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1 **CONVERGENT ALGORITHM BASED ON CARLEMAN ESTIMATES**
2 **FOR THE RECOVERY OF A POTENTIAL IN THE WAVE**
3 **EQUATION.***

4 LUCIE BAUDOUI[†], MAYA DE BUHAN[‡], AND SYLVAIN ERVEDOZA[§]

5 **Abstract.** This article develops the numerical and theoretical study of the reconstruction
6 algorithm of a potential in a wave equation from boundary measurements, using a cost functional built
7 on weighted energy terms coming from a Carleman estimate. More precisely, this inverse problem
8 for the wave equation consists in the determination of an unknown time-independent potential from
9 a single measurement of the Neumann derivative of the solution on a part of the boundary. While its
10 uniqueness and stability properties are already well known and studied, a constructive and globally
11 convergent algorithm based on Carleman estimates for the wave operator was recently proposed in
12 [BdBE13]. However, the numerical implementation of this strategy still presents several challenges,
13 that we propose to address here.

14 **Key words.** wave equation, inverse problem, reconstruction, Carleman estimates.

15 **AMS subject classifications.** 93B07, 93C20, 35R30.

16 **1. Introduction and algorithms.**

17 **1.1. Setting and previous results.** Let Ω be a smooth bounded domain of
18 \mathbb{R}^d , $d \geq 1$ and $T > 0$. This article focuses on the reconstruction of the potential in a
19 wave equation according to the following inverse problem:

20 Given the source terms f and f_∂ and the initial data (w_0, w_1) , con-
21 sidering the solution of

$$(1) \quad \begin{cases} \partial_t^2 W - \Delta W + QW = f, & \text{in } (0, T) \times \Omega, \\ W = f_\partial, & \text{on } (0, T) \times \partial\Omega, \\ W(0) = w_0, \quad \partial_t W(0) = w_1, & \text{in } \Omega, \end{cases}$$

23 can we determine the unknown potential $Q = Q(x)$, assumed to
24 depend only on $x \in \Omega$, from the additional knowledge of the flux of
25 the solution through a part Γ_0 of the boundary $\partial\Omega$, namely

$$(2) \quad \mathcal{M} = \partial_n W, \quad \text{on } (0, T) \times \Gamma_0 ?$$

27 Beyond the preliminary questions about the uniqueness and stability of this inverse
28 problem, already very well documented as we will detail below, we are interested in
29 the actual reconstruction of the potential Q from the extra information given by the
30 measurement of the flux \mathcal{M} of the solution on a part of the boundary. This issue was
31 already addressed theoretically in our previous work [BdBE13] based on Carleman
32 estimates. However, the algorithm proposed in [BdBE13], proved to be convergent,
33 cannot be implemented in practice as it involves minimization processes of function-
34 als containing too large exponential terms. Therefore, our goal is to address here the

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35 numerical challenges induced by that approach.

36

37 Before going further, let us recall that if $Q \in L^\infty(\Omega)$, $f \in L^1(0, T; L^2(\Omega))$,
 38 $f_\partial \in H^1((0, T) \times \partial\Omega)$, $w_0 \in H^1(\Omega)$ and $w_1 \in L^2(\Omega)$, and assuming the compati-
 39 bility condition $f_\partial(0, x) = w_0(x)$ for all $x \in \partial\Omega$, the Cauchy problem (1) is well-posed
 40 in $C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$, and the normal derivative $\partial_n W$ is well-defined
 41 as an element of $L^2((0, T) \times \partial\Omega)$, see e.g. [Lio88, LLT86].

42

43 Our results will require the following geometric conditions (sometimes called “mul-
 44 tiplier condition” or “ Γ -condition”):

45 $\exists x_0 \notin \overline{\Omega}$, such that

46 (3) $\Gamma_0 \supset \{x \in \partial\Omega, (x - x_0) \cdot \vec{n}(x) \geq 0\},$

47 (4) $T > \sup_{x \in \Omega} |x - x_0|.$

48 Space and time conditions (3)–(4) are natural from the observability point of view, and
 49 appear naturally in the context of the multiplier techniques developed in [Ho86, Lio88].
 50 They are more restrictive than the well-known observability results [BLR92] by Bar-
 51 dos Lebeau Rauch based on the behavior of the rays of geometric optics, but the
 52 geometric conditions (3)–(4) yield much more robust results, and this will be of pri-
 53 mary importance in our approach.

54

55 In fact, under the regularity assumption

56 (5) $W \in H^1(0, T; L^\infty(\Omega)),$

57 the positivity condition

58 (6) $\exists \alpha > 0$ such that $|w_0| \geq \alpha$ in Ω ,

59 the knowledge of an *a priori* bound $m > 0$ such that

60 (7) $\|Q\|_{L^\infty(\Omega)} \leq m, \quad \text{i.e.} \quad Q \in L_{\leq m}^\infty(\Omega) = \{q \in L^\infty(\Omega), \|q\|_{L^\infty(\Omega)} \leq m\},$

61 and the multiplier conditions (3)–(4), the results in [Baufr] (and in [Yam99] under
 62 more regularity hypothesis) state the Lipschitz stability of the inverse problem con-
 63 sisting in the determination of the potential Q in (1) from the measurement of the
 64 flux \mathcal{M} in (2).

65

66 We will introduce our work by describing what was done in our former article
 67 [BdBE13], in order to highlight stage by stage the main challenges when performing
 68 numerical implementations.

69 In [BdBE13], we proposed a prospective algorithm to recover the potential Q from
 70 the measurement \mathcal{M} on $(0, T) \times \Gamma_0$, that we briefly recall below. We assume that
 71 conditions (3)–(4) are satisfied for some $x_0 \notin \overline{\Omega}$, and we set $\beta \in (0, 1)$ such that

72 (8) $\beta T > \sup_{x \in \Omega} |x - x_0|.$

73 We then define, for $(t, x) \in (-T, T) \times \Omega$, the Carleman weight functions

74 (9) $\varphi(t, x) = |x - x_0|^2 - \beta t^2, \quad \text{and for } \lambda > 0, \quad \psi(t, x) = e^{\lambda(\varphi(t, x) + C_0)},$

75 where $C_0 > 0$ is chosen such that $\varphi + C_0 \geq 1$ in $(-T, T) \times \Omega$ and $\lambda > 0$ is large
 76 enough. The chore of the algorithm in [BdBE13] is the minimization of a functional
 77 $K_{s,q}[\mu]$ given for $s > 0$, $q \in L_{\leq m}^\infty(\Omega)$ and $\mu \in L^2((0, T) \times \Gamma_0)$ by

$$78 \quad (10) \quad K_{s,q}[\mu](z) = \frac{1}{2} \int_0^T \int_\Omega e^{2s\psi} |\partial_t^2 z - \Delta z + qz|^2 dx dt + \frac{s}{2} \int_0^T \int_{\Gamma_0} e^{2s\psi} |\partial_n z - \mu|^2 d\sigma dt,$$

79 set on the trajectories $z \in L^2(0, T; H_0^1(\Omega))$ such that $\partial_t^2 z - \Delta z + qz \in L^2((0, T) \times \Omega)$,
 80 $\partial_n z \in L^2((0, T) \times \Gamma_0)$ and $z(0, \cdot) = 0$ in Ω . Note in particular that [BdBE13] shows
 81 that there exists a unique minimizer of the above functional under the aforementioned
 assumptions. The algorithm then reads as follows:

Algorithm 1 (see [BdBE13])

Initialization: $q^0 = 0$ (or any guess in $L_{\leq m}^\infty(\Omega)$).

Iteration: From k to $k+1$

- *Step 1* - Given q^k , we set $\mu^k = \partial_t (\partial_n w[q^k] - \partial_n W[Q])$ on $(0, T) \times \Gamma_0$, where $w[q^k]$ denotes the solution of (1) with the potential q^k and $\partial_n W[Q]$ is the measurement given in (2).

- *Step 2* - Minimize $K_{s,q^k}[\mu^k]$ (defined in (10)) on the trajectories $z \in L^2(0, T; H_0^1(\Omega))$ such that $\partial_t^2 z - \Delta z + q^k z \in L^2((0, T) \times \Omega)$, $\partial_n z \in L^2((0, T) \times \Gamma_0)$ and $z(0, \cdot) = 0$ in Ω . Let Z^k be the unique minimizer of the functional $K_{s,q^k}[\mu^k]$.

- *Step 3* - Set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0}, \quad \text{in } \Omega,$$

where w_0 is the initial condition in (1) (recall assumption (6)).

- *Step 4* - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{with } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| > m, \end{cases}$$

where m is the a priori bound in (7).

82

83 Algorithm 1 comes along with the following convergence result:

84 **THEOREM 1** ([BdBE13, Theorem 1.5]). *Under assumptions (3)-(4)-(5)-(6)-(7)-*
 85 *(8), there exist constants $C > 0$, $s_0 > 0$ and $\lambda > 0$ such that for all $s \geq s_0$, Algorithm*
 86 *1 is well-defined and the iterates q^k constructed by Algorithm 1 satisfy, for all $k \in \mathbb{N}$,*

$$87 \quad (11) \quad \int_\Omega |q^{k+1} - Q|^2 e^{2s\psi(0)} dx \leq \frac{C \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2}{s^{1/2}\alpha^2} \int_\Omega |q^k - Q|^2 e^{2s\psi(0)} dx.$$

88 *In particular, for s large enough, the sequence q^k strongly converges towards Q as*
 89 *$k \rightarrow \infty$ in $L^2(\Omega)$.*

This algorithm presents the advantage of being convergent for any initial guess $q^0 \in L_{\leq m}^\infty(\Omega)$ without any a priori guess except for the knowledge of m . This is why we call this algorithm *globally convergent*. However, while this algorithm is theoretically satisfactory as at each iteration, it simply consists in the minimization of the strictly convex and coercive quadratic functional $K_{s,q}$, it nevertheless contains several flaws and drawbacks in its numerical implementation. In particular, we underline that the functional $K_{s,q}$ involves two exponentials, namely

$$\exp(s\psi) = \exp(s \exp(\lambda(\varphi + C_0))),$$

with a choice of parameters s and λ large enough and whose sizes are difficult to estimate. In particular, for $s = \lambda = 3$ - which are not so large of course - $\Omega = (0, 1)$, $x_0 \simeq 0^-$, $T \simeq 1^+$ and $\beta \simeq 1^-$, the ratio

$$\frac{\max_{(0,T) \times \Omega} \{\exp(2s\psi)\}}{\min_{(0,T) \times \Omega} \{\exp(2s\psi)\}}$$

90 is of the order of 10^{340} ! The numerical implementation of Algorithm 1 therefore
91 seems doomed.

92 The goal of this article is to improve the above algorithm so that it can fruitfully be
93 implemented. This will be achieved following several stages: working on the construc-
94 tion of the cost functional (specifically on the Carleman weight function), considering
95 the preconditioning of the cost functional, and adapting the new cost functional to
96 the discrete setting used for the numerics.

97 Before going further, let us mention that the inverse problem under consideration
98 has been well-studied in the literature, starting with the uniqueness result in the
99 celebrated article [BK81], see also [Kli92], which introduced the use of Carleman
100 estimates for these studies. Later on, stability issues were obtained for the wave
101 equation, first based on the so-called observability properties of the wave equation
102 [PY96, PY97] and then refined with the use of Carleman estimates, among which
103 [IY01a, IY01b, IY03, KY06]. In fact, a great part of the literature in this area, con-
104 cerning uniqueness, stability and reconstruction of coefficient inverse problems for
105 evolution partial differential equations can be found in the survey article [Kli13] and
106 we refer the interested reader to it. A slightly different approach can also be found in
107 the recent article [SU13] based on more geometric insights.

108 Let us also emphasize that we are interested in the case in which one performs only
109 one measurement. The question of determining coefficients from the Dirichlet to
110 Neumann map is different and we refer for instance to the boundary control method
111 proposed in [Bel97] or to methods based on the complex geometric optics, see [Isa91].
112 Here, as we said, we will focus on the reconstruction of the potential in the wave equa-
113 tion (1) from the flux \mathcal{M} in (2). This question has been studied only recently, though
114 the first investigation [KI95] appears in 1995, and we shall in particular point out
115 the most recent works of Beilina and Klibanov [KB12], [BK15], who study the recon-
116 struction of a coefficient in a hyperbolic equation from the use of a Carleman weight
117 function for the design of the cost functional. However, these techniques differ from
118 ours as they work on the functions obtained after a Laplace transform of the equation.
119

120 In what follows, we propose to develop a numerical algorithm in the spirit of
121 the one in [BdBE13], study its convergence and his implementation. Before going
122 further, let us also mention the fact that one can find in [CFCM13] some numerical
123 experiments based on the minimization of a quadratic functional similar to the one in
124 (10), but with s and λ rather small, namely $s = 1$ and $\lambda = 0.1$, see [CFCM13, Section
125 4]. Our goal is to overcome this restriction on the size of the Carleman parameters,
126 as we request them to be large for the convergence of the algorithm.

127 1.2. New weight functions, new cost functionals, and a new algorithm.

128 In a first stage, we aim at removing one exponential from the cost functional $K_{s,q}$ in
129 (10). Similarly to [BdBE13], looking again for a cost functional based on a Carleman
130 estimate for the wave equation, we will work with the Carleman weight function

131 $\exp(s\varphi)$ instead of $\exp(s \exp(\lambda(\varphi + C_0)))$. This requires an adaptation of the proof
 132 of [BdBE13] with such a weight function and the use of the Carleman estimates
 133 developed in [LRS86] (see also [IY01b]), that we will briefly recall in Section 2.

134 In particular, instead of minimizing $K_{s,q}[\mu]$ introduced in (10) as in Step 2 of
 135 Algorithm 1, we will perform a minimization process on a new functional $J_{s,q}[\tilde{\mu}]$,
 136 to be defined later in (13), based on the simplified weight function $\exp(s\varphi)$. Before
 137 introducing that functional, we shall define the following restricted set \mathcal{O} :
 138

$$139 \quad (12) \quad \mathcal{O} = \{(t, x) \in (0, T) \times \Omega, \beta t > |x - x_0|\}$$

$$140 \quad \quad \quad = \{(t, x) \in (0, T) \times \Omega, |\partial_t \varphi(t, x)| \geq |\nabla \varphi(t, x)|\},$$

142 which is depicted in Figure 1.

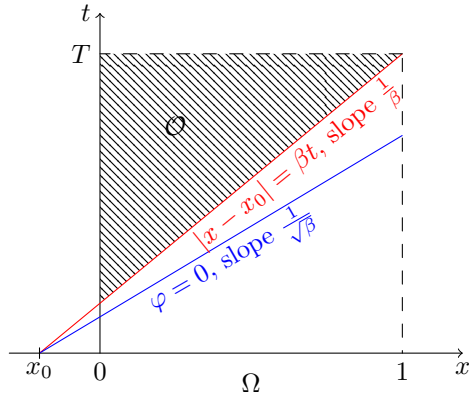


Fig. 1: Illustration of domain \mathcal{O} in the case $\Omega = (0, 1)$.

