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# CHAOTIC ATTRACTORS WITH THE SYMMETRY OF A TETRAHEDRON

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**Abstract**—Functions that are equivariant with respect to the symmetries of a tetrahedron are determined. Linear combinations of these functions that give rise to chaotic attractors are used to create images in 3-space of attractors with the symmetry of the tetrahedron. These attractors are visually appealing because of the tension between the pattern forced by their symmetry and the randomness arising from their chaotic behavior.

## 1. INTRODUCTION

Chaotic attractors arising from the iterative solution to systems of differential equations have been the subject of study since Lorenz noticed strange behavior in a model related to weather prediction [1]. Even the iteration of simple functions such as the logistic function can be used to illustrate dynamics including chaotic behavior; see, for example Ref. [2]. More recently, attractors arising from the iteration of functions in the plane have been studied and equivariant functions have been used to create attractors that appear chaotic while having rotational and/or reflectional symmetries [3–5]. Examples of attractors with the symmetry of the cube in 3-space and  $n$ -cube in  $n$ -space have also been created [6][7]; in addition, examples of attractors in 3-space with dueling planar symmetries have been studied [8]. In this paper we create attractors in 3-space which empirically appear to be chaotic but which have the symmetry of the tetrahedron.

Discussions of the symmetry groups of various geometric shapes can be found in a variety of places. In particular, Coxeter [9] describes the symmetry groups of the cube and tetrahedron. There are twelve rotations that preserve the four vertices of a regular tetrahedron. These include two third-turns that fix any one vertex of the tetrahedron yielding a total of eight third-turns. See Fig. 1. Also, there are three half-turns about the line connecting midpoints of opposing edges that switch two pairs of vertices. Lastly, the identity function also preserves the tetrahedron giving the twelve desired rotations. The results of composing these rotations can readily be observed geometrically. If reflections are also allowed, then the size of the symmetry group is twice as much: 24. While Fig. 1 shows the tetrahedron oriented with a vertex at a peak, it will be more convenient for us to consider the tetrahedron embedded in the cube by selecting alternate vertices of the cube. See Fig. 2. This representation makes it

clear that the rotational symmetry group of the tetrahedron is a subgroup of the rotational symmetry group of the cube. Notice the rotational symmetry group of the cube has quarter-turns (and some half-turns) that do not preserve the tetrahedron.

The next section develops the mathematics we need in order to construct our chaotic attractors with the symmetry of the tetrahedron. In particular, functions equivariant with respect to those rotations are determined. The last section gives illustrations of such attractors selected for visual appeal and diversity and describes our computations.

## 2. FUNCTIONS WITH THE SYMMETRY OF A TETRAHEDRON

Let  $T$  denote the tetrahedron with vertices  $(1,1,1)$ ,  $(1,-1,-1)$ ,  $(-1,1,-1)$  and  $(-1,-1,1)$ . One can check by direct computation that all twelve of the rotational symmetries of the tetrahedron are generated by composition from the two rotations  $\sigma\langle x,y,z\rangle\langle -x,-y,z\rangle$  and  $\tau\langle x,y,z\rangle\langle z,x,y\rangle$ . Notice that  $\tau$  is a third-turn which fixes the first vertex of  $T$ . The convenience of our choice of  $T$  is given by the fact that we can describe this third turn on  $T$  without adding coordinates or introducing a factor of  $\sqrt{3}$  into the description of the rotations. The rotation  $\sigma$  is a half-turn that interchanges two pairs of vertices of  $T$ . Moreover, all the reflectional and rotational symmetries of the tetrahedron are generated by  $\sigma$  and  $\tau$ , along with the reflection  $\nu$  where  $\nu\langle x,y,z\rangle\langle y,x,z\rangle$  as can be verified by checking these generate the required 24 functions.

In order to generate attractors with the symmetry of the tetrahedron, we need functions from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  that preserve those rotations and reflections in a certain manner. In particular, a function  $f$  is said to be *equivariant* with respect to the group of symmetries of  $T$  if for all  $\alpha$  in that group of symmetries and  $X \in \mathbb{R}^3$ ,  $f(\alpha(X)) = \alpha(f(X))$ . If  $\alpha$  is a rotation and  $f$  is  $\alpha$ -equivariant then the iterates of the rotation of a

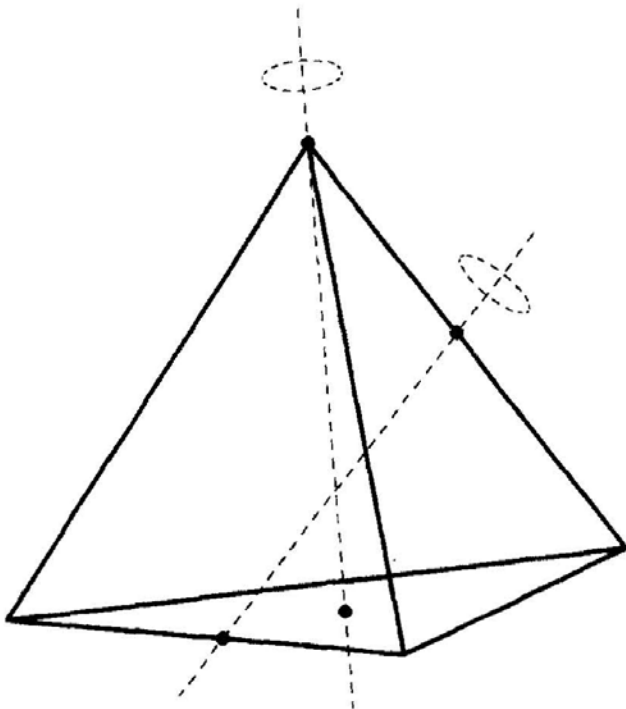


Fig. 1. A tetrahedron and axes of rotation for a third-turn and a half-turn.

