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# A general existence result for conservation laws with spatial heterogeneities

Benedetto Piccoli <sup>\*</sup>      Magali Tournus <sup>†</sup>

## Abstract

We consider a scalar conservation law with a flux containing spatial heterogeneities of bounded variation, where the number of discontinuities may be infinite. We address the question of existence of an adapted entropy solution in the BV framework. A sufficient key condition guaranteeing existence is identified and new BV estimates are given. This provides the most general BV theory available. Moreover, we show with a counter-example that if this hypothesis is violated, the problem may be ill-posed in the BV framework.

**Key-words.** Scalar conservation laws, Discontinuous flux with Bounded Variations, Front-Tracking Method, Adapted entropies.

**AMS subject classifications.** 35L65, 35B44, 35A01.

## 1 Introduction

### 1.1 Context

We are interested in scalar conservation laws with a flux containing spatial heterogeneities. The conservation law is written as

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} A(\rho, x) = 0, & t > 0, x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \quad (1)$$

where  $A(\rho, \cdot)$  may a priori be discontinuous in an infinite number of points  $x$ , and where  $\rho_0 \geq 0$ .

There are many models that lead to consider hyperbolic conservation laws with a flux function discontinuous in the state space, arising in fields as various as vehicular traffic flows [4] [13] [17], two-phase flows porous medias [2] (see also [16]), sedimentation [12], kidney physiology [23], cell dynamics [7], and others.

When the space dependence is sufficiently smooth, well-posedness of the scalar equation (1) is well-known and the commonly admitted criterion to select a physical solution was described by Kruřkov [21]. For discontinuous fluxes, however, it is less clear how to select the appropriate weak solution. In most theories, authors restrict to fluxes with separated variables, i.e. of the shape  $A(\rho, x) = a(\rho)f(x)$ , see [3], [8], [24], where  $f$  has a finite number of discontinuities, usually only one. Uniqueness theory then mostly relies on some given constraints imposed on the solution at the points where the flux is discontinuous (e.g. using germ theory [5]), whereas, elsewhere, the usual

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entropy formulation holds, see [1], [11]. For example, a natural condition to impose, at the points  $x$  where the flux is discontinuous, is the conservation of the flux through the heterogeneity

$$A(\rho(x^-, t), x^-) = A(\rho(x^+, t), x^+), \quad t > 0, \quad (2)$$

but it is not sufficient to isolate a unique solution (see the counter example in [6]). Interface entropy conditions may depend on which solution we aim to select and may thereby vary with the context. Notice that to impose specific entropy conditions at discontinuities of  $f$ , the use of traces across this line [26] is mandatory, which requires the solution to be regular enough.

The existence of solutions is then usually obtained either by using specific Riemann solvers related to the entropy formulation ([14], [18]), either by approximating the flux by regular ones ([19]) or by passing to the limit robust and convergent numerical schemes, sometimes established independently of any entropy criterion ([20], [25]). Note that the determination of the appropriate Riemann solver is a necessary tool to define a Godunov scheme [22]. In all of the works cited above, in addition to the separated variable framework, strong modeling assumptions were done regarding the flux  $a$  (convexity, concavity, monotony, ...) and on the initial conditions  $\rho_0$ .

To get rid of the separated variable framework, a new uniqueness theory for conservation laws with BV flux was introduced by Baiti-Jenssen [9] in 1997, and then developed by Audusse-Perthame [6] in 2008. The novelty of the approach of Baiti-Jensen [9] is that their framework does not distinguish the discontinuity points one by one, allowing them to be infinite. Their idea is to modify the usual Kruřkov entropy formulation, which makes no sense at the points where the flux is discontinuous, and to adapt the entropies to the discontinuous flux. The theory provides a framework in which the solution of (1) can be uniquely defined provided that the flux  $A$  satisfies the following assumptions

- A-1. For all  $\rho \in \mathbb{R}^+$ , the flux  $A(\rho, \cdot)$  is continuous, except maybe on a negligible subset  $N$  of  $\mathbb{R}_x$ .
- A-2. For all  $x \in \mathbb{R} \setminus N$ ,  $A(\cdot, x)$  is locally Lipschitz and invertible.
- A-3. There are two continuous functions  $A_- \geq 0, A_+ \geq 0$  (with  $A_\pm(\rho) > 0$  for  $\rho > 0$ ), such that  $A(\pm\infty) = +\infty$ , and  $A_-(\rho) \leq |A(\rho, x)| \leq A_+(\rho)$  for all  $x \in \mathbb{R}$ .

Under hypotheses (A-1,2,3), the usual Kruřkov [21] entropies  $(|\rho - k|)_{k \in \mathbb{R}}$  are replaced by the adapted entropies  $(|\rho - k_p(x)|)_{p \in \mathbb{R}}$ , where  $k_p$  is the unique solution of

$$A(k_p(x), x) = p \quad a.e. \quad x \in \mathbb{R}. \quad (3)$$

The existence and uniqueness of the  $k^p(x)$ , defined in (3), is guaranteed by the fact that  $A(\cdot, x)$  is one-to-one. Let us notice that the functions  $k_p$  are the solutions to the stationary problem

$$\frac{d}{dx} A(k_p, x) = 0.$$

When  $A$  does not depend on  $x$ , the functions  $k_p$  are then constant, as in the classical Kruřkov formalism. This framework leads to the following definition of entropy solutions :

**Definition 1** (Adapted Entropy solution). *A function  $\rho \in L^\infty([0, T] \times \mathbb{R}) \cap C^0([0, T], L^1_{loc}(\mathbb{R}))$  is an adapted entropy solution of (1), on  $[0, T] \times \mathbb{R}$  if for all the  $(k_p)_{p \in \mathbb{R}}$  satisfying (3) and for all non positive test function  $\varphi \in C^1_K([0, T] \times \mathbb{R})$ , the following inequality holds in the sense of distributions*

$$\frac{\partial}{\partial t} |\rho(x, t) - k_p(x)| + \frac{\partial}{\partial x} \left\{ \text{sign}(\rho(x, t) - k_p(x)) [A(\rho(x, t), x) - A(k_p(x), x)] \right\} \leq 0. \quad (4)$$

This formulation is well-adapted to guarantee uniqueness and  $L^1$  stability with respect to initial conditions, since the technique of doubling the variables can be applied. However, existence of such an entropy solution for (1) has only been shown under the specific assumption

$$A(\rho, x) = F(\rho, v(x)), \quad (5)$$

where  $v \in \text{BV}(\mathbb{R})$  and  $F \in \mathcal{C}^2(\mathbb{R}^2)$ , with the problem of existence for a general flux still open.

## 1.2 Framework

We set here the assumptions we make about the regularity and the shape of the flux. For modeling purposes and simplicity of exposition, we adopt the following framework

$$\boxed{\text{H0}} \quad \rho \mapsto A(\rho, x) \text{ is concave, strictly increasing, and } A(0, x) = 0, \text{ for every } x \in \mathbb{R}.$$

We impose that there exist constants  $c > 0$ ,  $C > 0$ , not depending on  $\rho$  or  $x$ , and a continuous function  $M$  such that

$$\boxed{\text{H1}} \quad \begin{aligned} c |\rho - \tilde{\rho}| &\leq |A(\rho, x) - A(\tilde{\rho}, x)| \leq C |\rho - \tilde{\rho}|, & \rho, \tilde{\rho} > 0, \\ \text{TV}.(A(\rho, \cdot)) &\leq M(\rho), & x \in \mathbb{R}. \end{aligned} \quad (6)$$

Let us recall that

$$\text{TV}.f := \sup \left\{ \sum_{j=1}^N |f(x_j) - f(x_{j-1})|, x_0 < x_1 < \dots < x_N \right\}, \quad (7)$$

where the sup is taken over all  $N \geq 1$  and over all  $N + 1$ -uplets  $(x_0, \dots, x_N)$ .

*Comments on the assumptions.* The surjectivity of  $A(\cdot, x)$  is crucial to guarantee the existence of the adapted entropies associated to (1) and also to guarantee the existence of a solution to the Riemann problem. The injectivity prevents any ambiguity on the solution, and may be relaxed, but this leads to a more complex definition of entropy solutions [6]. Notice that the injectivity of the flux is actually implied by the bi-Lipschitz assumption  $\boxed{\text{H1}}$ . The concavity is a technical assumption that allows us to significantly reduce complexity when solving Riemann problems. It may as well be relaxed in a future work. Unlike the assumption  $\boxed{\text{H0}}$  whose main purpose is to fix the ideas, the conditions  $\boxed{\text{H1}}$  we impose on the regularity of the flux are crucial for existence and uniqueness. The combination of assumptions  $\boxed{\text{H0}}$  and  $\boxed{\text{H1}}$  guarantees  $A - 3$ . Before we go further, we mention here that even if  $A$  is in a broader class than the separated variables functions, not any function  $A$  is allowed. For any flux  $A$ , we define

$$\text{Disc}(A) = \{x \in \mathbb{R} \mid \exists \rho \in \mathbb{R}^+, A(\rho, x^-) \neq A(\rho, x^+)\}. \quad (8)$$

**Lemma 1.** *Assuming  $\boxed{\text{H1}}$ ,  $\text{Disc}(A)$  is at most countable.*

$$(9)$$

The proof of Lemma 1 is postponed to Appendix. It allows us to approximate fluxes with infinitely many discontinuities by fluxes with a finite number of discontinuities. The key technical argument to guarantee existence of the solution is the fact that

$\boxed{H2}$  There exist a continuous function  $R : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $f \in BV(\mathbb{R})$  such that

$$|A(\rho, x) - A(\rho, y)| \leq R(\rho)|f(x) - f(y)|, \quad \text{for every } \rho \geq 0, \text{ and } (x, y) \in \mathbb{R}^2.$$

We impose  $R$  to be continuous, but what matters is only the fact that it is locally bounded at any point, including  $\rho = 0$ . Assumption  $\boxed{H2}$  is not a consequence of  $\boxed{H0}$ ,  $\boxed{H1}$ , and this leads to a deep obstacle in building the theory in the general case, as seen later.

