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Maurice A. Biot

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Mechanics of Incremental Deformations

Theory of Elasticity and Viscoelasticity
of Initially Stressed Solids and Fluids,
Including Thermodynamic Foundations and
Applications to Finite Strain

MAURICE A. BIOT

New York, New York

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Theory of Elasticity and Viscoelasticity of Initially Stressed Solids and Fluids, Including Thermodynamics Foundation and Applications to Finite Strain

By Maurice A. Biot (1905-1985)



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PREFACE

This book embodies an approach to non-linear elasticity which marks a fundamental departure from classical and current trends. The basic theory was first published between the years 1934 and 1940 in seven papers listed at the end of this Preface. In addition to a systematic treatment of the general theory and extensions to viscoelasticity, the book includes comprehensive new developments and applications, many of which are presented here for the first time.

The work is characterized by the use of cartesian concepts and of elementary mathematical methods that do not require a knowledge of the tensor calculus or other more specialized techniques. The explicit introduction of a local rotation field in the three-dimensional equations leads to a theory which separates the physics from the geometry and is equally valid for elastic and non-elastic materials, using either rectangular or curvilinear coordinates.

As this book demonstrates, the scope of problems solved by these new methods goes far beyond the results which it has been possible to obtain by the more elaborate and less general traditional approach. New insights, leading to many discoveries and a unified outlook have been brought into such widely diversified areas as rubber elasticity, internal gravity waves in a fluid and tectonic folding in geodynamics.

The theory provides rigorous and completely general equations governing the dynamics and stability of solids and fluids under initial stress in the context of small perturbations. It does not require that the medium be elastic or isotropic but is applicable to anisotropic, viscoelastic, or plastic media. No assumptions are introduced regarding the physical process by which the initial stress has been generated. The treatment of viscoelasticity, which constitutes a substantial portion of the book, incorporates some of the results established in my previous work on non-equilibrium thermodynamics.

Non-linear theories of deformation and applications to problems of finite strain are obtained by extension of the concept of incremental deformation in a medium under initial stress. In contrast to the

presentation in the papers listed at the end of this Preface, the concepts and methods are developed primarily in the context of the linearized mechanics of continuous media under initial stress as an independent theory.

In its earlier phase this work was interrupted by the Second World War. My interest in the subject was revived some fifteen years ago in connection with geological problems. Because the theory is valid for non-elastic media, it was found applicable to problems in geodynamics where it has opened a new phase and provided new and fruitful methods of analysis. Although the basic theory has been available in the scientific literature for more than twenty-five years and has been used occasionally by a few investigators in technological and geophysical problems, its potentialities seem to have been generally overlooked. This is perhaps due to a prevalent emphasis on tensor formalism. For many years it has been my feeling that, between the formalistic approach of the mathematician and the more pragmatic treatment of problems by the engineer, there is a need for a rigorous but intermediate theory based on cartesian concepts. It would extend to three-dimensional deformations the viewpoints and methods of what has come to be known as "Strength of Materials."

Classical approaches to non-linear elasticity have been handicapped in technological applications by a rigid formalism which obscures the physical significance of the analytical results. In the solution of complex problems encountered in practice an important requirement is the possibility of recognizing those factors which add considerably to the mathematical complexity and at the same time are not relevant to the physical problem and may be neglected. This cannot be achieved unless the analytical formulation itself is sufficiently simple and physically clear. One of the basic difficulties arising from the tensor theory is due to the use of the metric tensor as a measure of the finite strain. This requires the physical properties to be expressed in terms of the squares of the distances between material points. By its very nature this definition of the strain leads to a formulation which does not provide a clear distinction between the geometry of the deformation field and those properties which represent the physics of the material. Because it contains quadratic terms to begin with, the metric tensor is also the source of much confusion regarding the significance of second and higher order

elastic coefficients. In this respect illuminating contrast is provided by the simplified treatment of second order elasticity in Chapter 2 (section 9).

The formal conciseness of the tensor calculus is deceptive, since it leaves to the engineer and the physicist the burden of expressing physical properties of materials by means of non-cartesian concepts which are not essential and generally complicate the task. In fact, it may be said that the overemphasis on tensor methods in this case provides a prime example of mathematical techniques which in some areas have slowed progress and led occasionally to false physical interpretations.

The approach presented in this book is essentially free of these limitations and difficulties. A small region of the medium is considered to undergo a "pure deformation" followed by a solid rotation. The order in which these two transformations are applied is important and is chosen so that the strain components are referred to axes which have been rotated with the material. Thus a local rotation is introduced which varies from point to point and provides a separation of the purely geometric properties of the deformation field from those which depend on the physics of the material. Correspondingly the stress is also defined relative to these locally rotated axes. A dual representation is introduced by referring the stress to areas before or after deformation. This provides considerable freedom in the formulation of physical properties and permits the incorporation of thermodynamic principles in the stress-strain relations. On the other hand, problems may be formulated with equal ease when, for example, it is necessary to introduce a hydrostatic stress.

To be sure, the separation between rotation and pure deformation is not unique. Mathematically speaking, no restriction is imposed on how this separation is to be made. It depends entirely on the nature of the problem considered. Although in the general theory of Chapter 1 the pure deformation is defined by a linear transformation with symmetric coefficients which implies no rotation of the strain axes, the formulation is by no means restricted to this choice.

On the other hand, this very arbitrariness in the definition of the pure deformation and, at the same time, the use of a dual representation of the stress lead to greater flexibility. This is particularly important in applications where substantial simplifications are

achieved by direct and *ad hoc* solutions specifically tailored to the problem. In many cases it is preferable to carry out the analysis by a specialized approach which is not handicapped from the start by rigid methods and by the burden of invariance and excessive generality. Specialized and simple methods applied to typical problems which embody all essential features will generally bring out more clearly the fundamental physical properties.

These points are illustrated by the treatment of plates and rods in Chapter 2 (section 10) and Chapter 3 (sections 2 and 3) where remarkable simplification and physical clarity are achieved through a choice of variables which are not tensors and are tailored to the specific asymmetry of the physics and the geometry. On the other hand, the analysis of isotropic media and rubber elasticity in Chapter 2 (section 8) provides a good example of the use of alternative definitions of the stress which lead to new results and insights. For the purpose of comparison I have also derived these new results in a separate paper (see page 95) by a method of tensor invariants showing that the latter procedure is considerably more elaborate and tends to conceal the physics as well as potential algebraic simplifications.

The general analysis of stability in the presence of hydrostatic stress which is developed in Chapter 3 clears up some fundamental paradoxes and further illustrates the advantages provided by the alternative representations of the stress.

In later years it became apparent that the methods which I had developed earlier in the context of the theory of elasticity could be extended to stability problems of viscous and viscoelastic media. In fact, this realization has opened an entirely new phase in problems of deformation of the earth's crust and tectonic folding of geological structures. A similar extension is applicable to problems of acoustic propagation in viscoelastic media under initial stress. Incremental deformations of a medium initially at rest and in a given state of stress may be considered as thermodynamic perturbations of an equilibrium state. Hence the mechanics of such a medium may be analyzed by introducing the thermodynamics of irreversible processes as a unifying background. This systematic theory, which is developed in Chapter 6, includes many new results and theorems which are presented here for the first time. For vanishing initial stress, new results are also obtained in linear viscoelasticity as a particular case. The simultaneous treatment of elasticity and

viscoelasticity of initially stressed media under conditions which include the most general cases of anisotropy is a consequence of the separation of the physics from the geometry, in combination with a very general "principle of viscoelastic correspondence" (see pages 359 and 490). Some of the results may also be extended to plasticity by adding appropriate stress-strain relations to the equations which express only geometric and equilibrium properties.

Fluids at rest under initial stress are treated as a particular case of elasticity and viscoelasticity. This includes the theory of internal gravity waves and problems of stability and dynamics of viscous fluids in a gravity field.

The case of viscous fluids, which are not at rest under initial stress, requires special treatment. In particular, the conditions which determine the validity of viscoelastic correspondence in this case have been examined. At the same time a number of rather subtle difficulties associated with fundamental kinematic properties of the strain rate have been clarified.

A good deal of attention has been given in this book to variational methods and the principle of virtual work. They lead to the concept of generalized coordinates, generalized stresses and to Lagrangian equations. They are applicable to both elastic and non-elastic media and may be used to derive approximate solutions for complex problems. In addition, an important application of the principle of virtual work is its use to formulate general dynamical equations in curvilinear coordinates. This provides a simple technique based only on cartesian concepts which is applicable to all media regardless of their physical properties.

The methods and concepts used in the linearized theory of initially stressed media are directly applicable to non-linear theories and large deformations. The equations obtained in both of these cases are analogous. A brief outline of this is given in the Appendix. It is sometimes necessary to distinguish infinitesimal quantities of various orders in the mathematical sense and quantities which may be small but are not negligible in the physical context. This is particularly true in certain problems of elastic stability of thin plates and shells where some components of strain are very small while the corresponding stresses are not negligible. In such cases the linearized theory may not be adequate to determine practical stability. The so-called "post-buckling" behavior where similar effects must be

taken into account may also be analyzed by applying the results outlined in the Appendix.

Such limitations of the linearized perturbation theory in problems of elastic stability along with others of purely mathematical or academic interest are discussed in Chapter 3. There it is indicated how they may be clarified by considering non-elastic properties and non-linearity.

The equations discussed in the Appendix and developed in the papers listed at the end of this Preface provide a simple tool for non-linear analysis. In particular, I have shown (1934–38) that the separation of the deformation from the rotation leads to important simplifications when the strain remains small relative to the rotations. For the same reason it is possible to separate the non-linearity due to physical properties of the material from that due to the geometry of the deformation field. This type of non-linear analysis shows that in the vast majority of problems the essential features are adequately represented by expressions which involve a discriminating choice of suitable second and third order terms. These considerations are very important in theories of plates and shells.

Equations applicable to finite strain and expressed in terms of a velocity field and rate variables are readily obtained from the mechanics of incremental deformations by a trivial limiting process which introduces infinitesimal increments. This amounts to considering finite strain to be generated by a continuous sequence of incremental deformations.

It should be borne in mind that Applied Mathematics is an art as much as it is a science.* In physical theory it is of paramount importance to acquire an intimate grasp of the reality behind the mathematical symbols. The formalism alone or even numerical solutions do not by themselves bring to light the significant qualitative features which lead to deeper insight and constitute an essential part

* See the author's papers, Applied Mathematics an Art and a Science, *Journal of the Aeronautical Sciences*, Vol. 23, No. 5, pp. 406–410, p. 489, 1956; Are We Drowning in Complexity?, *Mechanical Engineering*, Vol. 85, No. 2, pp. 26–27, 1963; and Science and the Engineer, *Applied Mechanics Reviews*, Vol. 16, No. 2, pp. 89–90, 1963. The last paper has been reprinted in the following journals: *Journal of Engineering Education*, Vol. 54, No. 5, pp. 169–170, 1964; *Scientific World*, Vol. 7, No. 4, pp. 9–10, 1963; *Bulletin of Mechanical Engineering Education*, Vol. 2, No. 3, pp. 149–151, 1963; *Engineering and Science*, Vol. 26, No. 4, pp. 30–36, 1963; *Ciencia y Técnica*, (Spanish translation), Vol. 133, No. 673, 1964.

of any truly comprehensive theoretical treatment. The commonly accepted notion that all problems are solved once exact equations have been established which can be fed into automatic computers is a fundamental fallacy. The criterion of adequacy of a physical theory is not necessarily based on pure logical structure and generality. The theory must be associated with other advantages of a conceptual and pragmatic nature. This obviously involves a judgment of values which lies beyond the scope of mathematical principles.

There is no mathematical synthesis which guarantees the simplest and most direct solution to every type of problem. Some exceptional cases of large deformation may require the use of more specialized techniques. Many such problems are mainly of academic interest. They constitute a small fraction of the vast field of technological and physical problems which can be handled by more appropriate methods.

While stressing the practical limitations of the tensor calculus in problems of applied mechanics, one should of course recognize the well-established value of the tensor concept itself, particularly in its simpler cartesian form. The concept of cartesian tensor is implicit throughout the present work. However, as in the classical treatment of linear elasticity, it has not been found necessary to depend on the rules of tensor algebra as a separate mathematical discipline.

My efforts have been directed toward giving the engineer and physicist adequate tools with a sound mathematical foundation, and a minimum requirement in mathematical techniques. Procedures and viewpoints which tend to build up the mechanics of continuous media as an exercise in tensor formalism have been avoided. The emphasis has been put on methods which achieve a compromise between simplicity, generality, and usefulness. It is not intended to exclude other methods provided that the difference in emphasis is clearly understood and proper balance is maintained.

This book is divided into six chapters and an Appendix. The first chapter, parts of Chapters 2 and 5, and the Appendix are concerned mainly with the material originally developed during the years 1934 to 1940 in the seven papers listed at the end of this Preface. The presentation here is given in quite different form: the non-linear and large deformation theories are treated separately

in the Appendix as an extension of the linearized equations for a medium with initial stress. Chapter 2 deals primarily with the general theory of elasticity. The next two chapters are devoted to problems of elastic stability of isotropic and anisotropic media. The general dynamics of elastic media under initial stress is developed in Chapter 5; it includes problems of acoustic propagation, dynamic stability, and the theory of internal gravity waves in a fluid. The last chapter, which is by far the longest, is devoted exclusively to viscous and viscoelastic media under initial stress and includes a discussion and applications of the thermodynamics of irreversible processes.

For a detailed description of the contents and interrelations of various parts of the book the reader is referred to the introductory sections at the beginning of each chapter.

The basic theory contained in the seven papers listed below was developed while I was a member of the Applied Science Department of the University of Louvain, and of the Physics Department of Columbia University.

The preparation of the book itself and most of the research connected with the new developments, some of which have been presented in separate publications, were supported by the Air Force Office of Scientific Research under contracts AF 49 (638)-266, AF 49 (638)-837, and AF 49 (638)-1329.

Other original contributions incorporated in this book result from work sponsored by the Shell Development Company as part of a general research program in geodynamics.

I am indebted to Dr. A. Winzer for valuable assistance in proof-reading, in the preparation of the Index, and in some of the analytical derivations in section 8 of Chapter 3 and section 7 of Chapter 5.

The earlier papers (1934–1940) on which this book is based are:

1. M. A. Biot, Sur la stabilité de l'équilibre élastique. Equations de l'élasticité d'un milieu soumis à tension initiale, *Annales de la Société Scientifique de Bruxelles*, Vol. 54, Ser. B, part I, pp. 18–21, 1934.
2. M. A. Biot, Theory of Elasticity with Large Displacements and Rotations, in *Proceedings of the Fifth International Congress for Applied Mechanics* (Cambridge, Mass., September 1938), pp. 117–122, John Wiley & Sons, Inc., New York, Chapman & Hall Ltd., London 1939.
3. M. A. Biot, Théorie de l'élasticité du second ordre avec application à la théorie du

- flambage, *Annales de la Société Scientifique de Bruxelles*, Vol. 59, Ser. I, pp. 104–112, 1939.
4. M. A. Biot, Non-linear Theory of Elasticity and the Linearized Case for a Body under Initial Stress, *Philosophical Magazine*, Vol. 27, Ser. 7, pp. 468–489, 1939.
 5. M. A. Biot, Elastizitätstheorie zweiter Ordnung mit Anwendungen, *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 20, No. 2, pp. 89–99, 1940.
 6. M. A. Biot, Increase of Torsional Stiffness of a Prismatical Bar due to Axial Tension, *Journal of Applied Physics*, Vol. 10, No. 12, pp. 860–864, 1939.
 7. M. A. Biot, The Influence of Initial Stress on Elastic Waves, *Journal of Applied Physics*, Vol. 11, No. 8, pp. 522–530, 1940.

In order to avoid undue repetition, these papers are referred to in the book by their numbers as listed here.

MAURICE A. BIOT

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CHAPTER ONE

Statics and Kinematics of Incremental Stresses and Strains

1. INTRODUCTION

It is well known that a state of initial stress in a deformable medium induces mechanical properties which depend mainly on the magnitude of the stress and are quite distinct from those associated with the rigidity of the material itself.

This is best illustrated by the example of a string under tension. A perfectly flexible string is stretched under a tension T between two fixed points A and B (Fig. 1.1). If a vertical load F is applied to the string at point C , the deflection w at that point is determined entirely by the laws of static equilibrium.

The static analysis is simplified by assuming that the slope of the string is small and may be treated mathematically as a quantity of the first order. The deflection of the string is

$$w = \frac{F}{k} \tag{1.1}$$

with

$$k = T \left(\frac{1}{AC} + \frac{1}{CB} \right) \tag{1.2}$$

Although no elastic properties of the material itself are involved, the deflection is the same as if the load were acting on a spring whose rigidity is measured by a modulus k .

The same analogy extends to variational and energy methods. When the string deflects, the elastic potential energy in the string increases by the amount

$$W = T(AD + DB - AB) \quad (1.3)$$

To the second order we may write

$$\begin{aligned} AD &= \sqrt{(AC)^2 + w^2} = AC + \frac{1}{2} \frac{w^2}{AC} \\ DB &= \sqrt{(CB)^2 + w^2} = CB + \frac{1}{2} \frac{w^2}{CB} \end{aligned} \quad (1.4)$$

Hence

$$W = \frac{1}{2}T \left(\frac{1}{AC} + \frac{1}{CB} \right) w^2 = \frac{1}{2}kw^2 \quad (1.5)$$

This expression also represents the potential energy stored in a spring of modulus k . The deflection of such a spring under a force F may be obtained by the principle of virtual work:

$$F\delta w = \delta W = kw\delta w \quad (1.6)$$

The deflection w derived from (1.6) coincides with the value (1.1).

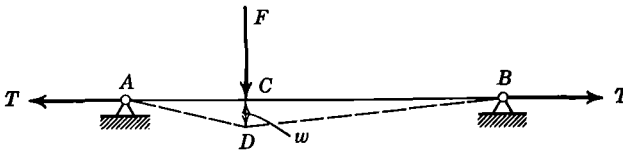


Figure 1.1 Deflection of a string under tension as an example of the apparent rigidity of a system under initial stress.

This simple example brings forth the important point that the equations derived from direct balance of the forces involve only first order terms in the geometry of the deformation. By contrast the corresponding variational principle involves second order geometry.

The next step which comes to mind is represented by problems in which the elastic properties of the material and the initial stress both contribute to the over-all rigidity of the structure. For example, we may ask what happens when the string considered in the previous example is not perfectly flexible and possesses an elastic rigidity in bending.

Problems of this type have been treated extensively in the past mostly in the context of engineering and the particular branch of science usually referred to as "Strength of Materials." One of the early developments is Euler's theory of buckling of a thin rod under axial compression. The presence of an initial stress may increase or decrease the over-all rigidity of an elastic structure. In a rod under axial compression the initial stress produces a decrease in lateral stiffness. For increasing values of the compression this decrease will overcome the natural bending rigidity of the rod, producing an instability known as buckling. On the other hand, a cable hanging under its own weight is under an initial tension which increases its rigidity. This effect is used in the design of suspension bridges.

These problems have usually been treated by approximate methods restricted to special structures such as slender rods and thin plates. The viewpoints and methods used in these approximate theories lead to a formulation which brings out explicitly in the equations the particular terms which contain the initial stress and are responsible for the characteristic features due to the presence of this stress. The same viewpoint can be maintained to develop a systematic and rigorous three-dimensional theory of the initially stressed continuum using elementary methods and following exactly the same procedures as in the classical theory of linear elasticity.

The presentation of this theory along the lines developed by the author is the objective of this chapter. It is essentially an analysis of the statics and kinematics for incremental deformations in the presence of initial stress. The concepts and equations are developed entirely with reference to the geometry of the deformation and the equilibrium of the stresses. At no time is any reference made to the physical properties of the material. The results of this chapter are therefore applicable to any type of continuum, whether it be a fluid or a solid, with elastic, plastic, or viscoelastic properties.

As shown in the simple example of the string under tension, a complete analysis of the problem requires an understanding of the geometry of the deformation which includes both first and second order terms. The second order terms are required in the linear theory in order to formulate the corresponding variational principles. We have therefore analyzed the concept of strain from this viewpoint in the two initial sections of this chapter. This is done by considering first a state of finite strain. A small region around a material point

undergoes a translation, a solid body rotation, and a "pure" homogeneous strain. The pure strain is defined by using the property that there are three orthogonal directions in the medium which remain orthogonal after deformation. This also leads to a unique definition of the solid rotation of an element. This particular definition of finite strain is well known and leads generally to transcendental equations due to the introduction of a solid body rotation, equations which in three dimensions involve intricate relations for the transformation of the coordinate axes.

However, this difficulty disappears in two important cases. One of them is represented by the classical theory of infinitesimal deformations of the *first order*. The other, which is the one considered in this chapter, involves an evaluation of the strain with an approximation of the *second order*. This second order analysis provides immediately the required quadratic expression for the strain energy leading to a variational formulation of the theory which is developed in detail in the next chapter. The non-linear expressions derived for the strain also clarify the validity of first order approximations. In addition, as shown by the author, they lead to a simplified non-linear theory of elasticity.*

We have treated separately the two-dimensional deformation (in section 2) and the three-dimensional deformation (in section 3). This separation of the two cases is maintained throughout because the physical significance is more easily explained and illustrated in two dimensions, whereas the mathematical symmetry of the equations is more readily emphasized in three dimensions. In connection with the three-dimensional kinematics of strain we have used the so-called "dummy index" rule. This is a notation of considerable conciseness used as a standard procedure in the tensor calculus. It has been used extensively in this book whenever needed either to avoid unnecessary writing or to bring out the mathematical structure of the equations. It should be remembered that, although this is a notation of the tensor calculus, no use is made of the tensor calculus itself and the mathematical procedures remain entirely elementary.

The following sections are devoted to the *linear* mechanics of a continuum under initial stress under conditions of static equilibrium.

* See references 3, 4, and 5 at the end of the Preface. A brief discussion of the non-linear theory is also given in the Appendix.

The original coordinates refer to the medium in the state of initial stress. A small displacement field is then superimposed. The strain associated with this deformation is infinitesimal and is described by the classical components for small strain. These classical considerations do not apply to the stress. The significance of incremental stresses is analyzed in sections 4 and 5. Of importance is the introduction of incremental stress components referred to axes whose directions are obtained by rotating the original coordinates by an amount equal to the local rotation of the material. The stress is thus referred to axes whose orientation varies from point to point. The purpose of representing the stress in this way is that its components now depend only on the physical properties of the material; *thus the physics is separated from the geometry* and the solid body rotation is eliminated from the relations between stress and strain. This feature is particularly useful in studies of viscoelastic and plastic materials.

It should be kept in mind that the linear theory is strictly applicable only if the stress variation is a small fraction of the initial stress. Smallness of the deformation in the physical sense does not always guarantee this condition to be fulfilled, as in certain types of problems of thin plates and shells. Such problems must be handled by specialized methods. The non-linear theories developed earlier by the author* also provide a basis for a fundamental but elementary approach to such problems which is similar to the linear theory.

The last two sections of this chapter are devoted to the derivation of equilibrium equations and boundary conditions using the foregoing definition of the incremental stresses. One important characteristic of these equations is that they are *intrinsic*, i.e., they depend on the local geometry of the deformation and at the same time retain the *cartesian representation* of the stress. This has the advantage of clarifying the physical significance of the mathematics and constitutes the reason for the usefulness of this form of the equations.

The equations derived in this chapter are restricted to cartesian coordinates. Their formulation for *curvilinear coordinates* has been relegated to Chapter 2 as an application of the variational principles.

Finally it should also be remarked that this approach does not require any knowledge of the physical process by which the initial

* See references 3, 4, and 5 at the end of the Preface.

stress has been generated. We should remember that the physics of initial stresses can be very different from that of incremental stresses. For example, a gas may be in isothermal equilibrium under gravity and incremental acoustic waves may propagate through it adiabatically. The same considerations apply to rapid elastic deformations in the earth where the initial stress is associated with a slow process of creep due to viscous and plastic deformations.

2. THE KINEMATICS OF TWO-DIMENSIONAL STRAIN

We consider a homogeneous deformation in the plane x, y such that a square S is transformed into a rectangle R while the sides keep fixed orientations I and II (Fig. 2.1). Such a deformation is represented by the linear transformation with symmetric coefficients:

$$\begin{aligned}\xi &= (1 + \varepsilon_{11})x + \varepsilon_{12}y \\ \eta &= \varepsilon_{21}x + (1 + \varepsilon_{22})y\end{aligned}\quad (2.1)$$

where

$$\varepsilon_{12} = \varepsilon_{21} \quad (2.2)$$

A point P of coordinates x, y is transformed into a point P' of coordinates ξ, η . The coefficients ε_{ij} define a *pure deformation*. The reason for this appellation is the existence of two directions I and II,

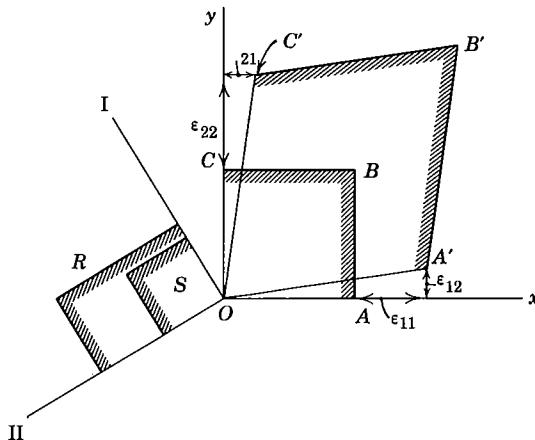


Figure 2.1 Representation of a pure deformation. The square S is transformed into the rectangle R .

perpendicular to each other, called *principal directions*, whose orientation remains fixed during the transformation. The deformation represented by equations 2.1 is therefore always equivalent to positive or negative elongations in the principal directions.

The existence of the principal directions is a consequence of the symmetry property (2.2). This can be shown by considering the quadratic form

$$\phi = \frac{1}{2}(1 + \varepsilon_{11})x^2 + \varepsilon_{12}xy + \frac{1}{2}(1 + \varepsilon_{22})y^2 \quad (2.3)$$

Because of the symmetry relation (2.2) we may write the transformation (2.1) as

$$\begin{aligned} \xi &= \frac{\partial \phi}{\partial x} \\ \eta &= \frac{\partial \phi}{\partial y} \end{aligned} \quad (2.4)$$

Therefore the vector ξ, η is parallel to the gradient of ϕ , i.e., normal to the conic section whose equation is

$$\phi = \text{const.} \quad (2.5)$$

and which passes through the point x, y . Obviously the axes of this conic section are the principal directions of the deformation.

Let us now write the general linear homogeneous transformation in the plane x, y , i.e.,

$$\begin{aligned} \xi &= (1 + a_{11})x + a_{12}y \\ \eta &= a_{21}x + (1 + a_{22})y \end{aligned} \quad (2.6)$$

where the coefficients a_{ij} may or may not be symmetric, i.e., where in general

$$a_{12} \neq a_{21}$$

In matrix form we write

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (2.7)$$

A pure rigid rotation is a particular case of transformation (2.6). A clockwise rotation through the angle θ transforms the coordinates x', y' into ξ, η by the linear relations:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \quad (2.8)$$

The question immediately arises whether the transformation (2.6) is always equivalent to two successive transformations, namely, *first* a pure deformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & 1 + \varepsilon_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (2.9)$$

with $\varepsilon_{12} = \varepsilon_{21}$ followed by a pure rotation (2.8). The successive application of these two transformations leads to

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & 1 + \varepsilon_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (2.10)$$

This transformation must be equivalent to transformation (2.7). Performing the matrix multiplication and equating the corresponding matrix elements in equations 2.7 and 2.10, we derive the four equations

$$\begin{aligned} (1 + \varepsilon_{11}) \cos \theta - \varepsilon_{21} \sin \theta &= 1 + a_{11} \\ (1 + \varepsilon_{11}) \sin \theta + \varepsilon_{21} \cos \theta &= a_{21} \\ (1 + \varepsilon_{22}) \cos \theta + \varepsilon_{12} \sin \theta &= 1 + a_{22} \\ -(1 + \varepsilon_{22}) \sin \theta + \varepsilon_{12} \cos \theta &= a_{12} \end{aligned} \quad (2.11)$$

We may solve the first two of these equations for $1 + \varepsilon_{11}$ and ε_{21} and the last two for $1 + \varepsilon_{22}$ and ε_{12} . We find

$$\begin{aligned} \varepsilon_{21} &= a_{21} \cos \theta - (1 + a_{11}) \sin \theta \\ \varepsilon_{12} &= a_{12} \cos \theta + (1 + a_{22}) \sin \theta \\ 1 + \varepsilon_{11} &= (1 + a_{11}) \cos \theta + a_{21} \sin \theta \\ 1 + \varepsilon_{22} &= (1 + a_{22}) \cos \theta - a_{12} \sin \theta \end{aligned} \quad (2.12)$$

Because $\varepsilon_{12} = \varepsilon_{21}$ expressions on the right side of the first two of relations (2.12) are equal, and we derive

$$\tan \theta = \frac{a_{21} - a_{12}}{2 + a_{11} + a_{22}} \quad (2.13)$$

This yields the magnitude of the pure rotation contained in transformation (2.7). It is a counterclockwise rotation through an angle θ . Knowing θ , we are able to calculate from relations (2.12) the

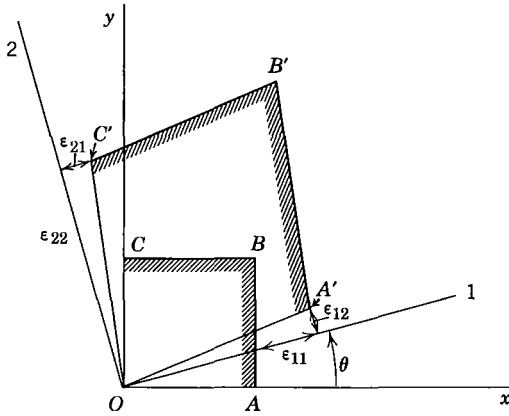


Figure 2.2 Superposition of a pure deformation and a rotation θ .

coefficients ϵ_{ij} of the pure deformation (2.9) contained in equation 2.7. They are

$$\begin{aligned} \epsilon_{21} &= \epsilon_{12} = \frac{1}{2}(a_{21} + a_{12}) \cos \theta + \frac{1}{2}(a_{22} - a_{11}) \sin \theta \\ \epsilon_{11} &= a_{11} \cos \theta + a_{21} \sin \theta + \cos \theta - 1 \\ \epsilon_{22} &= a_{22} \cos \theta - a_{12} \sin \theta + \cos \theta - 1 \end{aligned} \quad (2.14)$$

The first of equations 2.14 is obtained by adding the first two of equations 2.12 and dividing by 2.

By transformation (2.6) the unit square $OABC$ in Figure 2.2 is transformed into the parallelogram $OA'B'C'$. We have just shown that this is equivalent to rotating the square counterclockwise through an angle θ , then submitting it to a pure deformation. This pure deformation is defined by the coefficients ϵ_{ij} , i.e., by a symmetric transformation (2.1) where the coordinates x, y are now referred to axes 1, 2 which are rotated by the angle θ from their original direction. It is important to note that this is a consequence of the fact that we have first applied the transformation (2.9) and then the rotation (2.8). The sequence of these two transformations is not arbitrary since the multiplication of the two matrices in equation 2.10 is not commutative; that is, we find a different result if we reverse the order of multiplication. If we reverse the order of transformations (2.8) and (2.9), i.e., if we first apply a rotation and then perform a pure deformation, we find the same expression θ for the angle of rotation but different values for the coefficients ϵ_{ij} . There is, of course, no contradiction here because the ϵ_{ij} represent the same pure deformation,

referred this time to the original *unrotated axes* instead of the rotated axes 1, 2.

In the present theory the definition (2.12) of the pure deformation is adopted; that is, *we shall refer the pure deformation to rotated axes.*

Until now we have considered *finite deformations*. We now introduce an assumption of “*smallness*”; that is, we shall arbitrarily consider the coefficients a_{ij} to be small quantities of the first order,

$$a_{ij} \ll 1 \quad (2.15)$$

Letting

$$\omega = \frac{1}{2}(a_{21} - a_{12}) \quad (2.16)$$

we may write to the first order

$$\theta \cong \omega \quad (2.17)$$

Furthermore we obtain expressions for the coefficients ε_{ij} which are correct to the second order if in relations (2.14) we replace

$$\begin{aligned} \sin \theta & \text{ by } \omega \\ \cos \theta & \text{ by } 1 \\ 1 - \cos \theta & \text{ by } \frac{1}{2}\omega^2 \end{aligned} \quad (2.18)$$

Hence to the second order we find

$$\begin{aligned} \varepsilon_{21} = \varepsilon_{12} &= \frac{1}{2}(a_{21} + a_{12}) + \frac{1}{2}(a_{22} - a_{11})\omega \\ \varepsilon_{11} &= a_{11} + a_{21}\omega - \frac{1}{2}\omega^2 \\ \varepsilon_{22} &= a_{22} - a_{12}\omega - \frac{1}{2}\omega^2 \end{aligned} \quad (2.19)$$

In the text below we have used an equivalent form of these expressions which introduces explicitly the quantity $\frac{1}{2}(a_{21} + a_{12})$. We may write the identities

$$\begin{aligned} a_{21} &= \frac{1}{2}(a_{21} + a_{12}) + \omega \\ a_{12} &= \frac{1}{2}(a_{21} + a_{12}) - \omega \end{aligned} \quad (2.20)$$

and the coefficients (2.19) become

$$\begin{aligned} \varepsilon_{21} = \varepsilon_{12} &= \frac{1}{2}(a_{21} + a_{12}) + \frac{1}{2}(a_{22} - a_{11})\omega \\ \varepsilon_{11} &= a_{11} + \frac{1}{2}(a_{21} + a_{12})\omega + \frac{1}{2}\omega^2 \\ \varepsilon_{22} &= a_{22} - \frac{1}{2}(a_{21} + a_{12})\omega + \frac{1}{2}\omega^2 \end{aligned} \quad (2.21)$$

The next and last step in the analysis is to consider an *inhomogeneous* deformation, i.e., such that a point P of coordinates x, y is transformed into a point P' of coordinates

$$\begin{aligned}\xi &= x + u \\ \eta &= y + v\end{aligned}\tag{2.22}$$

The displacement field is represented by the vector of components

$$\begin{aligned}u &= u(x, y) \\ v &= v(x, y)\end{aligned}\tag{2.23}$$

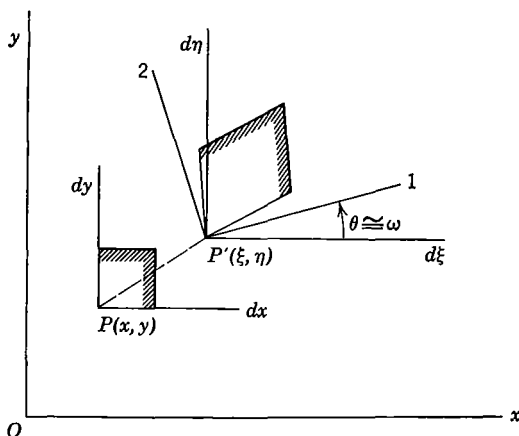
both functions of the initial coordinates x, y . The differential relations

$$\begin{aligned}d\xi &= \left(1 + \frac{\partial u}{\partial x}\right) dx + \frac{\partial u}{\partial y} dy \\ d\eta &= \frac{\partial v}{\partial x} dx + \left(1 + \frac{\partial v}{\partial y}\right) dy\end{aligned}\tag{2.24}$$

represent a linear transformation of the infinitesimal vector of components dx, dy in the vicinity of point P into an infinitesimal vector of components $d\xi, d\eta$ in the vicinity of point P' . In other words, relations (2.24) define a homogeneous transformation of the infinitesimal area around P into an infinitesimal area around P' (Fig. 2.3). Such a transformation is identical with the homogeneous transformation (2.6), and the previous analysis is immediately applicable. The coefficients a_{ij} become the partial derivatives,

$$\begin{aligned}a_{11} &= \frac{\partial u}{\partial x} & a_{12} &= \frac{\partial u}{\partial y} \\ a_{21} &= \frac{\partial v}{\partial x} & a_{22} &= \frac{\partial v}{\partial y}\end{aligned}\tag{2.25}$$

It is possible therefore to define a *local rotation* θ of the material which varies from point to point; it is given by expression (2.13). The pure deformation of the infinitesimal region around point P is represented by the coefficients ϵ_{ij} given by expressions (2.14). We must remember that these coefficients represent a pure deformation which is defined relative to the locally rotated directions 1, 2 (Fig. 2.3).

Figure 2.3 Deformation in the vicinity of a point P' originally at P .

From equation 2.16 we see that the magnitude of the local rotation is given to the first order by

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (2.26)$$

For convenience we introduce the notation

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} & e_{yy} &= \frac{\partial v}{\partial y} \\ e_{xy} &= e_{yx} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \quad (2.27)$$

To the second order the coefficients ϵ_{ij} or *strain components* are given by equations 2.21. With the notation (2.27) we may write*

$$\begin{aligned} \epsilon_{12} &= e_{xy} + \frac{1}{2}(e_{yy} - e_{xx})\omega \\ \epsilon_{11} &= e_{xx} + e_{xy}\omega + \frac{1}{2}\omega^2 \\ \epsilon_{22} &= e_{yy} - e_{xy}\omega + \frac{1}{2}\omega^2 \end{aligned} \quad (2.28)$$

The quantities e_{xx} , e_{yy} , e_{xy} are the first order strain components of the classical theory of elasticity.

To the first order we may write

$$\epsilon_{11} = e_{xx} \quad \epsilon_{22} = e_{yy} \quad \epsilon_{12} = e_{xy} \quad (2.29)$$

* Expressions (2.28) were derived by the author in 1938 (reference 2 at the end of the Preface).

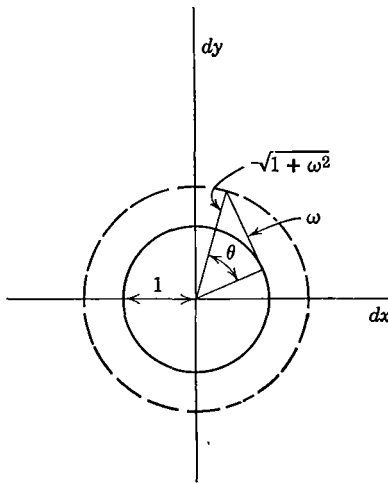


Figure 2.4 Uniform dilatation represented by linear transformation (2.32).

Another interesting consequence of relations (2.28) arises when the classical strain components are zero:

$$e_{xx} = e_{yy} = e_{xy} = 0 \tag{2.30}$$

In this case there is a second order strain,

$$\varepsilon_{12} = 0 \quad \varepsilon_{11} = \varepsilon_{22} = \frac{1}{2}\omega^2 \tag{2.31}$$

which corresponds to an isotropic extension of magnitude $\frac{1}{2}\omega^2$. This is easily verified directly by considering the transformation (2.24) which in this case becomes

$$\begin{aligned} d\xi &= dx - \omega dy \\ d\eta &= \omega dx + dy' \end{aligned} \tag{2.32}$$

All points on a unit circle centered at the origin are transformed by a displacement ω in a direction tangent to the circle. Hence the radius of the circle is enlarged by a factor

$$\sqrt{1 + \omega^2} \cong 1 + \frac{1}{2}\omega^2 \tag{2.33}$$

The circle, of course, also rotates through an angle θ given by

$$\tan \theta = \omega \tag{2.34}$$

This is illustrated in Figure 2.4.

An interesting feature of the pure deformation as defined above is that two successive pure deformations do not combine to give a pure deformation. In mathematical language we say that pure deformations do not constitute a group. In order to show this let us consider the two following pure deformations

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & 1 + \varepsilon_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad (2.35)$$

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon'_{11} & \varepsilon'_{12} \\ \varepsilon'_{12} & 1 + \varepsilon'_{22} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \quad (2.36)$$

The latter transformation is equivalent to a resultant transformation:

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad (2.37)$$

The matrix of this resultant transformation is obtained by multiplication of the matrices defining the transformations (2.35) and (2.36); hence

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon'_{11} & \varepsilon'_{12} \\ \varepsilon'_{12} & 1 + \varepsilon'_{22} \end{bmatrix} \begin{bmatrix} 1 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & 1 + \varepsilon_{22} \end{bmatrix} \quad (2.38)$$

The elements of this matrix are

$$\begin{aligned} c_{11} &= 1 + \varepsilon'_{11} + \varepsilon_{11} + \varepsilon'_{11}\varepsilon_{11} + \varepsilon'_{12}\varepsilon_{12} \\ c_{12} &= \varepsilon'_{12} + \varepsilon_{12} + \varepsilon'_{11}\varepsilon_{12} + \varepsilon'_{12}\varepsilon_{22} \\ c_{21} &= \varepsilon'_{12} + \varepsilon_{12} + \varepsilon'_{12}\varepsilon_{11} + \varepsilon'_{22}\varepsilon_{12} \\ c_{22} &= 1 + \varepsilon'_{22} + \varepsilon_{22} + \varepsilon'_{22}\varepsilon_{22} + \varepsilon'_{12}\varepsilon_{12} \end{aligned} \quad (2.39)$$

The resultant transformation (2.37) will represent a pure deformation only if

$$c_{12} = c_{21} \quad (2.40)$$

This relation which in general will not be fulfilled is equivalent to

$$\frac{\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{22}} = \frac{\varepsilon'_{12}}{\varepsilon'_{11} - \varepsilon'_{22}} \quad (2.41)$$

The significance of this relation appears if we consider the principal directions of strain. These directions are given by the principal

axes of the conic represented by equation 2.5. By a simple calculation the angle α of these principal directions with the x axis is found to satisfy the relation

$$\tan 2\alpha = \frac{2\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{22}} \quad (2.42)$$

We conclude that the necessary and sufficient condition for two successive pure deformations to represent also a pure deformation is that their principal directions coincide.

Condition (2.40) is always fulfilled if we neglect the second order quantities such as $\varepsilon'_{11}\varepsilon_{12}$ and $\varepsilon'_{12}\varepsilon_{22}$. Hence for *infinitesimal strain the combination of two pure deformations yields a pure deformation*.

It is also interesting to note that if relation (2.41) is not satisfied the resultant transformation (2.37) contains a higher order rotation of angle θ defined by equation 2.13. For instance, the successive application of two pure deformations of the first order produces a rotation of the second order. The sign of this rotation is reversed if we reverse the sequence of the pure deformations (2.35) and (2.36).

3. THE KINEMATICS OF THREE-DIMENSIONAL STRAIN

The concepts developed in the preceding section for two-dimensional strain may be extended readily to three dimensions. It is not necessary to repeat all the arguments in detail, and we start immediately with the general non-homogeneous transformation. The point P of initial coordinates x, y, z , is transformed into a point P' of coordinates

$$\begin{aligned} \xi &= x + u \\ \eta &= y + v \\ \zeta &= z + w \end{aligned} \quad (3.1)$$

The displacement field is represented by the vector of components

$$\begin{aligned} u &= u(x, y, z) \\ v &= v(x, y, z) \\ w &= w(x, y, z) \end{aligned} \quad (3.2)$$

In the vicinity of point P the continuum undergoes the linear transformation

$$\begin{aligned}d\xi &= \left(1 + \frac{\partial u}{\partial x}\right) dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \\d\eta &= \frac{\partial v}{\partial x} dx + \left(1 + \frac{\partial v}{\partial y}\right) dy + \frac{\partial v}{\partial z} dz \\d\zeta &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \left(1 + \frac{\partial w}{\partial z}\right) dz\end{aligned}\tag{3.3}$$

This is a homogeneous transformation. In this transformation a point of coordinates dx, dy, dz in the vicinity of point P is transformed into a point of coordinates $d\xi, d\eta, d\zeta$ in the vicinity of P' . As in the two-dimensional case discussed above, we shall show that the transformation (3.3) is equivalent to a pure deformation followed by a pure solid rotation. In three dimensions the kinematics of solid rotation is considerably more involved, and we shall therefore approach the analysis from a different viewpoint.

Let us introduce the symmetric and linear transformation

$$\begin{aligned}d\xi' &= (1 + \varepsilon_{11}) dx + \varepsilon_{12} dy + \varepsilon_{13} dz \\d\eta' &= \varepsilon_{21} dx + (1 + \varepsilon_{22}) dy + \varepsilon_{23} dz \\d\zeta' &= \varepsilon_{31} dx + \varepsilon_{32} dy + (1 + \varepsilon_{33}) dz\end{aligned}\tag{3.4}$$

with

$$\varepsilon_{23} = \varepsilon_{32} \quad \varepsilon_{31} = \varepsilon_{13} \quad \varepsilon_{12} = \varepsilon_{21}$$

That this transformation represents a pure deformation can be seen by using the same arguments as for two-dimensional strain and writing the quadratic form

$$\begin{aligned}\phi &= \frac{1}{2}(1 + \varepsilon_{11})x^2 + \frac{1}{2}(1 + \varepsilon_{22})y^2 + \frac{1}{2}(1 + \varepsilon_{33})z^2 \\&\quad + \varepsilon_{23}yz + \varepsilon_{31}zx + \varepsilon_{12}xy\end{aligned}\tag{3.5}$$

The three axes of the quadric

$$\phi = \text{const.}\tag{3.6}$$

represent the three principal directions of strain.* These three

* A more extended discussion of homogeneous strain will be found in Love's treatise, *The Mathematical Theory of Elasticity*, Fourth Edition, pp. 66–73, Cambridge University Press (reprinted by Dover Publications, New York, 1944).

directions are mutually perpendicular and do not change when the medium undergoes the symmetric transformation (3.4). A cube whose edges are originally oriented along these three directions becomes a rectangular parallelepiped, with its edges oriented along the same directions. We are therefore justified in defining the symmetric transformation (3.4) as a pure deformation. The coefficients in the transformation (3.4) are represented by the symmetric matrix

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \quad (3.7)$$

These coefficients are called the *strain components* of the pure deformation.

Immediately there arises the problem of finding out under what condition the more general transformation (3.3) will contain the same deformation as the symmetric transformation (3.4). The necessary and sufficient condition for this to occur is obviously that the distance between any pairs of points remain the same for both transformations. It may be expressed mathematically as follows. A pair of points whose vectorial distance is represented by dx, dy, dz acquire an absolute distance ds by the transformation (3.3) and an absolute distance ds' by the transformation (3.4). These distances are given by

$$\begin{aligned} ds^2 &= d\xi'^2 + d\eta'^2 + d\zeta'^2 \\ ds'^2 &= d\xi'^2 + d\eta'^2 + d\zeta'^2 \end{aligned} \quad (3.8)$$

The condition that the two transformations contain the same deformation is that the relation

$$ds^2 = ds'^2 \quad (3.9)$$

be verified identically for all values dx, dy, dz .

In order to carry out this identification we introduce the notation

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} & e_{yz} &= e_{zy} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \omega_x &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\ e_{yy} &= \frac{\partial v}{\partial y} & e_{zx} &= e_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \omega_y &= \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \\ e_{zz} &= \frac{\partial w}{\partial z} & e_{xy} &= e_{yx} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \omega_z &= \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \end{aligned} \quad (3.10)$$

With this notation the transformation (3.3) becomes

$$\begin{aligned} d\xi &= (1 + e_{xx}) dx + (e_{xy} - \omega_z) dy + (e_{zx} + \omega_y) dz \\ d\eta &= (e_{xy} + \omega_z) dx + (1 + e_{yy}) dy + (e_{yz} - \omega_x) dz \\ d\zeta &= (e_{zx} - \omega_y) dx + (e_{yz} + \omega_x) dy + (1 + e_{zz}) dz \end{aligned} \quad (3.11)$$

The length element for this transformation is written

$$\begin{aligned} ds^2 &= (1 + 2g_{xx}) dx^2 + (1 + 2g_{yy}) dy^2 + (1 + 2g_{zz}) dz^2 \\ &\quad + 4g_{yz} dy dz + 4g_{zx} dz dx + 4g_{xy} dx dy \end{aligned} \quad (3.12)$$

with the definitions

$$\begin{aligned} g_{xx} &= e_{xx} + \frac{1}{2}e_{xx}^2 + \frac{1}{2}(e_{xy} + \omega_z)^2 + \frac{1}{2}(e_{zx} - \omega_y)^2 \\ g_{yy} &= e_{yy} + \frac{1}{2}e_{yy}^2 + \frac{1}{2}(e_{yz} + \omega_x)^2 + \frac{1}{2}(e_{xy} - \omega_z)^2 \\ g_{zz} &= e_{zz} + \frac{1}{2}e_{zz}^2 + \frac{1}{2}(e_{zx} + \omega_y)^2 + \frac{1}{2}(e_{yz} - \omega_x)^2 \\ g_{yz} &= e_{yz} + \frac{1}{2}(e_{xy} - \omega_z)(e_{zx} + \omega_y) + \frac{1}{2}e_{yy}(e_{yz} - \omega_x) + \frac{1}{2}e_{zz}(e_{yz} + \omega_x) \\ g_{zx} &= e_{zx} + \frac{1}{2}(e_{yz} - \omega_x)(e_{xy} + \omega_z) + \frac{1}{2}e_{zz}(e_{zx} - \omega_y) + \frac{1}{2}e_{xx}(e_{zx} + \omega_y) \\ g_{xy} &= e_{xy} + \frac{1}{2}(e_{zx} - \omega_y)(e_{yz} + \omega_x) + \frac{1}{2}e_{xx}(e_{xy} - \omega_z) + \frac{1}{2}e_{yy}(e_{xy} + \omega_z) \end{aligned} \quad (3.13)$$

These quantities represent the classical definition of finite strain. The length element for the pure deformation (3.4) is

$$\begin{aligned} ds'^2 &= (1 + 2\gamma_{11}) dx^2 + (1 + 2\gamma_{22}) dy^2 + (1 + 2\gamma_{33}) dz^2 \\ &\quad + 4\gamma_{23} dy dz + 4\gamma_{31} dz dx + 4\gamma_{12} dx dy \end{aligned} \quad (3.14)$$

with

$$\begin{aligned} \gamma_{11} &= \varepsilon_{11} + \frac{1}{2}(\varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{31}^2) \\ \gamma_{22} &= \varepsilon_{22} + \frac{1}{2}(\varepsilon_{22}^2 + \varepsilon_{23}^2 + \varepsilon_{12}^2) \\ \gamma_{33} &= \varepsilon_{33} + \frac{1}{2}(\varepsilon_{33}^2 + \varepsilon_{31}^2 + \varepsilon_{23}^2) \\ \gamma_{23} &= \varepsilon_{23} + \frac{1}{2}(\varepsilon_{12}\varepsilon_{31} + \varepsilon_{22}\varepsilon_{23} + \varepsilon_{33}\varepsilon_{23}) \\ \gamma_{31} &= \varepsilon_{31} + \frac{1}{2}(\varepsilon_{23}\varepsilon_{12} + \varepsilon_{33}\varepsilon_{31} + \varepsilon_{11}\varepsilon_{31}) \\ \gamma_{12} &= \varepsilon_{12} + \frac{1}{2}(\varepsilon_{31}\varepsilon_{23} + \varepsilon_{11}\varepsilon_{12} + \varepsilon_{22}\varepsilon_{12}) \end{aligned} \quad (3.15)$$

Now, as already pointed out, the pure deformation (3.4) can be made to represent exactly the same state of strain as that produced by the transformation (3.3) provided the length elements ds and ds' are identical after the transformation, i.e., provided that relation (3.9) is satisfied identically. This condition is expressed analytically by the six equations

$$\begin{aligned} g_{xx} &= \gamma_{11} & g_{yz} &= \gamma_{23} \\ g_{yy} &= \gamma_{22} & g_{zx} &= \gamma_{31} \\ g_{zz} &= \gamma_{33} & g_{xy} &= \gamma_{12} \end{aligned} \quad (3.16)$$

These equations determine the six strain components (3.7) as functions of the nine coefficients (3.10) appearing in transformation (3.3).