point are the same as the rotation of the iterates of the point. This means that the attractor associated with  $f$  tends to have the desired symmetries; however, it is possible that the attractor has only the symmetry given by an admissible subgroup of symmetries [10, 11] or it may have the desired symmetry only in a trivial manner. Thus, even after identifying the appropriate equivariant functions one still needs to do some work to find examples that highlight the chaos and symmetry.

It is straightforward to check that a function that is equivariant with respect to the generators of a group will be equivariant with respect to every element of the group. With that observation, we are ready to determine the polynomial functions equivariant with respect to the symmetries of  $T$ .

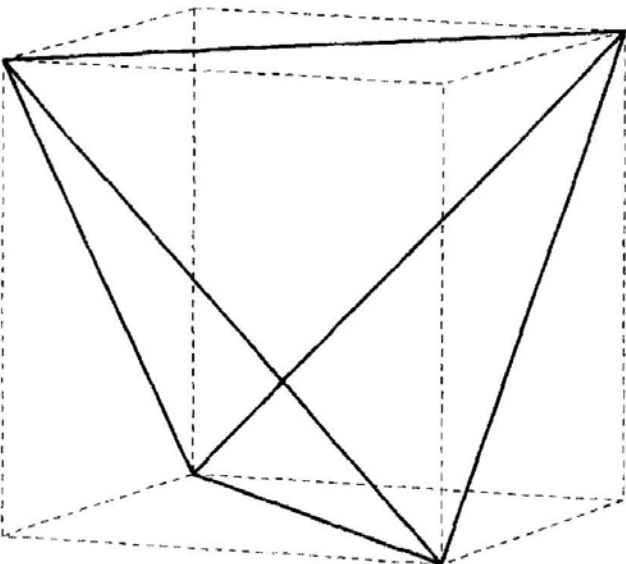


Fig. 2. A regular tetrahedron embedded in a cube.

**Theorem.** (i) The polynomial functions that are equivariant with respect to the rotational symmetries of the tetrahedron  $T$  are linear combinations of functions of the form

$$T_{ijk}(x, y, z) = \langle x^i y^j z^k, x^k y^i z^j, x^j y^k z^i \rangle$$

where  $j$  and  $k$  have the same parity (even or odd) which is different from the parity of  $i$ .

(ii) The polynomial functions that are equivariant with respect to the rotational and reflectional symmetries of the tetrahedron  $T$  are linear combinations of functions of the form

$$T_{ijk}(x, y, z) + T_{ikj}(x, y, z)$$

where  $j$  and  $k$  have the same parity which is different from the parity of  $i$ .

**Proof.** A general polynomial function,  $P$ , from  $R^3$  to  $R^3$  has the form

$$P(x, y, z) = \left\langle \sum_{0 \leq i, j, k \leq N} a_{ijk} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} b_{ijk} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} c_{ijk} x^i y^j z^k \right\rangle$$

where  $N$  is the maximal degree of any coordinate that appears.  $P$  is equivariant with respect to the rotational symmetries of the tetrahedron if it is equivariant with respect to  $\sigma$  and  $\tau$ . In particular, we must have  $\tau(P(x, y, z)) = P(\tau(x, y, z))$ . Now

$$\tau(P(x, y, z)) = \left\langle \sum_{0 \leq i, j, k \leq N} c_{ijk} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} a_{ijk} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} b_{ijk} x^i y^j z^k \right\rangle$$

while

$$\begin{aligned} P(\tau(x, y, z)) &= \left\langle \sum_{0 \leq i, j, k \leq N} a_{ijk} z^i x^j y^k, \sum_{0 \leq i, j, k \leq N} b_{ijk} z^i x^j y^k, \sum_{0 \leq i, j, k \leq N} c_{ijk} z^i x^j y^k \right\rangle \\ &= \left\langle \sum_{0 \leq i, j, k \leq N} a_{kij} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} b_{kij} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} c_{kij} x^i y^j z^k \right\rangle \end{aligned}$$

where the second equality holds by renaming the indices. Equating like coefficients yields

$$c_{ijk} = a_{kij}, \quad a_{ijk} = b_{kij}, \quad b_{ijk} = c_{kij}$$

Thus any nonzero term in any coordinate will be associated with a term with the same coefficient in