### 1.3 Main Results

This paper is pulling together the several theories explored for scalar conservation laws with BV fluxes and is providing a general framework. Moreover, we identify what are the key-assumptions on the flux which are necessary to build this theory, and provide in that sense optimal general estimates on the total variation of the solution. We prove that there exists a solution that can be built via solving Riemann problems for an approximated system, and then passing the sequence of solutions to the limit via the Wave Front-Tracking method. The function we obtain at the limit coincides with the adapted entropy solution (Definition 1). This result is a generalization of [9] in the sense that we consider fluxes not included in their framework.

**Theorem 1** (Existence of an entropy solution for (1)). *Assume  $\boxed{H0}$ ,  $\boxed{H1}$ ,  $\boxed{H2}$ , and  $\rho_0 \in BV(\mathbb{R})$ , then there exists an adapted entropy solution  $\rho \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$  to the Cauchy problem (1). Moreover the solution is obtained as almost everywhere limit of approximate Wave Front Tracking solutions, and it satisfies*

$$\|\rho\|_{L^\infty([0,T] \times \mathbb{R})} \leq \frac{C}{c} \|\rho_0\|_{L^\infty(\mathbb{R})}, \quad (10)$$

and

$$TV.(\rho(\cdot, t)) \leq \frac{C}{c} TV.(\rho_0) + \frac{2}{c} \max_{\rho \in [0, M]} R(\rho) TV.f, \quad M = \frac{C}{c} \|\rho_0\|_\infty. \quad (11)$$

Next Theorem explains why the technical assumption  $\boxed{H2}$  is crucial for a BV theory.

**Theorem 2** (Non-Existence of a BV solution in the general case). *Assume  $\boxed{H0}$  and  $\boxed{H1}$  but not  $\boxed{H2}$ , then we can build an example where starting from an initial condition  $\rho_0 \in BV(\mathbb{R})$ , the total variation of the solution blows up at  $t = 0^+$ .*

The correspondence between formulation (1) and the formulation provided by Baiti-Jenssen is not trivial for non-separated variables. In other words, for a given flux  $A$ , it may not be possible to find  $F$  and  $v$  such that (5) holds. For separated variables, the formal equivalence is clear by picking  $f = v$  and  $F(u, v) = a(u)v$  provided that  $a \in \mathcal{C}^2(\mathbb{R})$ , then, even in that context, our result of existence is a slight generalization to the theory developed in [9] since we allow the flux to be only Lipschitz in  $\rho$ , whereas their framework requires it to be at least  $\mathcal{C}^2$ . Notice that the framework considered by [9] implies that  $\boxed{H2}$  is satisfied since

$$|A(\rho, x) - A(\rho, y)| = |F(\rho, v(x)) - F(\rho, v(y))| \leq \|F'\|_\infty |v(x) - v(y)|, \quad (12)$$

where  $\|F'\|_\infty$  is the Lipschitz constant of  $F$ , independent of  $\rho$ . Thus, our results generalizes [9] also for non-separated variables.

The paper is organized as follows. In Section 2, we consider an approximated version of (1) with piecewise constant flux  $A$ , which enables us to define a solution via the use of an adapted Riemann solver [18]. We check that this solution is an adapted entropy solution (Definition 1). In Section 3,

we derive uniform a priori estimates on the solution. Then, we approximate the general BV flux by a sequence of piecewise constant approximated fluxes and pass to the limit using the uniform BV estimates. In Section 4, we detail a counter-example preventing to generalize this theory to generic fluxes and settling the importance of assumption  $\boxed{H2}$ .

## 2 Building the solution using a Riemann solver - (One jump for $A$ , one jump for $\rho_0$ .)

Let us define the Riemann solver we use. The initial condition is a piecewise constant function defined by

$$\rho_0(x) = \begin{cases} \rho_\ell, & x < 0, \\ \rho_r, & x > 0, \end{cases} \quad (13)$$

where  $\rho_\ell \geq 0$ ,  $\rho_r \geq 0$ . We assume here that the discontinuities in  $A(\rho, \cdot)$  and  $\rho_0$  are localized at the same point  $x = 0$  since at the points where  $A(\rho, \cdot)$  is continuous, the Riemann problems are usual Riemann problems. The flux is written

$$A(\rho, x) = \begin{cases} a_-(\rho), & x < 0, \\ a_+(\rho), & x > 0, \end{cases} \quad (14)$$

where  $a_-$  and  $a_+$  are any strictly increasing and concave functions. We solve this unusual problem using a specific Riemann solver based on the theory introduced in [18] and detailed below. A Riemann solver is a function  $R : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  which provides the left and right traces  $\rho^-$  and  $\rho^+$  at the boundary  $x = 0$  of a solution to the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} a_-(\rho) = 0, & x < 0, \\ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} a_+(\rho) = 0, & x > 0, \\ \rho(0, x) = \rho_0(x), \end{cases} \quad (15)$$

where  $\rho_0$  defined in (13). We write  $R(\rho_\ell, \rho_r) = (\rho^-, \rho^+)$ . There is an equivalence between a Riemann solver  $R$  and a solution to the associated Riemann problem (15).

**Definition 2** (Riemann solver for discontinuous flux [18]). *A function  $\rho$  is said to be a weak solution to the Riemann problem (13), (15) if the traces  $(\rho^-, \rho^+)$  at  $x = 0$  are some constants that do not depend on time, and which satisfy*

*R-1. The wave  $(\rho_\ell, \rho^-)$  has a negative speed,*

*R-2. The wave  $(\rho^+, \rho_r)$  has a positive speed,*

*R-3. The flux is preserved through the discontinuity  $x = 0$ , i.e.  $A(\rho^-, 0^-) = A(\rho^+, 0^+)$ .*

**Proposition 1** (Existence of a bounded solution). *There exists a unique weak solution  $\rho \in L^\infty([0, T] \times \mathbb{R})$  to the Riemann problem (13), (15) provided by the Riemann solver defined in Definition 2. It is an entropy solution in the sense of Definition 1. If moreover  $a_- < a_+$ , it satisfies the maximum principle*

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|\rho_0\|_{L^\infty(\mathbb{R})}, \quad t \in [0, T]. \quad (16)$$

*Proof.* **Building  $\rho^-$ .**

We first claim that  $\rho^- = \rho_\ell$ . Let us first assume by contradiction that  $\rho_\ell < \rho^-$ . The flux  $a_-$  is concave for  $x < 0$  which implies that the wave  $(\rho_\ell, \rho_-)$  is a shock of velocity given by the Rankine-Hugoniot formula  $\sigma = (a_-(\rho^-) - a_-(\rho_\ell))/(\rho^- - \rho_\ell)$ . Since  $a_-$  is increasing, the velocity  $\sigma$  is positive, which contradicts  $R - 1$ . Suppose now, still by contradiction, that  $\rho_\ell > \rho^-$ . The wave  $(\rho_\ell, \rho_-)$  is then a rarefaction wave (concavity of the flux) and the velocity of its slowest front is  $a'_-(\rho^-) > 0$ . This also contradicts  $R - 1$ , and thus  $\rho^- = \rho_\ell$ .

**Building  $\rho^+$ .**

The trace  $\rho^+$  is uniquely determined by the relation  $a_-(\rho_\ell) = a_+(\rho^+)$  since  $a_+$  is one-to-one. We point out that the condition  $a_- < a_+$  implies that  $\rho^+ \leq \rho_\ell$ , which guarantee that inequality (16) is satisfied.

**The wave  $(\rho^+, \rho_r)$  is entropic.** The wave  $(\rho^+, \rho_r)$  (shock or rarefaction wave) only propagates for  $x > 0$ , and is entropic in the sense given by Kruzkov [21]. As a consequence, it is entropic in the sense of the adapted entropies.