Transformations (3.3) and (3.4) thus related represent the same state of strain and can differ only by a rigid body rotation. The rigid body rotation that we must add to transformation (3.4) in order to obtain transformation (3.3) will be called the *local rotation* of the material. Transformation (3.3) contains nine independent coefficients, while the state of strain is determined by only six quantities. There are therefore three degrees of freedom leaving unchanged the length element ds and corresponding to the rigid body rotation contained in the general transformation (3.3).

The finite strain components (3.7) have the advantage that they are linearly related to the actual changes of length in the material, whereas the classical components (3.13) are linearly related to the change of the square of the length. On the other hand, the components (3.7) have the disadvantage that they cannot be expressed rationally by means of the nine quantities (3.10). However, this disadvantage vanishes when we assume *the nine quantities (3.10) to be small of the first order and when we consider only the first and second order terms* in the expressions for the strain components (3.7) as a function of the nine quantities (3.10). A solution of equations 3.16 is obtained immediately as follows.

We notice from equations 3.16 that e_{xx} and ϵ_{11} differ only by a second order quantity; the same is true for e_{xy} and ϵ_{12} , etc., so that we may write with an error of only the third order

$$\begin{aligned}
 e_{xx}^2 + e_{xy}^2 + e_{zx}^2 &= \epsilon_{11}^2 + \epsilon_{12}^2 + \epsilon_{31}^2 \\
 e_{yy}^2 + e_{yz}^2 + e_{xy}^2 &= \epsilon_{22}^2 + \epsilon_{23}^2 + \epsilon_{12}^2 \\
 e_{zz}^2 + e_{zx}^2 + e_{yz}^2 &= \epsilon_{33}^2 + \epsilon_{31}^2 + \epsilon_{23}^2 \\
 e_{xy}e_{zx} + e_{yy}e_{yz} + e_{zz}e_{yz} &= \epsilon_{12}\epsilon_{31} + \epsilon_{22}\epsilon_{23} + \epsilon_{33}\epsilon_{23} \\
 e_{yz}e_{xy} + e_{zz}e_{zx} + e_{xx}e_{zx} &= \epsilon_{23}\epsilon_{12} + \epsilon_{33}\epsilon_{31} + \epsilon_{11}\epsilon_{31} \\
 e_{zx}e_{yz} + e_{xx}e_{xy} + e_{yy}e_{xy} &= \epsilon_{31}\epsilon_{23} + \epsilon_{11}\epsilon_{12} + \epsilon_{22}\epsilon_{12}
 \end{aligned}
 \tag{3.17}$$

Introducing the approximate relations (3.17) into equations 3.16, we find for the strain components with an error of only the third order*

* Equations 3.18 were derived by the author in 1939, in references 3 and 4 at the end of the Preface. They were applied subsequently in reference 5.

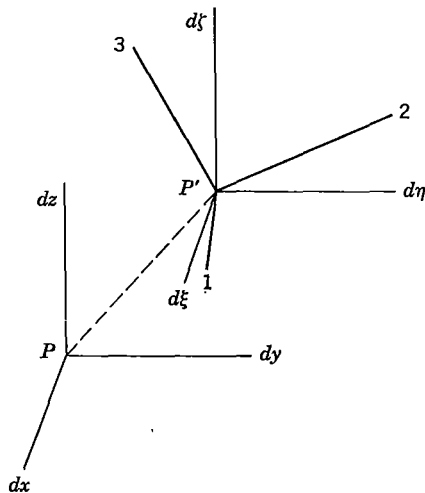


Figure 3.1 Local rotated coordinate system (1, 2, 3) and unrotated coordinate system ($d\xi, d\eta, d\zeta$) in the vicinity of a point P' initially at P .

$$\begin{aligned}
 \varepsilon_{11} &= e_{xx} + e_{xy}\omega_z - e_{zx}\omega_y + \frac{1}{2}(\omega_z^2 + \omega_y^2) \\
 \varepsilon_{22} &= e_{yy} + e_{yz}\omega_x - e_{xy}\omega_z + \frac{1}{2}(\omega_x^2 + \omega_z^2) \\
 \varepsilon_{33} &= e_{zz} + e_{zx}\omega_y - e_{yz}\omega_x + \frac{1}{2}(\omega_y^2 + \omega_x^2) \\
 \varepsilon_{23} &= e_{yz} + \frac{1}{2}\omega_x(e_{zz} - e_{yy}) + \frac{1}{2}\omega_y e_{xy} - \frac{1}{2}\omega_z e_{zx} - \frac{1}{2}\omega_y \omega_z \\
 \varepsilon_{31} &= e_{zx} + \frac{1}{2}\omega_y(e_{xx} - e_{zz}) + \frac{1}{2}\omega_z e_{yz} - \frac{1}{2}\omega_x e_{xy} - \frac{1}{2}\omega_z \omega_x \\
 \varepsilon_{12} &= e_{xy} + \frac{1}{2}\omega_z(e_{yy} - e_{xx}) + \frac{1}{2}\omega_x e_{zx} - \frac{1}{2}\omega_y e_{yz} - \frac{1}{2}\omega_x \omega_y
 \end{aligned} \tag{3.18}$$

At this point it is important to stress the physical significance of these components of strain. If we look at the homogeneous transformation (3.3) of a small region in the vicinity of a point attached to the material, we see that it can be obtained as follows (Fig. 3.1).

1. The material is translated as a rigid body so that point P coincides with P' .

2. We rotate this region as a rigid body. (We show below that this rotation is defined to the first order by the vector $\omega_x, \omega_y, \omega_z$.)

3. A system of rectangular coordinates with its origin at point P' and parallel with the x, y, z directions is rigidly rotated by the same amount as the material and becomes thereby a system we call (1, 2, 3). With respect to this coordinate system (1, 2, 3) we then perform the pure deformation (3.4) with strain components (3.18).

Therefore we may look upon strain components (3.18) as representing the pure deformation referred to a rectangular frame (1, 2, 3) originally parallel with the x, y, z directions and undergoing the same rotation as the material. The strain field is thus referred to a field of rectangular axes whose orientation varies from point to point according to the local rotation of the material. It is important to bear this in mind when considering the stress, because to correlate stress and strain we must refer them to the same set of axes.

Let us now examine the solid body rotation. We have mentioned above that to the first order it is represented by a vector of components $\omega_x, \omega_y, \omega_z$. This may easily be verified as follows. If we denote the components $\varepsilon_{11}, \varepsilon_{12}$, etc., by e_{ij} and e_{zx}, e_{yy} , etc., by e_{ij} , we derive from relations (3.18) that the components of the pure deformation are represented by e_{ij} if we neglect second order terms. In other words, to the first order

$$\varepsilon_{ij} = e_{ij}$$

Therefore to the same order the pure deformation is represented by the transformation in matrix form

$$\begin{bmatrix} d\xi' \\ d\eta' \\ d\zeta' \end{bmatrix} = \begin{bmatrix} 1 + e_{xx} & e_{xy} & e_{zx} \\ e_{xy} & 1 + e_{yy} & e_{yz} \\ e_{zx} & e_{yz} & 1 + e_{zz} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \quad (3.19)$$

On the other hand, let us add a second transformation of $d\xi', d\eta', d\zeta'$ into $d\xi, d\eta, d\zeta$

$$\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} 1 & -\omega_z & \omega_y \\ \omega_z & 1 & -\omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix} \begin{bmatrix} d\xi' \\ d\eta' \\ d\zeta' \end{bmatrix} \quad (3.20)$$

By substituting transformation (3.19) into (3.20), we must perform the matrix multiplication. If we do this and keep only the first order terms, we obtain transformation (3.11). Hence transformation (3.20) represents to the first order the solid rotation. The matrix which represents this rotation may be written by introducing double indices as follows.

$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \quad (3.21)$$

with

$$\begin{aligned}
 \omega_{11} &= \omega_{22} = \omega_{33} = 0 \\
 \omega_{32} &= -\omega_{23} = \omega_x \\
 \omega_{13} &= -\omega_{31} = \omega_y \\
 \omega_{21} &= -\omega_{12} = \omega_z
 \end{aligned} \tag{3.22}$$

With general indices these matrix elements are written ω_{ij} . Hence

$$\begin{aligned}
 \omega_{ij} &= 0 && \text{for } i = j \\
 \omega_{ij} &= -\omega_{ji} && \text{for } i \neq j
 \end{aligned} \tag{3.23}$$

It will also be found convenient to introduce general indices for the coordinates and displacements by putting

$$\begin{aligned}
 x &= x_1 & y &= x_2 & z &= x_3 \\
 u &= u_1 & v &= u_2 & w &= u_3
 \end{aligned} \tag{3.24}$$

We may then write the more concise general expressions

$$\begin{aligned}
 e_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
 \omega_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
 \end{aligned} \tag{3.25}$$

By such definition we may write relations (3.18) for the strain in a form which is completely symmetric and also much more concise:

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2} \sum_{\mu}^{\mu} (e_{i\mu} \omega_{\mu j} + e_{j\mu} \omega_{\mu i}) + \frac{1}{2} \sum_{\mu}^{\mu} \omega_{i\mu} \omega_{j\mu} \tag{3.26}$$

This form may be further abbreviated, using the so-called *dummy index rule* by which summation signs are dropped altogether. We then write

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2} (e_{i\mu} \omega_{\mu j} + e_{j\mu} \omega_{\mu i}) + \frac{1}{2} \omega_{i\mu} \omega_{j\mu} \tag{3.27}$$

By this notation, which is standard procedure in the tensor calculus, summations are taken for all possible values of the indices which appear more than once in the same term.

Another form of the strain components is found directly in terms of the gradients of u_i by substituting expressions (3.25) in equation 3.27. This yields

$$\begin{aligned}
 \varepsilon_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{8} \left(3 \frac{\partial u_{\mu}}{\partial x_i} \frac{\partial u_{\mu}}{\partial x_j} \right. \\
 &\quad \left. - \frac{\partial u_j}{\partial x_{\mu}} \frac{\partial u_i}{\partial x_{\mu}} - \frac{\partial u_{\mu}}{\partial x_i} \frac{\partial u_j}{\partial x_{\mu}} - \frac{\partial u_{\mu}}{\partial x_j} \frac{\partial u_i}{\partial x_{\mu}} \right)
 \end{aligned} \tag{3.28}$$

This form, which is convenient in some mathematical derivations, obscures the physical significance of the expression.

Dummy Index Rule Not To Be Confused with the Tensor Calculus. Use of a dummy index as a conventional notation to replace the summation sign is extremely helpful not only for the purpose of abbreviation but also because it brings out the hidden symmetry in the formulas. It is used throughout this book whenever convenient. Although the dummy index is generally associated with the tensor calculus, it is in fact quite independent of it. The treatment of continuum mechanics in this book is carried out without recourse to the tensor calculus at any time.

4. INCREMENTAL STRESSES IN TWO DIMENSIONS

We now turn our attention to the analysis of the stress field. It differs essentially from the strain analysis. The fact that the continuum is already deformed in the initial state is irrelevant for the definition of the incremental strain. This is not so for the incremental stress, and we shall see that the state of initial stress must be considered in the analysis.

In order to bring out more clearly the concepts and methods we consider first a two-dimensional stress field. We start by recalling some elementary definitions and properties. The two-dimensional stress at a point in the plane is defined by the three components

$$\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{array} \quad (4.1)$$

referred to orthogonal axes x and y . The physical significance of these components is obtained by considering the plane x, y to represent a slab of unit thickness. The stress components represent the forces in the x, y plane, acting per unit area on the sides of an infinitesimal element of size dx, dy cut out of the slab. The condition that the tangential component σ_{xy} be the same on both sides dx and dy of the element is a consequence of the fact that the total torque resulting from the stresses on the element must be zero. This feature of the stress components is referred to as the *symmetry* property.

However, there are exceptional cases in which this property will

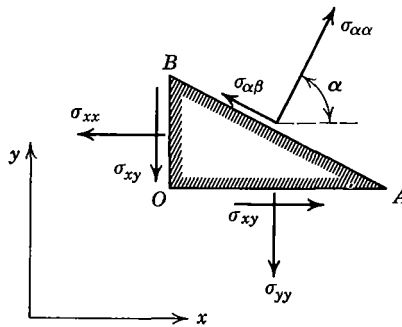


Figure 4.1 Variation of normal stress $\sigma_{\alpha\alpha}$ and tangential stress $\sigma_{\alpha\beta}$ with the normal direction.

not be verified. This occurs if the body force contains a *moment per unit volume* or if adequate representation of the internal stresses requires the introduction of *couples per unit area*. Such cases are excluded from the present treatment.

If we cut a small right-angled triangle OAB out of the slab (Fig. 4.1), the normal and tangential forces per unit area acting on the side AB are found by writing the equation of equilibrium of this element in the x and y directions. We derive

$$\begin{aligned}\sigma_{\alpha\alpha} &= \sigma_{xx} \cos^2 \alpha + \sigma_{yy} \sin^2 \alpha + \sigma_{xy} \sin 2\alpha \\ \sigma_{\alpha\beta} &= \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) \sin 2\alpha + \sigma_{xy} \cos 2\alpha\end{aligned}\quad (4.2)$$

The angle α measures the inclination of the normal to AB with the x direction. Relations (4.2) yield immediately the stress components with respect to axes 1, 2, which are rotated clockwise by an angle α from the original directions x, y .* The new components (Fig. 4.2)

$$\begin{array}{cc}\sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22}\end{array}\quad (4.3)$$

are found by substituting the values α and $\alpha + \pi/2$ in relations (4.2). They are

$$\begin{aligned}\sigma_{11} &= \sigma_{xx} \cos^2 \alpha + \sigma_{yy} \sin^2 \alpha + \sigma_{xy} \sin 2\alpha \\ \sigma_{22} &= \sigma_{xx} \sin^2 \alpha + \sigma_{yy} \cos^2 \alpha - \sigma_{xy} \sin 2\alpha \\ \sigma_{12} &= \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) \sin 2\alpha + \sigma_{xy} \cos 2\alpha\end{aligned}\quad (4.4)$$

* For further discussion see, for example, S. Timoshenko, *Theory of Elasticity*, p. 16, McGraw-Hill Book Co., New York, 1934.

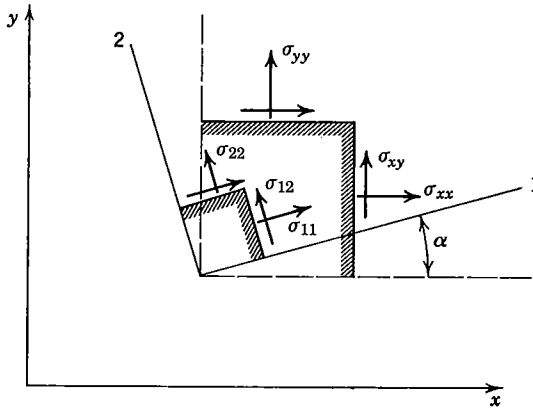


Figure 4.2 Representation of the same stress field relative to the directions x, y and the rotated axes 1, 2.

The last equation shows immediately that there is always a direction $\alpha = \alpha_1$ for which the tangential component σ_{12} or *shear stress* vanishes. The angle α_1 is given by

$$\tan 2\alpha_1 = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{4.5}$$

The stress components referred to this direction reduce to normal components σ_{11} and σ_{22} and are called *principal stresses*.

Inversely, by replacing α by $-\alpha$ we may express the stresses σ_{xx} , etc., in terms of the components σ_{11} , etc. We find

$$\begin{aligned} \sigma_{xx} &= \sigma_{11} \cos^2 \alpha + \sigma_{22} \sin^2 \alpha - \sigma_{12} \sin 2\alpha \\ \sigma_{yy} &= \sigma_{11} \sin^2 \alpha + \sigma_{22} \cos^2 \alpha + \sigma_{12} \sin 2\alpha \\ \sigma_{xy} &= \frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\alpha + \sigma_{12} \cos 2\alpha \end{aligned} \tag{4.6}$$

We consider now an initial stress field

$$\begin{matrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{matrix} \tag{4.7}$$

These components define the initial stress at a point P of coordinates x, y in the plane (Fig. 4.3). If the plane continuum is deformed, any

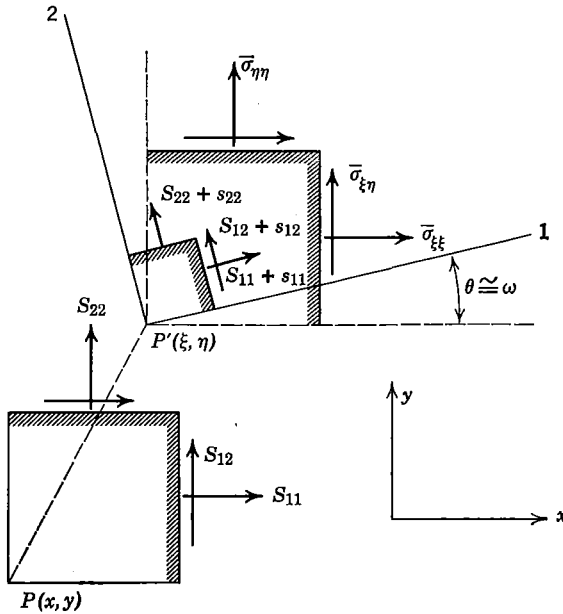


Figure 4.3 Representation of the initial stresses S_{11} , S_{22} , S_{12} and the incremental stresses s_{11} , s_{22} , s_{12} .

point P is displaced to a point P' of coordinates ξ , η , and the stress at this point P' acquires a new value defined by the components

$$\begin{aligned}\bar{\sigma}_{\xi\xi} &= S_{11} + \bar{s}_{\xi\xi} \\ \bar{\sigma}_{\eta\eta} &= S_{22} + \bar{s}_{\eta\eta} \\ \bar{\sigma}_{\xi\eta} &= S_{12} + \bar{s}_{\xi\eta}\end{aligned}\quad (4.8)$$

These components are referred to the fixed directions x , y . The components $\bar{s}_{\xi\xi}$, $\bar{s}_{\eta\eta}$, $\bar{s}_{\xi\eta}$ represent the increment of the total stress at the displaced point P' of coordinates ξ and η after deformation.

We introduce now an important consideration in the whole procedure, namely, that the incremental components $\bar{s}_{\xi\xi}$, $\bar{s}_{\eta\eta}$, $\bar{s}_{\xi\eta}$ are due not only to the strain but also to the fact that the initial stress field has been rotated by a certain angle when moving from P to P' . In other words, if there were no deformation at all, but simply a translation equal to the vector $\overrightarrow{PP'}$ followed by a solid rotation, there would be incremental stress components $\bar{s}_{\xi\xi}$, $\bar{s}_{\eta\eta}$, $\bar{s}_{\xi\eta}$ due to this

rotation, hence of *purely geometric origin*. In addition, if the material undergoes a strain there is an incremental stress of purely physical nature. It is therefore essential to separate the geometry from the physics in expressing the incremental stress components. This can be accomplished if, instead of referring the stress components to the original directions x, y , we refer them to new directions 1, 2. These new directions are rotated with respect to the original directions by an angle θ which is equal to the local rotation of the material. This angle has been evaluated in section 2 and is given by expressions (2.13) and (2.17). Its approximate value to the first order is

$$\theta \cong \omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (4.9)$$

The stress components referred to these rotated axes are

$$\begin{aligned} \sigma_{11} &= S_{11} + s_{11} \\ \sigma_{22} &= S_{22} + s_{22} \\ \sigma_{12} &= S_{12} + s_{12} \end{aligned} \quad (4.10)$$

The quantities s_{11}, s_{22}, s_{12} are the increments of stress referred to axes which rotate with the medium. It is possible to express the stresses $\bar{\sigma}_{\xi\xi}, \bar{\sigma}_{\eta\eta}, \bar{\sigma}_{\xi\eta}$ in terms of the stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ by using the transformation formulas 4.6 in which we replace $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ by $\bar{\sigma}_{\xi\xi}, \bar{\sigma}_{\eta\eta}$, and $\bar{\sigma}_{\xi\eta}$, and the angle α by $\omega \cong \alpha$. We write

$$\begin{aligned} \bar{\sigma}_{\xi\xi} &= \sigma_{11} \cos^2 \omega + \sigma_{22} \sin^2 \omega - \sigma_{12} \sin 2\omega \\ \bar{\sigma}_{\eta\eta} &= \sigma_{11} \sin^2 \omega + \sigma_{22} \cos^2 \omega + \sigma_{12} \sin 2\omega \\ \bar{\sigma}_{\xi\eta} &= \frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\omega + \sigma_{12} \cos 2\omega \end{aligned} \quad (4.11)$$

We shall assume that the *incremental stresses and the rotation are quantities of the first order*.

To the first order we put

$$\begin{aligned} \cos \omega &= \cos 2\omega \cong 1 \\ \sin \omega &= \frac{1}{2} \sin 2\omega \cong \omega \end{aligned} \quad (4.12)$$

Substituting expressions (4.8) and (4.10) in equations 4.11 and retaining only first order quantities, we find

$$\begin{aligned} \bar{s}_{\xi\xi} &= s_{11} - 2S_{12}\omega \\ \bar{s}_{\eta\eta} &= s_{22} + 2S_{12}\omega \\ \bar{s}_{\xi\eta} &= s_{12} + (S_{11} - S_{22})\omega \end{aligned} \quad (4.13)$$

These equations bring out the terms representing that portion of the incremental stresses which is due to the rotation alone. The first terms s_{11} , s_{22} , and s_{12} represent the stress due to the deformation and depend only on the *physical properties of the material*.

5. INCREMENTAL STRESSES IN THREE DIMENSIONS

We shall extend the preceding definitions to a three-dimensional stress field and consider a state of stress represented by the components

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{matrix} \tag{5.1}$$

Let us cut out of the medium a small tetrahedron of sides OA , OB , OC parallel to the axes x , y , z , and such that the triangular face ABC has a unit area (Fig. 5.1).

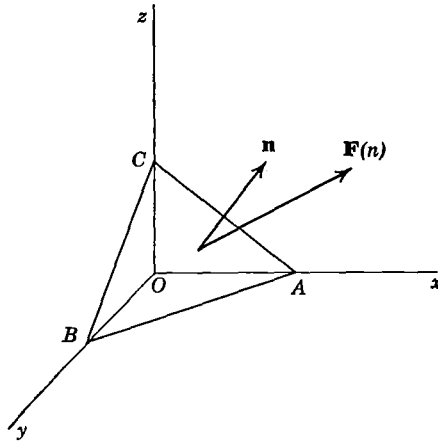


Figure 5.1 Force $\mathbf{F}(n)$ acting per unit area on a surface of normal direction \mathbf{n} in a stress field.

The orientation of the triangular face ABC is defined by a vector \mathbf{n} of unit length directed positively outward of the tetrahedron. The vector \mathbf{n} is called a *unit vector*. The cartesian components of this unit vector are the *directional cosines* of the direction \mathbf{n} . These

directional cosines are the cosines of the angle between the positive direction of \mathbf{n} and the three coordinate axes, i.e.,

$$\cos(n, x), \cos(n, y), \cos(n, z) \quad (5.2)$$

If the stress field (5.1) is acting in the tetrahedral element, from the condition of equilibrium of the element we may derive the force $\mathbf{F}(n)$ acting per unit area on the face ABC .

The cartesian components of this force are

$$\begin{aligned} F_x(n) &= \sigma_{xx} \cos(n, x) + \sigma_{xy} \cos(n, y) + \sigma_{zx} \cos(n, z) \\ F_y(n) &= \sigma_{xy} \cos(n, x) + \sigma_{yy} \cos(n, y) + \sigma_{yz} \cos(n, z) \\ F_z(n) &= \sigma_{zx} \cos(n, x) + \sigma_{yz} \cos(n, y) + \sigma_{zz} \cos(n, z) \end{aligned} \quad (5.3)$$

Because of the symmetry of the stress system (5.1) we may associate with these relations a quadratic form

$$\phi = \sigma_{xx}x^2 + \sigma_{yy}y^2 + \sigma_{zz}z^2 + 2\sigma_{xy}xy + 2\sigma_{yz}yz + 2\sigma_{zx}zx \quad (5.4)$$

If we identify the coordinates x, y, z with the directional cosines (5.2), we see that the vector \mathbf{F} is parallel to the gradient of ϕ , i.e., to the vector,

$$\text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \quad (5.5)$$

This gradient is normal to the quadric surface, called the *stress quadric*,*

$$\phi = \text{const.} \quad (5.6)$$

If the unit vector \mathbf{n} is directed along any one of the three axes of this quadric, the force $\mathbf{F}(n)$ is parallel to the vector \mathbf{n} , hence normal to the face ABC . We derive from this the existence of three *principal directions of stress*, i.e., directions for which the tangential components of stress $\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ vanish. The corresponding normal stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are the *principal stresses*.

The preceding equations also lead to expressions for the stress for a system of coordinate axes which are different from x, y, z . Let us consider a system of rectangular axes 1, 2, 3 with its origin at the same point as the original system x, y, z .

* For other properties of the stress quadric see Love's treatise, *The Mathematical Theory of Elasticity*, Fourth Edition, pp. 80-81, Cambridge University Press (reprinted by Dover Publications, New York, 1944).

The directional cosines of axes 1, 2, 3 relative to x, y, z are

$$\begin{aligned} \cos(1, x) \quad \cos(1, y) \quad \cos(1, z) \\ \cos(2, x) \quad \cos(2, y) \quad \cos(2, z) \\ \cos(3, x) \quad \cos(3, y) \quad \cos(3, z) \end{aligned} \quad (5.7)$$

We denote the stress components referred to the new axes by

$$\begin{aligned} \sigma_{11} \quad \sigma_{12} \quad \sigma_{31} \\ \sigma_{12} \quad \sigma_{22} \quad \sigma_{23} \\ \sigma_{31} \quad \sigma_{23} \quad \sigma_{33} \end{aligned} \quad (5.8)$$

The stress component σ_{11} , for instance, may be found by orienting the normal direction \mathbf{n} of the face ABC along axis 1. The force $\mathbf{F}(1)$ acting on this face is then projected on axis 1. We find

$$\sigma_{11} = F_x(1) \cos(1, x) + F_y(1) \cos(1, y) + F_z(1) \cos(1, z) \quad (5.9)$$

Substituting the values (5.3) for the components of $\mathbf{F}(1)$ yields

$$\begin{aligned} \sigma_{11} = \sigma_{xx} \cos^2(1, x) + \sigma_{yy} \cos^2(1, y) + \sigma_{zz} \cos^2(1, z) \\ + 2\sigma_{yz} \cos(1, y) \cos(1, z) + 2\sigma_{zx} \cos(1, z) \cos(1, x) \\ + 2\sigma_{xy} \cos(1, x) \cos(1, y) \end{aligned} \quad (5.10)$$

The other components are expressed in the same way; using the dummy index rule, we may write

$$\sigma_{\mu\nu} = \sigma_{ij} \cos(\mu, i) \cos(\nu, j) \quad (5.11)$$

We put $\sigma_{ij} = \sigma_{ji}$, hence also $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. Note that these expressions yield just as well the stress components (5.1) in terms of the components (5.8). This amounts to commuting the indices x, y, z with 1, 2, 3 in relations (5.10) and (5.11). Relations (5.11) then become

$$\sigma_{ij} = \sigma_{\mu\nu} \cos(i, \mu) \cos(j, \nu) \quad (5.12)$$

Explicitly this is written

$$\begin{aligned} \sigma_{xx} = \sigma_{11} \cos^2(x, 1) + \sigma_{22} \cos^2(x, 2) + \sigma_{33} \cos^2(x, 3) \\ + 2\sigma_{23} \cos(x, 2) \cos(x, 3) + 2\sigma_{31} \cos(x, 3) \cos(x, 1) \\ + 2\sigma_{12} \cos(x, 1) \cos(x, 2), \\ \text{etc.} \end{aligned} \quad (5.13)$$

We now go back to a three-dimensional deformation. The kinematics was analyzed in section 3. A point P originally of coordinates x, y, z is transported to a point P' of coordinates ξ, η, ζ .

The medium is under a state of initial stress. The components of initial stress at point P are

$$\begin{array}{ccc} S_{11} & S_{12} & S_{31} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{array} \quad (5.14)$$

At point $P'(\xi, \eta, \zeta)$ after deformation the stresses referred to axes parallel to x, y, z become

$$\begin{array}{ll} \bar{\sigma}_{\xi\xi} = S_{11} + \bar{s}_{\xi\xi} & \bar{\sigma}_{\eta\zeta} = S_{23} + \bar{s}_{\eta\zeta} \\ \bar{\sigma}_{\eta\eta} = S_{22} + \bar{s}_{\eta\eta} & \bar{\sigma}_{\zeta\xi} = S_{31} + \bar{s}_{\zeta\xi} \\ \bar{\sigma}_{\zeta\zeta} = S_{33} + \bar{s}_{\zeta\zeta} & \bar{\sigma}_{\xi\eta} = S_{12} + \bar{s}_{\xi\eta} \end{array} \quad (5.15)$$

Following the procedure of section 4 for two-dimensional stresses, we shall refer the stresses to rectangular axes 1, 2, 3 obtained by rotating locally with the material a rectangular system ξ, η, ζ originally parallel to x, y, z and with its origin at the displaced point P' . The rotation is defined to the first order by the vector $\omega_x, \omega_y, \omega_z$ as given by expressions (3.10). As in section 3 where we discussed the kinematics of three-dimensional strain, an infinitesimal vector $d\xi, d\eta, d\zeta$ in the unrotated coordinate system ξ, η, ζ is represented by the components $d\xi', d\eta', d\zeta'$ in the rotated axes 1, 2, 3. The relation between those two vectors is given to the *first order* by the equations

$$\begin{array}{l} d\xi = d\xi' - \omega_z d\eta' + \omega_y d\zeta' \\ d\eta = \omega_z d\xi' + d\eta' - \omega_x d\zeta' \\ d\zeta = -\omega_y d\xi' + \omega_x d\eta' + d\zeta' \end{array} \quad (5.16)$$

These equations are derived from the kinematics of rigid bodies. They yield the displacement field for a small solid rotation represented by the vector $\omega_x, \omega_y, \omega_z$. On the other hand, transformation (5.16) may also be considered a coordinate transformation from the axes 1, 2, 3 to axes ξ, η, ζ . The change of coordinates is represented by the equations

$$\begin{array}{l} d\xi = d\xi' \cos(\xi, 1) + d\eta' \cos(\xi, 2) + d\zeta' \cos(\xi, 3) \\ d\eta = d\xi' \cos(\eta, 1) + d\eta' \cos(\eta, 2) + d\zeta' \cos(\eta, 3) \\ d\zeta = d\xi' \cos(\zeta, 1) + d\eta' \cos(\zeta, 2) + d\zeta' \cos(\zeta, 3) \end{array} \quad (5.17)$$

Comparing relations (5.16) and (5.17), we derive the following first order approximation for the directional cosines.

$$\begin{aligned} \cos(\xi, 1) &= 1 & \cos(\xi, 2) &= -\omega_z & \cos(\xi, 3) &= \omega_y \\ \cos(\eta, 1) &= \omega_z & \cos(\eta, 2) &= 1 & \cos(\eta, 3) &= -\omega_x \\ \cos(\zeta, 1) &= -\omega_y & \cos(\zeta, 2) &= \omega_x & \cos(\zeta, 3) &= 1 \end{aligned} \quad (5.18)$$

The stress components referred to the locally rotated axes are denoted by

$$\begin{aligned} \sigma_{11} &= S_{11} + s_{11} & \sigma_{23} &= S_{23} + s_{23} \\ \sigma_{22} &= S_{22} + s_{22} & \sigma_{31} &= S_{31} + s_{31} \\ \sigma_{33} &= S_{33} + s_{33} & \sigma_{12} &= S_{12} + s_{12} \end{aligned} \quad (5.19)$$

The quantities s_{11} , s_{22} , etc., now designate the stress increments relative to the rotated axes. The transformation relations from one set of stresses to the other are easily established by applying the results obtained above. We use relations (5.12) and (5.13), replacing x , y , z by ξ , η , ζ and $\sigma_{xx}\sigma_{xy}$, etc., by $\bar{\sigma}_{\xi\xi}\bar{\sigma}_{\xi\eta}$, etc. We then substitute in these relations the approximate values (5.18) for the directional cosines, and expressions (5.19) for the stresses σ_{11} , σ_{22} , etc. Retaining only quantities of the first order in s_{11} , s_{22} , etc., and ω_x , ω_y , ω_z , we find

$$\begin{aligned} \bar{\sigma}_{\xi\xi} &= S_{11} + s_{11} + 2S_{31}\omega_y - 2S_{12}\omega_z \\ \bar{\sigma}_{\eta\eta} &= S_{22} + s_{22} + 2S_{12}\omega_z - 2S_{23}\omega_x \\ \bar{\sigma}_{\zeta\zeta} &= S_{33} + s_{33} + 2S_{23}\omega_x - 2S_{31}\omega_y \\ \bar{\sigma}_{\eta\zeta} &= S_{23} + s_{23} + (S_{22} - S_{33})\omega_x - S_{12}\omega_y + S_{31}\omega_z \\ \bar{\sigma}_{\zeta\xi} &= S_{31} + s_{31} + (S_{33} - S_{11})\omega_y - S_{23}\omega_z + S_{12}\omega_x \\ \bar{\sigma}_{\xi\eta} &= S_{12} + s_{12} + (S_{11} - S_{22})\omega_z - S_{31}\omega_x + S_{23}\omega_y \end{aligned} \quad (5.20)$$

In abbreviated notation equations 5.20 may be written

$$\bar{\sigma}_{ij} = S_{ij} + s_{ij} + S_{\mu j}\omega_{i\mu} + S_{i\mu}\omega_{j\mu} \quad (5.21)$$

In this expression we designate by S_{ij} the initial stress components (5.14) with the convention $S_{ij} = S_{ji}$ (also $\bar{\sigma}_{ij} = \bar{\sigma}_{ji}$). Note that the subscripts i, j stand for ξ, η, ζ on the left side and for 1, 2, 3 on the right side. The quantities ω_{ij} are defined by the elements of the matrix (3.21).

The term $S_{\mu j}\omega_{i\mu} + S_{i\mu}\omega_{j\mu}$ in equation 5.21 represents that portion

of the incremental stresses due to the rotation alone. The term s_{ij} represents the stress increment due to the deformation and depends therefore only on the *physical properties of the material*.

6. EQUILIBRIUM EQUATIONS FOR THE STRESS FIELD IN TWO DIMENSIONS

We now establish the equations which must be verified by the incremental stress field under the condition of static equilibrium.

For the sake of clarity we proceed first with the analysis of a two-dimensional field. In the plane x, y we consider a body outlined by a contour C . The initial stresses in the body are S_{11}, S_{22}, S_{12} . If X and Y are the components of the body force per unit mass, and if ρ is the mass density of the medium before deformation, the initial stress components must satisfy the well-known equilibrium conditions

$$\begin{aligned} \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \rho X(x, y) &= 0 \\ \frac{\partial S_{12}}{\partial x} + \frac{\partial S_{22}}{\partial y} + \rho Y(x, y) &= 0 \end{aligned} \tag{6.1}$$

We have assumed here that the body force per unit mass is a fixed field in space, a function only of the coordinates x, y . In practice this is generally the case; however, there are exceptions as, for instance, in some geophysical problems where the gravity field depends also on the deformation itself. For simplicity we exclude

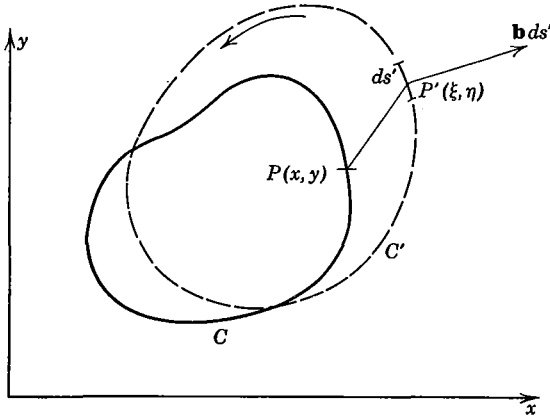


Figure 6.1 Forces on the boundary C' of a deformed body.

this case for the present. It may, however, be included in the equations, and we shall indicate briefly in the next section how this may be done.

A point P of the material originally of coordinates x, y moves to a point P' of coordinates ξ, η after deformation (Fig. 6.1). We denote by b_x and b_y the x and y components of the force \mathbf{b} acting at point P' of the boundary *per unit area after deformation*. We look upon this force as that acting on the solid inside the contour C' . The force on a line element ds' of the contour C' is $\mathbf{b} ds'$, where ds' is chosen positive in a counterclockwise direction on the contour C' . With these definitions and by considering the equilibrium of a triangular element adjacent to the boundary as shown in Figure 6.2, we may write

$$\begin{aligned} b_x ds' &= \bar{\sigma}_{\xi\xi} d\eta - \bar{\sigma}_{\xi\eta} d\xi \\ b_y ds' &= \bar{\sigma}_{\xi\eta} d\eta - \bar{\sigma}_{\eta\eta} d\xi \end{aligned} \quad (6.2)$$

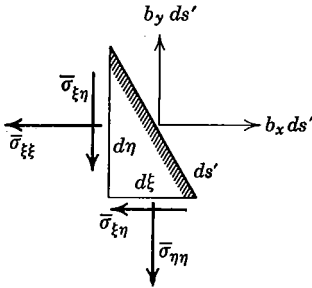


Figure 6.2 Boundary forces and stresses on a boundary element.

This force is the external force acting at the boundary on the solid lying inside the contour C' . The other external force acting on this solid is the body force. Consider an element of the solid of area dS at point x, y and of mass density ρ before deformation. After deformation it has moved to a point of coordinates ξ, η , its area has become dS' , and its new density is now ρ' . Because of the law of conservation of mass we may write

$$\rho dx dy = \rho dS = \rho' dS' \quad (6.3)$$

Let us now write the condition of equilibrium for the solid inside the contour by stating that the resultant of the boundary forces and the body forces acting on the solid vanishes. This condition is

$$\begin{aligned} \int_{C'} b_x ds' + \int \int_{S'} X(\xi, \eta) \rho' dS' &= 0 \\ \int_{C'} b_y ds' + \int \int_{S'} Y(\xi, \eta) \rho' dS' &= 0 \end{aligned} \quad (6.4)$$

The contour integrations are performed counterclockwise. We may

change the variables of integration in these integrals to x and y by using transformation (2.24); that is,

$$\begin{aligned} d\xi &= \left(1 + \frac{\partial u}{\partial x}\right) dx + \frac{\partial u}{\partial y} dy \\ d\eta &= \frac{\partial v}{\partial x} dx + \left(1 + \frac{\partial v}{\partial y}\right) dy \end{aligned} \quad (6.5)$$

With these expressions, relations (6.2) become

$$b_x ds' = \left[\bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial x} - \bar{\sigma}_{\xi\eta} \left(1 + \frac{\partial u}{\partial x}\right) \right] dx + \left[\bar{\sigma}_{\xi\xi} \left(1 + \frac{\partial v}{\partial y}\right) - \bar{\sigma}_{\xi\eta} \frac{\partial u}{\partial y} \right] dy \quad (6.6)$$

$$b_y ds' = \left[\bar{\sigma}_{\xi\eta} \frac{\partial v}{\partial x} - \bar{\sigma}_{\eta\eta} \left(1 + \frac{\partial u}{\partial x}\right) \right] dx + \left[\bar{\sigma}_{\xi\eta} \left(1 + \frac{\partial v}{\partial y}\right) - \bar{\sigma}_{\eta\eta} \frac{\partial u}{\partial y} \right] dy$$

We substitute these expressions in the equilibrium conditions (6.4), also replacing $\rho' ds'$ by $\rho dx dy$ according to relations (6.3). By applying Green's theorem to the contour integrals, they are transformed to surface integrals and equations 6.4 become

$$\begin{aligned} \iint_S \left\{ \frac{\partial}{\partial x} \left[\bar{\sigma}_{\xi\xi} \left(1 + \frac{\partial v}{\partial y}\right) - \bar{\sigma}_{\xi\eta} \frac{\partial u}{\partial y} \right] \right. \\ \left. - \frac{\partial}{\partial y} \left[\bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial x} - \bar{\sigma}_{\xi\eta} \left(1 + \frac{\partial u}{\partial x}\right) \right] + X(\xi, \eta) \rho \right\} dx dy = 0 \\ \iint_S \left\{ \frac{\partial}{\partial x} \left[\bar{\sigma}_{\xi\eta} \left(1 + \frac{\partial v}{\partial y}\right) - \bar{\sigma}_{\eta\eta} \frac{\partial u}{\partial y} \right] \right. \\ \left. - \frac{\partial}{\partial y} \left[\bar{\sigma}_{\xi\eta} \frac{\partial v}{\partial x} - \bar{\sigma}_{\eta\eta} \left(1 + \frac{\partial u}{\partial x}\right) \right] + Y(\xi, \eta) \rho \right\} dx dy = 0 \end{aligned} \quad (6.7)$$

Since these relations must be verified for any arbitrary contour C , i.e., for any arbitrary domain of integration S , the integrands must vanish. Therefore the equilibrium condition of the stress field becomes the differential equations

$$\begin{aligned} \frac{\partial \bar{\sigma}_{\xi\xi}}{\partial x} + \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial y} + \frac{\partial}{\partial x} \left(\bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial y} - \bar{\sigma}_{\xi\eta} \frac{\partial u}{\partial y} \right) \\ - \frac{\partial}{\partial y} \left(\bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial x} - \bar{\sigma}_{\xi\eta} \frac{\partial u}{\partial x} \right) + X(\xi, \eta) \rho = 0 \\ \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial x} + \frac{\partial \bar{\sigma}_{\eta\eta}}{\partial y} + \frac{\partial}{\partial x} \left(\bar{\sigma}_{\xi\eta} \frac{\partial v}{\partial y} - \bar{\sigma}_{\eta\eta} \frac{\partial u}{\partial y} \right) \\ - \frac{\partial}{\partial y} \left(\bar{\sigma}_{\xi\eta} \frac{\partial v}{\partial x} - \bar{\sigma}_{\eta\eta} \frac{\partial u}{\partial x} \right) + Y(\xi, \eta) \rho = 0 \end{aligned} \quad (6.8)$$

These equations do not involve any approximation. They include the conditions of equilibrium (6.1) for the initial stress since they reduce to these equations for $u = v = 0$. Note also that the quantity ρ in these equations is the mass density *before deformation*, hence is a given function of x and y .

At this point we are interested in introducing stress components which depend only on the strain and do not change when we superimpose a solid rotation on the medium. Such stress components were introduced in section 4. From equations 4.8 and 4.13 we write

$$\begin{aligned}\bar{\sigma}_{\xi\xi} &= S_{11} + s_{11} - 2S_{12}\omega \\ \bar{\sigma}_{\eta\eta} &= S_{22} + s_{22} + 2S_{12}\omega \\ \bar{\sigma}_{\xi\eta} &= S_{12} + s_{12} + (S_{11} - S_{22})\omega\end{aligned}\quad (6.9)$$

The stress components s_{11} , s_{22} , s_{12} are the incremental stresses projected on axes which rotate with the material by an angle ω defined by expression 2.26. The equations 6.9 are *approximate to the first order*. We shall now substitute these values in equations 6.8 and take into account the equilibrium conditions (6.1) for the initial stress. After these substitutions, keeping only the terms of the first order, we find

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho[X(\xi, \eta) - X(x, y)] \\ - 2 \frac{\partial}{\partial x} (S_{12}\omega) + \frac{\partial}{\partial y} [(S_{11} - S_{22})\omega] \\ + \frac{\partial}{\partial x} \left(S_{11} \frac{\partial v}{\partial y} - S_{12} \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(S_{11} \frac{\partial v}{\partial x} - S_{12} \frac{\partial u}{\partial x} \right) = 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho[Y(\xi, \eta) - Y(x, y)] \\ + \frac{\partial}{\partial x} [(S_{11} - S_{22})\omega] + 2 \frac{\partial}{\partial y} (S_{12}\omega) \\ + \frac{\partial}{\partial x} \left(S_{12} \frac{\partial v}{\partial y} - S_{22} \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(S_{12} \frac{\partial v}{\partial x} - S_{22} \frac{\partial u}{\partial x} \right) = 0\end{aligned}\quad (6.10)$$

In these equations we may write the incremental body force as

$$\begin{aligned}\Delta X &= X(\xi, \eta) - X(x, y) \\ \Delta Y &= Y(\xi, \eta) - Y(x, y)\end{aligned}\quad (6.11)$$

Equations 6.10 may be transformed by making use of identities derived from equations 2.26 and 2.27:

$$\begin{aligned}\frac{\partial u}{\partial x} &= e_{xx} & \frac{\partial v}{\partial y} &= e_{yy} \\ \frac{\partial v}{\partial x} &= e_{xy} + \omega & \frac{\partial u}{\partial y} &= e_{xy} - \omega\end{aligned}\quad (6.12)$$

When we introduce these expressions into equations 6.10, they become

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \Delta X - \frac{\partial}{\partial x} (S_{12}\omega) - \frac{\partial}{\partial y} (S_{22}\omega) \\ + \frac{\partial}{\partial x} (S_{11}e_{yy} - S_{12}e_{xy}) - \frac{\partial}{\partial y} (S_{11}e_{xy} - S_{12}e_{xx}) = 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho \Delta Y + \frac{\partial}{\partial x} (S_{11}\omega) + \frac{\partial}{\partial y} (S_{12}\omega) \\ + \frac{\partial}{\partial x} (S_{12}e_{yy} - S_{22}e_{xy}) - \frac{\partial}{\partial y} (S_{12}e_{xy} - S_{22}e_{xx}) = 0\end{aligned}\quad (6.13)$$

These equations may be further simplified if we take into account the following identities derived from (6.12).

$$\begin{aligned}\frac{\partial e_{xx}}{\partial y} &= \frac{\partial}{\partial x} (e_{xy} - \omega) \\ \frac{\partial e_{yy}}{\partial x} &= \frac{\partial}{\partial y} (e_{xy} + \omega)\end{aligned}\quad (6.14)$$

Introducing these identities into equations 6.13 and again taking into account the equilibrium conditions (6.1) for the initial stress, they are transformed to

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \Delta X + \rho \omega Y(x, y) - 2S_{12} \frac{\partial \omega}{\partial x} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} \\ + \frac{\partial S_{11}}{\partial x} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{12}}{\partial x} \right) e_{xy} + \frac{\partial S_{12}}{\partial y} e_{xx} = 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho \Delta Y - \rho \omega X(x, y) + 2S_{12} \frac{\partial \omega}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} \\ + \frac{\partial S_{22}}{\partial y} e_{xx} - \left(\frac{\partial S_{22}}{\partial x} + \frac{\partial S_{12}}{\partial y} \right) e_{xy} + \frac{\partial S_{12}}{\partial x} e_{yy} = 0\end{aligned}\quad (6.15)$$

These equations are the equilibrium conditions for the two-dimensional stress field expressed in terms of the incremental stresses s_{11} , s_{22} , s_{12} . Equations 6.15 were first derived by the author in 1938.* It was also shown that the various terms have a simple physical interpretation, as explained at the end of this section.

If we again take into account the equilibrium conditions (6.1) of the initial stress field, equations 6.15 may be written in an alternative and more symmetric form:

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \Delta X + \rho \omega Y(x, y) - \rho e X(x, y) \\ - 2S_{12} \frac{\partial \omega}{\partial x} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} \\ - \frac{\partial S_{11}}{\partial x} e_{xx} - \frac{\partial S_{12}}{\partial y} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{12}}{\partial x} \right) e_{xy} = 0 \end{aligned} \quad (6.16)$$

$$\begin{aligned} \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho \Delta Y - \rho \omega X(x, y) - \rho e Y(x, y) \\ + 2S_{12} \frac{\partial \omega}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} \\ - \frac{\partial S_{22}}{\partial y} e_{yy} - \frac{\partial S_{12}}{\partial x} e_{xx} - \left(\frac{\partial S_{22}}{\partial x} + \frac{\partial S_{12}}{\partial y} \right) e_{xy} = 0 \end{aligned}$$

We have put $e = e_{xx} + e_{yy}$.