143 For $s > 0$, $q \in L^\infty(\Omega)$ and $\tilde{\mu} \in L^2((0, T) \times \Gamma_0)$, we then introduce the functional
 144 $J_{s,q}[\tilde{\mu}]$ defined by
 145

$$146 \quad (13) \quad J_{s,q}[\tilde{\mu}](z) = \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 z - \Delta z + qz|^2 dx dt$$

$$147 \quad \quad \quad + \frac{s}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z - \tilde{\mu}|^2 d\sigma dt + \frac{s^3}{2} \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 dx dt,$$

148

to be compared with $K_{s,q}[\mu]$ in (10), on the trajectories $z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ such that $\partial_t^2 z - \Delta z + qz \in L^2((0, T) \times \Omega)$ and $z(0, \cdot) = 0$ in Ω .

This functional $J_{s,q}[\tilde{\mu}]$ is quadratic, and as we will show later in Section 2.3, under conditions (3)–(4)–(8), it is strictly convex and coercive, therefore enjoying similar properties as the functional $K_{s,q}[\mu]$. Nevertheless, let us once more emphasize that the functional $J_{s,q}[\tilde{\mu}]$ is less stiff than the functional $K_{s,q}[\mu]$ as now the weights are of the form $\exp(2s\varphi)$ instead of $\exp(2s\psi) = \exp(2s \exp(\lambda(\varphi + C_0)))$ in (10). This already indicates the possible gain we could have by working with the functional $J_{s,q}[\tilde{\mu}]$ in (13) instead of $K_{s,q}[\mu]$ in (10).

It may appear surprising to note $\tilde{\mu}$ instead of μ . These slightly different notations come from the fact that the functional $K_{s,q}[\mu]$ tries to find an optimal solution Z of

$$\partial_t^2 Z - \Delta Z + qZ \simeq 0 \text{ in } (0, T) \times \Omega, \quad \text{and} \quad \partial_n Z \simeq \mu \text{ in } (0, T) \times \Gamma_0,$$

while the functional $J_{s,q}[\tilde{\mu}]$ tries to find an optimal solution \tilde{Z} of

$$\partial_t^2 \tilde{Z} - \Delta \tilde{Z} + q \tilde{Z} \simeq 0 \text{ in } (0, T) \times \Omega, \quad \partial_n \tilde{Z} \simeq \tilde{\mu} \text{ in } (0, T) \times \Gamma_0, \quad \text{and} \quad \tilde{Z} \simeq 0 \text{ in } \mathcal{O}.$$

149 Therefore, as \tilde{Z} is sought after such that it is small in \mathcal{O} , it is natural to introduce a
150 smooth cut-off function $\eta \in C^2(\mathbb{R})$ such that $0 \leq \eta \leq 1$ and

$$151 \quad (14) \quad \eta(\tau) = 0, \text{ if } \tau \leq 0, \quad \text{and} \quad \eta(\tau) = 1, \text{ if } \tau \geq d_0^2 := d(x_0, \Omega)^2,$$

(recall that $d_0^2 > 0$ according to Assumption 3) see Figure 2. Next, the idea is that if

$$\tilde{\mu} = \eta(\varphi)\mu, \text{ in } (0, T) \times \Gamma_0.$$

152 and if Z denotes the minimizer of the functional $K_{s,q}[\mu]$ in (10), then the minimizer
153 \tilde{Z} of $J_{s,q}[\tilde{\mu}]$ in (13) should be close to $\eta(\varphi)Z$ in $(0, T) \times \Omega$ and in particular at $t = 0$
this should yield, due to the choice of η in (14), $\partial_t \tilde{Z}(0) \simeq \partial_t Z(0)$ in Ω .

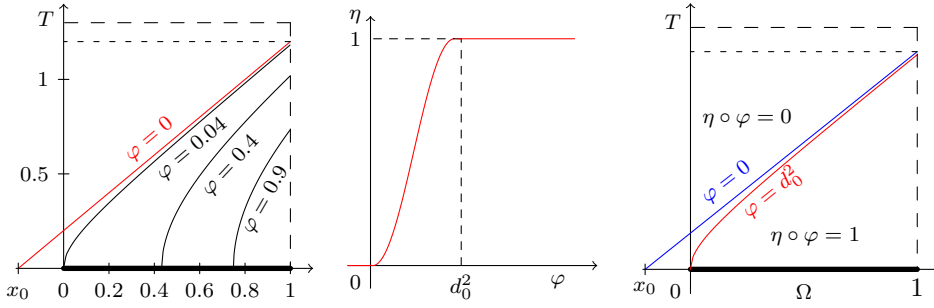


Fig. 2: Isovalues of the function φ ($x_0 = -0.2$, $\beta = 1$). Definition and application of the cut-off function η .

154

155

156 We are then led to propose a revised version of our reconstruction algorithm,
157 detailed in Algorithm 2 given below.

158 Of course, if one compares Algorithm 2 with Algorithm 1, the major difference
159 is in Step 2 in which one minimizes the functional $J_{s,q^k}[\tilde{\mu}]$ in (13) instead of the
160 functional $K_{s,q^k}[\mu]$ in (10). And as we have said above, the two functionals should
161 have minimizers that are close at $t = 0$. In fact, similarly as Theorem 1, we will obtain
162 the following result:

163 **THEOREM 2.** *Under assumptions (3)-(4)-(5)-(6)-(7)-(8), there exist positive con-*
164 *stants C and s_0 such that for all $s \geq s_0$, Algorithm 2 is well-defined and the iterates*
165 *q^k constructed by Algorithm 2 satisfy, for all $k \in \mathbb{N}$,*

$$166 \quad (17) \quad \int_{\Omega} |q^{k+1} - Q|^2 e^{2s\varphi(0)} dx \leq \frac{C \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2}{s^{1/2}\alpha^2} \int_{\Omega} |q^k - Q|^2 e^{2s\varphi(0)} dx.$$

167 *In particular, for s large enough, the sequence q^k strongly converges towards Q as*
168 *$k \rightarrow \infty$ in $L^2(\Omega)$.*

169 The proof of Theorem 2 is given in Section 2 and closely follows the one of Theorem 1
170 in [BdBE13]. The main difference is that the starting point of our analysis, instead

Algorithm 2

Initialization: $q^0 = 0$ (or any guess $q^0 \in L_{\leq m}^\infty(\Omega)$).

Iteration: From k to $k+1$

• *Step 1* - Given q^k , we set $\tilde{\mu}^k = \eta(\varphi)\partial_t(\partial_n w[q^k] - \partial_n W[Q])$ on $(0, T) \times \Gamma_0$, where $w[q^k]$ denotes the solution of

$$(15) \quad \begin{cases} \partial_t^2 w - \Delta w + q^k w = f, & \text{in } (0, T) \times \Omega, \\ w = f_\partial, & \text{on } (0, T) \times \partial\Omega, \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases}$$

corresponding to (1) with the potential q^k and $\partial_n W[Q]$ is the measurement in (2).

• *Step 2* - We minimize the functional $J_{s,q^k}[\tilde{\mu}^k]$ defined in (13), for some $s > 0$ that will be chosen independently of k , on the trajectories $z \in C^0([-T, T]; H_0^1(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$ such that $\partial_t^2 z - \Delta z + q^k z \in L^2((0, T) \times \Omega)$, $\partial_n z \in L^2((0, T) \times \Gamma_0)$ and $z(0, \cdot) = 0$ in Ω . Let \tilde{Z}^k be the unique minimizer of the functional $J_{s,q^k}[\tilde{\mu}^k]$.

• *Step 3* - Set

$$(16) \quad \tilde{q}^{k+1} = q^k + \frac{\partial_t \tilde{Z}^k(0)}{w_0}, \quad \text{in } \Omega,$$

where w_0 is the initial condition in (15) (or (1)).

• *Step 4* - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{with } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| \geq m, \end{cases}$$

where m is the a priori bound in (7).

171 of being the Carleman estimate in [Im02], is the Carleman estimate in [LRS86].
 172 The main improvement with respect to Algorithm 1 is the fact that the functional
 173 $J_{s,q}[\tilde{\mu}]$ in (13) contains weight functions with only one exponential, making the prob-
 174 lem less difficult to implement. However, it is still numerically challenging to use
 175 such functionals, especially as the convergence of Algorithm 2 gets better for large
 176 parameter s . We propose below two ideas to make it numerically tractable.

177 1.3. Preconditioning, processing and discretizing the cost functional.

178 When considering the functional $J_{s,q}[\tilde{\mu}]$ in (13), one easily sees that exponentials
 179 factors can be removed if considering the unknown $ze^{s\varphi}$ instead of z . Such transfor-
 180 mation corresponds to a preconditioning of the functional $J_{s,q}[\tilde{\mu}]$. Indeed, that way,
 181 exponential factors do not appear anymore when computing the gradient of the cost
 182 functional $J_{s,q}[\tilde{\mu}]$. Nevertheless, there are still exponentials factors appearing in the
 183 measurements. We therefore also develop a progressive algorithm in the resolution of
 184 the minimization process. The idea is to consider intervals in which the weight func-
 185 tion φ does not significantly change, allowing to preserve numerical accuracy despite
 186 the possible large values of s . Details will be given in Section 3.

187

188 When implementing the above strategy numerically, one has to discretize the
 189 wave equation under consideration, and to adapt the functional $J_{s,q}[\tilde{\mu}]$ to the discrete
 190 setting. As it is well-known [Tre82, Zua05], most of the numerical schemes exhibit
 191 some pathologies at high-frequency, namely discrete rays propagating at velocity 0 or

192 blow up of observability estimates. Therefore, we need to take some care to adapt the
 193 functional $J_{s,q}[\tilde{\mu}]$ to the discrete setting. In particular, following ideas well-developed
 194 in the context of the observability of discrete waves (see [Zua05]), we will introduce a
 195 naive discrete version of $J_{s,q}[\tilde{\mu}]$ and penalize the high-frequencies.

196 To simplify the presentation of these penalized frequency functionals, we will introduce
 197 it in full details on a space semi-discrete and time continuous 1d wave equations, where
 198 the space semi-discretization is done using the finite-difference method on a uniform
 199 mesh. In this case, our approach, even at the discrete level, can be made completely
 200 rigorous by adapting the arguments in the continuous setting and the discrete Carle-
 201 man estimates obtained in [BE11] (recently extended to a multi-dimensional setting
 202 in [BEO15]). We refer to Section 4 for extensive details.

203

204 Section 5 then presents numerical results illustrating our method on several ex-
 205 amples. In particular, we will illustrate the good convergence of the algorithm when
 206 the parameter s is large. We shall also discuss the cases in which the measurement
 207 is blurred by some noise and the case in which the initial datum w_0 is not positive
 208 everywhere.

209

210 *Outline.* Section 2 is devoted to the proof of the convergence of Algorithm 2. In
 211 Section 3 we explain how the minimization process of the functional $J_{s,q}$ in (13) can
 212 be strongly simplified. Section 4 then makes precise the new difficulties arising when
 213 discretizing the functional $J_{s,q}$, and Section 5 presents several numerical experiments.

214

2. Study of Algorithm 2.

215 **2.1. Main ingredients.** The goal of this section is to prove Theorem 2. As
 216 mentioned in the introduction, the proof will closely follows the one of Theorem 1
 217 in [BdBE13]. The main difference is that, instead of using the Carleman estimate
 218 developed in [Im02, Baufr], we will base our proof on the following one:

219 **THEOREM 3.** *Assume the multiplier conditions (3)-(4) and $\beta \in (0, 1)$ as in (8).
 220 Define the weight function φ as in (9). Then there exist $s_0 > 0$ and a positive constant
 221 M such that for all $s \geq s_0$:*

222

$$223 \quad (18) \quad s \int_{-T}^T \int_{\Omega} e^{2s\varphi} (|\partial_t z|^2 + |\nabla z|^2 + s^2 |z|^2) \, dxdt \leq M \int_{-T}^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 z - \Delta z|^2 \, dxdt$$

$$224 \quad \quad \quad + Ms \int_{-T}^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z|^2 \, d\sigma dt + Ms^3 \iint_{(|t|,x) \in \mathcal{O}} e^{2s\varphi} |z|^2 \, dxdt,$$

225

226 for all $z \in C^0([-T, T]; H_0^1(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$ with $\partial_t^2 z - \Delta z \in L^2((-T, T) \times \Omega)$,
 227 where the set \mathcal{O} satisfies (12).

228 Furthermore, if $z(0, \cdot) = 0$ in Ω , one can add to the left hand-side of (18), the following
 229 term:

$$230 \quad (19) \quad s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx.$$

231 The Carleman estimate of Theorem 3 is quite classical and can be found in the liter-
 232 ature in several places, among which [LRS86, Isa06, Zha00, FYZ07, Bel08]. For the
 233 convenience of the reader, we briefly sketch the proof in Section 2.2. However, the
 234 proof of the fact that the term (19) can be added in the left hand side of (18) when
 235 $z(0, \cdot) = 0$ in Ω is not explicitly written in the aforementioned references, although

275 where we have set $\alpha = 2d - 2$, d being the space dimension. Based on the estimate

$$\begin{aligned}
276 \quad & 2 \int_{-T}^T \int_{\Omega} P_1 y P_2 y \, dx dt \leq \int_{-T}^T \int_{\Omega} (|P_1 y|^2 + |P_2 y|^2) \, dx dt + 2 \int_{-T}^T \int_{\Omega} P_1 y P_2 y \, dx dt \\
277 \quad (23) \quad & \leq 2 \int_{-T}^T \int_{\Omega} |\mathcal{L}_s y|^2 \, dx dt + 2 \int_{-T}^T \int_{\Omega} |Ry|^2 \, dx dt,
\end{aligned}$$

the main part of the proof consists in the computation and bound from below of the cross-term

$$I = \int_{-T}^T \int_{\Omega} P_1 y P_2 y \, dx dt.$$

278 Tedious computations and integrations by parts yield

$$\begin{aligned}
279 \quad I &= s \int_{-T}^T \int_{\Omega} |\partial_t y|^2 (\partial_t^2 \varphi + \Delta \varphi - \alpha) \, dx dt + s \int_{-T}^T \int_{\Omega} |\nabla y|^2 (\partial_t^2 \varphi - \Delta \varphi + \alpha + 4) \, dx dt \\
280 \quad &+ s^3 \int_{-T}^T \int_{\Omega} |y|^2 [\partial_t (\partial_t \varphi (|\partial_t \varphi|^2 - |\nabla \varphi|^2)) + \alpha (|\partial_t \varphi|^2 - |\nabla \varphi|^2) \\
281 \quad &\quad - \nabla \cdot (\nabla \varphi (|\partial_t \varphi|^2 - |\nabla \varphi|^2))] \, dx dt \\
282 \quad &- s \left[\int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2) \partial_t \varphi \, dx \right]_{-T}^T + 2s \left[\int_{\Omega} \partial_t y (\nabla y \cdot \nabla \varphi) \, dx \right]_{-T}^T \\
283 \quad &- s^3 \left[\int_{\Omega} y^2 (|\partial_t \varphi|^2 - |\nabla \varphi|^2) \partial_t \varphi \, dx \right]_{-T}^T + \alpha s \left[\int_{\Omega} \partial_t y y \, dx \right]_{-T}^T \\
284 \quad &- s \int_{-T}^T \int_{\partial \Omega} |\partial_n y|^2 \partial_n \varphi \, d\sigma dt.
\end{aligned}$$

285 Let us now briefly explain how each term can be estimated.

286

287 • We focus on the terms in $s|\partial_t y|^2$ and $s|\nabla y|^2$ in order to insure that they are
288 strictly positive. Taking $\alpha = 2d - 2$, this means

$$\begin{aligned}
289 \quad & \partial_t^2 \varphi + \Delta \varphi - \alpha = -2\beta + 2d - \alpha = 2(1 - \beta) \quad \text{and} \\
290 \quad & \partial_t^2 \varphi - \Delta \varphi + \alpha + 4 = -2\beta - 2d + \alpha + 4 = 2(1 - \beta),
\end{aligned}$$

291 that are positive thanks to the assumption $\beta \in (0, 1)$.