**The stationary shock  $(\rho_\ell, \rho^+)$  is entropic.** Let us now check that the shock  $(\rho_\ell, \rho^+)$  is entropic in the sense of the adapted entropies. The function  $k_p$  defined by the formula (3) are expressed in that context as

$$k_p(x) = \begin{cases} a_-^{-1}(p), & x < 0, \\ a_+^{-1}(p), & x > 0. \end{cases} \quad (17)$$

The solution to the Riemann problem can be written as

$$\rho(x, t) = \rho_\ell(1 - H(x)) + \rho^+H(x), \quad x < \sigma t, \quad (18)$$

where  $\sigma$  is the velocity of the shock  $(\rho^+, \rho_r)$  if  $\rho^+ < \rho_r$  or the velocity to the backward front of the rarefaction wave  $(\rho^+, \rho_r)$  if  $\rho^+ > \rho_r$ , and  $H$  is the usual Heaviside function. We compute the dissipation of entropy as

$$D(x, t) = \frac{\partial}{\partial t} |\rho(x, t) - k_p(x)| + \frac{\partial}{\partial x} \left\{ \text{sign}(\rho(x, t) - k_p(x)) \{A(\rho(x, t), x) - p\} \right\}. \quad (19)$$

We have

$$D(x, t) = \frac{\partial}{\partial x} \left[ \text{sign}(\rho_\ell(1 - H(x)) + \rho^+H(x) - k_p(x)) \{a_-(\rho_\ell)(1 - H(x)) + a_+(\rho^+)H(x) - p\} \right] \quad (20)$$

which is

$$D(x, t) = \frac{\partial}{\partial x} E(x), \quad (21)$$

where

$$\begin{aligned} E(x) &= \text{sign}(\rho_\ell(1 - H(x)) + \rho^+H(x) - k_p(x)) \{a_-(\rho_\ell)(1 - H(x)) + a_+(\rho^+)H(x) - p\} \\ &= \begin{cases} \text{sign}(\rho_\ell - a_-^{-1}(p)) (a_-(\rho_\ell) - p), & \text{if } x < 0 \\ \text{sign}(\rho_r - a_+^{-1}(p)) (a_+(\rho_r) - p) & \text{if } x > 0. \end{cases} \end{aligned}$$

Since  $a_-$  and  $a_+$  are increasing, we have

$$E(x) = \begin{cases} |a_-(\rho_\ell) - p|, & \text{if } x < 0 \\ |a_+(\rho^+) - p| & \text{if } x > 0. \end{cases} \quad (22)$$

Condition  $R-3$  then implies that the function  $E$  is constant in a neighborhood of  $x = 0$ , and then  $D(x, t) = 0$  in a neighborhood of  $x = 0$ , which means that the entropy inequality (4) is trivially satisfied. □

**Remark 1.** *The stationary shock  $(\rho_\ell, \rho^+)$  does not dissipate entropy. In that sense, we consider it as a non-classical shock and will distinguish it in the following from classical shocks.*

### 3 Existence of a solution

To prove the existence of an adapted entropy solution to (1), we use the Wave-Front-Tracking algorithm [15]. We approximate the initial datum  $\rho_0 \in \text{BV}(\mathbb{R})$  and the flux  $x \rightarrow A(\rho, x)$  by piecewise constant functions  $\rho_{0,m}$  and  $A_n(\rho, \cdot)$  having respectively  $(m - 1)$  and  $(n - 1)$  discontinuities, such that  $\text{TV}\{\rho_{0,m}\} \leq \text{TV}\{\rho_0\}$ . We solve the Riemann problems at any discontinuity points of  $\rho_{0,m}$  and  $A_n$ , using the Riemann solver defined in Section 2. Depending whether the discontinuities of the initial profile are decreasing or increasing, they are evolving as shocks or rarefaction waves. We split rarefaction waves into rarefaction fans formed by rarefaction shocks of strength  $1/m$  (i.e. non entropic shocks) so that the profile stays piecewise constant [10]. When two waves interact or a wave interacts with a discontinuity of the medium, we solve a new Riemann problem. Notice that the number of waves can only increase at  $t = 0$ , so that the total number of waves is finite for all  $n, m \geq 0$  and for every time. Thus, we can define  $\rho_{n,m}$  for all  $n, m$  and for every time which provides a Wave Front Tracking approximate solution (in fact it is a weak solution violating the entropy condition by a quantity going to zero with  $m \rightarrow \infty$ ; see [10]). We first pass  $m$  to the limit to obtain a solution for  $\rho_0 \in \text{BV}(\mathbb{R})$  and  $n$  finite. Then, passing to the limit on the sequence of approximate solution as well as on the approximated sequence of fluxes, we prove the existence of a solution to the Cauchy problem (1) for  $\rho_0 \in \text{BV}(\mathbb{R})$ . The key point is to obtain a uniform BV estimate for the sequence of fluxes  $A_n(\rho_n(\cdot, t), \cdot)$ , where  $\rho_n$  are Wave-Front Tracking approximations. In the general case, it is not possible to obtain BV estimates directly for  $\rho_n$ , since interactions of waves with the junction can increase the total variation of the conserved quantity [13].

Equation (1) governs the evolution of the profile made of elementary waves, and we need to understand how two elementary waves interact to estimate the time evolution of the total variation. In our context, since the flux depends on  $x$ , besides shock and rarefaction waves, another kind of elementary wave may be created: Non-classical shocks. We define them and describe their evolution after interaction with another classical wave.

**Definition 3** (Elementary waves).

- A *Classical Shock (CS)* arising from  $x = x_d$  is a discontinuity  $(\rho_\ell, \rho_r)$  which satisfies  $\rho_\ell < \rho_r$  and moves forward with velocity  $s = \frac{A(\rho_r, x_d^+) - A(\rho_\ell, x_d^+)}{\rho_r - \rho_\ell}$  until it meets another elementary wave.
- A *Non Classical Shock (NCS)* is a stationary discontinuity  $(\rho_\ell, \rho_r)$  at  $x = x_d$  which satisfies

$$\frac{\partial}{\partial t} |G(x, t) - k_p(x)| + \frac{\partial}{\partial x} \left[ \text{sign} \left( G(x, t) - k_p(x) \right) \{A(G(x, t), x) - p\} \right] = 0, \quad (23)$$

where  $G(x, t) = \rho_\ell(1 - H(x - x_d)) + \rho_r H(x - x_d)$ .



- A Rarefaction Shock (RS) arising from  $x = x_d$  is a discontinuity which satisfies  $\rho_\ell > \rho_r$ . Its forward front moves with velocity  $\frac{\partial A}{\partial \rho}(\rho_r, x_d^+)$  and its backward front moves with velocity  $\frac{\partial A}{\partial \rho}(\rho_\ell, x_d^+)$ , until it meets another elementary wave. It is a piecewise constant approximation of a rarefaction wave.

### 3.1 The number of discontinuities is finite.

In that part, we state and prove the existence of a solution for a piecewise constant flux  $A(\rho, \cdot)$ .

**Proposition 2.** Under assumptions  $\boxed{H0}$ ,  $\boxed{H1}$ ,  $\boxed{H2}$  for  $A(\rho, \cdot)$  uniformly piecewise constant, i.e. such that there exists  $(z_1, \dots, z_n)$  such that  $\text{Disc}(A) \subset \{z_1, \dots, z_n\}$ , and for  $\rho_0 \in BV(\mathbb{R})$  we can build a solution in  $L^\infty([0, T] \times \mathbb{R})$  to the Cauchy problem (1). In addition, the solution  $\rho$  satisfies

$$\|\rho\|_{L^\infty([0, T] \times \mathbb{R})} \leq \frac{C}{c} \|\rho_0\|_{L^\infty(\mathbb{R})}, \quad (24)$$

and

$$\text{TV}(\rho(\cdot, t)) \leq \frac{C}{c} \text{TV}\{\rho_0\} + \frac{2}{c} \max_{[0, M]} R(\cdot) \text{TV}\{f\}, \quad M = \frac{C}{c} \|\rho_0\|_\infty. \quad (25)$$

The rest of this section is devoted to the proof of Proposition 2. Let us denote by  $\{z_i, i = 1 \dots n\}$  the set of points of discontinuity of  $A(\rho, \cdot)$ , and by  $\{y_j, j = 1 \dots m\}$  the points of discontinuity of  $\rho_0$  (to simplify notations, we drop the  $m$ - and  $n$ -dependence in the notation for  $\rho$ ,  $A$  and  $\rho_0$ ). We can assume without any restriction that  $\{z_i, i = 1 \dots n\} \cap \{y_j, j = 1 \dots m\} = \emptyset$ . We prove in this part that the total variation of the solution is bounded independently of  $m$  and  $n$ . Since the approximation  $\rho$  is piecewise constant, the total variation of  $\rho(\cdot, t)$  is defined as

$$\text{TV}(\rho(\cdot, t)) = \sum_i |\rho(x_i^+, t) - \rho(x_i^-, t)|, \quad x_i \text{ are the discontinuity points of } \rho. \quad (26)$$

The points of discontinuity  $x_i$  actually depend on  $t$ , but we drop this dependence for simplifying the notations. We distinguish the discontinuities of  $\rho$  as follows

$$\text{TV}(\rho(\cdot, t)) = \text{TV}_C(\rho(\cdot, t)) + \text{TV}_{NC}(\rho(\cdot, t)), \quad (27)$$

where

**Definition 4** (Total variation of Non-Classical and Classical Shocks). We define for all time  $t > 0$ , except for the times at which two waves interact:

$$\text{TV}_C(\rho(\cdot, t)) := \sum_{j=1}^I |\rho(x_j^+, t) - \rho(x_j^-, t)|, \quad (x_1, \dots, x_I) \in \mathbb{R}^I \mid x_j \notin \{z_i, i = 1 \dots n\}, \quad j = 1 \dots I, \quad (28)$$

$$\text{TV}_{NC}(\rho(\cdot, t)) := \sum_{i=1}^n |\rho(z_i^+, t) - \rho(z_i^-, t)|. \quad (29)$$

The total variation  $\text{TV}(\rho(\cdot, t))$  evolves with time 1) when two waves interact, and 2) at time  $t = 0$  because of the discontinuous medium. The interactions waves that can occur are of 3 types:

1. When a classical shock arrives at  $x = z_i$ , we have an interaction of type CS/NCS, and this gives rise to a Classical Shock and a Non-Classical Shock.