Some interesting properties of these equations are immediately apparent. If there is no body force ($X = Y = 0$) and if the initial state of stress is uniform, i.e., independent of x and y , equations 6.16 assume the simpler form

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - 2S_{12} \frac{\partial \omega}{\partial x} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} = 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + 2S_{12} \frac{\partial \omega}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} = 0 \end{aligned} \quad (6.17)$$

If the initial stress is hydrostatic, i.e., if

$$S_{11} = S_{22} \quad S_{12} = 0$$

* In reference 2 at the end of the Preface.

equations 6.17 reduce to

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} &= 0 \end{aligned} \tag{6.18}$$

which are the classical conditions of equilibrium for a stress field when there is no initial stress. The same equations (6.18) are obtained for a solid body rotation, i.e., for

$$\omega = \text{const.} \tag{6.19}$$

In the solution of specific problems we must be able to express certain boundary conditions. Let us therefore turn our attention to the force acting at the boundary. Consider a portion AB of the

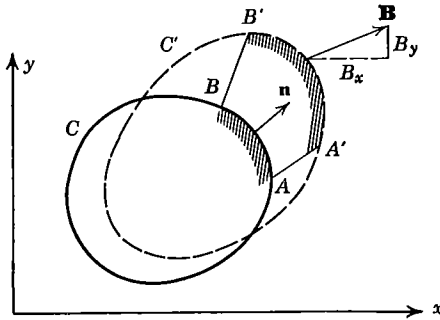


Figure 6.3 Integrated boundary force \mathbf{B} on a finite portion $A'B'$ of the deformed boundary.

contour C (Fig. 6.3). After deformation it becomes the part of the contour C' designated $A'B'$. The external force \mathbf{B} acting on the solid boundary $A'B'$ after deformation has the cartesian components

$$\begin{aligned} B_x &= \int_{A'}^{B'} b_x ds' = \int_{A'}^{B'} (\bar{\sigma}_{\xi\xi} d\eta - \bar{\sigma}_{\xi\eta} d\xi) \\ B_y &= \int_{A'}^{B'} b_y ds' = \int_{A'}^{B'} (\bar{\sigma}_{\xi\eta} d\eta - \bar{\sigma}_{\eta\eta} d\xi) \end{aligned} \tag{6.20}$$

These expressions are derived from equations 6.2. The integration is performed in the counterclockwise direction along the deformed line $A'B'$, and the solid is lying to the left when one moves from A' to

B' . By relations (6.6) these integrals may be transformed to line integrals along the line AB lying on the original contour before deformation. Substituting relations (6.6) in the integrals, we find

$$\begin{aligned}
 B_x &= \int_A^B \left[\bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial x} - \bar{\sigma}_{\xi n} \left(1 + \frac{\partial u}{\partial x} \right) \right] dx \\
 &\quad + \int_A^B \left[\bar{\sigma}_{\xi\xi} \left(1 + \frac{\partial v}{\partial y} \right) - \bar{\sigma}_{\xi n} \frac{\partial u}{\partial y} \right] dy \\
 B_y &= \int_A^B \left[\bar{\sigma}_{\xi n} \frac{\partial v}{\partial x} - \bar{\sigma}_{nn} \left(1 + \frac{\partial u}{\partial x} \right) \right] dx \\
 &\quad + \int_A^B \left[\bar{\sigma}_{\xi n} \left(1 + \frac{\partial v}{\partial y} \right) - \bar{\sigma}_{nn} \frac{\partial u}{\partial y} \right] dy
 \end{aligned} \tag{6.21}$$

If the line AB is an *element of the initial contour* of components dx, dy , the force $d\mathbf{B}$ acting on this element after deformation is represented by the components

$$\begin{aligned}
 dB_x &= \left(-\bar{\sigma}_{\xi n} + \bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial x} - \bar{\sigma}_{\xi n} \frac{\partial u}{\partial x} \right) dx \\
 &\quad + \left(\bar{\sigma}_{\xi\xi} + \bar{\sigma}_{\xi\xi} \frac{\partial v}{\partial y} - \bar{\sigma}_{\xi n} \frac{\partial u}{\partial y} \right) dy \\
 dB_y &= \left(-\bar{\sigma}_{nn} + \bar{\sigma}_{\xi n} \frac{\partial v}{\partial x} - \bar{\sigma}_{nn} \frac{\partial u}{\partial x} \right) dx \\
 &\quad + \left(\bar{\sigma}_{\xi n} + \bar{\sigma}_{\xi n} \frac{\partial v}{\partial y} - \bar{\sigma}_{nn} \frac{\partial u}{\partial y} \right) dy
 \end{aligned} \tag{6.22}$$

The boundary force may be expressed in terms of the stresses referred to the rotated axes by substituting expressions (6.9) into equations 6.22. In doing so we retain only the first order terms. We also substitute expressions (6.12) for the partial derivatives. We find

$$\begin{aligned}
 dB_x &= -(S_{12} + s_{12} - S_{22}\omega - S_{11}e_{xy} + S_{12}e_{xx}) dx \\
 &\quad + (S_{11} + s_{11} - S_{12}\omega + S_{11}e_{yy} - S_{12}e_{xy}) dy \\
 dB_y &= -(S_{22} + s_{22} + S_{12}\omega - S_{12}e_{xy} + S_{22}e_{xx}) dx \\
 &\quad + (S_{12} + s_{12} + S_{11}\omega + S_{12}e_{yy} - S_{22}e_{xy}) dy
 \end{aligned} \tag{6.23}$$

We note that the differential dx, dy must represent an element of arc positive counterclockwise, the solid lying on the left side of the element. These expressions may also be given an equivalent form

by introducing a unit vector \mathbf{n} normal to the original contour and chosen positive in a direction away from the solid (Fig. 6.3). We have the relations

$$\begin{aligned} -dx &= ds \cos(n, y) \\ dy &= ds \cos(n, x) \end{aligned} \quad (6.24)$$

where $\cos(n, x)$, $\cos(n, y)$ are the directional cosines of the direction \mathbf{n} normal to the original contour and ds is the absolute value of the element of arc,

$$ds = \sqrt{dx^2 + dy^2}$$

Furthermore we define a force \mathbf{f} of components

$$\begin{aligned} f_x &= \frac{dB_x}{ds} \\ f_y &= \frac{dB_y}{ds} \end{aligned} \quad (6.25)$$

This is the boundary force *per unit initial area of the boundary*. With these definitions we may write the boundary force as

$$\begin{aligned} f_x &= (S_{11} + s_{11} - S_{12}\omega + S_{11}e_{yy} - S_{12}e_{xy}) \cos(n, x) \\ &\quad + (S_{12} + s_{12} - S_{22}\omega - S_{11}e_{xy} + S_{12}e_{xx}) \cos(n, y) \\ f_y &= (S_{12} + s_{12} + S_{11}\omega + S_{12}e_{yy} - S_{22}e_{xy}) \cos(n, x) \\ &\quad + (S_{22} + s_{22} + S_{12}\omega - S_{12}e_{xy} + S_{22}e_{xx}) \cos(n, y) \end{aligned} \quad (6.26)$$

It is also possible to introduce *incremental boundary forces*, i.e., the difference between the actual boundary forces after deformation and their initial value before deformation. Expressed per unit initial area, these incremental boundary forces are

$$\begin{aligned} \Delta f_x &= (s_{11} - S_{12}\omega + S_{11}e_{yy} - S_{12}e_{xy}) \cos(n, x) \\ &\quad + (s_{12} - S_{22}\omega - S_{11}e_{xy} + S_{12}e_{xx}) \cos(n, y) \\ \Delta f_y &= (s_{12} + S_{11}\omega + S_{12}e_{yy} - S_{22}e_{xy}) \cos(n, x) \\ &\quad + (s_{22} + S_{12}\omega - S_{12}e_{xy} + S_{22}e_{xx}) \cos(n, y) \end{aligned} \quad (6.27)$$

We shall end this section with a few remarks on the significance of these results. Let us first look at the incremental body force ΔX , ΔY defined by equations 6.11. These expressions represent the change in body force per unit mass when we move from the original point x, y to the displaced point ξ, η after deformation. If this body force

is represented by a fixed field, function only of the coordinates, we may linearize expressions (6.11) and write

$$\begin{aligned}\Delta X &= \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) X(x, y) \\ \Delta Y &= \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) Y(x, y)\end{aligned}\tag{6.28}$$

At the beginning of this section we referred to the possibility that the field may depend not only on the fixed coordinates but also on the deformation itself. This might occur, for instance, if we were interested in the deformation of a large gravitational body such as a planet. In this case the configuration of the gravitational field would depend on the deformation itself. Additional terms $\Delta' X$, $\Delta' Y$ must then be added to represent the latter contribution to the incremental body force, and we should write

$$\begin{aligned}\Delta X &= \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) X(x, y) + \Delta' X \\ \Delta Y &= \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) Y(x, y) + \Delta' Y\end{aligned}\tag{6.29}$$

These additional terms can be evaluated only by solving the complete problem. In many applications when we are dealing with a uniform gravity field the incremental body force vanishes altogether.

Our next remark deals with the physical significance of equations 6.15. Let us rewrite the first of equations 6.15.

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \Delta X + \rho \omega Y(x, y) \\ - 2S_{12} \frac{\partial \omega}{\partial x} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} \\ + \frac{\partial S_{11}}{\partial x} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{22}}{\partial x} \right) e_{xy} + \frac{\partial S_{12}}{\partial y} e_{xx} = 0\end{aligned}\tag{6.30}$$

Let us look at the terms on the second line. They contain $\partial\omega/\partial x$ and $\partial\omega/\partial y$ and are different from zero only if the deformation is inhomogeneous. In the original publications they were referred to as the *curvature terms*.* By contrast the terms on the third line are

* See references 2, 3, and 4 at the end of the Preface.

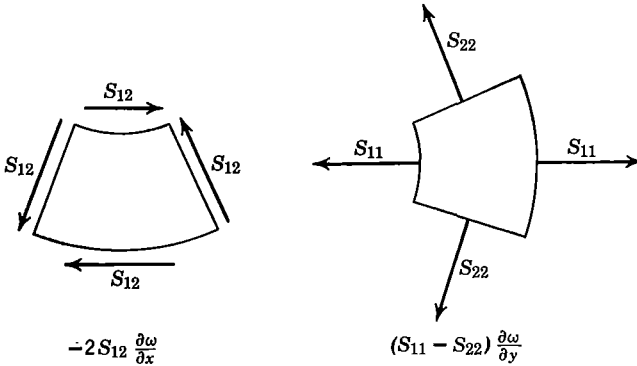


Figure 6.4 Physical interpretation of the “curvature terms.”

different from zero only if the initial stress is inhomogeneous. The physical significance of the curvature terms is illustrated by Figure 6.4. The horizontal resultant of the forces indicated in the figure are due to the curvatures and changes of areas of a deformed element. Note also that equation 6.30 is an *equilibrium condition referred to locally rotated axes*. This is consistent with the appearance of the term $\rho\omega Y$ which represents a projection of the body force on rotated axes. Equations 6.15 may therefore be considered an *intrinsic* form of the local equilibrium conditions.

Attention should also be called to the significance of equations 6.8. They could have been derived exactly by writing the equilibrium conditions for the stress field in terms of the coordinates ξ, η . These conditions are

$$\begin{aligned} \frac{\partial \bar{\sigma}_{\xi\xi}}{\partial \xi} + \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial \eta} + \rho'(\xi, \eta)X(\xi, \eta) &= 0 \\ \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial \xi} + \frac{\partial \bar{\sigma}_{\eta\eta}}{\partial \eta} + \rho'(\xi, \eta)Y(\xi, \eta) &= 0 \end{aligned} \tag{6.31}$$

The unknown mass density after deformation is ρ' .

We may transform these equations by using the differential relations (6.5) and express all partial derivatives $\partial/\partial\xi, \partial/\partial\eta$ in terms of $\partial/\partial x$ and $\partial/\partial y$. If we perform this transformation in equations 6.31, we obtain equations 6.8. This method of derivation was used in some of the earlier work, and it further illustrates the significance

of the formulas. However, the method used above has the advantage of providing at the same time suitable expressions for the boundary conditions.

7. EQUILIBRIUM EQUATIONS FOR THE STRESS FIELD IN THREE DIMENSIONS

The analysis of the equilibrium conditions for the two-dimensional field presented above may be extended to the three-dimensional case by following a similar procedure.

We start with a state of initial stress represented by the components (5.14). They satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \frac{\partial S_{31}}{\partial z} + \rho X(x, y, z) &= 0 \\ \frac{\partial S_{12}}{\partial x} + \frac{\partial S_{22}}{\partial y} + \frac{\partial S_{23}}{\partial z} + \rho Y(x, y, z) &= 0 \\ \frac{\partial S_{31}}{\partial x} + \frac{\partial S_{23}}{\partial y} + \frac{\partial S_{33}}{\partial z} + \rho Z(x, y, z) &= 0 \end{aligned} \quad (7.1)$$

The components of the body force per unit mass are X, Y, Z , and the mass density at the point x, y, z , is $\rho(x, y, z)$.

Consider a volume V bounded by a surface S before deformation. After deformation this surface becomes S' , and it encloses a volume V' . The x component B_x of the force acting on the boundary S' is

$$B_x = \iint_{S'} (\bar{\sigma}_{\xi\xi} d\eta d\zeta + \bar{\sigma}_{\xi\eta} d\zeta d\xi + \bar{\sigma}_{\xi\zeta} d\xi d\eta) \quad (7.2)$$

The surface integral is extended to the boundary after deformation, and $\bar{\sigma}_{\xi\xi}, \bar{\sigma}_{\xi\eta}, \bar{\sigma}_{\xi\zeta}$ are stress components at a point of coordinates ξ, η, ζ . We remember that the symmetry of stress components implies $\bar{\sigma}_{\xi\zeta} = \bar{\sigma}_{\zeta\xi}, \bar{\sigma}_{\eta\xi} = \bar{\sigma}_{\xi\eta}$, and $\bar{\sigma}_{\zeta\eta} = \bar{\sigma}_{\eta\zeta}$. We shall therefore pay no attention to the order of the indices and choose it for convenience. The variables of integration in equation 7.2 may be changed to the original coordinates x, y, z . The two sets of variables are related by relations (3.1)

$$\begin{aligned} \xi &= x + u \\ \eta &= y + v \\ \zeta &= z + w \end{aligned} \quad (7.3)$$

By known methods of transformation the integral B_x becomes

$$\begin{aligned}
 B_x = & \iint_S \bar{\sigma}_{\xi\zeta} \left[\frac{d(\eta, \zeta)}{d(y, z)} dy dz + \frac{d(\eta, \zeta)}{d(z, x)} dz dx + \frac{d(\eta, \zeta)}{d(x, y)} dx dy \right] \\
 & + \iint_S \bar{\sigma}_{\xi\eta} \left[\frac{d(\zeta, \xi)}{d(y, z)} dy dz + \frac{d(\zeta, \xi)}{d(z, x)} dz dx + \frac{d(\zeta, \xi)}{d(x, y)} dx dy \right] \\
 & + \iint_S \bar{\sigma}_{\zeta\xi} \left[\frac{d(\xi, \eta)}{d(y, z)} dy dz + \frac{d(\xi, \eta)}{d(z, x)} dz dx + \frac{d(\xi, \eta)}{d(x, y)} dx dy \right] \quad (7.4)
 \end{aligned}$$

The surface integrals are now extended to the same *material boundary* S before deformation. In this expression $[d(\eta, \zeta)]/[d(y, z)]$, etc., are the partial Jacobians of the transformation of x, y, z into ξ, η, ζ . For instance, we write

$$\frac{d(\eta, \zeta)}{d(y, z)} = \begin{vmatrix} \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{vmatrix} \quad (7.5)$$

These Jacobians are the cofactors of the determinant of the differential transformation (3.3). In order to abbreviate the writing let us put

$$\begin{aligned}
 A_{xx} &= \bar{\sigma}_{\xi\zeta} \frac{d(\eta, \zeta)}{d(y, z)} + \bar{\sigma}_{\xi\eta} \frac{d(\zeta, \xi)}{d(y, z)} + \bar{\sigma}_{\zeta\xi} \frac{d(\xi, \eta)}{d(y, z)} \\
 A_{xy} &= \bar{\sigma}_{\xi\zeta} \frac{d(\eta, \zeta)}{d(z, x)} + \bar{\sigma}_{\xi\eta} \frac{d(\zeta, \xi)}{d(z, x)} + \bar{\sigma}_{\zeta\xi} \frac{d(\xi, \eta)}{d(z, x)} \\
 A_{xz} &= \bar{\sigma}_{\xi\zeta} \frac{d(\eta, \zeta)}{d(x, y)} + \bar{\sigma}_{\xi\eta} \frac{d(\zeta, \xi)}{d(x, y)} + \bar{\sigma}_{\zeta\xi} \frac{d(\xi, \eta)}{d(x, y)}
 \end{aligned} \quad (7.6)$$

The surface integral (7.4) is then written

$$B_x = \iint_S (A_{xx} dy dz + A_{xy} dz dx + A_{xz} dx dy) \quad (7.7)$$

The mass density ρ at a point x, y, z before deformation becomes ρ' after deformation at the displaced point ξ, η, ζ . The x component of the resultant body force acting on the volume V' is

$$\iiint_{V'} X(\xi, \eta, \zeta) \rho' dV' = \iiint_V X(\xi, \eta, \zeta) \rho dV \quad (7.8)$$

This equation results from the conservation of mass, namely,

$$\rho' dV' = \rho dV \quad (7.9)$$

The total external force acting in the x direction is the sum of expressions (7.7) and (7.8). For equilibrium it must vanish; hence

$$\iint_S A_{xx} dy dz + A_{xy} dz dx + A_{xz} dx dy + \iiint_V X(\xi, \eta, \zeta)\rho dV = 0 \quad (7.10)$$

The surface integral is transformed to a volume integral by Green's theorem, and equation 7.10 becomes

$$\iiint_V \left[\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z} + X(\xi, \eta, \zeta)\rho \right] dV = 0 \quad (7.11)$$

Since this equation must be satisfied for any arbitrary volume V , it implies the differential equation

$$\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z} + X(\xi, \eta, \zeta)\rho = 0 \quad (7.12)$$

There are two other such equations for the y and z directions; they are obtained by cyclic permutation of the coordinate axes.

These equilibrium conditions (7.12) for the stress field contain the initial state as a particular case. Putting $u = v = w = 0$, they coincide with the equilibrium equations (7.1) for the initial stress.

We shall now introduce first order approximations in equations 7.12. To do this it is convenient to introduce some abbreviated notation. We denote the Jacobians by

$$\begin{aligned} M_{11} &= \frac{d(\eta, \zeta)}{d(y, z)} & M_{12} &= \frac{d(\eta, \zeta)}{d(z, x)} & M_{13} &= \frac{d(\eta, \zeta)}{d(x, y)} \\ M_{21} &= \frac{d(\zeta, \xi)}{d(y, z)} & M_{22} &= \frac{d(\zeta, \xi)}{d(z, x)} & M_{23} &= \frac{d(\zeta, \xi)}{d(x, y)} \\ M_{31} &= \frac{d(\xi, \eta)}{d(y, z)} & M_{32} &= \frac{d(\xi, \eta)}{d(z, x)} & M_{33} &= \frac{d(\xi, \eta)}{d(x, y)} \end{aligned} \quad (7.13)$$

We also write

$$\begin{aligned} A_{ij} &\text{ for } A_{xx}, A_{xy}, \text{ etc.} \\ \bar{\sigma}_{ij} &\text{ for } \bar{\sigma}_{\xi\xi}, \bar{\sigma}_{\xi\eta}, \text{ etc.} \quad (\bar{\sigma}_{ij} = \bar{\sigma}_{ji}) \end{aligned} \quad (7.14)$$

Relations 7.6 may then be written

$$A_{ij} = \bar{\sigma}_{ik} M_{kj} \quad (7.15)$$

The summation is performed for all values of the index k in accordance with the dummy index rule explained in section 3. If we denote x, y, z by x_i and ξ, η, ζ by ξ_i and the body force X, Y, Z by X_i , the equilibrium equations 7.12 may be written

$$\frac{\partial A_{ij}}{\partial x_j} + \rho X_i(\xi_i) = 0 \quad (7.16)$$

We keep only first order terms, and the approximate values for the Jacobians are

$$\begin{aligned} M_{11} &= 1 + e_{yy} + e_{zz} & M_{12} &= -e_{xy} - \omega_z & M_{13} &= -e_{zx} + \omega_y \\ M_{21} &= -e_{xy} + \omega_z & M_{22} &= 1 + e_{xx} + e_{zz} & M_{23} &= -e_{yz} - \omega_x \\ M_{31} &= -e_{zx} + \omega_y & M_{32} &= -e_{yz} + \omega_x & M_{33} &= 1 + e_{xx} + e_{yy} \end{aligned} \quad (7.17)$$

In abbreviated notation these relations take the form

$$M_{ij} = (1 + e)\delta_{ij} - e_{ij} + \omega_{ij} \quad (7.18)$$

where e_{ij} and ω_{ij} are the matrix elements in equations 3.19 and 3.21. We denote by δ_{ij} the Kronecker symbol defined as

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{for } i = j \\ \delta_{ij} &= 0 \quad \text{for } i \neq j \end{aligned} \quad (7.19)$$

and we put

$$e = e_{xx} + e_{yy} + e_{zz} = \delta_{ij}e_{ij} \quad (7.20)$$

We also make use of relation (5.21) for the stress components $\bar{\sigma}_{ij}$ expressed in terms of the initial stress and the stress increments s_{ij} referred to rotated axes. These relations are

$$\bar{\sigma}_{ij} = S_{ij} + s_{ij} + S_{\mu j}\omega_{i\mu} + S_{i\mu}\omega_{j\mu} \quad (7.21)$$

We now substitute into equation 7.15 the approximate values for M_{ij} and $\bar{\sigma}_{ij}$ as given by expressions (7.18) and (7.21). In doing so we drop all terms of order higher than the first, namely, those which are squares and products of the quantities s_{ij} , e_{ij} , and ω_{ij} . We find

$$\begin{aligned} A_{ij} &= (S_{ik} + s_{ik} + S_{\mu k}\omega_{i\mu} + S_{i\mu}\omega_{k\mu})\delta_{kj} \\ &\quad + S_{ik}(e\delta_{kj} - e_{kj} + \omega_{kj}) \end{aligned} \quad (7.22)$$

Because of the significance of δ_{kj} we may write

$$\begin{aligned} s_{ik}\delta_{kj} &= s_{ij} \\ S_{ik}\delta_{kj} &= S_{ij} \\ S_{\mu k}\omega_{i\mu}\delta_{kj} &= S_{\mu j}\omega_{i\mu} \\ S_{i\mu}\omega_{k\mu}\delta_{kj} &= S_{i\mu}\omega_{j\mu} \end{aligned} \tag{7.23}$$

Furthermore, because μ is a dummy index we may replace it by k , and the last two expressions can be written

$$\begin{aligned} S_{\mu j}\omega_{i\mu} &= S_{kj}\omega_{ik} \\ S_{i\mu}\omega_{j\mu} &= S_{ik}\omega_{jk} \end{aligned} \tag{7.24}$$

By taking into account identities (7.23) and (7.24), relation (7.22) becomes

$$A_{ij} = S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ik}\omega_{jk} + S_{ij}e - S_{ik}e_{kj} + S_{ik}\omega_{kj} \tag{7.25}$$

Further simplification is obtained if we take into account the property of antisymmetry of the matrix ω_{jk} (equation 3.21). This property is expressed by

$$\omega_{jk} = -\omega_{kj} \tag{7.26}$$

Hence

$$S_{ik}\omega_{jk} + S_{ik}\omega_{kj} = 0 \tag{7.27}$$

Taking the last identity into account, we finally obtain

$$A_{ij} = S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{kj} \tag{7.28}$$

Substitution of A_{ij} into equation 7.16 yields

$$\frac{\partial}{\partial x_j} [S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{kj}] + \rho X_i(\xi_1) = 0 \tag{7.29}$$

Note that ρ is the original mass density at the point x_i before deformation. Equations 7.29 are the three-dimensional equilibrium equations for the incremental stress field s_{ij} . As in the two-dimensional case examined in the previous section, they may be simplified by taking into account additional relations and identities. First we may take into account the equilibrium conditions (7.1) satisfied by the initial stress field. They may also be written

$$\frac{\partial S_{ij}}{\partial x_j} + \rho X_i(x_1) = 0 \tag{7.30}$$

By introducing this condition into equations 7.29 they become

$$\frac{\partial s_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j} [S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{kj}] + \rho \Delta X_i = 0 \quad (7.31)$$

Since $S_{kj} = S_{jk}$ and $e_{kj} = e_{jk}$ we improve the symmetry of the notation by writing this equation as

$$\frac{\partial s_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j} [S_{jk}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}] + \rho \Delta X_i = 0 \quad (7.32)$$

We have put

$$\Delta X_i = X_i(\xi_i) - X_i(x_i) \quad (7.33)$$

Hence ΔX_i represents the increment in body force per unit mass from the initial point to the displaced point. Equations 7.32 correspond to equations 6.13 for the two-dimensional case. As before, we may further simplify these equations by using identities between strain and rotation similar to equations 6.14 for the particular case of two dimensions. Such relations are derived by starting from the identities

$$\frac{\partial^2 u_l}{\partial x_j \partial x_k} = \frac{\partial^2 u_l}{\partial x_k \partial x_j} \quad (7.34)$$

Because of definitions (3.25) these identities may be written

$$\frac{\partial}{\partial x_j} (e_{ik} + \omega_{ik}) = \frac{\partial}{\partial x_k} (e_{ij} + \omega_{ij}) \quad (7.35)$$

Let us multiply this equation by δ_{ij} . Since

$$\begin{aligned} e_{ij}\delta_{ij} &= e \\ \omega_{ij}\delta_{ij} &= 0 \\ (e_{ik} + \omega_{ik})\delta_{ij} &= e_{jk} + \omega_{jk} \end{aligned} \quad (7.36)$$

we derive

$$\frac{\partial}{\partial x_j} (e_{jk} + \omega_{jk}) = \frac{\partial e}{\partial x_k} \quad (7.37)$$

Multiplying the last equation by S_{ik} , we obtain

$$S_{ik} \frac{\partial e_{jk}}{\partial x_j} = S_{ik} \frac{\partial e}{\partial x_k} - S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} \quad (7.38)$$

Because k is a dummy index we may also write this relation as

$$S_{ik} \frac{\partial e_{jk}}{\partial x_j} = S_{ij} \frac{\partial e}{\partial x_j} - S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} \quad (7.39)$$

Introducing the last identity into equations 7.32, we obtain

$$\begin{aligned} \frac{\partial S_{ij}}{\partial x_j} + S_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} + \omega_{ik} \frac{\partial S_{jk}}{\partial x_j} \\ + e \frac{\partial S_{ij}}{\partial x_j} - e_{jk} \frac{\partial S_{ik}}{\partial x_j} + \rho \Delta X_i = 0 \end{aligned} \quad (7.40)$$

Finally we again make use of the equilibrium condition (7.30) of the initial stress field which may be written

$$\begin{aligned} \frac{\partial S_{jk}}{\partial x_j} &= -\rho X_k(x_i) \\ \frac{\partial S_{ij}}{\partial x_j} &= -\rho X_i(x_i) \end{aligned} \quad (7.41)$$

Substituting these expressions, equations 7.40 become*

$$\begin{aligned} \frac{\partial S_{ij}}{\partial x_j} + \rho \Delta X_i - \rho \omega_{ik} X_k(x_i) - \rho e X_i(x_i) \\ + S_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S_{ik}}{\partial x_j} = 0 \end{aligned} \quad (7.42)$$

In two dimensions these equations reduce to the form (6.16). The same remarks may be made here as discussed for the two-dimensional case at the end of the previous section. The incremental body force may be expressed as

$$\Delta X_i = u_j \frac{\partial X_i}{\partial x_j} + \Delta' X_i \quad (7.43)$$

where the first term represents a linearizing of the increment of body force due to the displacement alone, and $\Delta' X_i$ is the increment of the over-all field due to the deformation of the body as a whole. As already stated, the latter part would arise for the gravity field generated by a large deforming medium.

* Equations 7.42 were derived in this particular form by the author in references 3 and 5 at the end of the Preface. An alternative derivation was given in reference 7.

Variation of the Gravitational Field Due to Deformation. In problems of planetary and astrophysical dynamics dealing with large gravitational bodies we must take into account the term $\Delta'X_i$, which represents the variation of the gravitational field due to the deformation itself. The initial gravitational potential U satisfies the equation

$$\nabla^2 U = 4\pi G\rho \tag{7.43a}$$

where G is the gravitational constant and ρ the initial mass density distribution. The deformation changes the local mass distribution by the amount

$$\Delta\rho = -\frac{\partial\rho}{\partial x_i} u_i - \rho e \tag{7.43b}$$

Hence the increment U' of the gravitational potential satisfies the equation

$$\nabla^2 U' = -4\pi G\left(\frac{\partial\rho}{\partial x_i} u_i + \rho e\right) \tag{7.43c}$$

and the corresponding incremental field is

$$\Delta X'_i = -\frac{\partial U'}{\partial x_i} \tag{7.43d}$$

Density discontinuities are taken into account by adding at the surfaces of discontinuity a mass distribution of surface density

$$-(\rho_1 - \rho_2)u_n \tag{7.43e}$$

where u_n is the displacement normal to the surface and $\rho_1 - \rho_2$ is the density discontinuity in the positive direction of u_n . This includes, of course, the free surface of a solid medium. These surface distributions of mass appear in boundary conditions for the solutions of equation 7.43b and involve discontinuities of the normal derivatives of the potential.

We may also distinguish two types of terms in equation 7.42. One group

$$S_{jk} \frac{\partial\omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial\omega_{jk}}{\partial x_j} \tag{7.44}$$

may be referred to as the curvature terms. Another group

$$\rho \Delta X_i - \rho\omega_{ik}X_k - \rho eX_i - e_{jk} \frac{\partial S_{ik}}{\partial x_j} \tag{7.45}$$

depends essentially on the body force and the initial stress gradient and is of a different nature. The significance of the curvature terms was discussed in section 6 for two-dimensional strain. A similar discussion in three dimensions introduces the twist of an element.*

* See references 2 and 4 at the end of the Preface and equation 7.49 below.

The curvature terms disappear if the initial stress is hydrostatic, since in this case

$$S_{ij} = S\delta_{ij} \tag{7.46}$$

and

$$\begin{aligned} S_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} &= S \frac{\partial}{\partial x_j} (\delta_{jk} \omega_{ik} + \delta_{ik} \omega_{jk}) \\ &= S \frac{\partial}{\partial x_j} (\omega_{ij} + \omega_{ji}) = 0 \end{aligned} \tag{7.47}$$

The latter expression vanishes because of relation (7.26). The terms (7.45), on the other hand, vanish if the initial stress field is uniform.

Equations 7.16 are equivalent to the equilibrium equations

$$\frac{\partial \bar{\sigma}_{ij}}{\partial \xi_j} + \rho'(\xi_i) X_i(\xi_i) = 0 \tag{7.48}$$

expressed by means of the stress components $\bar{\sigma}_{ij}$ and the displaced coordinates ξ_i . These equations are transformed to the form (7.16) by changing the independent variables from ξ_i to x_i . This property has already been mentioned in connection with equations 6.31 for the two-dimensional case.

Equations 7.42 stand for *three* distinct equations. We shall write the first one explicitly in terms of the cartesian coordinates. This first equation obtained by putting $i = 1$ is*

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} + \rho \Delta X + \rho(\omega_z Y - \omega_y Z) - \rho e X \\ - e_{xx} \frac{\partial S_{11}}{\partial x} - e_{yy} \frac{\partial S_{12}}{\partial y} - e_{zz} \frac{\partial S_{31}}{\partial z} - e_{yz} \left(\frac{\partial S_{31}}{\partial y} + \frac{\partial S_{12}}{\partial z} \right) \\ - e_{zx} \left(\frac{\partial S_{11}}{\partial z} + \frac{\partial S_{31}}{\partial x} \right) - e_{xy} \left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} \right) \\ + (S_{11} - S_{22}) \frac{\partial \omega_z}{\partial y} - 2S_{12} \frac{\partial \omega_z}{\partial x} + S_{12} \frac{\partial \omega_x}{\partial z} \\ + (S_{33} - S_{11}) \frac{\partial \omega_y}{\partial z} + 2S_{31} \frac{\partial \omega_y}{\partial x} - S_{31} \frac{\partial \omega_x}{\partial y} \\ + S_{23} \left(\frac{\partial \omega_y}{\partial y} - \frac{\partial \omega_z}{\partial z} \right) = 0 \end{aligned} \tag{7.49}$$

* This explicit form of the equilibrium equations was derived by the author in 1938 (see references 2 and 4 in the Preface).

The two other equations are obtained by cyclic permutation of $x, y, z, 1, 2, 3$, and X, Y, Z . The last term in these equations represents something peculiar to the three-dimensional case, namely, the effect of the twist of an element.

We shall now turn our attention to the boundary conditions. Consider the integral (7.7) expressing the force B_x acting on an area S of the deformed body. Equation 7.7 is written for the case where S represents the whole surface boundary, but it is of course valid where B_x represents the force acting on any portion S of the boundary. We denote by n_x, n_y, n_z the directional cosines of the normal outward direction of the boundary *before deformation*. Then the surface integral (7.7) extended to the *initial surface* may be written

$$B_x = \iint_S (A_{xx}n_x + A_{xy}n_y + A_{xz}n_z) dS \tag{7.50}$$

or in more general form using the dummy index summation rule

$$B_i = \iint_S A_{ij}n_j dS \tag{7.51}$$

The important feature of this formula lies in the interpretation of the integrand. If we consider a unit elementary area of the *initial* boundary, then *after deformation* the x component of the force acting on that same material element is

$$f_x = A_{xx}n_x + A_{xy}n_y + A_{xz}n_z \tag{7.52}$$

The three components of this force are expressed by the general formula

$$f_i = A_{ij}n_j \tag{7.53}$$

If we replace A_{ij} by their first order approximations (7.28), we derive

$$f_i = [S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}]n_j \tag{7.54}$$

Equations 7.53 and 7.54 may be looked upon as boundary conditions.

Since $S_{ij}n_j$ is the force acting at the boundary in the initially stressed but undeformed state, we may also write

$$f_i = S_{ij}n_j + \Delta f_i \tag{7.55}$$

where Δf_i represents an *incremental boundary force*,

$$\Delta f_i = (s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk})n_j \tag{7.56}$$

The x component of the incremental boundary force may be written explicitly as

$$\begin{aligned} \Delta f_x = & (s_{11} - S_{12}\omega_z + S_{31}\omega_y + S_{11}e - S_{11}e_{xx} - S_{12}e_{xy} - S_{31}e_{zx})n_x \\ & + (s_{12} - S_{22}\omega_z + S_{23}\omega_y + S_{12}e \\ & \qquad \qquad \qquad - S_{11}e_{xy} - S_{12}e_{yy} - S_{31}e_{yz})n_y \\ & + (s_{31} - S_{23}\omega_z + S_{33}\omega_y + S_{31}e \\ & \qquad \qquad \qquad - S_{11}e_{zx} - S_{12}e_{yz} - S_{31}e_{zz})n_z \end{aligned} \quad (7.57)$$

The other two components are obtained by cyclic permutation of x, y, z and 1, 2, 3.

Hydrostatic Boundary Condition. The boundaries of the medium may be submerged in a heavy fluid of density ρ_f . The hydrostatic stress in this fluid is a function $S(x_i)$ of the coordinates. Denoting by $X_i(x_i)$ the body force per unit mass acting on this fluid and assuming that it is in hydrostatic equilibrium, we find that the hydrostatic stress satisfies the equation

$$\frac{\partial S}{\partial x_i} + \rho_f X_i = 0 \quad (7.58)$$

The force acting on an element of boundary of the solid is at all instants the same as that acting on the same element of the undisturbed fluid. We therefore apply equation 7.56 to the undisturbed fluid. The incremental stress is

$$s_{ij} = [S(\xi_i) - S(x_i)]\delta_{ij} \quad (7.59)$$

The increment of stress in the fluid when moving from the initial point to the displaced point is to the first order

$$S(\xi_i) - S(x_i) = \frac{\partial S}{\partial x_k} u_k \quad (7.60)$$

Hence

$$s_{ij} = \frac{\partial S}{\partial x_k} u_k \delta_{ij} \quad (7.61)$$

The initial stress in the fluid is

$$S_{ij} = S(x_i)\delta_{ij} \quad (7.62)$$

Substituting these values of s_{ij} and S_{ij} into expression (7.56) yields

$$\Delta f_i = \left(\frac{\partial S}{\partial x_k} u_k + Se \right) n_i - S \frac{\partial u_j}{\partial x_i} n_j \quad (7.63)$$

Using the equilibrium condition (7.58) for the fluid, we also write

$$\Delta f_i = (-\rho_f X_k u_k + Se) n_i - S \frac{\partial u_j}{\partial x_i} n_j \quad (7.64)$$

By equating the two values (7.64) and (7.56) for Δf_i we obtain the boundary condition at the fluid-solid interface.

Inserting expressions (7.61) and (7.62) into equations 7.54, we find

$$f_i = \left(S + Se + \frac{\partial S}{\partial x_k} u_k \right) n_i - S \frac{\partial u_j}{\partial x_i} n_j \quad (7.65)$$

This is the total force per unit initial area exerted by the fluid on the deformed boundary. It is a vector perpendicular to the deformed boundary at the displaced point. When we put $S = 1$, the vector f_i becomes the unit normal n'_i for the deformed surface multiplied by the ratio $d\sigma'/d\sigma$ of areas after and before deformation; that is,

$$n'_i \frac{d\sigma'}{d\sigma} = (1 + e) n_i - \frac{\partial u_j}{\partial x_i} n_j \quad (7.66)$$

This expression is obviously identical with those already used at the beginning of this section in connection with surface integral transformations.

Curvilinear Coordinates. The equilibrium equations derived in this chapter are expressed in cartesian coordinates. Elementary procedures for the derivation of similar equations in curvilinear coordinates will be developed in Chapter 2 as a particular application of variational principles.

CHAPTER TWO

Elasticity Theory of a Medium under Initial Stress

1. INTRODUCTION

This chapter introduces the physics of the material in the form of linear relations between the incremental stresses and the strain. Such linear relations do not necessarily imply that the material is elastic. They are applicable to non-elastic media undergoing an incremental deformation in the vicinity of a prestressed condition.

In addition, we shall impose a restriction on these linear relations by assuming that we may define a strain energy potential for the incremental deformation. In this sense the incremental deformation is not only linear but also elastic.

More precisely, we consider deformations which are *elastic for the incremental deformations* alone, irrespective of the manner by which the state of initial stress has been generated. For example, in geophysical applications the state of initial stress in the earth is the result of a slow but highly irreversible process of a viscous or plastic nature. Rapid deformations, however, may be approximately elastic.

In order to derive appropriate stress-strain relations for the elastic material it is convenient to introduce an alternative definition of the incremental stresses by referring the stresses to areas before deformation. The reason for the alternative definition* is that the stress

* This was discussed in earlier works by the author (see references 3, 4, and 5 in the Preface).

components are conjugate to the strain components. They are therefore useful in deriving the expression for the strain energy, the properties of the incremental moduli, and variational principles. In general, they are more convenient for expressing physical properties of materials.

The alternative definition of the incremental stress is first introduced in section 2. Relations are derived between these stress components and the components s_{ij} of Chapter 1. The two sets of stress components differ by terms which depend on the initial stress. The stress-strain relations for incremental elastic deformations are derived in section 3 for a two-dimensional field, and in section 4 for the three-dimensional field. The two cases are discussed separately in order to bring out more clearly the physical significance of the general equations.

A variational principle for incremental deformations is formulated in section 5 by the introduction of an incremental strain energy. The principle is shown to be equivalent to the equilibrium equations for the stress field derived in the preceding chapter. The analysis in section 5 follows very closely the author's previous work.*

An important application of this variational principle is the derivation of *equilibrium equations in curvilinear coordinates*. Actually, when formulated as a principle of virtual work, variational equations do not require the existence of an elastic potential. They involve only the principles of statics and lead to equilibrium conditions for the stress field in any kind of continuum.

The elastic properties of a medium of orthotropic symmetry are the subject of a detailed discussion in section 6. A number of physically measurable incremental elastic coefficients are introduced. Special attention is devoted to one of these coefficients which was introduced by the author as the *slide modulus*.

In section 7 the properties of an elastic medium isotropic, in finite strain, are investigated with reference to small deformations superposed on an initial state of finite strain. Values of the incremental elastic coefficients are derived in terms of the initial finite stress and strain.

The incremental properties of incompressible elastic media are analyzed in section 8. The results are applied to the special cases of

* As in references 3, 4, and 5 of the Preface.

isotropic media and rubber elasticity. They involve the remarkable property of certain materials to retain their isotropy in a state of finite strain for plane incremental deformations. The question of incremental properties in second order elasticity is treated in section 9. The non-linear stress-strain relations involve five elastic constants. Values are derived for the incremental elastic coefficients by means of these five elastic constants as linear functions of the initial strain.

As a specific application of the general theory, section 10 presents an analysis of the torsional stiffness of a bar under axial tension. We consider first a material which is either isotropic or transverse isotropic with the axis of elastic symmetry along the axis of the bar. The solution for this case is identical with the result derived by the author many years ago.* These results are further extended to the case of an orthotropic non-homogeneous bar, under a non-homogeneous initial stress. The theory leads to a simple solution which brings out the particular features due to the presence of initial stress and is considerably more general than the result derived in the earlier paper mentioned above.

2. THE INCREMENTAL STRESSES REFERRED TO INITIAL AREAS

The stress components s_{11} , s_{22} , s_{12} considered in Chapter 1 are the stresses at point P' referred not only to rotated axes but also to unit areas *after deformation*. See Figure 2.1, which reproduces part of Figure 2.3 of Chapter 1, namely, the region around the displaced point P' . A square of unit size in the plane x, y , with its sides oriented along x and y , becomes the parallelogram $P'ABC$ after deformation. In this parallelogram we cut out a square of unit size whose sides are oriented along the rotated directions 1 and 2. The forces acting on the sides of the unit cube thus defined and projected on directions 1 and 2 are the total stress components

$$S_{11} + s_{11}, \quad S_{22} + s_{22}, \quad S_{12} + s_{12} \quad (2.1)$$

Instead of considering the forces acting on this unit cube, we may consider those acting on the sides of a parallelepiped defined by the

* See reference 6 at the end of the Preface.

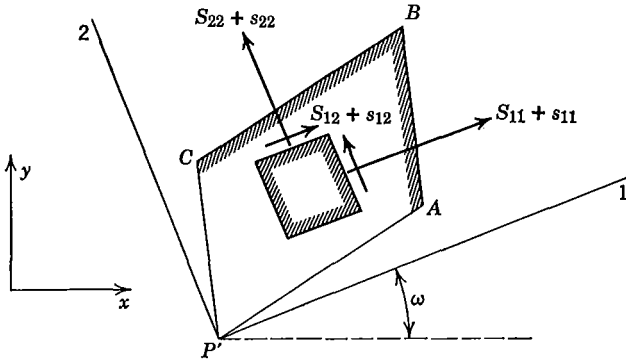


Figure 2.1 The total stress field referred to rotated axes 1, 2 in the vicinity of the displaced point P' .

parallelogram $P'ABC$ (Fig. 2.1). In order to do this let us consider the force $d\mathbf{F}$ acting on an element $d\xi', d\eta'$ (Fig. 2.2). The element $d\xi', d\eta'$ is looked upon as an oriented vector with the material lying on the left side of the element. The force $d\mathbf{F}$ acting on the material has components along directions 1 and 2 given by

$$\begin{aligned} dF_1 &= (S_{11} + s_{11}) d\eta' - (S_{12} + s_{12}) d\xi' \\ dF_2 &= (S_{12} + s_{12}) d\eta' - (S_{22} + s_{22}) d\xi' \end{aligned} \tag{2.2}$$

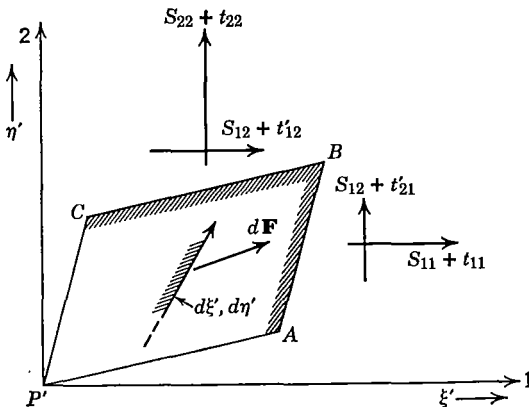


Figure 2.2 The "alternative" stress components referred to initial areas and rotated axes.

The coordinates of points A, B, C , relative to P' as origin are

	ξ'	η'	
A	$1 + \varepsilon_{11}$	ε_{12}	
B	$1 + \varepsilon_{11} + \varepsilon_{12}$	$1 + \varepsilon_{22} + \varepsilon_{12}$	(2.3)
C	ε_{12}	$1 + \varepsilon_{22}$	

We may now evaluate the force $S_{11} + t_{11}$ on side AB in direction 1. It is given by

$$S_{11} + t_{11} = \int_A^B dF_1 \quad (2.4)$$

Substituting dF_1 from equations 2.2 and performing the line integration, we derive

$$S_{11} + t_{11} = (S_{11} + s_{11})(1 + \varepsilon_{22}) - (S_{12} + s_{12})\varepsilon_{12} \quad (2.5)$$

Similarly we define the other components as

$$\begin{aligned} S_{12} + t'_{21} &= \int_A^B dF_2 \\ S_{12} + t'_{12} &= \int_B^C dF_1 \\ S_{22} + t_{22} &= \int_B^C dF_2 \end{aligned} \quad (2.6)$$

and obtain the expressions

$$\begin{aligned} t_{11} &= s_{11} + (S_{11} + s_{11})\varepsilon_{22} - (S_{12} + s_{12})\varepsilon_{12} \\ t'_{21} &= s_{12} + (S_{12} + s_{12})\varepsilon_{22} - (S_{22} + s_{22})\varepsilon_{12} \\ t'_{12} &= s_{12} + (S_{12} + s_{12})\varepsilon_{11} - (S_{11} + s_{11})\varepsilon_{12} \\ t_{22} &= s_{22} + (S_{22} + s_{22})\varepsilon_{11} - (S_{12} + s_{12})\varepsilon_{12} \end{aligned} \quad (2.7)$$

These expressions may be simplified if we neglect products such as $s_{12}\varepsilon_{22}$, $s_{22}\varepsilon_{12}$ as quantities of a higher order. Then they become

$$\begin{aligned} t_{11} &= s_{11} + S_{11}\varepsilon_{22} - S_{12}\varepsilon_{12} \\ t'_{21} &= s_{12} + S_{12}\varepsilon_{22} - S_{22}\varepsilon_{12} \\ t'_{12} &= s_{12} + S_{12}\varepsilon_{11} - S_{11}\varepsilon_{12} \\ t_{22} &= s_{22} + S_{22}\varepsilon_{11} - S_{12}\varepsilon_{12} \end{aligned} \quad (2.8)$$

The quantities

$$\begin{array}{cc} t_{11} & t'_{12} \\ t'_{21} & t_{22} \end{array} \quad (2.9)$$

may be considered stress components since they represent forces acting on an element of unit dimensions *before deformation*.

As later considerations will show, the primary reason for introducing the alternative definition of the stress is a physical one. It is convenient to use in finding the physical relation between incremental stresses and strain and in expressing the strain energy. This is a consequence of the fact that, if ε_{ij} is the strain defined in Chapter 1, the product $t_{ij}\varepsilon_{ij}$ is an exact expression for the work done by the incremental stresses. In this sense we may say that the variables t_{ij} and ε_{ij} are *conjugate*. The stresses (2.8) also provide in some cases a physical meaning for certain equations.

The striking feature of the representation (2.8) for the stresses is, of course, that the components are not symmetric, that is,

$$t'_{12} \neq t'_{21} \quad (2.10)$$

In fact, we have

$$t'_{21} - t'_{12} = S_{12}(\varepsilon_{22} - \varepsilon_{11}) + (S_{11} - S_{22})\varepsilon_{12} \quad (2.11)$$

The difference between these two stress components is not mysterious; it simply expresses the fact that the total torque acting on the deformed element $P'ABC$ must be zero.

For our purpose here it is not the actual components t'_{12} and t'_{21} which are useful, but rather their average,

$$t_{12} = \frac{1}{2}(t'_{12} + t'_{21}) \quad (2.12)$$

The components of this symmetric part of the stress are

$$\begin{aligned} t_{11} &= s_{11} + S_{11}\varepsilon_{22} - S_{12}\varepsilon_{12} \\ t_{22} &= s_{22} + S_{22}\varepsilon_{11} - S_{12}\varepsilon_{12} \\ t_{12} &= s_{12} + \frac{1}{2}S_{12}(\varepsilon_{11} + \varepsilon_{22}) - \frac{1}{2}(S_{11} + S_{22})\varepsilon_{12} \end{aligned} \quad (2.13)$$

As shown in Chapter 1, the strain components ε_{ij} are identical with e_{ij} to a first order of approximation. Therefore, to the first order, expressions (2.13) may also be written

$$\begin{aligned} t_{11} &= s_{11} + S_{11}e_{yy} - S_{12}e_{xy} \\ t_{22} &= s_{22} + S_{22}e_{xx} - S_{12}e_{xy} \\ t_{12} &= s_{12} + \frac{1}{2}S_{12}(e_{xx} + e_{yy}) - \frac{1}{2}(S_{11} + S_{22})e_{xy} \end{aligned} \quad (2.14)$$

The last expressions are chosen to represent the *alternative stress system*.