292

293 • The terms in $s^3|y|^2$ can be rewritten as follows (since $\nabla^2 \varphi = 2\text{Id}$):

$$\begin{aligned}
294 \quad & \partial_t (\partial_t \varphi (|\partial_t \varphi|^2 - |\nabla \varphi|^2)) + \alpha (|\partial_t \varphi|^2 - |\nabla \varphi|^2) - \nabla \cdot (\nabla \varphi (|\partial_t \varphi|^2 - |\nabla \varphi|^2)) \\
295 \quad &= (\partial_t^2 \varphi - \Delta \varphi + \alpha) (|\partial_t \varphi|^2 - |\nabla \varphi|^2) + 2|\partial_t \varphi|^2 \partial_t^2 \varphi + 2\nabla^2 \varphi \cdot \nabla \varphi \cdot \nabla \varphi \\
296 \quad &= (-6\beta - 2d + \alpha) (|\partial_t \varphi|^2 - |\nabla \varphi|^2) + 4(1 - \beta) |\nabla \varphi|^2 \\
297 \quad &= -(2 + 6\beta) (|\partial_t \varphi|^2 - |\nabla \varphi|^2) + 4(1 - \beta) |\nabla \varphi|^2.
\end{aligned}$$

This quantity is bounded from below by a strictly positive constant in the region of $(-T, T) \times \Omega$ in which

$$|\partial_t \varphi(t, x)|^2 - |\nabla \varphi(t, x)|^2 \leq 0 \iff \beta t \leq |x - x_0|,$$

299 i.e. the complementary of the set $\{(t, x) \in (-T, T) \times \Omega \text{ with } (|t|, x) \in \mathcal{O}\}$ where \mathcal{O}
 300 satisfies (12).

301

302 • We now estimate the boundary terms in time¹ appearing at time $t = T$ and
 303 $t = -T$. We focus on the terms at time T , as the ones at time $-T$ can be handled
 304 similarly. Let us first collect them:

305

$$306 \quad I_T := 2s\beta T \int_{\Omega} (|\partial_t y(T)|^2 + |\nabla y(T)|^2) dx + 8s^3\beta T \int_{\Omega} |y(T)|^2 (\beta^2 T^2 - |x - x_0|^2) dx$$

$$307 \quad + 4s \int_{\Omega} \partial_t y(T) \left(\nabla y(T) \cdot (x - x_0) + \frac{\alpha}{4} y(T) \right) dx.$$

308

309 The first and second terms are obviously positive (under Condition (8) for the second
 310 one), so we only need to check that they are sufficiently positive to absorb the last
 311 term, whose sign is unknown. We remark that

$$312 \quad \int_{\Omega} \left| \nabla y(T) \cdot (x - x_0) + \frac{\alpha}{4} y(T) \right|^2 dx$$

$$313 \quad = \int_{\Omega} |\nabla y(T) \cdot (x - x_0)|^2 dx + \frac{\alpha}{4} \int_{\Omega} (x - x_0) \cdot \nabla (|y(T)|^2) dx + \frac{\alpha^2}{16} \int_{\Omega} |y(T)|^2 dx$$

$$314 \quad = \int_{\Omega} |\nabla y(T) \cdot (x - x_0)|^2 dx + \left(\frac{\alpha^2}{16} - \frac{\alpha d}{4} \right) \int_{\Omega} |y(T)|^2 dx$$

$$315 \quad \leq \sup_{\Omega} \{|x - x_0|^2\} \int_{\Omega} |\nabla y(T)|^2 dx,$$

316 since $\alpha = 2d - 2$ gives $\alpha^2 - 4\alpha d = -4(d - 1)(d + 1) \leq 0$. This inequality allows to
 317 deduce, by Cauchy-Schwarz inequality, that

318

$$319 \quad 4s \int_{\Omega} \partial_t y(T) \left(\nabla y(T) \cdot (x - x_0) + \frac{\alpha}{4} y(T) \right) dx$$

$$320 \quad \leq 2s \sup_{\Omega} \{|x - x_0|\} \left(\int_{\Omega} (|\partial_t y(T)|^2 + |\nabla y(T)|^2) dx \right).$$

321

322 Using again Condition (8), we easily obtain $I_T \geq 0$.

323

324 Gathering these informations, and using the geometric condition (3) on Γ_0 , it
 325 yields that there exists a constant $M > 0$ independent of s such that

326

$$327 \quad \int_{-T}^T \int_{\Omega} P_1 y P_2 y dx dt \geq Ms \int_{-T}^T \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) dx dt$$

$$328 \quad - Ms \int_{-T}^T \int_{\Gamma_0} |\partial_n y|^2 d\sigma dt - Ms^3 \iint_{(|t|, x) \in \mathcal{O}} |y|^2 dx dt.$$

329

¹The authors acknowledge Xiaoyu Fu for having pointed out to us the fact that these boundary terms have positive signs.

330 From (23), we easily derive

331

$$\begin{aligned}
332 \quad & s \int_{-T}^T \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) \, dxdt + \int_{-T}^T \int_{\Omega} (|P_1 y|^2 + |P_2 y|^2) \, dxdt \\
333 \quad & \leq M \int_{-T}^T \int_{\Omega} |\mathcal{L}_s y|^2 \, dxdt + Ms^2 \int_{-T}^T \int_{\Omega} |y|^2 \, dxdt \\
334 \quad & \quad + Ms \int_{-T}^T \int_{\Gamma_0} |\partial_n y|^2 \, d\sigma dt + Ms^3 \iint_{(|t|,x) \in \mathcal{O}} |y|^2 \, dxdt. \\
335
\end{aligned}$$

336 We take now s_0 large enough in order to make sure that the term in $s^2 |y|^2$ of the right
337 hand side is absorbed by the dominant term in $s^3 |y|^2$ of the left hand side as soon as
338 $s \geq s_0$ and we obtain

339

$$\begin{aligned}
340 \quad (24) \quad & s \int_{-T}^T \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) \, dxdt + \int_{-T}^T \int_{\Omega} (|P_1 y|^2 + |P_2 y|^2) \, dxdt \\
341 \quad & \leq M \int_{-T}^T \int_{\Omega} |\mathcal{L}_s y|^2 \, dxdt + Ms \int_{-T}^T \int_{\Gamma_0} |\partial_n y|^2 \, d\sigma dt + Ms^3 \iint_{(|t|,x) \in \mathcal{O}} |y|^2 \, dxdt \\
342
\end{aligned}$$

343 We then deduce (18) by substituting $y = ze^{s\varphi}$.

344

345 Furthermore, under the additional condition $z(0, \cdot) = 0$ in Ω , we get $y(0, \cdot) = 0$
346 in Ω . We then choose $\rho : t \mapsto \rho(t)$ a smooth function such that $\rho(0) = 1$ and ρ
347 vanishes close to $t = -T$. We multiply $P_1 y$ by $\rho \partial_t y$ and integrate over $(-T, 0) \times \Omega$,
348 to get

$$\begin{aligned}
349 \quad & \int_{-T}^0 \int_{\Omega} P_1 y \rho \partial_t y \, dxdt = \int_{-T}^0 \int_{\Omega} (\partial_t^2 y - \Delta y + s^2 ((\partial_t \varphi)^2 - |\nabla \varphi|^2) y) \rho \partial_t y \, dxdt \\
350 \quad & = \frac{1}{2} \int_{-T}^0 \int_{\Omega} \rho \partial_t (|\partial_t y|^2 + |\nabla y|^2) \, dxdt + \frac{s^2}{2} \int_{-T}^0 \int_{\Omega} \rho (|\partial_t \varphi|^2 - |\nabla \varphi|^2) \partial_t (y^2) \, dxdt \\
351 \quad & = \frac{1}{2} \int_{\Omega} |\partial_t y(0)|^2 \, dx - \frac{1}{2} \int_{-T}^0 \int_{\Omega} \partial_t \rho (|\partial_t y|^2 + |\nabla y|^2) + s^2 \partial_t (\rho (|\partial_t \varphi|^2 - |\nabla \varphi|^2)) y^2 \, dxdt \\
352 \quad & \geq \frac{1}{2} \int_{\Omega} |\partial_t y(0)|^2 \, dx - M \int_{-T}^0 \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) \, dxdt.
\end{aligned}$$

By Cauchy-Schwarz inequality, this implies

$$s^{1/2} \int_{\Omega} |\partial_t y(0)|^2 \, dx \leq \int_{-T}^T \int_{\Omega} |P_1 y|^2 \, dxdt + Ms \int_{-T}^T \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) \, dxdt.$$

353 Using (24) and $y = ze^{s\varphi}$, we easily deduce the estimate of term (19) and conclude the
354 proof of Theorem 3. \square

355 From this proof of Theorem 3, we can directly exhibit (see (24)) the following ‘‘con-
356 jugate’’ Carleman estimate, of practical interest later on:

357 **COROLLARY 4.** *Assume the multiplier condition (3)-(4) and $\beta \in (0, 1)$ as in (8).
358 Define the weight function φ as in (9). Then there exist constants $M > 0$ and $s_0 > 0$*

359 such that for all $s \geq s_0$,
 360

$$361 \quad (25) \quad s \int_{-T}^T \int_{\Omega} (|\partial_t y|^2 + |\nabla y|^2 + s^2 |y|^2) \, dxdt + \int_{-T}^T \int_{\Omega} (|P_1 y|^2 + |P_2 y|^2) \, dxdt$$

$$362 \quad \leq M \int_{-T}^T \int_{\Omega} |\mathcal{L}_s y|^2 \, dxdt + Ms \int_{-T}^T \int_{\Gamma_0} |\partial_n y|^2 \, d\sigma dt + Ms^3 \iint_{(|t|,x) \in \mathcal{O}} |y|^2 \, dxdt$$

364 for all $y \in C^0([-T, T]; H_0^1(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$, with $\mathcal{L}_s y \in L^2((-T, T) \times \Omega)$,
 365 where \mathcal{L}_s , P_1 and P_2 are defined in (22).

366 Furthermore, if $y(0, \cdot) = 0$ in Ω , one can add the term $s^{1/2} \int_{\Omega} |\partial_t y(0)|^2 \, dx$ to the left
 367 hand-side of (25).

368 2.3. Proof of the convergence theorem.

369 *Proof of Theorem 2.* Let us first begin by showing that Algorithm 2 is well-
 370 defined. We introduce

$$372 \quad \mathcal{T}_q = \left\{ z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \right.$$

$$373 \quad \left. \text{with } \partial_t^2 z - \Delta z + qz \in L^2((0, T) \times \Omega) \text{ and } z(0, \cdot) = 0 \text{ in } \Omega \right\},$$

375 endowed with the norm
 376

$$377 \quad \|z\|_{\text{obs},s,q}^2 = \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 z - \Delta z + qz|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z|^2 \, d\sigma dt$$

$$378 \quad + s^3 \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 \, dxdt.$$

380 The proof that this quantity is a norm on \mathcal{T}_q stems from the Carleman estimate of
 381 Theorem 3 applied to $z_e(t, x) = z(t, x)$ for $t \in [0, T]$ and $z_e(t, x) = -z(-t, x)$ for
 382 $t \in [-T, 0]$, $x \in \Omega$. Indeed, (18) applied to z_e yields for all $s \geq s_0$,

$$383 \quad s^3 \int_0^T \int_{\Omega} e^{2s\varphi} |z|^2 \, dxdt \leq 2M \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 z - \Delta z + qz|^2 \, dxdt$$

$$384 \quad + 2M \|q\|_{L^\infty(\Omega)}^2 \int_0^T \int_{\Omega} e^{2s\varphi} |z|^2 \, dxdt$$

$$385 \quad + Ms \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z|^2 \, d\sigma dt + Ms^3 \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 \, dxdt,$$

386 so that $\|\cdot\|_{\text{obs},s,q}$ is a norm on \mathcal{T}_q provided s is large enough, and then for all $s > 0$ as
 387 the weight functions are bounded on $[0, T] \times \bar{\Omega}$. This immediately implies that $J_{s,q}[\tilde{\mu}]$
 388 defined in (13) is coercive and strictly convex on the set \mathcal{T}_q , so that it admits a unique
 389 minimizer and as a consequence, Algorithm 2 is well-defined.

390 Moreover, this shows that the class \mathcal{T}_q , which was *a priori* dependent of q , is in
 391 fact independent of q (for $q \in L^\infty(\Omega)$) and is simply given by

$$393 \quad \mathcal{T} = \left\{ z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \right.$$

$$394 \quad \left. \text{with } \partial_t^2 z - \Delta z \in L^2((0, T) \times \Omega) \text{ and } z(0, \cdot) = 0 \text{ in } \Omega \right\}.$$

395

396 In order to show estimate (17), instead of considering only functionals of the
 397 form $J_{s,q}[\tilde{\mu}]$, we introduce slightly more general functionals $J_{s,q}[\tilde{\mu}, g]$ given for $s > 0$,
 398 $q \in L^\infty(\Omega)$, $\tilde{\mu} \in L^2((0, T) \times \Gamma_0)$, $g \in L^2((0, T) \times \Omega)$ and for all $z \in \mathcal{T}$, by:

399

$$400 \quad (26) \quad J_{s,q}[\tilde{\mu}, g](z) = \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 z - \Delta z + qz - g|^2 dxdt$$

$$401 \quad + \frac{s}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z - \tilde{\mu}|^2 d\sigma dt + \frac{s^3}{2} \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 dxdt.$$

403 With the same argument as above, the functional $J_{s,q}[\tilde{\mu}, g]$ is coercive in the norm
 404 $\|\cdot\|_{\text{obs},s,q}$ and strictly convex, so that it admits a unique minimizer for each $\tilde{\mu} \in$
 405 $L^2((0, T) \times \Gamma_0)$ and $g \in L^2((0, T) \times \Omega)$.