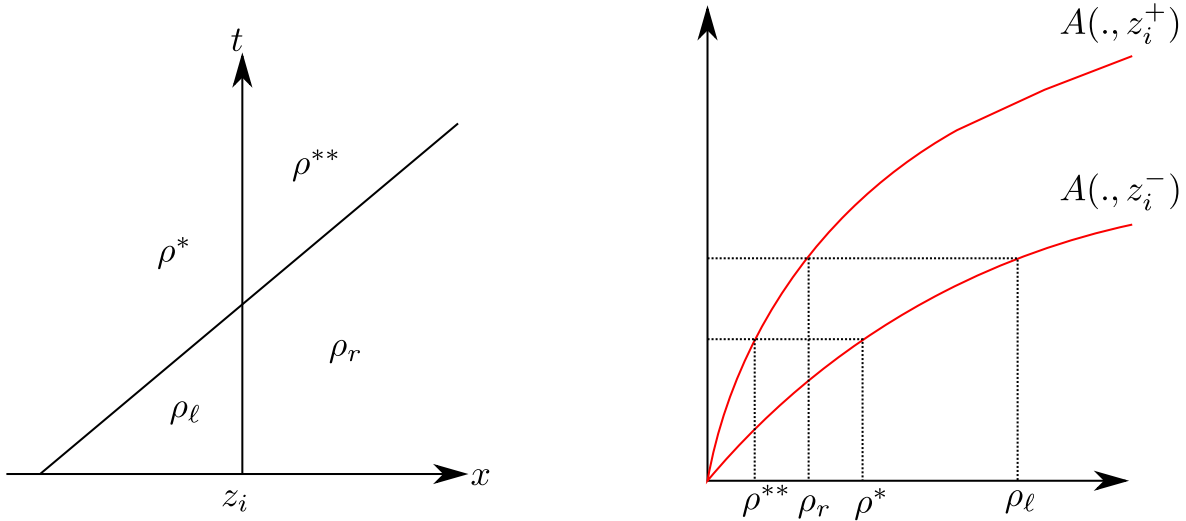


Figure 1: Waves interaction. A wave  $(\rho^*, \rho_\ell)$  meets a Non-Classical Shock  $(\rho_\ell, \rho_r)$  This gives rise to a new state  $\rho^{**}$  (Left) obtained by the Rankine Hugoniot conditions (Right).

2. When a Rarefaction Shock arrives at  $x = z_i$ , we have an interaction of type RS/NCS, and this gives rise to another Rarefaction Shock and a Non-Classical Shock.
3. Two waves of type CS/RS, CS/CS, RS/CS can interact and it gives rise to one wave of type CS or RS (usual Riemann problem).

See Figure 1 for an illustration. We now describe the waves creations at time  $t = 0$ . Let us remind that  $\rho_0$  is continuous at the points where the medium is discontinuous, so that  $\rho_0(z_i)$  is well-defined. At time  $t = 0$ ,

1. At the points where  $\rho_0$  is discontinuous, a Classical Wave is created. It may be a classical shock or a rarefaction shock depending on whether  $\rho_0$  is increasing or decreasing.
2. At the points  $z_i$  of discontinuity of  $A(\rho, \cdot)$ , we have the creation of a classical wave  $(\rho_i^*, \rho_0(z_i))$  and creation of Non-Classical-Shock  $(\rho_0(z_i), \rho_i^*)$ .

See Figure 2 for an illustration. At time  $t = 0$ , the discontinuities in  $\rho_0$  can neither be classified as Classical nor Non Classical Shocks so that we do not distinguish  $\text{TV}_C.\{\rho_0\}$  and  $\text{TV}_{NC}.\{\rho_0\}$ , but we merely have

$$\text{TV}.\{\rho_0\} = \sum_{j=1}^m |\rho_0(y_j^+) - \rho_0(y_j^-)|. \quad (30)$$

The sequence of the following lemmas ends in the proof of Proposition 2.

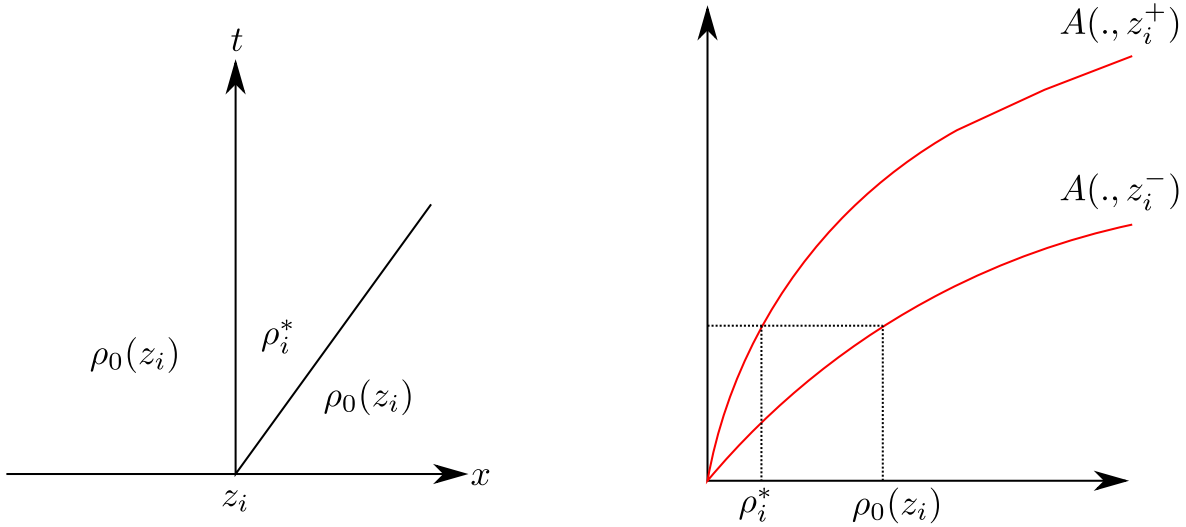


Figure 2: Wave creation. Two waves  $(\rho_0(z_i), \rho_i^*)$  and  $(\rho_i^*, \rho_0(z_i))$  emerge at time  $t = 0$ . The new state  $\rho_i^*$  (Left) is obtained by the Rankine Hugoniot conditions (Right).

**Lemma 2** (TV. bounds on the fluxes). *We denote by  $\rho$  the approximated solution to (1), where the flux  $A$  is piecewise constant. Then, for almost every time (except interaction times), we define*

$$\begin{aligned} TV_C.(A(\rho(\cdot, t), \cdot)) &= TV.(A(\rho_0(\cdot), \cdot)), \\ TV_{NC}.(A(\rho(\cdot, t), \cdot)) &= 0. \end{aligned} \quad (31)$$

*Proof.* We have

$$\begin{aligned} TV_C.(A(\rho(\cdot, t), \cdot)) &= \sum_{j=1}^I |A(\rho(x_j, t), x_j) - A(\rho(x_{j-1}, t), x_{j-1})|, (x_1, \dots, x_I) \in \mathbb{R}^I \mid x_j \neq z_i, \\ & \quad j = 1 \dots I, i = 1 \dots n, \end{aligned} \quad (32)$$

The Total Variation of  $\rho(\cdot, t)$  is not modified when two Classical Waves interact, and thus, the Total Variation of  $A(\rho(\cdot, t), \cdot)$  is not modified either since Classical waves can only interact at  $x \neq z_i$ . Let us assume that a Classical wave interacts with a Non Classical Shock at time  $t$  and at  $x = z_i$ , then we have (see Figure 1)

$$TV_C.(A(\rho(\cdot, t^+), \cdot)) - TV_C.(A(\rho(\cdot, t^-), \cdot)) = \left( |A(\rho^*, z_i^-) - A(\rho_\ell, z_i^-)| \right) - \left( |A(\rho^{**}, z_i^+) - A(\rho_r, z_i^+)| \right), \quad (33)$$

where the Rankine-Hugoniot conditions impose that

$$\begin{aligned} A(\rho^*, z_i^-) &= A(\rho^{**}, z_i^+) \\ A(\rho_\ell, z_i^-) &= A(\rho_r, z_i^+). \end{aligned} \quad (34)$$

Hence, as we have

$$\text{TV}_C.(A(\rho(\cdot, t^-), \cdot)) = \text{TV}_C.(A(\rho(\cdot, t^+), \cdot)), \quad (35)$$

which implies that the total variation of the flux does not evolve after an interaction NCS-CS or NCS-RS, i.e.