A similar stress system may be considered in three dimensions. This is conveniently formulated by considering equation 7.53 obtained in Chapter 1 for the forces at the boundary. We have found the expression

$$f_i = A_{ij}n_j \quad (2.15)$$

representing the force acting on an area originally equal to unity. The force f_i is represented by its components on the fixed axes x, y, z . It is clear that, for a deformation without rotation, relations (2.15) yield immediately the nine components of the forces acting on the faces of a parallelepiped which is a unit cube oriented along x, y, z before deformation. These nine force components are simply the expressions A_{ij} in which we put $\omega_{ij} = 0$. We write these forces as

$$S_{ij} + t'_{ij} = A'_{ij} \quad (2.16)$$

with

$$A'_{ij} = S_{ij} + s_{ij} + S_{ij}e - S_{ik}e_{jk} \quad (2.17)$$

Hence the nine incremental force components are

$$t'_{ij} = s_{ij} + S_{ij}e - S_{ik}e_{jk} \quad (2.18)$$

If there is a rotation, these forces are now referred to directions 1, 2, 3 which rotate locally with the material as defined in section 5 of Chapter 1. The components t'_{ij} are in general not symmetric; that is,

$$t'_{ij} \neq t'_{ji} \quad (2.19)$$

The difference is

$$t'_{ij} - t'_{ji} = S_{jk}e_{ik} - S_{ik}e_{jk} \quad (2.20)$$

The difference is zero if the initial stress is hydrostatic. As in the two-dimensional case, the symmetric part of t'_{ij} may be chosen as an alternative representation of the stress. We define this stress as

$$t_{ij} = \frac{1}{2}(t'_{ij} + t'_{ji}) \quad (2.21)$$

or

$$t_{ij} = s_{ij} + S_{ij}e - \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (2.22)$$

In two dimensions these relations become identical with equations 2.14 obtained above.

We may express the equilibrium conditions for the stress field in terms of t_{ij} . Solving equation 2.22 for s_{ij} and substituting these values in the equilibrium equations 7.29 of Chapter 1 we find

$$\frac{\partial}{\partial x_j} [S_{ij} + t_{ij} + S_{kj}\omega_{ik} - \frac{1}{2}S_{ik}e_{jk} + \frac{1}{2}S_{jk}e_{ik}] + \rho X_i(\xi_i) = 0 \quad (2.23)$$

An alternative form analogous to equations 7.32 of Chapter 1 is obtained by making use of the equilibrium condition of the initial stress field.*

$$\frac{\partial t_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j} [S_{kj}\omega_{ik} - \frac{1}{2}S_{ik}e_{jk} + \frac{1}{2}S_{jk}e_{ik}] + \rho \Delta X_i = 0 \quad (2.24)$$

Equivalent equations using alternative stresses t'_{ij} were obtained by Biezeno and Hencky† who considered the equilibrium of a deformed infinitesimal element. This equivalence provides a physical interpretation of the various terms in equations 2.24.

3. TWO-DIMENSIONAL RELATIONS BETWEEN STRAIN AND INCREMENTAL STRESS

We consider an elastic continuum in an initial state of stress whose components in the x, y plane are

$$\begin{array}{cc} S_{11} & S_{12} \\ S_{12} & S_{22} \end{array} \quad (3.1)$$

The displacements u, v of the medium are also in the x, y plane and produce a state of plane strain defined to the first order by the components

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} \\ e_{yy} &= \frac{\partial v}{\partial y} \\ e_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \quad (3.2)$$

* Equations in the form (2.24) were first derived by the author in references 3, 4, and 5 at the end of the Preface.

† C. B. Biezeno and H. Hencky, On a General Theory of Elastic Stability, *Proceedings of the Royal Academy, Amsterdam*, Vol. 31, pp. 569-592, 1928, and Vol. 32, pp. 444-456, 1929.

The incremental stresses relative to locally rotated axes, as defined in section 4 of Chapter 1, are

$$\begin{array}{cc} s_{11} & s_{12} \\ s_{12} & s_{22} \end{array} \quad (3.3)$$

We have shown in Chapter 1, section 2, that, if the deformation is small, the strain components (3.2) are identical to the first order with components ε_{ij} of the deformation referred to rotated axes. This means that under the same conditions of small incremental deformations the incremental stress components (3.3) must be functions only of the strain components (3.2). It is important to note that this is due to the fact that both the incremental stresses and the strain have been referred to rotated axes, so that the influence of the rotation on the relation between stress and strain has been eliminated.

For small incremental deformations we may also assume that the incremental stress-strain relations are linear. We may write these relations as

$$\begin{aligned} s_{11} &= B_{11}^{11}e_{xx} + B_{11}^{22}e_{yy} + 2B_{11}^{12}e_{xy} \\ s_{22} &= B_{22}^{11}e_{xx} + B_{22}^{22}e_{yy} + 2B_{22}^{12}e_{xy} \\ s_{12} &= B_{12}^{11}e_{xx} + B_{12}^{22}e_{yy} + 2B_{12}^{12}e_{xy} \end{aligned} \quad (3.4)$$

We have introduced the factor 2 before coefficients B_{11}^{12} , B_{22}^{12} , and B_{12}^{12} . The reason will become clearer when we use the "dummy index rule" in the next section.

An important physical property of elastic continua must be introduced at this stage, namely, the existence of a strain energy potential. Strictly speaking, the existence of the strain energy may be derived from thermodynamics and it is a rigorous notion for isothermal or adiabatic deformations. In this sense the coefficients B_{11}^{11} , B_{11}^{22} , etc., appearing in the stress-strain relations (3.4) may be either isothermal or adiabatic coefficients. Actually, for many problems the use of isothermal coefficients constitutes a satisfactory approximation. Further clarification of this point will be found in section 4 of Chapter 5.

In order to introduce the concept of strain energy we must first go back to an alternative expression of the stresses. These are the stress components t'_{ij} introduced in section 2. They are

$$\begin{array}{cc} t_{11} & t'_{12} \\ t'_{21} & t_{22} \end{array} \quad (3.5)$$

and they represent the *incremental forces*, acting on the actual areas of an element after deformation. Furthermore, these forces are referred to the rotated axes. The total forces are, of course,

$$\begin{aligned} S_{11} + t_{11} & S_{12} + t'_{12} \\ S_{12} + t'_{21} & S_{22} + t_{22} \end{aligned} \quad (3.6)$$

Let us now consider a virtual deformation represented by variations $\delta\varepsilon_{11}$, $\delta\varepsilon_{22}$, $\delta\varepsilon_{12}$ of the strain components. The work done by the forces (3.6) on an element of the medium undergoing this virtual deformation is

$$\begin{aligned} \delta V = (S_{11} + t_{11})\delta\varepsilon_{11} + (S_{22} + t_{22})\delta\varepsilon_{22} \\ + (t'_{12} + t'_{21} + 2S_{12})\delta\varepsilon_{12} \end{aligned} \quad (3.7)$$

It is important to remember here that we are dealing with the energy associated with a linear theory. It is a general rule in such cases that expressions for the energy must include all second order terms. To be correct we must therefore express the virtual work in terms of the components ε_{ij} defined in Chapter 1, including first and second order quantities. An immediate simplification may be introduced in expression (3.7) since t'_{ij} is already of the first order. Therefore, in multiplying these quantities, we may replace ε_{ij} by its first order approximation e_{ij} . The virtual work then becomes

$$\begin{aligned} \delta V = t_{11}\delta e_{xx} + t_{22}\delta e_{yy} + (t'_{12} + t'_{21})\delta e_{xy} \\ + S_{11}\delta\varepsilon_{11} + S_{22}\delta\varepsilon_{22} + 2S_{12}\delta\varepsilon_{12} \end{aligned} \quad (3.8)$$

This expression depends only on the symmetric part of the stress components. Writing as before

$$t_{12} = \frac{1}{2}(t'_{12} + t'_{21}) \quad (3.9)$$

we obtain

$$\begin{aligned} \delta V = t_{11}\delta e_{xx} + t_{22}\delta e_{yy} + 2t_{12}\delta e_{xy} \\ + S_{11}\delta\varepsilon_{11} + S_{22}\delta\varepsilon_{22} + 2S_{12}\delta\varepsilon_{12} \end{aligned} \quad (3.10)$$

The existence of a strain energy potential is expressed by the condition that this expression is an exact differential. The initial stress is assumed given at any point, and only t_{ij} depends on the strain. The condition that expression (3.10) be an exact differential

is expressed by the three relations

$$\begin{aligned}\frac{\partial t_{11}}{\partial e_{yy}} &= \frac{\partial t_{22}}{\partial e_{xx}} \\ \frac{\partial t_{11}}{\partial e_{xy}} &= 2 \frac{\partial t_{12}}{\partial e_{xx}} \\ \frac{\partial t_{22}}{\partial e_{xy}} &= 2 \frac{\partial t_{12}}{\partial e_{yy}}\end{aligned}\quad (3.11)$$

Linear relations between the stresses t_{ij} and the strains may be written

$$\begin{aligned}t_{11} &= C_{11}^{11}e_{xx} + C_{11}^{22}e_{yy} + 2C_{11}^{12}e_{xy} \\ t_{22} &= C_{22}^{11}e_{xx} + C_{22}^{22}e_{yy} + 2C_{22}^{12}e_{xy} \\ t_{12} &= C_{12}^{11}e_{xx} + C_{12}^{22}e_{yy} + 2C_{12}^{12}e_{xy}\end{aligned}\quad (3.12)$$

These equations must satisfy conditions (3.11); hence

$$\begin{aligned}C_{11}^{22} &= C_{22}^{11} \\ C_{11}^{12} &= C_{12}^{11} \\ C_{22}^{12} &= C_{12}^{22}\end{aligned}\quad (3.13)$$

The stress components t_{ij} are related to s_{ij} by equations 2.14:

$$\begin{aligned}t_{11} &= s_{11} + S_{11}e_{yy} - S_{12}e_{xy} \\ t_{22} &= s_{22} + S_{22}e_{xx} - S_{12}e_{xy} \\ t_{12} &= s_{12} + \frac{1}{2}S_{12}(e_{xx} + e_{yy}) - \frac{1}{2}(S_{11} + S_{22})e_{xy}\end{aligned}\quad (3.14)$$

Substituting in equations 3.12 yields stress-strain relations in the form (3.4); that is,

$$\begin{aligned}s_{11} &= C_{11}^{11}e_{xx} + (C_{11}^{22} - S_{11})e_{yy} + (2C_{11}^{12} + S_{12})e_{xy} \\ s_{22} &= (C_{22}^{11} - S_{22})e_{xx} + C_{22}^{22}e_{yy} + (2C_{22}^{12} + S_{12})e_{xy} \\ s_{12} &= (C_{12}^{11} - \frac{1}{2}S_{12})e_{xx} + (C_{12}^{22} - \frac{1}{2}S_{12})e_{yy} \\ &\quad + (2C_{12}^{12} + \frac{1}{2}S_{11} + \frac{1}{2}S_{22})e_{xy}\end{aligned}\quad (3.15)$$

Comparing equations 3.4 and 3.15, we derive

$$\begin{aligned}B_{11}^{11} &= C_{11}^{11} & B_{11}^{22} &= C_{11}^{22} - S_{11} & B_{11}^{12} &= C_{11}^{12} + \frac{1}{2}S_{12} \\ B_{22}^{11} &= C_{22}^{11} - S_{22} & B_{22}^{22} &= C_{22}^{22} & B_{22}^{12} &= C_{22}^{12} + \frac{1}{2}S_{12} \\ B_{12}^{11} &= C_{12}^{11} - \frac{1}{2}S_{12} & B_{12}^{22} &= C_{12}^{22} - \frac{1}{2}S_{12} & B_{12}^{12} &= C_{12}^{12} + \frac{1}{4}S_{11} + \frac{1}{4}S_{22}\end{aligned}\quad (3.16)$$

These coefficients do not satisfy the same symmetry relations as do equations 3.13, but the following are derived from them:

$$\begin{aligned} B_{11}^{22} + S_{11} &= B_{22}^{11} + S_{22} \\ B_{11}^{12} - \frac{1}{2}S_{12} &= B_{12}^{11} + \frac{1}{2}S_{12} \\ B_{22}^{12} - \frac{1}{2}S_{12} &= B_{12}^{22} + \frac{1}{2}S_{12} \end{aligned} \quad (3.17)$$

It is interesting to note that coefficients B become symmetric, i.e., will satisfy the symmetry relations of type (3.13) if

$$S_{11} = S_{22}, \quad S_{12} = 0 \quad (3.18)$$

In this case the initial stress is two-dimensionally isotropic.

4. THREE-DIMENSIONAL RELATIONS BETWEEN STRAIN AND INCREMENTAL STRESS

In order to deal with the stress-strain relations in three dimensions we shall make use of the more compact notation resulting from the dummy index summation rule. However, in this particular case it is important first to clarify a possible source of confusion connected with this notation.

Let us go back to the two-dimensional case considered in the previous section. We may obviously write the three equations 3.12 in the form

$$\begin{aligned} t_{11} &= C_{11}^{11}e_{xx} + C_{11}^{22}e_{yy} + C_{11}^{12}e_{xy} + C_{11}^{21}e_{yx} \\ t_{22} &= C_{22}^{11}e_{xx} + C_{22}^{22}e_{yy} + C_{22}^{12}e_{xy} + C_{22}^{21}e_{yx} \\ t_{12} &= C_{12}^{11}e_{xx} + C_{12}^{22}e_{yy} + C_{12}^{12}e_{xy} + C_{12}^{21}e_{yx} \\ t_{21} &= C_{21}^{11}e_{xx} + C_{21}^{22}e_{yy} + C_{21}^{12}e_{xy} + C_{21}^{21}e_{yx} \end{aligned} \quad (4.1)$$

These are the same as equations 3.12 provided that we put

$$\begin{aligned} e_{xy} &= e_{yx} & t_{12} &= t_{21} \\ C_{11}^{12} &= C_{11}^{21} & C_{12}^{11} &= C_{21}^{11} \\ C_{22}^{12} &= C_{22}^{21} & C_{12}^{22} &= C_{21}^{22} \\ C_{12}^{12} &= C_{12}^{21} & C_{12}^{12} &= C_{21}^{21} \end{aligned} \quad (4.2)$$

Thus we have not introduced any new coefficients but simply a new notation. Because of relations (3.13) and the purely formal ones

(4.2), the matrix of coefficients in equations 4.1 is symmetric; that is, we may write

$$C_{ij}^{\mu\nu} = C_{\mu\nu}^{ij} \quad (4.3)$$

This may also be expressed by the equation

$$\frac{\partial t_{ij}}{\partial e_{\mu\nu}} = \frac{\partial t_{\mu\nu}}{\partial e_{ij}} \quad (4.4)$$

which is the same as equations 3.11. The reader will note that the disappearance of the factor 2 from these equations is due specifically to the interpretation of the partial derivative. The variable e_{xy} in equations 3.11 and 4.4 are not the same when considered from the standpoint of partial derivation. In equations 3.11, when we vary e_{xy} , we vary at the same time e_{yx} since the two variables are put equal to each other a priori. This is in contrast with equations 4.4 when the variables e_{xy} and e_{yx} are considered *independent variables before differentiation* and put equal to each other only after the operation has been performed. This is the origin of the factor 2 in the earlier equations. These points should be carefully kept in mind when applying the more general notation.

Let us now consider a state of three-dimensional stress. The six components of the initial stress referred to x, y, z axes are

$$S_{ij} = \left\| \begin{array}{ccc} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{array} \right\| \quad (4.5)$$

with the condition $S_{ij} = S_{ji}$. The total stress referred to axes which have rotated with the material is

$$\sigma_{ij} = S_{ij} + s_{ij}$$

where the incremental stress is

$$s_{ij} = \left\| \begin{array}{ccc} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{array} \right\| \quad (4.6)$$

Again we put $s_{ij} = s_{ji}$. Finally, the infinitesimal strain components referred to the rotated axes are

$$e_{ij} = \left\| \begin{array}{ccc} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{array} \right\| \quad (4.7)$$

with $e_{ij} = e_{ji}$.

Instead of using the stress components s_{ij} , we may represent the incremental stresses by the alternative components t_{ij} referred to initial areas. The relations between the two representations of the stress are expressed by equation 2.22. The advantage of the t_{ij} components lies in the property that, when multiplied by the corresponding strain components, they lead to the expression for the work done by the stresses on an element of the medium. In other words, t_{ij} and ε_{ij} are conjugate variables.

Generalized to three dimensions, the linear relations (4.1) between t_{ij} and e_{ij} are written

$$t_{ij} = C_{ij}^{\mu\nu} e_{\mu\nu} \quad (4.8)$$

with coefficients satisfying the relations

$$C_{ij}^{\mu\nu} = C_{ji}^{\mu\nu} = C_{ij}^{\nu\mu} \quad (4.9)$$

The indices in equations 4.8 assume all values 1, 2, 3 and x, y, z . As pointed out in section 1, a linear relation like (4.8) may of course be assumed to occur under certain conditions for anelastic media. If the medium is elastic, the existence of a strain energy potential restricts the choice of coefficients. In analogy with equation 3.8 for the two-dimensional case, we write for the virtual work of the total forces $t'_{ij} + S_{ij}$ on an element of the medium

$$\delta V = t'_{ij} \delta e_{ij} + S_{ij} \delta \varepsilon_{ij} \quad (4.10)$$

As explained in section 3, in order to ensure that this expression be correct to the second order we have used the strain components ε_{ij} defined in terms of first and second order quantities. We have used here the actual force $t'_{ij} + S_{ij}$ with non-symmetric components. However, because of the symmetry of e_{ij} , i.e., because

$$e_{ij} = e_{ji} \quad (4.11)$$

this virtual work depends only on the symmetric part of t'_{ij} , namely,

$$t_{ij} = \frac{1}{2}(t'_{ij} + t'_{ji}) \quad (4.12)$$

Hence we may write

$$\delta V = t_{ij} \delta e_{ij} + S_{ij} \delta \varepsilon_{ij} \quad (4.13)$$

The existence of a strain energy potential requires that this expression be an exact differential, and therefore condition (4.3) must be satisfied; that is, coefficients $C_{ij}^{\mu\nu}$ must obey the symmetry relations

$$C_{ij}^{\mu\nu} = C_{\mu\nu}^{ij} \quad (4.14)$$

We may also use the stresses s_{ij} and write linear relations between these incremental stresses and the strains as

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu} \quad (4.15)$$

with

$$B_{ij}^{\mu\nu} = B_{ji}^{\mu\nu} = B_{ij}^{\nu\mu} \quad (4.16)$$

In order to compare coefficients $B_{ij}^{\mu\nu}$ and $C_{ij}^{\mu\nu}$ let us substitute into equations 4.8 the value of t_{ij} in terms of s_{ij} as given by equations 2.22. We derive

$$s_{ij} = C_{ij}^{\mu\nu} e_{\mu\nu} - S_{ij} e + \frac{1}{2}(S_{ik} e_{jk} + S_{jk} e_{ik}) \quad (4.17)$$

In order to bring out the hidden symmetry of this expression it is essential to write it in a modified form as follows. We first write the identities

$$\begin{aligned} e &= e_{\mu\nu} \delta_{\mu\nu} \\ S_{ik} e_{jk} &= S_{iv} e_{jv} \\ S_{jk} e_{ik} &= S_{jv} e_{iv} \end{aligned} \quad (4.18)$$

These relations are based on the property of dummy indices and of the symbol δ_{ij} ; that is,

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{for } i = j \\ \delta_{ij} &= 0 \quad \text{for } i \neq j \end{aligned} \quad (4.19)$$

Furthermore, because of this same property of δ_{ij} , we may put

$$\begin{aligned} e_{jv} &= \delta_{\mu j} e_{\mu v} \\ e_{iv} &= \delta_{\mu i} e_{\mu v} \end{aligned} \quad (4.20)$$

With these results the last two of relations (4.18) become

$$\begin{aligned} S_{ik} e_{jk} &= S_{iv} \delta_{\mu j} e_{\mu v} \\ S_{jk} e_{ik} &= S_{jv} \delta_{\mu i} e_{\mu v} \end{aligned} \quad (4.21)$$

Finally we utilize the fact that

$$e_{\mu\nu} = e_{\nu\mu} \quad (4.22)$$

Hence

$$\begin{aligned} S_{ik} e_{jk} &= \frac{1}{2}(S_{iv} \delta_{\mu j} + S_{i\mu} \delta_{vj}) e_{\mu\nu} \\ S_{jk} e_{ik} &= \frac{1}{2}(S_{jv} \delta_{\mu i} + S_{j\mu} \delta_{vi}) e_{\mu\nu} \end{aligned} \quad (4.23)$$

When we introduce the last values and the first of expressions (4.18)

into equation 4.17, it becomes

$$s_{ij} = C_{ij}^{\mu\nu} e_{\mu\nu} - S_{ij} \delta_{\mu\nu} e_{\mu\nu} + \frac{1}{4}(S_{i\nu} \delta_{\mu j} + S_{i\mu} \delta_{\nu j} + S_{j\nu} \delta_{\mu i} + S_{j\mu} \delta_{\nu i}) e_{\mu\nu} \quad (4.24)$$

Comparing the coefficients in the stress-strain relations (4.15) and (4.24), we conclude that

$$B_{ij}^{\mu\nu} = C_{ij}^{\mu\nu} - S_{ij} \delta_{\mu\nu} + \frac{1}{4}(S_{i\nu} \delta_{\mu j} + S_{i\mu} \delta_{\nu j} + S_{j\nu} \delta_{\mu i} + S_{j\mu} \delta_{\nu i}) \quad (4.25)$$

This yields a relation between the coefficients in the two alternative forms (4.8) and (4.15) of the stress-strain relations. Applying equations 4.25 to the two-dimensional case yields equations 3.16, derived by a direct procedure.

Other properties of these coefficients may be brought out by separating the terms into two parts. We write

$$B_{ij}^{\mu\nu} = B_{ij}^{(s)\mu\nu} + B_{ij}^{(a)\mu\nu} \quad (4.26)$$

with

$$\begin{aligned} B_{ij}^{(s)\mu\nu} &= C_{ij}^{\mu\nu} - \frac{1}{2}(S_{ij} \delta_{\mu\nu} + S_{\mu\nu} \delta_{ij}) \\ &\quad + \frac{1}{4}(S_{i\nu} \delta_{\mu j} + S_{i\mu} \delta_{\nu j} + S_{j\nu} \delta_{\mu i} + S_{j\mu} \delta_{\nu i}) \quad (4.27) \\ B_{ij}^{(a)\mu\nu} &= \frac{1}{2}(S_{\mu\nu} \delta_{ij} - S_{ij} \delta_{\mu\nu}) \end{aligned}$$

Because of relations (4.3) for $C_{ij}^{\mu\nu}$, it is easily seen that we have the symmetry property

$$B_{ij}^{(s)\mu\nu} = B_{\mu\nu}^{(s)ij} \quad (4.28)$$

and the antisymmetry property

$$B_{ij}^{(a)\mu\nu} = -B_{\mu\nu}^{(a)ij} \quad (4.29)$$

In general, therefore, we shall have the inequality

$$B_{ij}^{\mu\nu} \neq B_{\mu\nu}^{ij} \quad (4.30)$$

We conclude from this that the existence of a potential strain energy does not necessarily lead to a symmetric matrix for the coefficients $B_{ij}^{\mu\nu}$ in the incremental stress-strain relations. The existence of a potential strain energy does, however, lead to conditions of a more general type to be satisfied by the coefficients. The condition for the coefficients is

$$B_{ij}^{\mu\nu} - B_{\mu\nu}^{ij} = S_{\mu\nu} \delta_{ij} - S_{ij} \delta_{\mu\nu} \quad (4.31)$$

The coefficients $B_{ij}^{\mu\nu}$ will be symmetric if the right side of this equation vanishes; i.e., if

$$S_{\mu\nu}\delta_{ij} - S_{ij}\delta_{\mu\nu} = 0 \quad (4.32)$$

This can only occur if the initial stress is of the form

$$S_{ij} = S\delta_{ij} \quad (4.33)$$

i.e., if the initial stress is isotropic.

The properties of the coefficients $B_{ij}^{\mu\nu}$ may also be illustrated by putting

$$D_{ij}^{\mu\nu} = \frac{1}{4}(S_{iv}\delta_{\mu j} + S_{iu}\delta_{\nu j} + S_{jv}\delta_{\mu i} + S_{j\mu}\delta_{\nu i}) \quad (4.34)$$

This expression is completely symmetric; that is,

$$D_{ij}^{\mu\nu} = D_{ji}^{\mu\nu} = D_{ij}^{\nu\mu} = D_{ji}^{\nu\mu} \quad (4.35)$$

Equation 4.25 may now be written

$$B_{ij}^{\mu\nu} + S_{ij}\delta_{\mu\nu} = C_{ij}^{\mu\nu} + D_{ij}^{\mu\nu} = Z_{ij}^{\mu\nu} \quad (4.36)$$

where $Z_{ij}^{\mu\nu}$ is now a completely symmetric coefficient obeying the same relations as those in equations 4.9, 4.14, and 4.35; that is,

$$Z_{ij}^{\mu\nu} = Z_{ji}^{\mu\nu} = Z_{ij}^{\nu\mu} = Z_{ji}^{\nu\mu} \quad (4.37)$$

The value of $B_{ij}^{\mu\nu}$ may be written

$$B_{ij}^{\mu\nu} = Z_{ij}^{\mu\nu} - S_{ij}\delta_{\mu\nu} \quad (4.38)$$

With this value the stress-strain relations (4.15) become

$$s_{ij} = (Z_{ij}^{\mu\nu} - S_{ij}\delta_{\mu\nu})e_{\mu\nu} \quad (4.39)$$

Since the volume dilatation is

$$e = \delta_{\mu\nu}e_{\mu\nu} \quad (4.40)$$

we finally obtain the stress-strain relations in the form

$$s_{ij} = Z_{ij}^{\mu\nu}e_{\mu\nu} - S_{ij}e \quad (4.41)$$

The non-symmetric terms have now been reduced to a very simple expression which contains only the dilatation e .

An interesting consequence of this result is the *disappearance of the non-symmetric terms for an incompressible medium* ($e = 0$).

5. VARIATIONAL PRINCIPLES

Variational procedures* to formulate the linear mechanics of continua under initial stress are appreciably more involved than in the corresponding classical problem with a stress-free initial state. In the classical problem the treatment calls for the use of linear kinematics only. In the case of a prestressed medium we must introduce *the concept of non-linear strain* and take into account the second order quantities. This will bring into play the expressions obtained in Chapter 1 for the strain components ε_{ij} which are accurate to the second order. Expression (3.27) of Chapter 1 for the strain components is

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2}(e_{i\mu}\omega_{\mu j} + e_{j\mu}\omega_{\mu i}) + \frac{1}{2}\omega_{i\mu}\omega_{j\mu} \quad (5.1)$$

In order to obtain the potential strain energy for a medium under initial stress we must evaluate this energy correctly up to second order terms. We shall again make use of the stress system t_{ij} which represents forces acting on the deformed faces of an element which is originally before deformation a cube of unit size oriented along a direction parallel to the coordinate axes. The forces t_{ij} are components projected on the rotated axes. The strain components ε_{ij} are similarly the components along the same axes. Since the total forces are $t_{ij} + S_{ij}$, the virtual work of these forces on the deformed element is

$$\delta V = t_{ij}\delta e_{ij} + S_{ij}\delta \varepsilon_{ij} \quad (5.2)$$

This expression is the same as equation 4.13, and it includes all second order quantities. The assumption of the existence of a strain energy potential requires, as we have seen, that relations (4.14) be satisfied. The term $t_{ij}\delta e_{ij}$ is then an exact differential.

Consider now the linear relations (4.8) between t_{ij} and e_{ij} .

$$t_{ij} = C_{ij}^{\mu\nu}e_{\mu\nu} \quad (5.3)$$

Let us write

$$t_{ij}e_{ij} = C_{ij}^{\mu\nu}e_{\mu\nu}e_{ij} \quad (5.4)$$

* The variational formulation of the problem of incremental deformation was introduced by the author in reference 1 at the end of the Preface, and later developed in references 3, 4, and 5.

Then, taking the differential of this expression, we obtain

$$\delta(t_{ij}e_{ij}) = C_{ij}^{\mu\nu}e_{\mu\nu}\delta e_{ij} + C_{ij}^{\mu\nu}e_{ij}\delta e_{\mu\nu} \quad (5.5)$$

Since

$$C_{ij}^{\mu\nu} = C_{\mu\nu}^{ij} \quad (5.6)$$

we may also write

$$\delta(t_{ij}e_{ij}) = 2C_{ij}^{\mu\nu}e_{\mu\nu}\delta e_{ij} = 2t_{ij}\delta e_{ij} \quad (5.7)$$

Introducing this result into equation 5.2, we obtain an exact differential. We write

$$V = \frac{1}{2}t_{ij}e_{ij} + S_{ij}e_{ij} \quad (5.8)$$

This is the strain energy potential per unit initial volume. We write explicitly the value of e_{ij} as given by equation 5.1, and V becomes

$$V = \frac{1}{2}t_{ij}e_{ij} + S_{ij}[e_{ij} + \frac{1}{2}e_{i\mu}\omega_{\mu j} + \frac{1}{2}e_{j\mu}\omega_{\mu i} + \frac{1}{2}\omega_{i\mu}\omega_{j\mu}] \quad (5.9)$$

From the general principles of mechanics it should be possible to show that equilibrium conditions are equivalent to a variational principle applied to the total potential energy. Let us therefore consider the total strain energy potential of a volume τ . This volume is defined in terms of the initial geometry before deformation. The total strain energy of this body is

$$\mathcal{V} = \iiint_{\tau} V d\tau \quad (5.10)$$

We shall now evaluate the variation of \mathcal{V} for variations δu_i of the displacement field of the continuum, i.e., for variations δu , δv , δw of the three cartesian components of this field. In performing this evaluation we recall the definition of e_{ij} , ω_{ij} (equation 3.25, Chapter 1).

$$\begin{aligned} e_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ \omega_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \end{aligned} \quad (5.11)$$

Let us first evaluate δV . Taking into account expression (5.7) for $\delta(t_{ij}e_{ij})$, we derive

$$\begin{aligned} \delta V &= (t_{ij} + S_{ij})\delta e_{ij} \\ &+ \frac{1}{2}S_{ij}e_{i\mu}\delta\omega_{\mu j} + \frac{1}{2}S_{ij}\omega_{\mu j}\delta e_{i\mu} \\ &+ \frac{1}{2}S_{ij}e_{j\mu}\delta\omega_{\mu i} + \frac{1}{2}S_{ij}\omega_{\mu i}\delta e_{j\mu} \\ &+ \frac{1}{2}S_{ij}\omega_{i\mu}\delta\omega_{j\mu} + \frac{1}{2}S_{ij}\omega_{j\mu}\delta\omega_{i\mu} \end{aligned} \quad (5.12)$$

We notice that we may exchange indices i and j in these various terms because $S_{ij} = S_{ji}$; that is,

$$\begin{aligned} S_{ij}e_{i\mu}\delta\omega_{\mu j} &= S_{ij}e_{j\mu}\delta\omega_{\mu i} \\ S_{ij}\omega_{\mu j}\delta e_{i\mu} &= S_{ij}\omega_{\mu i}\delta e_{j\mu} \\ S_{ij}\omega_{i\mu}\delta\omega_{j\mu} &= S_{ij}\omega_{j\mu}\delta\omega_{i\mu} \end{aligned} \quad (5.13)$$

Therefore the expression for δV simplifies to

$$\delta V = (t_{ij} + S_{ij})\delta e_{ij} + S_{ij}(e_{i\mu}\delta\omega_{\mu j} + \omega_{\mu i}\delta e_{j\mu} + \omega_{i\mu}\delta\omega_{j\mu}) \quad (5.14)$$

From equation 5.11 we obtain the relations

$$\begin{aligned} \omega_{\mu i}\delta e_{j\mu} + \omega_{i\mu}\delta\omega_{j\mu} &= \omega_{\mu i}\delta(e_{\mu j} + \omega_{\mu j}) \\ &= \omega_{\mu i}\frac{\partial}{\partial x_j}\delta u_{\mu} \end{aligned} \quad (5.15)$$

and, finally, changing the dummy indices,

$$S_{ij}(\omega_{\mu i}\delta e_{j\mu} + \omega_{i\mu}\delta\omega_{j\mu}) = S_{kj}\omega_{ik}\frac{\partial}{\partial x_j}\delta u_i \quad (5.16)$$

We also derive

$$S_{ij}e_{i\mu}\delta\omega_{\mu j} = \frac{1}{2}S_{ij}\left(e_{i\mu}\frac{\partial}{\partial x_j}\delta u_{\mu} - e_{i\mu}\frac{\partial}{\partial x_{\mu}}\delta u_j\right) \quad (5.17)$$

and, by a change of dummy indices,

$$S_{ij}e_{i\mu}\delta\omega_{\mu j} = \frac{1}{2}(S_{kj}e_{ki} - S_{ki}e_{kj})\frac{\partial}{\partial x_j}\delta u_i \quad (5.18)$$

Hence we may write

$$\delta V = (t_{ij} + S_{ij} + S_{kj}\omega_{ik} + \frac{1}{2}S_{kj}e_{ki} - \frac{1}{2}S_{ki}e_{kj})\frac{\partial}{\partial x_j}\delta u_i \quad (5.19)$$

Let us now evaluate the variation of the total strain energy potential of the volume τ . Consider the quantity in parentheses in equation 5.19. We may put

$$A_{ij} = t_{ij} + S_{ij} + S_{kj}\omega_{ik} + \frac{1}{2}S_{kj}e_{ki} - \frac{1}{2}S_{ki}e_{kj} \quad (5.20)$$

If we substitute the value of t_{ij} from equation 2.22, we obtain

$$A_{ij} = s_{ij} + S_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj} \quad (5.21)$$

This is the same expression A_{ij} defined in Chapter 1 by equation 7.28. The variation δV is therefore

$$\delta V = A_{ij}\frac{\partial}{\partial x_j}\delta u_i \quad (5.22)$$

The variation of the total strain energy potential of the initial volume τ is

$$\delta\mathcal{V} = \iiint_{\tau} \delta V \, d\tau = \iiint_{\tau} A_{ij} \frac{\partial}{\partial x_j} \delta u_i \, d\tau \quad (5.23)$$

By integration by parts this may be written in the equivalent form

$$\delta\mathcal{V} = - \iiint_{\tau} \frac{\partial A_{ij}}{\partial x_j} \delta u_i \, d\tau + \iint_S A_{ij} n_j \delta u_i \, dS \quad (5.24)$$

where the double integral is extended to the boundary S of the volume τ . The unit vector normal to the boundary and positive outward is denoted by n_j . The surface integral brings out a quantity which we have also encountered previously. By equation 7.53 of Chapter 1 we have

$$f_i = A_{ij} n_j \quad (5.25)$$

This represents the x, y, z components of the force acting on the boundary per unit initial area.

We shall now write a variational equation by expressing that the virtual work of all the external forces, i.e., forces f_i at the boundary and body forces X_i , is equal to the variation of the strain-energy potential of the volume τ . The variational equation is

$$\delta\mathcal{V} = \iiint_{\tau} X_i(\xi_i) \delta u_{i\rho} \, d\tau + \iint_S f_i \delta u_i \, dS \quad (5.26)$$

In the volume integral, representing the virtual work of the body force, the force components $X_i(\xi_i)$ are taken at the displaced points ξ_i . However, ρ and $d\tau$ represent the mass density and the volume element before deformation. Whether $\rho \, d\tau$ is expressed before or after deformation is, of course, immaterial because of the invariance of this quantity which follows from the principle of mass conservation. We have already made use of this property in connection with equation 7.8 in Chapter 1.

Making use of the identity (5.24), we may transform the variational equation (5.26) into

$$\iiint_{\tau} \left[\frac{\partial A_{ij}}{\partial x_j} + \rho X_i(\xi_i) \right] \delta u_i \, d\tau + \iint_S (f_i - A_{ij} n_j) \delta u_i \, dS = 0 \quad (5.27)$$

This equation must be verified for all arbitrary variations of the displacement field; hence we must have

$$\begin{aligned} \frac{\partial A_{ij}}{\partial x_j} + \rho X_i(\xi_i) &= 0 \\ f_i &= A_{ij}n_j \end{aligned} \tag{5.28}$$

With values A_{ij} given by equation 5.20, the first equations are equivalent to the equilibrium conditions (2.23) in terms of the t_{ij} components. With values of A_{ij} in terms of s_{ij} , i.e., given by expressions (5.21), the first equations are identical with the equilibrium conditions (7.29) in Chapter 1. The second of equations 5.28 gives the boundary conditions.

We have therefore established that the variational principle (5.26) is equivalent to the equilibrium equations for the stresses.

The variational principle (5.26) may be formulated in a different form by taking into account the condition that the initial stresses and initial boundary forces are in a state of equilibrium. Let us introduce an incremental strain energy

$$\Delta V = \frac{1}{2}t_{ij}e_{ij} + \frac{1}{2}S_{ij}(e_{i\mu}\omega_{\mu j} + e_{j\mu}\omega_{\mu i} + \omega_{i\mu}\omega_{j\mu}) \tag{5.29}$$

With this definition we may write

$$V = S_{ij}e_{ij} + \Delta V \tag{5.30}$$

Furthermore, by definitions (7.33) and (7.55) of Chapter 1, we also write

$$X_i(\xi_i) = X_i(x_i) + \Delta X_i \tag{5.31}$$

$$f_i = S_{ij}n_j + \Delta f_i \tag{5.32}$$

These equations involve the incremental body force ΔX_i and the incremental boundary force Δf_i . Substituting expressions (5.30), (5.31), and (5.32) into the variational principle (5.26), we obtain

$$\begin{aligned} &\iiint_{\tau} [S_{ij}\delta e_{ij} + \delta \Delta V] d\tau \\ &= \iiint_{\tau} [X_i(x_i) + \Delta X_i]\rho\delta u_i d\tau + \iint_S (S_{ij}n_j + \Delta f_i)\delta u_i dS \end{aligned} \tag{5.33}$$

On the other hand, the condition that the initial stress and the

initial boundary forces are in equilibrium is expressed by the variational principle

$$\iiint_{\tau} S_{ij} \delta e_{ij} d\tau = \iiint_{\tau} X_i(x_i) \rho \delta u_i d\tau + \iint_S S_{ij} n_j \delta u_i dS \quad (5.34)$$

That this is effectively the case is easily shown after integration by parts of the integral on the left side. Equation 5.34 then becomes

$$\iiint_{\tau} \left(\frac{\partial S_{ij}}{\partial x_j} + \rho X_i \right) \delta u_i d\tau = 0 \quad (5.35)$$

This relation is verified because the initial stress must satisfy the equilibrium conditions

$$\frac{\partial S_{ij}}{\partial x_j} + \rho X_i = 0 \quad (5.36)$$

We now subtract equation 5.34 from equation 5.33 and obtain

$$\iiint_{\tau} \delta \Delta V d\tau = \iiint_{\tau} \Delta X_i \rho \delta u_i d\tau + \iint_S \Delta f_i \delta u_i dS \quad (5.37)$$

In this form the variational principle states that the variation of incremental strain energy is equal to the virtual work of the incremental body forces and the incremental boundary forces. The variational principle is thus formulated in *incremental form*.

Attention should be called to the significance of the incremental energy ΔV as defined by equation 5.30. The local increment of strain energy is actually V and not ΔV . However, the additional term $S_{ij} e_{ij}$ cancels out of the final equations when the equilibrium conditions and the boundary forces in the initial state are taken into account, as shown by equation 5.34.

If we denote by

$$\Delta \mathcal{V} = \iiint_{\tau} \Delta V d\tau \quad (5.38)$$

the total incremental strain energy, the variational principle (5.37) may be written

$$\delta \Delta \mathcal{V} = \iiint_{\tau} \Delta X_i \rho \delta u_i d\tau + \iint_S \Delta f_i \delta u_i dS \quad (5.39)$$

Another alternative formulation is obtained if the body force is derived from a potential. Then

$$X_i = - \frac{\partial U}{\partial x_i} \quad (5.40)$$

where $U(x_i)$ is a potential function of the coordinates. For example, U may be a gravitational or centrifugal potential. According to equation 7.43 of Chapter 1,

$$\Delta X_i = u_j \frac{\partial X_i}{\partial x_j} + \Delta' X_i \quad (5.41)$$

We may put $\Delta' X_i = 0$ since X_i is a point function of the coordinates. Hence

$$\Delta X_i = -u_j \frac{\partial^2 U}{\partial x_j \partial x_i} \quad (5.42)$$

The following incremental body force potential is introduced:

$$\Delta U = \frac{1}{2} \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \quad (5.43)$$

Then the variation of this quantity is

$$\delta \Delta U = \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \delta u_i = -\Delta X_i \delta u_i \quad (5.44)$$

When we use the last relation, the variational principle (5.37) becomes

$$\iiint_{\tau} \delta(\Delta V + \rho \Delta U) d\tau = \iint_S \Delta f_i \delta u_i dS \quad (5.45)$$

The variational principle in this case takes the form

$$\delta \mathcal{P}_{\tau} = \iint_S \Delta f_i \delta u_i dS \quad (5.46)$$

where

$$\mathcal{P}_{\tau} = \iiint_{\tau} (\Delta V + \rho \Delta U) d\tau \quad (5.47)$$

is the total incremental potential, i.e., the sum of the incremental strain energy and the incremental body force potential of the original volume τ .

Equilibrium Equations in Curvilinear Coordinates. The variational principle (5.46) has been derived in the context of elasticity by assuming the existence of a strain energy. However, this assumption is not necessary. It is possible to state the variational principle in such a way that it yields the equilibrium equations independently of any physical properties of the medium. This is

done by applying the principle of virtual work. In the present case it takes the form*

$$\iiint_{\tau} (t_{ij} + S_{ij}) \delta \varepsilon_{ij} d\tau = \iiint_{\tau} \rho X_i(\xi_i) \delta u_i d\tau + \iint_S f_i \delta u_i dS \quad (5.48)$$

Since the medium is in equilibrium in the initial state of stress, the principle also applies in that state. Hence

$$\iiint_{\tau} S_{ij} \delta e_{ij} d\tau = \iiint_{\tau} \rho X_i(x_i) \delta u_i d\tau + \iint_S S_{ij} n_j \delta u_i dS \quad (5.49)$$

The difference of the two equations 5.48 and 5.49 is

$$\iiint_{\tau} (t_{ij} \delta e_{ij} + S_{ij} \delta \eta_{ij}) d\tau = \iiint_{\tau} \rho \Delta X_i \delta u_i d\tau + \iint_S \Delta f_i \delta u_i dS \quad (5.50)$$

In writing this difference we have substituted $t_{ij} \delta e_{ij}$ for $t_{ij} \delta \varepsilon_{ij}$ since it does not affect the significant terms, namely, those of the first and second order. We have also put

$$\eta_{ij} = \frac{1}{2}(e_{iu} \omega_{uj} + e_{ju} \omega_{ui} + \omega_{iu} \omega_{ju}) \quad (5.51)$$

That the variational equation (5.50) is equivalent to the equilibrium condition may be verified by following exactly the procedure used for equation 5.26.

The principle of virtual work expressed in the form of equation 5.50 is of particular interest in deriving the equilibrium equations in orthogonal curvilinear coordinates. In these equations the initial stress is referred to local orthogonal axes tangent to the coordinate lines. The same local axes are used to represent the other variables. The body force ΔX_i and the displacements u_i are projected on these axes without difficulty.† For the strain and the rotation we must take into account the curvature of the coordinate system. The differential element ds is written

$$ds^2 = h_1^2 dq_1 + h_2^2 dq_2 + h_3^2 dq_3 \quad (5.52)$$

where h_1, h_2, h_3 are functions of the curvilinear coordinates q_1, q_2, q_3 . When the medium is deformed, the coordinates of a particle become

$$\begin{aligned} \bar{q}_1 &= q_1 + \alpha_1 \\ \bar{q}_2 &= q_2 + \alpha_2 \\ \bar{q}_3 &= q_3 + \alpha_3 \end{aligned} \quad (5.53)$$

* This form was used by the author as the basis for a non-linear theory of elasticity (see reference 5 in the Preface).

† For further elaboration see p. 492.

The displacement, represented by $\alpha_1, \alpha_2, \alpha_3$, is assumed to be small. The first order strain components referred to local axes are then

$$\begin{aligned} e_{11} &= \frac{1}{2} \frac{\Delta h_1^2}{h_1^2} + \frac{\partial \alpha_1}{\partial q_1} \\ e_{12} &= \frac{1}{2} \left(\frac{h_1}{h_2} \frac{\partial \alpha_1}{\partial q_2} + \frac{h_2}{h_1} \frac{\partial \alpha_2}{\partial q_1} \right) \\ &\text{etc.} \end{aligned} \quad (5.54)$$

with

$$\Delta h_1^2 = \frac{\partial h_1^2}{\partial q_1} \alpha_1 + \frac{\partial h_1^2}{\partial q_2} \alpha_2 + \frac{\partial h_1^2}{\partial q_3} \alpha_3$$

The rotation is

$$\omega_{21} = -\omega_{12} = \frac{1}{2} \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial q_1} (h_2^2 \alpha_2) - \frac{\partial}{\partial q_2} (h_1^2 \alpha_1) \right], \quad \text{etc.} \quad (5.55)$$

Other components of e_{ij} and ω_{ij} are derived by cyclic permutation of indices. The particle displacement u_i considered as a first order quantity and *projected on the local cartesian axes* is

$$\begin{aligned} u_1 &= h_1 \alpha_1 \\ u_2 &= h_2 \alpha_2 \\ u_3 &= h_3 \alpha_3 \end{aligned} \quad (5.56)$$

Substitution of these values of e_{ij} and ω_{ij} in the variational equation 5.50 yields the differential equations of equilibrium of the stress field in curvilinear coordinates.

A detailed derivation of the values in equations 5.54 and 5.55 for the strain and rotation components in curvilinear coordinates is given by Temple.* An outline of his derivation of the strain components follows.

Expressions (5.54) for the strain components may be obtained by considering the value of ds^2 in the deformed state. It becomes

$$ds^2 = (h_1^2 + \Delta h_1^2) d\bar{q}_1^2 + (h_2^2 + \Delta h_2^2) d\bar{q}_2^2 + (h_3^2 + \Delta h_3^2) d\bar{q}_3^2 \quad (5.56a)$$

with

$$\begin{aligned} \Delta h_1^2 &= \frac{\partial h_1^2}{\partial \alpha_1} \alpha_1 + \frac{\partial h_1^2}{\partial \alpha_2} \alpha_2 + \frac{\partial h_1^2}{\partial \alpha_3} \alpha_3 \\ d\bar{q}_1 &= \left(1 + \frac{\partial \alpha_1}{\partial q_1} \right) dq_1 + \frac{\partial \alpha_1}{\partial q_2} dq_2 + \frac{\partial \alpha_1}{\partial q_3} dq_3 \\ &\text{etc.} \end{aligned} \quad (5.56b)$$

* G. Temple, *Cartesian Tensors*, p. 84, Methuen & Co. Ltd., London, John Wiley & Sons, Inc., New York, 1960.

The value of ds^2 may be written as a quadratic form in dq_i . Dropping terms of higher order than α_i , we find

$$ds^2 = \left(h_1^2 + \Delta h_1^2 + 2h_1^2 \frac{\partial \alpha_1}{\partial q_1} \right) dq_1^2 + 2 \left(h_1^2 \frac{\partial \alpha_1}{\partial q_2} + h_2^2 \frac{\partial \alpha_2}{\partial q_1} \right) dq_1 dq_2 + \text{etc.} \quad (5.56c)$$

We introduce the differential lengths along the local coordinates:

$$\begin{aligned} dx_1 &= h_1 dq_1 \\ dx_2 &= h_2 dq_2 \\ dx_3 &= h_3 dq_3 \end{aligned} \quad (5.56d)$$

Using these differentials, we may identify the quadratic form with expression (3.12) of Chapter 1 and thus obtain the strain components (5.54).

The rotation (5.55) may also be derived by application of Stokes' theorem. We evaluate the circulation of the displacement u_i on a closed contour along coordinate lines and divide by the area enclosed.

6. INCREMENTAL ELASTIC COEFFICIENTS FOR AN ORTHOTROPIC MEDIUM

At this point it is important to clarify the physical significance of the incremental coefficients. There are two cases of particular interest which readily come to mind. We may assume the unstressed medium to be either of *orthotropic** or *isotropic* symmetry in the original unstressed condition.

In this section we shall discuss orthotropic symmetry. The medium with isotropic properties will be considered in the next section.

A medium of orthotropic symmetry is such that its elastic properties are symmetric with respect to three orthogonal planes. For convenience we choose the coordinate axes to be oriented along the directions of symmetry.

We also assume that the principal components of the initial stress are directed along the planes of elastic symmetry of the medium. Then the symmetry of the material is not modified by the initial stress, and it applies equally to the incremental deformation.

* In crystallography the term orthorhombic is used instead of orthotropic for this type of symmetry.

The state of initial stress is represented by its three principal components S_{11}, S_{22}, S_{33} along the coordinate axes. Since the incremental stress-strain relations possess orthotropic symmetry, they must necessarily assume the form

$$\begin{aligned}
 s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} + B_{13}e_{zz} \\
 s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} + B_{23}e_{zz} \\
 s_{33} &= B_{31}e_{xx} + B_{32}e_{yy} + B_{33}e_{zz} \\
 s_{23} &= 2Q_1e_{yz} \\
 s_{31} &= 2Q_2e_{zx} \\
 s_{12} &= 2Q_3e_{xy}
 \end{aligned} \tag{6.1}$$

These relations describe a particular case of the general equations 4.15 obtained by reducing the number of indices for the coefficients.

