or digraph G is a mapping λ from the set of vertices $V(G)$ of G to the set of labels $\{1,\dots,r\}$. An automorphism ϕ of G is *λ-preserving* if $\lambda(\phi(u)) = \lambda(u)$ for every vertex u of G. An r-labeling λ of G is *distinguishing* if the only λ-preserving automorphism of G is the identity, that is, the labeling λ breaks all the symmetries of G. In [4], Albertson and Collins introduced the *distinguishing number* of G, denoted $D(G)$, defined as the smallest r for which G admits a distinguishing r-labeling.

A digraph G is *rigid* (or *asymmetric*) if the only automorphism of G is the identity. Therefore, $D(G) = 1$ if and only if G is rigid.

In the last decade, distinguishing numbers have been studied for several families of graphs, such as trees [11, 23], interval graphs [12], planar graphs [5], hypercubes [6], Cartesian products of graphs [1, 9, 13, 16, 18] and of complete graphs [14, 15], Kneser graphs [2, 10], or infinite graphs [7, 17, 22] for instance. On the other hand, distinguishing numbers of digraphs have been less frenquently considered since the paper of Albertson and Collins [3] (see [19, 20, 21]).

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We denote by $\mathbb{Z}_n$ the group of residues modulo n. Let p be an integer, $p \geq 1$, and S be a subset of $\mathbb{Z}_{2p+1} \setminus \{0\}$ such that for every $k \in \mathbb{Z}_{2p+1} \setminus \{0\}$, $|S \cap \{k, 2p+1-k\}| = 1$. The *cyclic tournament* $T = T(2p+1; S)$ is the tournament of order $2p+1$ defined by $V(T) = \mathbb{Z}_{2p+1}$ and ij is an arc in T if and only if $j - i \in S$. Cyclic tournaments are sometimes called *circulant* or *rotational* tournaments in the literature.

Albertson and Collins proposed the following conjecture [3]:

Conjecture 1 (Albertson-Collins) *For every cyclic tournament $T = T(2p+1; S)$, $D(T) = 2$. Moreover, the 2-labeling λ^*, given by $\lambda^*(i) = 1$ if $0 \leq i \leq p$ and $\lambda^*(i) = 2$ otherwise, is distinguishing.*

In the rest of this paper, we will call the above defined 2-labeling λ^* the *canonical labeling*.

Symmetry breaking in tournaments has been studied by Lozano [21] who considered other ways of distinguishing vertices, namely by means of determining sets [8] (sometimes called fixing sets) or resolving sets.

In this paper, we study the distinguishing number of cyclic tournaments and prove that several classes of cyclic tournaments satisfy Albertson-Collins conjecture. We first give some definitions, notation and basic results in Section 2. We then consider the so-called pseudo-cyclic tournaments in Section 3 and prove our main results in Section 4. We finally propose some directions for future work in Section 5.

2 Preliminaries

We denote by $V(G)$ and $E(G)$ the set of vertices and the set of arcs of a digraph G, respectively. Let G be a digraph and u a vertex of G. The *outdegree* of u in G, denoted $d_G^+(u)$, is the number of arcs in $E(G)$ of the form uv, and the *indegree* of u in G, denoted $d_G^-(u)$, is the number of arcs in $E(G)$ of the form vu. The *degree* of u, denoted $d_G(u)$, is then defined by $d_G(u) = d_G^+(u) + d_G^-(u)$. If uv is an arc in G, u is an *in-neighbour* of v and v is an *out-neighbour* of u. Let v and w be two neighbours of u. We will say that v and w *agree on* u if either both v and w are in-neighbours of u, or both v and w are out-neighbours of u, and that v and w *disagree on* u otherwise.

An *automorphism* of a digraph G is an arc-preserving permutation of its vertices, that is, a one-to-one mapping $\phi : V(G) \to V(G)$ such that $\phi(u)\phi(v)$ is an arc in G whenever uv is an arc in G. The set of automorphisms of G is denoted $\mathrm{Aut}(G)$. The *order* of an automorphism ϕ is the smallest integer $k > 0$ for which $\phi^k = Id$, where Id denotes the identity. An automorphism ϕ of a digraph G is *nontrivial* if $\phi \neq Id$. A vertex u of G is *fixed* by ϕ if $\phi(u) = u$, and a subdigraph H of G is *fixed* by ϕ if every vertex of H is fixed by ϕ. If u is a vertex of G, the *orbit* of u with respect to ϕ is the set $\{u, \phi(u), \ldots, \phi^{q-1}(u)\}$, where q is the *order of u with respect to ϕ*, that is the smallest integer for which $\phi^q(u) = u$ (note that the order of u necessarily divides the order of ϕ). A tournament cannot admit an automorphism of order 2 (such an automorphism would interchange the ends of some arc) and thus the automorphism group of a tournament has odd order and every orbit of an automorphism of a tournament contains either 1 or at least 3 elements. Note also that if v is fixed by an automorphism ϕ and the orbit of u with respect to ϕ is of size at least 3, then all vertices in the orbit of u agree on v. Moreover, if the orbits of u and u' with respect to ϕ are of size at least 3 (these two orbits being not necessarily distinct), then all vertices in the orbit of u have the same number of in-neighbours (and thus of out-neighbours) in the orbit of u'.

The *transitive tournament* of order n, denoted TT_n, is defined by $V(TT_n) = \{0, \ldots, n-1\}$ and, for every $i, j \in V(TT_n)$, ij is an arc whenever $i < j$. Clearly, every transitive tournament is rigid (all its vertices have distinct indegrees) and thus $D(TT_n) = 1$ for every n. The *almost*

transitive tournament of order n, denoted TT_n^*, is obtained from TT_n by reversing the arc from 0 to $n-1$. The tournament TT_n^* is thus Hamiltonian. It is also known that every almost transitive tournament is rigid, and thus $D(TT_n^*) = 1$ for every n.

Let p be an integer, $p \geq 1$, and S be a subset of $\mathbb{Z}_{2p+1} \setminus \{0\}$ such that for every $k \in \mathbb{Z}_{2p+1} \setminus \{0\}$, $|S \cap \{k, 2p+1-k\}| = 1$. Let

$$S^+ = S \cap \{1, \ldots, p\}, \quad \text{and} \quad S^- = \{1, \ldots, p\} \setminus S^+.$$

We call the elements of S^+ the *positive connectors* of the cyclic tournament $T(2p+1, S)$ and the elements of S^- the *negative connectors* of $T(2p+1, S)$. Note that knowing either S^+ or S^- is enough to determine S since

$$S = S^+ \cup \left\{-s \ : \ s \in \{1, \ldots, p\} \setminus S^+\right\}.$$

Therefore, we will preferably denote the cyclic tournament $T = T(2p+1; S)$ by $T = T(2p+1; S^-)$ whenever we deal with an explicit set S. In that case, for every $i, j \in \mathbb{Z}_{2p+1}$, $i < j$, ij is an arc in $T = T(2p+1; S^-)$ if and only if $j - i \notin S^-$.

Note that for any cyclic tournament $T = T(2p+1; S)$ and every vertex $u \in T$, $d_T^-(u) = d_T^+(u) = p = |S|$.

The *converse* T^c of a tournament T is obtained from T be reversing all the arcs. Clearly, the converse of any cyclic tournament $T = T(2p+1; S)$ is the cyclic tournament $T^c = T(2p+1; S^c)$ with $S^c = \{1, \ldots, 2p\} \setminus S$. Moreover, T and T^c are isomorphic, via the mapping γ defined by $\gamma(0) = 0$ and $\gamma(i) = 2p+1-i$ for every $i \in V(T) \setminus \{0\}$.

3 Pseudo-cyclic tournaments

Let $T = T(2p+1; S)$ be a cyclic tournament and i, j two vertices of T with $i < j$. We denote by $T_{i,j}$ the subtournament of T induced by the set of vertices $\{i, i+1, \ldots, j\}$. Note that the subtournament $T_{i,j}$ is not necessarily a cyclic tournament. Hence, the canonical 2-labeling λ^* defined in Conjecture 1 assigns label 1 to vertices of $T_{0,p}$ and label 2 to vertices of $T_{p+1,2p}$.

Since the tournament $T = T(2p+1; S)$ is cyclic, the subtournaments $T_{p+1,2p}$ and $T_{0,p-1}$ are isomorphic. Recall that the set S is characterized by the set S^- of its negative connectors. In the following, we will thus study (not necessarily cyclic) tournaments of the following form:

Definition 2 Let N be a subset of $\mathbb{Z}_{p+1} \setminus \{0\}$, $p \geq 2$. The *pseudo-cyclic* tournament $P = P(p; N)$ is the tournament of order $p+1$ defined by $V(P) = \mathbb{Z}_{p+1}$ and ij, $i > j$, is an arc in P if and only if $i - j \in N$.

Note that if $T = T(2p+1; S)$ is a cyclic tournament then $T_{0,p} = P(p; S^-)$, and $T_{p+1,2p} = T_{0,p-1} = P(p-1; S^-)$ (if $p \notin S^-$) or $T_{p+1,2p} = T_{0,p-1} = P(p-1; S^- \setminus \{p\})$ (if $p \in S^-$).

We first prove that $T = T(2p+1; S)$ satisfies Albertson-Collins Conjecture whenever $T_{0,p}$ or $T_{p+1,2p}$ is rigid.

Proposition 3 *Let $T = T(2p+1; S)$ be a cyclic tournament. If $T_{0,p}$ is rigid or $T_{p+1,2p}$ is rigid then the canonical 2-labeling λ^* of T is distinguishing.*

Proof. Let ϕ be a λ^*-preserving automorphism of T, that is, $\lambda^* \circ \phi = \lambda^*$, and let ϕ_1 and ϕ_2 denote the restriction of ϕ to $T_{0,p}$ and $T_{p+1,2p}$, respectively. Since $\lambda^* \circ \phi = \lambda^*$, both ϕ_1 and ϕ_2 are automorphisms. Moreover, since $T_{0,p}$ is rigid or $T_{p+1,2p}$ is rigid, we get $\phi_1 = Id$ or $\phi_2 = Id$. We will prove that we necessarily have $\phi_1 = \phi_2 = Id$, which gives $\phi = Id$ so that λ^* is a distinguishing labeling of T.

Suppose first that $\phi_1 = Id$ and assume to the contrary that $\phi_2 \neq Id$. Let $a_1 \in \{p+1, \ldots, 2p\}$ be the "smallest" non-fixed vertex of $T_{p+1,2p}$. Since T is cyclic, we can assume without loss of generality $a_1 = p+1$. (If $a_1 \neq p+1$, by using the "shift" automorphism $\alpha : i \mapsto i+a_1-p-1$, the subtournaments $T_{0,p}$ and $T_{p+1,2p}$ are shifted to T_{a_1-p-1,a_1-1} and T_{a_1,a_1-p-2}, respectively. The two restricted automorphisms ϕ_1 and ϕ_2 become $\phi_1' = \alpha\phi_1\alpha^{-1}$ and $\phi_2' = \alpha\phi_2\alpha^{-1}$, respectively, and we still have $\phi_1' = Id$.) Since the order of a_1 with respect to ϕ_2 is odd, a_1 necessarily belongs to a cycle in $T_{p+1,2p}$ and thus to a 3-cycle in $T_{p+1,2p}$, say $a_1a_2a_3$.