$$\text{TV}_C.(A(\rho(\cdot, t), \cdot)) = \text{TV}_C.(A(\rho(\cdot, 0^+), \cdot)), \quad t > 0. \quad (36)$$

Let us now explore how the Total Variation is modified initially. At time  $t = 0$ ,  $n$  Classical Waves  $(\rho_i^*, \rho_0(x_i))$  are simultaneously created from the  $n$  discontinuities of the medium, and the new states  $\rho_i^*$  satisfy  $A(\rho_0(z_i), z_i^-) = A(\rho_i^*, z_i^+)$ . At time  $t = 0$ , we have

$$\text{TV}.(A(\rho_0(\cdot), \cdot)) = \sum_{i=1}^n |A(\rho_0(z_i), z_i^+) - A(\rho_0(z_i), z_i^-)| + \sum_{j=1}^m |A(\rho_0(y_j^-), y_j) - A(\rho_0(y_j^+), y_j)|. \quad (37)$$

At time  $t = 0^+$ , we have  $N + J$  classical waves and  $N$  non-classical shocks

$$\begin{aligned} \text{TV}_C.(A(\rho(\cdot, 0^+), \cdot)) &= \sum_{i=1}^n |A(\rho_i^*, z_i^+) - A(\rho_0(z_i), z_i^+)| + \sum_{j=1}^m |A(\rho_0(y_j^-), y_j) - A(\rho_0(y_j^+), y_j)| \\ &= \text{TV}.(A(\rho_0(\cdot), \cdot)), \end{aligned} \quad (38)$$

and the first statement of Lemma 2 is proved combining (36) and (38). The result for Non-Classical Shocks holds because of the flux conservation through the discontinuities  $z_i$ .  $\square$

In the following lemma, we state that the conservation of the flux implies that the solution stays bounded for all time

**Lemma 3** ( $L^\infty$  bound). *We denote by  $\rho$  the approximated solution to (1), where the flux  $A$  is piecewise constant. Then*

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{c} \|\rho_0\|_{L^\infty(\mathbb{R})}. \quad (39)$$

*Proof.* We have, using  $\boxed{H1}$

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{c} \|A(\rho(\cdot, t), \cdot)\|_{L^\infty(\mathbb{R})}. \quad (40)$$

The flux conservation through both classical and non-classical waves implies

$$\|A(\rho(\cdot, t), \cdot)\|_{L^\infty(\mathbb{R})} = \|A(\rho_0(\cdot), \cdot)\|_{L^\infty(\mathbb{R})}. \quad (41)$$

Then

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{c} \|A(\rho_0(\cdot), \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{c} \|\rho_0\|_{L^\infty(\mathbb{R})}. \quad (42)$$

$\square$

Notice that this is not a maximum principle since it may happen that  $C > c$ .

**Lemma 4** (TV. bounds for  $\rho_n$ ). *We denote by  $\rho$  the approximated solution to (1), where the flux  $A$  is piecewise constant. Then,*

$$\begin{aligned} \text{TV}_C.(\rho(\cdot, t)) &\leq \frac{1}{c} \left( \max_{[0, \|\rho_0\|_\infty]} R(\cdot) \text{TV}\{f\} + C \text{TV}\{\rho_0\} \right), \\ \text{TV}_{NC}.(\rho(\cdot, t)) &\leq \frac{1}{c} \max_{[0, M]} R(\cdot) \text{TV}\{f\}, \quad \text{where } M = \frac{C}{c} \|\rho_0\|_\infty. \end{aligned} \quad (43)$$

*Proof.* The solution  $\rho$  is piecewise constant, so that its Total Variation is uniquely determined by its discontinuities.

**First Step :  $\mathbf{TV}_C(\rho(\cdot, t))$ .** At time  $t$ , the number of classical waves is at most  $N + J$ . Let us denote by  $(\rho_j^-, \rho_j^+)$  the wave number  $j$  localized at  $x = x_j$ . The localization  $x_j$  does not coincide with a discontinuity point  $z_i$  (except maybe for a finite number of times we do not pick here). We first control the Total Variation of  $\rho(\cdot, t)$  by the Total Variation of  $A(\rho(\cdot, t), \cdot)$

$$\mathbf{TV}_C(\rho(\cdot, t)) = \sum_{j=1}^{n+m} |\rho_j^+ - \rho_j^-| \leq \frac{1}{c} \sum_{j=1}^{n+m} |A(\rho_j^+, x_j) - A(\rho_j^-, x_j)| \leq \frac{1}{c} \mathbf{TV} \cdot (A(\rho(\cdot, t), \cdot)). \quad (44)$$

We now use Lemma 2 to obtain

$$\mathbf{TV}_C(\rho(\cdot, t)) \leq \frac{1}{c} \mathbf{TV} \cdot (A(\rho_0(\cdot), \cdot)). \quad (45)$$

For all  $(K + 1)$ -uplets  $x_0 < x_1 < \dots < x_K$ , for all  $\rho \in \mathbf{BV}(\mathbb{R})$ , using  $\boxed{H1}$  and  $\boxed{H2}$ ,

$$\begin{aligned} \sum_{j=1}^K |A(\rho(x_j), x_j) - A(\rho(x_{j-1}), x_{j-1})| &\leq \sum_{j=1}^K |A(\rho(x_j), x_j) - A(\rho(x_j), x_{j-1})| \\ &\quad + \sum_{j=1}^K |A(\rho(x_j), x_{j-1}) - A(\rho(x_{j-1}), x_{j-1})| \\ &\leq \sum_{j=1}^K R(\rho(x_j)) |f(x_j) - f(x_{j-1})| + \sum_{j=1}^K C |\rho(x_j) - \rho(x_{j-1})|. \end{aligned} \quad (46)$$

Hence,

$$\mathbf{TV} \cdot (A(\rho_0(\cdot), \cdot)) \leq \max_{[0, \|\rho_0\|_\infty]} R(\cdot) \mathbf{TV} \cdot \{f\} + C \mathbf{TV} \cdot \{\rho_0\}. \quad (47)$$

The combination of (45) and (47) gives us the first line of Lemma 4.

**Second Step :  $\mathbf{TV}_{NC}(\rho(\cdot, t))$ .** The Non-Classical Shocks are the waves  $(\rho(z_i^-, t), \rho(z_i^+, t))$  such that

$$A(\rho(z_i^-, t), z_i^-) = A(\rho(z_i^+, t), z_i^+). \quad (48)$$

Thus, we have, using  $\boxed{H1}$

$$\mathbf{TV}_{NC} \cdot \{\rho(\cdot, t)\} = \sum_{i=1}^n |\rho(z_i^+, t) - \rho(z_i^-, t)| \leq \frac{1}{c} \sum_{i=1}^n |A(\rho(z_i^+, t), z_i^+) - A(\rho(z_i^-, t), z_i^-)| \quad (49)$$

and then by (48)

$$\mathbf{TV}_{NC} \cdot \{\rho(\cdot, t)\} \leq \frac{1}{c} \sum_{i=1}^n |A(\rho(z_i^-, t), z_i^-) - A(\rho(z_i^-, t), z_i^+)|, \quad (50)$$

which is, using  $\boxed{H2}$  and Lemma 3

$$\mathbf{TV}_{NC} \cdot \{\rho(\cdot, t)\} \leq \frac{1}{c} \sum_{i=1}^n R(\rho(z_i^-, t)) |f(z_i^-) - f(z_i^+)|, \quad (51)$$

□

which ends the proof of Lemma 4, since  $M \geq \|\rho_0\|_\infty$ . Proposition 2 is obtained by approximating  $\rho_0$  by piecewise a constant function  $\rho_{0,m}$  such that  $\text{TV}\{\rho_{0,m}\} \leq \text{TV}\{\rho_0\}$ , and by letting  $m \rightarrow \infty$  (classical Wave Front Tracking algorithm).

### 3.2 The number of discontinuities is infinite.

We describe here how we pass to the limit and ends the proof of Theorem 1.

