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Thomas Sauvaget. A new formula for  $\zeta(2n + 1)$  (and how not to prove that  $\zeta(5)$  is irrational). 2016. hal-01352764v2

**HAL Id: hal-01352764**

**<https://hal.science/hal-01352764v2>**

Preprint submitted on 28 Oct 2016 (v2), last revised 14 Dec 2016 (v4)

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# A NEW FORMULA FOR $\zeta(2n+1)$ AND (AND HOW NOT TO PROVE THAT $\zeta(5)$ IS IRRATIONAL)

THOMAS SAUVAGET

ABSTRACT. Using a new polylogarithmic identity, we express the values of  $\zeta$  at odd integers  $2n+1$  as integrals over unit  $n$ -dimensional hypercubes of simple functions involving products of logarithms. We also prove an useful property of those functions as some of their variables are raised to a power. Finally, we discuss how one attempt to adapt Beukers's integral-based proof of the irrationality of  $\zeta(2)$  and  $\zeta(3)$  to the case of  $\zeta(5)$  fails.

## 1. INTRODUCTION

Are all values of Riemann's  $\zeta$  function irrational numbers when the argument is a positive integer ? This question goes back to the XVIIIth century when Euler published in 1755,  $n$  being a positive integer, that  $\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$  (where  $B_{2n} \in \mathbb{Q}$  is an even Bernoulli number) and Lambert proved in 1761 that  $\pi$  is irrational [11].

On the other hand, only in 1978 did Apéry [3] famously proved that  $\zeta(3)$  is irrational. This was later reproved in a variety of ways by several authors, in particular Beukers [6] who devised a simple approach involving certain intergrals over  $[0, 1]^3$  (which will be recalled in section 3). The reader should consult Fichler's very informative Bourbaki Seminar [10] for more details and references. In the early 2000s, an important work of Rivoal [15] and Ball and Rivoal [4] determined that an infinity of values of  $\zeta$  at odd integers are irrational, and the work of Zudililn [18] proved that at least one among  $\zeta(5), \zeta(7), \zeta(9)$  and  $\zeta(11)$  is irrational. Despite these advances, to this day no value of  $\zeta(2n+1)$  with  $2n+1 > 3$  is known to be irrational.

One dimensional integral formulas for  $\zeta(2n+1)$  have been known for a long time, for instance the 1965 monograph of Abramowitz and Stegun [1] gives:

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot(\pi x) dx$$

While it bears a striking structural analogy with Euler's formula for  $\zeta(2n)$ , it is not obvious how one might try to prove or disprove that these numbers are irrational.

On the other hand, multidimensional integral formulas for  $\zeta(2n+1)$  are more recent: as mentionned by Baumard in his PhD Thesis [5], quoting Zagier [17], it is Kontsevich in the early 1990s who found such a type of formula for Multiple Zeta Values, which in the case of simple zeta boils down, for any odd or even  $k$ , to:

$$\zeta(k) = \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \dots \int_0^{x_{k-2}} \frac{dx_{k-1}}{x_{k-1}} \int_0^{x_{k-1}} \frac{dx_k}{1-x_k}$$

This is easily proved by expanding the integrand in geometric series and integrating. As remarked by Silagadze [16], this can be rewritten more simply as a multidimensional integral over a unit hypercube:

$$\zeta(k) = \int_{[0;1]^k} \frac{dx_1 \dots dx_k}{1-x_1 \dots x_k}$$

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The author would like to thank the numerous contributors to useful freely available online knowledge resources, in particular the arXiv, Wikipedia, the SagemathCloud, WolframAlpha, and Stack Exchange sites.

This is much closer to the type of integrals that Beukers used, yet it is not clear how it might be adapted directly to prove that zeta is irrational at odd integers.

One should mention that Brown [7] has in the past few years outlined a geometric approach to proving the irrationality of all  $\zeta(2n+1)$ , which involves a generalization of Beukers method, see also the recent work of Dupont [9]. This is an active subject of current research, indeed a Hot Topics Workshop on this circle of ideas is due to be organized at MSRI in 2017.