Since $a_1a_2a_3$ is a 3-cycle, we get $\{a_2-a_1, a_3-a_2, a_1-a_3\} \subseteq S$. If $a_2 < a_3$, let $w = a_1-a_3+a_2$, so that wa_1 is an arc of T. We then have $w \in V(T_{0,p})$ and thus, since w is fixed by ϕ, wa_1 and wa_2 are both arcs of T, a contradiction since $a_2 - w = a_3 - a_1$ and $a_3 - a_1 \notin S$. If $a_2 > a_3$, we get a similar contradiction by considering the vertex $w = a_1 + a_3 - a_2$: a_1w is an arc of T, again $w \in V(T_{0,p})$, which implies that a_1w and a_3w are both arcs of T, a contradiction since $w - a_3 = a_1 - a_2 \notin S$.

The case $\phi_2 = Id$ is similar. □

It should be noticed that the condition in Proposition 3 is sufficient for a cyclic tournament to satisfy Albertson-Collins Conjecture but not necessary, as shown by the following example.

Example 4 Consider the cyclic tournament $T = T(13; \{2, 5, 6\})$. The automorphism group of T only contains rotations (that is, mappings $\phi : i \mapsto i + b$, $0 \leq b \leq 12$), so that the canonical 2-labeling λ^* is clearly distinguishing, and thus T satisfies Albertson-Collins Conjecture. However, none of the subtournaments $T_{0,6}$ and $T_{7,12}$ is rigid. The subtournament $T_{0,6} = P(6; \{2, 5, 6\})$ admits an automorphism of order 3, namely $\phi' = (0, 3, 6)$, and the subtournament $T_{7,12} = P(5; \{2, 5\})$ also admits an automorphism of order 3, namely $\phi'' = (0, 1, 2)(3, 4, 5)$.

Moreover, we have the following:

Proposition 5 *Let $T = T(2p+1; S)$ be a cyclic tournament and λ^* be the canonical 2-labeling of T. If ϕ is a nontrivial λ^*-preserving automorphism of T, then neither $T_{0,p}$ nor $T_{p+1,2p}$ has a unique orbit with respect to ϕ.*

Proof. Suppose first that $T_{0,p}$ has a unique orbit with respect to ϕ, which implies that p is even. No vertex a of $T_{p+1,2p}$ can be fixed by ϕ, since otherwise we would have either $d_T^-(a) \geq p+1$ or $d_T^+(a) \geq p+1$, in contradiction with the definition of T. Therefore, there is an even number of orbits in $T_{p+1,2p}$, of respective odd sizes $p_1, \ldots, p_{2\ell}$, $\ell \geq 1$, with $p_i \geq 3$ for every i, $1 \leq i \leq 2\ell$, and $\sum_{i=1}^{2\ell} p_i = p$. Since $p_1 < p$, the automorphism $\phi_1 = \phi^{p_1}$ is λ^*-preserving and $T_{0,p}$ has a unique orbit with respect to ϕ_1. But ϕ_1 fixes p_1 vertices of $T_{p+1,2p}$, contradicting the above remark.

Suppose now that $T_{p+1,2p}$ has a unique orbit, which implies that p is odd. As before, we claim that no vertex of $T_{0,p}$ is fixed by ϕ. Assume to the contrary that a is such a vertex. We then have either $d_T^-(a) = 0$ or $d_T^-(a) = p$, which implies $|S^-| = 0$ or $|S^-| = p$, respectively. In both cases, we get that $T_{0,p}$ is transitive, and thus rigid, which implies $\phi = Id$ by Proposition 3, a contradiction. Therefore, there is an odd number of orbits of size at least 3 in $T_{0,p}$ (this number is at least 3, as proved before), and a contradiction arises as in the previous case. □

Let $P = P(p; N)$ be a pseudo-cyclic tournament. The *indegree sequence* of P is the sequence defined by:

$$\sigma(P) = (d_P^-(0), \ldots, d_P^-(p)).$$

The value of $d_P^-(i)$ for any vertex i of P is given by the following:

Proposition 6 *The indegree $d_P^-(i)$ of any vertex i of the pseudo-cyclic tournament $P = P(p; N)$, $0 \leq i \leq p$, is given by*

$$d_P^-(i) = \begin{cases} i + |N \cap \{i+1, \ldots, p-i\}| & \text{if } 0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor, \\ p - d_P^-(p-i) & \text{otherwise.} \end{cases}$$

Proof. By definition of P, there is an arc from j to i, $j > i$, if and only if $j - i \in N$. Hence,

$$d_P^-(i) = (i - |N \cap \{1, \ldots, i\}|) + |N \cap \{1, \ldots, p-i\}|.$$

Therefore, we get

$$d_P^-(i) = i + |N \cap \{i+1, \ldots, p-i\}|$$

if $0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor$ (with, in particular, $d_P^-(\frac{p}{2}) = \frac{p}{2}$ if p is even), and

$$d_P^-(i) = i - |N \cap \{p-i+1, \ldots, i\}| = p - d_P^-(p-i)$$

if $\left\lceil \frac{p}{2} \right\rceil \leq i \leq p$. □

From Proposition 6, we get that for every pseudo-cyclic tournament $P = P(p; N)$, $d_P^-(0) = |N|$ and $d_P^-(p) = p - |N|$, and that the indegree sequence $\sigma(P)$ admits a central symmetry. Moreover, the difference of the indegrees of any two consecutive vertices is either -1, 0 or 1:

Proposition 7 *For any two consecutive vertices i and $i+1$ of the pseudo-cyclic tournament $P = P(p; N)$, $0 \leq i \leq p-1$, $d_P^-(i+1) - d_P^-(i) \in \{-1, 0, 1\}$. More precisely, we have*

1. $d_P^-(i+1) - d_P^-(i) = 1 - |N \cap \{i+1, p-i\}|$,
2. *if p is odd, $d_P^-\left(\left\lceil \frac{p}{2} \right\rceil\right) - d_P^-\left(\left\lfloor \frac{p}{2} \right\rfloor\right) = 1 - 2\left|N \cap \left\{\left\lfloor \frac{p}{2} \right\rfloor + 1\right\}\right|$.*

Proof. By Proposition 6, if $0 \leq i < \left\lfloor \frac{p}{2} \right\rfloor$ we have

$$\begin{aligned} d_P^-(i+1) - d_P^-(i) &= i + 1 + |N \cap \{i+2, \ldots, p-i-1\}| \\ &\quad - i - |N \cap \{i+1, \ldots, p-i\}| \\ &= 1 - |N \cap \{i+1, p-i\}|. \end{aligned}$$

Since $0 \leq |N \cap \{i+1, p-i\}| \leq 2$, we get $d_P^-(i+1) - d_P^-(i) \in \{-1, 0, 1\}$.

If $\left\lceil \frac{p}{2} \right\rceil \leq i < p$ then, by Proposition 6, we get

$$\begin{aligned} d_P^-(i+1) - d_P^-(i) &= p - d^-(p-i-1) - p + d^-(p-i) \\ &= d^-(p-i) - d^-(p-i-1) \\ &= 1 - |N \cap \{p-i-1+1, p-p+i+1\}| \\ &= 1 - |N \cap \{i+1, p-i\}|. \end{aligned}$$

Finally, if p is odd, we have

$$\begin{aligned} d_P^-\left(\left\lceil \frac{p}{2} \right\rceil\right) - d_P^-\left(\left\lfloor \frac{p}{2} \right\rfloor\right) &= p - \left(\left\lfloor \frac{p}{2} \right\rfloor + \left|N \cap \left\{\left\lfloor \frac{p}{2} \right\rfloor + 1\right\}\right|\right) - \left(\left\lfloor \frac{p}{2} \right\rfloor + \left|N \cap \left\{\left\lfloor \frac{p}{2} \right\rfloor + 1\right\}\right|\right) \\ &= 1 - 2\left|N \cap \left\{\left\lfloor \frac{p}{2} \right\rfloor + 1\right\}\right|. \end{aligned}$$

This completes the proof. □

Note that is p is odd, the value of $d_P^-\left(\left\lceil \frac{p}{2} \right\rceil\right) - d_P^-\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$ is either 1 or -1.

We can then define three types of vertices as follows:

Definition 8 Let $P = P(p; N)$ be a pseudo-cyclic tournament. A vertex $i \in V(P)$, $0 \leq i \leq p-1$, is said to be:

1. an *ascent-vertex* if $d_P^-(i+1) = d_P^-(i) + 1$, that is, $N \cap \{i+1, p-i\} = \emptyset$,
2. a *descent-vertex* if $d_P^-(i+1) = d_P^-(i) - 1$, that is, $\{i+1, p-i\} \subseteq N$, or
3. a *plateau-vertex* if $d_P^-(i+1) = d_P^-(i)$, that is, $|N \cap \{i+1, p-i\}| = 1$.

A pseudo-cyclic tournament $P = P(p; N)$ is cyclic if and only if all its vertices have the same indegree, which implies p is even and $|N| = \frac{p}{2}$, and the set N is such that $i \in N$ if and only $p+1-i \notin N$. In other words:

Observation 9 *A pseudo-cyclic tournament $P = P(p; N)$ is cyclic if and only if every vertex $i \in V(P)$, $0 \leq i \leq p-1$, is a plateau-vertex.*

Moreover, we have the following:

Observation 10 *Let $P^c = (p; N^c)$ be the converse pseudo-cyclic tournament of $P = (p; N)$, that is, $N^c = \{1, \ldots, p\} \setminus N$. If i is an ascent-vertex (resp. a descent-vertex, a plateau-vertex) in P then i is a descent-vertex (resp. an ascent-vertex, a plateau-vertex) in P^c.*

This observation directly follows from the fact that $d_{P^c}^-(i) = p - d_P^-(i)$ for every vertex i.

We will denote by $\alpha(P)$, $\delta(P)$ and $\pi(P)$ the number of ascent-vertices, descent-vertices and plateau-vertices in $\sigma(P)$, respectively. Moreover, according to the three above types of vertices, we define three types of subsequences of the indegree sequence $\sigma(P)$:

Definition 11 Let $P = P(p; N)$ be a pseudo-cyclic tournament. A sequence of $k \geq 2$ consecutive vertices $(i, \ldots, i+k-1)$, $i \leq n-k+1$, of P is called

1. an *ascent of size k* if $d_P^-(i+j+1) = d_P^-(i+j) + 1$ for every j, $0 \leq j \leq k-2$,
2. a *descent of size k* if $d_P^-(i+j+1) = d_P^-(i+j) - 1$ for every j, $0 \leq j \leq k-2$,
3. a *plateau of size k* if $d_P^-(i+j+1) = d_P^-(i+j)$ for every j, $0 \leq j \leq k-2$.