The existence of a strain energy requires the coefficients to satisfy relations (4.31), which in this case are simplified to

$$\begin{aligned}
 B_{12} - B_{21} &= S_{22} - S_{11} \\
 B_{23} - B_{32} &= S_{33} - S_{22} \\
 B_{31} - B_{13} &= S_{11} - S_{33}
 \end{aligned} \tag{6.2}$$

The stress-strain relations (6.1) may also be written in a form which is more convenient for practical use:

$$\begin{aligned}
 s_{11} &= C_{11}e_{xx} + (C_{12} - S_{11})e_{yy} + (C_{31} - S_{11})e_{zz} \\
 s_{22} &= (C_{12} - S_{22})e_{xx} + C_{22}e_{yy} + (C_{23} - S_{22})e_{zz} \\
 s_{33} &= (C_{31} - S_{33})e_{xx} + (C_{23} - S_{33})e_{yy} + C_{33}e_{zz} \\
 s_{23} &= 2Q_1e_{yz} \\
 s_{31} &= 2Q_2e_{zx} \\
 s_{12} &= 2Q_3e_{xy}
 \end{aligned} \tag{6.3}$$

The coefficients in these equations automatically satisfy relations (6.2). They correspond to the form (4.17) of the general equations. In the last three of equations 6.3 the coefficients are only subject to a change of notation. Equations 6.3 may be written by means of the alternative stress components t_{ij} . Applying equations 2.22, we find

$$\begin{aligned}
 t_{11} &= s_{11} + S_{11}(e_{yy} + e_{zz}) \\
 t_{22} &= s_{22} + S_{22}(e_{zz} + e_{xx}) \\
 t_{33} &= s_{33} + S_{33}(e_{xx} + e_{yy}) \\
 t_{23} &= s_{23} \quad t_{31} = s_{31} \quad t_{12} = s_{12}
 \end{aligned} \tag{6.4}$$

When we substitute equations 6.4, equations 6.3 become

$$\begin{aligned}
 t_{11} &= C_{11}e_{xx} + C_{12}e_{yy} + C_{31}e_{zz} \\
 t_{22} &= C_{12}e_{xx} + C_{22}e_{yy} + C_{23}e_{zz} \\
 t_{33} &= C_{31}e_{xx} + C_{23}e_{yy} + C_{33}e_{zz} \\
 t_{23} &= 2Q_1e_{yz} \\
 t_{31} &= 2Q_2e_{zx} \\
 t_{13} &= 2Q_3e_{xy}
 \end{aligned} \tag{6.5}$$

These equations correspond to the form (4.8) of the general equations where the coefficients are $C_{ij}^{\mu\nu}$. The six coefficients

$$\begin{array}{ccc}
 C_{11} & C_{12} & C_{31} \\
 C_{12} & C_{22} & C_{23} \\
 C_{31} & C_{23} & C_{33}
 \end{array} \tag{6.6}$$

constitute a *symmetric matrix*, thereby satisfying the symmetry relations (4.14) of the general theory, namely,

$$C_{ij}^{\mu\nu} = C_{\mu\nu}^{ij} \tag{6.7}$$

Note that the stress-strain relations (6.3) contain nine independent coefficients which are functions of the initial deformation.

Let us now examine the physical significance and measurements involved in the six elastic coefficients (6.6). Denote the original coordinates of the medium in the unstressed state by X, Y, Z . In the state of homogeneous initial stress, considered here, they become

$$\begin{aligned}
 x &= \lambda_1 X \\
 y &= \lambda_2 Y \\
 z &= \lambda_3 Z
 \end{aligned} \tag{6.8}$$

The extension ratios $\lambda_1, \lambda_2, \lambda_3$ represent the lengths acquired by the sides of a cube originally of unit dimension oriented along the directions of orthotropic symmetry (the same as x, y, z).

We denote by

$$W = W(\lambda_1, \lambda_2, \lambda_3) \tag{6.9}$$

the strain energy of this deformed cube. The normal forces acting on each face of the deformed cube are

$$f_1 = \frac{\partial W}{\partial \lambda_1} \quad f_2 = \frac{\partial W}{\partial \lambda_2} \quad f_3 = \frac{\partial W}{\partial \lambda_3} \tag{6.10}$$

and the normal stresses are

$$S_{11} = \frac{1}{\lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_1} \quad S_{22} = \frac{1}{\lambda_3 \lambda_1} \frac{\partial W}{\partial \lambda_2} \quad S_{33} = \frac{1}{\lambda_1 \lambda_2} \frac{\partial W}{\partial \lambda_3} \quad (6.11)$$

A small additional deformation along the same principal direction will produce incremental stresses which may be identified with the differentials. We write

$$s_{11} = dS_{11} = \frac{\partial S_{11}}{\partial \lambda_1} d\lambda_1 + \frac{\partial S_{11}}{\partial \lambda_2} d\lambda_2 + \frac{\partial S_{11}}{\partial \lambda_3} d\lambda_3 \quad (6.12)$$

and two analogous expressions for s_{22} and s_{33} . Evaluating the differentials, we find the first three of relations (6.3). In doing this we must take into account the following definitions of the incremental strains,

$$e_{xx} = \frac{d\lambda_1}{\lambda_1} \quad e_{yy} = \frac{d\lambda_2}{\lambda_2} \quad e_{zz} = \frac{d\lambda_3}{\lambda_3} \quad (6.13)$$

The elastic coefficients are

$$\begin{aligned} C_{11} &= \frac{\lambda_1}{\lambda_2 \lambda_3} \frac{\partial f_1}{\partial \lambda_1} = \frac{\lambda_1}{\lambda_2 \lambda_3} \frac{\partial^2 W}{\partial \lambda_1^2} \\ C_{22} &= \frac{\lambda_2}{\lambda_3 \lambda_1} \frac{\partial f_2}{\partial \lambda_2} = \frac{\lambda_2}{\lambda_3 \lambda_1} \frac{\partial^2 W}{\partial \lambda_2^2} \\ C_{33} &= \frac{\lambda_3}{\lambda_1 \lambda_2} \frac{\partial f_3}{\partial \lambda_3} = \frac{\lambda_3}{\lambda_1 \lambda_2} \frac{\partial^2 W}{\partial \lambda_3^2} \\ C_{23} &= \frac{1}{\lambda_1} \frac{\partial f_2}{\partial \lambda_3} = \frac{1}{\lambda_1} \frac{\partial f_3}{\partial \lambda_2} = \frac{1}{\lambda_1} \frac{\partial^2 W}{\partial \lambda_2 \partial \lambda_3} \\ C_{31} &= \frac{1}{\lambda_2} \frac{\partial f_3}{\partial \lambda_1} = \frac{1}{\lambda_2} \frac{\partial f_1}{\partial \lambda_3} = \frac{1}{\lambda_2} \frac{\partial^2 W}{\partial \lambda_3 \partial \lambda_1} \\ C_{12} &= \frac{1}{\lambda_3} \frac{\partial f_1}{\partial \lambda_2} = \frac{1}{\lambda_3} \frac{\partial f_2}{\partial \lambda_1} = \frac{1}{\lambda_3} \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \end{aligned} \quad (6.14)$$

The Slide Modulus and Its Physical Significance. Our next discussion will be concerned with the significance of the shear coefficients Q_1, Q_2, Q_3 . Let us impose a shear displacement parallel to the x direction. The displacement components are

$$u = \gamma y \quad v = w = 0 \quad (6.15)$$

This is a plane deformation parallel to the x, y plane (Fig. 6.1). A rectangle $ABDC$ in this plane is deformed into the parallelogram $ABD'C'$. In order to produce this deformation we must apply to the face CD a tangential force which may be evaluated from equations 6.27 of Chapter 1. When we put $S_{12} = 0$ in these equations, they become

$$\begin{aligned} \Delta f_x &= (s_{11} + S_{11}e_{yy}) \cos(n, x) \\ &\quad + (s_{12} - S_{22}\omega - S_{11}e_{xy}) \cos(n, y) \\ \Delta f_y &= (s_{12} + S_{11}\omega - S_{22}e_{xy}) \cos(n, x) \\ &\quad + (s_{22} + S_{22}e_{xx}) \cos(n, y) \end{aligned} \quad (6.16)$$

These expressions represent the incremental forces acting on a face

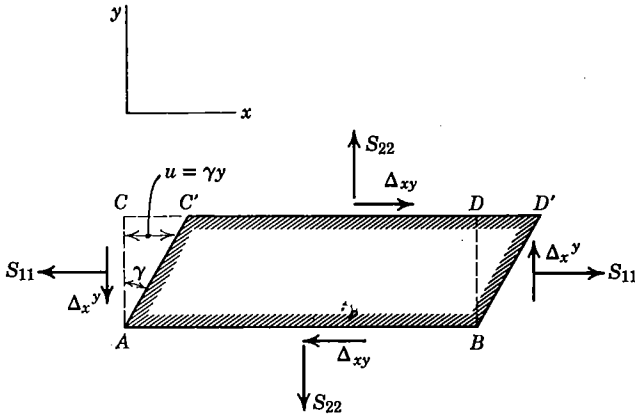


Figure 6.1 Physical significance of the slide modulus L_{12} .

of the material when the normal direction is defined by the directional cosines, $\cos(n, x)$ and $\cos(n, y)$. For the shear displacement (equations 6.15) we may write

$$\begin{aligned} e_{xy} &= -\omega = \frac{1}{2}\gamma \\ e_{xx} &= e_{yy} = e_{zz} = 0 \\ e_{yz} &= e_{zx} = 0 \end{aligned} \quad (6.17)$$

From the stress-strain relations (6.3) we derive

$$\begin{aligned} s_{11} &= s_{22} = 0 \\ s_{12} &= Q_3 \gamma \end{aligned} \quad (6.18)$$

Hence expressions (6.16) become

$$\begin{aligned}\Delta f_x &= \Delta_{xy} \cos(n, y) \\ \Delta f_y &= \Delta_x^y \cos(n, x)\end{aligned}\quad (6.19)$$

where

$$\begin{aligned}\Delta_{xy} &= (Q_3 + \frac{1}{2}S_{22} - \frac{1}{2}S_{11})\gamma \\ \Delta_x^y &= (Q_3 - \frac{1}{2}S_{22} - \frac{1}{2}S_{11})\gamma\end{aligned}\quad (6.20)$$

The quantity Δ_{xy} denotes the tangential stress applied to the face CD in order to produce the shear displacement defined by the angle γ . The relation

$$\Delta_{xy} = L_{12}\gamma \quad (6.21)$$

introduces a *measurable slide modulus*

$$L_{12} = Q_3 + \frac{1}{2}(S_{22} - S_{11}) \quad (6.22)$$

This relation may also be considered an experimental definition for Q_3 . To emphasize the distinction between these two quantities we shall refer to Q_3 as a *shear modulus*. Note that the quantity Δ_x^y defines the vertical stress which is applied to the sides AC and BD . From equations 6.20 we derive

$$\Delta_x^y = \Delta_{xy} - S_{22}\gamma \quad (6.23)$$

This equation turns out to express the condition for equilibrium of moments in the x, y plane.

The procedure may be repeated for the other coordinate planes, thus relating coefficients Q_1 and Q_2 to a shear test. The process, however, is not unique, and we may define more than three such experimental coefficients. Coefficients of the type L_{12} play an important role in applications. They must, however, be suitably defined in each specific case by referring to a particular configuration such as shown in Figure 6.1.

An alternative interpretation of the coefficient L_{12} may be given by considering a strip of material initially subject to an axial stress F and a uniform hydrostatic pressure p_f (Fig. 6.2). The initial stress components here are

$$\begin{aligned}S_{11} &= F - p_f \\ S_{22} &= -p_f\end{aligned}\quad (6.24)$$

The specimen is immersed in a fluid at pressure p_f while it is pulled

by an axial stress F in addition to the hydrostatic stress. This state of initial stress is then disturbed by a tangential force Δ_{xy} producing a shearing deformation γ . The coefficient L_{12} is the ratio Δ_{xy}/γ . Note that in this resolution of the stresses the *same force* Δ_{xy} must also be added on AC and BD in order to balance the moments.

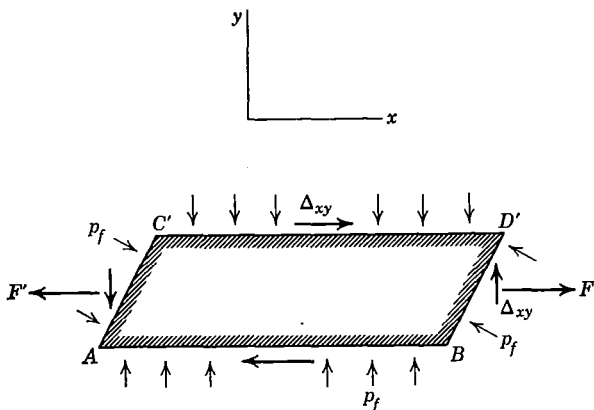


Figure 6.2 Alternative interpretation of the slide modulus L_{12} with a test specimen under hydrostatic pressure p_f .

The preceding definition of the slide modulus L_{12} refers not only to the x, y plane but also to both the x direction and the x, y plane. It is not symmetric with respect to subscripts 1 and 2. This is illustrated by the interpretation in Figure 6.2, where the shearing deformation is in the x, y plane while the force F superimposed on the hydrostatic pressure is acting along the x direction. Hence it is possible to define two slide moduli for each coordinate plane. They are

$$\begin{aligned}
 L_{23} &= Q_1 + \frac{1}{2}(S_{33} - S_{22}) & L_{32} &= Q_1 + \frac{1}{2}(S_{22} - S_{33}) \\
 L_{31} &= Q_2 + \frac{1}{2}(S_{11} - S_{33}) & L_{13} &= Q_2 + \frac{1}{2}(S_{33} - S_{11}) \\
 L_{12} &= Q_3 + \frac{1}{2}(S_{22} - S_{11}) & L_{21} &= Q_3 + \frac{1}{2}(S_{11} - S_{22})
 \end{aligned} \tag{6.25}$$

Note the relations

$$\begin{aligned}
 Q_1 &= \frac{1}{2}(L_{23} + L_{32}) \\
 Q_2 &= \frac{1}{2}(L_{31} + L_{13}) \\
 Q_3 &= \frac{1}{2}(L_{12} + L_{21})
 \end{aligned} \tag{6.26}$$

Hence only three of the slide moduli are independent.

7. INCREMENTAL ELASTIC COEFFICIENTS FOR AN ISOTROPIC MEDIUM

We now consider the important case of a medium for which the finite stress-strain law is isotropic. By this we mean that the stress is related to the finite strain by relations which are independent of the orientation of the stress field. From the viewpoint of incremental stresses the medium is isotropic in the vicinity of the unstressed state, but it will become anisotropic in a state of finite strain considered as the initial state. The coordinate system may be chosen to coincide with the principal directions of the initial stress which is then represented by its three principal components S_{11}, S_{22}, S_{33} . Since the medium is isotropic in finite strain, the principal directions of stress define three planes of symmetry for the incremental elastic properties. In other words, the incremental stress-strain relations must possess orthotropic symmetry.

In general it is possible to distinguish between three cases for the elastic symmetry of a given material. The symmetry may refer to properties of finite deformations. It may also apply to small deformations of a medium originally stress free or to small incremental deformations in the vicinity of a state of initial stress. The following terminology may be used to emphasize the distinction between various cases.

Finite Isotropy: the finite stress-strain relations are independent of the stress orientation.

Finite Orthotropy: the elastic symmetry is orthotropic in finite strain.

Incremental Orthotropy: the elastic symmetry is orthotropic for incremental stresses.

Induced Orthotropy: the orthotropy is caused by the initial stress.

Intrinsic Orthotropy: the orthotropy is not induced.

Related expressions whose meaning follows from these definitions are incremental symmetry, incremental anisotropy, intrinsic symmetry, etc.

The case to be discussed in this section is that of finite isotropy with induced incremental orthotropy. The incremental stresses are

governed by the same equations (6.1 and 6.3) discussed in the preceding section. However, there is one difference. Because of the finite isotropy the nine coefficients in these equations cannot be chosen arbitrarily. We shall show that the coefficients Q_1, Q_2, Q_3 are given by important and very simple formulas which, except for some special degenerate cases, involve only the finite initial strain and the corresponding stresses.

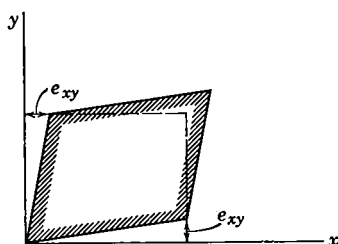


Figure 7.1 Incremental shear deformation corresponding to equations 7.1.

To show this let us start from coordinates X, Y, Z of the medium in the original unstressed state. We shall call this state (a). By applying the principal stresses S_{11}, S_{22}, S_{33} along the coordinate axes the medium goes into the state of initial stress denoted as state (b). The coordinates x, y, z in state (b) are given by equations 6.8. A small incremental shear strain e_{xy} parallel to the x, y plane is now superimposed on state (b), and it deforms the medium into state (c) (Fig. 7.1). In this state the coordinates of the medium become

$$\begin{aligned}\xi &= x + e_{xy}y \\ \eta &= e_{xy}x + y \\ \zeta &= z\end{aligned}\tag{7.1}$$

Substituting the values (6.8) for $x, y,$ and z yields

$$\begin{aligned}\xi &= \lambda_1 X + \lambda_2 e_{xy} Y \\ \eta &= \lambda_1 e_{xy} X + \lambda_2 Y \\ \zeta &= \lambda_3 Z\end{aligned}\tag{7.2}$$

These equations represent the total transformation of coordinates from state (a) to state (c). In the following analysis we need to

consider only the first two of equations 7.2, which represent the transformation in the x, y plane. These two equations are equivalent to the transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad (7.3)$$

By referring to section 2 of Chapter 1, the reader will notice that this represents a solid rotation through an angle θ in the plane followed by a pure deformation. We may find the coefficients b_{ij} and the rotation θ by identifying the four coefficients in the two sets of transformations (7.2 and 7.3). Following a procedure identical with the solution of equations 2.11 of Chapter 1, we find

$$\begin{aligned} \tan \theta &= e_{xy} \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \\ b_{11} &= \lambda_1 \cos \theta - e_{xy} \lambda_2 \sin \theta \\ b_{22} &= e_{xy} \lambda_1 \sin \theta + \lambda_2 \cos \theta \\ 2b_{12} &= e_{xy} (\lambda_1 + \lambda_2) \cos \theta + (\lambda_1 - \lambda_2) \sin \theta \end{aligned} \quad (7.4)$$

Since e_{xy} is assumed to be a small quantity of the first order, θ is also of the same order. Hence, neglecting second order terms, we may write

$$\begin{aligned} b_{11} &= \lambda_1 \\ b_{22} &= \lambda_2 \\ b_{12} &= e_{xy} \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 + \lambda_2} \end{aligned} \quad (7.5)$$

The symmetric matrix b_{ij} defines two principal directions of strain. We have discussed this in Chapter 1 and have shown that these directions are obtained by using equations 2.4 of that chapter. Here these equations become

$$\begin{aligned} b \cos \alpha &= b_{11} \cos \alpha + b_{12} \sin \alpha \\ b \sin \alpha &= b_{12} \cos \alpha + b_{22} \sin \alpha \end{aligned} \quad (7.6)$$

where α represents the angle of the principal direction with the x axis and b is the principal elongation (Fig. 7.2). Solving equations 7.6 for α and b yields

$$\begin{aligned} \frac{1}{2} \tan 2\alpha &= \frac{b_{12}}{b_{11} - b_{22}} \\ b &= b_{11} \cos^2 \alpha + b_{22} \sin^2 \alpha + b_{12} \sin 2\alpha \end{aligned} \quad (7.7)$$

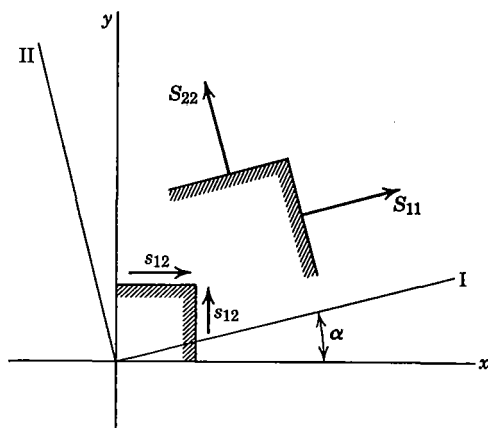


Figure 7.2 Resolution of stresses for incremental shear.

There are two principal directions. Let us denote by I the direction corresponding to the smallest angle α . Since b_{12} is of the first order, α is of the same order. Hence to the first order

$$\alpha = \frac{b_{12}}{b_{11} - b_{22}} \quad (7.8)$$

and, because of equations 7.5,

$$\alpha = e_{xy} \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \quad (7.9)$$

The elongation in direction I is the value b_I of b obtained by substituting this value of α into the second of equations 7.7. With an error of the second order we find

$$b_I = b_{11} = \lambda_1 \quad (7.10)$$

Principal direction II makes an angle $\alpha + (\pi/2)$ with the x axis. With an error of the second order the elongation in that direction is

$$b_{II} = b_{22} = \lambda_2 \quad (7.11)$$

To the preceding approximation, the principal elongations in state (c) are therefore equal to the values λ_1 , λ_2 , and λ_3 in state (b). They are, however, oriented at an angle α with the principal directions of state (b) in the x, y plane.

Here we introduce the important assumption of *isotropy in finite strain*. Since these finite stress-strain relations are independent of the orientation of the stress, the principal stresses in state (c) must (except for second order terms) be equal to the stresses S_{11} and S_{22} of state (b) and must be oriented at an angle α with their direction in state (b). This is illustrated in Figure 7.2.

We may resolve the principal stresses S_{11} and S_{22} oriented along directions I and II into x and y components. This may be done by using equations 4.6 of Chapter 1. We find a shear component

$$s_{12} = \frac{1}{2}(S_{11} - S_{22}) \sin 2\alpha \tag{7.12}$$

which to the first order is

$$s_{12} = (S_{11} - S_{22})\alpha \tag{7.13}$$

Substituting the value (7.9) for α , we finally obtain

$$s_{12} = 2Q_3 e_{xy} \tag{7.14}$$

where Q_3 is now

$$Q_3 = \frac{1}{2}(S_{11} - S_{22}) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \tag{7.15}$$

This expression is the incremental shear coefficient in the last of equations 6.3.

By repeating the same calculation for incremental shear strain in the other coordinate planes we derive*

$$\begin{aligned} Q_1 &= \frac{1}{2}(S_{22} - S_{33}) \frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2} \\ Q_2 &= \frac{1}{2}(S_{33} - S_{11}) \frac{\lambda_3^2 + \lambda_1^2}{\lambda_3^2 - \lambda_1^2} \end{aligned} \tag{7.16}$$

We have therefore obtained the announced result that for an isotropic medium in finite strain the incremental shear coefficients may be expressed in terms of the initial stresses and strains. It is interesting

* These results were obtained by the author as published in two recent papers: M. A. Biot, Incremental Elastic Coefficients of an Isotropic Medium in Finite Strain, Air Force Office of Scientific Research Technical Report No. 1772, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 151-167, 1963; Internal Buckling under Initial Stress in Finite Elasticity, *Proceedings of the Royal Society, A*, Vol. 273, pp. 306-328, 1963.

to note that they do not involve any physical incremental change. This is in contrast with the coefficients C_{ij} of equations 6.3.

However, the last remark applies only if the initial stresses are all different. If they are not, there arises an indeterminacy. Consider the case where

$$\begin{aligned} S_{11} &= S_{22} \\ \lambda_1 &= \lambda_2 \end{aligned} \quad (7.17)$$

Expression (7.15) for Q_3 becomes indeterminate. To find its true value we introduce a differential increase $d\lambda_1$ of λ_1 and write

$$\begin{aligned} d(S_{11} - S_{22}) &= s_{11} - s_{22} \\ \lambda_1^2 - \lambda_2^2 &= 2\lambda_1 d\lambda_1 \\ e_{xx} &= \frac{d\lambda_1}{\lambda_1} \end{aligned} \quad (7.18)$$

Substitution in equation 7.15 yields the limiting value for $\lambda_1 = \lambda_2$,

$$Q_3 = \frac{1}{2} \frac{s_{11} - s_{22}}{e_{xx}} \quad (7.19)$$

Putting $e_{yy} = e_{zz} = 0$ in equations 6.1 and subtracting the first two equations, we obtain

$$s_{11} - s_{22} = (B_{11}^* - B_{21})e_{xx} \quad (7.20)$$

Hence, comparing equations 7.19 and 7.20, we derive

$$B_{11} = B_{21} + 2Q_3 \quad (7.21)$$

Similarly, by varying λ_2 or more simply by considerations of symmetry, we also derive

$$B_{22} = B_{12} + 2Q_3 \quad (7.22)$$

Because $S_{11} = S_{22}$, equations 6.2 yield

$$B_{12} = B_{21} \quad (7.23)$$

and hence also

$$B_{11} = B_{22} = B_{12} + 2Q_3 \quad (7.24)$$

Moreover, axial symmetry about the z direction implies the relations

$$B_{12} = B_{23} \quad B_{31} = B_{32} \quad B_{13} = B_{23} \quad Q_1 = Q_2 = Q \quad (7.25)$$

Therefore the stress-strain relations (6.1) become

$$\begin{aligned}
 s_{11} &= 2Q_3e_{xx} + B_{12}(e_{xx} + e_{yy}) + B_{13}e_{zz} \\
 s_{22} &= 2Q_3e_{yy} + B_{12}(e_{xx} + e_{yy}) + B_{13}e_{zz} \\
 s_{33} &= B_{31}(e_{xx} + e_{yy}) + B_{33}e_{zz} \\
 s_{23} &= 2Qe_{yz} \\
 s_{31} &= 2Qe_{zx} \\
 s_{12} &= 2Q_3e_{xy}
 \end{aligned}
 \tag{7.26}$$

It is seen that the coefficient Q_3 in this case involves the incremental properties for extensions in the x and y directions. We note the relations

$$\begin{aligned}
 B_{13} &= C_{31} - S_{11} \\
 B_{31} &= C_{31} - S_{33}
 \end{aligned}
 \tag{7.27}$$

and the inequality

$$B_{13} \neq B_{31}
 \tag{7.28}$$

Equations 7.26 represent a medium with *transverse isotropic* symmetry about the z direction. The equations remain invariant for a rotation about the z axis. In particular, equations 7.26 are valid for finite isotropy when the initial stress is *uniaxial* ($S_{11} = S_{22} = 0$ and $S_{33} \neq 0$).

Values of the coefficients B_{ij} as related to the finite stress-strain relations of an isotropic medium will be discussed further in section 7 of Chapter 5.

The Method of Tensor Invariants. Expressions (7.15) and (7.16) for the incremental moduli of a medium with finite isotropy may be obtained by an alternative method which uses the three fundamental invariants of the strain. These invariants, usually denoted by I_1, I_2, I_3 , are functions of the finite strain components and remain unchanged when the medium undergoes a rigid rotation. In a medium of finite isotropy the strain energy is a function only of these three invariants. Hence it is possible to use this property in order to derive the results obtained in this section. The procedure, however, turns out to be much more elaborate than the one presented above. In addition, the physics of the phenomenon remains obscure and no interpretation of the result is obtained. This has been shown in the author's paper* where these new results are derived by both methods for the purpose of comparison.

* M. A. Biot, Incremental Elastic Coefficients of an Isotropic Medium in Finite Strain, Air Force Office of Scientific Research Report TN 1772, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 151-167, 1963.

The theory of tensor invariants has also been applied by some authors* to an isotropic elastic medium to derive equations which govern small deformations superposed on a state of finite strain. The usefulness of this procedure is restricted by its elaborate formalism.

8. INCREMENTAL STRESSES IN INCOMPRESSIBLE MEDIA; APPLICATION TO RUBBER ELASTICITY

The special case of an incompressible elastic medium is of particular interest. We shall consider first the orthotropic symmetry with coordinate axes lying in the planes of symmetry. A unit cube with sides oriented along the x, y, z axes subject to normal stresses S_{11}, S_{22}, S_{33} on its three faces becomes a rectangular parallelepiped whose edges acquire lengths $\lambda_1, \lambda_2, \lambda_3$.

These extension ratios represent the initial finite strain. The difference between this and the more general case considered in section 6 arises from the fact that the volume remains constant. In other words, the three extension ratios must satisfy the constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1 \quad (8.1)$$

This means that there are only two independent strain variables for extensions along the coordinate axes.

In expressing the strain energy of the original unit cube we may consider it a function of any pair of independent variables, λ_1 and λ_2 for example. It may be written

$$W = W(\lambda_1, \lambda_2) \quad (8.2)$$

The normal forces acting on the faces of the solid being denoted by f_1, f_2, f_3 , conservation of energy requires the following differential relation to be satisfied:

$$dW = f_1 d\lambda_1 + f_2 d\lambda_2 + f_3 d\lambda_3 \quad (8.3)$$

In this relation the differentials $d\lambda_1, d\lambda_2, d\lambda_3$ are not independent but satisfy an equation obtained by differentiating the condition (8.1) for constant volume; that is,

$$\frac{d\lambda_1}{\lambda_1} + \frac{d\lambda_2}{\lambda_2} + \frac{d\lambda_3}{\lambda_3} = 0 \quad (8.4)$$

* A. E. Green, R. S. Rivlin, and R. T. Shield, General Theory of Small Elastic Deformations Superposed on Large Elastic Deformations, *Proceedings of the Royal Society, A*, Vol. 21, pp. 128-154, 1952.

Solving this equation for $d\lambda_3$ and substituting the value in equation 8.3, we obtain

$$dW = \left(f_1 - \frac{\lambda_3}{\lambda_1} f_3 \right) d\lambda_1 + \left(f_2 - \frac{\lambda_3}{\lambda_2} f_3 \right) d\lambda_2 \quad (8.5)$$

Since the differentials $d\lambda_1$ and $d\lambda_2$ are now independent, equation 8.5 implies

$$\begin{aligned} f_1 - \frac{\lambda_3}{\lambda_1} f_3 &= \frac{\partial W}{\partial \lambda_1} \\ f_2 - \frac{\lambda_3}{\lambda_2} f_3 &= \frac{\partial W}{\partial \lambda_2} \end{aligned} \quad (8.6)$$

or

$$\begin{aligned} \lambda_1 f_1 - \lambda_3 f_3 &= \lambda_1 \frac{\partial W}{\partial \lambda_1} \\ \lambda_2 f_2 - \lambda_3 f_3 &= \lambda_2 \frac{\partial W}{\partial \lambda_2} \end{aligned} \quad (8.7)$$

The normal stresses on each face of the deformed solid are

$$\begin{aligned} S_{11} &= \frac{1}{\lambda_2 \lambda_3} f_1 = \lambda_1 f_1 \\ S_{22} &= \frac{1}{\lambda_3 \lambda_1} f_2 = \lambda_2 f_2 \\ S_{33} &= \frac{1}{\lambda_1 \lambda_2} f_3 = \lambda_3 f_3 \end{aligned} \quad (8.8)$$

Hence equations 8.7 may be written

$$\begin{aligned} S_{11} - S_{33} &= \lambda_1 \frac{\partial W}{\partial \lambda_1} \\ S_{22} - S_{33} &= \lambda_2 \frac{\partial W}{\partial \lambda_2} \end{aligned} \quad (8.9)$$

These are the finite stress-strain relations of the medium. They lead to the important conclusions that an *isotropic stress produces no strain* and that the *superposition of an isotropic stress on existing stresses generates no incremental deformation*.

The incremental stress-strain relations are obtained by taking differentials of the finite stresses.

$$\begin{aligned} d(S_{11} - S_{33}) &= s_{11} - s_{33} \\ d(S_{22} - S_{33}) &= s_{22} - s_{33} \end{aligned} \quad (8.10)$$

Differentiating the finite stress-strain relations (8.9), we obtain

$$\begin{aligned} d(S_{11} - S_{33}) &= \left(\frac{\partial W}{\partial \lambda_1} + \lambda_1 \frac{\partial^2 W}{\partial \lambda_1^2} \right) d\lambda_1 + \lambda_1 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} d\lambda_2 \\ d(S_{22} - S_{33}) &= \lambda_2 \frac{\partial^2 W}{\partial \lambda_2 \partial \lambda_1} d\lambda_1 + \left(\frac{\partial W}{\partial \lambda_2} + \lambda_2 \frac{\partial^2 W}{\partial \lambda_2^2} \right) d\lambda_2 \end{aligned} \quad (8.11)$$

Introducing expressions (6.13) for the incremental strains, we derive

$$\begin{aligned} s_{11} - s_{33} &= a_1 e_{xx} + a_2 e_{yy} \\ s_{22} - s_{33} &= a_2 e_{xx} + a_3 e_{yy} \end{aligned} \quad (8.12)$$

with

$$\begin{aligned} a_1 &= \lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} \\ a_2 &= \lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \\ a_3 &= \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} \end{aligned} \quad (8.13)$$

Equations 8.12 are the incremental stress-strain relations. They may be written in a more symmetric form by taking into account the condition for incompressibility:

$$e - e_{xx} + e_{yy} + e_{zz} = 0 \quad (8.14)$$

Relations (8.12) then become

$$\begin{aligned} s_{11} - s_{33} &= (a_1 - a_2)e_{xx} - a_2 e_{zz} \\ s_{22} - s_{33} &= (a_3 - a_2)e_{yy} - a_2 e_{zz} \end{aligned} \quad (8.15)$$

We now put

$$\begin{aligned} a_1 - a_2 &= A \\ a_3 - a_2 &= B \\ a_2 &= C \end{aligned} \quad (8.16)$$

With this notation the stress-strain relations (8.15) take the form

$$\begin{aligned} s_{22} - s_{33} &= B e_{yy} - C e_{zz} \\ s_{33} - s_{11} &= C e_{zz} - A e_{xx} \\ s_{11} - s_{22} &= A e_{xx} - B e_{yy} \end{aligned} \quad (8.17)$$

These relations contain three incremental elastic coefficients A , B ,

and C . To these relations for the normal incremental stresses must be added those giving the shear stresses, which are independent of compressibility properties and are the same as the last three of equations 6.3, that is,

$$\begin{aligned} s_{23} &= 2Q_1 e_{yz} \\ s_{31} &= 2Q_2 e_{zx} \\ s_{12} &= 2Q_3 e_{xy} \end{aligned} \quad (8.18)$$

The six equations 8.17 and 8.18 are the complete incremental stress-strain relations for the orthotropic incompressible medium when the principal directions of the initial stress are located in the planes of elastic symmetry. Comparing with equations 6.1 for the general case, we note that because of incompressibility the number of elastic coefficients is reduced from nine to six.

It is also worth noting that for the incompressible orthotropic medium considered here the initial stress does not appear explicitly in the stress-strain relations (8.17) and (8.18). *They are formally identical with those of an initially unstressed medium.* This is a consequence of the general property discussed in connection with equation 4.41, which shows that the initial stress does not appear explicitly in the stress-strain relations for an incompressible medium.

Equations 8.17 for the normal stresses may be written in an alternative form by introducing the average *three-dimensional* stress component

$$s_3 = \frac{1}{3}(s_{11} + s_{22} + s_{33}) \quad (8.19)$$

Equations 8.17 become

$$\begin{aligned} 3(s_{11} - s_3) &= 2Ae_{xx} - Be_{yy} - Ce_{zz} \\ 3(s_{22} - s_3) &= -Ae_{xx} + 2Be_{yy} - Ce_{zz} \\ 3(s_{33} - s_3) &= -Ae_{xx} - Be_{yy} + 2Ce_{zz} \end{aligned} \quad (8.20)$$

The expressions in parentheses on the left side represent the incremental *stress deviator in three dimensions*. As pointed out below, it is not generally the same as the two-dimensional deviator.

It is interesting to establish the limiting process by which the stress-strain relations (6.3) for a compressible material can be made to coincide with those for an incompressible material. We need consider only the first three of equations 6.3. We write

$$\begin{aligned} C_{11} &= C'_{11} - S_{11} \\ C_{22} &= C'_{22} - S_{22} \\ C_{33} &= C'_{33} - S_{33} \end{aligned} \quad (8.21)$$

and these equations become

$$\begin{aligned} s_{11} &= C'_{11}e_{xx} + C_{12}e_{yy} + C_{31}e_{zz} - S_{11}e \\ s_{22} &= C_{12}e_{xx} + C'_{22}e_{yy} + C_{23}e_{zz} - S_{22}e \\ s_{33} &= C_{31}e_{xx} + C_{23}e_{yy} + C'_{33}e_{zz} - S_{33}e \end{aligned} \quad (8.22)$$

We now substitute:

$$\begin{aligned} 3C'_{11} &= 3K + A \\ 3C'_{22} &= 3K + B \\ 3C'_{33} &= 3K + C \\ 3C_{12} &= 3K - A - B \\ 3C_{23} &= 3K - B - C \\ 3C_{31} &= 3K - C - A \end{aligned} \quad (8.23)$$

If we assume that K tends to infinity while e tends to zero in such a way that

$$\lim_{K \rightarrow \infty} Ke = s_3 \quad (8.23a)$$

equations 8.22 become identical with equations 8.20.

Plane Strain. In many applications, *plane strain* for incremental deformation is of particular importance. This incremental strain may be applied to an initial state which may itself be a state of plane strain or a state of triaxial strain. Incremental plane strain in the x, y plane corresponds to putting

$$e_{zz} = 0 \quad (8.24)$$

in the preceding results. The condition of incompressibility becomes

$$e_{xx} + e_{yy} = 0 \quad (8.25)$$

If we let

$$\begin{aligned} A + B &= 4N \\ Q_3 &= Q \end{aligned} \quad (8.26)$$

equations 8.17 and 8.18 yield

$$\begin{aligned} s_{11} - s_{22} &= 4Ne_{xx} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (8.27)$$

These two-dimensional stress-strain relations may be written in equivalent form by introducing an additional variable s .

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} \\ s_{22} - s &= 2Ne_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (8.28)$$

In this case we must consider the condition of incompressibility (8.25) as an additional equation. Combining equations 8.28 and 8.25, we derive

$$s = \frac{1}{2}(s_{11} + s_{22}) \quad (8.29)$$

This value of s is then a consequence of equations 8.25 and 8.28.

Relations (8.28) could, of course, be derived directly from the property that the plane strain components depend only on the *two-dimensional deviator* ($s_{ij} - \delta_{ij}s$). The three components of this deviator appear on the left side of equations 8.28. It should be pointed out that it is generally different from the *three-dimensional deviator* ($s_{ij} - \delta_{ij}s_3$) used in equations 8.20. This can be shown by adding the first two of these equations after putting $e_{zz} = 0$. Along with the third equation this yields

$$\begin{aligned} 6(s - s_3) &= (A - B)e_{xx} \\ 3(s_{33} - s_3) &= (B - A)e_{xx} \end{aligned} \quad (8.30)$$

For $A = B$ we obtain

$$s = s_3 = s_{33} \quad (8.31)$$

However, if $A \neq B$, equation 8.31 will not be verified.

The Incompressible Medium in Plane Strain as a Limiting Case.

It is of interest to derive equations 8.28 by introducing the condition of incompressibility into the general stress-strain relations in two dimensions. Putting $e_{zz} = e_{yz} = e_{zx} = 0$ into equations 6.1 and writing Q instead of Q_3 , we obtain

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (8.31a)$$

This is the general form of the incremental stress in a compressible medium with two-dimensional orthotropy. The principal stresses are S_{11} and S_{22} with principal directions along the coordinate axes. According to equations 6.2, we must satisfy the condition

$$B_{12} + S_{11} = B_{21} + S_{22} \quad (8.31b)$$

or

$$B_{12} = B_{21} + P \quad (8.31c)$$

with

$$P = S_{22} - S_{11} \quad (8.31d)$$

We put

$$\begin{aligned} B_{11} &= K + N + P \\ B_{12} &= K - N + P \\ B_{21} &= K - N \\ B_{22} &= K + N \end{aligned} \quad (8.31e)$$

When we substitute these values into equations 8.31a they become

$$\begin{aligned} s_{11} &= N(e_{xx} - e_{yy}) + K(e_{xx} + e_{yy}) + P(e_{xx} + e_{yy}) \\ s_{22} &= N(e_{yy} - e_{xx}) + K(e_{xx} + e_{yy}) \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (8.31f)$$

We now put

$$e_{xx} + e_{yy} = 0 \quad (8.31g)$$

and make K tend to infinity in such a way that

$$\lim_{K \rightarrow \infty} K(e_{xx} + e_{yy}) = s \quad (8.31h)$$

Equations 8.31a become

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} \\ s_{22} - s &= 2Ne_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (8.31i)$$

Hence in the limiting case equations 8.31a become identical with equations 8.28.

Incremental Isotropy. Another particular case of special interest occurs when the medium under initial stress is *isotropic for incremental plane strain*. This is expressed by putting

$$N = Q = \mu \quad (8.32)$$

and the stress-strain relations (8.28) are written

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx} \\ s_{22} - s &= 2\mu e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned} \quad (8.33)$$

In the particular x, y plane considered, the medium is then characterized by a single incremental shear modulus whose value depends on the state of initial strain.

In the terminology of section 7 this represents *incremental isotropy* in plane strain. It is conceivable that a medium of finite orthotropy could exhibit such incremental isotropy for a particular state of initial stress. This would then be induced isotropy.

A more interesting case, however, is that of a medium of finite isotropy which retains incremental isotropy in the x, y plane for all values of the initial stress in that plane. The discussion of this case and that of rubber elasticity which follows is taken from the author's paper.*

In order to show that there exists a material with such property, we consider an incompressible elastic medium isotropic in finite strain. Assuming the finite strain to be a state of plane strain, we write

$$\lambda = \lambda_1 = \frac{1}{\lambda_2} \quad \lambda_3 = 1 \quad (8.34)$$

We may write the finite stress-strain law as

$$S_{11} - S_{22} = F(\lambda) \quad (8.35)$$

The total differential yields incremental stresses

$$s_{11} - s_{22} = \frac{dF}{d\lambda} d\lambda \quad (8.36)$$

or

$$s_{11} - s_{22} = \lambda \frac{dF}{d\lambda} e_{xx} \quad (8.37)$$

Comparing these equations with equations 8.27, we derive

$$4N = \lambda \frac{dF}{d\lambda} \quad (8.38)$$

On the other hand, expression (7.15) for the coefficient Q becomes

$$Q = \frac{1}{2} F \frac{\lambda^4 + 1}{\lambda^4 - 1} \quad (8.39)$$

Equating N and Q yields the differential equation

$$\lambda \frac{dF}{d\lambda} = 2F \frac{\lambda^4 + 1}{\lambda^4 - 1} \quad (8.40)$$

* M. A. Biot, Surface Instability of Rubber in Compression, Air Force Office of Scientific Research Report T.N. 1771, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 168-181, 1963.

By integration we find

$$F = S_{11} - S_{22} = \mu_0 \left(\lambda^2 - \frac{1}{\lambda^2} \right) \quad (8.41)$$

This turns out to be the finite stress-strain relation for rubber-type elasticity. The constant of integration μ_0 represents the shear modulus in the original unstressed state.

Rubber Elasticity. These properties also apply to initial triaxial strain for rubber-type elasticity. Such a material is incompressible and isotropic in finite strain. It was shown by Treloar* that a typical expression for the strain energy in rubber-type media is

$$W = \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad (8.42)$$

with

$$\lambda_1\lambda_2\lambda_3 = 1 \quad (8.43)$$

The coefficient μ_0 represents the shear modulus in the unstressed original state. In order to apply the general formulas derived above we write the strain energy in the form

$$W(\lambda_1\lambda_2) = \frac{1}{2}\mu_0 \left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3 \right) \quad (8.44)$$

Applying equations 8.9, we obtain the stress-strain relations

$$\begin{aligned} S_{22} - S_{33} &= \mu_0(\lambda_2^2 - \lambda_3^2) \\ S_{33} - S_{11} &= \mu_0(\lambda_3^2 - \lambda_1^2) \\ S_{11} - S_{22} &= \mu_0(\lambda_1^2 - \lambda_2^2) \end{aligned} \quad (8.45)$$

For finite plane strain ($\lambda_3 = 1$), the last equation is identical with relation (8.41) derived from the condition that the medium retains incremental plane strain isotropy.

The total differentials of these equations according to the procedure followed in the general case yields the incremental stresses (see equations 8.11).

$$\begin{aligned} s_{22} - s_{33} &= 2\mu_0(\lambda_2^2 e_{yy} - \lambda_3^2 e_{zz}) \\ s_{33} - s_{11} &= 2\mu_0(\lambda_3^2 e_{zz} - \lambda_1^2 e_{xx}) \\ s_{11} - s_{22} &= 2\mu_0(\lambda_1^2 e_{xx} - \lambda_2^2 e_{yy}) \end{aligned} \quad (8.46)$$

* L. R. G. Treloar, Large Elastic Deformation in Rubberlike Materials, in *Deformation and Flow of Solids*, pp. 208–217, Springer, Berlin, 1956 (IUTAM Colloquium, Madrid, 1955).

Hence

$$\begin{aligned} A &= 2\mu_0\lambda_1^2 \\ B &= 2\mu_0\lambda_2^2 \\ C &= 2\mu_0\lambda_3^2 \end{aligned} \quad (8.47)$$

Since the medium is isotropic in finite strain, the incremental shear moduli are given by equations 7.15 and 7.16. Combining those equations with expressions (8.45) for the finite initial stresses, we derive

$$\begin{aligned} Q_1 &= \frac{1}{2}\mu_0(\lambda_2^2 + \lambda_3^2) \\ Q_2 &= \frac{1}{2}\mu_0(\lambda_3^2 + \lambda_1^2) \\ Q_3 &= \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2) \end{aligned} \quad (8.48)$$

The shear stresses are therefore written

$$\begin{aligned} s_{23} &= \mu_0(\lambda_2^2 + \lambda_3^2)e_{yz} \\ s_{31} &= \mu_0(\lambda_3^2 + \lambda_1^2)e_{zx} \\ s_{12} &= \mu_0(\lambda_1^2 + \lambda_2^2)e_{xy} \end{aligned} \quad (8.49)$$

Putting $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we see that the constant μ_0 represents the shear modulus in the original unstressed state, as already stated.

The six slide moduli defined by equations 6.25 become

$$\begin{aligned} L_{23} &= L_{13} = \mu_0\lambda_3^2 = \frac{1}{2}C \\ L_{31} &= L_{21} = \mu_0\lambda_1^2 = \frac{1}{2}A \\ L_{12} &= L_{32} = \mu_0\lambda_2^2 = \frac{1}{2}B \end{aligned} \quad (8.50)$$

We consider now an incremental plane strain in the x, y plane. Application of relations (8.46) and (8.49) shows that the incremental stress-strain relations in the x, y plane are expressed by equations 8.33 with a shear modulus

$$\mu = \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2) \quad (8.51)$$

We may repeat this derivation for the two other coordinate planes. *In each of these planes the incremental elastic properties are isotropic.* There will be three elastic moduli dependent on the initial strain and characterizing the incremental elasticity in each plane.

The values of the slide moduli in the x, y plane are L_{12} and L_{21} , obtained in equations 8.50.

Stress-strain relations represented by equations 8.46 and 8.49 were

considered by Green* in 1839, for a medium initially stress-free, with reference to an analogy between the transverse elastic waves and the propagation of light in a crystal. However, although the presence of an initial stress does not affect the form of the stress-strain relations, it does influence the propagation through the additional terms containing the initial stress which appear in the dynamical equations (see Chapter 5).

Mooney Material. Rubber-like elasticity and the property of incremental isotropy are not restricted to materials represented by the strain energy (8.42).

We have shown that, in the particular case of finite plane strain for initial deformation, the condition of incremental isotropy in that plane implies that the finite stress-strain law is equation 8.41. We find that this corresponds to the strain energy (8.42) if we put

$$\begin{aligned}\lambda_1 &= \lambda \\ \lambda_2 &= \frac{1}{\lambda} \\ \lambda_3 &= 1\end{aligned}\tag{8.52}$$

Now we consider the more general case of three-dimensional initial strain with extension ratios $\lambda_1, \lambda_2, \lambda_3$ along the coordinate axes. The condition of incompressibility for the initial strain is

$$\lambda_1 \lambda_2 \lambda_3 = 1\tag{8.53}$$

For the incremental strain it is

$$e_{xx} + e_{yy} + e_{zz} = 0\tag{8.54}$$

For an incremental strain restricted to the x, y plane we put $e_{zz} = 0$ and condition (8.54) becomes

$$e_{xx} + e_{yy} = 0\tag{8.55}$$

* G. Green, On the Propagation of Light in Crystallized Media, *Transactions of the Cambridge Philosophical Society*, Vol. 7, 1839, or *Mathematical Papers*, London, 1871, p. 293. See also A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, p. 299, Fourth Edition, Cambridge University Press, 1927 (reprinted by Dover Publications, New York, 1944).

With this condition, equations 8.17 and 8.18 for the x, y components become

$$\begin{aligned} s_{11} - s_{22} &= (A + B)e_{xx} \\ s_{12} &= 2Q_3e_{xy} \end{aligned} \quad (8.56)$$

As can be seen from equations 8.26 and 8.32, the condition for isotropy in the x, y plane is

$$\frac{1}{4}(A + B) = Q_3 \quad (8.57)$$

The expressions on each side of this equation may be written in terms of a strain energy $W(\lambda_1, \lambda_2)$ function of the two extension ratios λ_1 and λ_2 . Because of the assumption of incompressibility, the third extension ratio λ_3 is a function of λ_1 and λ_2 determined by the condition (8.53). Relations (8.13) and (8.16) yield

$$\begin{aligned} A + B &= \lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_2 \frac{\partial W}{\partial \lambda_2} \\ &+ \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} - 2\lambda_1\lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \end{aligned} \quad (8.58)$$

and from relations (7.15) and (8.9) we derive

$$Q_3 = \frac{1}{2} \left(\lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} \right) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \quad (8.59)$$

Hence the condition (8.57) for incremental isotropy in the x, y plane is a linear partial differential equation for W . A solution of this equation is

$$\begin{aligned} W &= C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \\ &+ C_2 \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3 \right) \end{aligned} \quad (8.60)$$

with $\lambda_3 = 1/\lambda_1\lambda_2$. This expression of the strain energy has been proposed by Mooney* to represent the elastic properties of rubber. There are two elastic constants, C_1 and C_2 . The term containing C_1

* M. Mooney, A Theory of Large Elastic Deformations, *Journal of Applied Physics*, Vol. 11, pp. 582-592, 1940.

is the same as expression (8.42), discussed earlier. The incremental stress-strain relations (8.56) become

$$\begin{aligned} s_{11} - s_{22} &= 4\mu_3 e_{xx} \\ s_{12} &= 2\mu_3 e_{xy} \end{aligned} \quad (8.61)$$

with

$$\mu_3 = C_1(\lambda_1^2 + \lambda_2^2) + C_2 \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \quad (8.62)$$

The stress-strain relations (8.61) are equivalent to

$$\begin{aligned} s_{11} - s &= 2\mu_3 e_{xx} \\ s_{22} - s &= 2\mu_3 e_{yy} \\ s_{12} &= 2\mu_3 e_{xy} \end{aligned} \quad (8.63)$$

The property of incremental isotropy is valid in each of the three principal planes of the initial strain. The incremental stress-strain relations (8.63) in the planes y, z and z, x are obtained from equations 8.63 by cyclic permutation with the coefficients:

$$\begin{aligned} \mu_1 &= C_1(\lambda_2^2 + \lambda_3^2) + C_2 \left(\frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \right) \\ \mu_2 &= C_1(\lambda_3^2 + \lambda_1^2) + C_2 \left(\frac{1}{\lambda_3^2} + \frac{1}{\lambda_1^2} \right) \end{aligned} \quad (8.64)$$

By putting $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we obtain

$$\mu_1 = \mu_2 = \mu_3 = 2(C_1 + C_2) \quad (8.65)$$

which represents the shear modulus in the unstressed state.