In this work, we go along another path and prove new polylogarithmic identities which then allow to write each  $\zeta(2n+1)$  as an alternating sign  $(-1)^{n+1}$  times a multiple integral over a  $n$ -dimensional unit hypercube of certain functions involving logarithms (rather unsigned integrals over a  $(2n+1)$ -dimensional unit hypercube as in the previously mentioned formulas). These functions are shown to have an interesting property: raising some of the variables to a power leads to a fractional multiple of  $\zeta(2n+1)$  that belongs to the interval  $]0, \zeta(2n+1)[$ . We also investigate a related family of integrals for  $n=2$  for which we formulate some conjectures on their closed-form expression. These results and conjectures are the main aim of this work. Additionally we show how, assuming this on the one hand, and making some technical modifications to the framework that Beukers had devised to prove the irrationality of  $\zeta(2)$  and  $\zeta(3)$  on the other hand, one still cannot obtain even a conditional proof of the irrationality of  $\zeta(5)$  in that setting.

It is rather curious that these precise identities and integrals seem not to have been considered before, despite their simplicity. A search through the litterature did not return them (we have used the treatise of Lewin [13] as well as the relevant page on *functions.wolfram.com*[14]) : identities involving polylogarithms of different degrees are rather scarce, all the more so when all variables must be integers, and representations of  $\zeta(s)$  as multiple integrals over bounded domains, including some that have been worked out very recently by Alzer and Sondow [2], only go as far as a double integrals. The idea to consider the formulas presented below came to the author in a fortunate way after studying and trying to generalize an integral representation of  $\zeta(3) = \frac{1}{2} \int_0^1 \frac{\log(x)\log(1-x)}{x(1-x)} dx$  established by Janous [12] (and mentioned by Alzer and Sondow, where the author first learned about it), while the idea of trying to prove the irrationality of  $\zeta(5)$  was a reaction to a footnote in a section of the fine undergraduate book of Colmez [8] devoted to Nesterenko's proof of the irrationality of  $\zeta(3)$ .

## 2. VALUES OF $\zeta$ AT ODD INTEGERS AS MULTIDIMENSIONAL INTEGRALS ON UNIT HYPERCUBES

Recall that the polylogarithm function of order  $s \in \mathbb{R}$  is defined for  $z \in \{z \in \mathbb{C}, |z| < 1\}$  by  $\text{Li}_s(z) := \sum_{k=1}^{+\infty} \frac{z^k}{k^s}$  (and is extended by analytic continuation to the whole complex plane). In particular it satisfies the following well-known properties:

**Lemma 2.1.** *For any positive integer  $n$  we have  $\text{Li}_n(1) = \zeta(n)$ .*

**Lemma 2.2.** *For any integer  $n > 2$  and real number  $x \in ]0, +\infty[$  we have  $\frac{\partial \text{Li}_n}{\partial x}(x) = \frac{\text{Li}_{n-1}(x)}{x}$ . Moreover,  $\text{Li}_1(x) = -\log(1-x)$ .*

The aim of this section is to establish the following results (which, to the best of our knowledge, are new).

**Theorem 2.3.** *Let  $n$  be a positive integer, and for any integer  $1 \leq k \leq n$  define  $D_{k,n}$  to be the set of all ordered  $k$ -uplets  $j_1 < \dots < j_k$  of distinct integers taken in  $\{1, \dots, n\}$ . So  $\#D_{k,n} = \binom{n}{k}$ .*

*Define for any  $(x_1, \dots, x_n) \in ]0, 1[^n$  (the open unit hypercube of dimension  $n$ ) the function  $M_n$  as*

$$M_n(x_1, \dots, x_n) := \sum_{i=1}^n (-1)^i \left( \sum_{J \in D_{n-i,n}} \prod_{j \in J} \log(x_j) \right) \text{Li}_{n+i} \left( \prod_{u=1}^n x_u \right) + (-1)^{n+1} \text{Li}_{2n+1} \left( \prod_{u=1}^n x_u \right)$$

*Then we have*

$$\frac{\partial^n}{\partial x_1 \partial x_2 \dots \partial x_n} M_n(x_1, \dots, x_n) = \left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log \left( 1 - \prod_{i=1}^n x_i \right)$$

**Corollary 2.4.** *Let  $n$  be a positive integer. Then the value of Riemann's  $\zeta$  function at odd integers is :*

$$\zeta(2n+1) = (-1)^{n+1} \int \cdots \int_{[0;1]^n} \left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log \left( 1 - \prod_{i=1}^n x_i \right) dx_1 \cdots dx_n$$

*Proof.* For any positive integer  $n$ , the function  $M_n$  is at least  $\mathcal{C}^n$  away from 1. As for the corresponding (generalized) multiple integral, its integrand is  $\mathcal{C}^\infty$  away from 0 and 1, and the following will show the integral exists.

The corollary follows immediately from lemma 2.1 and the property  $\log(1) = 0$ . As for theorem 2.3, it is a tedious undergraduate exercise in differentiation to which we now turn.