Note here that an ascent, a descent or a plateau of size k contains $k-1$ ascent-vertices, descent-vertices or plateau-vertices, respectively.

Example 12 The pseudo-cyclic tournament $P = P(8; \{2, 4, 5\})$ is depicted on Figure 1(a) (only arcs corresponding to negative connectors are drawn, every missing arc is thus directed from left to right). The indegree sequence of P is given by

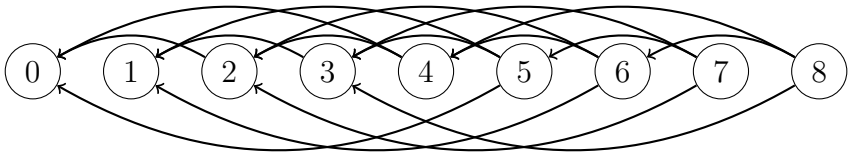
$$\sigma(P) = (3, 4, 4, 5, 4, 3, 4, 4, 5)$$

and is represented by the (centrally symmetric) *indegree path* of P depicted on Figure 1(b). We then have $\pi(P) = 2$, $\delta(P) = 2$ and $\alpha(P) = 4$.

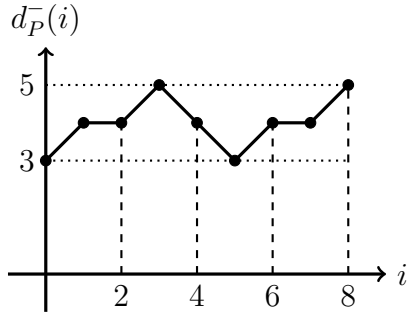
The number of descent-vertices and the number of plateau-vertices are related to the cardinality of the set N as follows:

Proposition 13 *For every pseudo-cyclic tournament $P = P(p; N)$,*

$$\delta(P) + \frac{1}{2}\pi(P) = |N|.$$



(a) The pseudo-tournament $P(8;\{2,4,5\})$



(b) The indegree path of $P(8;\{2,4,5\})$

Figure 1: The pseudo-tournament $P(8;\{2,4,5\})$ and its indegree path

Proof. If $N=\emptyset$ then $\sigma(P)=(0,1,\ldots,p)$ (P is transitive), so that $\delta(P)=\pi(P)=0$ and we are done.

Otherwise, let $N=\{a_1,\ldots,a_q\}$, so that $|N|=q\geq 1$. We claim that each a_i generates either a descent-vertex or two plateau-vertices. Indeed, for each a_i, either $p+1-a_i\notin N$, which implies that both vertices a_i-1 and $p-a_i$ are plateau-vertices, or $p+1-a_i\in N$, which implies that a_i-1 is a descent-vertex. We thus get $|N|=\delta(P)+\frac{1}{2}\pi(P)$, as required. □

The following proposition shows that vertices with same indegree in a pseudo-cyclic tournament are necessarily "not too far" from each other:

Proposition 14 *Let $P=P(p;N)$ be a pseudo-cyclic tournament. If i and j are two vertices of P with $i<j$ and $d_P^-(i)=d_P^-(j)$ then:*

1. *if $0\leq i<j\leq\lfloor\frac{p}{2}\rfloor$ or $\lceil\frac{p}{2}\rceil\leq i<j\leq p$, then $j\leq i+|N|$,*
2. *if $0\leq i\leq\lfloor\frac{p}{2}\rfloor<j\leq p$, then $j\leq i+2|N|$.*

Proof. Let i and j be such that $0\leq i<j\leq\lfloor\frac{p}{2}\rfloor$ and $j>i+|N|$. By Proposition 6, we have

$$\begin{aligned} d_P^-(j)-d_P^-(i) &= j+|N\cap\{j+1,\ldots,p-j\}|)-i-|N\cap\{i+1,\ldots,p-i\}|) \\ &= j-i-|N\cap(\{i+1,j\}\cup\{p-j+1,p-i\})| \\ &> |N|-|N|=0. \end{aligned}$$

Therefore, $d_P^-(i)\neq d_P^-(j)$. Since the indegree path of P is centrally symmetric, the result also holds when $\lceil\frac{p}{2}\rceil\leq i<j\leq p$.

Suppose now that $0\leq i\leq\lfloor\frac{p}{2}\rfloor<j\leq p$ and $j>i+2|N|$. By Proposition 6, we have

$$\begin{aligned} d_P^-(j)-d_P^-(i) &= j-|N\cap\{p-j+1,\ldots,j\}|)-i-|N\cap\{i+1,\ldots,p-i\}|) \\ &= j-i-(|N\cap\{p-j+1,\ldots,j\}|+|N\cap\{i+1,\ldots,p-i\}|) \\ &> 2|N|-(|N|+|N|)=0. \end{aligned}$$

Again, we get $d_P^-(i) \neq d_P^-(j)$. □

Finally, a property that will be useful in the sequel is the following:

Proposition 15 *Let $P = P(p; N)$ be a pseudo-cyclic tournament. If a vertex i is fixed by every automorphism of P, then the vertex $p - i$ is also fixed by every automorphism of P.*

Proof. Suppose to the contrary that i is fixed by every automorphism of P and that there exists an automorphism ϕ of P such that $\phi(p - i) \neq p - i$.

Let $\phi^* : V(P) \longrightarrow V(P)$ be the mapping defined by $\phi^*(i) = p - \phi(p - i)$ for every $i \in V(P)$.

We claim that $\phi^* \in Aut(P)$. Let ij be an arc in P with $i > j$, that is $i - j \in N$. We then have

$$\phi^*(i) - \phi^*(j) = p - \phi(p - i) - p + \phi(p - j) = \phi(p - j) - \phi(p - i).$$

Since $p - j > p - i$ and $p - j - p + i = i - j \in N$, $(p - j)(p - i)$ is an arc in P. Since $\phi \in Aut(P)$, $\phi(p - j)\phi(p - i)$ is an arc in P.

Suppose first that $\phi(p - j) > \phi(p - i)$, which implies $\phi(p - j) - \phi(p - i) \in N$. In that case $\phi^*(j) = p - \phi(p - j) < p - \phi(p - i) = \phi^*(i)$ and, since $\phi^*(i) - \phi^*(j) = \phi(p - j) - \phi(p - i) \in N$, $\phi^*(i)\phi^*(j)$ is an arc in P.

Suppose now that $\phi(p - j) < \phi(p - i)$, which implies $\phi(p - i) - \phi(p - j) \notin N$. In that case $\phi^*(i) = p - \phi(p - i) < p - \phi(p - j) = \phi^*(j)$ and, since $\phi^*(j) - \phi^*(i) = \phi(p - i) - \phi(p - j) \notin N$, $\phi^*(i)\phi^*(j)$ is an arc in P.

The case of an arc ij with $i < j$ is similar, and thus $\phi^* \in Aut(P)$.

We then have $\phi^*(i) = p - \phi(p - i) \neq i$, contradicting the fact that i is fixed by every automorphism of P. □

This immediately gives the following:

Corollary 16 *Let $P = P(p; N)$ be a pseudo-cyclic tournament. If all vertices $i \leq \left\lfloor \frac{p}{2} \right\rfloor$ of P are fixed by every automorphism of P then P is rigid.*

4 Cyclic tournaments satisfying Albertson-Collins Conjecture

In this section, we will prove that several classes of cyclic tournaments $T(2p + 1; S)$ satisfy Albertson-Collins Conjecture. We first propose a few sufficient conditions for a cyclic tournament to satisfy Albertson-Collins Conjecture and then consider several specific classes of cyclic tournaments, depending on the structure of the set S of connectors.

4.1 Simple sufficient conditions

By Proposition 5, for every nontrivial λ^*-preserving automorphism ϕ of $T = T(2p+1; S)$, none of $T_{0,p}$ and $T_{p+1,2p}$ has a unique orbit with respect to ϕ. Using this property, we can prove the following:

Lemma 17 *The cyclic tournament $T = T(2p + 1; S)$ satisfies Albertson-Collins Conjecture whenever one of the following conditions is satisfied:*

1. *$Aut(T)$ is of order $2p + 1$,*
2. *p is even, $|S^-| = \frac{p}{2}$, $p + 1 - s \notin S^-$ for every $s \in S^-$, and $|Aut(T_{0,p})| = p + 1$,*
3. *p is odd, $|S^- \setminus \{p\}| = \frac{p-1}{2}$, $p - s \notin S^-$ for every $s \in S^-$, and $|Aut(T_{p+1,2p})| = p$.*

Proof. Recall that for every cyclic tournament T_n of order n, the n rotations of T_n form a subgroup of $Aut(T_n)$.

In Case 1, $Aut(T)$ contains only rotations, so that λ^* is clearly distinguishing. In Case 2, $T_{0,p}$ is a cyclic tournament and every automorphism of $T_{0,p}$ is a rotation. Therefore, the restriction ϕ_1 to $T_{0,p}$ of any λ^*-preserving automorphism ϕ of T is a rotation and thus $\phi_1 = Id$. The result then follows by Proposition 5. Case 3 is similar, using $T_{p+1,2p}$ instead of $T_{0,p}$. □

Let T be a tournament. We denote by $V_d(T)$, $d \geq 0$, the set of vertices of T with indegree d. Since any automorphism maps every vertex to a vertex with same degree, we have the following:

Lemma 18 *Let $P = P(n; N)$ be a pseudo-cyclic tournament. If the subtournament $P[V_d(P)]$ of P, induced by $V_d(T)$, is rigid for every d, $d \geq 0$, then P is rigid.*

This lemma directly gives the following results:

Theorem 19 *Let $T = T(2p + 1; S)$ be a cyclic tournament, $P_1 = T_{0,p}$ and $P_2 = T_{p+1,2p}$. If $P_1[V_d(P_1)]$ is rigid for every d, or $P_2[V_d(P_2)]$ is rigid for every d, $d \geq 0$, then T satisfies Albertson-Collins Conjecture.*

Proof. The result directly follows from Lemma 18 and Proposition 3. □

Theorem 20 *Let $T = T(2p + 1; S)$ be a cyclic tournament with $S^- \neq \emptyset$ and let $\min(S^-)$ denote the smallest negative connector of S. If $\min(S^-) \geq 2|S^-|$ then T satisfies Albertson-Collins Conjecture.*

Proof. Consider the pseudo-cyclic subtournament $P = T_{0,p}$. By Proposition 14, we know that if two vertices i and j, $i < j$, have the same indegree in P then $j - i \leq 2|S^-|$. Since $\min(S^-) \geq 2|S^-|$, we get that the subtournament $P[V_d(P)]$ is either transitive or almost transitive, and thus rigid, for every d, $d \geq 0$. The result then follows by Theorem 19. □

4.2 Cyclic tournaments with few negative connectors

We consider in this subsection the case of cyclic tournaments with at most two negative connectors. We prove that every such tournament satisfies Albertson-Collins Conjecture.