Table 1. Details of J code

<pre>T=:2: 'm.&amp;* @ ((*/@: ^^1)&amp;((-i.3) ."0_n.) )'</pre>			NB. define T
<pre>f=:3 T 3 2 4+5 T 1 0 2</pre>			NB. define f as above
<pre>f':(i.4) 0.5 0.4 0.5</pre>			NB. compute some iterates
0.5	0.4	0.5	
0.62875	0.503	0.4024	
0.514	1.00391	0.514	
0.70765	1.38212	2.69946	

each coordinate (but the exponents will be rearranged). Thus, each polynomial function equivariant with respect to  $\tau$  is a linear combination of functions of the form

$$T_{ijk}\langle x, y, z \rangle = \langle x^i y^j z^k, x^k y^i z^j, x^j y^k z^i \rangle$$

Now  $P$  being equivariant with respect to  $\sigma$  means

$$\sigma(P\langle x, y, z \rangle) = \left\langle - \sum_{0 \leq i, j, k \leq N} a_{ijk} x^i y^j z^k, - \sum_{0 \leq i, j, k \leq N} b_{ijk} x^i y^j z^k, \sum_{0 \leq i, j, k \leq N} c_{ijk} x^i y^j z^k \right\rangle$$

is the same as

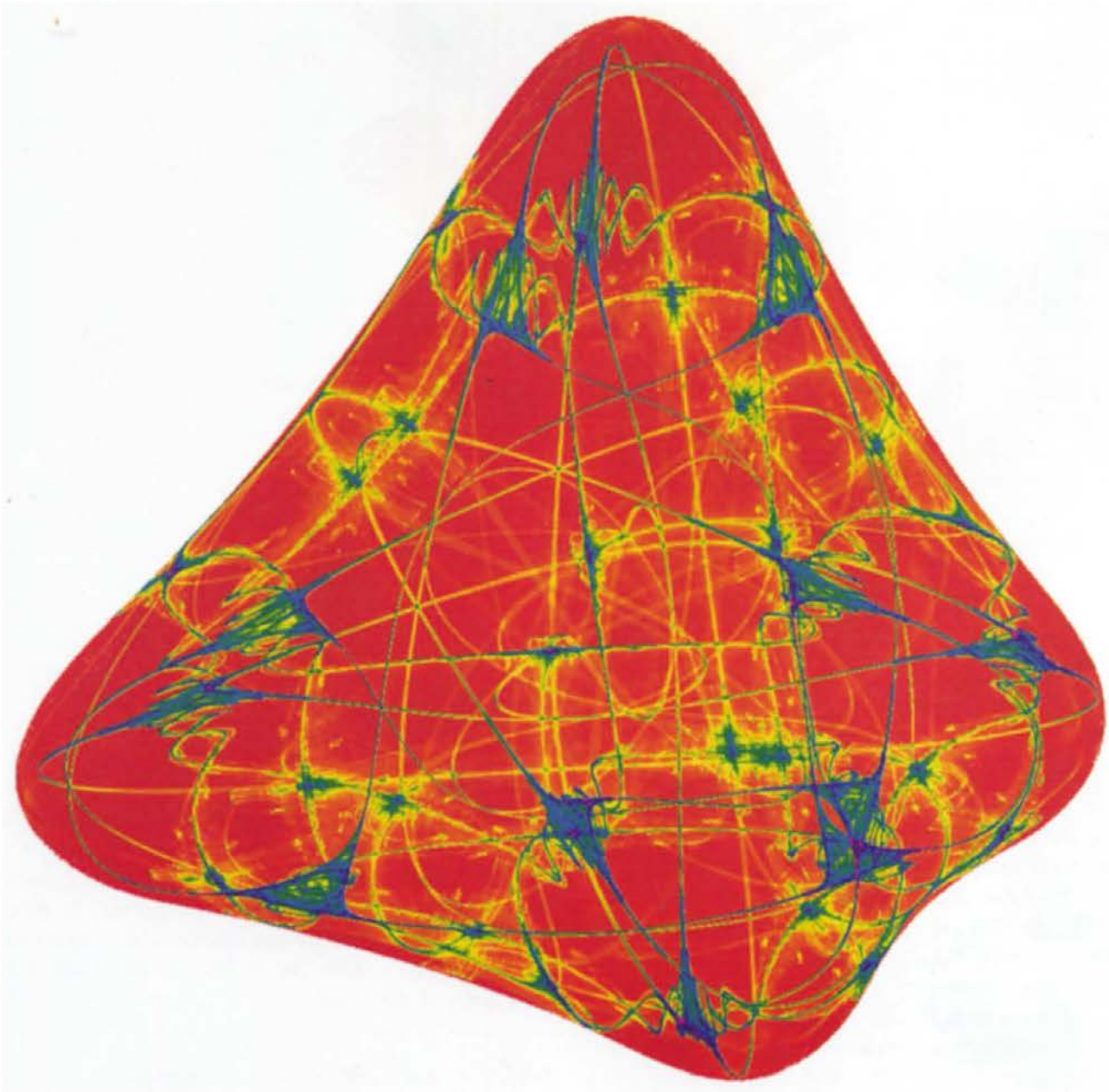


Fig. 3. An attractor with tetrahedral shape and symmetry.