406 We then observe that $\tilde{z}^k := \eta(\varphi)z^k$, where z^k satisfies (20) (recall the definitions
 407 of η in (14) and φ in (9), pictured in Figure 2), is the minimizer of $J_{s,q^k}[\tilde{\mu}^k, \tilde{g}^k]$ with

$$408 \quad (27) \quad \tilde{g}^k = \eta(\varphi)(Q - q^k)\partial_t W[Q] + [\partial_t^2 - \Delta, \eta(\varphi)]z^k,$$

409 since it solves:

$$410 \quad (28) \quad \begin{cases} \partial_t^2 \tilde{z}^k - \Delta \tilde{z}^k + q^k \tilde{z}^k = \tilde{g}^k, & \text{in } (0, T) \times \Omega, \\ \tilde{z}^k = 0, & \text{on } (0, T) \times \partial\Omega, \\ \tilde{z}^k(0) = 0, \quad \partial_t \tilde{z}^k(0) = \eta(\varphi(0, \cdot))z_1^k, & \text{in } \Omega, \end{cases}$$

411 and $\partial_n \tilde{z}^k = \tilde{\mu}^k = \eta(\varphi)\partial_t(\partial_n w[q^k] - \partial_n W[Q])$ on $(0, T) \times \Gamma_0$.

412 We shall then compare \tilde{Z}^k and \tilde{z}^k , the minimizers of the functionals $J_{s,q^k}[\tilde{\mu}^k, 0]$
 413 and $J_{s,q^k}[\tilde{\mu}^k, \tilde{g}^k]$ respectively, especially at the time $t = 0$ corresponding to the set in
 414 which the information on $(Q - q^k)$ is encoded. The result is stated as follows:

415 **PROPOSITION 5.** *Assume the geometric and time conditions (3)-(4) on Γ_0 and T ,
 416 that β is chosen as in (8), and let $\mu \in L^2((0, T) \times \Gamma_0)$ and $g^a, g^b \in L^2((0, T) \times \Omega)$.
 417 Assume also that q belongs to $L_{\leq m}^\infty(\Omega)$ for $m > 0$.*

418 *Let Z^j be the unique minimizer of the functional $J_{s,q}[\mu, g^j]$ on \mathcal{T} for $j \in \{a, b\}$. Then
 419 there exist positive constants $s_0(m)$ and $M = M(m)$ such that for $s \geq s_0(m)$ we have:*

$$420 \quad (29) \quad s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t Z^a(0) - \partial_t Z^b(0)|^2 dx \leq M \int_0^T \int_\Omega e^{2s\varphi} |g^a - g^b|^2 dxdt.$$

421 where φ and $s_0(m)$ are chosen so that Theorem 3 holds.

422 We postpone the proof of Proposition 5 to the end of the section and first show how
 423 it can be used for the proof of Theorem 2.

424

425 Recall now that $\partial_t \tilde{z}^k(0, \cdot) = (Q - q^k)w^0$. Setting \tilde{q}^{k+1} as in (16), we get from
 426 Proposition 5 applied to $Z^a = \tilde{Z}^k$ and $Z^b = \tilde{z}^k$ that

$$427 \quad (30) \quad s^{1/2} \int_\Omega e^{2s\varphi(0)} |\tilde{q}^{k+1} - Q|^2 |w^0|^2 dx \leq M \int_0^T \int_\Omega e^{2s\varphi} |\tilde{g}^k|^2 dxdt.$$

428 The next step is to get an estimates of \tilde{g}^k defined by (27). Using the fact that

429 $[\eta(\varphi), \partial_t^2 - \Delta] z^k$ has support in a region where $\varphi \leq d_0^2 := d(x_0, \Omega)^2$, we obtain

$$\begin{aligned}
430 \quad & \int_0^T \int_{\Omega} e^{2s\varphi} |\tilde{g}^k|^2 dxdt \leq M \int_0^T \int_{\Omega} e^{2s\varphi} |\eta(\varphi)(Q - q^k) \partial_t W[Q]|^2 dxdt \\
431 \quad & \quad + M \int_0^T \int_{\Omega} e^{2s\varphi} |[\eta(\varphi), \partial_t^2 - \Delta] z^k|^2 dxdt \\
432 \quad & \leq M \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2 \int_{\Omega} e^{2s\varphi(0)} |q^k - Q|^2 dx \\
433 \quad & \quad + M e^{2sd_0^2} \int_0^T \int_{\Omega} (|\nabla z^k|^2 + |\partial_t z^k|^2 + |z^k|^2) dxdt.
\end{aligned}$$

434 Usual *a priori* energy estimates for z^k solution of equation (21) also yields

$$\begin{aligned}
435 \quad (31) \quad & \|z^k\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\partial_t z^k\|_{L^\infty(0,T;L^2(\Omega))} \leq M (\|z_1^k\|_{L^2(\Omega)} + \|g^k\|_{L^1(0,T;L^2(\Omega))}) \\
436 \quad & \leq M \|Q - q^k\|_{L^2(\Omega)} (\|w_0\|_{L^\infty(\Omega)} + \|\partial_t W[Q]\|_{L^1(0,T;L^\infty(\Omega))}) \\
437 \quad & \leq M \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))} \|Q - q^k\|_{L^2(\Omega)}, \\
438 \quad &
\end{aligned}$$

439 so that combining the above estimates, we get

$$\begin{aligned}
440 \quad & \\
441 \quad & s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\tilde{q}^{k+1} - Q|^2 |w^0|^2 dx \leq M \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2 \int_{\Omega} e^{2s\varphi(0)} |q^k - Q|^2 dx \\
442 \quad & \quad + M \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2 e^{2sd_0^2} \|Q - q^k\|_{L^2(\Omega)}^2. \\
443 \quad &
\end{aligned}$$

444 Using $\varphi(0, x) \geq d_0^2$ for all x in Ω and Assumption (6), we deduce

$$(32) \quad s^{1/2} \alpha^2 \int_{\Omega} e^{2s\varphi(0)} |\tilde{q}^{k+1} - Q|^2 dx \leq M \|W[Q]\|_{H^1(0,T;L^\infty(\Omega))}^2 \int_{\Omega} e^{2s\varphi(0)} |q^k - Q|^2 dx.$$

446 Now, using the *a priori* assumption (7), i.e. $Q \in L_{\leq m}^\infty(\Omega)$, we easily check that this
447 estimate cannot deteriorate in step 4 of Algorithm 2, which is there only to ensure
448 that the sequence q^k stays in $L_{\leq m}^\infty(\Omega)$ for all $k \in \mathbb{N}$. This completes the proof of
449 Theorem 2. \square

450 It only remains to prove the former proposition.

451 *Proof of Proposition 5.* Let us write the Euler Lagrange equations satisfied by
452 Z^j , for $j \in \{a, b\}$. For all $z \in \mathcal{T}$, we have

$$\begin{aligned}
453 \quad (33) \quad & \int_0^T \int_{\Omega} e^{2s\varphi} (\partial_t^2 Z^j - \Delta Z^j + q Z^j - g^j) (\partial_t^2 z - \Delta z + qz) dxdt \\
454 \quad & \quad + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} (\partial_n Z^j - \mu) \partial_n z d\sigma dt + s^3 \iint_{\mathcal{O}} e^{2s\varphi} Z^j z dxdt = 0. \\
455 \quad & \\
456 \quad &
\end{aligned}$$

457 Applying (33) for $j = a$ and $j = b$ to $z = Z = Z^a - Z^b$ and subtracting the two
458 identities, we obtain:

$$\begin{aligned}
459 \quad & \\
460 \quad & \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 Z - \Delta Z + qZ|^2 dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n Z|^2 d\sigma dt + s^3 \iint_{\mathcal{O}} e^{2s\varphi} |Z|^2 dxdt \\
461 \quad & \quad = \int_0^T \int_{\Omega} e^{2s\varphi} (g^a - g^b) (\partial_t^2 Z - \Delta Z + qZ) dxdt. \\
462 \quad &
\end{aligned}$$

463 This implies

464

$$465 \quad (34) \quad \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 Z - \Delta Z + qZ|^2 dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n Z|^2 d\sigma dt$$

$$466 \quad \quad \quad + s^3 \iint_{\mathcal{O}} e^{2s\varphi} |Z|^2 dxdt \leq \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |g^a - g^b|^2 dxdt.$$

467

468 Since the left hand side of (34) is precisely the right hand side of the Carleman
469 estimate (18), applying Theorem 3 to Z , we immediately deduce (29). \square

470 **3. Technical issues on the minimization of the cost functional.** The goal
471 of this section is to give several details about the actual construction of an efficient
472 numerical algorithm based on Algorithm 2. The main step in Algorithm 2 is to
473 minimize the functional $J_{s,q}[\tilde{\mu}]$, that we recall here for convenience,

$$474 \quad J_{s,q}[\tilde{\mu}](z) = \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t^2 z - \Delta z + qz|^2 dxdt + \frac{s}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_n z - \tilde{\mu}|^2 d\sigma dt$$

$$475 \quad \quad \quad + \frac{s^3}{2} \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 dxdt,$$

476 and which is minimized on the set

477

$$478 \quad (35) \quad \mathcal{T} = \left\{ z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \right.$$

$$479 \quad \quad \quad \left. \text{with } \partial_t^2 z - \Delta z \in L^2((0, T) \times \Omega) \text{ and } z(0) = 0 \text{ in } \Omega \right\}.$$

480

481 Due to the presence of large exponential factors in the functional, the minimization
482 of $J_{s,q}[\tilde{\mu}]$ is not a straightforward task from the numerical point of view, even if, as
483 we emphasized earlier, the minimization of $J_{s,q}[\tilde{\mu}]$ is much less stiffer than the one of
484 $K_{s,q}[\mu]$ defined in (10) [BdBE13]. We therefore propose the two following ideas:

- 485 • Work on the conjugate variable $y = ze^{s\varphi}$. This change of unknown acts as a
486 preconditioner. Details are given in Section 3.1.
- 487 • A progressive algorithm to minimize the functional $J_{s,q}[\tilde{\mu}]$ in subdomains in
488 which the variations of the exponential factors are small, see Section 3.2.

3.1. Conjugate variable. For z in \mathcal{T} , we set $y = ze^{s\varphi}$, so that y satisfies the
following equation:

$$\begin{cases} \partial_t^2 y - \Delta y + qy - 2s\partial_t\varphi\partial_t y + 2s\nabla\varphi \cdot \nabla y \\ \quad - s(\partial_t^2\varphi - \Delta\varphi)y + s^2(|\partial_t\varphi|^2 - |\nabla\varphi|^2)y = e^{s\varphi}(\partial_t^2 - \Delta + q)z, & \text{in } (0, T) \times \Omega, \\ y = 0, & \text{on } (0, T) \times \partial\Omega, \\ y(0) = 0, \quad \partial_t y(0) = z_1 e^{s\varphi(0)}, & \text{in } \Omega, \end{cases}$$

489 where $\partial_t\varphi = -2\beta t$, $\nabla\varphi = 2(x - x_0)$, $\partial_t^2\varphi = -2\beta$ and $\Delta\varphi = 2d$. We set $\mathcal{L}_{s,q}$ defined
490 by $\mathcal{L}_{s,q} = e^{s\varphi}(\partial_t^2 - \Delta + q)e^{-s\varphi}$:

$$491 \quad \mathcal{L}_{s,q}y = \partial_t^2 y - \Delta y + qy - 2s\partial_t\varphi\partial_t y + 2s\nabla\varphi \cdot \nabla y - s(\partial_t^2\varphi - \Delta\varphi)y$$

$$492 \quad \quad \quad + s^2(|\partial_t\varphi|^2 - |\nabla\varphi|^2)y$$

$$493 \quad \quad \quad = \partial_t^2 y - \Delta y + qy + 4s\beta t\partial_t y + 4s(x - x_0) \cdot \nabla y + 2s(\beta + d)y$$

$$494 \quad (36) \quad \quad \quad + 4s^2(\beta^2 t^2 - |x - x_0|^2)y.$$

Thus, minimizing $J_{s,q}[\tilde{\mu}]$ in (13) on the set \mathcal{T} is equivalent to minimize the functional $\tilde{J}_{s,q}[\tilde{\mu}]$ defined by

$$\tilde{J}_{s,q}[\tilde{\mu}](y) = \frac{1}{2} \int_0^T \int_{\Omega} |\mathcal{L}_{s,q}y|^2 dxdt + \frac{s}{2} \int_0^T \int_{\Gamma_0} |\partial_n y - \tilde{\mu}e^{s\varphi}|^2 d\sigma dt + \frac{s^3}{2} \iint_{\mathcal{O}} y^2 dxdt$$

on the same set \mathcal{T} . The minimization process for $\tilde{J}_{s,q}[\tilde{\mu}]$ is then equivalent to the resolution of the following variational formulation:

Find $Y \in \mathcal{T}$ such that for all $y \in \mathcal{T}$,

$$(37) \quad \int_0^T \int_{\Omega} \mathcal{L}_{s,q}Y \mathcal{L}_{s,q}y dxdt + s \int_0^T \int_{\Gamma_0} \partial_n Y \partial_n y d\sigma dt + s^3 \iint_{\mathcal{O}} Yy dxdt = s \int_0^T \int_{\Gamma_0} e^{s\varphi} \tilde{\mu} \partial_n y d\sigma dt.$$

From the Carleman estimate (25) applied to y extended for negative times t by $y(t) = -y(-t)$, the left-hand side of (37) defines a coercive quadratic form, while the exponentials now appear only in the right hand side of (37). Therefore, no exponential factor appears anymore in the computation of the gradient of the functional $\tilde{J}_{s,q}[\tilde{\mu}]$. Our next goal is to deal with the exponential factor still in front of $\tilde{\mu}$.

3.2. Progressive process. The idea to tackle the exponential factor in the right hand side of (37) is to develop a progressive process to compute the minimizer of $\tilde{J}_{s,q}[\tilde{\mu}]$ as the aggregation of several problems localized in subdomains in which the exponential factors are all of the same order.