9. ELASTIC COEFFICIENTS IN SECOND ORDER ELASTICITY

We now consider an elastic medium of finite isotropy. Let us assume the initial strain to be small but let us take into account both first and second order terms in the stress-strain relations.

Because of isotropy the principal directions of stress and strain coincide. We denote by

$$\begin{aligned} \lambda_1 &= 1 + \varepsilon_{11} \\ \lambda_2 &= 1 + \varepsilon_{22} \\ \lambda_3 &= 1 + \varepsilon_{33} \end{aligned} \quad (9.1)$$

the extension ratios in these principal directions. A cube originally of unit size becomes a rectangular parallelepiped with edges of lengths $\lambda_1, \lambda_2, \lambda_3$. The forces acting on its faces are denoted by $\tau_{11}, \tau_{22}, \tau_{33}$. These quantities are not stresses in the usual sense, but forces referred to unit areas measured in the unstrained state.

To the second order in the quantities ε_{ij} , these forces must be given by the expressions

$$\begin{aligned}\tau_{11} &= 2\mu\varepsilon_{11} + \lambda\varepsilon + D\varepsilon_{11}^2 + F(\varepsilon_{22}^2 + \varepsilon_{33}^2) \\ &\quad + F'\varepsilon_{11}(\varepsilon_{22} + \varepsilon_{33}) + G\varepsilon_{22}\varepsilon_{33} \\ \tau_{22} &= 2\mu\varepsilon_{22} + \lambda\varepsilon + D\varepsilon_{22}^2 + F(\varepsilon_{33}^2 + \varepsilon_{11}^2) \\ &\quad + F'\varepsilon_{22}(\varepsilon_{33} + \varepsilon_{11}) + G\varepsilon_{33}\varepsilon_{11} \\ \tau_{33} &= 2\mu\varepsilon_{33} + \lambda\varepsilon + D\varepsilon_{33}^2 + F(\varepsilon_{11}^2 + \varepsilon_{22}^2) \\ &\quad + F'\varepsilon_{33}(\varepsilon_{11} + \varepsilon_{22}) + G\varepsilon_{11}\varepsilon_{22}\end{aligned}\quad (9.2)$$

where

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\quad (9.3)$$

The first of equations 9.2 is obtained by writing all linear and quadratic terms and imposing the condition that directions 2 and 3 may be interchanged. The other two equations are obtained by cyclic permutation.

In addition, because of the existence of a strain energy, the expressions must satisfy the relations[†]

$$\begin{aligned}\frac{\partial\tau_{11}}{\partial\varepsilon_{22}} &= \frac{\partial\tau_{22}}{\partial\varepsilon_{11}} \\ \frac{\partial\tau_{22}}{\partial\varepsilon_{33}} &= \frac{\partial\tau_{33}}{\partial\varepsilon_{22}} \\ \frac{\partial\tau_{33}}{\partial\varepsilon_{11}} &= \frac{\partial\tau_{11}}{\partial\varepsilon_{33}}\end{aligned}\quad (9.4)$$

This requires

$$2F = F'\quad (9.5)$$

and relations (9.2) become

$$\begin{aligned}\tau_{11} &= 2\mu\varepsilon_{11} + \lambda\varepsilon + D\varepsilon_{11}^2 \\ &\quad + F[\varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\varepsilon_{11}(\varepsilon_{22} + \varepsilon_{33})] + G\varepsilon_{22}\varepsilon_{33} \\ \tau_{22} &= 2\mu\varepsilon_{22} + \lambda\varepsilon + D\varepsilon_{22}^2 \\ &\quad + F[\varepsilon_{33}^2 + \varepsilon_{11}^2 + 2\varepsilon_{22}(\varepsilon_{33} + \varepsilon_{11})] + G\varepsilon_{33}\varepsilon_{11} \\ \tau_{33} &= 2\mu\varepsilon_{33} + \lambda\varepsilon + D\varepsilon_{33}^2 \\ &\quad + F[\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{33}(\varepsilon_{11} + \varepsilon_{22})] + G\varepsilon_{11}\varepsilon_{22}\end{aligned}\quad (9.6)$$

These equations show that *five coefficients* are needed to describe elastic properties of the first and second order in an isotropic medium. The procedure by which they are derived here is the one used by the author in the original paper.* The existence of five coefficients for this case may also be established from the theory of tensor invariants.† The coefficients obtained by this method are different from D , F , G . The derivation given here has the advantage of simplicity and physical clarity.

The incremental coefficients are immediately derived by applying the methods developed above. We are interested in deriving the first order corrections of these coefficients due to the initial strain.

We denote by σ_{11} , σ_{22} , σ_{33} the stresses, i.e., the forces *per unit area after deformation*. The forces defined above are

$$\begin{aligned}\tau_{11} &= \lambda_2 \lambda_3 \sigma_{11} \\ \tau_{22} &= \lambda_3 \lambda_1 \sigma_{22} \\ \tau_{33} &= \lambda_1 \lambda_2 \sigma_{33}\end{aligned}\tag{9.7}$$

In a state of initial stress the stress components are written

$$\begin{aligned}\sigma_{11} &= S_{11} \\ \sigma_{22} &= S_{22} \\ \sigma_{33} &= S_{33}\end{aligned}\tag{9.8}$$

If we superimpose small incremental stresses in the same principal directions, they may be represented by the differential:

$$\begin{aligned}s_{11} &= d\sigma_{11} \\ s_{22} &= d\sigma_{22} \\ s_{33} &= d\sigma_{33}\end{aligned}\tag{9.9}$$

We now take the total differentials of equations 9.7, considering the quantities to be functions of λ_1 , λ_2 , λ_3 . For example, the first equation yields

$$\begin{aligned}(\lambda_3 d\lambda_2 + \lambda_2 d\lambda_3)\sigma_{11} + \lambda_2 \lambda_3 d\sigma_{11} \\ = \frac{\partial \tau_{11}}{\partial \lambda_1} d\lambda_1 + \frac{\partial \tau_{11}}{\partial \lambda_2} d\lambda_2 + \frac{\partial \tau_{11}}{\partial \lambda_3} d\lambda_3\end{aligned}\tag{9.10}$$

* Published in 1940 (see reference 5 in the Preface).

† See F. D. Murnaghan, Finite Deformations of an Elastic Solid, *American Journal of Mathematics*, Vol. 59, p. 235, 1937.

Using the values (6.13) for the incremental strain and taking into account relations (9.8) and (9.9), we derive

$$\begin{aligned}
 s_{11} + S_{11}(e_{yy} + e_{zz}) &= \frac{\lambda_1}{\lambda_2 \lambda_3} \frac{\partial \tau_{11}}{\partial \varepsilon_{11}} e_{xx} + \frac{1}{\lambda_3} \frac{\partial \tau_{11}}{\partial \varepsilon_{22}} e_{yy} + \frac{1}{\lambda_2} \frac{\partial \tau_{11}}{\partial \varepsilon_{33}} e_{zz} \\
 s_{22} + S_{22}(e_{zz} + e_{xx}) &= \frac{1}{\lambda_3} \frac{\partial \tau_{22}}{\partial \varepsilon_{11}} e_{xx} + \frac{\lambda_2}{\lambda_3 \lambda_1} \frac{\partial \tau_{22}}{\partial \varepsilon_{22}} e_{yy} + \frac{1}{\lambda_1} \frac{\partial \tau_{22}}{\partial \varepsilon_{33}} e_{zz} \\
 s_{33} + S_{33}(e_{xx} + e_{yy}) &= \frac{1}{\lambda_2} \frac{\partial \tau_{33}}{\partial \varepsilon_{11}} e_{xx} + \frac{1}{\lambda_1} \frac{\partial \tau_{33}}{\partial \varepsilon_{22}} e_{yy} + \frac{\lambda_3}{\lambda_1 \lambda_2} \frac{\partial \tau_{33}}{\partial \varepsilon_{33}} e_{zz}
 \end{aligned} \tag{9.11}$$

We recognize here the first three equations of the stress-strain relations (6.5). The symmetric matrix on the right side is identical with the matrix (6.6) of the coefficients C_{ij} .

Identification with equations 6.1 yields the coefficients B_{ij} . By writing only the constant and first order terms in the initial strain we obtain the expressions

$$\begin{aligned}
 B_{11} &= (2\mu + \lambda)(1 + \varepsilon_{11} - \varepsilon_{22} - \varepsilon_{33}) + 2D\varepsilon_{11} \\
 &\quad + 2F(\varepsilon_{22} + \varepsilon_{33}) \\
 B_{22} &= (2\mu + \lambda)(1 + \varepsilon_{22} - \varepsilon_{33} - \varepsilon_{11}) + 2D\varepsilon_{22} \\
 &\quad + 2F(\varepsilon_{33} + \varepsilon_{11}) \\
 B_{33} &= (2\mu + \lambda)(1 + \varepsilon_{33} - \varepsilon_{11} - \varepsilon_{22}) + 2D\varepsilon_{33} \\
 &\quad + 2F(\varepsilon_{11} + \varepsilon_{22})
 \end{aligned} \tag{9.12}$$

$$\begin{aligned}
 B_{23} + S_{22} &= B_{32} + S_{33} \\
 &= \lambda(1 - \varepsilon_{11}) + 2F(\varepsilon_{22} + \varepsilon_{33}) + G\varepsilon_{11} \\
 B_{31} + S_{33} &= B_{13} + S_{11} \\
 &= \lambda(1 - \varepsilon_{22}) + 2F(\varepsilon_{33} + \varepsilon_{11}) + G\varepsilon_{22} \\
 B_{12} + S_{11} &= B_{21} + S_{22} \\
 &= \lambda(1 - \varepsilon_{33}) + 2F(\varepsilon_{11} + \varepsilon_{22}) + G\varepsilon_{33}
 \end{aligned} \tag{9.13}$$

The incremental shear coefficients are derived by applying equations 7.15 and 7.16.

$$Q_1 = \frac{1}{2} \left(\frac{\lambda_2 \tau_{22} - \lambda_3 \tau_{33}}{\lambda_1 \lambda_2 \lambda_3} \right) \left(\frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2} \right) \tag{9.14}$$

Cancelling the factor $\varepsilon_{22} - \varepsilon_{33}$ in numerator and denominator and neglecting terms of higher order than the first, we obtain an expression for Q_1 which is linear in the initial strain. We proceed similarly for Q_2 and Q_3 and obtain

$$\begin{aligned} 2Q_1 &= 2\mu + (\mu + \lambda + D - F)(\varepsilon_{22} + \varepsilon_{33}) \\ &\quad + (\lambda - 2\mu + 2F - G)\varepsilon_{11} \\ 2Q_2 &= 2\mu + (\mu + \lambda + D - F)(\varepsilon_{33} + \varepsilon_{11}) \\ &\quad + (\lambda - 2\mu + 2F - G)\varepsilon_{22} \\ 2Q_3 &= 2\mu + (\mu + \lambda + D - F)(\varepsilon_{11} + \varepsilon_{22}) \\ &\quad + (\lambda - 2\mu + 2F - G)\varepsilon_{33} \end{aligned} \quad (9.15)$$

The coefficients given by equations 9.12, 9.13, and 9.15 furnish all incremental elastic properties of the medium in terms of the five basic coefficients λ , μ , D , F , G . The coefficients λ and μ are, of course, identical with the Lamé coefficients of the classical theory of elasticity of the first order. Equations 9.12, 9.13, and 9.15 for the incremental coefficients are the same as those obtained earlier by the author.*

10. TORSIONAL STIFFNESS OF A BAR UNDER AXIAL TENSION

As an application of the theory of elasticity of a solid under initial stress we shall consider the problem of a cylindrical bar of infinite length under axial tension. The author has treated this particular application of the theory in a paper published in 1939.† The treatment here, which constitutes a rigorous solution of the problem, is fundamentally the same, but the derivation is simplified and a clearer interpretation of the result is given in terms of the slide modulus. Moreover, it will be shown at the end of this section that the solution which was first derived for the homogeneous medium of isotropic or transverse isotropic symmetry may readily be extended

* M. A. Biot, Incremental Elastic Coefficients of an Isotropic Medium in Finite Strain, Air Force Office of Scientific Research Report T.N. 1772, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 151–167, 1963.

† See reference 6 in the Preface.

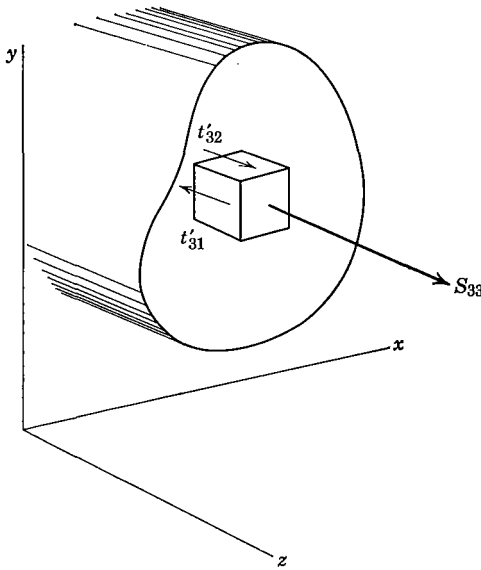


Figure 10.1 Cylindrical bar under axial stress.

to a non-homogeneous bar of orthotropic properties. We shall consider first a homogeneous material with transverse isotropy.

Homogeneous Bar with Transverse Isotropy. The z axis is oriented along the axis of the bar. The cross section is in the x, y plane (Fig. 10.1). The initial stress is reduced to the component

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{33} \end{vmatrix} \tag{10.1}$$

namely, a z component normal to the cross section. This initial stress represents an axial tension when positive, and an axial compression when negative. It is assumed that S_{33} is constant; hence it is independent of z and uniformly distributed. The more general case where S_{33} varies over the cross section will be discussed at the end of this section.

Equations 7.49 of Chapter 1, which express the equilibrium conditions of the incremental stress field s_{ij} , become

$$\begin{aligned}
\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} + S_{33} \frac{\partial \omega_y}{\partial z} &= 0 \\
\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - S_{33} \frac{\partial \omega_x}{\partial z} &= 0 \\
\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} + S_{33} \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) &= 0
\end{aligned} \tag{10.2}$$

We denote by u, v, w the displacement components of the solid. The rotations ω_x and ω_y are

$$\begin{aligned}
\omega_x &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\
\omega_y &= \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)
\end{aligned} \tag{10.3}$$

The strain components are

$$\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} & e_{yz} &= \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \\
e_{yy} &= \frac{\partial v}{\partial y} & e_{zx} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
e_{zz} &= \frac{\partial w}{\partial z} & e_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\end{aligned} \tag{10.4}$$

The incremental forces at the boundary are given by equations 7.56 of Chapter 1. In the present case they become

$$\begin{aligned}
\Delta f_x &= s_{11}n_x + s_{12}n_y + s_{31}n_z + S_{33}\omega_y n_z \\
\Delta f_y &= s_{21}n_x + s_{22}n_y + s_{23}n_z - S_{33}\omega_x n_z \\
\Delta f_z &= s_{31}n_x + s_{23}n_y + s_{33}n_z + S_{33}(e_{xx} + e_{yy})n_z \\
&\quad - S_{33}e_{zx}n_x - S_{33}e_{yz}n_y
\end{aligned} \tag{10.5}$$

The directional cosines of the unit normal to the boundary directed outwardly are n_x, n_y, n_z .

The next step is to consider the stress-strain relations of the material. An elastic bar will generally be anisotropic as a result of the manufacturing process, whether it be rolling, extension, or forging. Moreover, even if the material is isotropic in the unstressed state, anisotropy will be generated by the initial stress itself. Let us therefore assume that the symmetry of the material is of the same kind as

the initial stress, i.e., that properties are invariant under a rotation about the axis of the bar. Such elastic symmetry is usually referred to as *transverse isotropy*. The stress-strain relations for this case were derived above and are given by equations 7.26.

In the terminology of section 7 the material may be said to exhibit incremental transverse isotropy. This transverse isotropy may be induced if the medium has finite isotropy, or it may be intrinsic if the anisotropy does not vanish with the initial stress.

The problem of torsional deformation of the bar is solved by showing that a displacement field of the type

$$\begin{aligned} u &= -\theta yz \\ v &= \theta xz \\ w &= w(x, y) \end{aligned} \tag{10.6}$$

when substituted in the stress-strain relations (7.26) leads to a solution of the field equations (10.2) and can be made to satisfy the boundary conditions. The quantity θ is a constant characteristic of the twist. The solution follows the well-known Saint-Venant theory for the torsion of an initially unstressed bar.* The displacement field (equations 10.6) yields for the strain the values

$$\begin{aligned} e_{xx} &= e_{yy} = e_{zz} = e_{xy} = 0 \\ e_{yz} &= \frac{1}{2} \left(\frac{\partial w}{\partial y} + \theta x \right) \\ e_{zx} &= \frac{1}{2} \left(\frac{\partial w}{\partial x} - \theta y \right) \end{aligned} \tag{10.7}$$

The rotations (10.3) are now

$$\begin{aligned} \omega_x &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \theta x \right) \\ \omega_y &= \frac{1}{2} \left(\frac{\partial w}{\partial x} + \theta y \right) \end{aligned} \tag{10.8}$$

* See, for instance, A. E. H. Love, *The Mathematical Theory of Elasticity*, Fourth Edition, Chapter XIV, Cambridge University Press (reprinted by Dover Publications, New York, 1944).

The stresses derived from equations 7.26 become

$$\begin{aligned} s_{11} &= s_{22} = s_{33} = s_{12} = 0 \\ s_{23} &= 2Qe_{yz} = Q \left(\frac{\partial w}{\partial y} + \theta x \right) \\ s_{31} &= 2Qe_{zx} = Q \left(\frac{\partial w}{\partial x} - \theta y \right) \end{aligned} \quad (10.9)$$

These equations may be simplified by introducing the variables

$$\begin{aligned} t'_{32} &= s_{23} - S_{33}e_{yz} \\ t'_{31} &= s_{31} - S_{33}e_{zx} \end{aligned} \quad (10.10)$$

The physical significance of these quantities is brought out by verifying that they are the same as the alternative stress components given by expression (2.18). They are shown in Fig. 10.1 as the forces acting in the z direction on the faces of an elementary cube. Using equations 10.7 and 10.8, we may also write

$$\begin{aligned} s_{23} - S_{33}\omega_x &= t'_{32} + S_{33}\theta x \\ s_{31} + S_{33}\omega_y &= t'_{31} - S_{33}\theta y \end{aligned} \quad (10.11)$$

With these results the equilibrium equations (10.2) are reduced to*

$$\frac{\partial t'_{31}}{\partial x} + \frac{\partial t'_{32}}{\partial y} = 0 \quad (10.12)$$

and the stress strain relations (10.9) become

$$\begin{aligned} t'_{32} &= 2Le_{yz} = L \left(\frac{\partial w}{\partial y} + \theta x \right) \\ t'_{31} &= 2Le_{zx} = L \left(\frac{\partial w}{\partial x} - \theta y \right) \end{aligned} \quad (10.13)$$

with a coefficient

$$L = Q - \frac{1}{2}S_{33} \quad (10.13a)$$

The condition that the surface of the bar be free of stress is fulfilled by equating to zero the force components given by expressions (10.5). Since $n_z = 0$ at the surface, this reduces to the condition

$$t'_{31}n_x + t'_{23}n_y = 0 \quad (10.14)$$

to be satisfied on the boundary of the cross section.

* Equation 10.12 is a particular case of the more general equations derived in section 3 of Chapter 3.

We have now reduced the problem to the solution of equations 10.12 and 10.13 with the boundary condition (10.14). These equations are formally the same as in the classical Saint-Venant problem of torsion. They are solved in the same way. Equation 10.12 is satisfied identically by

$$\begin{aligned} t'_{32} &= -\frac{\partial\psi}{\partial x} \\ t'_{31} &= \frac{\partial\psi}{\partial y} \end{aligned} \quad (10.15)$$

with $\psi(x, y)$ an unknown function over the cross section. Introducing these values in equations 10.13 and eliminating w , we obtain

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2L\theta \quad (10.16)$$

This equation must be solved for ψ with the boundary condition (10.14). Because of relations (10.15) the boundary condition may be written

$$\psi = \text{const.} \quad (10.17)$$

on the contour of the cross section.

The solution up to this point has not referred to the initial stress. It appears explicitly if we evaluate the torque over the cross section. In order to do this we must introduce the forces Δf_x and Δf_y acting over the cross section. They are obtained by putting $n_x = n_y = 0$ in equations 10.5. We find

$$\begin{aligned} \Delta f_x &= t'_{31} - S_{33}\theta y \\ \Delta f_y &= t'_{32} + S_{33}\theta x \end{aligned} \quad (10.18)$$

The moment of these forces about the origin of the coordinates is

$$T = \iint (xt'_{32} - yt'_{31}) dx dy + S_{33}\theta \iint (x^2 + y^2) dx dy \quad (10.19)$$

with surface integrals over the cross section. It can be seen that the resultant forces obtained by integrating the distributed forces (10.18) over the cross section will vanish if the origin of the coordinates is located at the center of gravity of the cross section. If this is the case,

the forces acting over the cross section are reduced to a pure torque

$$T = T_{SV} + S_{33}I_G\theta \quad (10.20)$$

where I_G is the polar moment of inertia of the cross section with respect to its center of gravity and

$$T_{SV} = \iint (xt'_{32} - yt'_{31}) dx dy \quad (10.21)$$

The term T_{SV} is identical with the torque derived from the Saint-Venant theory for an isotropic and initially unstressed bar whose shear modulus is L . This coefficient L is identical with the slide modulus defined and discussed in section 6.

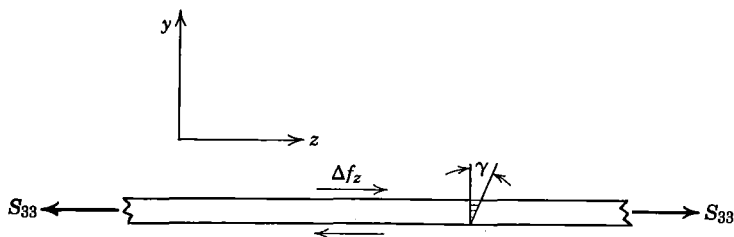


Figure 10.2 Physical significance of the slide modulus $L = \Delta f_z/\gamma$ under an axial stress S_{33} .

Here the coefficient L is derived by considering a thin slice of material cut parallel to the axis of the bar (Fig. 10.2). This slice is assumed to be under an axial tension S_{33} . While this axial tension is maintained we apply a tangential force Δf_z per unit area to the flat sides. The slide modulus L is defined as the ratio $\Delta f_z/\gamma$, where γ is the shear angle produced by the tangential force.

As an example let us consider a rod of circular cross section of radius a . The Saint-Venant torque is

$$T_{SV} = LI_G\theta \quad (10.22)$$

with a polar moment of inertia

$$I_G = \frac{1}{2}\pi a^4 \quad (10.23)$$

Hence the torque under initial stress is*

$$T = (L + S_{33})I_G\theta \quad (10.24)$$

* If the bar is under compression $P = -S_{33}$, the torsional stiffness disappears for $L = P$. This is the condition for internal instability discussed in section 3 of Chapter 4. It appears here as a torsional buckling.

If the value of L is small compared to the axial stress S_{33} , the effect of the initial stress on the torsional rigidity will be large. In particular we may visualize a bar resembling a cable made up of thin steel wires parallel to the axis and bonded together by a very elastic material. The axial stress can then be very large compared to the slide modulus L . We may even go to the limit when the rigidity of the bonding material goes to zero ($L = 0$). In this case the torsional rigidity is entirely due to the initial stress, and we may write

$$T = S_{33}I_G\theta \quad (10.25)$$

Finite Isotropy. If the material of the bar is isotropic for finite strain, the anisotropy is induced by the initial axial stress. Then the value of Q is obtained by applying equations 7.16. Hence the shear modulus is

$$Q = \frac{1}{2}S_{33} \frac{\lambda_3^2 + \lambda_2^2}{\lambda_3^2 - \lambda_2^2} \quad (10.26)$$

where λ_3 is the extension ratio under the stress S_{33} along the axis and $\lambda_2 = \lambda_1$, the transverse extension ratio, the same in all directions about the axis. The slide modulus is

$$L = S_{33} \frac{\lambda_2^2}{\lambda_3^2 - \lambda_2^2} \quad (10.27)$$

For a circular cross section the torque (10.24) becomes

$$T = S_{33} \frac{\lambda_3^2}{\lambda_3^2 - \lambda_2^2} I_G\theta \quad (10.28)$$

Note that the polar moment of inertia (10.23) may be written

$$I_G = \frac{1}{2}\pi a_o^4 \lambda_2^4 \quad (10.28a)$$

where a_o denotes the original radius in the unstrained state.

These formulas lead to a remarkably simple result for rubber-type elasticity. The medium is incompressible and we put

$$\begin{aligned} \lambda_3 &= \lambda \\ \lambda_2 &= \lambda_1 = \frac{1}{\sqrt{\lambda}} \end{aligned} \quad (10.29)$$

Also, the stress S_{33} derived from equations 8.45 is

$$S_{33} = \mu_0 \left(\lambda^2 - \frac{1}{\lambda} \right) \quad (10.30)$$

Hence

$$T = \frac{1}{2}\mu_0\pi a_0^4\theta \quad (10.31)$$

This result leads to the remarkable conclusion that, for a circular rod with rubber-type elasticity governed by equation 10.30, the torsional rigidity is independent of the axial stress. This may be attributed to a compensation effect resulting from the decrease in cross section when the axial tension is increased.

Orthotropic and Non-homogeneous Bar. In the foregoing analysis we have assumed the material to be transverse isotropic around the axis of the bar. The analysis is easily extended to the more general case of orthotropy where the axis of the bar is parallel to one of the planes of elastic symmetry. We may choose the coordinate planes as planes of symmetry. The derivation is entirely similar to that for transverse isotropy except that the stress-strain relations (10.13) are replaced by

$$\begin{aligned} t'_{32} &= 2L_{32}e_{yz} \\ t'_{31} &= 2L_{31}e_{zx} \end{aligned} \quad (10.32)$$

The coefficients L_{32} and L_{31} are slide moduli corresponding to equations 6.25. Their physical significance is analogous to that of L illustrated by Figure 10.2. Equation 10.16 is replaced by

$$\frac{\partial}{\partial x} \left(\frac{1}{L_{32}} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{L_{31}} \frac{\partial \psi}{\partial y} \right) = -2\theta \quad (10.33)$$

The torque is then given by the same formula (10.20), but T_{sv} now represents the torque in an orthotropic bar in the absence of initial stress. The stress-free case is a classical problem.*

Another generalization of the problem concerns a non-homogeneous material where the elastic properties and the initial stress vary over the cross section but remain independent of z . Here again the previous derivation may be repeated. The first two of equations 10.2 remain unchanged, but the last one is replaced by

* Problems of this type have been treated by C. I. Bors, *La méthode de la fonction de tension dans le problème de la torsion des barres anisotropes non-homogènes*, IUTAM Symposium on Nonhomogeneity in Elasticity and Plasticity (Warsaw 1958), pp. 95–100, Pergamon Press, New York, 1959.

$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} - e_{yz} \frac{\partial S_{33}}{\partial y} - e_{zx} \frac{\partial S_{33}}{\partial x} + S_{33} \frac{\partial \omega_y}{\partial x} - S_{33} \frac{\partial \omega_x}{\partial y} = 0 \quad (10.34)$$

With definitions (10.10) for t'_{32} and t'_{31} , this leads to the same equation (10.12). Proceeding as above, we express the total torque as

$$T = T_{SV} + \theta \iint S_{33}(x^2 + y^2) dx dy \quad (10.35)$$

The stress $S_{33}(x, y)$ is now variable over the cross section and T_{SV} is the generalized Saint-Venant torque obtained by solving equation 10.33 with variable elastic moduli $L_{32}(x, y)$ and $L_{31}(x, y)$. The forces over the cross section constitute a pure torque if the origin is located so that

$$\iint S_{33}x dx dy = \iint S_{33}y dx dy = 0 \quad (10.36)$$

Expressions like (10.35) are known in the engineering literature where they are derived by the usual approximate methods of "strength of materials." They are of considerable importance in problems of aeroelastic stability of thin supersonic wings in the presence of thermal stresses.* In this connection it is of particular interest to point out that equation 10.35 is not an approximation but a rigorous consequence of the theory of elasticity.

In section 3 of the next chapter it will be shown that the solution presented here for torsional rigidity may be derived as a particular case of the general equations for a rod under axial stress.

The exact results obtained here are closely related to the approximate theory of secondary stresses in twist as originally treated by K. Weber† by considering the elongation of longitudinal fibers. The soundness and accuracy of Weber's theory have been confirmed in a recent paper by Goodier and Shaw.‡

* See, for instance, M. A. Biot, Influence of Thermal Stresses on the Aeroelastic Stability of Supersonic Wings, *Journal of the Aeronautical Sciences*, Vol. 24, No. 6, pp. 418-421, June 1957.

† K. Weber, Die Lehre der Drehungsfestigkeit, *Forschungsarbeiten*, Heft 249, Berlin, 1921. See also S. Timoshenko, *Strength of Materials*, Part 1, p. 87, D. Van Nostrand Company Inc., Princeton, N.J., 1930.

‡ J. N. Goodier and W. A. Shaw, Nonlinear Effects in Elastic Torsion of Bars of Slender Section, *Journal of Mechanics and Physics of Solids*, Vol. 10, pp. 35-52, 1962.

CHAPTER THREE

Theory of Elastic Stability and Its Application to Isotropic Media

1. INTRODUCTION

In the preceding chapter, general equations were derived for the incremental deformation of an elastic medium in equilibrium under initial stress. We now apply these equations to the problem of stability of such an equilibrium state.

Stability is defined here in terms of purely static concepts and is restricted to external forces which are derived from a potential energy. It has the same significance as in the traditional treatment of buckling phenomena. Addition of inertia forces and viscoelastic properties in Chapters 5 and 6 clearly illustrates the significance of the stability as an immediate natural extension of the purely static concept and without any fundamental change of properties.

Special cases which involve non-conservative boundary forces or gyroscopic and Coriolis effects are not considered here.

There is also an area of practical importance for which the stability concept must be extended to include both linear and quadratic terms of the incremental deformation, because they are of the same magnitude although they may remain small in a physical sense; for example, some problems of stability of thin shells and plates whose shape before or after deformation is close to a ruled surface. These problems must be treated by special methods which are not included in the present analysis. However, they lie within the scope of the

broader non-linear theories which were developed by the author in earlier papers* and are intimately related to the general mechanics of incremental deformations.

In order to illustrate the physical significance of the stability equations they are discussed first in section 2 for plane strain and related to the classical equations of buckling of the theory of structures. In section 3 new equations are derived for the stability of plates and rods. Although they are exact equations, they are of the same form as those in the classical approximate theories and they have the advantage of providing a simple intuitive interpretation.

The variational formulation of stability in section 4 presents the general theory in terms of a condition of positive definiteness of the incremental strain energy. Special attention is given to a paradox which arises in connection with the effect of hydrostatic stress on the value of the potential energy. The total energy includes that of the conservative boundary force. An interesting new form of this potential energy is derived for the case where the elastic material is in contact with a frictionless rigid boundary in terms of the curvature of this boundary. The question of stability in the presence of hydrostatic stress often arises. Such questions as the buckling of a column while submerged in a fluid under pressure have remained obscure because of the lack of a general theory. This fundamental theory is developed in section 5.

The last three sections of this chapter are devoted to applications of the stability theory to specific problems for the case where the material is incompressible and has the property of isotropy for incremental plane strain. Actually the problems are treated for rubber-like materials which, in addition to being partially isotropic for incremental strains, are also isotropic in finite strain. This includes the so-called "Mooney material." However, the results are not restricted to materials of finite isotropy.

Section 6 discusses the problem of surface instability of a homogeneous half-space under compression and brings out the apparent softening of the surface as a function of the compression.

Section 7 considers the buckling of a thick slab in the complete range of thickness-to-length ratios and derives the range of validity of the classical Euler theory. The variational principle is also

* See references 4 and 5 in the Preface and a short outline in the Appendix.

applied to this case, and it yields an approximate solution which is quite accurate for the complete range of thickness-to-length ratios.

With respect to surface instability, an interesting question is the influence of non-homogeneity, particularly when the non-homogeneity varies in a continuous way. In section 8 this problem is examined for a half-space of horizontal surface where the rigidity decreases exponentially with depth; the influence of the weight of the material is also taken into account.

The derivations in the last three sections follow closely the treatment of the same problems in earlier papers by the author.

2. PHYSICAL SIGNIFICANCE OF THE STABILITY EQUATIONS IN PLANE STRAIN

As an example of instability in plane strain we consider an elastic plate of thickness h and infinite extent. Let the surfaces of this plate be the planes $y = \pm h/2$. We first derive some general equations for plane strain deformation in the x, y plane (Fig. 2.1).

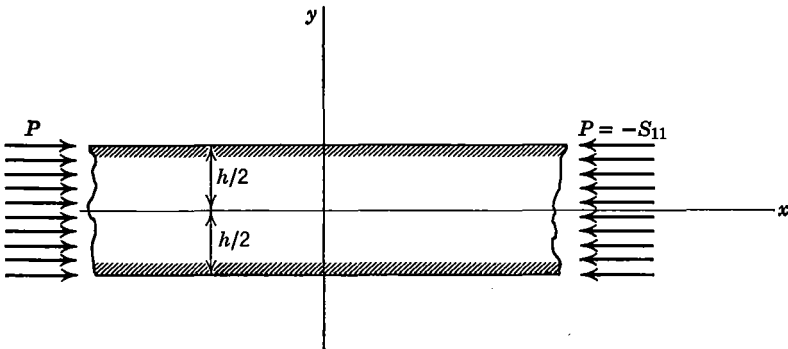


Figure 2.1 Plate under initial compression P viewed across the thickness.

We assume a uniform initial compression $P = -S_{11}$ acting along the x direction. An initial stress may also be present in the z direction; however, it does not appear explicitly in the plane strain analysis.

By substituting $S_{11} = -P$, $S_{22} = S_{11} = 0$ into the equilibrium equations 6.17 of Chapter 1, those equations become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0 \end{aligned} \tag{2.1}$$

The physical significance of these equations is brought to light if we write them in a different form by introducing the alternative stress

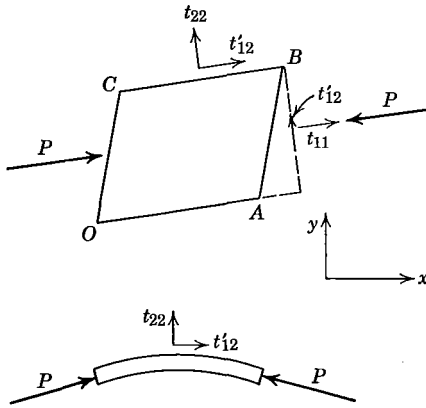


Figure 2.2 Illustration of the physical significance of equations 2.3.

components discussed in section 2 of Chapter 2. Following equations 2.8 and 2.14 of that chapter, we write

$$\begin{aligned} t_{11} &= s_{11} - Pe_{yy} \\ t_{22} &= s_{22} \\ t'_{12} &= s_{12} + Pe_{xy} \end{aligned} \tag{2.2}$$

Taking these relations into account, we may now write equations 2.1 in the form

$$\begin{aligned} \frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} &= 0 \\ \frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} &= P \frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{2.3}$$

The significance of the alternative stresses (2.2) is illustrated in Figure 2.2. A square element of unit size initially oriented along x and y becomes the parallelogram $OABC$ after deformation. The

quantity t'_{12} is the tangential force acting on the side CB , while t_{11} and t_{22} are the forces acting on the sides AB and CB in directions parallel and perpendicular respectively to the side OA . That this is effectively the case may be verified by inspecting Figure 2.2 of Chapter 2. When $S_{22} = S_{12} = 0$, the force components t_{11} , t'_{12} , and t_{22} considered small first order quantities are the same as in Figure 2.2 of this section.

The significance of the stresses (2.2) may also be brought out by relating them to the strain. Assuming the plate to be of orthotropic symmetry about x and y , equations 6.1 of Chapter 2 for the x, y plane become

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (2.4)$$

Substituting the values (2.2), we obtain

$$\begin{aligned} t_{11} &= C_{11}e_{xx} + C_{12}e_{yy} \\ t_{22} &= C_{12}e_{xx} + C_{22}e_{yy} \\ t'_{12} &= 2Le_{xy} \end{aligned} \quad (2.5)$$

with

$$L = Q + \frac{1}{2}P \quad (2.6)$$

The latter coefficient is the slide modulus whose significance was discussed in detail in sections 6 and 10 of Chapter 2.

These results lead to some remarkable and rigorous equations which are identical in form with those derived by the classical approximations of the buckling theories of thin plates. Integrating equations 2.3 across the thickness after multiplying the first one by y , we obtain

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial x} - \mathcal{N} + m_0 &= 0 \\ \frac{\partial \mathcal{N}}{\partial x} + q_0 &= Ph \frac{\partial^2 v_a}{\partial x^2} \end{aligned} \quad (2.7)$$

The first of equations 2.3 requires an integration by parts. We have put

$$\begin{aligned} \mathcal{M} &= \int_{-h/2}^{+h/2} t_{11} y \, dy \\ \mathcal{N} &= \int_{-h/2}^{+h/2} t'_{12} \, dy \\ q_0 &= [t_{22}]_{y=h/2} - [t_{22}]_{y=-h/2} \\ m_0 &= [t'_{12} y]_{y=h/2} - [t'_{12} y]_{y=-h/2} \end{aligned} \quad (2.8)$$

The quantity v_a defined as

$$v_a = \frac{1}{h} \int_{-h/2}^{+h/2} v \, dy \quad (2.9)$$

represents the vertical displacement averaged over the thickness. The sum of the vertical forces applied to the top and bottom and per unit initial length is represented by q_0 . The clockwise moment due to the tangential forces t'_{12} acting on the faces per unit initial length is represented by m_0 .

Note that equations 2.7 do not involve any approximations or any assumption regarding the elastic properties. Elimination of \mathcal{N} in equations 2.7 yields

$$\frac{\partial^2 \mathcal{M}}{\partial x^2} + \frac{\partial m_0}{\partial x} + q_0 = Ph \frac{\partial^2 v_a}{\partial x^2} \quad (2.10)$$

This exact equation is of the same form as that derived from the classical but approximate theory of bending of thin plates under initial stress. The deflection of the middle plane (at $y = 0$) which appears in the classical equations is replaced here by the average deflection v_a . The significance of the forces and moments in equations 2.7 and 2.10 is illustrated in Figure 2.3.

If the lateral loads and moments are zero, equation 2.10 reduces to

$$\frac{\partial^2 \mathcal{M}}{\partial x^2} = Ph \frac{\partial^2 v_a}{\partial x^2} \quad (2.11)$$

This is the familiar form of the equation for the bending moment \mathcal{M} as obtained from approximate buckling theories of thin plates. The procedure by which the equilibrium conditions (2.7) for the bending

moment \mathcal{M} and the resultant shear \mathcal{N} have been derived is suggested by the similar method commonly used in the theory of plates initially

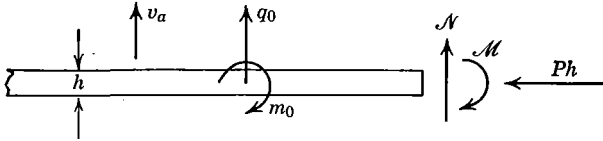


Figure 2.3 Forces and moments appearing in equations 2.7 and 2.10.

stress-free. It was used by Cauchy* and more recently by Mindlin† in his theory of shear-bending deformation of plates.

3. SPECIAL EQUATIONS FOR THE STABILITY OF RODS AND PLATES

In the preceding section some special and interesting forms of the stability equations were derived for plates in plane strain. Equations of this type may be generalized to three-dimensional problems of stability of rods and plates.

Rod under Axial Stress. Consider a rod of arbitrary cross section with its axis along the z direction. Assume that the state of initial stress is reduced to a single component S_{33} along the z direction. The equilibrium condition requires that S_{33} be independent of z . However, it may vary from point to point over the cross section. We may write

$$S_{33} = S_{33}(x, y) \quad (3.1)$$

Such a state of initial stress has already been considered in the problem of torsional stiffness of a rod. The equilibrium conditions

* A. L. Cauchy, Sur l'équilibre et le mouvement d'une plaque solide, *Exercices de mathématique*, Vol. 3, Bure Frères; Paris, 1828.

† R. D. Mindlin, Influence of Rotary Inertia and Shear on Flexural Vibrations of Isotropic Plates, *Journal of Applied Mechanics*, Vol. 18, pp. 31–38, 1951.

for this case are given by equations 10.2 and 10.34 of Chapter 2; that is,

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} + S_{33} \frac{\partial \omega_y}{\partial z} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - S_{33} \frac{\partial \omega_x}{\partial z} &= 0 \\ \frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} + S_{33} \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) \\ &\quad - e_{yz} \frac{\partial S_{33}}{\partial y} - e_{zx} \frac{\partial S_{33}}{\partial x} = 0 \end{aligned} \tag{3.2}$$

Let us introduce some alternative stress components. The following two were used in section 10 of Chapter 2.

$$\begin{aligned} t'_{31} &= s_{31} - S_{33}e_{zx} \\ t'_{32} &= s_{23} - S_{33}e_{yz} \end{aligned} \tag{3.3}$$

In addition we also introduce the component

$$t_{33} = s_{33} + S_{33}(e_{xx} + e_{yy}) \tag{3.4}$$

When these alternative stress components are substituted into equations 3.2, they become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial t'_{31}}{\partial z} + S_{33} \frac{\partial^2 u}{\partial z^2} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial t'_{32}}{\partial z} + S_{33} \frac{\partial^2 v}{\partial z^2} &= 0 \\ \frac{\partial t'_{31}}{\partial x} + \frac{\partial t'_{32}}{\partial y} + \frac{\partial t_{33}}{\partial z} &= 0 \end{aligned} \tag{3.5}$$

The interest of these equations lies in their physical interpretation. Consider a fiber of material along the z direction (Fig. 3.1). The stress components t'_{31} and t'_{32} are tangential forces per unit area acting on the fiber in the z direction. The component t_{33} is the increment of normal force in the axial direction. These forces are referred to unit initial areas. The stress components s_{11} , s_{12} , and s_{22} are defined as usual in classical infinitesimal theories. *If the terms containing S_{33} are omitted, the equilibrium equations 3.5 are the same as in the classical infinitesimal theory of unstressed media.* Hence the effect of the initial axial stress in this case is equivalent to a classical problem

with the introduction of fictitious body forces $S_{33} \partial^2 u / \partial z^2$ and $S_{33} \partial^2 v / \partial z^2$ acting normally to the fiber and proportional to its curvature. If the initial stress is a compression $P = -S_{33}$, the additional curvature terms are similar to those of the usual buckling theories.

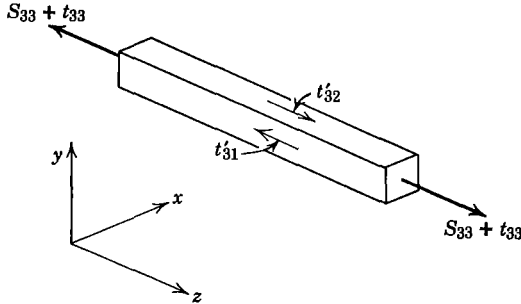


Figure 3.1 Illustration of the stress components t'_{32} , t'_{31} , and t_{33} acting on an axial fiber of the rod.

The stress-strain relations for the alternative stresses are of the same form as in a material initially stress-free. This can be shown as follows. We shall assume that the material is orthotropic for incremental deformations and that the coordinate axes coincide with the directions of elastic symmetry. The properties are also assumed to be independent of z ; however, they may depend on x and y . Hence in the plane of a cross section the medium may be non-homogeneous. The incremental stress-strain relations of such a material are represented by equations 6.3 of Chapter 2. When we introduce the alternative stress components (3.3) and (3.4), the stress-strain relations become

$$\begin{aligned}
 s_{11} &= C_{11}e_{xx} + C_{12}e_{yy} + C_{31}e_{zz} \\
 s_{22} &= C_{12}e_{xx} + C_{22}e_{yy} + C_{23}e_{zz} \\
 t_{33} &= C_{31}e_{xx} + C_{23}e_{yy} + C_{33}e_{zz} \\
 t'_{32} &= 2L_{32}e_{yz} \\
 t'_{31} &= 2L_{31}e_{zx} \\
 s_{12} &= 2Q_3e_{xy}
 \end{aligned} \tag{3.6}$$

These equations introduce two slide moduli,

$$\begin{aligned}
 L_{32} &= Q_1 - \frac{1}{2}S_{33} \\
 L_{31} &= Q_2 - \frac{1}{2}S_{33}
 \end{aligned} \tag{3.7}$$

They were defined by equations 6.25 and 10.32 of Chapter 2. Their physical significance is quite simple, and as already shown they represent the slide rigidity of the fiber under initial axial stress where tangential forces are applied at the surface along its axis. (See Fig. 6.2 and Fig. 10.2 of Chapter 2.)

The stress-strain relations (3.6) are formally the same as if the material were unstressed. The matrix of coefficients is symmetric. These equations include the particular case of a material with isotropic stress-strain relations for finite deformations. In this case we must apply equations 7.15 and 7.16 of Chapter 2.

Torsional Rigidity. Equations 3.5 and 3.6 lead immediately to the solution of the problem of torsional rigidity as derived in section 10 of Chapter 2. The curvature terms disappear, and the problem is formally the same as in classical elasticity theory.

Variational Principle. Equations 3.5 lead immediately to a variational principle for problems of deformation of a rod under axial stress. The variational principle is obtained by applying the general equations in section 5 of Chapter 2 with the following expression for the incremental energy density.

$$\begin{aligned} \Delta V = & \frac{1}{2}(s_{11}e_{xx} + s_{22}e_{yy} + t_{33}e_{zz}) \\ & + t'_{32}e_{yz} + t'_{31}e_{zx} + s_{12}e_{xy} \\ & + \frac{1}{2}S_{33}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}S_{33}\left(\frac{\partial v}{\partial z}\right)^2 \end{aligned} \tag{3.7a}$$

The effect of the initial stress is embodied in the last two terms. They are the same as for a string under a tension S_{33} .

Application to Non-elastic Materials. The equilibrium equations 3.5 do not involve any material property and are therefore applicable to non-elastic materials including those with plastic properties. The same remark applies to equations (3.15) below.

Plate under Initial Stress. Another particular case of interest is that of a plate of constant thickness h . The z axis is chosen normal to the faces. It is assumed that an initial state of uniform plane stress is present; that is,

$$S_{33} = S_{23} = S_{31} = 0 \tag{3.8}$$

The remaining stress components S_{11}, S_{22}, S_{12} are constant and are parallel to the faces (Fig. 3.2).

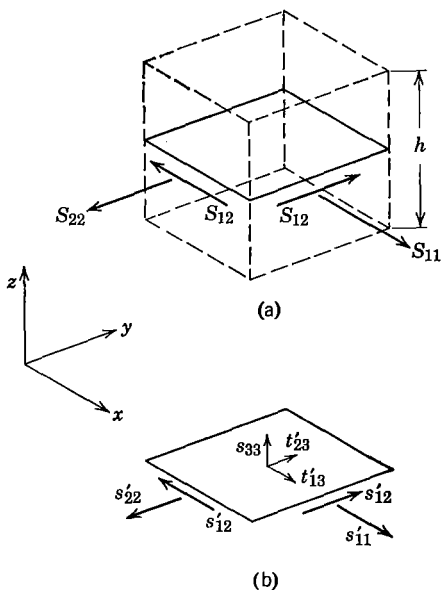


Figure 3.2 (a) Initial stresses in the plate. (b) Incremental membrane stresses.