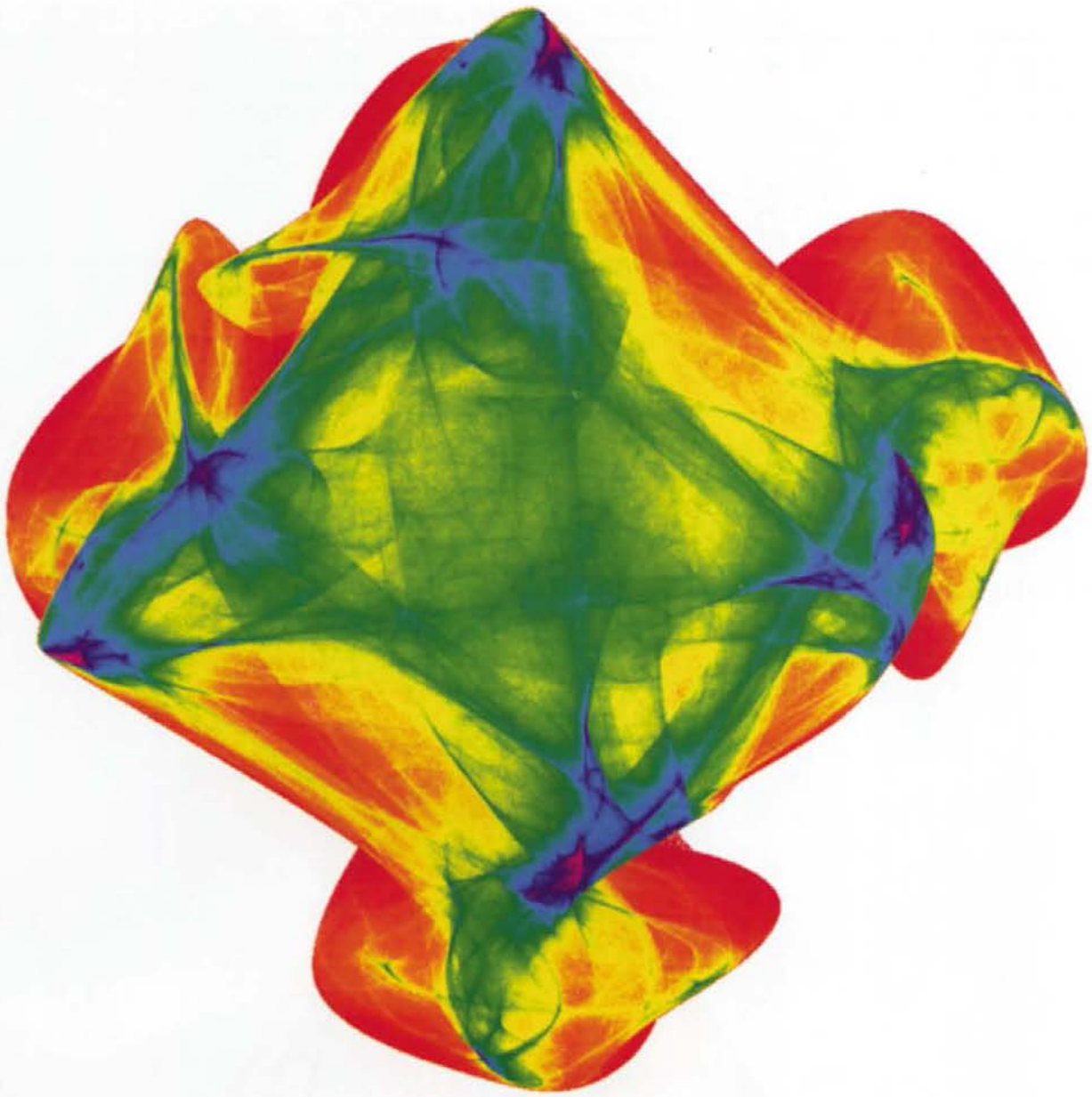


Fig. 4. An attractor with tetrahedral symmetry and opposing ears.

$$P(\sigma\langle x, y, z \rangle) = \left\langle \sum_{0 \leq i, j, k \leq N} (-1)^{i+j} a_{ijk} x^i y^j z^k, \right. \\ \left. \sum_{0 \leq i, j, k \leq N} (-1)^{i+j} b_{ijk} x^i y^j z^k, \right. \\ \left. \sum_{0 \leq i, j, k \leq N} (-1)^{i+j} c_{ijk} x^i y^j z^k \right\rangle$$

which implies that a nonzero term  $a_{ijk} x^i y^j z^k$  can occur only when the parity of  $i$  and  $j$  are different. Moreover, such a nonzero term corresponds to  $c_{jki} x^j y^k z^i$  in the third coordinate and  $\sigma$ -equivariance implies  $j$  and  $k$  have the same parity. Thus, the only way for a function to be  $\tau$  and  $\sigma$ -equivariant is if it is a linear combination of  $T_{ijk}$ 's where  $j$  and  $k$  have the same parity which is different from the parity of  $i$ .