Theorem 21 *If $T = T(2p + 1; S)$ is a cyclic tournament with $p \geq 2$ and $|S^-| \leq 2$ then T satisfies Albertson-Collins Conjecture.*

Proof. We will prove that one of the two subtournaments $P_1 = T_{0,p}$ or $P_2 = T_{p+1,2p}$ is rigid, so that the result follows by Proposition 3.

If $S^- = \emptyset$ then P_1 is transitive and thus rigid.

If $|S^-| = 1$, say $S^- = \{a\}$, then, by Proposition 6, the indegree of any vertex i of P_1 is as follows:

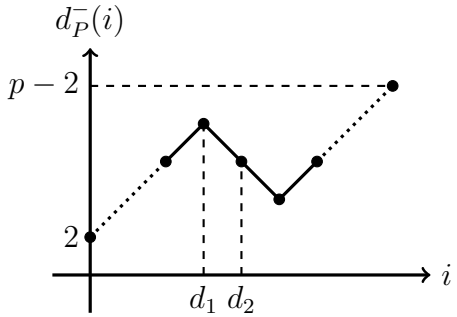
$$d_P^-(i) = \begin{cases} i & \text{if } a \notin \{i+1, \ldots, p-i\} \text{ and } 0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor, \\ i+1 & \text{if } a \in \{i+1, \ldots, p-i\} \text{ and } 0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor, \\ p - d_P^-(p-i) & \text{otherwise.} \end{cases}$$

Therefore, $d^-(i) = d^-(j)$, $i < j \leq \left\lfloor \frac{p}{2} \right\rfloor$, if and only if $j = i + 1$ and $j = a$. If $a \neq \left\lfloor \frac{p}{2} \right\rfloor$ then the number of vertices in P_1 with degree d is at most 2 for every d. In that case, P_1 satisfies the conditions of Lemma 18 and is thus rigid. If $a = \left\lfloor \frac{p}{2} \right\rfloor$ the same property holds except if p is even (in that case, we have $|V_{\frac{p}{2}}(P_1)| = 3$ and $|V_d(P_1)| = 1$ for every $d \neq \frac{p}{2}$). But then, P_2 satisfies the conditions of Lemma 18 (since $p - 1$ is odd) and we are done too.

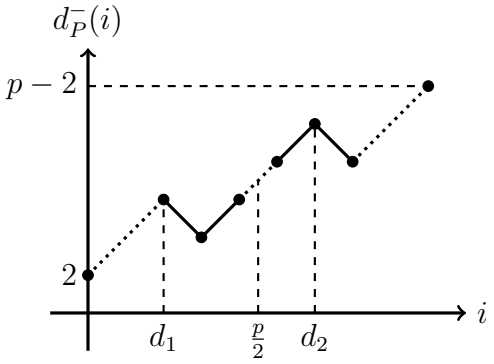
Assume now that $|S^-| = 2$ and let $S^- = \{a_1, a_2\}$. If $p = 2$ then P_2 has order 2 and is thus rigid. If $p = 3$ then either $S^- = \{1, 2\}$, $S^- = \{1, 3\}$ or $S^- = \{2, 3\}$. It is then easily checked that $|V_1(P_1)| = |V_2(P_1)| = 2$ in all cases, so that P_1 satisfies the conditions of Lemma 18 and we are done.

Suppose thus that $p > 3$ and, moreover, that p is even (if p is odd we simply consider P_2 instead of P_1). By Proposition 13, we know that $\delta(P_1) + \frac{1}{2}\pi(P_1) = 2$. We thus have three cases to consider:

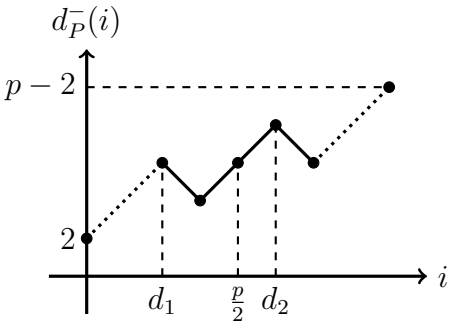
1. $\delta(P_1) = 2$ and $\pi(P_1) = 0$.
   In that case, the sequence $\sigma(P)$ has no plateau and exactly two descent-vertices, say d_1 and d_2. Since $\sigma(P_1)$ is centrally symmetric and p is even, we necessarily have $d_2 = p - d_1 - 1$ which implies in particular that the difference $d_2 - d_1$ is necessarily odd.
   If $d_2 - d_1 = 1$, we necessarily have $d_2 = \frac{p}{2}$ (see Figure 2(a)). Moreover, since $p > 3$, the vertex 0 cannot be a descent-vertex (this would imply $\frac{p}{2} = 1$) and thus $1 \notin S^-$. The only set $V_d(P_1)$ containing more than two elements is thus $V_{\frac{p}{2}}(P_1) = \{\frac{p}{2} - 2, \frac{p}{2}, \frac{p}{2} + 2\}$. But since the vertex $\frac{p}{2} - 1$ is fixed by every automorphism of P_1 and vertices $\frac{p}{2} - 2$ and $\frac{p}{2}$ disagree on $\frac{p}{2} - 1$ (as $1 \notin N$), $\frac{p}{2} - 2$ and $\frac{p}{2}$ are also fixed so that P_1 is rigid by Corollary 16.
   If $d_2 - d_1 > 3$ (see Figure 2(b)), then every set $V_d(P_1)$ contains at most two elements and Lemma 18 again allows us to conclude.
   Suppose finally that $d_2 - d_1 = 3$ (see Figure 2(c)). This case is similar to the case $d_2 - d_1 = 1$: again, the only set $V_d(P_1)$ containing more than two elements is $V_{\frac{p}{2}}(P_1) = \{\frac{p}{2} - 2, \frac{p}{2}, \frac{p}{2} + 2\}$ while the vertex $\frac{p}{2} - 1$ is fixed by every automorphism of P_1 and vertices $\frac{p}{2} - 2$ and $\frac{p}{2}$ disagree on $\frac{p}{2} - 1$. Again, P_1 is rigid thanks to Corollary 16.
2. $\delta(P_1) = 1$ and $\pi(P_1) = 2$.
   Since $\delta(P_1) = 1$, we cannot have two consecutive descent-vertices in P_1. Therefore, since $\sigma(P_1)$ is centrally symmetric, if i is a descent-vertex then $p - (i+1)$ is also a descent-vertex. As $\delta(P_1) = 1$, we must have $i = p - (i + 1)$, in contradiction with our assumption on the parity of p. Hence, this case cannot happen.
3. $\delta(P_1) = 0$ and $\pi(P_1) = 4$.
   Since p is even and $\sigma(P_1)$ is centrally symmetric, we have four possibilities: (a) P_1 contains four plateaus of size 2, (b) P_1 contains two plateaus of size 3, (c) P_1 contains one plateau of size 3 (centered at $\frac{p}{2}$) and two plateaus of size 2, or (d) P_1 contains a unique plateau of size 5 (centered at $\frac{p}{2}$), see Figure 3(a), (b), (c) or (d), respectively.
   (a) If P_1 contains four plateaus of size 2, then P_1 satisfies the conditions of Lemma 18 and we are done.
   (b) If P_1 contains two plateaus of size 3, then the plateau-vertices are necessarily of the form i, $i + 1$, $p - i - 2$ and $p - i - 1$, with $i \leq \frac{p}{2} - 3$, so that $N = \{a_1, a_2\}$, with $a_1 \in \{i + 1, p - i\}$, $a_2 \in \{i + 2, p - i - 1\}$, and $d^-_{P_1}(i) = 2 + i$. We then have $V_{2+i}(P_1) = \{i, i + 1, i + 2\}$ and all other sets $V_d(P_1)$, $2 \leq d \leq \frac{p}{2}$, $d \neq 2 + i$, are singletons.
   If $i > 0$ and $i + 1 \in N$ then, since $i \notin N$, vertices i and $i + 1$ disagree on 0. Since vertex 0 is fixed by every automorphism of P_1, i and $i + 1$ cannot belong to the same orbit and thus, by Corollary 16, P_1 is rigid. If $i > 0$ and $p - i \in N$, we get the same conclusion since, in that case, either $i + 2 \in N$, which implies that i and $i + 2$ disagree on 0, or $p - i - 1 \in N$, which implies, since $p - i - 2 \notin N$, that i and $i + 2$ disagree on p (and p is fixed by every automorphism of P_1). Suppose finally that $i = 0$. Since p is even, the vertex 3 is not a plateau-vertex and thus is fixed by every automorphism of P_1. If 0, 1 and 2 disagree on 3, these three vertices cannot belong to the same orbit and P_1 is rigid thanks to Corollary 16. Otherwise, since 3 cannot be an in-neighbour



(a) $\delta(P_1) = 2$, $\pi(P_1) = 0$, $d_2 - d_1 = 1$



(b) $\delta(P_1) = 2$, $\pi(P_1) = 0$, $d_2 - d_1 > 3$



(c) $\delta(P_1) = 2$, $\pi(P_1) = 0$, $d_2 - d_1 = 3$

Figure 2: Indegree paths for the proof of Theorem 21

of 0, 1 and 2 (this would imply $|N| \geq 3$), 3 is an out-neighbour of 0, 1 and 2, so that $1, 2 \notin N$, which implies that $P_1[\{0,1,2\}]$ is transitive, and P_1 is rigid by Lemma 18.

(c) if P_1 contains one plateau of size 3 and two plateaus of size 2, then the plateau-vertices are necessarily of the form i, $\frac{p}{2}-1$, $\frac{p}{2}$ and $p-i-1$, with $i \leq \frac{p}{2}-3$, so that $N = \{a_1, a_2\}$, with $a \in \{i+1, p-i\}$, $a_2 \in \{\frac{p}{2}, \frac{p}{2}+1\}$, and $d^-_{P_1}(\frac{p}{2}) = \frac{p}{2}$. Therefore, the only set $V_d(P_1)$ containing more than two elements is $V_{\frac{p}{2}}(P_1) = \{\frac{p}{2}-1, \frac{p}{2}, \frac{p}{2}+1\}$. Since N contains either $\frac{p}{2}$ or $\frac{p}{2}+1$, vertices $\frac{p}{2}$ and $\frac{p}{2}+1$ disagree on 0. Since vertex 0 is fixed by every automorphism of P_1, $\frac{p}{2}-1$, $\frac{p}{2}$ and $\frac{p}{2}+1$ are also fixed and thus, by Corollary 16, P_1 is rigid.