**Lemma 5.** *Given a flux  $A$  which satisfies  $\boxed{H0}$ ,  $\boxed{H1}$ ,  $\boxed{H2}$ , we can build a sequence of piecewise constant fluxes  $A_n$  (with  $(n-1)$  discontinuities) satisfying  $\boxed{H0}$ ,  $\boxed{H1}$ ,  $\boxed{H2}$  as well and such that*

1. *the fluxes  $A_n$  approximate the flux  $A$  in the sense that*

$$\forall \rho \in \mathbb{R}, \quad \|A_n(\rho, \cdot) - A(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq \varepsilon(n)R(\rho), \quad \varepsilon(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (52)$$

2. *If we denote by  $k_n^p$  and  $k^p$  the functions which respectively satisfy  $A_n(k_n^p, x) = p$  and  $A(k^p, x) = p$ , then*

$$\lim_{n \rightarrow \infty} \|k_n^p - k^p\|_{L^1(\mathbb{R})} = 0. \quad (53)$$

*Proof. First Step. Building the  $A_n$ .* The construction of the piecewise approximation of the flux  $A$  is based on the constructive proof of Lemma 2.2, [10], which states that any BV function can be approximated in  $L^\infty$  by a sequence of piecewise constant functions. We adapt the proof in our case, basically by showing that we can perform the same construction independently of  $\rho$ . Let us define the right-continuous function

$$F(x) = \text{TV}\{f|_{(-\infty, x]}\} = \sup \left\{ \sum_{j=1}^N |f(x_j) - f(x_{j-1})|; N \geq 1, x_0 < x_1 < \dots < x_N = x \right\}, \quad (54)$$

where  $f$  is defined in  $\boxed{H2}$  and assumed to be right-continuous (with no restriction). Given  $\varepsilon > 0$ , let  $n$  be the largest integer such that  $n\varepsilon \leq \text{TV}\{f\}$ , and consider the points

$$z_0 = -\infty, \quad z_n = +\infty, \quad z_j = \min\{x; F(x) \geq j\varepsilon\}, \quad j = 1 \dots n-1. \quad (55)$$

Defining

$$f_n(x) = f(z_j), \quad A_n(\rho, x) = A(\rho, z_j), \quad \text{for } x \in [z_j, z_{j+1}), \quad (56)$$

we guarantee that the piecewise constant flux  $A_n$  satisfies  $\boxed{H0}$  and  $\boxed{H1}$  with the same constants than  $A$ , and  $\boxed{H2}$  is satisfied as well since there exists  $i, j \in [0, N]$  such that

$$|A_n(\rho, x) - A_n(\rho, y)| = |A(\rho, z_i) - A(\rho, z_j)| \leq R(\rho)|f(z_i) - f(z_j)| = R(\rho)|f_n(x) - f_n(y)|, \quad (57)$$

and

$$\text{TV}\{f_n\} \leq \text{TV}\{f\}. \quad (58)$$

Finally, the estimate (52) is guaranteed since for  $x \in [z_j, z_{j+1})$ ,

$$|A(\rho, x) - A_n(\rho, x)| = |A(\rho, x) - A(\rho, z_j)| \leq R(\rho)|f(z_j) - f(x)| \leq R(\rho) \varepsilon(n). \quad (59)$$

**Second Step. Convergence of the  $(k_n^p)_n$ .** We first claim that the  $k_n^p$  are uniformly bounded in  $L^\infty(\mathbb{R})$ . Indeed, we have, using  $\boxed{H1}$

$$|k_n^p(x)| \leq \frac{1}{c}|A(k_n^p(x), x)| = \frac{|p|}{c}, \quad x \in \mathbb{R}. \quad (60)$$

Secondly, we claim that the  $k_n^p$  are uniformly bounded in  $BV(\mathbb{R})$ . Indeed, the functions  $k_n^p$  are piecewise constant and we have

$$\text{TV}\{k_n^p\} = \sum_{i=1}^n |k_n^p(z_i^+) - k_n^p(z_i^-)| \leq \frac{1}{c} \sum_{i=1}^n |A(k_n^p(z_i^+), z_i^+) - A(k_n^p(z_i^-), z_i^-)|. \quad (61)$$

Since  $A(k_n^p(z_i^+), z_i^+) = p = A(k_n^p(z_i^-), z_i^-)$ , we obtain

$$\begin{aligned} \text{TV}\{k_n^p\} &\leq \frac{1}{c} \sum_{i=1}^n |A(k_n^p(z_i^-), z_i^-) - A(k_n^p(z_i^-), z_i^+)| \leq \frac{1}{c} \sum_{i=1}^n R(k_n^p(z_i^-)) |f_n(z_i^-) - f_n(z_i^+)| \\ &\leq \frac{1}{c} \max_{\rho \in [0, |p|/c]} R(\rho) \text{TV}\{f\}. \end{aligned} \quad (62)$$

Helly theorem assures that there exists  $k^p \in L^1(\mathbb{R})$  such that  $\|k_n^p - k^p\|_{L^1(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$ . Let us now check that  $A(k^p(x), x) = p$ . We have

$$\begin{aligned} |A(k^p(x), x) - p| &= |A(k^p(x), x) - A_n(k_n^p(x), x)| \\ &\leq |A(k^p(x), x) - A_n(k^p(x), x)| + |A_n(k^p(x), x) - A_n(k_n^p(x), x)|. \end{aligned} \quad (63)$$

According to (52),

$$|A(k^p(x), x) - A_n(k^p(x), x)| \leq \max_{\rho \in [0, |p|/c]} R(\rho) \varepsilon(n), \quad (64)$$

and using (53)

$$|A_n(k^p(x), x) - A_n(k_n^p(x), x)| \leq C |k^p(x) - k_n^p(x)| \xrightarrow{n \rightarrow \infty} 0, \quad a.e., \quad (65)$$

which ends the proof of Lemma 5.  $\square$

**Proof of Theorem (1).** We approximate the flux  $x \rightarrow A(\rho, x)$  by  $A_n(\rho, \cdot)$  given by Lemma 5. According to Proposition 2, for all  $n \geq 0$ , there exists  $\rho_n \in L^\infty([0, T] \times \mathbb{R})$  uniformly bounded in  $BV$  and such that we have in the distributional sense

$$\partial_t \left| \rho_n(x, t) - k_n^p(x) \right| + \text{sign} \left( \rho_n(x, t) - k_n^p(x) \right) \left\{ A_n(\rho_n(x, t), x) - A_n(k_n^p(x), x) \right\} \leq 0. \quad (66)$$

Clearly, the  $\rho_n$  are uniformly Lipschitz in time with values in  $L^1(\mathbb{R})$ . Then, Proposition 2 combined with Theorem 2.6 of [10] gives us that  $\rho_n$  is uniformly bounded in  $BV([0, T] \times \mathbb{R})$  and thus there exists  $\rho \in L^1_{\text{loc}}([0, T] \times \mathbb{R})$  such that after extraction of a subsequence,  $\|\rho_n - \rho\|_{L^1([0, T] \times \mathbb{R})} \rightarrow 0$  as  $n \rightarrow \infty$ . We combine this with (53), and

$$\begin{aligned} |A_n(\rho_n(x, t), x) - A(\rho(x, t), x)| &\leq |A_n(\rho_n(x, t), x) - A_n(\rho(x, t), x)| + |A_n(\rho(x, t), x) - A(\rho(x, t), x)| \\ &\leq C |\rho_n(x, t) - \rho(x, t)| + \varepsilon(n) R(\rho(x, t)), \end{aligned} \quad (67)$$

so that  $A_n(\rho_n(x, t), x)$  converges strongly toward  $A(\rho(x, t), x)$ , after extracting a subsequence. We can now pass to the limit in the entropy inequality (66), and we get that  $\rho$  satisfies

$$\partial_t \left| \rho(x, t) - k^p(x) \right| + \text{sign} \left( \rho(x, t) - k^p(x) \right) \left\{ A(\rho(x, t), x) - A(k^p(x), x) \right\} \leq 0, \quad \mathcal{D}'(\mathbb{R}), \quad (68)$$

which means that  $\rho$  is the unique entropy solution of (1).  $\square$

## 4 Non-Existence in the general case : A counter example.

This section is devoted to the proof of Theorem 2. We notice (see proof of Lemma 4) that assumption  $\boxed{H1}$  and  $\boxed{H2}$  imply that

$$\forall \rho \in \text{BV}(\mathbb{R}), \text{TV}.A(\rho(\cdot), \cdot) < +\infty. \quad (69)$$

Based on a diagonal argument, we show that if (69) is violated, discontinuities in  $x$  may immediately generate a blow up: even in the initial condition has a finite total variation at time  $t = 0$ , the total variation can become infinite at time  $t = 0^+$ . We then exhibit a flux  $A$  satisfying  $\boxed{H0}$  and  $\boxed{H1}$ , and an initial datum  $\rho_0 \in \text{BV}(\mathbb{R})$  such as

$$\sum_{n=1}^{+\infty} |A(\rho_0(n^+), n^+) - A(\rho_0(n^-), n^-)| = +\infty. \quad (70)$$

Our proof is divided in two steps:

1. Step 1: Picking the initial datum  $\rho_0 \in \text{BV}(\mathbb{R})$  and a flux  $A$  satisfying  $\boxed{H0}$ ,  $\boxed{H1}$  such that the key point (69) is violated, the total variation of the solution  $\rho(\cdot, 0^+)$  is unbounded.
2. Step 2: There exists  $\rho_0 \in \text{BV}(\mathbb{R})$  and a flux  $A$  satisfying  $\boxed{H0}$ ,  $\boxed{H1}$  such that the key point (69) is violated.

### 4.1 Step 1. A necessary condition to prevent blow up at time $t = 0^+$ .

In this subsection, we focus on Step 1. We assume that the couple  $(A, \rho_0)$  satisfies (70). Without loss of assumption, we can have that  $\rho_0$  is continuous. We have

$$\begin{aligned} \text{TV}.\{\rho(\cdot, 0^+)\} &= \sum_{n=1}^{+\infty} \left| \rho(n^+, 0^+) - \rho(n^-, 0^+) \right| \geq \frac{1}{C} \sum_{n=1}^{+\infty} \left| A(\rho(n^+, 0^+), n^+) - A(\rho(n^-, 0^+), n^+) \right| \\ &= \frac{1}{C} \sum_{n=1}^{+\infty} \left| A(\rho_0(n), n^-) - A(\rho_0(n), n^+) \right|, \end{aligned} \quad (71)$$

since  $A(\rho(n^+, 0^+), n^+) = A(\rho(n^-, 0^+), n^-) = A(\rho_0(n), n^-)$  (See Figure 1 ). Then, the total variation of  $\rho(\cdot, 0^+)$  is unbounded.