In this objective, from the smooth cut-off function η equal to 1 for $\tau \geq d_0^2$ defined in (14), we introduce N cut-off functions $\{\eta_j\}_{1 \leq j \leq N}$ (these ones are not necessarily smooth) such that

$$(38) \quad \forall \tau \in \mathbb{R}, \quad \sum_{j=1}^N \eta_j(\tau) = \eta(\tau),$$

as illustrated in Figure 3.

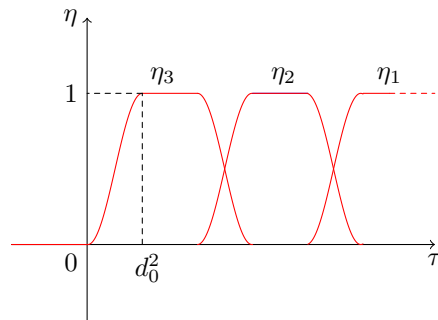


Fig. 3: Example of cut-off functions η_j for $1 \leq j \leq 3$.

Therefore, the target flux $\tilde{\mu} = \eta(\varphi)\mu$ can be decomposed as follows:

$$(39) \quad \tilde{\mu} = \eta(\varphi)\mu = \sum_{j=1}^N \tilde{\mu}_j,$$

where $\tilde{\mu}_j(t, x) = \eta_j(\varphi(t, x))\mu(t, x)$, $\forall(t, x) \in (0, T) \times \Gamma_0$, $\forall j \in \{1, \dots, N\}$.

As the variational formulation in (37) is linear in $\tilde{\mu}$, one immediately gets that, for each $j \in \{1, \dots, N\}$, we denote by Y_j the minimizer of $\tilde{J}_{s,q}[\tilde{\mu}_j]$ on \mathcal{T} , then the minimizer Y of $\tilde{J}_{s,q}[\tilde{\mu}]$ is simply given by

$$Y = \sum_{j=1}^N Y_j.$$

518 The interest of this approach is that the target flux $\tilde{\mu}_j$ involves exponential terms in
519 φ on the support of $\eta_j(\varphi(t, x))$. This becomes particularly interesting if we impose
520 that for each $j \in \{1, \dots, N\}$,

$$521 \quad (40) \quad \text{Supp } \eta_j \subset [a_j, b_j] \text{ with } b_j - a_j \leq C,$$

for some constant $C > 0$. Indeed, in that case, we get

$$\frac{\sup_{\text{Supp } \eta_j(\varphi)} e^{s\varphi}}{\inf_{\text{Supp } \eta_j(\varphi)} e^{s\varphi}} \leq e^{sC},$$

522 so that if $C \simeq 1/s$, all the exponentials are of the same order when computing $\tilde{\mu}_j$.
523 Consequently, under the conditions (38)–(39)–(40), for all $j \in \{1, \dots, N\}$, the mini-
524 mization of $\tilde{J}_{s,q}[\tilde{\mu}_j]$ over \mathcal{T} is easier numerically than the direct minimization of $\tilde{J}_{s,q}[\tilde{\mu}]$
525 over \mathcal{T} . Besides, this approach can be used, at least theoretically, to parallelize the
526 minimization of $\tilde{J}_{s,q}[\tilde{\mu}]$ over the set \mathcal{T} .
527

Let us present one possible way to construct the functions η_j in practice, precisely the ones we used in our numerical experiments (where we chose to use C^∞ functions, even if it is not necessary). We set

$$d_0^2 = \inf_{\Omega} |x - x_0|^2 \quad \text{and} \quad L_0^2 = \sup_{\Omega} |x - x_0|^2.$$

Let us then choose an integer $N \in \mathbb{N}^*$ and set $\varepsilon_0 = d_0^2/N$. Next, define the cut-off function η as follows:

$$f(t) = \exp\left(\frac{-1}{t(\varepsilon_0 - t)}\right), \quad \text{and} \quad \eta(\tau) = \begin{cases} 0, & \text{if } \tau \leq 0, \\ 1 - \frac{\int_{\tau}^{\varepsilon_0} f(t)dt}{\int_0^{\varepsilon_0} f(t)dt}, & \text{if } 0 < \tau < \varepsilon_0, \\ 1, & \text{if } \tau \geq \varepsilon_0. \end{cases}$$

528 Thus we introduce the cut-off functions η_j defined by the formula

$$529 \quad \eta_0(\tau) = \eta(\tau - L_0^2), \quad \text{and for } j \in \{1, \dots, N\}, \\ 530 \quad \eta_j(\tau) = \eta\left(\tau - L_0^2 \frac{N-j}{N}\right) - \eta\left(\tau - L_0^2 \frac{N-(j-1)}{N}\right).$$

We then easily verify (38), $\text{Supp } \eta_0 \subset]L_0^2, +\infty[$, and that

$$\forall j \in \{1, \dots, N\}, \quad \text{Supp } \eta_j \subset \left]L_0^2 \left(1 - \frac{j}{N}\right), L_0^2 \left(1 - \frac{j-1}{N}\right) + \frac{d_0^2}{N}\right].$$

In particular, we have $\eta_0(\varphi(t, x)) = 0$ for all $(t, x) \in (-T, T) \times \Omega$ as $\varphi(t, x) \leq L_0^2$ for all $(t, x) \in (-T, T) \times \Omega$, so that we can omit $\eta_0(\varphi)$ in our approach.

By construction, the support of each η_j for $j \in \{1, \dots, N\}$ is included in an interval of size $(L_0^2 + d_0^2)/N$. We can then try to optimize the number N of intervals in the progressive algorithm so that on each interval the weight function $\exp(s\varphi)$ varies of less than 5 order of magnitude, for instance by taking N as a function of s as follows:

$$N = \left\lfloor \frac{s(L_0^2 + d_0^2)}{10} \right\rfloor + 1,$$

531 where $\lfloor \cdot \rfloor$ denotes the integer part.

532 **4. Discrete setting for the algorithm.** In this section, we present the tech-
533 nical solutions we have developed to implement numerically the algorithm. In order
534 to simplify the presentation, from now on we focus on the one-dimensional case
535 $\Omega = (0, L)$ and $\Gamma_0 = \{x = L\}$. We consider a semi-discrete in space and time-
536 continuous approximation of our system, with a space discretization based on a finite-
537 difference approximation method on a uniform mesh. In this restrictive setting, all our
538 assertions can be fully proved rigorously by adapting the arguments in [BE11, BEO15].
539 Though this might seem very restrictive, we believe that our approach can be gener-
540 alized to fully discrete models and in higher dimensions for quasi-uniform meshes.

541 To begin with, we introduce some notations for this 1-d space semi-discrete frame-
542 work. The appropriate discrete Carleman estimate will follow. We will finally briefly
543 present how we approximate the functional $J_{s,q}[\tilde{\mu}]$ in (13).

544 **4.1. Notations.** In our framework, the space variable $x \in [0, L]$ is taking values
545 on a discrete mesh $[0, L]_h$ indexed by the number of points $N \in \mathbb{N}$. To be more precise,
546 for $N \in \mathbb{N}$, we set $h = L/(N+1)$, $x_j = jh$ for $j \in \{0, \dots, N+1\}$, and $[0, L]_h = \{x_j, j \in$
547 $\{0, \dots, N+1\}\}$. For convenience, we will also note $(0, L)_h$, respectively $[0, L)_h$, the
548 set of discrete points $\{x_j, j \in \{1, \dots, N\}\}$, respectively $\{x_j, j \in \{0, \dots, N\}\}$.
549 Below, we will use the subscript h for discrete functions f_h defined on a mesh of the
550 form $[0, L]_h$ for some N , i.e. $f_h = (f_j)_{j \in \{0, \dots, N+1\}}$. Analogously with the continuous
551 case, we write:

$$552 \quad (41) \quad \int_{(0,L)_h} f_h = h \sum_{j=1}^N f_j, \quad \int_{[0,L)_h} f_h = h \sum_{j=0}^N f_j.$$

553 We also make use of the following notation for the discrete operators:

$$554 \quad (\partial_h v_h)_j = \frac{v_{j+1} - v_{j-1}}{2h}; \quad (\partial_h^+ v_h)_j = (\partial_h^- v_h)_{j+1} = \frac{v_{j+1} - v_j}{h};$$

$$555 \quad (\Delta_h v_h)_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}.$$

556 By analogy with the definition of $\mathcal{L}_{s,q}$ in (36), we finally introduce, for $s > 0$ and q_h
557 a discrete potential, the conjugate operator $\mathcal{L}_{s,q_h,h}$ defined by

$$558 \quad (42) \quad \mathcal{L}_{s,q_h,h} y_h = e^{s\varphi} (\partial_t^2 - \Delta_h + q_h) (e^{-s\varphi} y_h),$$

559 for y_h functions of $t \in (-T, T)$ and $x \in \{x_j, j \in \{1, \dots, N\}\}$.

560 Before going further, let us emphasize that the discrete operator $\mathcal{L}_{s,q_h,h}$ is different
561 from the operator $\widetilde{\mathcal{L}}_{s,q_h,h}$ obtained by a naive discretization of $\mathcal{L}_{s,q}$ in (36) as follows:

$$562 \quad \widetilde{\mathcal{L}}_{s,q_h,h} y_h = \partial_t^2 y_h - \Delta_h y_h + q_h y_h + 4s\beta t \partial_t y_h + 4s(x - x_0) \partial_h y_h$$

$$563 \quad (43) \quad + 2s(\beta + 1) y_h + 4s^2(\beta^2 t^2 - |x - x_0|^2) y_h,$$

564 for any function y_h defined on $(-T, T) \times \{x_j, j \in \{1, \dots, N\}\}$.

565 **4.2. A discrete Carleman estimate for the discrete wave operator.** In
566 this section, we provide the counterpart of Corollary 4 at the discrete level.

567 **THEOREM 6.** *Assume the multiplier condition (3)-(4) and $\beta \in (0, 1)$ as in (8).
568 Let $L > 0$, take $x_0 < 0$, and define the weight function φ as in (9). Then there exist
569 $s_0 > 0$, $N_0 > 0$, $\varepsilon_0 > 0$ and a positive constant M such that for all $s \in [s_0, \varepsilon_0/h]$ and
570 for all $N \geq N_0$,*

$$572 \quad (44) \quad s \int_{-T}^T \int_{(0,L)_h} (|\partial_t y_h|^2 + |\partial_h^+ y_h|^2 + s^2 |y|^2) dt$$

$$573 \quad \leq M \int_{-T}^T \int_{(0,L)_h} |\mathcal{L}_{s,0,h} y_h|^2 dt + Ms \int_{-T}^T |\partial_h^- y_{N+1}(t)|^2 dt$$

$$574 \quad + Ms^3 \int_{-T}^T \int_{(0,L)_h} \mathbf{1}_{(|t,x_j| \in \mathcal{O})} |y_h|^2 dt + Msh^2 \int_{-T}^T \int_{(0,L)_h} |\partial_t \partial_h^+ y_h|^2 dt,$$

576 for all y_h such that $y_j \in H^2(-T, T)$ for all $j \in \{1, \dots, N\}$, where \mathcal{O} is defined in
577 (12).

578 Furthermore, if $y_h(0) = 0$ in $(0, L)_h$, the term $s^{1/2} \int_{(0,L)_h} |\partial_t y_h(0)|^2$ can be added to
579 the left hand-side of (44).

580 The proof of Theorem 6 is left to the reader as it follows step by step the proof of
581 Theorem 3 using discrete rules of integration by parts, which can be found in [BE11,
582 Lemma 2.6]. It is actually particularly simple as the coefficients of $\mathcal{L}_{s,0,h}$ depend only
583 on time or only on space variables.

584

585 Let us now briefly comment Theorem 6. First, compared with Corollary 4, we
586 see that the right-hand side of (44) contains one more term than (25), namely

$$587 \quad (45) \quad Msh^2 \int_{-T}^T \int_{(0,L)_h} |\partial_t \partial_h^+ y_h|^2 dt.$$

588 This is a high-frequency term. Indeed, as $h\partial_h^+$ is of the order of $h|\xi|$ for frequencies
589 ξ , this term can be absorbed for large s by the left hand-side of (44) for frequencies
590 $\xi = o(1/h)$. However, for frequencies of the order of the mesh-size h , this term cannot
591 be absorbed anymore by the left hand-side of (44). This is not surprising in view
592 of the lack of uniform observability for discrete waves, see [Zua05], and the various
593 comments done in [BE11] on the discrete Carleman estimates for the wave equation
594 with weight functions $\exp(s\psi) = \exp(s \exp(\lambda(\varphi + C_0)))$.

595 Let us also point out that as in [BE11], the parameter s in Theorem 6 cannot be made
596 arbitrarily large as in Theorem 3, but is limited to some ε_0/h . Roughly speaking, this
597 condition comes from the following fact:

$$598 \quad (46) \quad \|\exp(s\varphi)\partial_h(\exp(-s\varphi)) + s\partial_x \phi\|_{L^\infty((0,T) \times \Omega)} \leq Csh,$$

599 so that the coefficients of $\mathcal{L}_{s,0,h}$ in (42) and $\widetilde{\mathcal{L}}_{s,0,h}$ in (43) are close only for sh small
600 enough.

601 We end up this section with a warning. If we were considering the operator $\widetilde{\mathcal{L}}_{s,0,h}$
602 in (43) instead of $\mathcal{L}_{s,0,h}$ in (42), the restriction on the size of the parameter s could

603 be removed as the errors done in the conjugation process, for instance in (46), are
 604 inexistent. However, when conjugating back the discrete operator $\widetilde{\mathcal{L}}_{s,0,h}$, one would
 605 not obtain the discretization of the wave operator $\partial_{tt} - \Delta_h$, and this would yield
 606 inaccuracies in our numerical experiments.