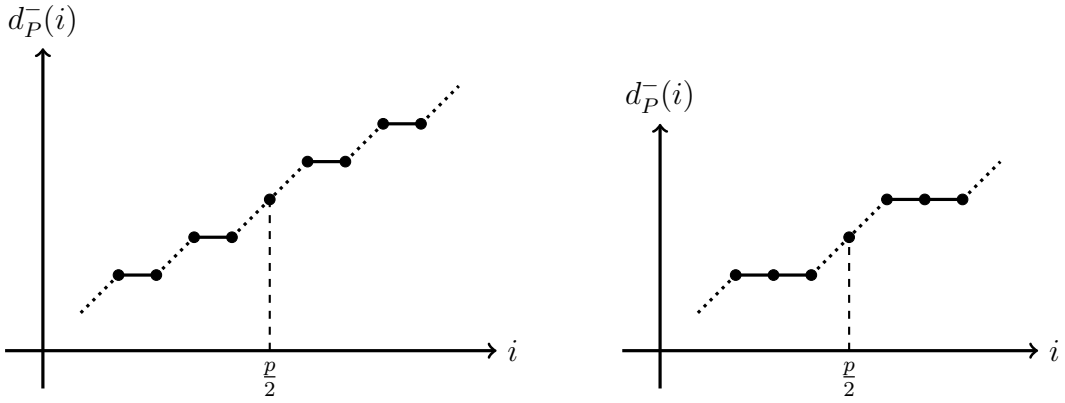
(d) if P_1 contains a unique plateau of size 5, then the plateau-vertices are necessarily $\frac{p}{2}-2$, $\frac{p}{2}-1$, …, $\frac{p}{2}+1$, so that $N = \{a_1, a_2\}$, with $a_1 \in \{\frac{p}{2}-1, \frac{p}{2}+2\}$, $a_2 \in \{\frac{p}{2}, \frac{p}{2}+1\}$, and $d^-_{P_1}(\frac{p}{2}) = \frac{p}{2}$. Moreover, if $p > 4$ then every set $V_d(P_1)$ except $V_{\frac{p}{2}}(P_1)$ is a singleton.
Suppose first that $p > 4$, so that $1 \notin N$ (otherwise, this would imply $1 = \frac{p}{2}-1$).
If $\{2,3\} \cap N = \emptyset$ (which happens in particular if $p \geq 10$), the subtournament $V_{\frac{p}{2}}(P_1)$ is either transitive or almost transitive, and thus rigid, so that P_1 is rigid by Lemma 18 (this also follows from Theorem 20).
Suppose now that $p \in \{6, 8\}$, so that vertices 0 and p are fixed by every automorphism of P_1. If all vertices belonging to the plateau are fixed by every automorphism of P_1 then P_1 is rigid by Lemma 18. Otherwise, some of these vertices may form an orbit, say O, of size 3 or 5. Considering the structure of the set N, we get that the vertex 0 has three out-neighbours and two in-neighbours in $\{\frac{p}{2}-2, \frac{p}{2}-1, \frac{p}{2}, \frac{p}{2}+1, \frac{p}{2}+2\}$. Therefore, the size of the orbit O must be 3. Moreover, since $0(\frac{p}{2}-x)$ is an arc if and only if $(\frac{p}{2}+x)p$ if an arc for every $x \in \{0,1,2\}$, we get that this orbit, if it exists, is necessarily $O = \{\frac{p}{2}-2, \frac{p}{2}, \frac{p}{2}+2\}$, and thus $N = \{\frac{p}{2}-1, \frac{p}{2}+1\}$. Moreover, O must induce a 3-cycle in P_1, which implies either $2 \in N$ and $4 \notin N$, or $2 \notin N$ and $4 \in N$. This forces $N = \{4, 6\}$ and thus $\{2,3\} \cap N = \emptyset$ for which we have seen before that P_1 is rigid.
Suppose finally that $p = 4$. Without loss of generality, we can suppose that $1 \notin N$ (otherwise, we consider P^c instead of P and we have $|N^c| = |N|$, so that either $N = \{2,4\}$ or $N = \{3,4\}$ and, in both cases, P_1 is a cyclic tournament. Due to our initial assumption on the parity of p, this case occurs either if $T = T(9; N)$, in which case $P(4; N) = P_1$, or $T = T(11; N)$, in which case $P(4; N) = P_2$, so that $P_1 = P(5; N)$.
In the former case, the pseudo-tournament P_2 is then either $P(3; \{2\})$ or $P(3; \{3\})$, respectively. In both cases, P_2 is rigid since $|N| = 1$.
In the latter case, we will prove that $P = P(5; N)$ is rigid. We consider two subcases, according to the set N:

i. $N = \{2,4\}$.
In that case, we have $V_2(P) = \{0,2,4\}$ and $V_3(P) = \{1,3,5\}$. Since both subtournaments $P[V_2(P)]$ and $P[V_3(P)]$ are transitive, P is rigid by Lemma 18.

ii. $N = \{3,4\}$.
In that case, we have $V_2(P) = \{0,3,4\}$ and $V_3(P) = \{1,2,5\}$. Again, both subtournaments $P[V_2(P)]$ and $P[V_3(P)]$ are transitive, P is rigid by Lemma 18.

This completes the proof. □

For proving Theorem 21, we showed that whenever $|S^-| \leq 2$, at least one of the two subtournaments $T_{0,p}$ or $T_{p+1,2p}$ of the cyclic tournament $T = T(2p+1; S)$ is rigid. It should be noticed that this property does not hold in general when $|S^-| \geq 3$, as shown by the cyclic tournament $T = T(13; \{2,5,6\})$ given in Example 4.



(a) Four plateaus of size 2

(b) Two plateaus of size 3

(c) Two plateaus of size 2

(d) One plateau of size 5

Figure 3: Indegree paths for the proof of Theorem 21 (Cont.)

Since any tournament T is isomorphic to its converse, Theorem 21 gives the following:

Corollary 22 *If $T = T(2p+1; S)$ is a cyclic tournament, such that $|S^-| \geq p-1$, then T satisfies Albertson-Collins Conjecture.*

4.3 The set S^- forms an interval

Observe first the following easy result:

Theorem 23 *For every $p \geq 1$, if $S = \{1, \ldots, p\}$ or $S = \{1, \ldots, p-1\}$ then the cyclic tournament $T(2p+1; S)$ satisfies Albertson-Collins conjecture.*

Proof. This directly follows from Theorem 21 since we have either $|S^-| = 0$ or $|S^-| = 1$. □

In fact, we can prove that whenever the set S^- of negative connectors forms an interval of integers, the corresponding cyclic tournament $T(2p+1; S)$ satisfies Albertson-Collins Conjecture.

We first prove two lemmas. The first one says that for every pseudo-cyclic tournament P, if vertices $0, \ldots, i-1$ are fixed by every automorphism of P, i is the first vertex of a plateau whose corresponding negative connectors form an interval and every vertex outside the plateau having the same indegree as the vertices of the plateau is fixed by every automorphism of P, then all vertices of the plateau are also fixed by every automorphism of P.

Lemma 24 *Let $P = P(p; N)$ be a pseudo-cyclic tournament, and i a vertex of P with $i < \frac{p}{2}$, satisfying all the following conditions:*

(i) *every vertex j, $0 \leq j < i$, is fixed by every automorphism of P,*

(ii) *$i-1$ is an ascent vertex,*

(iii) *$(i, i+1, \ldots, i+k-1)$ is a plateau of size $k \geq 2$ such that either*

  (a) *$i+k-1 < \frac{p}{2}$ and either $[i+1, i+k-1] \subseteq N$, or $[p-i-k+2, p-i] \subseteq N$, or*

  (b) *$i+k-1 > \frac{p}{2}$ and either $[i+1, \frac{p}{2}] \subseteq N$, or $[\frac{p}{2}+1, p-i] \subseteq N$,*

(iv) *$|V_d(P)| \geq k$, for $d = d_P^-(i)$, and, if $k > |V_d(P)|$, every vertex in $V_d(P) \backslash \{i, i+1, \ldots, i+k-1\}$ is fixed by every automorphism of P.*

Then, every vertex in $\{i, i+1, \ldots, i+k-1\}$ is fixed by every automorphism of P.

Proof. Suppose that P and i satisfy the conditions of the lemma. According to condition (iv), it is enough to prove that no two vertices in $\{i, i+1, \ldots, i+k-1\}$ can belong to the same orbit with respect to any automorphism of P. Observe that $i \notin N$ since $i-1$ is an ascent-vertex.

We consider two cases, according to the position of the plateau:

1. $i+k-1 < \frac{p}{2}$.
   We consider two subcases, depending on the corresponding interval of negative connectors:

   (a) $[i+1, i+k-1] \subseteq N$.
   In that case, P contains arcs $0i$, $(i+1)0$, …, $(i+k-1)0$, so that i and all other vertices of the plateau disagree on 0, and thus i cannot belong to the same orbit of another vertex of the plateau with respect to any automorphism of P. Similarly, for every $j = i+1, \ldots, i+k-1$ (in that order), j and vertices $j+1, \ldots, i+k-1$ disagree on $j-i$ (which is fixed by every automorphism of P), so that j cannot belong to the same orbit of another vertex of the plateau with respect to any automorphism of P. Therefore, all vertices of the plateau are fixed by every automorphism of P.

(b) $[p-i-k+2, p-i] \subseteq N$.
If N contains no negative connector $a < i$, then the subtournament induced by the plateau is transitive (since we then have $\min(N) > i+k-1$), so that all vertices of the plateau are fixed by every automorphism of P.
Otherwise, let a be the largest negative connector such that $a < i$. We then have $[a+1, i+k-1] \cap N = \emptyset$, so that i and all other vertices of the plateau disagree on $i-a$, and thus i cannot belong to the same orbit of another vertex of the plateau with respect to any automorphism of P Similarly, for every $j = i+1, \ldots, i+k-1$ (in that order), j and vertices $j+1, \ldots, i+k-1$ disagree on $j-a$ (which is fixed by every automorphism of P), so that cannot belong to the same orbit of another vertex of the plateau with respect to any automorphism of P. Therefore, all vertices of the plateau are fixed by every automorphism of P.