### 4.2 Step 2. Building $A$ and $\rho_0$ .

We exhibit in this part a flux  $A$  such that  $\text{Disc}(A) \subset \mathbb{N}$ , where  $\mathbb{N}$  denotes here the set of strictly positive integers, and an initial condition  $\rho_0 \in \text{BV}([1, +\infty))$  such that (69) is violated. Let us first consider the bounded initial condition

$$\rho_0(x) = \frac{1}{x}, \quad x \in [1, +\infty], \quad (72)$$

which satisfies

$$\text{TV}.\{\rho_0\} = \int_1^{+\infty} \frac{dx}{x^2} < \infty. \quad (73)$$

We then consider the flux  $A(\rho, x)$  defined as, for all  $n$  even

$$A(\rho, x) = \begin{cases} a_n(\rho), & n \leq x < n+1, \\ \bar{a}_n(\rho), & n-1 \leq x < n. \end{cases} \quad (74)$$



where  $a_n$  and  $\bar{a}_n$  are defined as

$$a_n(\rho) = \begin{cases} 2\rho, & \rho \in \left(0, \frac{1}{2n}\right), \\ \frac{2}{3}\left(\rho + \frac{1}{n}\right), & \rho \in \left(\frac{1}{2n}, \infty\right), \end{cases} \quad \bar{a}_n(\rho) = \begin{cases} \rho, & \rho \in \left(0, \frac{2}{n}\right), \\ \frac{2}{3}\left(\rho + \frac{1}{n}\right), & \rho \in \left(\frac{2}{n}, \infty\right). \end{cases} \quad (75)$$

(see Figure 3, Left).

**1. The flux  $A$  satisfies the requirements H-1,2,3.** For all  $(\rho, x) \in \mathbb{R}^2$ ,  $\rho \rightarrow A(\rho, x)$  is strictly increasing and concave, and uniformly Lipschitz in the sense that

$$\frac{2}{3}|\rho - \tilde{\rho}| \leq |A(\rho, x) - A(\tilde{\rho}, x)| \leq 2|\rho - \tilde{\rho}|, \quad x \in [1, +\infty). \quad (76)$$

The function  $\rho \mapsto A(\rho, x)$  is piecewise linear for the sake of simplicity, but we can render it  $C^\infty$  by a smoothing process. The lack of regularity of  $\rho \mapsto A(\rho, x)$  is not a key point here. Let us now check that the spatial discontinuities of  $A$  are BV. We have

$$\text{TV}.A(\rho, \cdot) = \sum_{n=1}^{+\infty} \left| A(\rho, n^+) - A(\rho, n^-) \right|, \quad (77)$$

where

$$|A(\rho, n^+) - A(\rho, n^-)| = \begin{cases} |a_n(\rho) - \bar{a}_n(\rho)|, & n \text{ even}, \\ |\bar{a}_{n+1}(\rho) - a_{n-1}(\rho)|, & n \text{ odd}. \end{cases} \quad (78)$$

**Case 1 :**  $\rho > 2$ . We have  $|a_{n+1}(\rho) - \bar{a}_{n-1}(\rho)| = \frac{2}{3}\left(\rho + \frac{1}{n+1}\right) - \frac{2}{3}\left(\rho + \frac{1}{n-1}\right)$ , and  $|a_n(\rho) - \bar{a}_n(\rho)| = 0$ , then

$$\text{TV}.A(\rho, \cdot) = \sum_{\substack{n=3 \\ n \text{ odd}}}^{+\infty} \left| \frac{2}{3}\left(\rho + \frac{1}{n+1}\right) - \frac{2}{3}\left(\rho + \frac{1}{n-1}\right) \right| \leq \frac{4}{3} \sum_{n=3}^{+\infty} \frac{1}{(n-1)(n+1)} < \frac{2\pi^2}{9}. \quad (79)$$

**Case 2 :** There exists  $m \in \mathbb{N}$  such that  $\rho \in \left(\frac{2}{m+1}, \frac{2}{m}\right]$ .

$$\text{TV}.A(\rho, \cdot) = \sum_{n=1}^{\lfloor \frac{m}{4} \rfloor} \left| A(\rho, n^+) - A(\rho, n^-) \right| + \sum_{n=\lfloor \frac{m}{4} \rfloor + 1}^m \left| A(\rho, n^+) - A(\rho, n^-) \right| + \sum_{n=m+1}^{+\infty} \left| A(\rho, n^+) - A(\rho, n^-) \right| \quad (80)$$

We start by estimating the third term. For  $n \geq m+1$ , we have  $\frac{1}{2n} \leq \frac{2}{n} \leq \frac{2}{m+1} \leq \rho$ , and then  $(a_n(\rho) - \bar{a}_n(\rho)) = 0$ . The third term is then equal to

$$\begin{aligned} \sum_{n=m+1}^{+\infty} \left| A(\rho, n^+) - A(\rho, n^-) \right| &= \sum_{\substack{n=m+1 \\ n \text{ odd}}}^{+\infty} \left| A(\rho, n^+) - A(\rho, n^-) \right| = \sum_{\substack{n=m+1 \\ n \text{ odd}}}^{+\infty} \left| \frac{2}{3}\left(\rho + \frac{1}{n+1}\right) - \frac{2}{3}\left(\rho + \frac{1}{n-1}\right) \right| \\ &\leq \frac{4}{3} \sum_{n=2}^{+\infty} \frac{1}{(n-1)(n+1)} < \frac{2\pi^2}{9}. \end{aligned} \quad (81)$$

We deal with the first term the following way. For  $n \leq \frac{m}{4}$ , we have  $\rho \leq \frac{2}{m} \leq \frac{1}{2n}$  which implies  $|A(\rho, n^+) - A(\rho, n^-)| = |2\rho - \rho| = \rho$ . Thus

$$\sum_{n=1}^{\lfloor \frac{m}{4} \rfloor} |A(\rho, n^+) - A(\rho, n^-)| = \sum_{n=1}^{\lfloor \frac{m}{4} \rfloor} \rho \leq \frac{m}{4} \frac{2}{m} = \frac{1}{2}. \quad (82)$$

Let us finally deal with the second term. We pick  $n \in [\lfloor \frac{m}{4} \rfloor + 1, m]$ . We have

$$\begin{aligned} |a_n(\rho) - \bar{a}_n(\rho)| &\leq a_n(\rho) + \bar{a}_n(\rho) \leq 2\rho + \frac{2}{3n} + \rho + \frac{2}{3n} \\ |\bar{a}_{n+1}(\rho) - a_{n-1}(\rho)| &\leq \bar{a}_{n+1}(\rho) + a_{n-1}(\rho) \leq 2\rho + \frac{2}{3(n+1)} + \rho + \frac{2}{3(n-1)} \end{aligned} \quad (83)$$

We get then, after separating the even and odd  $n$

$$\sum_{n=\lfloor \frac{m}{4} \rfloor + 1}^m |A(\rho, n^+) - A(\rho, n^-)| = \sum_{\substack{n=\lfloor \frac{m}{4} \rfloor + 1 \\ n \text{ even}}}^m \left(3\rho + \frac{4}{3n}\right) + \sum_{\substack{n=\lfloor \frac{m}{4} \rfloor + 1 \\ n \text{ odd}}}^m \left(3\rho + \frac{2}{3(n+1)} + \frac{2}{3(n-1)}\right). \quad (84)$$

Let us focus on the term corresponding to  $n$  even. On the first hand, since  $\rho \leq \frac{2}{m}$  we have

$$\sum_{\substack{n=\lfloor \frac{m}{4} \rfloor + 1 \\ n \text{ even}}}^m 3\rho \leq 3\left(m - \frac{m}{4} - 1\right) \frac{2}{m} \leq \frac{9}{2}, \quad (85)$$

and on the other hand, for  $m > 4$

$$\sum_{\substack{n=\lfloor \frac{m}{4} \rfloor + 1 \\ n \text{ even}}}^m \frac{4}{3n} \leq \frac{4}{3} \int_{\lfloor \frac{m}{4} \rfloor}^m \frac{dx}{x} = \frac{4}{3} \left( \log(m) - \log(\lfloor \frac{m}{4} \rfloor) \right) \leq \frac{4}{3} \log\left(\frac{4m}{m-4}\right) \leq \frac{4}{3} \log(20). \quad (86)$$

The sum is trivially finite for  $m = 1, 2, 3$  or  $4$ . The terms corresponding to  $n$  odd are treated the same way. As a consequence, the second line of [H1](#) is satisfied.