607 **4.3. Semi-discretization scheme and algorithm.** We now explain the dis-
 608 cretization in space of the variational problem (37).
 609 First, we have to discretize the set \mathcal{T} in (35). We thus introduce the set \mathcal{T}_h defined
 610 as follows:

$$612 \quad (47) \quad \mathcal{T}_h = \{z_h \in H^2(0, T; \mathbb{R}^{N+2}) \text{ with } z_{0,h}(t) = z_{N+1,h}(t) = 0 \text{ for all } t \in (0, T) \\ 613 \quad \text{and } z_{j,h}(0) = 0 \text{ for all } j \in \{1, \dots, N\}\}.$$

615 Following Theorem 6, it is natural to discretize the variational problem (37) as follows:
 616 Find $Y_h \in \mathcal{T}_h$ such that for all $y_h \in \mathcal{T}_h$,

$$617 \quad (48) \quad \int_0^T \int_{(0,L)_h} (\mathcal{L}_{s,q_h,h} Y_h)(\mathcal{L}_{s,q_h,h} y_h) dt + s \int_0^T \frac{Y_{N_h,h}}{h} \frac{y_{N_h,h}}{h} dt \\ 618 \quad \quad \quad + s^3 \int_{-T}^T \int_{(0,L)_h} \mathbf{1}_{(|t|, x_j) \in \mathcal{O}} Y_h y_h dt \\ 619 \quad \quad \quad + s h^2 \int_0^T \int_{[0,L)_h} (\partial_t \partial_h^+ Y_h)(\partial_t \partial_h^+ y_h) dt = s \int_0^T e^{s\varphi} \tilde{\mu} \left(\frac{-y_{N_h,h}}{h} \right) dt.$$

620 Actually we will use this variational formulation (48) in the numerical experiments.
 621 Compared with (37), we have added here the term

$$s h^2 \int_0^T \int_{[0,L)_h} (\partial_t \partial_h^+ Y_h)(\partial_t \partial_h^+ y_h),$$

622 which is of course the counterpart of the term (45) and aims at penalizing the spurious
 623 high-frequency waves which may appear in the discretization process. This term is
 624 indeed really helpful when considering noisy data, as we will illustrate in the numerical
 625 experiments in Section 5. But this term also guarantees that the variational problem
 626 in (48) is coercive uniformly with respect to the discretization parameter $h > 0$, as it
 627 can be deduced immediately from Theorem 6. In particular, it allows us to prove the
 628 convergence of the algorithm given afterwards.

629 In order to state it precisely, by analogy with (26), for $h > 0$, a discrete potential
 630 q_h , a parameter $s > 0$, and $\tilde{\mu} \in L^2(0, T)$, $\tilde{g}_h \in L^2(0, T; \mathbb{R}^N)$, $\tilde{\nu}_h \in L^2(0, T; \mathbb{R}^N)$, we
 631 introduce the discrete functional

$$632 \quad (49) \quad J_{s,q_h,h}[\tilde{\mu}, \tilde{g}_h, \tilde{\nu}_h](z_h) = \\ 633 \quad \frac{1}{2} \int_0^T \int_{(0,L)_h} e^{2s\varphi} |\partial_t^2 z_h - \Delta_h z_h + q_h z_h - \tilde{g}_h|^2 dt + \frac{s}{2} \int_0^T e^{2s\varphi} \left| \frac{-z_{N_h,h}}{h} - \tilde{\mu}(t) \right|^2 dt \\ 634 \quad \quad \quad + \frac{s^3}{2} \int_{-T}^T \int_{(0,L)_h} \mathbf{1}_{(|t|, x_j) \in \mathcal{O}} e^{2s\varphi} |z_h|^2 dt + \frac{s h^2}{2} \int_0^T \int_{[0,L)_h} e^{2s\varphi} |\partial_t \partial_h^+ z_h - \tilde{\nu}_h|^2 dt.$$

635 defined on the set \mathcal{T}_h . Of course, one easily checks that the solution $Y_h \in \mathcal{T}_h$ of the
 636 variational formulation in (48) corresponds to the minimizer Z_h of $J_{s,q_h,h}[\tilde{\mu}, 0, 0]$ over

639 \mathcal{T}_h through the formula $Y_h = e^{s\varphi} Z_h$.

640

641 For any mesh-size $h > 0$, we define the discrete functions $w_{0,h}, w_{1,h}$ approximating
642 the initial data w_0, w_1 , and the discrete functions f_h and $f_{\partial,h}$ approximating the source
643 terms f and f_{∂} . We construct Algorithm 3 as follows.

Algorithm 3

Initialization: $q_h^0 = 0$.

Iteration: From k to $k + 1$

- **Step 1** - Given q_h^k , we set

$$\tilde{\mu}_h^k(t) = \eta(\varphi(t, L)) \partial_t \left(\frac{w_{N+1,h}^k(t) - w_{N,h}^k(t)}{h} - \partial_n W[Q](t, L) \right), \quad \text{on } (0, T),$$

where w_h^k denotes the solution of

$$(50) \quad \begin{cases} \partial_t^2 w_h - \Delta_h w_h + q_h^k w_h = f_h, & \text{in } (0, T) \times (0, L)_h, \\ w_{0,h}(t) = f_{\partial}(t, 0), \quad w_{N+1,h}(t) = f_{\partial}(t, L), & \text{on } (0, T), \\ w_h(0) = w_{0,h}, \quad \partial_t w_h(0) = w_{1,h}, & \text{in } (0, L)_h, \end{cases}$$

corresponding to (15) with the potential q^k and $\partial_n W[Q]$ is the measurement in (2).
And then set

$$(51) \quad \tilde{v}_h^k = \partial_t \partial_h^+ (\eta(\varphi) \partial_t w_h[q_h^k]) \quad \text{in } (0, T) \times (0, L)_h.$$

- **Step 2** - We minimize the functional $J_{s, q_h^k, h}[\tilde{\mu}_h^k, 0, \tilde{v}_h^k]$ defined in (49), for some $s > 0$ that will be chosen independently of k , on the trajectories $z_h \in \mathcal{T}_h$. Let \tilde{Z}_h^k be the unique minimizer of the functional $J_{s, q_h^k, h}[\tilde{\mu}_h^k, 0, \tilde{v}_h^k]$.

- **Step 3** - Set

$$\tilde{q}_h^{k+1} = q_h^k + \frac{\partial_t \tilde{Z}_h^k(0)}{w_{0,h}}, \quad \text{in } (0, L)_h.$$

- **Step 4** - Finally, set

$$q_h^{k+1} = T_m(\tilde{q}_h^{k+1}), \quad \text{with } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| \geq m., \end{cases}$$

where m is the a priori bound in (7).

644 One can then state a convergence result provided several assumptions are satis-
645 fied, basically corresponding to (5)–(6)–(7) and the consistency of our approximation
646 schemes. Namely we assume:

647

648 (i) Assumptions (5)–(6)–(7) and (8) are satisfied.

649

650 (ii) There exists $\alpha > 0$ independent of h such that for all $h > 0$,

$$(52) \quad \inf_{(0, L)_h} |w_{0,h}| \geq \alpha.$$

651

652 (iii) There exists a sequence of discrete potential $(Q_h)_{h>0}$, each Q_h being

653 defined on $(0, L_h)$ such that:

654 1. For each $h > 0$, Q_h is bounded uniformly on $(0, L)_h$ by m :

$$655 \quad (53) \quad \sup_{(0, L)_h} |Q_h| \leq m.$$

656 2. The piecewise constant extensions of Q_h strongly converge in $L^2(0, L)$ to Q
657 when $h \rightarrow 0$.

658 3. For each $h > 0$, introducing $W_h[Q_h]$ the solution of

$$659 \quad (54) \quad \begin{cases} \partial_t^2 W_h - \Delta_h W_h + Q_h W_h = f_h, & \text{in } (0, T) \times (0, L)_h, \\ W_{0,h}(t) = f_{\partial,h}(t, 0), \quad W_{N+1,h}(t) = f_{\partial,h}(t, L), & \text{on } (0, T), \\ W_h(0) = w_{0,h}, \quad \partial_t W_h(0) = w_{1,h}, & \text{in } (0, L)_h, \end{cases}$$

660 we get

$$661 \quad (55) \quad \sup_{h>0} \int_0^T \left| \sup_{(0, L)_h} |\partial_t W_h[Q_h]| \right|^2 dt < \infty,$$

662 and the following consistency assumptions:

$$663 \quad (56) \quad \lim_{h \rightarrow 0} \left(\int_0^T \eta(\varphi(t, L))^2 \left| \frac{\partial_t W_{N+1,h}[Q_h] - \partial_t W_{N,h}[Q_h]}{h} - \partial_t \partial_n W[Q](t, L) \right|^2 dt \right) = 0,$$

$$\lim_{h \rightarrow 0} \left(\int_0^T \int_{[0, L)_h} |h \partial_h^+ \partial_t(\eta(\varphi) \partial_t W_h[Q_h])|^2 dt \right) = 0.$$

664 These are natural assumptions regarding the inverse problem at hand. They have been
665 widely discussed in [BE11, Section 4] and [BEO15, Section 4]. These two works give
666 sufficient conditions for the existence of a sequence of discrete potential Q_h satisfying
667 (53)–(55)–(56). They also proved that, under some further suitable assumptions on
668 the convergence of f_h , $f_{\partial,h}$, $w_{0,h}$, $w_{1,h}$, a sequence Q_h satisfying (53) and (56)_(1,2)
669 necessarily converges to the potential Q in $L^2(0, L)$ (after having been extended as
670 piecewise constant functions in a natural way).

671 We get the following result:

672 **THEOREM 7.** *Under assumptions (i)–(ii)–(iii) above, Algorithm 3 is well-posed*
673 *for all $h > 0$ small enough. Specifically, the discrete sequence q_h^k satisfies for some*
674 *constants $C_0, C_1 > 0$ independent of $s > 0$ and $h > 0$,*

$$675 \quad (57) \quad \int_{(0, L_h)} e^{2s\varphi} |q_h^{k+1} - Q_h|^2 \leq \frac{C_0}{\sqrt{s}} \int_{(0, L_h)} e^{2s\varphi} |q_h^k - Q_h|^2$$

$$677 \quad + C_1 s^{1/2} \int_0^T \int_{[0, L)_h} e^{2s\varphi} |h \partial_h^+ \partial_t(\eta(\varphi) \partial_t W_h[Q_h])|^2 dt$$

$$678 \quad + C_1 s^{1/2} \int_0^T e^{2s\varphi} \left| \frac{\partial_t W_{N+1,h}[Q_h] - \partial_t W_{N,h}[Q_h]}{h} - \partial_t \partial_n W[Q](t, L) \right|^2 dt.$$

679

680 In particular, for $s \geq 4C_0^2$, we get, for all $k \in \mathbb{N}$,

681

$$\begin{aligned}
682 \quad (58) \quad & \int_{(0,L_h)} e^{2s\varphi} |q_h^k - Q_h|^2 \leq \frac{1}{2^k} \int_{(0,L_h)} e^{2s\varphi} |Q_h|^2 \\
683 \quad & + 2C_1 s^{1/2} \int_0^T \int_{[0,L)_h} e^{2s\varphi} |h \partial_h^+ \partial_t (\eta(\varphi) \partial_t W_h[Q_h])|^2 dt \\
684 \quad & + 2C_1 s^{1/2} \int_0^T e^{2s\varphi} \left| \frac{\partial_t W_{N+1,h}[Q_h] - \partial_t W_{N,h}[Q_h]}{h} - \partial_t \partial_n W[Q](t, L) \right|^2 dt, \\
685
\end{aligned}$$

686 so that as $k \rightarrow \infty$, q_h^k enters a neighborhood of Q_h , whose size depends on h and s
687 and goes to zero as $h \rightarrow 0$ according to (56).

Proof. We focus on the proof of (57). As in the continuous case, it mainly consists in showing that \tilde{Z}_h^k is close to $\tilde{z}_h^k = \eta(\varphi) z_h^k$, where

$$z_h^k = \partial_t (w_h[q_h^k] - W_h[Q_h]).$$

The main idea is to remark that z_h^k satisfies

$$\begin{cases} \partial_t^2 z_h^k - \Delta_h z_h^k + q_h^k z_h^k = g_h^k, & \text{in } (0, T) \times (0, L)_h, \\ z_{0,h}^k = z_{N+1,h}^k = 0, & \text{on } (0, T), \\ z_h^k(0) = 0, \quad \partial_t z_h^k(0) = z_{1,h}^k, & \text{in } (0, L)_h, \end{cases}$$

with

$$g_h^k = (Q_h - q_h^k) \partial_t W_h[Q_h], \quad z_{1,h}^k = (Q_h - q_h^k) w_{0,h}.$$

688 In particular, \tilde{z}_h^k satisfies:

$$689 \quad (59) \quad \begin{cases} \partial_t^2 \tilde{z}_h^k - \Delta_h \tilde{z}_h^k + q_h^k \tilde{z}_h^k = \tilde{g}_h^k, & \text{in } (0, T) \times (0, L)_h, \\ \tilde{z}_{0,h}^k = \tilde{z}_{N+1,h}^k = 0, & \text{on } (0, T), \\ \tilde{z}_h^k(0) = 0, \quad \partial_t \tilde{z}_h^k(0) = z_{1,h}^k, & \text{in } (0, L)_h, \end{cases}$$

with

$$\tilde{g}_h^k = \eta(\varphi) (Q_h - q_h^k) \partial_t W_h[Q_h] + [\partial_t^2 - \Delta_h, \eta(\varphi)] z_h^k.$$

690 Moreover, one has the following boundary data

$$691 \quad (60) \quad \frac{-\tilde{z}_{N,h}^k(t)}{h} = \tilde{\mu}_h^k(t) - \delta_h(t) \quad \text{on } (0, T),$$

where

$$\delta_h(t) = \eta(\varphi(t, L)) \partial_t \left(\frac{W_{N+1,h}[Q_h](t) - W_{N,h}[Q_h](t)}{h} - \partial_n W[Q](t, L) \right).$$

692 Therefore, \tilde{z}_h^k is the minimizer of the functional $J_{s,q_h^k,h}[\tilde{\mu}_h^k - \delta_h, \tilde{g}_h^k, \tilde{\nu}_h^k - \hat{\nu}_h]$ where $\hat{\nu}_h$
693 is given by

$$694 \quad (61) \quad \hat{\nu}_h = \partial_h^+ \partial_t (\eta(\varphi) \partial_t W_h[Q_h]), \quad \text{in } (0, T) \times (0, L)_h.$$

695 But by construction, \tilde{Z}_h^k is the minimizer of $J_{s,q_h^k,h}[\tilde{\mu}_h^k, 0, \tilde{\nu}_h^k]$. We thus only need to
696 compare minimizers corresponding to the other coefficients (δ_h , \tilde{g}_h^k and $\hat{\nu}_h$). As in

697 the proof of Proposition 5, using Euler-Lagrange formulation and using the Carleman
698 estimate (44), one easily gets:

699

$$700 \quad (62) \quad s^{1/2} \int_{(0,L)_h} e^{2s\varphi(0)} |\partial_t Z_h^k(0) - \partial_t \tilde{z}_h^k(0)|^2 \leq C s \int_0^T e^{2s\varphi(t,L)} |\delta_h|^2 dt$$

$$701 \quad \quad \quad + C \int_0^T \int_{(0,L)_h} e^{2s\varphi} |\tilde{g}_h^k|^2 dt + C s h^2 \int_0^T \int_{(0,L)_h} e^{2s\varphi} |\hat{\nu}_h|^2 dt.$$

702

703 Following now the proof of Theorem 2 we can show that

$$704 \quad \int_0^T \int_{(0,L)_h} e^{2s\varphi} |\tilde{g}_h^k|^2 dt \leq C \int_{(0,L)_h} e^{2s\varphi} |q_h^k - Q_h|^2 dt,$$

705 while, by construction,

$$706 \quad s^{1/2} \int_{(0,L)_h} e^{2s\varphi(0)} |\partial_t Z_h^k(0) - \partial_t \tilde{z}_h^k(0)|^2 \geq s^{1/2} \alpha^2 \int_{(0,L)_h} e^{2s\varphi(0)} |\tilde{q}_h^{k+1} - Q_h|^2$$

$$707 \quad \quad \quad \geq s^{1/2} \alpha^2 \int_{(0,L)_h} e^{2s\varphi(0)} |q_h^{k+1} - Q_h|^2.$$

708 We then put together the two last estimates in (62). Recalling that δ_h and ν_h are
709 respectively given by (60) and (61), we immediately obtain (57).