2. $i+k-1 > \frac{p}{2}$.
In the case, p is necessarily even. Since $\sigma(P)$ is centrally symmetric, we have $(i, i+1, \ldots, i+k-1) = (\frac{p}{2}-q, \ldots, \frac{p}{2}+q)$, for some $q \geq 1$.
If $\left[\frac{p}{2}-q+1, \frac{p}{2}\right] \subseteq N$ then, using the same proof as in Case 1a above, we get that no two vertices in $\{\frac{p}{2}-q, \ldots, \frac{p}{2}\}$ can belong to a same orbit with respect to any automorphism of P. Symmetrically, starting with vertex p instead of vertex 0 (recall that $p-j$, $j \geq 0$, is fixed whenever j is fixed, by Proposition 15), we can prove similarly that no two vertices in $\{\frac{p}{2}, \ldots, \frac{p}{2}+q\}$ can belong to a same orbit with respect to any automorphism of P. Therefore, no vertex of the plateau can belong to an orbit of size at least 3, which implies that every vertex in $\{\frac{p}{2}-q, \ldots, \frac{p}{2}+q\}$ is fixed by every automorphism of P and we are done.
If $\left[\frac{p}{2}+1, \frac{p}{2}+q\right] \subseteq N$ then we proceed as in Case 1b above: if the vertices of the plateau do not induce a transitive tournament then (i) using the largest negative connector $a < \frac{p}{2}-q$, we get that no two vertices in $\{\frac{p}{2}-q, \ldots, \frac{p}{2}\}$ can belong to a same orbit with respect to any automorphism of P, and (ii) symmetrically, using the smallest negative connector $a' > \frac{p}{2}+q$, we get that no two vertices in $\{\frac{p}{2}, \ldots, \frac{p}{2}+q\}$ can belong to a same orbit with respect to any automorphism of P and thus, again, every vertex in $\{\frac{p}{2}-q, \ldots, \frac{p}{2}+q\}$ is fixed by every automorphism of P.

This concludes the proof. □

The second lemma says that every pseudo-cyclic tournament containing either ascent-vertices and no descent-vertices, or descent-vertices and no ascent-vertices, is rigid whenever every plateau, if any, is produced by an interval of negative connectors.

Lemma 25 *If $P = P(p; N)$ is a pseudo-cyclic tournament such that*

(i) $\alpha(P) + \delta(P) > 0$,

(ii) $\alpha(P)\delta(P) = 0$,

(iii) *for every plateau $(i, i+1, \ldots, i+k-1)$ of size $k \geq 3$, with $i+k-1 < \frac{p}{2}$, either $[i+1, i+k-1] \subseteq N$, or $[p-i-k+2, p-i] \subseteq N$, and*

(iv) *for every plateau $(\frac{p}{2}-q, \ldots, \frac{p}{2}+q)$, p even, $q \geq 1$, of size $2q+1$, either $\left[\frac{p}{2}-q+1, \frac{p}{2}\right] \subseteq N$, or $\left[\frac{p}{2}+1, \frac{p}{2}+q\right] \subseteq N$,*

then P is rigid.

Proof. If $\pi(P) = 0$, then P is a transitive tournament and is thus rigid.

Assume now that $\pi(P) > 0$ and suppose first that P has no descent-vertices, that is $\delta(P) = 0$ and $\alpha(P) > 0$. We will prove by induction on i that every vertex $i \leq \left\lfloor \frac{p}{2} \right\rfloor$ is fixed by every automorphism of P, so that the result follows by Corollary 16.

If 0 is an ascent-vertex or the first vertex of a plateau of size 2, then 0 is fixed by every automorphism of P. If 0 is the first vertex of a plateau $(0,\dots,k-1)$ of size $k \geq 3$, then the subtournament induced by this plateau is transitive since we have either $[1,k-1] \in N$ or $[p-k+2,p] \in N$. Therefore, all vertices $0,\dots,k-1$ are fixed by every automorphism of P.

Suppose now that all vertices $0,\dots,i-1 < \lfloor \frac{p}{2} \rfloor$ are fixed by every automorphism of P. If i is an ascent-vertex then i is fixed by every automorphism of P, otherwise i is the first vertex of a plateau and the result follows from Lemma 24.

The case $\alpha(P) = 0$ and $\delta(P) > 0$ follows from Observation 10, considering P^c instead of P (we then have $\delta(P^c) = 0$ and $\alpha(P^c) > 0$). □

Note that conditions (iii) and (iv) in Lemma 25 are both necessary. For instance, the pseudo-cyclic tournament $P = P(5;\{2,5\})$ has only ascent-vertices and plateau-vertices (we have $\sigma(P) = (2,2,2,3,3,3)$) while $(0,1,2)(3,4,5)$ is an automorphism of P so that P is not rigid, and thus condition (iii) is necessary. On the other hand, the pseudo-cyclic tournament $P = P(8;\{2,3,5\})$ has only ascent-vertices and plateau-vertices, but a unique central plateau (we have $\sigma(P) = (3,4,\dots,4,5)$) while $(1,4,7)$ is an automorphism of P so that P is not rigid, and thus condition (iv) is necessary.

We are now able to prove the following:

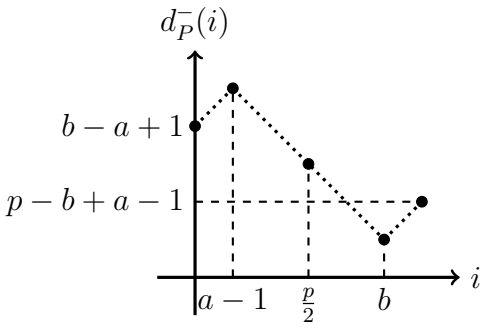
Theorem 26 *If $T = T(2p+1;S)$ is a cyclic tournament, such that $S^- = [a,b]$, $1 \leq a \leq b \leq p$, then T satisfies Albertson-Collins Conjecture.*

Proof. We will prove that $P = P(p;[a,b])$ is rigid whenever p is even. Therefore, considering either $T_{0,p}$ or $T_{p+1,2p}$ depending on the parity of p, the result will follow by Proposition 3.

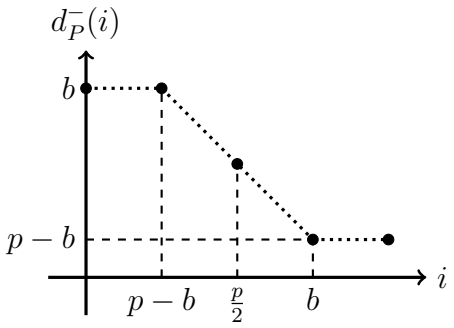
If $b \leq a+1$, the result follows from Theorem 21, so that we can assume $b \geq a+2$. Suppose now that $\frac{p}{2} \geq b$ or $\frac{p}{2} < a$. If $[a,b] \neq \left[1,\frac{p}{2}\right]$ and $[a,b] \neq \left[\frac{p}{2}+1,p\right]$ then $T_{0,p}$ is rigid by Lemma 25. Otherwise, $T_{p+1,2p}$ is rigid by Lemma 25 (since $\sigma(T_{p+1,2p}) = (\frac{p}{2},\dots,\frac{p}{2},\frac{p}{2}-1,\dots,\frac{p}{2}-1)$ if $[a,b] = \left[1,\frac{p}{2}\right]$, and $\sigma(T_{p+1,2p}) = (\frac{p}{2}-1,\dots,\frac{p}{2}-1,\frac{p}{2},\dots,\frac{p}{2})$ if $[a,b] = \left[\frac{p}{2}+1,p\right]$).

From now on, we thus assume that $a \leq \frac{p}{2} < b$. We consider three subcases, depending on the size of $[a,b]$.

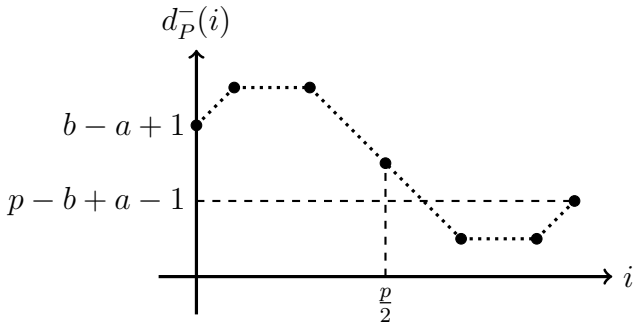
1. $b-a+1 > \frac{p}{2}$.
   In that case, $d_P^-(0) = b-a+1 > p-(b-a+1) = d_P^-(p)$. We consider three subcases, corresponding to the three possible forms of the indegree path of P.
   (a) $a+b = p+1$.
   In that case, no vertex in P can be a plateau-vertex, and thus the indegree path of P contains an ascent, a descent and an ascent (see Figure 4(a)). Therefore, each set $V_d(P)$ has cardinality at most 2 and P is rigid by Lemma 18.
   (b) $a = 1$ or $b = p$.
   In that case, vertices $0,\dots,p-b-1$ (if $a=1$) or $0,\dots,a-1$ (if $p=b$) are plateau-vertices, and thus the indegree path of P contains a plateau of size $p-b+1$ or $a+1$, a descent and a plateau of size $p-b+1$ or $a+1$ (see Figure 4(b) for the case $a=1$). In both cases, the subtournaments induced by these plateaus are transitive, so that every vertex of P is fixed by every automorphism of P and P is rigid by Corollary 16.
   (c) $1 < a$, $b < p$ and $a+b \neq p+1$.
   In that case, the indegree path of P contains an ascent, a plateau, a descent, a plateau and an ascent (see Figure 4(c)). Every vertex not belonging to a plateau belongs to a set $V_d(P)$ with cardinality 2 and is thus fixed by every automorphism of P. By Lemma 24, every vertex belonging to a plateau is also fixed by every automorphism of P. Using now Corollary 16, we get that P is rigid.



(a) $a+b=p+1$

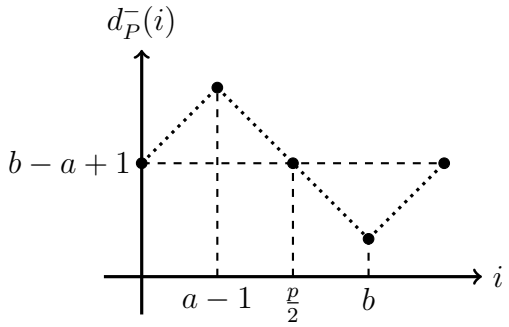


(b) $a=1$

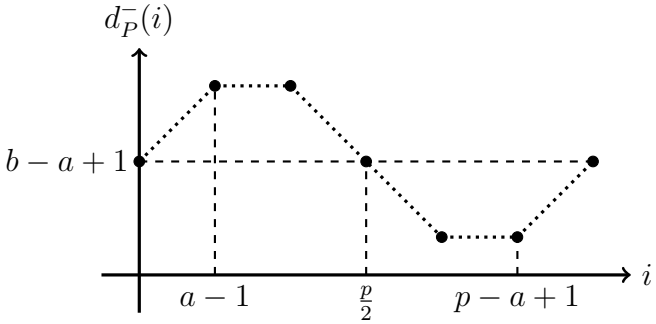


(c) $1<a$, $b<p$ and $a+b\neq p+1$

Figure 4: Indegree paths for the proof of Theorem 26, case $b-a+1>\frac{p}{2}$



(a) $a+b=p+1$

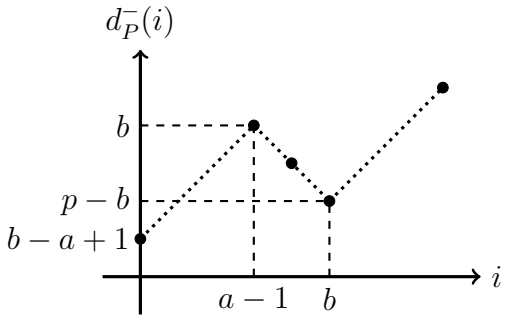


(b) $a+b\neq p+1$

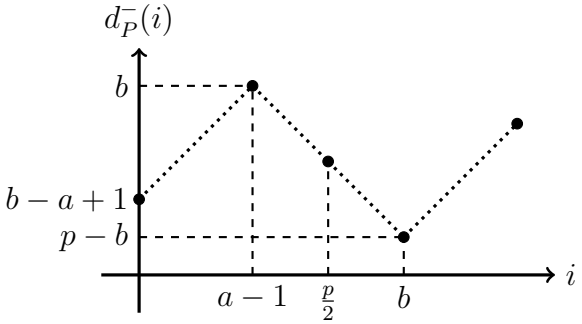
Figure 5: Indegree paths for the proof of Theorem 26, case $b-a+1=\frac{p}{2}$