**2. The flux violates assumption (69).** Using the expression (72) for the initial datum

$$\begin{aligned} \sum_{n=1}^{+\infty} |A(\rho_0(n^+), n^+) - A(\rho_0(n^-), n^-)| &= \sum_{n=1}^{+\infty} \left| A\left(\frac{1}{n}, n^+\right) - A\left(\frac{1}{n}, n^-\right) \right| \\ &= \sum_{\substack{n=1 \\ n \text{ even}}}^{+\infty} \left| a_n\left(\frac{1}{n}\right) - \bar{a}_n\left(\frac{1}{n}\right) \right| + \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \left| \bar{a}_{n+1}\left(\frac{1}{n}\right) - a_{n-1}\left(\frac{1}{n}\right) \right|. \end{aligned} \quad (87)$$

Using now expression (75), we find that the terms corresponding to the even integers are giving the sum

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{+\infty} \left| \frac{2}{3} \left( \frac{1}{n} + \frac{1}{n} \right) - \frac{1}{n} \right| = \frac{1}{3} \sum_{\substack{n=1 \\ n \text{ even}}}^{+\infty} \frac{1}{n} = +\infty, \quad (88)$$

and this ends the proof of Theorem (69).

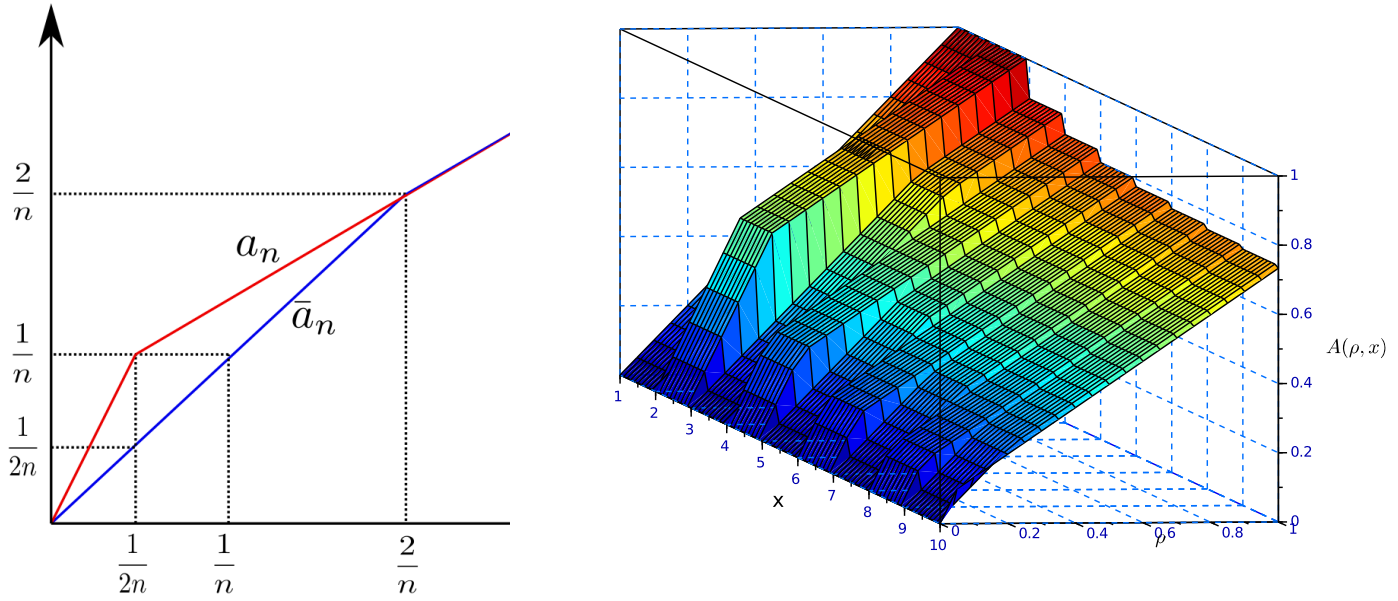


Figure 3: Left: The flux  $A(., x)$  depends on  $x$ . For  $x \in [n - 1, n)$  with  $n$  odd, for  $\rho \in (0, \frac{2}{n})$ ,  $A(\rho, x) = \bar{a}_n(\rho) = \rho$ , for  $\rho \in (\frac{2}{n}, +\infty)$ ,  $A(\rho, x) = \bar{a}_n(\rho) = \frac{2}{3}(\rho + \frac{1}{n})$ . For  $x \in [n, n + 1)$  with  $n$  even, for  $\rho \in (0, \frac{1}{2n})$ ,  $A(\rho, x) = a_n(\rho) = 2\rho$ , and for  $\rho \in (\frac{1}{2n}, +\infty)$ ,  $A(\rho, x) = a_n(\rho) = \frac{2}{3}(\rho + \frac{1}{n})$ . Right: Tri-dimensional visualization of the flux function  $A$  detailed in the counter-example.

**Remark 2.** We can assume the discontinuity points lay on a compact set, for example  $\text{Disc}(A) \subset \{\arctan(n), n \in \mathbb{N}\}$ . In this counter example, there is no  $R$  continuous such that  $\boxed{H2}$  is satisfied (Indeed  $R$  would blow up at  $\rho = 0$ ). We point out that  $\rho = 0$  does not play a specific role here, and that we can shift the flux so that the blow up for  $R$  occurs at any  $\rho > 0$ .

## 5 Conclusion

We focus on a spatially heterogeneous conservation law, and define a BV framework in which an adapted entropy solution defined by [9] exists for every Cauchy problem with BV initial datum. We also provide new BV estimates on the entropy solution. To prove our result, we introduces a key assumption on the flux, which is optimal since as soon as it is violated, we can exhibit counter-examples where, starting from a BV initial condition, the total variation of the solution blows up instantaneously. If the variables of the flux are separated, or are a combination of separated variables, which is the focus of most authors, the key condition  $\boxed{H2}$  is satisfied under conditions  $\boxed{H0}$  and  $\boxed{H1}$ .

## Appendix: Proof of Lemma 1.

*Proof.* Let us first introduce some notations. For all  $\rho > 0$ , we consider the following well-defined limits

$$A(\rho, x^-) = \lim_{\substack{y \rightarrow x \\ y < x}} A(\rho, y), \quad A(\rho, x^+) = \lim_{\substack{y \rightarrow x \\ y > x}} A(\rho, y), \quad (89)$$

and denote the jump of  $A$  at  $(\rho, x)$  by

$$J_\rho(x) = |A(\rho, x^+) - A(\rho, x^-)|. \quad (90)$$

We then write

$$\text{Disc}(A) = \bigcup_{N \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} I_{k,N}, \quad (91)$$

where

$$I_{k,N} = \{x \in \mathbb{R} \mid \exists \rho_x \in [0, N] \text{ such that } J_{\rho_x}(x) > \frac{1}{k}\} \quad (92)$$

We claim that for all  $N \geq 0, k \geq 0, I_{k,N}$  is finite, which makes  $\text{Disc}(A)$  at most countable. Let us assume by contradiction that there exists  $(k, N) \in \mathbb{N}^2$  such that  $I_{k,N}$  is infinite. Then

$$\exists \epsilon > 0 \mid \forall x \in I_{k,N}, |\rho - \rho_x| < \epsilon \implies J_\rho(x) > \frac{1}{2k}. \quad (93)$$

Indeed,  $\forall x \in I_{k,N}, \forall \rho \in \mathbb{R}^+$ ,

$$\begin{aligned} \frac{1}{k} < J_{\rho_x}(x) &\leq |A(\rho, x^+) - A(\rho_x, x^+)| + J_\rho(x) + |A(\rho, x^-) - A(\rho_x, x^-)| \\ &\leq 2C|\rho - \rho_x| + J_\rho(x) \end{aligned} \quad (94)$$

and (93) holds for  $\epsilon = \frac{1}{4Ck}$ . For all  $x \in I_{k,N}$ , we denote by

$$U_k(x) = [\rho_x - \epsilon, \rho_x + \epsilon]. \quad (95)$$

The compact set  $[0, N + \varepsilon]$  is then covered by an infinite number of sets  $U_k(x)$ , each of them being of size at least  $2\varepsilon$ . Thus, there exists  $\rho \in [0, N + \varepsilon]$  such that  $\rho$  belongs to an infinite number of  $U_k(x)$ . Indeed, by contradiction, if there is no such  $\rho$ , then, in particular,  $\varepsilon$  only belongs to a finite number of  $U_k(x)$ , which implies that  $[\varepsilon, N + \varepsilon]$  is covered by an infinite number of  $U_k(x)$ . In the same way,  $2\varepsilon$  only belongs to a finite number of  $U_k(x)$ , which implies that  $[2\varepsilon, N + \varepsilon]$  is covered by an infinite number of  $U_k(x)$ . By recursion,  $[N, N + \varepsilon]$  is covered by an infinite number of  $U_k(x)$ , and thus, any  $\rho \in [N, N + \varepsilon]$  belongs to an infinite number of  $U_k(x)$ , which contradicts the assumption. Thus, for that specific  $\rho$ ,

$$\text{TV}.A(\rho, \cdot) = \sum_{x \in I_{k,N}} |A(\rho, x^+) - A(\rho, x^-)| > \sum_{x \in I_{k,N}} \frac{1}{2k} = +\infty, \quad (96)$$

which contradicts  $\boxed{H1}$ . □

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