For sake of clarity let us first work out an example for a small value of  $n$ , say  $n = 4$ , to show the type of cancellations that occur. For that case we have :

$$\begin{aligned} M_4(x_1, x_2, x_3, x_4) &:= -\log(x_1)\log(x_2)\log(x_3)\log(x_4)\text{Li}_5(x_1x_2x_3x_4) \\ &+ \left( \log(x_1)\log(x_2)\log(x_3) + \log(x_1)\log(x_2)\log(x_4) + \log(x_1)\log(x_3)\log(x_4) + \log(x_2)\log(x_3)\log(x_4) \right) \text{Li}_6(x_1x_2x_3x_4) \\ &- \left( \log(x_1)\log(x_2) + \log(x_1)\log(x_3) + \log(x_1)\log(x_4) + \log(x_2)\log(x_3) + \log(x_2)\log(x_4) + \log(x_3)\log(x_4) \right) \text{Li}_7(x_1x_2x_3x_4) \\ &+ \left( \log(x_1) + \log(x_2) + \log(x_3) + \log(x_4) \right) \text{Li}_8(x_1x_2x_3x_4) - \text{Li}_9(x_1x_2x_3x_4) \end{aligned}$$

So we find that :

$$\begin{aligned} \frac{\partial M_4}{\partial x_1}(x_1, x_2, x_3, x_4) &= - \underbrace{\left( \frac{\log(x_2)\log(x_3)\log(x_4)}{x_1} \text{Li}_5(x_1x_2x_3x_4) + \log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_4(x_1x_2x_3x_4)}{x_1x_2x_3x_4} \times x_2x_3x_4 \right)}_{\text{term A}} \\ &+ \underbrace{\left( \left( \frac{\log(x_2)\log(x_3)}{x_1} + \frac{\log(x_2)\log(x_4)}{x_1} + \frac{\log(x_3)\log(x_4)}{x_1} \right) \text{Li}_6(x_1x_2x_3x_4) \right)}_{\text{term B}} \\ &+ \left( \log(x_1)\log(x_2)\log(x_3) + \log(x_1)\log(x_2)\log(x_4) + \log(x_1)\log(x_3)\log(x_4) + \underbrace{\log(x_2)\log(x_3)\log(x_4)}_{\text{cancels term A}} \right) \frac{\text{Li}_5(x_1x_2x_3x_4)}{x_1x_2x_3x_4} \times x_2x_3x_4 \\ &- \underbrace{\left( \left( \frac{\log(x_2)}{x_1} + \frac{\log(x_3)}{x_1} + \frac{\log(x_4)}{x_1} \right) \text{Li}_7(x_1x_2x_3x_4) \right)}_{\text{term C}} \\ &+ \left( \log(x_1)\log(x_2) + \log(x_1)\log(x_3) + \log(x_1)\log(x_4) + \underbrace{\log(x_2)\log(x_3) + \log(x_2)\log(x_4) + \log(x_3)\log(x_4)}_{\text{cancels term B}} \right) \frac{\text{Li}_6(x_1x_2x_3x_4)}{x_1x_2x_3x_4} \times x_2x_3x_4 \\ &+ \left( \underbrace{\frac{1}{x_1} \text{Li}_8(x_1x_2x_3x_4)}_{\text{term D}} + \left( \log(x_1) + \underbrace{\log(x_2) + \log(x_3) + \log(x_4)}_{\text{cancels term C}} \right) \frac{\text{Li}_7(x_1x_2x_3x_4)}{x_1x_2x_3x_4} \times x_2x_3x_4 \right) - \underbrace{\frac{\text{Li}_8(x_1x_2x_3x_4)}{x_1x_2x_3x_4} \times x_2x_3x_4}_{\text{cancels term D}} \\ &= -\log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_4(x_1x_2x_3x_4)}{x_1} \\ &+ \left( \log(x_1)\log(x_2)\log(x_3) + \log(x_1)\log(x_2)\log(x_4) + \log(x_1)\log(x_3)\log(x_4) \right) \frac{\text{Li}_5(x_1x_2x_3x_4)}{x_1} \\ &- \left( \log(x_1)\log(x_2) + \log(x_1)\log(x_3) + \log(x_1)\log(x_4) \right) \frac{\text{Li}_6(x_1x_2x_3x_4)}{x_1} + \log(x_1) \frac{\text{Li}_7(x_1x_2x_3x_4)}{x_1} \end{aligned}$$