Now consider the condition implied by the third coordinate of the equality:  $v(P\langle x, y, z \rangle) = P\langle v(x, y, z) \rangle$ . We see that

$$\sum_{0 \leq i, j, k \leq N} c_{ijk} x^i y^j z^k = \sum_{0 \leq i, j, k \leq N} c_{ijk} y^i x^j z^k \\ = \sum_{0 \leq i, j, k \leq N} c_{jik} x^i y^j z^k$$

from which we see

$$c_{ijk} = c_{jik} \text{ and hence } c_{jki} = c_{kji}$$

Now this means that any time there is a nonzero term in the third coordinate there are two such terms (just one if  $j=k$ ) and if we have equivariance with respect to the rotations then there are two corresponding terms in each other coordinate. In particular, we see that a function equivariant with respect to  $\sigma$ ,  $\tau$ , and  $v$  must be a linear combination of functions of the form

$$T_{ijk}\langle x, y, z \rangle + T_{ikj}\langle x, y, z \rangle$$

Moreover, we can directly check that such func-

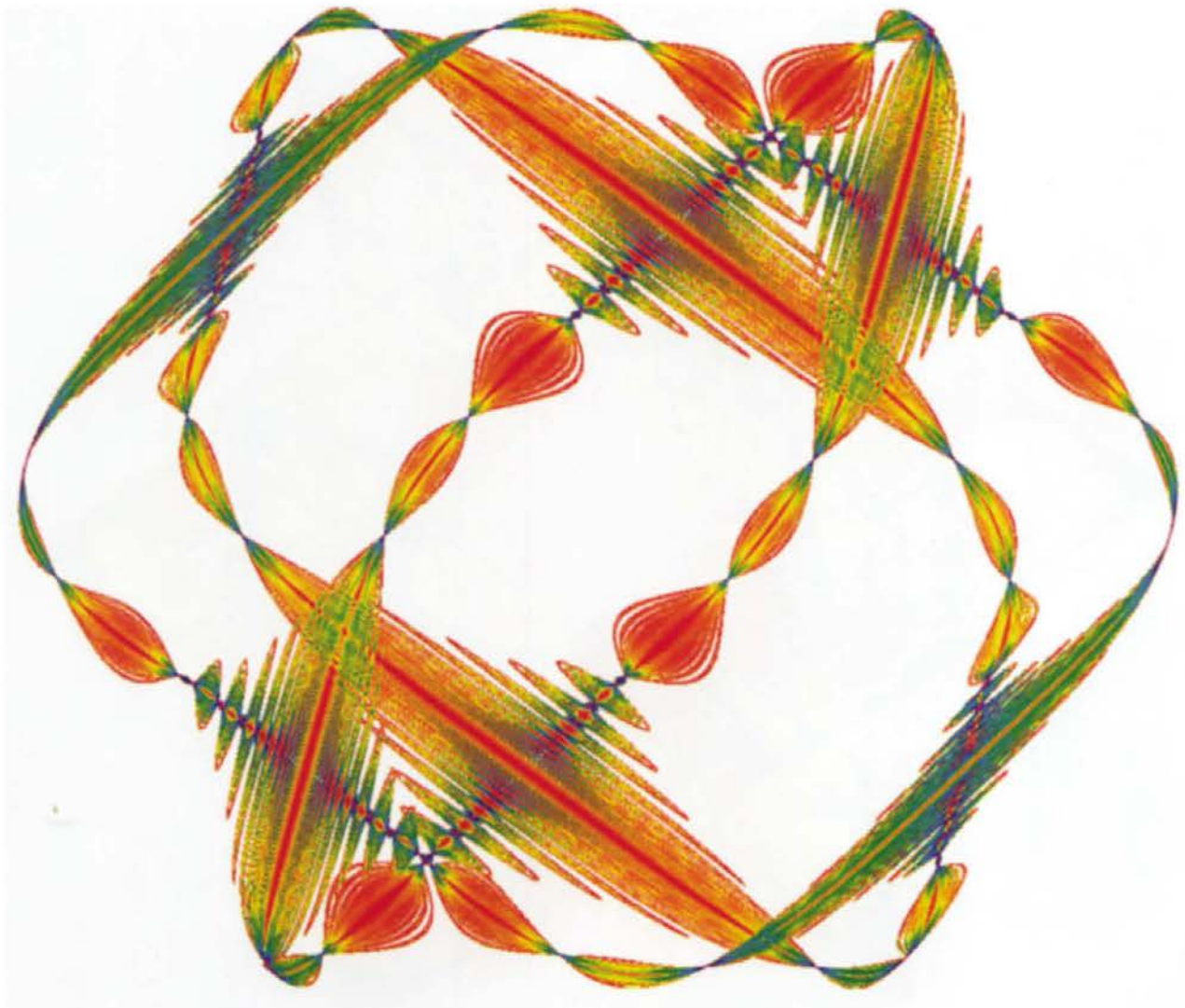


Fig. 5. A cube wrapping attractor that has only the symmetries of the tetrahedron.