2. $b - a + 1 = \frac{p}{2}$.
Note first that since $a \leq \frac{p}{2} < b$, we necessarily have $a > 1$ and $b < p$. The indegree path of P contains an ascent, possibly a plateau, a descent, possibly a plateau and an ascent. More precisely, $\sigma(P)$ contains a plateau if and only if $a+b \neq p+1$ (see Figure 5(a) and (b)). Moreover, since $d_P^-(0) = d_P^-(p) = \frac{p}{2}$, we get $V_{\frac{p}{2}}(P) = \{0, \frac{p}{2}, p\}$ (and the subtournament induced by $V_{\frac{p}{2}}(P)$ is a 3-cycle). Only two other sets $V_d(P)$ may have cardinality at least 3, namely those two sets corresponding to the two plateaus.
We consider three subcases, depending on the values of a and b.
   (a) $a > 2$ and $b < p - 1$.
   We first prove that every vertex in $V_{\frac{p}{2}}(P) = \{0, \frac{p}{2}, p\}$ is fixed by every automorphism of P. Since the vertex 1 is an ascent-vertex with $d_P^-(1) = \frac{p}{2} + 1$ and $|V_{\frac{p}{2}+1}| = 2$, it is fixed by every automorphism of P. Since $p - 1 \notin [a, b]$, vertices 0 and p disagree on 1, so that 0 and p cannot belong to a same orbit and we are done.
   Moreover, by Lemma 24, all vertices of the first plateau (whenever $a + b \neq p + 1$) are fixed by every automorphism of P. Hence, all vertices $i \leq \frac{p}{2}$ are fixed by every automorphism of P and the result follows by Corollary 16.
   (b) $b = p - 1$.
   In that case, we thus have $a = \frac{p}{2}$. The first plateau contains vertices 1 to $\frac{p}{2}-1$ and thus induces a transitive tournament, so that all its vertices are fixed by every automorphism of P. Now, since vertices $\frac{p}{2}$ and p disagree on $\frac{p}{2} - 1$, the three vertices $\{0, \frac{p}{2}, p\}$ are all fixed by every automorphism of P and the result follows by Corollary 16.
   (c) $a = 2$.
   In that case, we thus have $b = \frac{p}{2} + 1$. The first plateau contains vertices 1 to $\frac{p}{2} - 1$ and is isomorphic to the pseudo-cyclic tournament $PC = P(\frac{p}{2} - 2; [2, \frac{p}{2} - 2])$. Since PC^c is rigid by Theorem 21, PC is rigid. Now, since vertices 0 and p disagree on 1, the three vertices $\{0, \frac{p}{2}, p\}$ are all fixed by every automorphism of P and, again, the result follows by Corollary 16.

3. $b - a + 1 < \frac{p}{2}$.
The indegree path of P contains an ascent, possibly a plateau (if $a \neq p+1-b$), a descent, possibly a plateau and an ascent. If $a \leq p + 1 - b$, the first vertex of the first plateau is $a-1$ and its last vertex is $p-b$ (so that there is no plateau if $a = p+1-b$). If $a > p+1-b$, the first vertex of the first plateau is $p - b$ and its last vertex is $a - 1$.
We will prove that every vertex of P is fixed by every automorphism of P. Note that $d_P^-(0) = b - a + 1$, so that $d_P^-(p) = p - b + a - 1 > d_P^-(0)$ since $b - a + 1 < \frac{p}{2}$.
We consider two subcases, depending on whether $\sigma(P)$ contains a plateau or not.
   (a) $a = p + 1 - b$.
   In that case, $\sigma(P)$ does not contain any plateau, the first vertex of the descent is $a-1$ and its last vertex is b. Moreover, $d_P^-(a - 1) = b$ and $d_P^-(b) = p - b$. Let us consider the vertex b. If $d_P^-(b) = p-b \geq b-a+1 = d_P^-(0)$ (see Figure 6(a)) then $|V_{p-b}(P)| = 2$ and thus b is fixed by every automorphism of P. If $p-b > b-a+1$ (see Figure 6(b)) then $|V_{p-b}(P)| = 1$ and thus, again, b is fixed by every automorphism of P.
   Now, we claim that every set $V_d(P)$ with $|V_d(P)| = 3$ contains three fixed vertices. Each such set is of the form $\{i, j, k\}$, $i < j < k$, with $j + k = 2b$. Then, j and k disagree on b (since $b - j = k - b < p - b$, so that jb and bk are arcs in P) and, as b is fixed by every automorphism of P, i, j and k are also fixed by every automorphism of P. The result then follows by Corollary 16.
   (b) $a \neq p + 1 - b$.
   In that case, the vertex $a - 1$ is either the first or the last vertex of the first plateau



(a) $a+b=p+1$ and $p-b \geq b-a+1$



(b) $a+b=p+1$ and $p-b < b-a+1$

Figure 6: Indegree paths for the proof of Theorem 26, case $b-a+1 < \frac{p}{2}$

and $d_P^-(a-1) = b$ or $d_P^-(a-1) = p-a+1$, respectively. Similarly, the vertex b is either the first or the last vertex of the last plateau and $d_P^-(b) = p-b$ or $d_P^-(b) = a-1$, respectively.

We first claim that whenever 0 is fixed by every automorphism of P, all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P. Indeed, for any such set $\{i, j, k\}$ with $i < j < k$, we have $i < a-1$ and $a-1 < j < b$, so that i and j disagree on 0. Therefore, i, j and k are all fixed by every automorphism of P.

We now consider three subcases, depending on the values of $d_P^-(b)$ and $d_P^-(0)$.

i. $d_P^-(b) > d_P^-(0)$.
In that case (see Figure 7(a)), the vertex 0 is fixed by every automorphism of P, so that all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P.
If $a-1$ is the first vertex of the first plateau, let $x > \frac{p}{2}$ be the unique vertex with $d_P^-(x) = d_P^-(a-1)$. We then have $V_b(P) = \{a-1, a, \ldots, p-b, x\}$. Since $a-1$ and x are the only in-neighbours of 0 in $V_b(P)$, both of them are fixed by every automorphism of P. Therefore, by Lemma 24, every vertex belonging to the first plateau is fixed by every automorphism of P.
Suppose now that $a-1$ is the last vertex of the first plateau. Similarly, let $x < \frac{p}{2}$ be the unique vertex with $d_P^-(x) = d_P^-(b)$, so that $V_{a-1}(P) = \{x, p-a+1, \ldots, b\}$. Since x is the unique out-neighbour of 0 in $V_{a-1}(P)$ (as $x < a$ and $p-a+1 > a$, which gives $V_{a-1}(P) \cap S^- = \{p-a+1, \ldots, b\}$), x is fixed by every automorphism of P and, by Proposition 15, $p-x$ is also fixed by every automorphism of P. Therefore, by Lemma 24, every vertex belonging to the first plateau is fixed by every automorphism of P.
The result then follows by Corollary 16.

ii. $d_P^-(b) = d_P^-(0)$.
Suppose first that $a-1$ is the last vertex of the first plateau (that is, $a > p+1-b$), so that b is the last vertex of the second plateau. In that case, the vertices of the second plateau induce a transitive tournament and they all agree on 0. Therefore, the subtournament induced by vertices of $V_{b-a+1}(P)$ is rigid, so that all its vertices, and in particular 0, are fixed by every automorphism of P. Hence, all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P. By Proposition 15, all vertices of the first plateau are also fixed by every automorphism of P and, again, the result follows by Corollary 16.

Suppose now that $a-1$ is the first vertex of the first plateau (that is, $a < p+1-b$), so that b is the first vertex of the second plateau (see Figure 7(b)).
We first prove that the vertex 0 is fixed by every automorphism of P. Suppose to the contrary that with respect to some automorphism ϕ of P, 0 belongs to an orbit of size at least 3. Among the vertices of the second plateau, b is the only in-neighbour of 0. Therefore, the orbit must be of size 3 (recall that the subtournament induced by an orbit must be regular) and thus a 3-cycle $b0x$. Note that since xb is an arc of P, we then have $x - b \geq a$. Consider now the set $V_{b-a+2} = \{1, b-1, p-a+2\}$. The vertex 0 has only one in-neighbour in V_{b-a+2}, namely $b-1$, since $b-1 \in S^-$, $1 \notin S^-$ and $p-a+2 \notin S^-$ (since $p-a+2 > b$). On the other hand, b has at least two in-neighbours in V_{b-a+2}, namely 1 (since $b-1 \in S^-$) and $p-a+2$ (since $p-a+2-b > x-b \geq a$ and $p-a+2-b < \frac{p}{2} < b$). This implies that V_{b-a+2} cannot be an orbit with respect to ϕ, so that, in particular, 1 is fixed by ϕ. Now, note that 1 is an in-neighbour

of x (since $x-1 \geq b+a-1 > b$) and an out-neighbour of 0, a contradiction since all vertices in $\{0, b, x\}$ should agree on every vertex fixed by ϕ.
Therefore, 0 is fixed by every automorphism of P, so that all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P. By Proposition 15, p is also fixed by every automorphism of P and thus, by Lemma 24, every vertex of the first plateau is fixed by every automorphism of P. The result then follows by Corollary 16.

iii. $d_P^-(b) < d_P^-(0)$.
Suppose first that $a-1$ is the last vertex of the first plateau (that is, $a > p+1-b$), so that b is the last vertex of the second plateau (see Figure 7(c)). In that case, the vertices of each plateau induce a transitive tournament and thus all vertices of these plateaus are fixed by every automorphism of P. Consider the set $V_{b-a+1} = \{0, x, y\} = \{0, p-b+a-1, 2b-2a+2\}$. Vertices 0 and $p-b+a-1$ disagree on $p-a+1$ (since $p-a+1 \in S^-$ and $q = p-a+1-(p-b+a-1) < p-b < a-1$ so that $q \notin S^-$), which is fixed by every automorphism of P, and thus 0, $p-b+a-1$ and $2b-2a+2$ are all fixed by every automorphism of P.
Therefore, all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P, and the result follows by Corollary 16.