Taking now the derivative of this with respect to  $x_2$  we find :

$$\begin{aligned}
\frac{\partial^2 M_4}{\partial x_2 \partial x_1}(x_1, x_2, x_3, x_4) &= - \left( \underbrace{\frac{\log(x_1)\log(x_3)\log(x_4)}{x_2} \frac{\text{Li}_4(x_1 x_2 x_3 x_4)}{x_1}}_{\text{term E}} + \log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_3(x_1 x_2 x_3 x_4)}{x_1 x_2} \right) \\
&\quad + \left( \underbrace{\left( \frac{\log(x_1)\log(x_3)}{x_2} + \frac{\log(x_1)\log(x_4)}{x_2} \right) \frac{\text{Li}_5(x_1 x_2 x_3 x_4)}{x_1}}_{\text{term F}} \right. \\
&\quad \left. + \left( \log(x_1)\log(x_2)\log(x_3) + \log(x_1)\log(x_2)\log(x_4) + \underbrace{\log(x_1)\log(x_3)\log(x_4)}_{\text{cancels term E}} \right) \frac{\text{Li}_4(x_1 x_2 x_3 x_4)}{x_1 x_2} \right) \\
&\quad - \left( \underbrace{\left( \frac{\log(x_1)}{x_2} \frac{\text{Li}_6(x_1 x_2 x_3 x_4)}{x_1} \right)}_{\text{term G}} + \left( \log(x_1)\log(x_2) + \underbrace{\log(x_1)\log(x_3) + \log(x_1)\log(x_4)}_{\text{cancels term F}} \right) \frac{\text{Li}_5(x_1 x_2 x_3 x_4)}{x_1 x_2} \right) + \underbrace{\log(x_1) \frac{\text{Li}_6(x_1 x_2 x_3 x_4)}{x_1 x_2}}_{\text{cancels term G}} \\
&= -\log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_3(x_1 x_2 x_3 x_4)}{x_1 x_2} + \left( \log(x_1)\log(x_2)\log(x_3) + \log(x_1)\log(x_2)\log(x_4) \right) \frac{\text{Li}_4(x_1 x_2 x_3 x_4)}{x_1 x_2} \\
&\quad - \log(x_1)\log(x_2) \frac{\text{Li}_5(x_1 x_2 x_3 x_4)}{x_1 x_2}
\end{aligned}$$

Differentiation of that result with respect to  $x_3$  leads to :

$$\begin{aligned}
\frac{\partial^3 M_4}{\partial x_3 \partial x_2 \partial x_1}(x_1, x_2, x_3, x_4) &= - \left( \underbrace{\frac{\log(x_1)\log(x_2)\log(x_4)}{x_3} \frac{\text{Li}_3(x_1 x_2 x_3 x_4)}{x_1 x_2}}_{\text{term H}} + \log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_2(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3} \right) \\
&\quad \left( \underbrace{\left( \frac{\log(x_1)\log(x_2)}{x_3} \right) \frac{\text{Li}_4(x_1 x_2 x_3 x_4)}{x_1 x_2}}_{\text{term I}} + \left( \log(x_1)\log(x_2)\log(x_3) + \underbrace{\log(x_1)\log(x_2)\log(x_4)}_{\text{cancels term H}} \right) \frac{\text{Li}_3(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3} \right) \\
&\quad - \underbrace{\log(x_1)\log(x_2) \frac{\text{Li}_4(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3}}_{\text{cancels term I}} \\
&= -\log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_2(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3} + \log(x_1)\log(x_2)\log(x_3) \frac{\text{Li}_3(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3}
\end{aligned}$$

And finally differentiating this previous result with respect to  $x_4$  we find :

$$\begin{aligned}
\frac{\partial^4 M_4}{\partial x_4 \partial x_3 \partial x_2 \partial x_1}(x_1, x_2, x_3, x_4) &= - \left( \underbrace{\frac{\log(x_1)\log(x_2)\log(x_3)}{x_4} \frac{\text{Li}_2(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3}}_{\text{term J}} + \log(x_1)\log(x_2)\log(x_3)\log(x_4) \frac{\text{Li}_1(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3 x_4} \right) \\
&\quad + \underbrace{\log(x_1)\log(x_2)\log(x_3) \frac{\text{Li}_2(x_1 x_2 x_3 x_4)}{x_1 x_2 x_3 x_4}}_{\text{cancels term J}} = - \prod_{i=1}^4 \frac{\log(x_i)}{x_i} \text{Li}_1\left(\prod_{i=1}^4 x_i\right)
\end{aligned}$$

and by using the second statement of lemma 2.2 this concludes the proof of the  $n = 4$  case.