tions are equivariant with respect to  $\sigma$ ,  $\tau$ , and  $\nu$  giving the desired result.  $\square$

For example, we see the function

$$\begin{aligned} f\langle x, y, z \rangle &= 3T_{324}\langle x, y, z \rangle + 5T_{102}\langle x, y, z \rangle \\ &= \langle 3x^3y^2z^4 + 5xz^2, 3x^4y^3z^2 + 5x^2y, \\ &\quad 3x^2y^4z^3 + 5y^2z \rangle \end{aligned}$$

is equivariant with respect to the rotational symmetries of  $T$ . The function

$$\begin{aligned} f\langle x, y, z \rangle &= 5T_{324}\langle x, y, z \rangle + 5T_{342}\langle x, y, z \rangle \\ &= \langle 5x^3y^2z^4 + 5x^3y^4z^2, 5x^4y^3z^2 + 5x^2y^3z^4, \\ &\quad 5x^2y^4z^3 + 5x^4y^2z^3 \rangle \end{aligned}$$

is equivariant with respect to the rotational and reflectional symmetries of  $T$ . Also, anytime  $i$  and  $j$  have different parities and we take  $j=k$ , it follows that

$$T_{ijj}\langle x, y, z \rangle = \frac{1}{2}T_{ijj}\langle x, y, z \rangle + \frac{1}{2}T_{ijj}\langle x, y, z \rangle$$

is equivariant with respect to the rotational and reflectional symmetries of  $T$ .

### 3. ILLUSTRATIONS AND COMPUTATIONS

We noted above that linear combinations of the functions  $T_{ijk}$  for the appropriate choice of indices gives functions that are equivariant with respect to the symmetry group of  $T$ . We created linear combinations of those functions in the programming language J which is available from <http://www.jsoftware.com/>. The indices and coefficients were selected at random and the resulting function was tested to determine whether it created a nontrivial attractor (however, we always include the indices 1 0 0 since we want the fixed point at 0 0 0 to be repelling). If the attractor was nontrivial, then an image was created and observed. Images that appeared promising were mutated (parameters varied along a search direction [8]) so that variations on the promising attractors could be observed. Finally, higher quality images were created for several functions using parallel computations of the attractors [12]. While the details of J code often cannot be read by the uninitiated, we offer, in Table 1, three lines of the details so that readers interested in replicating our work have