Equilibrium equations 7.49 of Chapter 1 become

$$\begin{aligned}
 \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} + (S_{11} - S_{22}) \frac{\partial \omega_z}{\partial y} \\
 - 2S_{12} \frac{\partial \omega_z}{\partial x} + S_{12} \frac{\partial \omega_x}{\partial z} - S_{11} \frac{\partial \omega_y}{\partial z} = 0 \\
 \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} + (S_{11} - S_{22}) \frac{\partial \omega_z}{\partial x} \\
 + 2S_{12} \frac{\partial \omega_z}{\partial y} - S_{12} \frac{\partial \omega_y}{\partial z} + S_{22} \frac{\partial \omega_x}{\partial z} = 0 \\
 \frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} - S_{11} \frac{\partial \omega_y}{\partial x} \\
 + S_{22} \frac{\partial \omega_x}{\partial y} + S_{12} \left(\frac{\partial \omega_x}{\partial x} - \frac{\partial \omega_y}{\partial y} \right) = 0
 \end{aligned} \tag{3.9}$$

Let us introduce the composite stress system

$$\begin{aligned}
 t'_{23} &= s_{23} - S_{22}e_{yz} - S_{12}e_{zx} \\
 t'_{13} &= s_{31} - S_{11}e_{zx} - S_{12}e_{yz}
 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} s'_{11} &= s_{11} - 2S_{12}\omega_z + S_{11}e_{zz} \\ s'_{22} &= s_{22} + 2S_{12}\omega_z + S_{22}e_{zz} \\ s'_{12} &= s_{12} + (S_{11} - S_{22})\omega_z + S_{12}e_{zz} \end{aligned} \tag{3.11}$$

From equations 4.13 and 5.20 of Chapter 1 it can be seen that the terms

$$\begin{aligned} \bar{s}_{\xi\xi} &= s_{11} - 2S_{12}\omega_z \\ \bar{s}_{\eta\eta} &= s_{22} + 2S_{12}\omega_z \\ \bar{s}_{\xi\eta} &= s_{12} + (S_{11} - S_{22})\omega_z \end{aligned} \tag{3.12}$$

represent the incremental stresses referred to the original unrotated x, y, z directions. Since e_{zz} is the increment of thickness of a membrane of unit initial thickness, it follows that $s'_{11}, s'_{22}, s'_{12}$ may be considered two-dimensional stresses in this membrane. These membrane stresses are referred to unrotated axes and to unit length of the membrane after deformation.

The other stress components, t'_{23} and t'_{13} , are the incremental tangential forces acting on the upper surface of the membrane as shown in Figure 3.2.

By substituting the stress components (3.10) and (3.11), the equilibrium equations 3.9 become

$$\begin{aligned} \frac{\partial s'_{11}}{\partial x} + \frac{\partial s'_{12}}{\partial y} + \frac{\partial t'_{13}}{\partial z} &= 0 \\ \frac{\partial s'_{12}}{\partial x} + \frac{\partial s'_{22}}{\partial y} + \frac{\partial t'_{23}}{\partial z} &= 0 \\ \frac{\partial t'_{13}}{\partial x} + \frac{\partial t'_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} + S_{11} \frac{\partial^2 w}{\partial x^2} + 2S_{12} \frac{\partial^2 w}{\partial x \partial y} + S_{22} \frac{\partial^2 w}{\partial y^2} &= 0 \end{aligned} \tag{3.13}$$

We now proceed as in the previous section. For simplicity we assume that the faces of the plate are free of stresses; that is,

$$t'_{13} = t'_{23} = s_{33} = 0 \quad \text{for } z = \pm \frac{h}{2} \tag{3.14}$$

We integrate equations 3.13 across the thickness after multiplying the first two equations by z . We obtain

$$\begin{aligned}\frac{\partial \mathcal{M}_{11}}{\partial x} + \frac{\partial \mathcal{M}_{12}}{\partial y} &= \mathcal{N}_1 \\ \frac{\partial \mathcal{M}_{12}}{\partial x} + \frac{\partial \mathcal{M}_{22}}{\partial y} &= \mathcal{N}_2\end{aligned}\tag{3.15}$$

$$\frac{\partial \mathcal{N}_1}{\partial x} + \frac{\partial \mathcal{N}_2}{\partial y} + S_{11}h \frac{\partial^2 w_a}{\partial x^2} + 2S_{12}h \frac{\partial^2 w_a}{\partial x \partial y} + S_{22}h \frac{\partial^2 w_a}{\partial y^2} = 0$$

The bending and twisting moments are defined by

$$\mathcal{M}_{ij} = \int_{-h/2}^{+h/2} s'_{ij} z \, dz$$

The total shear forces are

$$\begin{aligned}\mathcal{N}_1 &= \int_{-h/2}^{+h/2} t'_{13} \, dz \\ \mathcal{N}_2 &= \int_{-h/2}^{+h/2} t'_{23} \, dz\end{aligned}\tag{3.16}$$

and

$$w_a = \frac{1}{h} \int_{-h/2}^{+h/2} w \, dz$$

is the normal deflection averaged across the plate thickness.

If the faces of the plate are not free, we must add terms representing distributed moments and normal loads analogous to m_0 and q_0 in equations 2.7.

Equations 3.15 are identical in form with the classical equations of equilibrium of thin plates under initial stress. Again we can see that they are exact equations where the moments are suitably defined and the middle plane deflection is replaced by the average normal deflection.

The procedure used in this section and in the previous one in order to derive equilibrium conditions for bending moments and resultant shear is similar to the one introduced by Cauchy* in his treatment of the theory of elastic plates without initial stresses.

* A. L. Cauchy, Sur l'équilibre et le mouvement d'une plaque solide, *Exercices de mathématique*, Vol. 3, Bure Frères, Paris, 1828.

4. VARIATIONAL FORMULATION OF STABILITY

In section 5 of Chapter 2 a variational principle was derived for incremental deformations of an elastic body under initial stress. Equation 5.47 of that chapter introduces the total incremental potential

$$\mathcal{P}_\tau = \iiint_{\mathcal{V}} (\Delta V + \rho \Delta U) d\tau \quad (4.1)$$

In this expression

$$\Delta V = \frac{1}{2} t_{ij} e_{ij} + \frac{1}{2} S_{ij} (e_{i\mu} \omega_{\mu j} + e_{j\mu} \omega_{\mu i} + \omega_{i\mu} \omega_{j\mu}) \quad (4.2)$$

is the incremental strain energy density. The incremental body force potential is

$$\Delta U = \frac{1}{2} \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \quad (4.3)$$

The variational principle (5.46) of Chapter 2 states that the variation $\delta \mathcal{P}_\tau$ is equal to the virtual work of the incremental boundary forces Δf_i ; that is,

$$\delta \mathcal{P}_\tau = \iint_A \Delta f_i \delta u_i dA \quad (4.4)$$

Obviously the type of forces applied on the boundary of the volume τ must have an important influence on stability problems. We shall distinguish two fundamental cases in which the right side of equation 4.4 does or does not vanish.

Stability for the Case $\iint_A \Delta f_i \delta u_i dA = 0$. Many stability problems involve boundary conditions such that the surface forces or the displacements are zero. Sometimes the boundary forces do not vanish, but their increments Δf_i are either zero or perform no work under a virtual displacement δu_i compatible with the constraints. Then

$$\iint_A \Delta f_i \delta u_i dA = 0 \quad (4.4a)$$

It follows from equation 4.4 that any displacement field such that

$$\delta \mathcal{P}_\tau = 0 \quad (4.5)$$

represents a possible equilibrium configuration in the vicinity of a state of initial stress.

Because \mathcal{P}_τ is a homogeneous function, any displacement field proportional to this equilibrium configuration will satisfy the condition

$$\mathcal{P}_\tau = 0 \quad (4.6)$$

Such a configuration is known as a *buckling mode*. In a first order theory the amplitude of the equilibrium configuration is arbitrary. It corresponds to a state of neutral equilibrium. A buckling mode may be stable or metastable. It will be metastable if there are other displacement fields for which $\mathcal{P}_\tau < 0$.

As a simple example let us consider the plane strain problem. We assume that there are no body forces. Hence we put $U = 0$ in equation 4.1. The general expression (4.2) for plane strain becomes

$$\begin{aligned} \Delta V = & \frac{1}{2}t_{11}e_{xx} + \frac{1}{2}t_{22}e_{yy} + t_{12}e_{xy} \\ & + S_{11}(e_{xy}\omega + \frac{1}{2}\omega^2) \\ & + S_{22}(-e_{xy}\omega + \frac{1}{2}\omega^2) \\ & + \frac{1}{2}S_{12}(e_{yy} - e_{xx})\omega \end{aligned} \quad (4.7)$$

It is recalled that the strain components and the rotation are given by

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} & e_{yy} &= \frac{\partial v}{\partial y} \\ e_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \omega &= \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \end{aligned} \quad (4.8)$$

where u, v are the displacements in the x, y plane. The alternative stress components are assumed to satisfy the elastic stress-strain relations (3.12) of Chapter 2. They may be written conveniently in the form

$$\begin{aligned} t_{11} &= C_{11}e_{xx} + C_{12}e_{yy} + 2C_{16}e_{xy} \\ t_{22} &= C_{12}e_{xx} + C_{22}e_{yy} + 2C_{26}e_{xy} \\ t_{12} &= C_{16}e_{xx} + C_{26}e_{yy} + 2C_{66}e_{xy} \end{aligned} \quad (4.9)$$

The coefficients are written here in conformity with a standard notation. The potential \mathcal{P}_τ is

$$\mathcal{P}_\tau = \iint_{\Omega} \Delta V \, dx \, dy \quad (4.10)$$

where the integral is extended to a two-dimensional domain Ω .

An interesting form of the strain energy is brought out by considering a state of initial stress reduced to a compression P acting in the x direction; that is,

$$S_{11} = -P \quad S_{22} = S_{12} = 0 \quad (4.11)$$

The expression for the strain energy becomes

$$\Delta V = \frac{1}{2}t_{11}e_{xx} + \frac{1}{2}t_{22}e_{yy} + t_{12}e_{xy} - P(e_{xy}\omega + \frac{1}{2}\omega^2) \quad (4.12)$$

Applying equations 2.14 of Chapter 2, we write

$$t_{12} = s_{12} + \frac{1}{2}Pe_{xy} \quad (4.13)$$

The stress t'_{12} defined by equations 2.2 is

$$t'_{12} = s_{12} + Pe_{xy} \quad (4.14)$$

Hence

$$t_{12} = t'_{12} - \frac{1}{2}Pe_{xy} \quad (4.15)$$

Substituting this value into ΔV , we obtain

$$\Delta V = \frac{1}{2}t_{11}e_{xx} + \frac{1}{2}t_{22}e_{yy} + t'_{12}e_{xy} - \frac{1}{2}P\left(\frac{\partial v}{\partial x}\right)^2 \quad (4.16)$$

For an orthotropic medium satisfying stress-strain relations (2.5) the strain energy becomes

$$\Delta V = \frac{1}{2}C_{11}e_{xx}^2 + \frac{1}{2}C_{22}e_{yy}^2 + C_{12}e_{xx}e_{yy} + 2Le_{xy}^2 - \frac{1}{2}P\left(\frac{\partial v}{\partial x}\right)^2 \quad (4.17)$$

This brings out the slide modulus L . Application of the variational principle using this expression for ΔV leads directly to the equilibrium equations in the form (2.3). The same procedure is applicable for the more general stress-strain relations (4.9).

Incompressible Material. An incompressible material requires special attention. The variational formulation in this case must take into account the constraint

$$e = 0 \quad (4.18)$$

When we introduce a Lagrangian multiplier λ , the variational principle must be replaced by

$$\delta \iiint_{\tau} (\lambda e + \Delta V + \rho \Delta U) d\tau = 0 \quad (4.19)$$

in which the variations are unconstrained.

In order to illustrate the procedure let us again consider plane strain in the x, y plane with an initial compressive stress P in the x direction. Let the medium be of orthotropic symmetry along the x and y directions. The stress-strain relations are given by equations 8.28 of Chapter 2, namely,

$$\begin{aligned} s_{11} - s &= 2N e_{xx} \\ s_{22} - s &= 2N e_{yy} \\ s_{12} &= 2Q e_{xy} \end{aligned} \quad (4.20)$$

Using equations 2.2 and the condition of incompressibility

$$e = e_{xx} + e_{yy} = 0 \quad (4.21)$$

we derive

$$\begin{aligned} t_{11} &= s + (2N + P)e_{xx} \\ t_{22} &= s + 2N e_{yy} \\ t'_{12} &= 2L e_{xy} \end{aligned} \quad (4.22)$$

With these values, taking into account relation (4.21), we find that the strain energy (4.16) takes the simple form

$$\Delta V = 2M e_{xx}^2 + 2L e_{xy}^2 - \frac{1}{2}P \left(\frac{\partial v}{\partial x} \right)^2 \quad (4.23)$$

The coefficient

$$M = N + \frac{1}{4}P \quad (4.24)$$

has the following physical significance. If we put $t_{22} = 0$ and $e_{xx} = -e_{yy}$ in equations 4.22, we derive

$$t_{11} = 4M e_{xx} \quad (4.25)$$

Hence $4M$ is the modulus measuring the incremental force t_{11} per unit initial area for a plane strain elongation in the x direction when the material is free to expand laterally in the y direction. It can be looked upon as a "tangent modulus" for plane strain and referred to areas in the initial state of stress.

The variational principle (4.19) is now written (with $\Delta U = 0$)

$$\delta \iint_{\Omega} \left[\mathcal{A}e + 2M e_{xx}^2 + 2L e_{xy}^2 - \frac{1}{2}P \left(\frac{\partial v}{\partial x} \right)^2 \right] dx dy = 0 \quad (4.26)$$

This yields the differential equations

$$\begin{aligned}\frac{\partial \Lambda}{\partial x} + 4M \frac{\partial e_{xx}}{\partial x} + 2L \frac{\partial e_{xy}}{\partial y} &= 0 \\ \frac{\partial \Lambda}{\partial y} + 2L \frac{\partial e_{xy}}{\partial x} - P \frac{\partial^2 v}{\partial x^2} &= 0\end{aligned}\tag{4.27}$$

By putting

$$t_{22} = \Lambda\tag{4.28}$$

we derive from relations (4.22)

$$t_{11} = \Lambda + 4M e_{xx}\tag{4.29}$$

Equations 4.27 then become identical with equilibrium equations 2.3.

Boundary conditions derived from the variational principle are found to express the absence of forces or the vanishing of incremental forces at the unconstrained boundaries.

The Effect of Adding Hydrostatic Pressure in the Variational Procedure. An apparent contradiction arises in the variational procedure if we consider the addition of a hydrostatic pressure to the initial stress. In the preceding example we considered the initial state of stress (4.11) where the initial stress is reduced to a compression $P = -S_{11}$ in the x direction. Let us add a uniform pressure p_f to the initial state of stress. We assume for simplicity that the medium is homogeneous and of finite isotropy. In this case the principal directions of the initial stress remain unchanged. The initial stresses become

$$\begin{aligned}S_{11} &= -P - p_f \\ S_{22} &= -p_f \\ S_{12} &= 0\end{aligned}\tag{4.30}$$

where

$$P = S_{22} - S_{11}\tag{4.31}$$

represents the difference of the principal stresses in the initial state. Substituting the values (4.30) into expression (4.7), we find

$$\Delta V = \frac{1}{2} t_{11} e_{xx} + \frac{1}{2} t_{22} e_{yy} + t_{12} e_{xy} - P(e_{xy} \omega + \frac{1}{2} \omega^2) - p_f \omega^2\tag{4.32}$$

Comparing the two expressions (4.12) and (4.32), we notice that they differ by the addition of a term $-p_f \omega^2$. On the other hand, for physical reasons the addition of an over-all hydrostatic pressure

should at first sight have no significant effect on the stability problem. For a compressible material a possible effect is a physical one due entirely to the change of the incremental elastic coefficients. This is because of the volume change caused by the increase in pressure. Another possible effect is a change of shape of the body in elastic anisotropy. However, for an incompressible material no effect should be observed. How can we reconcile this statement with the presence of a term $-p_f\omega^2$ in the incremental strain energy (4.32)?

In order to clarify this point we consider the stresses defined by equations 2.14 of Chapter 2. In this case they are

$$\begin{aligned} t_{11} &= s_{11} + S_{11}e_{yy} \\ t_{22} &= s_{22} + S_{22}e_{xx} \\ t_{12} &= s_{12} - \frac{1}{2}(S_{11} + S_{22})e_{xy} \end{aligned} \quad (4.33)$$

Substituting the initial stress (4.11) before the addition of the pressure p_f yields values of t_{ij} which we call $t_{ij}^{(r)}$; that is,

$$\begin{aligned} t_{11}^{(r)} &= s_{11} - Pe_{yy} \\ t_{22}^{(r)} &= s_{22} \\ t_{12}^{(r)} &= s_{12} + \frac{1}{2}Pe_{xy} \end{aligned} \quad (4.34)$$

If instead we substitute the initial stress (4.30) derived by adding the pressure p_f , we find

$$\begin{aligned} t_{11} &= s_{11} - Pe_{yy} - p_f e_{yy} \\ t_{22} &= s_{22} - p_f e_{xx} \\ t_{12} &= s_{12} + \frac{1}{2}Pe_{xy} + p_f e_{xy} \end{aligned} \quad (4.35)$$

or

$$\begin{aligned} t_{11} &= t_{11}^{(r)} - p_f e_{yy} \\ t_{22} &= t_{22}^{(r)} - p_f e_{xx} \\ t_{12} &= t_{12}^{(r)} + p_f e_{xy} \end{aligned} \quad (4.36)$$

If we insert these values into the incremental strain energy relation (4.32), it becomes

$$\begin{aligned} \Delta V &= \frac{1}{2}t_{11}^{(r)}e_{xx} + \frac{1}{2}t_{22}^{(r)}e_{yy} + t_{12}^{(r)}e_{xy} \\ &\quad - P(e_{xy}\omega + \frac{1}{2}\omega^2) - p_f(e_{xx}e_{yy} - e_{xy}^2 + \omega^2) \end{aligned} \quad (4.37)$$

The additional term is represented by the factor

$$e_{xx}e_{yy} - e_{xy}^2 + \omega^2 = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \quad (4.38)$$

This expression has an important geometrical significance. Refer to the linear transformation (2.24) of Chapter 1; its determinant

$$\begin{vmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} \end{vmatrix} = 1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \quad (4.39)$$

represents the relative change of volume. We may talk of volumes instead of areas if we keep in mind that the plane strain problem is obtained by considering the deformation of a slab of unit thickness in a direction perpendicular to the x, y plane. The linear terms are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = e_{xx} + e_{yy} = e \quad (4.40)$$

and the second order terms are identical with expression (4.38). When applying the variational principle we must integrate the expression over a certain area in the x, y plane. This integral must represent the second order change of volume of the whole body. It follows immediately that, if the boundary constraints are such that the total volume remains constant to the second order, this integral must vanish and the terms (4.38) may be dropped from the incremental strain energy.

This may be verified as follows. We write the identity

$$\begin{aligned} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} &= \frac{1}{2} \frac{\partial}{\partial x} \left[eu - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} \right] \\ &+ \frac{1}{2} \frac{\partial}{\partial y} \left[ev - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} \right] \end{aligned} \quad (4.41)$$

By integration over the two-dimensional domain Ω we derive

$$\begin{aligned} \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) dx dy \\ = \frac{1}{2} \oint_C \left(eu - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} \right) dy \\ - \frac{1}{2} \oint_C \left(ev - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} \right) dx \end{aligned} \quad (4.42)$$

The surface integral is thus transformed into a contour integral over the boundary. The quantities

$$\begin{aligned}
 u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \Delta u \\
 u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \Delta v
 \end{aligned}
 \tag{4.43}$$

are the second order vector components of the increment of the displacement u, v when we move from an initial point x, y to a displaced point of coordinates $x + u$ and $y + v$. We may write

$$\begin{aligned}
 \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) dx dy \\
 = \frac{1}{2} \oint_C (eu - \Delta u) dy - (ev - \Delta v) dx
 \end{aligned}
 \tag{4.44}$$

Consider the particular case where the medium is confined between rigid rectangular boundaries parallel to the x and y directions (Fig.

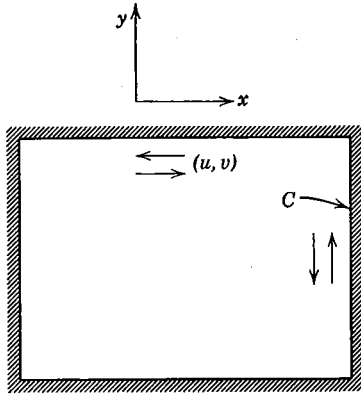


Figure 4.1 Medium confined within a rigid rectangular boundary C with perfect slip at the boundary.

4.1). Although perfect slip is assumed at the boundary, the displaced points remain on the original contour. Hence the vector dx, dy is parallel to the vector u, v and the vector $\Delta u, \Delta v$. Therefore on the contour $(eu - \Delta u) dy - (ev - \Delta v) dx$ is zero and the integral (4.44) vanishes. Hence the potential may be written

$$\mathcal{P}_r = \iiint_{\tau} \Delta V d\tau
 \tag{4.45}$$

where the additional term in expression (4.37) for ΔV has been dropped; that is,

$$\Delta V = \frac{1}{2}t_{11}^{(r)}e_{xx} + \frac{1}{2}t_{22}^{(r)}e_{yy} + t_{12}^{(r)}e_{xy} - P(e_{xy}\omega + \frac{1}{2}\omega^2) \quad (4.46)$$

In this expression P is now the principal stress difference (4.31). It is the same as that in equation 4.16 for an initial compression P in the x direction. In particular, expression (4.23) for ΔV in the case of an incompressible medium is also valid with the same boundary conditions for the more general case of initial stresses (4.30). The definition of M in equation 4.24 is also valid provided P represents the stress difference.

We note that in this example the boundary conditions are of a special type. Because of the assumption of perfect slip and the fact that the boundary is a rigid plane surface, the incremental forces and the virtual displacements remain perpendicular. Hence our assumption that

$$\iint_A \Delta f_i \delta u_i dA = 0 \quad (4.47)$$

is verified.

This simple example points to the important role of the boundary condition in stability problems. In particular, we had to take into consideration the fact that the boundaries are plane surfaces, indicating that the curvature of the boundary must enter into play. This will be further analyzed in the more general treatment which follows.

Conservative Boundary Forces. We now turn our attention to the case where the right side of equation 4.4 does not vanish. The incremental boundary forces may depend linearly on the deformation and may be such that

$$\delta \mathcal{P}_A = - \iint_A \Delta f_i \delta u_i dA \quad (4.48)$$

In this case they are derived from a potential \mathcal{P}_A and they are said to be conservative. The variational principle (4.4) becomes

$$\delta \mathcal{P} = 0 \quad (4.49)$$

where

$$\mathcal{P} = \mathcal{P}_i + \mathcal{P}_A \quad (4.50)$$

and

$$\mathcal{P}_\tau = \iiint_\tau (\Delta V + \rho \Delta U) d\tau \quad (4.51)$$

This is the total potential of the system, including the potential energy contributed by the elastic medium and by the boundary forces. Again, buckling modes correspond to neutral equilibrium and the buckling configuration satisfies the condition

$$\mathcal{P} = 0 \quad (4.52)$$

We shall examine a particular case of conservative boundary conditions which is of special importance. Here the elastic body is in contact with a rigid surface along which it slides without friction. It is possible to express the incremental boundary force by considerations similar to those of section 7 in Chapter 1 for the boundary condition when the body is immersed in a fluid. We denote by S the initial normal stress at a certain point of the boundary. After deformation this point has slipped along this boundary and the normal stress at this point is now $S + s$. Because the boundary force is normal to the surface, the incremental boundary force is the same as if the medium were a fluid with hydrostatic stress S and $S + s$ in the initial and deformed state.

We put

$$\begin{aligned} s_{ij} &= s\delta_{ij} \\ S_{ij} &= S\delta_{ij} \end{aligned} \quad (4.53)$$

in equation 7.56 of Chapter 1 and the incremental boundary force Δf_i becomes

$$\Delta f_i = (s + Se)n_i - S \frac{\partial u_j}{\partial x_i} n_j \quad (4.54)$$

The unit normal to the boundary at the *initial point* is n_i . It is directed positively in the outward direction from the elastic medium. Since Δf_i is zero on any free surface of the elastic body, we may write

$$\iint_A \Delta f_i \delta u_i dA = \iint_B \left[(s + Se)n_i - S \frac{\partial u_j}{\partial x_i} n_j \right] \delta u_i dA \quad (4.55)$$

where the surface integral on the right side is extended only to the rigid boundary B . We now introduce an important additional con-

dition for the boundary constraint. We assume that the displacements are tangent to the boundary at the initial point. Hence at this boundary we must satisfy the conditions

$$\begin{aligned} n_i u_i &= 0 \\ n_i \delta u_i &= 0 \end{aligned} \tag{4.56}$$

This is a *linear boundary condition*. It is justified by the fact that it is the same as that commonly used when solving the linear differential equations of the problem. Actually, of course, these linear constraints obey the exact boundary conditions only if the boundary is a plane surface. If the boundary is curved, the displaced point is assumed to be in the plane tangent to the surface at the initial point. This involves a second order displacement outside the actual boundary.

By inserting conditions (4.56), expression (4.55) is simplified to

$$\iint_A \Delta f_i \delta u_i dA = - \iint_B S \frac{\partial u_j}{\partial x_i} n_j \delta u_i dA \tag{4.57}$$

By further transformation of this expression we shall show that it is an exact differential under conditions (4.56). Consider the function $F(x_1, x_2, x_3)$ defined in such a way that the boundary is given by the equation

$$F(x_1, x_2, x_3) = 0 \tag{4.58}$$

We put

$$\phi = \pm \left[\left(\frac{\partial F}{\partial x_1} \right)^2 + \left(\frac{\partial F}{\partial x_2} \right)^2 + \left(\frac{\partial F}{\partial x_3} \right)^2 \right]^{-1/2} \tag{4.59}$$

The normal vector is then

$$n_i = \phi \frac{\partial F}{\partial x_i} \tag{4.60}$$

The sign in the definition of ϕ is chosen to correspond to the outward direction of n_i . Inserting the value (4.60) for n_i into the first of equations 4.56, we obtain

$$\phi \frac{\partial F}{\partial x_j} u_j = 0 \tag{4.61}$$

Hence

$$\frac{\partial F}{\partial x_j} u_j = 0 \tag{4.62}$$

Equation 4.61 is verified everywhere on the surface B . Therefore the gradient of $\phi(\partial F/\partial x_j)u_j$ is normal to this surface, and we may also write

$$\delta u_i \frac{\partial}{\partial x_i} \left(\phi \frac{\partial F}{\partial x_j} u_j \right) = 0 \quad (4.63)$$

or

$$\left(\frac{\partial \phi}{\partial x_i} \frac{\partial F}{\partial x_j} u_j + \phi \frac{\partial^2 F}{\partial x_j \partial x_i} u_j + \phi \frac{\partial F}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) \delta u_i = 0 \quad (4.64)$$

By virtue of equations 4.60 and 4.62 this relation is simplified to

$$n_j \frac{\partial u_j}{\partial x_i} \delta u_i = -\phi \frac{\partial^2 F}{\partial x_j \partial x_i} u_j \delta u_i \quad (4.65)$$

The right side of equation 4.65 is now an exact differential. By introducing this quantity into equation 4.57 we derive

$$\iint_A \Delta f_i \delta u_i dA = -\delta \mathcal{P}_B \quad (4.66)$$

with

$$\mathcal{P}_B = -\frac{1}{2} \iint_B S\phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j dA \quad (4.67)$$

The variational principle (4.49) then becomes

$$\delta(\mathcal{P}_v + \mathcal{P}_B) = 0 \quad (4.68)$$

The boundary forces in this case are represented by \mathcal{P}_B and are therefore conservative.

An interesting consequence of equation 4.67 is the particular form of the surface energy \mathcal{P}_B where the boundary is constituted by plane faces. The intersections of these planes are lines of infinite curvature, and in the limit the surface integrals degenerate into line integrals along these edges. However, as illustrated in the example of Figure 4.1 by considering displacements which vanish on these edges, the value of \mathcal{P}_B will also vanish.

Illustration of Conservative Boundary Forces. The significance of the preceding results is brought out more clearly by discussing some trivial examples. Consider a cylinder of a homogeneous isotropic elastic material, shown in cross section in Figure 4.2. It is confined in a cylindrical cavity of radius a , under a uniform pressure p_f . Perfect slip is assumed at the interface between the cylinder and the surrounding material. This system is obviously in neutral

equilibrium since the cylinder is free to undergo a rigid rotation about its axis.

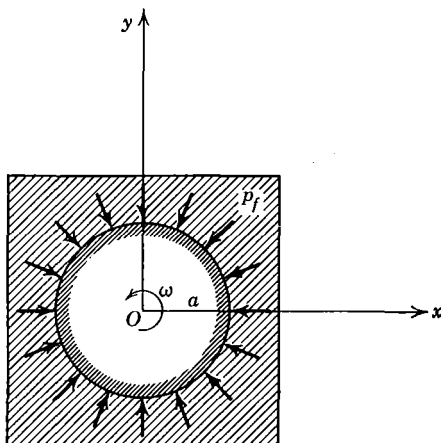


Figure 4.2 Elastic material confined in a cylindrical cavity of radius a under a uniform pressure p_f and with perfect slip at the boundary.

In a linear theory such a rotation is represented by the displacement field.

$$\begin{aligned} u &= -\omega y \\ v &= \omega x \end{aligned} \tag{4.69}$$

The displacement is tangent to the interface and satisfies the boundary conditions of the linear theory. We now apply the variational principle to this displacement. Expression (4.2) for ΔV is reduced to

$$\Delta V = -\frac{1}{2} p_f \delta_{ij} \omega_{i\mu} \omega_{j\mu} \tag{4.70}$$

or

$$\Delta V = -p_f \omega^2 \tag{4.71}$$

Since $\Delta U = 0$, the value (4.1) of \mathcal{P}_τ is

$$\mathcal{P}_\tau = -p_f \omega^2 \pi a^2 \tag{4.72}$$

The volume integral in this case is applied to a unit thickness and becomes a surface integral in the x, y plane extended to the area πa^2 of the cross section of the cylinder.

We must also evaluate the value (4.67) of \mathcal{P}_B . The equation of the boundary is

$$F(x, y) = x^2 + y^2 - a^2 = 0 \tag{4.73}$$

Hence (with $S = -p_f$)

$$\frac{1}{2}S\phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j = -\frac{1}{2}p_f a \omega^2 \tag{4.74}$$

The surface integral \mathcal{P}_B reduces in this case to a line integral over the circle of perimeter $2\pi a$; hence

$$\mathcal{P}_B = \pi p_f a^2 \omega^2 \tag{4.75}$$

Adding expressions (4.72) and (4.75), we find

$$\mathcal{P}_\tau + \mathcal{P}_B = 0 \tag{4.76}$$

This result verifies the variational principle (4.68). The equilibrium is effectively neutral.

Another interpretation of this result is obtained by writing

$$\mathcal{P}_\tau = -\frac{1}{2}\omega^2 a \cdot 2\pi a \cdot p_f \tag{4.77}$$

As shown by equation 2.33 of Chapter 1, the linear rotation field (4.69) is associated with a second order change of radius $\frac{1}{2}\omega^2 a$. The value (4.77) of \mathcal{P}_τ obviously represents the work done by the second order expansion against the pressure p_f . An actual rigid rotation does not contain such an expansion,

but this requires a non-linear second order constraint to be verified. The term \mathcal{P}_B compensates for this effect without having to introduce non-linear boundary conditions.

As a second illustration we shall discuss the stability problem illustrated in Figure 4.3. A rod constituted of two telescoping parts and an inner spring C is mounted so that it presses without friction against two curved surfaces at points A and B . A lateral translation of the rod brings its axis to the line $A'B'$. The displacement AA' and BB' are perpendicular to the axis AB as required by the linear constraint. Because of the curvature of the boundary, points A' and B' violate the actual boundary condition by a second order distance. Since the rod undergoes a rigid translation, the value of \mathcal{P}_τ is zero.

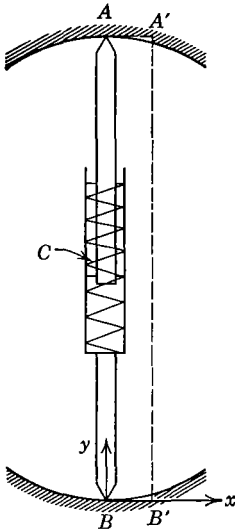


Figure 4.3 Stability of an extensible rod between curved surfaces.

The stability in this case is controlled entirely by the boundary condition. We shall assume the boundary at B to be defined by a parabola

$$F(x, y) = y - ax^2 = 0 \quad (4.78)$$

We write the displacement as

$$AA' = BB' = u \quad (4.79)$$

At point B we find

$$\frac{1}{2} \phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j = au^2 \quad (4.80)$$

On the other hand, the surface integral

$$\iint_B S dA = -P \quad (4.81)$$

represents the concentrated normal compression acting on the boundary due to the spring in the rod. Hence

$$\mathcal{P}_B = Pau^2 \quad (4.82)$$

It is seen that \mathcal{P}_B represents the work done by the compressive force P when point B is maintained along the actual curved surface and undergoes the vertical second order displacement $y = au^2$ along the parabola. The value (4.82) of \mathcal{P}_B must, of course, be multiplied by a factor 2 to take into account the equal contribution at point A . If $a > 0$, the value \mathcal{P}_B is positive and the system is stable. The system is stable or unstable under translational displacements depending on the sign of a . If $a < 0$, then $\mathcal{P}_B < 0$ and the system is unstable. This corresponds to a convex boundary. In a more complete analysis we must also consider possible rotation of the rod. In this case neutral equilibrium will arise as in the previous example if the radius of curvature of the boundary is equal to $\frac{1}{2}AB$, and stability will depend on whether it is larger or smaller than this value.

Stability Problems with Non-conservative Boundary Forces.

When the surface integral (4.48) is not an exact differential, the boundary forces are said to be non-conservative. Then it is generally not possible to derive stability criteria based on purely static considerations or the sign of a potential function. Usually it will be necessary to consider the dynamics of the systems and the actual

solutions of the equations of motion. Individual cases will require special treatment. Problems of this type will not be discussed here.

Remarks on the Validity of the Stability Criterion. It should be kept in mind that the present theory is a linearized theory valid only if the displacement gradients remain sufficiently small. There are problems where the non-linear terms become important although in a physical sense the deformations and rotations remain small. Such is the case, for instance, in problems of buckling of thin shells* and in a category of phenomena generally referred to as "oil canning." Such problems have to be treated as special cases, and suitable non-linear theories† have to be used. In a mathematical sense the deformation may always be assumed arbitrarily small in order to ensure the validity of the linearized equations. However, such a procedure does not guarantee the practical value of the results.

Another theoretical difficulty arises if the linearized equations have singular solutions such that the magnitude of the strain is infinite at these points. At such points the assumption of small displacement gradients breaks down. Therefore conclusions based on this assumption do not strictly apply. A similar situation arises in the classical linear theory of elasticity as illustrated by the occurrence of infinite stress concentrations in sharp corners. However, the difficulty is rather academic and has not affected the usefulness of the results, because the physical assumption of elastic behavior also breaks down in this case. Local plastic behavior and the presence of damping in the case of dynamic phenomena set a limit on the magnitude of the physical variables in the region of the mathematical singularity. In practice it has been possible to introduce suitable "corrections" based on physical considerations.

5. STABILITY IN THE PRESENCE OF HYDROSTATIC STRESS

In many problems, particularly in the field of geophysics, we are dealing with conditions in which the initial stress is partly hydrostatic.

* Th. von Kármán and H. S. Tsien, The Buckling of Thin Cylindrical Shells under Axial Compression, *Journal of Aeronautical Sciences*, Vol. 8, pp. 303-312, 1941.

† Non-linear three-dimensional equations suitable for the solutions of such problems have been derived by the author. They are closely related to the linearized theory for incremental stresses. This is briefly discussed in the Appendix.

In other types of problems the solid is in contact with a fluid in hydrostatic equilibrium. As will now be shown, it is possible in such cases to express the equilibrium condition in alternative forms which provide new physical insights and also lead to simplified methods of solution.

Consider a fluid of distributed density ρ_f in equilibrium under the action of a body force field X_i derived from a potential U . The body force is

$$X_i = - \frac{\partial U}{\partial x_i} \quad (5.1)$$

The hydrostatic stress S in the fluid satisfies the equilibrium condition

$$\frac{\partial S}{\partial x_i} - \rho_f \frac{\partial U}{\partial x_i} = 0 \quad (5.2)$$

Multiplying this equation by dx_i with the summation convention yields immediately

$$dS - \rho_f dU = 0 \quad (5.3)$$

This result shows that equipotential surfaces are surfaces of constant fluid pressure. Hence we may write

$$S = S(U) \quad (5.4)$$

and

$$\frac{d}{dU} S(U) = \rho_f \quad (5.5)$$

Therefore the equipotential surfaces are also surfaces of constant fluid density. Let us introduce a solid into the fluid. At any point in the solid we may consider the hydrostatic stress S and the density ρ_f which were present at that point before the introduction of the solid. The fields S and ρ_f are thereby defined throughout the solid, although they do not exist physically in that region.

The actual mass density of the solid is denoted by ρ . The initial stress field S_{ij} in the solid is assumed to be quite general, and it results from the simultaneous application of the body force X_i and arbitrary boundary forces part of which are the fluid pressures.

We now define, inside the solid two fictitious fields, a "residual density"

$$\rho' = \rho - \rho_f \quad (5.6)$$

and a "residual initial stress"

$$S'_{ij} = S_{ij} - S\delta_{ij} \quad (5.7)$$

The equilibrium condition of the initial stress in the solid is

$$\frac{\partial S_{ij}}{\partial x_j} - \rho \frac{\partial U}{\partial x_j} = 0 \quad (5.8)$$

Combining equations 5.2 and 5.8 yields

$$\frac{\partial S'_{ij}}{\partial x_j} - \rho' \frac{\partial U}{\partial x_j} = 0 \quad (5.9)$$

Hence *the residual stress satisfies equilibrium conditions for a solid with residual densities.*

Equations 7.42 of Chapter 1 which express the equilibrium conditions of incremental stresses in the solid are

$$\begin{aligned} \frac{\partial s_{ij}}{\partial x_j} + \rho \Delta X_i - \rho \omega_{ij} X_j - \rho e X_i \\ - S_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S_{ik}}{\partial x_j} = 0 \end{aligned} \quad (5.10)$$

The incremental body force ΔX_i in these equations will be linearized according to equation 5.42 of Chapter 2 by writing

$$\Delta X_i = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = \frac{\partial X_j}{\partial x_i} u_j \quad (5.11)$$

When we insert this expression into equation 5.10 along with the residual density (5.6) and the residual stresses (5.7), the equilibrium condition becomes

$$\frac{\partial s_{ij}}{\partial x_j} + \mathcal{F}_i + \mathcal{E}_i = 0 \quad (5.12)$$

The terms are written in two groups. One group is

$$\mathcal{E}_i = -\rho_f \frac{\partial^2 U}{\partial x_i \partial x_j} u_j - \rho_f \omega_{ij} X_j - \rho_f e X_i - e_{ji} \frac{\partial S}{\partial x_j} \quad (5.13)$$

By using equations 5.1, 5.2, and 5.11 it may also be written

$$\mathcal{E}_i = -\rho_f \frac{\partial^2 U}{\partial x_i \partial x_j} u_j + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} \quad (5.14)$$

The other terms are

$$\begin{aligned} \mathcal{F}_i = & -\rho' \frac{\partial^2 U}{\partial x_i \partial x_j} u_j - \rho' \omega_{ij} X_j - \rho' e X_i \\ & - S'_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S'_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S'_{ik}}{\partial x_j} \end{aligned} \quad (5.15)$$

The terms \mathcal{F}_i are the same as those for a solid of residual density ρ' and under the residual initial stress S'_{ij} . The terms \mathcal{E}_i , on the other hand, are due to the presence of the fluid. They may be transformed into an equivalent form which brings out a new and interesting physical interpretation. This may be shown as follows.

We further transform the value (5.14) of \mathcal{E}_i by using equations 5.1, 5.2, and 5.11. The terms \mathcal{E}_i become

$$\mathcal{E}_i = -\rho_f X_i e + \rho_f X_j \frac{\partial u_j}{\partial x_i} + \rho_f u_j \frac{\partial X_j}{\partial x_i} \quad (5.16)$$

or

$$\mathcal{E}_i = -\rho_f X_i e + \rho_f \frac{\partial}{\partial x_i} (u_j X_j) \quad (5.17)$$

Obviously this may also be written

$$\mathcal{E}_i = \frac{\partial}{\partial x_i} (\rho_f u_j X_j) - X_j \frac{\partial \rho_f}{\partial x_i} u_j - \rho_f e X_i \quad (5.18)$$

With this result the equilibrium equations 5.12 are now

$$\frac{\partial s_{ij}}{\partial x_j} + \mathcal{F}_i + \frac{\partial}{\partial x_i} (\rho_f u_j X_j) - X_j \frac{\partial \rho_f}{\partial x_i} u_j - \rho_f e X_i = 0 \quad (5.19)$$

Let us examine the significance of the new terms. We may write

$$-X_j \frac{\partial \rho_f}{\partial x_i} u_j = -n_i X \frac{\partial \rho_f}{\partial n} u_n \quad (5.20)$$

where n_i denotes the unit normal to the equipotential surface. The factors X , $\partial \rho_f / \partial n$, and u_n on the right side of equation 5.20 are the projections of the three vectors X_i , $\partial \rho / \partial x_i$, and u_i on this normal direction. Hence the term (5.20) represents a buoyancy force arising from the displacement and the density gradient. The other term, $\rho_f e X_i$, represents a buoyancy effect arising from the change of volume. The term

$$\Delta S = -\rho_f u_j X_j = u_j \frac{\partial S}{\partial x_j} \quad (5.21)$$

is the change of hydrostatic stress associated with the particle displacement in the hydrostatic field S .

If we put

$$s'_{ij} = s_{ij} - \delta_{ij} \Delta S \quad (5.22)$$

the equilibrium equations become

$$\frac{\partial s'_{ij}}{\partial x_j} - X_j \frac{\partial \rho_f}{\partial x_i} u_j - \rho_f e X_i + \mathcal{F}_i = 0 \quad (5.23)$$

We must also consider the boundary conditions. The incremental boundary forces as given by equations 7.56 of Chapter 1 are

$$\Delta f_i = (s_{ij} + S_{kj} \omega_{ik} + S_{ij} e - S_{ik} e_{jk}) n_j \quad (5.24)$$

Here again it is possible to introduce "residual" components. Let us imagine a "rigid" hydrostatic stress field which is a function of fixed coordinates and equal to $S\delta_{ij}$. It remains unchanged when any point of the solid moves through it. In this case the incremental stress s_{ij} at this moving point is due only to the particle displacement in this field. With the definition (5.21) for ΔS , it is given by

$$s_{ij} = \Delta S \delta_{ij} \quad (5.25)$$

The incremental boundary force $\Delta f_i^{(h)}$ due to such a hydrostatic field is obtained by substituting this value of s_{ij} and the value $S_{ij} = S\delta_{ij}$ into equation 5.24. We obtain

$$\begin{aligned} \Delta f_i^{(h)} &= (\Delta S \delta_{ij} + S \delta_{kj} \omega_{ik} + S \delta_{ij} e - S \delta_{ik} e_{jk}) n_j \\ &= (\Delta S + S e) n_i - S \frac{\partial u_j}{\partial x_i} n_j \end{aligned} \quad (5.26)$$

This result is also identical with expression (7.63) of Chapter 1, obtained for the boundary condition in the presence of a fluid.

The difference between the values (5.24) and (5.26) represents a residual boundary force $\Delta f'_i$. It is written

$$\begin{aligned} \Delta f'_i &= \Delta f_i - \Delta f_i^{(h)} \\ &= (s'_{ij} + S'_{kj} \omega_{ik} + S'_{ij} e - S'_{ik} e_{jk}) n_j \end{aligned} \quad (5.27)$$

If we compare this expression with equation 5.24, we see that it is obtained by substituting the residual stress fields s'_{ij} and S'_{ij} into the general formula as if the hydrostatic stress component did not exist.

The residual boundary force $\Delta f'_i$ is the force acting at the boundary in excess of that due to the fluid pressure at the displaced point.

Application of Modified Equilibrium Equations 5.19. In order to give more substance to the presentation we have assumed the actual presence of a fluid. In fact this is not necessary, and the separation of the initial stress into hydrostatic and residual components may be entirely arbitrary and fictitious. However, we shall not pursue further this more general viewpoint. Instead we shall call attention to three particular cases of special importance (*a*, *b*, *c*, below) which may also serve as an illustration of the usefulness of the separation of the initial stress into hydrostatic and residual components.

(*a*) *Compressible Solid with $\rho = \rho_f$.* If the initial density of the solid is constant along equipotential lines, a fluid of the same mass density is in equilibrium in the body force field. Hence we may put

$$\rho = \rho_f \quad (5.28)$$

The fluid may be fictitious or real. The total initial stress may be separated into a residual stress S'_{ij} and a hydrostatic component $S\delta_{ij}$ which represents the stress in a fluid with the same density distribution as the solid. The residual density vanishes:

$$\rho' = 0 \quad (5.29)$$

Equilibrium equations 5.19 become

$$\begin{aligned} \frac{\partial s_{ij}}{\partial x_j} + \frac{\partial}{\partial x_i} (\rho u_j X_j) - X_j \frac{\partial \rho}{\partial x_i} u_j - \rho e X_i \\ - S'_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S'_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S'_{ik}}{\partial x_j} = 0 \end{aligned} \quad (5.30)$$

The last three terms depend only on the residual initial stress.

The physical significance of the other terms containing the body force results from our previous discussion. (See equations 5.20 and 5.21.)

(*b*) *Incompressible Solid with $\rho = \rho_f$.* This is case (*a*) with the additional condition of incompressibility

$$e = 0 \quad (5.31)$$

Here the form (5.23) for the equilibrium equations is of special interest. They become

$$\frac{\partial s'_{ij}}{\partial x_j} - X_j \frac{\partial \rho}{\partial x_i} u_j - S'_{jk} \frac{\partial \omega_{ik}}{\partial x_j} + S'_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S'_{ik}}{\partial x_j} = 0 \quad (5.32)$$

Several important features are exhibited by this case. Putting

$$s_3 = \frac{1}{3}s_{ij}\delta_{ij} \quad s'_3 = \frac{1}{3}s'_{ij}\delta_{ij} \quad (5.33)$$

we derive from equation 5.22 the property

$$s_{ij} - s_3\delta_{ij} = s'_{ij} - s'_3\delta_{ij} \quad (5.34)$$

This means that the three-dimensional stress deviator discussed in section 8 of Chapter 2 is the same for the stresses s'_{ij} and s_{ij} . Hence the stress-strain relations of the material may be written by replacing the actual stresses s_{ij} by s'_{ij} .

On the other hand, the modified equilibrium equations 5.32 are the same as for a medium under the action of initial residual stresses S'_{ij} represented by the terms \mathcal{F}_i with the addition of a small body force $-X_j(\partial\rho/\partial x_i)u_j$ proportional to the displacement. Furthermore, the boundary condition is expressed by means of a boundary force (5.27) expressed by means of the residual stresses alone, as if the hydrostatic stress were not present, and it is equal to the force acting on the boundary in excess of the forces generated by the fluid at the displaced point.

This result may be interpreted by stating that the solid behaves as an *analog model* in which the hydrostatic stress may be ignored. The model is obtained by assuming that the only incremental and initial stresses acting in the body are the residual stresses s'_{ij} and S'_{ij} . An additional body force must be introduced in the model. This body force is proportional to the local displacement.

Similarly, for the boundary condition at a fluid-solid interface, we have pointed out that it may be formulated by ignoring the hydrostatic stress.

As we can see, the effect of the hydrostatic field in the model is represented entirely by the buoyancy term $-X_j(\partial\rho/\partial x_i)u_j$. We have already discussed the significance of this term through equation 5.20. It acts like an elastic force proportional to the displacement directed normally to the equipotential surfaces with a local "spring constant" proportional to the product of the body force magnitude and the density gradient. It may be stabilizing or destabilizing, depending on the sign of the spring constant. If the body is made up of regions of constant density separated by surfaces of density discontinuity along equipotential surfaces, the analog model is further simplified. In this case the restoring forces are localized at these surfaces. They

act normally to the surface and their magnitude per unit area is proportional to the normal displacement of the surface and to the density discontinuity. A free surface may be treated as a density discontinuity. We shall examine these points in more detail in section 5 of Chapter 5 dealing with internal gravity waves in a fluid and in Chapter 6 (p. 475). An example of this analog model is also provided by the stability problem of a non-homogeneous half-space analyzed in section 8 of this chapter.

(c) *Incompressible Solid of Uniform Density Submerged in a Fluid of the Same Density.* This is case (b) with the additional condition

$$\rho = \rho_f = \text{const.} \quad (5.35)$$

Here the analog restoring force disappears and equilibrium equations 5.32 are the same as if the body force did not exist. Therefore the stability problem is formulated by ignoring completely the presence of the fluid and the hydrostatic component.

Modified Variational Principle. The separation of the initial stress into hydrostatic and residual parts leads to an interesting form of the variational principle. In order to show this let us consider the alternative stress components given by equation 2.22 of Chapter 2.

$$t_{ij} = s_{ij} + S_{ij}e - \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (5.36)$$

Replacing the initial stress by

$$S_{ij} = S'_{ij} + S\delta_{ij} \quad (5.37)$$

we find

$$t_{ij} = t_{ij}^{(r)} + Se\delta_{ij} - Se_{ij} \quad (5.38)$$

The components

$$t_{ij}^{(r)} = s_{ij} + S'_{ij}e - \frac{1}{2}(S'_{ik}e_{jk} + S'_{jk}e_{ik}) \quad (5.39)$$

contain only the terms due to the residual initial stress. Introducing expression (5.37) and (5.38) into the strain energy (4.2) yields

$$\Delta V = \Delta' V + \mathcal{R} \quad (5.40)$$

with

$$\begin{aligned} \Delta' V &= \frac{1}{2}t_{ij}^{(r)}e_{ij} + \frac{1}{2}S'_{ij}(e_{i\mu}\omega_{j\mu} + e_{j\mu}\omega_{i\mu} + \omega_{i\mu}\omega_{j\mu}) \\ \mathcal{R} &= \frac{1}{2}Se^2 + \frac{1}{2}S(2e_{ij}\omega_{ij} + \omega_{ij}\omega_{ij} - e_{ij}e_{ij}) \end{aligned} \quad (5.41)$$

The latter expression may be simplified to

$$\mathcal{R} = \frac{1}{2}S \left(e^2 - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) \quad (5.42)$$

It represents the product of S and the second order increment of volume. The total incremental potential (4.1) becomes

$$\mathcal{P}_\tau = \iiint_\tau (\Delta'V + \mathcal{R} + \rho \Delta U) d\tau \quad (5.43)$$

By introducing the residual density (5.6) this expression becomes

$$\mathcal{P}_\tau = \iiint_\tau (\Delta'V + \rho' \Delta U + \mathcal{R} + \rho_f \Delta U) d\tau \quad (5.44)$$

The term $\Delta'V + \rho' \Delta U$ corresponds to a medium of residual density ρ' under the residual initial stress S'_{ij} . With the value (5.44) for \mathcal{P}_τ , the general variational principle (4.4) becomes

$$\begin{aligned} \delta \iiint_\tau (\Delta'V + \rho' \Delta U) d\tau + \delta \iiint_\tau (\mathcal{R} + \rho_f \Delta U) d\tau \\ = \iint_A \Delta f_j \delta u_i dA \end{aligned} \quad (5.45)$$

The differential equations derived from this principle are the equilibrium conditions (5.12) where \mathcal{E}_i is written in the particular form (5.14).