Notice in the previous computations the telescopic structure of the cancellations. The general case proceeds in a similar fashion to that of  $n = 4$ , except it is a bit cumbersome to now make the cancellations explicit. Namely, differentiating  $M_n$  with respect to  $x_1$  one finds :

$$\frac{\partial M_n}{\partial x_1}(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^i \left( \left( \sum_{J \in D_{n-i,n}^{*(1)}} \frac{1}{x_1} \prod_{j \in J} \log(x_j) \right) \text{Li}_{n+i} \left( \prod_{k=1}^n x_k \right) + \left( \sum_{J \in D_{n-i,n} \setminus D_{n-i,n}^{*(1)}} \prod_{j \in J} \log(x_j) \right) \frac{\text{Li}_{n+i-1} \left( \prod_{k=1}^n x_k \right)}{x_1} \right) + (-1)^{n+1} \frac{\text{Li}_{2n} \left( \prod_{k=1}^n x_k \right)}{x_1}$$

where  $D_{n-i,n}^{*(1)}$  denotes elements of the set  $D_{n-i,n}$  where 1 is not in the  $(n-i)$ -uplet, so this is also exactly the set of ordered  $(n-i-1)$ -uplets of distinct elements taken in the set  $\{2, \dots, n\}$ , and so we have the inclusion  $D_{n-i,n}^{*(1)} \subset D_{n-i-1,n}$ . Hence the telescopic cancellations witnessed in the case  $n=4$  occur between terms of two consecutive values of  $i$ . The remaining differentiations with respect to the other variables ultimately lead to the desired expression. Thus we have an explicit antiderivative of  $\left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log(1 - \prod_{i=1}^n x_i)$  and it is clear from its expression that the generalized integral exists. This finishes the proof.  $\square$

The second pair of results is an extension of the previous one:

**Theorem 2.5.** *Let  $n$  and  $r$  be positive integers, and for any integer  $1 \leq k \leq n$  define  $D_{k,n}$  to be the set of all ordered  $k$ -uplets  $j_1 < \dots < j_k$  of distinct integers taken in  $\{1, \dots, n\}$ . So  $\#D_{k,n} = \binom{n}{k}$ .*

*Define for any  $(x_1, \dots, x_n) \in ]0, 1[^n$  (the open unit hypercube of dimension  $n$ ) the function  $N_{n,r}$  as*

$$N_{n,r}(x_1, \dots, x_n) := \sum_{i=1}^n (-1)^i r^{n+1-i} \left( \sum_{J \in D_{n-i,n}} \prod_{j \in J} \log(x_j) \right) \text{Li}_{n+i} \left( \left( \prod_{u=1}^n x_u \right)^r \right) + (-1)^{n+1} \text{Li}_{2n+1} \left( \left( \prod_{u=1}^n x_u \right)^r \right)$$

*Then we have*

$$\frac{\partial^n}{\partial x_1 \partial x_2 \dots \partial x_n} N_{n,r}(x_1, \dots, x_n) = r^{2n} \left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log \left( 1 - \left( \prod_{i=1}^n x_i \right)^r \right)$$

**Corollary 2.6.** *Let  $n$  and  $r$  be positive integers. Then we have:*

$$\frac{\zeta(2n+1)}{r^{2n}} = (-1)^{n+1} \int_{[0;1]^n} \dots \int \left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log \left( 1 - \left( \prod_{i=1}^n x_i \right)^r \right) dx_1 \dots dx_n$$

*Proof.* It is a simple variant of the one for the first pair of results. We show simply the case  $n=2$ ,  $r > 1$  to illustrate the new cancelations.

By definition we have  $N_{2,r}(x, y) = -r^2 \log(x) \log(y) \text{Li}_3((xy)^r) + r(\log(x) + \log(y)) \text{Li}_4((xy)^r) - \text{Li}_5((xy)^r)$ .