710 The proof of estimate (58) easily follows from (57). Indeed, by recurrence, one
711 can easily show that, if $s \geq 4C_0^2$, for all $k \in \mathbb{N}$,

712

$$713 \quad \int_{(0,L)_h} e^{2s\varphi} |q_h^k - Q_h|^2 \leq \frac{1}{2^k} \int_{(0,L)_h} e^{2s\varphi} |Q_h|^2$$

$$714 \quad \quad \quad + \left(\sum_{j=0}^{k-1} \frac{1}{2^j} \right) C_1 s^{1/2} \int_0^T \int_{[0,L)_h} e^{2s\varphi} |h \partial_h^+ \partial_t (\eta(\varphi) \partial_t W_h[Q_h])|^2 dt$$

$$715 \quad \quad \quad + \left(\sum_{j=0}^{k-1} \frac{1}{2^j} \right) C_1 s^{1/2} \int_0^T e^{2s\varphi} \left| \frac{\partial_t W_{N+1,h}[Q_h] - \partial_t W_{N,h}[Q_h]}{h} - \partial_t \partial_n W[Q](t, L) \right|^2 dt,$$

716

717 which is slightly stronger than (58) and concludes the proof of Theorem 7. \square

718 Note that we presented the above theoretical results by restricting ourselves to the
719 1d case for the time continuous and space semi-discrete approximation of the inverse
720 problem. Though, this analysis can very likely be carried on in much more general set-
721 tings, for instance higher dimensions or fully discrete approximations. Of course, the
722 key missing point is then the counterpart of the Carleman estimate in Theorem 3. De-
723 spite important recent efforts for developing this powerful tool in the discrete setting,
724 see in particular [KS91, BHLR10a, BHLR10b, BHLR11, EdG11, BLR14] for discrete
725 elliptic and parabolic equations, and [BE11, BEO15] for discrete wave equations, the
726 validity of discrete Carleman estimates in the discrete settings remains mainly limited
727 to smooth deformations of cartesian grids for the finite-difference method.

728 We would like also to emphasize that Theorem 7 is not a proper convergence theorem,
729 as it only says that the sequence of discrete potentials q_h^k will enter a neighborhood

730 of Q_h as $k \rightarrow \infty$. The size of this neighborhood, given by
 731

$$732 \quad 2C_1 s^{1/2} \int_0^T \int_{[0,L)_h} e^{2s\varphi} |h \partial_h^+ \partial_t (\eta(\varphi) \partial_t W_h[Q_h])|^2 dt$$

$$733 \quad + 2C_1 s^{1/2} \int_0^T e^{2s\varphi} \left| \frac{\partial_t W_{N+1,h}[Q_h] - \partial_t W_{N,h}[Q_h]}{h} - \partial_t \partial_n W[Q](t, L) \right|^2 dt,$$
 734

735 see (58), is in fact very much related to the consistency error. It is nonetheless
 736 interesting to point out that choosing s large to improve the speed of convergence of
 737 the algorithm also increases the size of this neighborhood. One should keep in mind
 738 that remark, which also applies in the presence of noise.

739 **IMPORTANT REMARK 1.** *The choice made in (51) does not seem natural because*
 740 *it is not based on the difference between $w_h[q_h^k]$ and $W_h[Q_h]$, the latter being unknown.*
 741 *It is also important to mention that for some reasons that we still do not fully un-*
 742 *derstand the numerical results given by Algorithm 3 with this choice show numerical*
 743 *instabilities. Instead, we propose to replace Algorithm 3 by*

Algorithm 4

Everything as in Algorithm 3 except:

Iteration:

- Step 2: We minimize the functional $J_{s,q_h^k,h}[\tilde{\mu}_h^k, 0, 0]$ defined in (49), for some $s \geq 0$ that will be chosen independently of k , on the trajectories $z_h \in \mathcal{T}_h$. Let Z_h^k be the unique minimizer of the functional $J_{s,q_h^k,h}[\tilde{\mu}_h^k, 0, 0]$.
-

744 *With this choice, we do not know how to prove a convergence result of the algo-*
 745 *rithm similar to Theorem 7.*

746 *However, this choice coincides more with the insights we have on the algorithm as*
 747 *\tilde{z}_h^k in (59) is the minimizer of $J_{s,q_h^k,h}[\tilde{\mu}_h^k - \delta_h, 0, \tilde{\nu}_h^k - \hat{\nu}_h]$, and if convergence occurs,*
 748 *$\tilde{\nu}_h^k - \hat{\nu}_h$ should be small and converge to zero.*

749 *The numerical results presented in Section 5 will all be performed using Algorithm 4.*
 750 *As we will see, this will lead to good numerical results, in agreement with the above*
 751 *insights.*

752 **4.4. Full discretization.** When implementing Algorithm 3 numerically, one
 753 should of course consider fully discrete wave equations. We will not give all the details
 754 of this discretization process, but simply state how we implement the minimization
 755 process of the functional $J_{s,q_h,h}$.

756 First, we shall of course consider a fully discrete version $J_{s,q_h^k,h,\tau}$ of the functional
 757 $J_{s,q_h,h}$ in (49), in which we have implemented a time-discretization of $J_{s,q_h,h}$ of time-
 758 step τ . This implies in particular that:

- The minimization space \mathcal{T}_h has to be replaced by the set of time discrete functions $z_{h,\tau} \in \mathbb{R}^{N_t} \times \mathbb{R}^{N+2}$, with $N_t = \lceil T/\tau \rceil$ and the corresponding boundary conditions.
- The time continuous integral in (49) shall be replaced by discrete sums $\tau \sum_{t \in [0,T] \cap \tau\mathbb{Z}}$.
- The wave operator should be replaced by a time-discrete version of the space semi-discrete wave operator $\partial_{tt} - \Delta_h + q_h$. We simply choose to approximate ∂_{tt} by the usual 3-points difference operator Δ_τ (similar to Δ_h but applied

767 in time now). Similarly, the operator ∂_t in the last term of (49) will be
 768 replaced by the operator ∂_τ^+ which is the approximation of ∂_t computed with
 769 the subsequent time-step.

- 770 • The solution w_h of (50) has to be computed on a fully discrete version of
- 771 (50). We choose to discretize using an explicit Euler method.
- 772 • There is no need to add a new penalization term for high-frequency spurious
- 773 terms as we will impose a Courant-Friedrichs-Lax (CFL) type condition $\tau \leq$
- 774 h , so that the last term in (49) already penalizes the spurious high-frequency
- 775 solutions.

776 Of course, the strategies that have been presented in Section 3 to make the nu-
 777 merical implementation of the minimization of $J_{s,q}$ more efficient can be successfully
 778 applied to the functional $J_{s,q_h^k,h,\tau}$ as well. Namely, in the implementation of Algo-
 779 rithm 4, we will always work on the conjugated functional, i.e. the one given in the
 780 conjugated variable $y = e^{s\varphi}z$, and we will always decompose the domain using the
 781 progressive argument presented in Section 3.2.

782 We also point out that the minimization of the quadratic functional $J_{s,q_h,h,\tau}$
 783 obtained that way can be recast using a variational formulation similar to (37), which
 784 presents the advantage to underline the fact that we are actually solving a sparse
 785 linear system. We therefore use a Compressed Sparse Row (CSR) tool as sparse
 786 matrix storage format and solve the linear system thanks to an LU factorization.

787 The iterative process on the potential is supposed to reach convergence when the
 788 following stop criterion is satisfied

789

$$790 \quad (63) \quad \frac{\int_{(0,L)_h} |q_h^{k+1} - q_h^k|^2}{\int_{(0,L)_h} |q_h^1 - q_h^0|^2} \leq \epsilon_0 \quad \text{or}$$

$$791 \quad \frac{1}{\int_{[0,T)_\tau} |\partial_n W[Q](t, L)|^2} \int_{[0,T)_\tau} |(\partial_h^- w_h^k)_{N+1}(t) - \partial_n W[Q](t, L)|^2 \leq \epsilon_1.$$

792

793 for given choices of the parameters $\epsilon_0 > 0$ and $\epsilon_1 > 0$, in which the integrals have to
 794 be interpreted in the discrete sense.

795 **5. Numerical results.** This section is devoted to the presentation of some nu-
 796 merical examples to illustrate the properties of the reconstruction algorithm and its
 797 efficiency. All simulations are executed with the software SCILAB. The source codes
 798 are available on request.

799 **5.1. Synthetic noisy data.** In this article, we work with synthetic data. To
 800 discretize the wave equations with potential (1), we use a finite differences scheme
 801 in space and a θ -scheme in time. The space and time steps are denoted by h and τ
 802 respectively. We set $L = (N_x + 1)h$ and $T = N_t\tau$, and we define, for $0 \leq j \leq N_x + 1$
 803 and $0 \leq n \leq N_t$, W_j^n a numerical approximation of the solution $W(t^n, x_j)$ with

804 $t^n = n\tau$ and $x_j = jh$. It is solution of the following system:

$$805 \quad (64) \quad \begin{cases} \frac{W_j^{n+1} - 2W_j^n + W_j^{n-1}}{\tau^2} - \frac{\theta}{2}(\Delta_h W_h)_j^{n+1} - (1-\theta)(\Delta_h W_h)_j^n \\ \quad - \frac{\theta}{2}(\Delta_h W_h)_j^{n-1} + Q(x_j)W_j^n = f(t^n, x_j), \\ W_j^1 = w_0(x_j) + \tau w_1(x_j) + \frac{\tau^2}{2}((\Delta_h w_0)(x_j) - q(x_j)w_0(x_j) + f(0, x_j)), \\ W_j^0 = w_0(x_j) & 1 \leq j \leq N_x, \\ W_0^n = f_\partial(t^n, 0) \quad \text{and} \quad W_{N_x+1}^n = f_\partial(t^n, L), & 1 \leq n \leq N_t. \end{cases}$$

Then, we compute \mathcal{M}_τ the counterpart of the continuous measurement \mathcal{M} given in (2) as follows:

$$\mathcal{M}_\tau(t^n) = \frac{W_{N_x+1}^n - W_{N_x}^n}{h}, \quad 0 \leq n \leq N_t.$$

806 On the computed data, we may add a Gaussian noise:

$$807 \quad (65) \quad \mathcal{M}_\tau(t^n) \leftarrow (1 + \alpha \mathcal{N}(0, 0.5)) \mathcal{M}_\tau(t^n), \quad 0 \leq n \leq N_t$$

808 where $\mathcal{N}(0, 0.5)$ satisfies a centered normal law with deviation 0.5 and α is the level
809 of noise. Note that the model of noise, that we chose, is a multiplicative noise. It
810 allows to model the experimental error in the measurements.

811 One of the main drawbacks of the method presented in Algorithm 4 is that we
812 have to derive in time the observation flux. On Figure 4, we plot the flux \mathcal{M} with
813 respect to time (on the left hand side) and of its time derivative (on the right hand
814 side). For each of the graphs, the red line is the exact value and the black line the
815 generated noisy data. It shows that even a small perturbation on the observations
816 gives rise to a large perturbation on its derivative. In order to partially remedy to
817 this problem, we regularize the data thanks to a convolution process with a Gaussian:

$$818 \quad (66) \quad \mathcal{M}(t) \leftarrow \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathcal{M}(t-r) \exp\left(-\frac{r^2}{4}\right) dr.$$

819 The number of iterations in this regularization process must be chosen in accordance
820 to the *a priori* knowledge of the noise level. On Figure 4, the new regularized data
821 that we use as an entry for the algorithm is plotted in blue.

822 In order to avoid the *inverse crime*, we use neither the same schemes nor the
823 same meshes for the direct and the inverse problems. Hence, we solve (1) thanks to
824 an implicit scheme ($\theta = 1$) with $\tau = 0.00033$ and $h = 0.00025$ and we use an explicit
825 scheme ($\theta = 0$) for equation (15) in Algorithm 4, with $\tau = 0.01$ and $h = \frac{\tau}{\text{CFL}}$. Table
826 1 gathers the numerical values used for all the following examples, unless specified
827 otherwise where appropriate. In all the figures, the exact potential that we want to
828 recover is plotted by a red line, the numerical potential recovered by the algorithm is
represented by black crosses.

L	f	f_∂	w_0	w_1	x_0	β	T	s	m	CFL
1	0	2	$2 + \sin(\pi x)$	0	-0.3	0.99	1.3	100	3	0.9 or 1

Table 1: Numerical values for the variables.

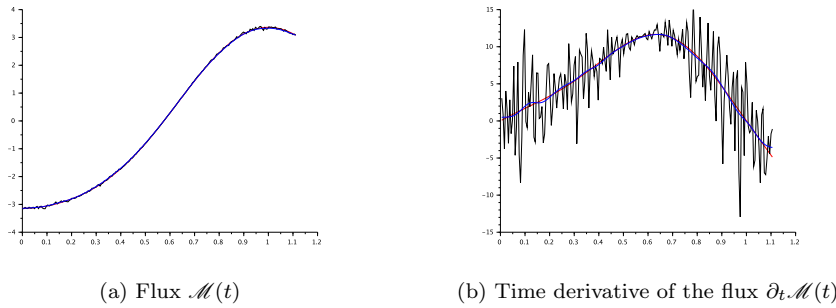


Fig. 4: The measurement \mathcal{M} in the presence of 2% noise.

830 **5.2. Simulations from data without noise.** In this subsection, we present
 831 the results obtained for $CFL = 1$. For that very special choice, the explicit scheme
 832 used to discretized (1) is of order 2. We observe that in this case, the additional
 833 regularization term (45) in the functional does not seem to be necessary and s can be
 834 chosen as large as wanted independently of the value of h to achieve convergence. The
 835 successive results at each iteration of Algorithm 4 in the case of the reconstruction of
 836 the potential $Q(x) = \sin(2\pi x)$ are presented in Figure 5. One can observe that in less
 837 than 3 iterations, the convergence criteria (63) for $\epsilon_0 = 10^{-5}$ is met.