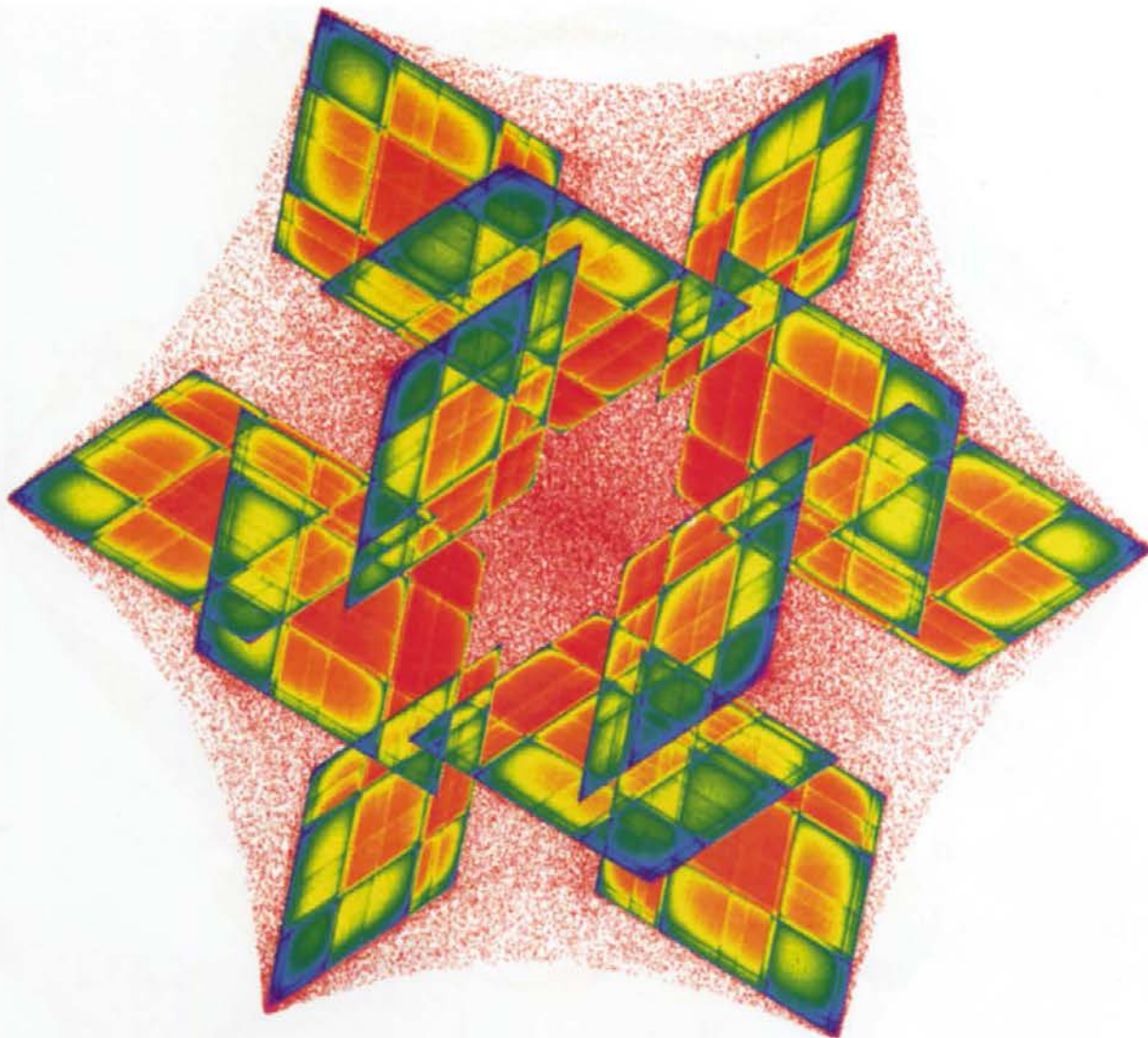


Fig. 6. An attractor with the symmetries of the tetrahedron and platelike components.

sufficient information to implement our functions in J for comparison or experimentation. The first line constructs the function builder (conjunction)  $T$ ; the second line creates a sample function  $f$  from the previous section and the third line shows how to create some iterates of that function.

Figure 3 shows the result of 200 million iterations of the function defined by

---


$$f =: \quad \_1.66332962 \ T \ 1 \ 0 \ 0 + 0.419143344 \ T \ 2 \ 5 \ 5 \\ + 0.542221824 \ T \ 3 \ 0 \ 0 + 1.00 \ T \ 0 \ 1 \ 1$$


---

Notice that the underlying shape of a tetrahedron is apparent, there are triple crossovers near the vertices and there are some hot spots nearby that are connected to each other in a complicated fashion. There are also hot spots near the edges that are interesting and this attractor also has reflectional symmetry. The color scheme shows pixels that are visited a small number of times in red; as the frequency of visits increases the colors run through the hues of the spectrum with magenta corresponding to the highest frequencies. Figure 4 also shows an attractor where a tetrahedron is also apparent. However, in this case notice that opposite each vertex there is a swirling ear/neck which lack

reflectional symmetry and the attractor seems to fill space much more than the attractor in Fig. 3. The function used for Fig. 4 is given by the following.

---


$$f =: \quad \_1.972379872 \ T \ 1 \ 0 \ 0 + \_1.788455548 \ T \ 0 \ 3 \ 1 \\ + 0.802610368 \ T \ 0 \ 1 \ 1 + 1.847505316 \ T \ 3 \ 0 \ 0$$


---

Figure 5 shows the attractor produced by the function

---


$$f =: \quad 1.672451176 \ T \ 1 \ 0 \ 0 + \_1.550560932 \ T \ 1 \ 4 \ 4 \\ + \_0.340028192 \ T \ 1 \ 4 \ 0 + \_0.416261108 \ T \ 3 \ 0 \ 0$$


---

which appears to roughly form three girdles about a cube. However, notice that the rotations around the central axes perpendicular to any one of the girdles require a half-turn to preserve the attractor. Quarter-turns cause the oscillations to be misoriented. The point being that although there is a cubical underlying form, the symmetries of the attractor are the symmetries of the tetrahedron and not the symmetries of the cube.

Figure 6 shows the attractor produced by the function

---


$$f =: \quad \_1.801862888 \ T \ 1 \ 0 \ 0 + 0.090473596 \ T \ 5 \ 0 \ 0 \\ + 0.589775748 \ T \ 3 \ 2 \ 2 + 0.36102616 \ T \ 1 \ 2 \ 2$$