So we get:

$$\begin{aligned} \frac{\partial N_{2,r}}{\partial x}(x, y) &= \underbrace{-\frac{r^2}{x} \log(y) \text{Li}_3((xy)^r)}_{\text{term A}} - r^2 \log(x) \log(y) \frac{\text{Li}_2((xy)^r)}{(xy)^r} r(xy)^{r-1} y \\ &+ \underbrace{\frac{r}{x} \text{Li}_4((xy)^r)}_{\text{term B}} + r \left( \log(x) + \underbrace{\log(y)}_{\text{cancels term A}} \right) \frac{\text{Li}_3((xy)^r)}{(xy)^r} r(xy)^{r-1} y - \underbrace{\frac{\text{Li}_4((xy)^r)}{x}}_{\text{cancels term B}} r \\ &= -r^3 \log(x) \log(y) \frac{\text{Li}_2((xy)^r)}{x} + r^2 \frac{\log(x)}{x} \text{Li}_3((xy)^r) \end{aligned}$$

And differentiating this with respect to  $y$  we do get  $-r^4 \frac{\log(x)}{x} \frac{\log(y)}{y} \log(1 - (xy)^r)$ .  $\square$

**Remark 2.7.** *If in 2.6 we choose  $r$  such that there exists an integer  $q$  satisfying  $r = 7^q$  then we have:*

$$0 < (-1)^{n+1} \int \cdots \int_{[0;1]^n} \left( \prod_{i=1}^n \frac{\log(x_i)}{x_i} \right) \log \left( 1 - \left( \prod_{i=1}^n x_i \right)^{7^q} \right) dx_1 \cdots dx_n = \frac{\zeta(2n+1)}{(7^q)^{2n}} < \left( \frac{1}{7} \right)^{nq}.$$

### 3. HOW NOT TO PROVE THE IRRATIONALITY OF $\zeta(5)$

First let us recall the standard irrationality criteria of Dirichlet :

**Lemma 3.1.** (*Dirichlet, 1848*)  $\alpha \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow \forall \epsilon > 0 \quad \exists p \in \mathbb{N} \exists q \in \mathbb{Q}$  such that  $|p\alpha - q| < \epsilon$ .

In 1979, shortly after Apéry presented his proof of the irrationality of  $\zeta(2)$  and  $\zeta(3)$ , Beukers [6] found another proof using Dirichlet's criteria applied to some particular integral representations of those two numbers. We quickly summarize the strategy as follows (the author also benefited from the extremely clear slides of Brown [7]).

- step 1: we have  $\int_0^1 \int_0^1 \frac{-\log(xy)}{1-xy} dx dy = 2\zeta(3)$  and for any integer  $r \geq 1$  we have  $\int_0^1 \int_0^1 \frac{-\log(xy)}{1-xy} (xy)^r dx dy = 2 \left( \zeta(3) - \frac{1}{1^3} - \cdots - \frac{1}{r^3} \right) \leq 2\zeta(3)$
- step 2: by denoting a Legendre-type polynomial  $P_k(x) := \frac{1}{k!} \left\{ \frac{d}{dx} \right\}^k x^k (1-x^k) \in \mathbb{Z}[X]$  and using the previous step we have  $I_k := \int_0^1 \int_0^1 \frac{-\log(xy)}{1-xy} P_k(x) P_k(y) dx dy = \frac{A_k + B_k \zeta(3)}{d_k^3}$  with  $A_k, B_k \in \mathbb{Z}$  and  $d_k := \text{lcm}(1, \dots, k)$
- step 3: by the Prime Number Theorem we have for any integer  $k \geq 1$  that  $d_k < 3^k$
- step 4: we have  $\int_0^1 \frac{1}{1-(1-xy)z} dz = -\frac{\log(xy)}{1-xy}$
- step 5: by integration by parts one finds also that  $I_k = \int_0^1 \int_0^1 \int_0^1 \left( \frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} \right)^k \frac{dx dy dz}{1-(1-xy)z}$
- step 6: we can bound uniformly for  $0 \leq x, y, z \leq 1$  one part of the integrand  $\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} \leq (\sqrt{2}-1)^4 < \frac{1}{2}$  (this is the reason for introducing  $P_k$  rather than working with the integrand of step 1 where  $(xy)^r$  can only be bounded by 1)
- step 7: by using most of the previous steps we find  $0 < \left| \frac{A_k + B_k \zeta(3)}{d_k^3} \right| \leq 2\zeta(3)(\sqrt{2}-1)^{4k}$
- step 8: using now the information on the growth of  $d_k$ , so of  $d_k^3$  too, we get  $0 < |A_k + B_k \zeta(3)| \leq \left( \frac{4}{5} \right)^k$ , which concludes the proof by Dirichlet's criteria.

Unfortunately in the ensuing years and decades no tweak to that strategy could be made to work for values of  $\zeta$  at other odd integers. In what follows we shall use our results from the previous section as well as some conjectures on a closed-form expression for another family of integrals, to show that inserting that with straightforward modifications into Beukers's strategy does not lead to a proof that  $\zeta(5)$  is irrational.