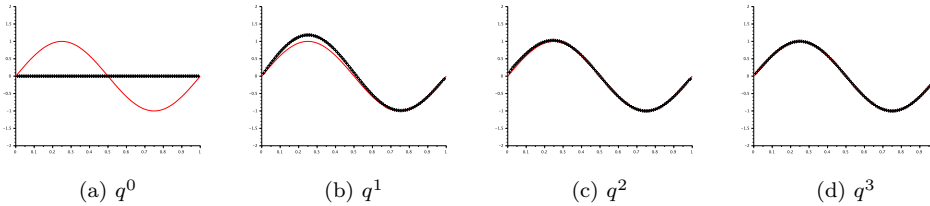


Fig. 5: Illustration of the convergence of the algorithm for $CFL = 1$ and $s = 100$.

Using the same target potential, Figure 6 illustrate the progressive process on the first iteration of Algorithm 4. From an initial data $q_0^0 = 0$, we represent successively

$$q_j^0 = q_{j-1}^0 + \frac{\partial_t Y_j^0(0)}{e^{s\varphi(0)} w_0}, \quad 1 \leq j \leq 5,$$

838 where Y_j^0 is the minimizer of $\tilde{J}_{s, q^0}[\tilde{\mu}_j^0]$.

839 In Figure 7, several results of reconstruction of potentials obtained using Algo-
 840 rithm 4 in the absence of noise are given.

841 We recall that in our approach, it is mandatory to know the *a priori* bound m
 842 such that $Q \in L_{\leq m}^\infty(\mathbb{R})$. On Figure 8, we illustrate the behavior of the algorithm
 843 in the case where an error is made on that bound. One can observe that the recovery of
 844 the potential is correct only in the zones where the potential Q is effectively bounded
 845 by m . In this situation, the convergence of the process doesn't occur. In practice, if

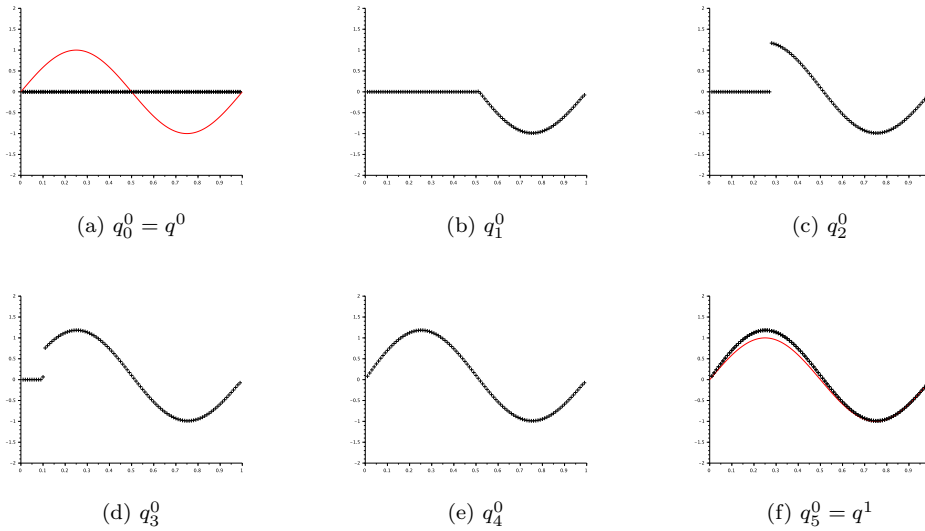


Fig. 6: Illustration of the progressive process for $Q(x) = \sin(2\pi x)$ for $s = 100$.

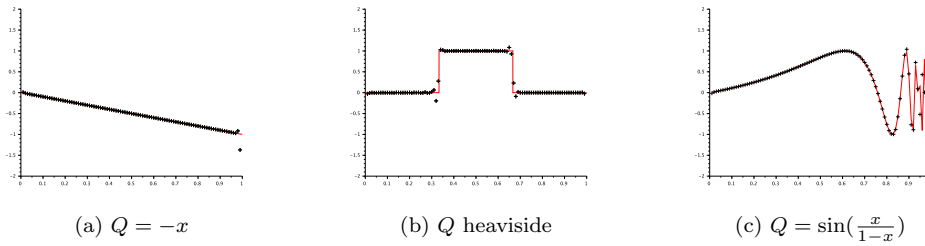


Fig. 7: Different examples of reconstruction for $CFL = 1$ and $s = 100$.

846 the retrieved potential meets the value of m in several points, it is recommended to
 847 repeat the reconstruction process after choosing a greater value of m .

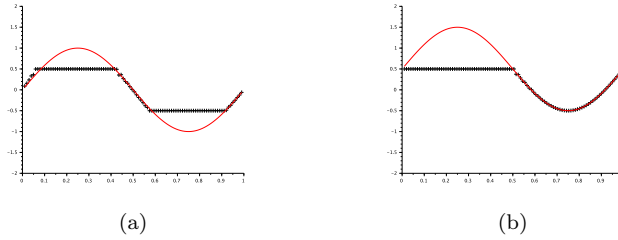


Fig. 8: Reconstruction of exact potentials with the wrong choice of the *a priori* bound $m = 0.5$, for $CFL = 1$ and $s = 100$.

848 **5.3. Simulations with several levels of noise.** If we slightly modify the sta-
 849 bility condition and take a CFL condition strictly smaller than 1, the explicit numeri-
 850 cal scheme used to solve (15) leads to a non negligible approximation error, acting as
 851 a noise. The presence of the additional regularization term (45) in the functional is
 852 therefore necessary. In that case, if the mesh size h (through τ) is given, it is not pos-
 853 sible to take s as large as desired. Nevertheless, even for smaller values of s , Algorithm
 854 4 gives good results, that can be improved by refining the mesh. In Figure 9, several
 855 results of reconstruction of potentials obtained for $\alpha = 0$, $CFL = 0.9$ and $s = 10$
 are presented. Figure 10 shows the results for $Q(x) = \sin(\pi x)$ with different level of

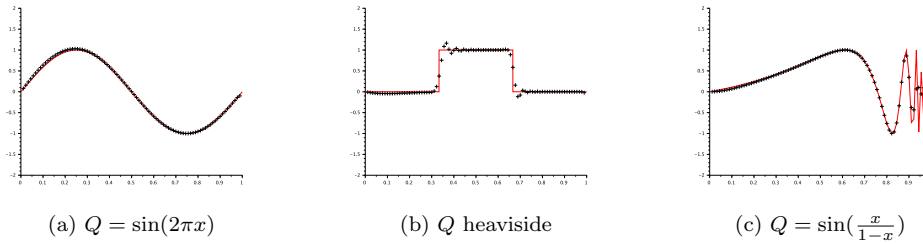


Fig. 9: Different examples of reconstruction for $CFL = 0.9$ and $s = 10$.

856 noise in the measurements ($\alpha = 1\%$, 5% and 10%). Here, we used the appropriate
 857 discretized functional constructed to deal with the discretization process.

859 Eventually, Figure 11 shows on the left hand side, an example of result obtained
 860 when the functional is discretized without taking into account the additional terms
 861 (45) requisite for its uniform coercivity with respect to the mesh size. Since the first
 862 iteration, severe oscillations occur and they amplify with the iterative process. On
 863 the right hand side, we illustrate the necessity of choosing a discretization space step
 864 small enough with respect to the value of the parameter s . Indeed, if the mesh size
 865 is too coarse, numerical instabilities appear.

866 **5.4. Simulations for initial datum not satisfying (6).** So far, we presented
 867 numerical simulations in which the positivity assumption (6) on w_0 was satisfied. In
 868 this section, we would like to briefly present what can be done in the case in which it

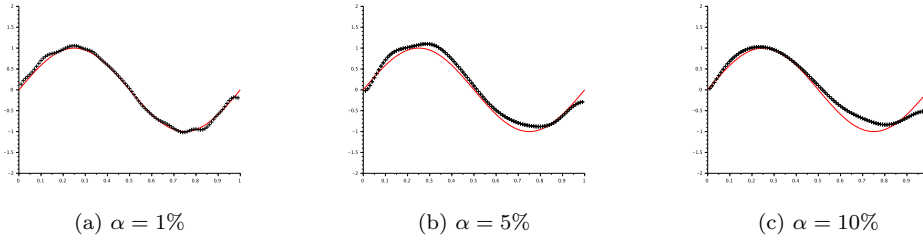


Fig. 10: Recovery of the potential $Q(x) = \sin(\pi x)$ in presence of noise in the data. The level of noise is denoted by α . Here, $CFL = 0.9$ and $s = 10$.

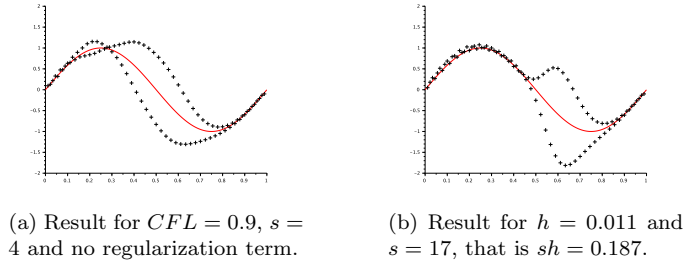


Fig. 11: Illustration of the need of the additional regularization term (45) in the functional (left). Illustration of the needed condition (46) between s and the space step h (right).

869 is not satisfied. In that case, Step 3 of Algorithm 4 can be replaced by :

(67)

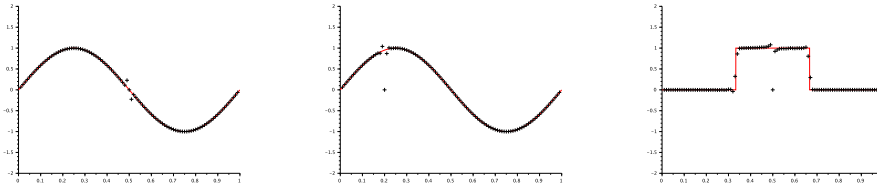
$$870 \quad \tilde{q}_h^{k+1}(x_j) = \begin{cases} q_h^k(x_j) + \frac{\partial_t \tilde{Z}_h^k(0, x_j)}{w_0(x_j)}, & \text{for } j \in \{1, \dots, N\} \text{ such that } |w_0(x_j)| \geq \alpha, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ is the constant appearing in (6). As an example, let us consider

$$w_0(x) = -a + x, \quad a \in (0, L),$$

871 which cancels at $x = a$ in a single isolated point. If we take $\alpha = 10^{-2}$, we obtain the
872 results given in Figure 12. Actually, the reconstruction is satisfactory outside a small
873 neighborhood around $x = a$.

874 Note that here, we made the choice to set 0 for the potential in the set $\{x \in$
875 $(0, L), |w_0(x)| \leq \alpha\}$. Of course, other choices are possible. Among them, one could
876 for instance simply do a linear interpolation between the values at the boundary of
877 the set $\{x \in (0, L), |w_0(x)| > \alpha\}$. Though, as illustrated in Figure 12, it seems that
878 Algorithm 4 converges anyway in the set $\{x \in (0, L), |w_0(x)| > \alpha\}$. One can therefore
879 perform any kind of interpolation process to complete the values of the potentials in
880 the set $\{x \in (0, L), |w_0(x)| > \alpha\}$ after the convergence has been achieved.



(a) $Q = \sin(2\pi x)$ and $a = 0.5$. (b) $Q = \sin(2\pi x)$ and $a = 0.2$. (c) Q heaviside and $a = 0.5$.

Fig. 12: Reconstructions for $w_0(x) = -a + x$ not satisfying (6), $CFL = 1$ and $s = 100$.

881 **5.5. Simulations in two dimensions.** We also performed some reconstructions
 882 in two dimensions where $\Omega = [0, 1]^2$, $x_0 = (-0.3, -0.3)$, $\Gamma_0 = \{x = 1\} \cup \{y = 1\}$,
 883 $w_0(x_1, x_2) = 2 + \sin(\pi x_1) \sin(\pi x_2)$, $w_1 = 0$, $f = 0$, $f_\partial = 2$, $\beta = 0.99$, $m = 2$ and
 884 $CFL = 0.5 \leq \frac{\sqrt{2}}{2}$. Figure 13 presents the results obtained for three different potentials.
 885 We took $s = 3$ and could not take it larger. Indeed, decreasing the space step
 886 h to ensure that sh remains small (condition (46)) leads to large systems (37) that
 887 exhaust the computational memory of SCILAB pretty fast. The preliminary results of
 888 Figure 13 are obtained in an ideal framework where both direct and inverse problems
 889 are solved with the same numerical scheme on the same mesh and there is no noise.
 890 All these simplifications will be removed in a forthcoming work where we wish to
 891 develop a convergent algorithm to reconstruct a non homogeneous wave speed from
 892 the information given by the flux \mathcal{M} .

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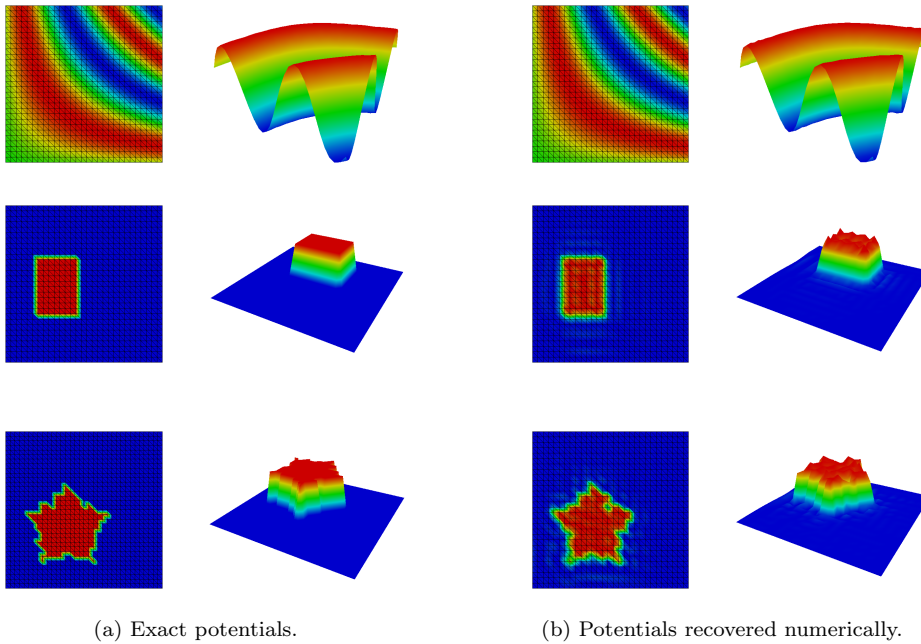


Fig. 13: Different examples of reconstruction in the 2d case.

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