Suppose now that $a-1$ is the first vertex of the first plateau (that is, $a < p+1-b$), so that b is the first vertex of the second plateau.
We first prove that the vertex 0 is fixed by every automorphism of P. Suppose to the contrary that with respect to some automorphism ϕ of P, 0 belongs to an orbit of size 3, say a 3-cycle $0xy$, with $a < x < b$ and $y > b+1$, so that $x0$, $0y$ and yx are arcs in P, implying in particular $a \leq y - i \leq b$. Consider now the set $V_{b-a+2} = \{1, x-1, y+1\}$. Since $1 < a$, $a \leq x-2$ (as $a-1 < \frac{p}{2} < x$) and $x < b$, we get that 01, $x1$, $(x-1)0$ and $(x-1)x$ are all arcs in P, so that 1 and $x-1$ disagree on the orbit of 0. This implies that V_{b-a+2} cannot be an orbit with respect to ϕ, so that, in particular, 1 is fixed by ϕ. Now, note that 1 is an in-neighbour of y (since $y-1 > b$) and an out-neighbour of 0, a contradiction since all vertices in $\{0, x, y\}$ should agree on every vertex fixed by ϕ.
Therefore, 0 is fixed by every automorphism of P, so that all vertices belonging to a set $V_d(P)$ with $|V_d(P)| = 3$ are also fixed by every automorphism of P. By Lemma 24, every vertex of the first plateau is fixed by every automorphism of P and the result follows by Corollary 16.

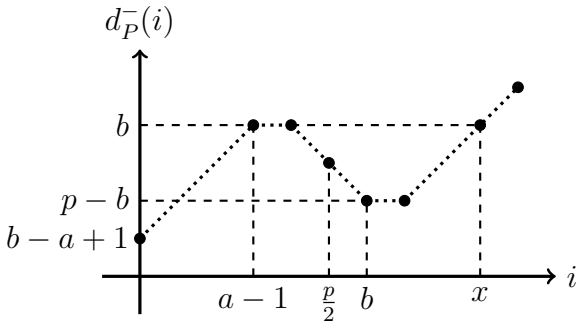
This concludes the proof. □

Again, since any tournament T is isomorphic to its converse, Theorem 26 gives the following:

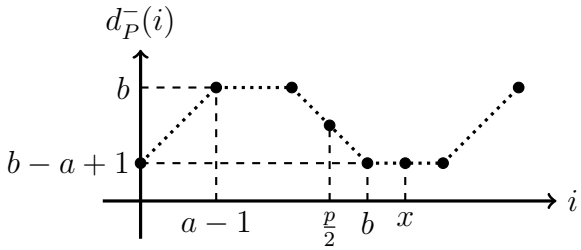
Corollary 27 *If $T = T(2p+1; S)$ is a cyclic tournament, such that $S^- = [1, a] \cup [b, p]$, $1 \leq a \leq b \leq p$, then T satisfies Albertson-Collins Conjecture.*

4.4 Paley tournaments

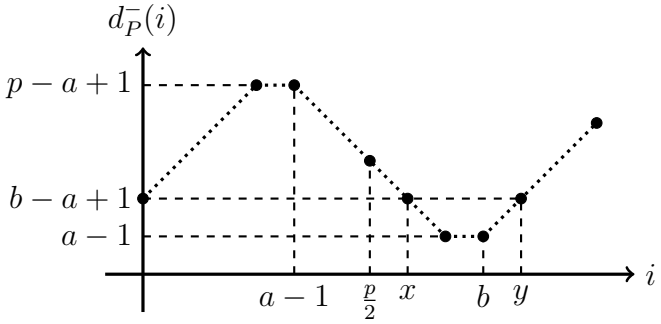
Let n be a prime, $n \equiv 3 \pmod 4$, so that $n = 2p+1$ with p odd. The *Paley tournament* is the cyclic tournament $QR_n = T(n; S_n)$ where S_n^+ is the set of non-zero quadratic residues of n. That is, for every i, j, $i < j$, ij is an arc in QR_n if and only if $j - i$ is a non-zero square in $\mathbb{Z}_n$. Paley tournaments are arc-transitive and the set $\mathrm{Aut}(QR_n)$ is the set of mappings $i \mapsto ai + b$, where $a, b \in \mathbb{Z}_n$ and a is a non-zero square. Since the number of non-zero squares in $\mathbb{Z}_n$ is $p = \lfloor \frac{n}{2} \rfloor$, we thus have $|Aut(QR_n)| = np$.



(a) $a+b \neq p+1$ and $p-b > b-a+1$



(b) $a+b \neq p+1$, $p-b = b-a+1$ and $a < p+1-b$



(c) $a+b \neq p+1$, $a-1 < b-a+1$ and $a > p+1-b$

Figure 7: Indegree paths for the proof of Theorem 26, case $b-a+1 < \frac{p}{2}$ (Cont.)

In [3], Albertson and Collins showed that the paley tournament QR_7 satisfies their conjecture. In fact, it is not difficult to see that every Paley tournament has distinguishing number 2. In addition, we prove that Paley tournaments satisfy Albertson-Collins Conjecture:

Theorem 28 *For every prime n, $n \equiv 3 \pmod 4$, $D(QR_n) = 2$. Moreover, QR_n satisfies Albertson-Collins Conjecture.*

Proof. Let λ be the 2-labeling defined by $\lambda(0) = \lambda(1) = 1$ and $\lambda(i) = 2$ for every $i \in \mathbb{Z}_n \setminus \{0, 1\}$. We claim that λ is distinguishing, which implies $D(QR_n) = 2$. Let $\phi : i \mapsto ai + b$ be any λ-preserving automorphism of QR_n. Since the only two vertices with label 1 are 0 and 1, we necessarily have $\phi(0) = 0$ and $\phi(1) = 1$, which gives $b = 0$ and $a = 1$, so that $\phi = Id$ and we are done.

In order to prove that QR_n satisfies Albertson-Collins Conjecture, we need to prove that the canonical labeling λ^* of QR_n is also a distinguishing 2-labeling. Suppose to the contrary that there exists a λ^*-preserving nontrivial automorphism $\phi : i \mapsto ai + b$, with $a, b \in \mathbb{Z}_n$ and a is a non-zero quadratic residue and let ℓ denote the order of ϕ.

Consider the action on $V(QR_n)$ of the group $H = <\phi> = \{\phi^k : 1 \leq k \leq \ell\}$. Note that $|H|$ divides $|Aut(QR_n)| = np$ and, since ϕ is λ^*-preserving, $|H|$ divides p. Moreover, p cannot be prime, since otherwise the subtournament T_2 of QR_n induced by vertices $\{p+1, \ldots, 2p\}$ would have a unique orbit with respect to ϕ, in contradiction with Proposition 5.

We claim that we necessarily have $b = 0$. Assume to the contrary that $b \neq 0$ and let r be the size of the orbit of 0 with respect to ϕ. We have $\phi(0) = b$, $\phi^2(0) = ab + b$ and so on, so that

$$\phi^r(0) = a^{r-1}b + a^{r-2}b + \cdots + ab + b = 0$$

and $\phi^{r+1}(0) = \phi(0) = b$. On one other hand, we have

$$\phi^{r+1}(0) = a^r b + a^{r-1}b + \cdots + ab + b = a^r b.$$

Therefore, $a^r b = b$ and thus $a^r = 1$. Since a is a non-zero quadratic residue in $\mathbb{Z}_n$, its order in the multiplicative group $\mathbb{Z}_n^*$ is p and thus $p \,|\, r$. Moreover, since ϕ is λ^*-preserving, we have $r = p$. Hence, the subtournament T_1 of QR_n induced by vertices $\{0, \ldots, p\}$ contains a fixed vertex x such that all vertices of the orbit of 0 agree on x, which implies either $d^+_{T_1}(x) = 0$ or $d^+_{T_1}(x) = p$ in contradiction with the definition of QR_n since both sets $\{1, \ldots, p\}$ and $\{p+1, \ldots, 2p\}$ contain non-zero quadratic residues.

We thus have $b = 0$, so that 0 is fixed by ϕ. Moreover, the orbit of 1 with respect to ϕ is $\{1, a, a^2, \ldots, a^{p-1}\}$ (recall that the order of a in the multiplicative group $\mathbb{Z}_n^*$ is p) and its size is p. Since ϕ is λ^*-preserving, the orbit of 1 is thus $\{1, \ldots, p\}$. Therefore, all vertices in $\{1, \ldots, p\}$ agree on 0 and we obtain a contradiction as above. □

5 Discussion

In this paper, we proposed several sufficient conditions for a cyclic tournament to satisfy Albertson-Collins Conjecture. Proposition 3 says that the cyclic tournament $T = T(2p + 1; S)$ satisfies Albertson-Collins Conjecture whenever any of the subtournaments $T_{0,p}$ or $T_{p+1,2p}$ is rigid (however, this condition is not necessary, as shown by Example 4). Using this property, we then got several sufficient conditions on the structure of the set S^- of negative connectors for at least one of these two subtournaments to be rigid.

We finally propose some directions for future research.

1. The main question following our work is obviously to prove, or disprove, Albertson-Collins Conjecture.

2. In [8], Collins introduced the notion of a *determining set* of a graph G, that is, a subset X of $V(G)$ such that, for every two automorphisms $\phi_1, \phi_2 \in Aut(G)$, $\phi_1 = \phi_2$ whenever $\phi_1(x) = \phi_2(x)$ for every vertex $x \in X$. Albertson and Boutin proved in [2] that $D(G) \leq d$ if and only if G has a determining set X with $D(X) \leq d-1$. Therefore $D(G) = 2$ if and only if the subgraph of G induced by X is rigid.

   For each cyclic tournament T of order $2p+1$ for which we proved Albertson-Collins Conjecture, we exhibited a rigid determining set of size at most $\lceil \frac{p}{2} \rceil$. We also proved that every Paley tournament has a rigid determining set of size 2. In [21], Lozano proved that every tournament contains a (not necessarily rigid) determining set of size $\lfloor \frac{p}{3} \rfloor$.

   It would thus be interesting to determine the minimal size of a rigid determining set in a cyclic tournament satisfying Albertson-Collins Conjecture. Note here that proving that every cyclic tournament contains such a rigid determining set would settle Albertson-Collins Conjecture.

3. In [9, 10], Boutin also defined the *cost of distinguishing a graph* G such that $D(G) = 2$, denoted $\rho(G)$, as the minimum size of a label class in a 2-distinguishing labeling of G. We proved in particular that $\rho(QR_n) = 2$ for every Paley tournament QR_n. It would be interesting to determine $\rho(T)$ for every cyclic tournament $T = T(2p+1; S)$ satisfying Albertson-Collins Conjecture, or to characterize cyclic tournaments T with $\rho(T) = 2$.

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