Indeed, we now make the following observations:

**Conjecture 3.2.** *Let  $n \geq 2$  and  $k \geq 1$  be integers. Then:*

(i) *there exists two sequences of positive integers  $(a_k)_{k \in \mathbb{N}^*}$  and  $(b_k)_{k \in \mathbb{N}^*}$  such that for all  $k \geq 3$  we have  $0 < a_k < b_k < ((2k+1)!)^{11}$  and  $\gcd(a_k, b_k) = 1$  and*

$$\int \cdots \int_{[0;1]^2} \frac{\log(x) \log(y) \log(1-(xy)^1)}{xy} \log(x) \log(y) (xy)^{2k+1} dx dy = \frac{4\pi^6}{3^3 \cdot 5 \cdot 7 \cdot (2k+1)} + \frac{4\pi^4}{3^2 \cdot 5 \cdot (2k+1)^3} + \frac{2\pi^2}{3 \cdot (2k+1)^5} + \frac{4\zeta(5)}{(2k+1)^2} + \frac{4\zeta(3)}{(2k+1)^4} - \frac{a_k}{b_k}$$

(ii) there exists two sequences of integers  $(c_k)_{k \in \mathbb{N}^*}$  and  $(d_k)_{k \in \mathbb{N}^*}$  such that for all  $k \geq 5$  we have  $0 < c_k < d_k < ((2k+1)!)^{11}$  and  $\gcd(c_k, d_k) = 1$  and

$$\int_{[0;1]^2} \dots \int \frac{\log(x)\log(y)\log(1-(xy)^2)}{xy} \log(x)\log(y)(xy)^{2k+1} dx dy =$$

$$\frac{\pi^6}{2^3 \cdot 3 \cdot 5 \cdot (2k+1)} + \frac{\pi^4}{2^2 \cdot 3 \cdot (2k+1)^3} + \frac{\pi^2}{(2k+1)^5} + \frac{8\log(2)}{(2k+1)^6} + \frac{31\zeta(5)}{4(2k+1)^2} + \frac{7\zeta(3)}{(2k+1)^4} - \frac{c_k}{d_k}$$

(iii) more generally for each integer  $r \geq 3$  there exists two sequences of integers  $(u_k^{(r)})_{k \in \mathbb{N}^*}$  and  $(v_k^{(r)})_{k \in \mathbb{N}^*}$  such that for all  $k \geq 2r+1$  we have  $0 < u_k^{(r)} < v_k^{(r)} < ((2k+1)!)^{11}$  and  $\gcd(u_k^{(r)}, v_k^{(r)}) = 1$  and:

$$\int_{[0;1]^2} \dots \int \frac{\log(x)\log(y)\log(1-(xy)^r)}{xy} \log(x)\log(y)(xy)^{2k+1} dx dy =$$

$$\frac{\alpha^{(r)}\pi^6}{\beta^{(r)} \cdot (2k+1)} + \frac{\gamma^{(r)}\pi^4}{\delta^{(r)}(2k+1)^3} + \frac{\eta^{(r)}\pi^2}{\kappa^{(r)}(2k+1)^5} + \frac{\lambda^{(r)}\log(r)}{\mu^{(r)}(2k+1)^6} + \frac{\nu^{(r)}\zeta(5)}{\rho^{(r)}(2k+1)^2} + \frac{\tau^{(r)}\zeta(3)}{\omega^{(r)}(2k+1)^4} - \frac{u_k^{(r)}}{v_k^{(r)}}$$

where the positive integers  $\alpha^{(r)}, \beta^{(r)}, \gamma^{(r)}, \delta^{(r)}, \eta^{(r)}, \kappa^{(r)}, \lambda^{(r)}, \mu^{(r)}, \nu^{(r)}, \rho^{(r)}, \tau^{(r)}$  and  $\omega^{(r)}$  depend only on  $r$ .

**Remark 3.3.** For cases (i) and (ii) this is based on observations for  $k$  up to 7, and (iii) is purely based on observation of features of the previous two. Unfortunately, it is beyond the mathematical ability of the author to establish any of them.