---



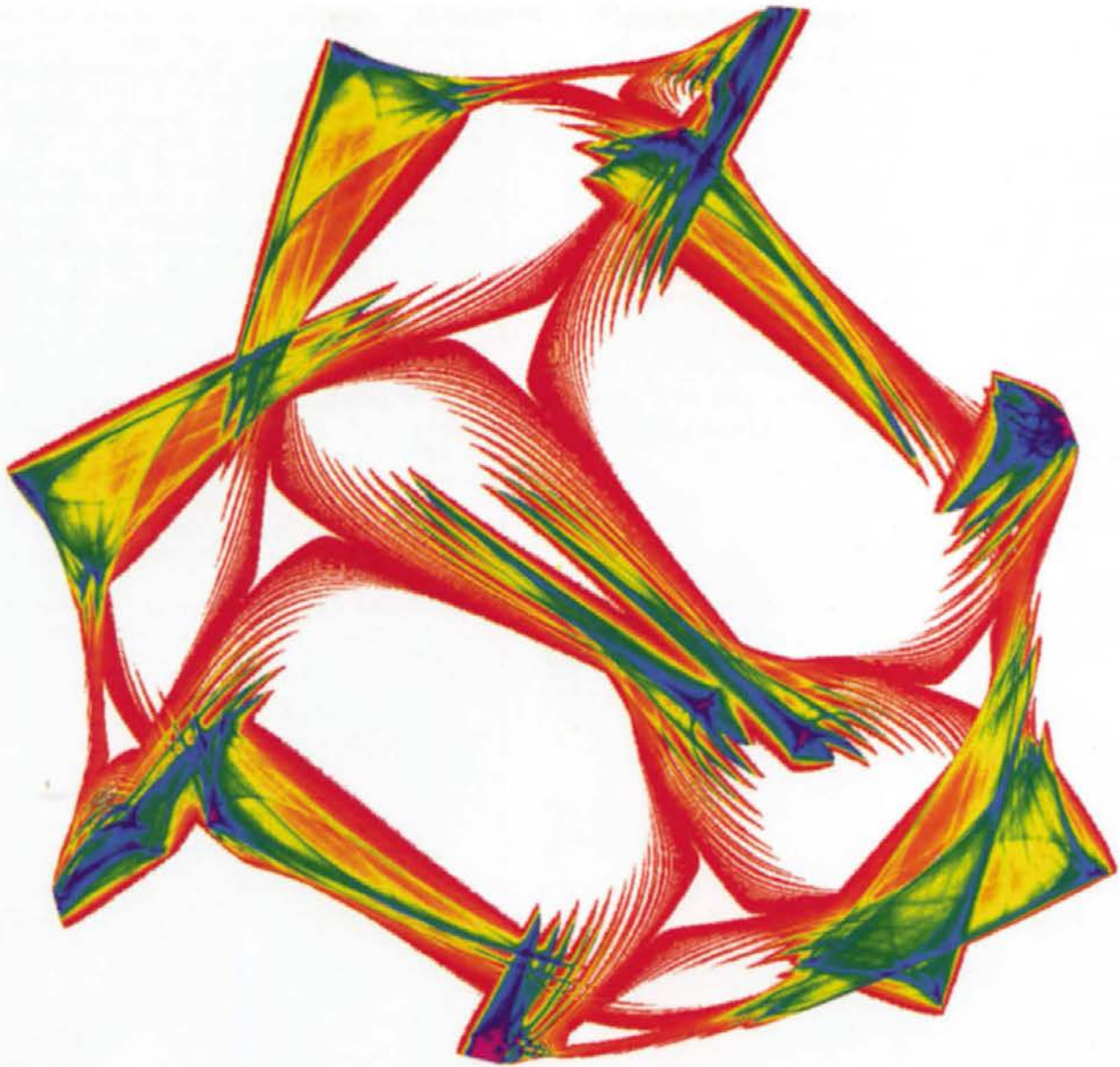


Fig. 7. An attractor with the symmetries of the tetrahedron and wings.

This figure is qualitatively quite different from the others since it seems to be constructed as a composite of many plates. This attractor also has reflectional symmetry. Figure 7 shows the attractor produced by the function

$$f =: \_1.131395052 \text{ T } 1 \ 0 \ 0 + 0.643411644 \text{ T } 2 \ 3 \ 1 \\ + 1.069549392 \text{ T } 0 \ 1 \ 5 + 1.166645204 \text{ T } 5 \ 2 \ 0$$

Here the vertices of the tetrahedron lie along an arc connecting two points and the connections are reminiscent of wings.

While it is difficult to prove results about the chaotic behavior of attractors, we can investigate this empirically via Ljapunov exponents [2, 13]. Table 2 shows the Ljapunov exponents for the attractors shown in our images. Each of these has at least one positive Ljapunov exponent: this is associated with chaotic behavior. It is interesting to note that Figs 4 and 6 have two positive Ljapunov exponents and these appear to be the heaviest attractors. Of course, these computations are not definitive. For example, the function

$$f =: \_1.66332962 \text{ T } 1 \ 0 \ 0 + \\ 0.419143344 \text{ T } 2 \ 5 \ 5 \\ + 0.542221824 \text{ T } 3 \ 0 \ 0 \\ + 0.997242248 \text{ T } 0 \ 1 \ 1$$

is closely related to the function used to produce Fig. 3; however, our experience with this function shows the iterates of points converging to a 126-cycle after tens of thousands of iterations. One cannot exclude the possibility that our examples do not likewise become trivial after some very large number of iterations.

Table 2. Ljapunov exponents for the symmetric attractors

Fig.	$\lambda_i$		
	1	2	3
3	0.111	-0.030	-0.271
4	0.447	0.256	-0.072
5	0.228	-0.297	-1.315
6	0.556	0.555	-0.003
7	0.238	0.114	-0.227



We have seen that by choosing a careful representation of a regular tetrahedron we could describe generators for the rotational and reflectional symmetries in terms of sign changes and coordinate rearrangements. This allowed us to describe the functions equivariant with respect these symmetries as linear combinations of some simple generating functions. Random linear combinations of these functions could then be investigated yielding examples of chaotic attractors with the desired symmetries. Often the tetrahedral symmetry leads to attractors in the form of a tetrahedron but the attractor can be in the shape of a cube with some oriented features that destroy the cubical symmetry and even more bizarre shapes can appear.

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