Our estimates for the bounds on  $b_k$  and  $d_k$  (and thus also  $v_k^{(r)}$ ) are very conservative and based on the following observations:

$k$	$b_k$	$d_k$
1	$2^4 3^6 = 3! \times 2^3 3^5$	$3^5 = 3! \times \frac{3^4}{2}$
2	$2^{10} 3^6 5^7 = 5! \times 2^7 3^5 5^6$	$3^6 5^7 = 5! \times \frac{3^5 5^6}{2^3}$
3	$2^{10} 3^6 5^6 7^7 = 7! \times 2^6 3^4 5^5 7^6$	$3^6 5^6 7^7 = 7! \times \frac{3^4 5^5 7^6}{2^4}$
4	$2^{16} 3^{13} 5^6 7^6 = 9! \times 2^9 3^9 5^5 7^5$	$3^{11} 5^6 7^6 = 9! \times \frac{3^7 5^5 7^5}{2^7}$
5	$2^{16} 3^{12} 5^5 7^6 11^7 = 11! \times 2^8 3^8 5^3 7^5 11^6$	$3^{12} 5^6 7^6 11^7 = 11! \times \frac{3^8 5^4 7^5 11^6}{2^8}$
6	$2^{16} 3^{12} 5^4 7^6 11^6 13^7 = 13! \times 2^6 3^7 5^2 7^5 11^5 13^6$	$3^{12} 5^6 7^6 11^6 13^7 = 13! \times \frac{3^7 5^4 7^5 11^5 13^6}{2^{10}}$
7	$2^{17} 3^{12} 5^5 7^7 11^6 13^6 = 15! \times 2^6 3^6 5^2 7^5 11^5 13^5$	

Now, define for positive integers  $q$  the numbers:

$$J_q := \int_{[0;1]^2} \dots \int \frac{\log(x)\log(y)\log(1-(xy)^{7^q})}{xy} \log(x)\log(y)(xy)^{2 \cdot 7^q + 1} dx dy$$

$$- \left( \frac{\alpha^{(7^q)}\pi^6}{\beta^{(7^q)} \cdot (2 \cdot 7^q + 1)} + \frac{\gamma^{(7^q)}\pi^4}{\delta^{(7^q)}(2 \cdot 7^q + 1)^3} + \frac{\eta^{(7^q)}\pi^2}{\kappa^{(7^q)}(2 \cdot 7^q + 1)^5} + \frac{\lambda^{(7^q)}\log(7^q)}{\mu^{(7^q)}(2 \cdot 7^q + 1)^6} + \frac{\tau^{(7^q)}\zeta(3)}{\omega^{(7^q)}(2 \cdot 7^q + 1)^4} \right)$$

From our evidence-based conjectures we have that  $J_q = \frac{\nu^{(7^q)}\zeta(5)}{\rho^{(7^q)}(2 \cdot 7^q + 1)^2} - \frac{u_k^{(7^q)}}{v_k^{(7^q)}}$ . So, using the triangle inequality on the definition of  $J_q$ , there also exists some real number  $T_{7^q} > 0$  such that one can bound as follows:



$$J_q \leq \underbrace{\text{Sup}\{|\log(x)\log(y)(xy)^{2 \cdot 7^q + 1}| \text{ where } 0 \leq x, y \leq 1\}}_{< \frac{1}{e} < 1} \times \left| \int \dots \int_{[0;1]^2} \frac{\log(x)\log(y)\log(1 - (xy)^{7^q})}{xy} dx dy \right| + \frac{T_{7^q}}{2 \cdot 7^q}$$

$$< \frac{\zeta(5)}{(7^q)^4} + \frac{T_{7^q}}{2 \cdot 7^q} = \frac{1}{7^q} \left( \frac{\zeta(5)}{(7^q)^3} + \frac{T_{7^q}}{2} \right) < \frac{1}{7^q} (1 + T_{7^q})$$

So recapitulating we have shown so far that:

$$0 < \left| \frac{v_k^{(7^q)} \nu^{(7^q)} \zeta(5) - \rho^{(7^q)} (2 \cdot 7^q + 1)^2 u_k^{(7^q)}}{\rho^{(7^q)} (2 \cdot 7^q + 1)^2 v_k^{(7^q)}} \right| < \frac{1}{7^q} (1 + T_{7^q})$$

To conclude on the irrationality of  $\zeta(5)$  one would thus need that for any given  $\epsilon \in ]0, 1[$  there exists some  $q$  such that:

$$\frac{\rho^{(7^q)} (2 \cdot 7^q + 1)^2 v_k^{(7^q)}}{7^q} (1 + T_{7^q}) < \epsilon$$

which is patently not the case (whatever the precise growth rate of  $T_{7^q}$  may be).

**Remark 3.4.** *There might still be a way to use those results and conjectures differently so as to obtain the irrationality of  $\zeta(5)$ , but the author was not able to find it.*

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*E-mail address:* thomasfsauvaget@gmail.com