We shall now derive an equivalent principle which corresponds to the alternative form (5.19) of the equilibrium equations.

This can be done by using the identity

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(S e \delta u_i + \frac{\partial S}{\partial x_j} u_j \delta u_i \right) - \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \delta u_i \right) \\ = \frac{\partial S}{\partial x_i} \delta(eu_i) + \frac{1}{2} S \delta \left(e^2 - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right) + \frac{1}{2} \frac{\partial^2 S}{\partial x_i \partial x_j} \delta(u_i u_j) \end{aligned} \quad (5.46)$$

In this identity S is an arbitrary function of the coordinates and δu_i is an arbitrary variation. If we identify S with the initial fluid stress, we derive from equations 5.1 and 5.2

$$\begin{aligned} \frac{\partial S}{\partial x_i} &= -\rho_f X_i \\ \frac{\partial^2 S}{\partial x_i \partial x_j} &= \rho_f \frac{\partial^2 U}{\partial x_i \partial x_j} - X_j \frac{\partial \rho_f}{\partial x_i} \end{aligned} \quad (5.47)$$

Substituting these values into the identity (5.46), we obtain

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(S e \delta u_i + \frac{\partial S}{\partial x_j} u_j \delta u_i \right) - \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \delta u_i \right) \\ = \delta(\mathcal{R} + \rho_f \Delta U - \mathcal{Y}) \end{aligned} \quad (5.48)$$

where

$$\mathcal{Y} = \rho_f X_j u_{j,e} + \frac{1}{2} X_j \frac{\partial \rho_f}{\partial x_i} u_i u_j \quad (5.49)$$

We recognize in equation 5.48 the stress increment ΔS defined by equation 5.21. When we integrate equation 5.48 over the volume τ , we obtain

$$\begin{aligned} \delta \iiint_{\tau} (\mathcal{R} + \rho_f \Delta U - \mathcal{Y}) d\tau \\ = \iint_A \left[(\Delta S + S_e) n_i - S \frac{\partial u_j}{\partial x_i} n_j \right] \delta u_i dA \end{aligned} \quad (5.50)$$

The factor in the surface integral on the right side is simply the vector $\Delta f_i^{(h)}$ defined by relation (5.26). Hence

$$\delta \iiint_{\tau} (\mathcal{R} + \rho_f \Delta U - \mathcal{Y}) d\tau = \iint_A \Delta f_i^{(h)} \delta u_i dA \quad (5.51)$$

When we substitute the last result into equation 5.45, the variational principle becomes

$$\delta \iiint_{\tau} (\Delta' V + \rho' \Delta U + \mathcal{Y}) d\tau = \iint_A \Delta f_i' \delta u_i dA \quad (5.52)$$

where $\Delta f_i'$ is the residual boundary force defined by equation 5.27.

The variational principle (5.52) is in a form corresponding to equilibrium equations 5.19. In the potential energy on the left side of equation 5.52 the terms $\Delta' V$ and $\rho' \Delta U$ contain the residual density and the residual initial stress. The effect of the hydrostatic component of the initial stress is represented in the term \mathcal{Y} . According to its definition 5.49, it is made up of two terms representing the work of the buoyancy forces discussed previously.

6. SURFACE INSTABILITY

The existence of surface instability for the homogeneous half-space was derived by the author* in the context of incremental theories of

* M. A. Biot, Stability Problems of Inhomogeneous Viscoelastic Media, *Non-homogeneity in Elasticity and Plasticity* (IUTAM Symposium, Warsaw, 1958), pp. 311-321, Pergamon Press, New York, 1959; Folding of a Layered Viscoelastic Medium Derived from an Exact Stability Theory of a Continuum under Initial Stress, *Quarterly of Applied Mathematics*, Vol. XVII, No. 2, pp. 185-204, 1959.

elasticity and viscoelasticity. In a later development* the elastic properties of an isotropic medium in finite strain were introduced explicitly.

Consider a rubber-like medium in a state of initial stress such that in the x direction the stress is a compression

$$P = -S_{11} \quad (6.1)$$

while in the y direction the stress is zero,

$$S_{22} = 0 \quad (6.2)$$

The material is incompressible and the extension ratios satisfy the condition of constant volume

$$\lambda_1 \lambda_2 \lambda_3 = 1 \quad (6.3)$$

Extensions along x and y are measured by λ_1 and λ_2 . The finite stress-strain relations (8.45) of Chapter 2 become in this case

$$\begin{aligned} -S_{33} &= \mu_0(\lambda_2^2 - \lambda_3^2) \\ S_{33} + P &= \mu_0(\lambda_3^2 - \lambda_1^2) \\ -P &= \mu_0(\lambda_1^2 - \lambda_2^2) \end{aligned} \quad (6.4)$$

The coefficient μ_0 is the shear modulus in the unstressed state. This initial state of stress is possible in a half-space whose free surface coincides with the coordinate plane $y = 0$.

We shall apply the general theory to the problem of stability of the free surface under the initial compression P and consider the incremental deformations in the x, y plane (Fig. 6.1). For such incremental plane strain in the x, y plane, we have seen that the material retains its isotropy under initial stress. The incremental stresses are given by equations 8.33 and 8.51 of Chapter 2; that is,

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx} \\ s_{22} - s &= 2\mu e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned} \quad (6.5)$$

with

$$\mu = \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2) \quad (6.6)$$

* M. A. Biot, Surface Instability of Rubber in Compression, Air Force Office of Scientific Research Report T.N. 1771, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 168-182, 1963.

To these equations we must add the condition of incompressibility

$$e_{xx} + e_{yy} = 0 \tag{6.7}$$

The equilibrium conditions for the incremental stress field are equations 2.1.

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0 \end{aligned} \tag{6.8}$$

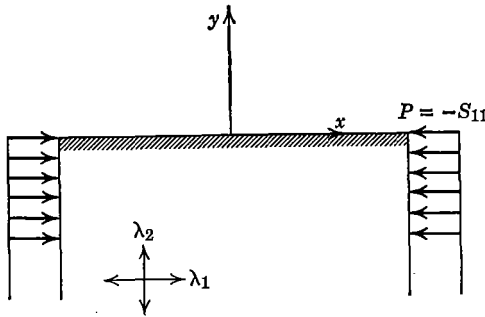


Figure 6.1 Elastic half-space under initial compressive stress P .

The condition (6.7) for incompressibility is satisfied by putting

$$\begin{aligned} u &= - \frac{\partial \phi}{\partial y} \\ v &= \frac{\partial \phi}{\partial x} \end{aligned} \tag{6.9}$$

Eliminating the stress components s_{ij} between equations 6.5 and 6.8, we obtain two equations with two unknowns s and ϕ .

$$\begin{aligned} \frac{\partial s}{\partial x} - \left(\mu + \frac{P}{2} \right) \frac{\partial}{\partial y} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= 0 \\ \frac{\partial s}{\partial y} + \left(\mu - \frac{P}{2} \right) \frac{\partial}{\partial x} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= 0 \end{aligned} \tag{6.10}$$

These equations imply that ϕ satisfies the equation

$$\left(\mu - \frac{P}{2} \right) \frac{\partial^4 \phi}{\partial x^4} + 2\mu \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \left(\mu + \frac{P}{2} \right) \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{6.11}$$

or, by factorization,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left[\left(\mu - \frac{P}{2}\right) \frac{\partial^2 \phi}{\partial x^2} + \left(\mu + \frac{P}{2}\right) \frac{\partial^2 \phi}{\partial y^2} \right] = 0 \quad (6.12)$$

We are looking for a solution which is sinusoidal along the x direction and vanishes at infinite depth ($y = -\infty$). Such a solution is

$$\begin{aligned} \phi &= \frac{1}{l^2} (C_1 e^{ly} + C_2 e^{kly}) \sin lx \\ s &= C_2 P k e^{kly} \cos lx \end{aligned} \quad (6.13)$$

The constants C_1 and C_2 are undetermined. We have put

$$\begin{aligned} k &= \sqrt{\frac{1-\zeta}{1+\zeta}} = \frac{\lambda_1}{\lambda_2} \\ \zeta &= \frac{P}{2\mu} = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2} \end{aligned} \quad (6.14)$$

We conclude that

$$0 < \zeta < 1 \quad (6.15)$$

Therefore k is real and also satisfies the same inequality,

$$0 < k < 1 \quad (6.16)$$

The boundary condition at the surface will now be considered. The x and y components of the forces applied at the surface are given by equations 6.27 of Chapter 1. In the present case they become

$$\begin{aligned} \Delta f_x &= s_{12} + P e_{xy} = (2Q + P) e_{xy} \\ \Delta f_y &= s_{22} = s + 2\mu e_{yy} \end{aligned} \quad (6.17)$$

The condition that the surface be free of forces requires that

$$\Delta f_x = \Delta f_y = 0 \quad (6.18)$$

at $y = 0$. Introducing solution (6.13) into this boundary condition (6.18) and eliminating the constants C_1, C_2 , we obtain the characteristic equation

$$(1 + \zeta)^2 k - 1 = 0 \quad (6.19)$$

When we rationalize this equation, it becomes

$$\zeta^3 + 2\zeta^2 - 2 = 0 \quad (6.20)$$

This cubic has only one real root,

$$\zeta = 0.839 \quad (6.21)$$

For this value of ζ the surface is unstable. Since the characteristic equation is independent of l , all wavelengths are equally unstable. This means that the half-space remains in neutral equilibrium for plane deformations such that the free boundary becomes an arbitrary cylindrical surface. At instability, the value of k is

$$k = 0.295 \quad (6.22)$$

When introduced in the term e^{kly} of the general solution (6.13), it measures the depth to which the surface deformation is felt inside the body. This depth is about three times larger than for the initially unstressed medium. By equations 6.14 the parameter ζ is related to the extension ratios λ_1 and λ_2 which determine the state of initial strain.

We shall consider two particular cases. In the first we put

$$\begin{aligned} \lambda_3 &= 1 \\ \lambda_2 &= \frac{1}{\lambda_1} \end{aligned} \quad (6.23)$$

This is the case where the finite initial strain is applied to a material whose extension is restrained in the direction perpendicular to the x, y plane. Hence

$$\zeta = \frac{1 - \lambda_1^4}{1 + \lambda_1^4} \quad (6.24)$$

The extension ratio corresponding to instability for this case is

$$\lambda_1 = 0.544 \quad (6.25)$$

and the compressive stress is

$$P = 3.08\mu_0 \quad (6.26)$$

In the other case the initial stress is applied by allowing the material to expand freely in the lateral direction. Hence we put

$$\begin{aligned} S_3 &= 0 \\ \lambda_2 &= \lambda_3 = \frac{1}{\sqrt{\lambda_1}} \end{aligned} \quad (6.27)$$

The value of ζ is then

$$\zeta = \frac{1 - \lambda_1^3}{1 + \lambda_1^3} \quad (6.28)$$

and the extension ratio for which instability appears is

$$\lambda_1 = 0.444 \quad (6.29)$$

The corresponding compressive stress is

$$P = 2.05\mu_0 \quad (6.30)$$

To state that the surface is unstable amounts to saying that the apparent surface rigidity must vanish at the critical load. It is of interest to investigate how this apparent rigidity varies as a function of the initial stress. More specifically, we may consider plane initial strain, $\lambda_3 = 1$, for which the critical values are given by equations 6.25 and 6.26. Let us apply a sinusoidally distributed normal load. The boundary conditions (6.18) must be replaced by

$$\begin{aligned} \Delta f_x &= 0 \\ \Delta f_y &= q_0 \cos lx \end{aligned} \quad (6.31)$$

The problem is solved by substituting solution (6.13) into the boundary conditions (6.31). We obtain two equations which determine the constants C_1 and C_2 .

The surface deflection at $y = 0$ is written

$$v = V \cos lx \quad (6.32)$$

The relation between the surface load q_0 and the deflection V is found to be

$$V = \frac{q_0}{2\mu l \varphi} \quad (6.33)$$

with

$$\varphi = \frac{1}{\zeta} [k(1 + \zeta)^2 - 1] \quad (6.34)$$

When the medium is not under initial stress, i.e., for $P = \zeta = 0$, the true value of φ is unity, and the deflection is given by

$$V = \frac{q_0}{2\mu_0 l} \quad (6.35)$$

The effect of the initial strain on the surface deflection amounts to replacing the shear modulus μ_0 of the unstressed state by an "effective modulus" $\mu\varphi$. Note that the incremental modulus in the present case ($\lambda_3 = 1$) is

$$\mu = \frac{1}{2}\mu_0 \left(\lambda_1^2 + \frac{1}{\lambda_1^2} \right) \quad (6.36)$$

It is a minimum for $\lambda_1 = 1$. On the other hand, the factor φ tends to zero as we approach the critical state. The increase of incremental modulus μ is measured by μ/μ_0 , while the effective surface rigidity is measured by $\mu\varphi/\mu_0$. Values of these factors were given in the author's paper* and are shown in Table 1 as a function of the extension ratio λ_1 of the initial strain.

Table 1
 Variation of incremental modulus μ and "effective surface rigidity" $\mu\varphi$ as a function of the extension ratio λ_1 of the initial strain for the case $\lambda_3 = 1$

λ_1	μ/μ_0	$\mu\varphi/\mu_0$
2.00	2.125	2.275
1.50	1.347	1.518
1.20	1.067	1.193
1.00	1.000	1.000
0.90	1.022	0.893
0.80	1.101	0.758
0.70	1.266	0.567
0.60	1.569	0.262
0.544	1.838	0

The values in the table include the case $\lambda_1 > 1$ for which the initial stress is a tension. As the tension decreases, going over into an increasing compression, there is a continuous drop in surface rigidity. It vanishes at the critical value $\lambda_1 = 0.544$ for which $\varphi = 0$.

Mooney Material. The results of this section have been derived for a material of rubber-like elasticity satisfying the finite stress-strain relations (6.4). However, they are immediately applicable to materials of a more general type such as the Mooney material whose incremental deformation was analyzed in section 8 of Chapter 2. For this material the incremental stress-strain relations (6.5) remain valid with a value of the incremental shear modulus μ given by

$$\mu = C_1(\lambda_1^2 + \lambda_2^2) + C_2 \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \tag{6.37}$$

* M. A. Biot, Surface Instability of Rubber in Compression, Air Force Office of Scientific Research Report T.N. 1771, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 168-182, 1963.

where C_1 and C_2 are two elastic constants. The value of ζ is given by the same equation (6.14) with the value (6.37) for μ .

7. BUCKLING OF A THICK SLAB

A solution based on elasticity theory for the stability of a thick slab under axial compression was derived by the author in 1938.* This solution leads immediately to the characteristic equation (7.12) for the particular case of an incompressible medium. The results are also a particular case of the more general problem of instability of a slab embedded in an infinite medium which was analyzed† in the context of the viscoelasticity theory and leads to the same characteristic equation (7.12).

More recently the author developed a more detailed analysis based on these results‡ by introducing explicitly the finite deformation properties of the medium. The material presented here is a shortened version of this paper, and the reader is referred to it for further data such as plots of the stress distribution over the cross section during buckling. An elastic slab of length \mathcal{L}_0 and thickness h_0 is shown in the unstressed state in Figure 7.1. The width in the direction perpendicular to the figure is infinite. This slab is then compressed to a length \mathcal{L} and a thickness h by a compressive stress P acting along its axis (Fig. 7.2). The compression is exerted by two rigid frictionless blocks a and b (Fig. 7.1). Over-all slippage

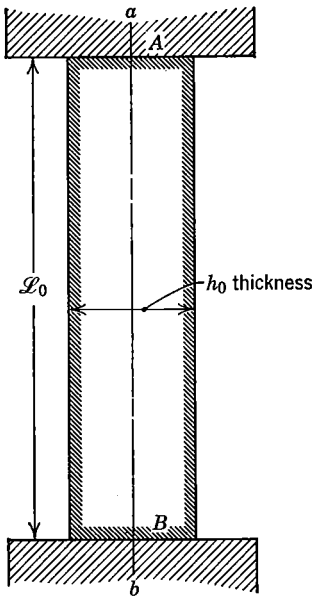


Figure 7.1 Elastic slab in the original unstressed state. The slab is viewed across the thickness.

between the slab and the blocks is prevented by pinning the slab to the blocks at points A and B on the axis.

* See reference 2 of the Preface.

† See references * on p. 159.

‡ M. A. Biot, Exact Theory of Buckling of a Thick Slab, Air Force Office of Scientific Research Report T.N. 1770, 1961, and *Applied Scientific Research*, A, Vol. 12, pp. 182–198, 1963.

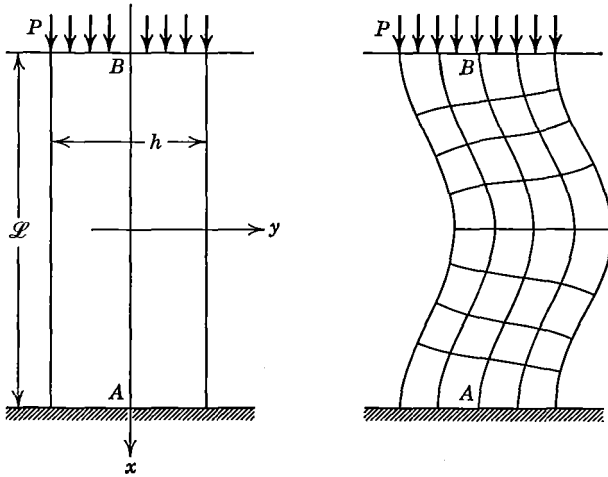


Figure 7.2 Elastic slab in the initial state of compression and in the buckled state, viewed across the thickness.

The material of the slab is assumed to satisfy stress-strain relations (6.4) which are typical of a rubber-like solid. In the initial state of finite strain the slab may be restrained in a direction perpendicular to the plane x, y of the figure. In this case $\lambda_3 = 1$. The compression is then

$$P = \mu_0 \left(\frac{1}{\lambda_1^2} - \lambda_1^2 \right) \tag{7.1}$$

and the incremental modulus in the x, y plane is

$$\mu = \frac{1}{2}\mu_0 \left(\lambda_1^2 + \frac{1}{\lambda_1^2} \right) \tag{7.2}$$

The extension ratio λ_1 ($\lambda_1 < 1$) measures the finite compressive strain along the axis. The length and thickness of the slab in the compressed state are, respectively,

$$\mathcal{L} = \mathcal{L}_0 \lambda_1 \quad h = \frac{h_0}{\lambda_1} \tag{7.3}$$

On the other hand, the slab may be free to expand perpendicularly

to the x, y plane. In this case $S_{33} = 0$ and the only initial stress component is the compressive stress P given by

$$P = \mu_0 \left(\frac{1}{\lambda_1} - \lambda_1^2 \right) \quad (7.4)$$

The corresponding incremental modulus is

$$\mu = \frac{1}{2}\mu_0 \left(\lambda_1^2 + \frac{1}{\lambda_1} \right) \quad (7.5)$$

The length and width of the plate in the compressed state are

$$\mathcal{L} = \mathcal{L}_0 \sqrt{\lambda_1} \quad h = \frac{h_0}{\sqrt{\lambda_1}} \quad (7.6)$$

It is also possible to consider a state of partial restraint intermediate between those considered above with a suitable value of the initial stress component S_{33} . In this case the value of μ is given by equation 6.6.

Let us now consider a plane strain perturbation from the initial state. This incremental deformation is assumed to be parallel to the x, y plane.

The equations for this incremental deformation are the same as in the previous section. We must satisfy equations 6.10. The solution of the equations which are to be considered here and which correspond to buckling is

$$\begin{aligned} \phi &= \frac{1}{l^2} (C_1 \cosh ly + C_2 \cosh kly) \sin lx \\ s &= C_2 P k \sinh kly \cos lx \end{aligned} \quad (7.7)$$

where C_1 and C_2 are undetermined constants. It is readily seen that the frictionless constraints imposed at the rigid blocks are satisfied if we put $l = 2\pi/\mathcal{L}$. Note that for this solution the points of attachment A and B do not remain on the y axis during buckling, but this does not give rise to any difficulty since we can always superimpose an arbitrary translation. As before, we put

$$\begin{aligned} k &= \sqrt{\frac{1-\zeta}{1+\zeta}} \\ \zeta &= \frac{P}{2\mu} \end{aligned} \quad (7.8)$$

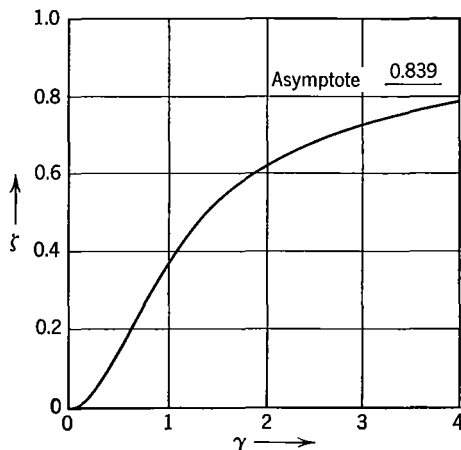


Figure 7.3 Value of ζ as a function of γ under buckling conditions as determined by the characteristic equation (7.12).

The next step is to satisfy the boundary conditions at the free surfaces of the slab. This implies

$$\Delta f_x = \Delta f_y = 0 \quad (7.9)$$

at $y = \pm h/2$. Because of the symmetry of solution (7.7) it is sufficient to satisfy this condition at one of the faces, for example, $y = h/2$. The expressions for Δf_x and Δf_y are given by equations 6.17. Substituting solution (7.7) into the boundary conditions (7.9), we obtain

$$\begin{aligned} 2C_1 \cosh \gamma + C_2(1 + k^2) \cosh k\gamma &= 0 \\ C_1(1 + k^2) \sinh \gamma + 2C_2k \sinh k\gamma &= 0 \end{aligned} \quad (7.10)$$

with

$$\gamma = \frac{1}{2}kh = \frac{\pi h}{\mathcal{L}} \quad (7.11)$$

We derive the characteristic equation

$$4k \tanh k\gamma - (1 + k^2)^2 \tanh \gamma = 0 \quad (7.12)$$

This is a relation between ζ and γ . It represents in non-dimensional form the relation between the buckling load P and the length \mathcal{L} of the slab.

A plot of ζ versus γ is shown in Figure 7.3. The parameter γ is

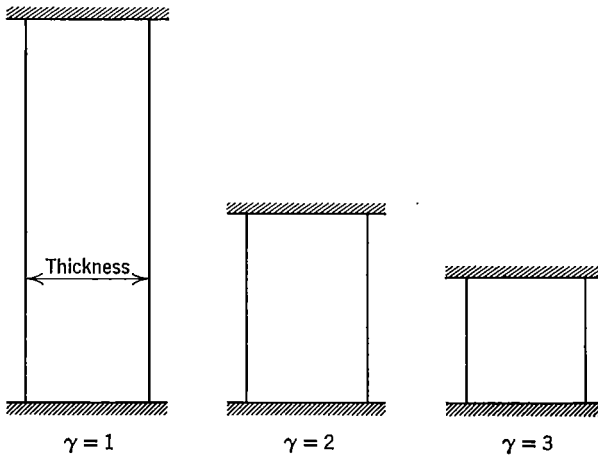


Figure 7.4 Slab slenderness in the state of initial compression illustrating the significance of γ for $\gamma = 1, 2, 3$.

inversely proportional to the slab slenderness in the state of initial compression. Its significance is illustrated in Figure 7.4. As γ tends to infinity, the value of ζ approaches the asymptotic value

$$\zeta = 0.839 \quad (7.13)$$

A physical interpretation of this asymptotic behavior is obtained by substituting $\gamma = \infty$ in the characteristic equation (7.12). It becomes

$$4k - (1 + k^2)^2 = 0 \quad (7.14)$$

or

$$(1 + \zeta)^2 k - 1 = 0 \quad (7.15)$$

which coincides with condition (6.19) for surface instability. The real root ζ of this equation is the asymptotic value (7.13). This means that for a very short slab the buckling degenerates into a surface instability. At the other extreme we have a very slender slab which is represented by the behavior of the curve, ζ versus γ , in the vicinity of $\gamma = 0$. The approximate relationship in this case is obtained by expanding the hyperbolic functions of equation 7.12 in power series of γ . Limiting the expression to the third power, we write

$$\begin{aligned} \tanh k\gamma &= k\gamma - \frac{1}{3}k^3\gamma^3 \\ \tanh \gamma &= \gamma - \frac{1}{3}\gamma^3 \end{aligned} \quad (7.16)$$

Substituting these approximate values into equation 7.12, we obtain

$$\zeta = \frac{2\gamma^2}{3 + \gamma^2} \quad (7.17)$$

To the first order in γ^2 the buckling load is given by

$$\zeta = \frac{2}{3}\gamma^2 \quad (7.18)$$

or

$$P = \frac{1}{3}\mu l^2 h^2 \quad (7.19)$$

Let us show that this value coincides with the Euler buckling load obtained from thin plate theory. The equation for the transversal deflection v of an elastic plate of thickness h under an axial compressive stress P is

$$\frac{E}{1 - \nu^2} \cdot \frac{h^3}{12} \frac{d^2v}{dx^2} + Phv = 0 \quad (7.20)$$

Young's modulus is represented by E , and Poisson's ratio by ν . In an incompressible material the elastic coefficient is

$$\frac{E}{1 - \nu^2} = 4\mu \quad (7.21)$$

Substituting a sinusoidal deflection

$$v = V \cos lx \quad (7.22)$$

into equation 7.20, we find for the buckling load

$$P = \frac{1}{3}\mu l^2 h^2 \quad (7.23)$$

which coincides with the value (7.19).

This approximate value of ζ given by equation 7.18 is plotted in Figure 7.5 and compared with the exact value. It can be seen that it constitutes a good approximation in the range

$$0 < \gamma < 0.3 \quad (7.24)$$

We conclude that the Euler theory of buckling yields a satisfactory value of the buckling load in the range

$$\frac{\mathcal{L}}{h} > 10 \quad (7.25)$$

For this conclusion to be valid the material must, of course, retain its elastic properties under the applied initial compression.

An interesting analogy appears when we consider the behavior of the slab as a function of the buckling wavelength. At large wavelengths the buckling is represented by a plate bending and, as the wavelength decreases, the buckling degenerates into a surface instability. Similarly the vibration of a plate which is represented by bending waves at large wavelength degenerates into Rayleigh waves propagating at the surface when the wavelength becomes small relative to the plate thickness.

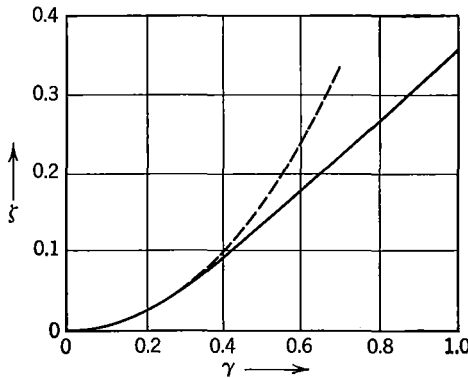


Figure 7.5 Portion of the graph of Figure 7.3 near the origin and comparison with the Euler theory. Solid curve, exact value; broken curve, $\zeta = \frac{2}{3}\gamma^2$ (Euler theory).

The problem may also be solved by use of the variational principles discussed in section 4. The strain energy per unit volume given by equation 4.23 is

$$\Delta V = 2Me_{xx}^2 + 2Le_{xy}^2 - \frac{1}{2}P \left(\frac{\partial v}{\partial x} \right)^2 \tag{7.26}$$

In the present case the elastic coefficients are

$$N = Q = \mu \tag{7.27}$$

Hence

$$M = \mu + \frac{1}{4}P \tag{7.28}$$

$$L = \mu + \frac{1}{2}P$$

The potential energy of a length \mathcal{L} of the slab is

$$\mathcal{P}_\tau = \int_0^{\mathcal{L}} dx \int_{-h/2}^{+h/2} \Delta V dy \tag{7.29}$$

It corresponds to equation 4.10. Since the rigid boundaries are plane frictionless surfaces, the variational principle reduces to putting equal to zero the variation of \mathcal{P}_τ . In evaluating \mathcal{P}_τ we introduce the approximate displacement

$$\begin{aligned} u &= u_1 y \sin lx \\ v &= v_0 \cos lx \end{aligned} \quad (7.30)$$

An additional assumption is also introduced when performing the integration in equation 7.29. We note that the term $2Le_{xy}^2$ in expression (7.26) is equal to $t'_{12}e_{xy}$ where t'_{12} is a shearing stress which vanishes at the free surfaces of the plate. On the other hand, the approximation (7.30) yields a constant value of t'_{12} over the cross section. This does not actually occur in the exact solution. Therefore it is more accurate to average out the integral by assuming t'_{12} to be constant in the interval

$$-\frac{1}{2}Kh < y < \frac{1}{2}Kh \quad (7.31)$$

and zero outside. This amounts to integrating the term $2Le_{xy}^2$ between the limits $\pm Kh/2$. With this procedure the value of \mathcal{P}_τ is given by

$$\frac{2\mathcal{P}_\tau}{\mathcal{L}} = \frac{1}{8}Ml^2h^3u_1^2 + \frac{1}{2}LhK(u_1 - lv_0)^2 - \frac{1}{2}Phl^2v_0^2 \quad (7.32)$$

The condition for buckling is obtained by writing that the value of \mathcal{P}_τ is stationary; that is,

$$\frac{\partial \mathcal{P}_\tau}{\partial u_1} = 0 \quad \frac{\partial \mathcal{P}_\tau}{\partial v_0} = 0 \quad (7.33)$$

or

$$\begin{aligned} \frac{1}{8}Ml^2h^3u_1 + LK(u_1 - lv_0) &= 0 \\ LK(u_1 - lv_0) + Plv_0 &= 0 \end{aligned} \quad (7.34)$$

Elimination of u_1 and v_0 yields the characteristic equation

$$\frac{1}{8}l^2h^2M \left(1 - \frac{P}{KL} \right) = P \quad (7.35)$$

By introducing the variables ζ and γ defined by equations 7.8 and 7.11 and the values (7.28) for L and M , the buckling condition (7.35) becomes

$$\gamma^2 = \frac{3\zeta}{2 + \zeta} \frac{1}{\left[1 - \frac{2\zeta}{K(1 + \zeta)} \right]} \quad (7.36)$$

If we choose the value $K = 0.91$, this equation yields a curve which cannot be distinguished from the exact one when plotted in Figure 7.3.

The results are also applicable to the more general case of a *Mooney material* and for a slab with arbitrary extension ratios λ_1 and λ_2 along the x and y directions. In this material the incremental shear modulus μ takes the value (6.37) and the value of ζ is given by equation 6.14.

8. INSTABILITY OF A NON-HOMOGENEOUS HALF-SPACE

The previous examples have been restricted to homogeneous materials and a uniform initial stress field.

We now consider a problem of stability in a medium which is continuously inhomogeneous and in which the initial stress is not uniform. The problem considered here was treated by the author

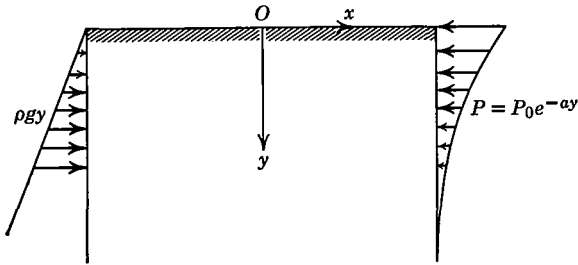


Figure 8.1 Non-homogeneous elastic half-space. The initial stress is the hydrostatic pressure ρgy and a horizontal compression P .

for the more general case of the viscoelastic medium.* The half-space is limited by a free surface which is a horizontal plane located at $y = 0$. It is under the action of a uniform gravity field, and the material is incompressible and of uniform mass density ρ . In order to conform with the treatment of the problem in the paper cited above, the y axis is oriented positively downward (Fig. 8.1). Consider the stresses in the x, y plane (Fig. 8.1). We shall assume that the initial stress field in this plane is

* M. A. Biot, *Instability of a Continuously Inhomogeneous Viscoelastic Half-Space under Initial Stress*, *Journal of the Franklin Institute*, Vol. 270, No. 3, pp. 190–201, 1960.

$$\begin{aligned}
 S_{11} &= -P(y) - \rho gy \\
 S_{22} &= -\rho gy \\
 S_{12} &= 0
 \end{aligned}
 \tag{8.1}$$

The gravity acceleration is denoted by g , and we have put

$$P = P_0 e^{-ay} \tag{8.2}$$

The field is composed of two parts, a term ρgy increasing linearly with depth, and a horizontal compression $P_0 \exp(-ay)$ decreasing exponentially with depth.

A stress S_{33} may act normally to the figure, but it will not play any role in the problem considered since we shall assume the incremental deformation to be a plane strain in the x, y plane.

Let us show how such a state of initial stress may be generated by a *finite homogeneous* deformation of the elastic medium. We put

$$\mu_0 = \mu_{0s} e^{-ay} \tag{8.3}$$

in the stress-strain relations (8.45) of Chapter 2 for a rubber-like medium. The quantity μ_{0s} is the shear modulus of the unstressed material at the surface. Such a medium is incompressible, isotropic, and elastically non-homogeneous. Its rigidity decreases exponentially with depth with a gradual change from solid to liquid properties.

The finite stress-strain relations (8.45) of Chapter 2 become

$$\begin{aligned}
 S_{22} - S_{33} &= \mu_{0s} e^{-ay} (\lambda_2^2 - \lambda_3^2) \\
 S_{33} - S_{11} &= \mu_{0s} e^{-ay} (\lambda_3^2 - \lambda_1^2) \\
 S_{11} - S_{22} &= \mu_{0s} e^{-ay} (\lambda_1^2 - \lambda_2^2)
 \end{aligned}
 \tag{8.4}$$

The uniform extension ratios $\lambda_1, \lambda_2, \lambda_3$ which represent the homogeneous deformation are measured along the coordinate axes and are independent of the location. They satisfy the condition of incompressibility

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{8.5}$$

In addition we must satisfy the initial condition of equilibrium in the gravity field; that is,

$$\frac{\partial S_{22}}{\partial y} + \rho g = 0 \tag{8.6}$$

Equations 8.1 and 8.6 are satisfied by the initial stress field (8.4) if we put

$$P_0 = \mu_{0s}(\lambda_2^2 - \lambda_1^2) \quad (8.7)$$

This equation represents the horizontal compression at the surface. The value of the stress S_{33} is not relevant here. It is given by either of the first two of equations (8.4).

In order to analyze the stability we apply equations 6.16 of Chapter 1 for an incremental strain in the x, y plane. In the present case they become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho g \omega - P \frac{\partial \omega}{\partial y} + \left(\frac{\partial P}{\partial y} + \rho g \right) e_{xy} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} + \rho g e_{yy} &= 0 \end{aligned} \quad (8.8)$$

In writing these equations we must take into account the condition of incompressibility,

$$e = e_{xx} + e_{yy} = 0 \quad (8.9)$$

The incremental stress-strain relations are given by equations 6.5:

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx} \\ s_{22} - s &= 2\mu e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned} \quad (8.10)$$

However, in this case the incremental modulus (6.6) is written

$$\mu = \frac{1}{2}\mu_{0s}e^{-ay}(\lambda_1^2 + \lambda_2^2) \quad (8.11)$$

Hence it also decreases exponentially with depth.

The boundary conditions at the free surface are the same as equations 6.18; that is, we must have

$$\begin{aligned} e_{xy} &= 0 \\ s_{22} &= 0 \end{aligned} \quad (8.12)$$

for $y = 0$.

We have thus formulated the problem mathematically. The equations to be solved are 8.8, 8.9, and 8.10, with the boundary conditions (8.12) at the surface and the additional condition that the solution vanish at infinite depth.

Before deriving such a solution it is useful to transform the equations following the general method of section 5. We are dealing

here with a stability problem in the presence of a hydrostatic stress field. It was shown that in such a case the problem may be reformulated into an equivalent one by an analog model in which the hydrostatic stress has been eliminated. The procedure is readily applied to the present case by putting

$$\begin{aligned} s'_{11} &= s_{11} + \rho g v \\ s'_{22} &= s_{22} + \rho g v \\ s' &= s + \rho g v \end{aligned} \quad (8.13)$$

Equations 8.8 become

$$\begin{aligned} \frac{\partial s'_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} + \frac{\partial P}{\partial y} e_{xy} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s'_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0 \end{aligned} \quad (8.14)$$

The stress-strain relations (8.10) are transformed into an identical form,

$$\begin{aligned} s'_{11} - s' &= 2\mu e_{xx} \\ s'_{22} - s' &= 2\mu e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned} \quad (8.15)$$

However, the boundary conditions (8.12) at the surface are changed to

$$\begin{aligned} e_{xy} &= 0 \\ s'_{22} &= \rho g v \end{aligned} \quad (8.16)$$

It can be seen that the equations are now those of the same medium *free of gravity stresses* and subject to an initial horizontal compression $P(y)$. The influence of gravity has been replaced by the application at the surface of a normal force $\rho g v$ acting in a direction opposite to the displacement and proportional to the local change of altitude. In order to solve these equations we first satisfy the condition of incompressibility (8.9) by putting as before

$$\begin{aligned} u &= - \frac{\partial \phi}{\partial y} \\ v &= \frac{\partial \phi}{\partial x} \end{aligned} \quad (8.17)$$

We further particularize the solution by looking for deformations which are sinusoidal along x . We therefore put

$$\begin{aligned}\phi &= \varphi(y) \sin lx \\ s' &= P_0 f(y) e^{-ay} \cos lx\end{aligned}\tag{8.18}$$

Substitution of these expressions into equations 8.14 and 8.15 yields two ordinary differential equations for the unknown functions φ and f ; that is,

$$\begin{aligned}\frac{1}{2} \frac{1 + \zeta}{\zeta} (\varphi''' - \varphi' l^2 - a\varphi'' - a\varphi l^2) + lf &= 0 \\ \frac{1}{2} \frac{1 - \zeta}{\zeta} (\varphi l^2 - \varphi'') l + \frac{1}{\zeta} a\varphi' l - f' + af &= 0\end{aligned}\tag{8.19}$$

The primes represent derivatives with respect to y , and we have put

$$\begin{aligned}\zeta &= \frac{P}{2\mu} = \frac{P_0}{2\mu_s} \\ \mu_s &= \frac{1}{2}\mu_{0s}(\lambda_1^2 + \lambda_2^2)\end{aligned}\tag{8.20}$$

Referring to equations 8.7 and 8.11, we note that μ_s represents the incremental modulus at the free surface and that we may also write

$$\zeta = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2}\tag{8.21}$$

Solutions of equations 8.19 are of the form

$$\begin{aligned}\varphi &= C_1 e^{\beta_1 ay} + C_2 e^{\beta_2 ay} \\ f &= C_3 e^{\beta_1 ay} + C_4 e^{\beta_2 ay}\end{aligned}\tag{8.22}$$

Substitution of these expressions in equations 8.19 yields a characteristic equation for the values of β_1 and β_2 :

$$\begin{aligned}[\beta(\beta^2 - \delta^2) - (\beta^2 + \delta^2)](\beta - 1) + k^2 \delta^2 (\delta^2 - \beta^2) \\ + \frac{2\beta\delta^2}{1 + \zeta} = 0\end{aligned}\tag{8.23}$$

where

$$\begin{aligned}k^2 &= \frac{1 - \zeta}{1 + \zeta} \\ \delta &= \frac{l}{a}\end{aligned}\tag{8.24}$$

There are two roots, β_1 and β_2 , for β which have negative real parts. These two roots are used in the exponents of the solution (8.22). This ensures that the solution vanishes at infinite depth.

Equation 8.23 is of the fourth degree in β . However, it can easily be transformed into a quadratic equation in $\beta(\beta - 1)$. For example, we may write

$$\zeta = \frac{\delta^4 - \delta^2[2\beta(\beta - 1) - 1] + [\beta(\beta - 1)]^2}{\delta^4 - \delta^2 - [\beta(\beta - 1)]^2} \quad (8.25)$$

Solving this quadratic equation for $\beta(\beta - 1)$ and again a second quadratic equation for β , we find

$$\left. \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\} = \frac{1}{2} \left[1 - \sqrt{1 + \frac{4\delta}{1 + \zeta} (\delta \pm \sqrt{\delta^2 \zeta^2 - (1 + \zeta)^2})} \right] \quad (8.26)$$

The four constants in solution (8.22) are not arbitrary. The relations among them are found by substitution of the solution in the differential equations 8.19. This leaves only two of the constants arbitrary, and they may be eliminated by substitution of solution (8.22) into boundary conditions (8.16) at the free surface. This leads to the following condition for surface buckling:

$$1 + \zeta = \frac{2\delta^2[\delta^2 - \beta_1\beta_2 - G(\beta_1 + \beta_2)]}{\beta_1^2\beta_2^2 - \delta^4 + \delta^2(\beta_1 + \beta_2)(\beta_1 + \beta_2 - 2G)} \quad (8.27)$$

The parameter

$$G = \frac{\rho g}{P_0 a} \quad (8.28)$$

represents the influence of gravity in non-dimensional form.

Equation 8.27 may be considered a relation between ζ and δ where G plays the role of a parameter. This relationship is plotted in Figure 8.2 for several values of G . The variable δ is inversely proportional to the wavelength of the surface deformation. It represents a dimensionless wave number. It is seen that, for a given value of G , ζ goes through a minimum. This minimum determines the compression under which the surface will buckle. The value δ_a of δ , at which this minimum occurs, determines the buckling wavelength; that is,

$$\mathcal{L}_a = \frac{2\pi}{a\delta_a} \quad (8.29)$$

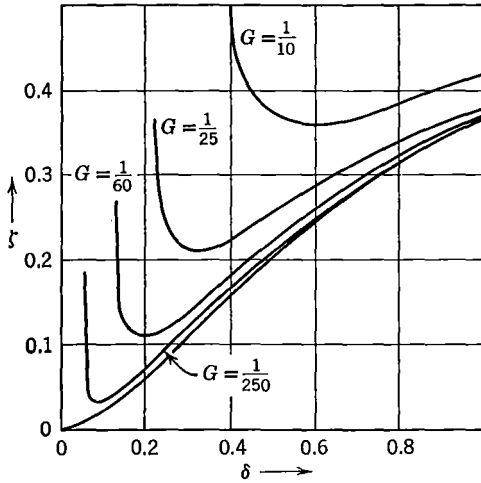


Figure 8.2 The stability variable ζ as a function of the dimensionless wave number δ and the gravity parameter G .

The value of δ_d as a function of G is found to be approximately given by the expression

$$\delta_d = 2.2G^{3/5} \tag{8.30}$$

The buckling wavelength tends to infinity for $G \rightarrow 0$; that is, for vanishing gravity forces.

Attention is called to the fact that in the range

$$\delta < \frac{1}{\zeta} + 1 \tag{8.31}$$

the roots β_1 and β_2 are complex so that solution 8.22 is an *oscillatory function* of depth for the range of values represented by the diagram of Figure 8.2. For example, at the critical buckling load and for $G = 1/250$ the roots are

$$\left. \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\} = -0.010 \pm 0.072i \tag{8.32}$$

Another point of interest is the existence of a vertical asymptote for the curves in Figure 8.2. The abscissa δ of the asymptote is a

function of G . More detailed discussion of these points will be found in the reference cited.*

A limiting case of interest is the asymptotic value of ζ corresponding to $\delta = \infty$, that is, at infinitely small wavelength. By putting $\delta = \infty$ in equation 8.27, we find that this asymptotic value must satisfy the equation

$$\zeta^3 + 2\zeta^2 - 2 = 0 \quad (8.33)$$

whose real root is

$$\zeta = 0.839 \quad (8.34)$$

Equation 8.33 coincides with the characteristic equation (6.20) already derived for the homogeneous half-space without gravity. This result becomes evident when we consider the fact that for vanishing wavelength the material must behave like a homogeneous medium. In addition, for vanishing dimensions the gravity forces obey a scaling factor which renders them negligible relative to the elastic stresses. The roots β_1 and β_2 for this case become

$$\begin{aligned} \beta_1 &= -\frac{l}{a} \\ \beta_2 &= -\frac{kl}{a} \end{aligned} \quad (8.35)$$

With these values solution (8.22) contains the same exponential factors as those in equations 6.13 obtained for the surface instability of the homogeneous gravity-free half-space.

The results are also applicable to the non-homogeneous half-space constituted by a *Mooney material*, provided the two elastic constants C_1 and C_2 describing this material are proportional to the *same* exponential factor e^{-ay} . In this case the incremental shear coefficient given by equation 6.37 is also proportional to this exponential factor.

* M. A. Biot, Instability of a Continuously Inhomogeneous Viscoelastic Half-Space under Initial Stress, *Journal of the Franklin Institute*, Vol. 270, No. 3, pp. 190-201, 1960.

CHAPTER FOUR

Elastic Stability of Anisotropic Media

1. INTRODUCTION

In the preceding chapter the applications of the theory of elastic stability were restricted to isotropic media. More specifically they were restricted to materials which retain the property of isotropy for incremental plane strain superposed on a state of initial stress.

We now consider problems of elastic stability for anisotropic media. The property of anisotropy referred to here means that (1) the medium may be anisotropic in the stress-free state, i.e. it may possess intrinsic anisotropy; (2) it may also be isotropic for finite deformations and may exhibit an induced anisotropy for incremental deformations in the vicinity of a state of initial stress. Problems of stability of anisotropic media require an analysis which is more elaborate than that for isotropic media considered in Chapter 3, because of the appearance in this case of new features related to the phenomenon of internal instability.

The problems treated in this chapter are restricted to incompressible materials. This provides drastic simplifications of the algebra without altering the essential features of the results.

In order to familiarize the reader with some significant mechanical properties of anisotropic media we shall examine in some detail in section 2 an anisotropic laminated medium. In particular, we derive the properties of a medium made by the superposition of thin layers

of two materials alternately soft and hard. This provides an interesting interpretation of the elastic coefficients for anisotropic media and particularly for the incremental coefficients in the presence of initial stress.

The important problem of internal instability is treated in section 3. There are two physically distinct cases: internal instability of the first kind, and internal instability of the second kind. From the mathematical viewpoint internal instability is due to the hyperbolic or mixed hyperbolic-elliptic nature of the equations for the incremental field beyond certain critical values of the initial stress.

The physical significance of internal instability is brought to light by applying the variational principles derived in earlier chapters and by analyzing the properties of the incremental strain energy.

The phenomenon of surface instability for the anisotropic medium is analyzed in section 4. A numerical table provides a key solution. For an isotropic medium with induced anisotropy this table yields immediately the condition for surface instability if the finite stress-strain law for the material is known.

The effect of gravity on the surface instability is also discussed. Results for this case are derived quite simply by adding an elastic surface force in accordance with the concept of the analog model as explained in sections 5 and 8 of Chapter 3.

As a basis for the solution of a large class of problems, section 5 derives six distinct matrix coefficients which are fundamental in the mechanics of a plate under initial stress. These coefficients provide the relations between the surface displacements of the plate and the incremental normal and tangential loads applied to these surfaces. They constitute the key to the mechanics of plates and multilayered media.

These results are applied in section 6 to the problem of buckling of a free and embedded plate with anisotropic properties. The particular case of rubber-like materials is also considered. Numerical solutions provide a description of the buckling phenomenon in the complete range of wavelength-to-thickness ratios. The range from large wavelengths, where Euler's theory is applicable, down to the smaller wavelengths, where the phenomenon degenerates into a surface instability, is covered. In the case of the embedded layer the buckling at small wavelengths degenerates into an interfacial instability.

The phenomena which emerge from this analysis are in complete analogy with those of wave propagation in elastic media. Internal instability is analogous to body waves; surface instability and interfacial instability are analogous to Rayleigh and Stoneley waves.

As in the case of wave propagation, a clear understanding of these features is necessary before more complex problems of elastic stability of heterogeneous systems can be solved.

An example of more complex stability problems is provided by analysis of multilayered media under initial stress in section 7. A very concise formulation of the problem is provided by using the six basic coefficients for a plate derived in section 5. The problem may be formulated by applying a variational principle to a single quadratic function of the interfacial displacements. These results lead to recurrence equations and a matrix multiplication procedure which are well suited for numerical calculations when a large number of layers is involved. The buckling problem is also solved for a rubber-like medium composed of alternate layers of two materials of different rigidity. The solution is remarkably simple. It is verified that, when the wavelength is large enough compared to the thickness of the layers, the buckling coincides with the phenomenon of internal instability of a continuous anisotropic medium derived in section 3.

2. A LAMINATED MEDIUM AS AN EXAMPLE OF ANISOTROPY

A good illustration of some of the properties of anisotropic media is provided by analyzing a laminated medium. Such a medium is obtained by superposition of thin adhering layers which are alternately hard and soft. The hard and soft materials occupy, respectively, fractions α_1 and α_2 of the total thickness. Within certain limits such a medium behaves like an elastic continuum with anisotropic properties, although the individual layers may be isotropic.

For simplicity let us assume first that the materials are incompressible and free of initial stress. In addition, we shall restrict the discussion to plane strain. With the y axis perpendicular to the plane

of the lamination, the stress-strain relations of the hard material are written

$$\begin{aligned} s_{11} - s &= 2N_1 e_{xx} \\ s_{22} - s &= 2N_1 e_{yy} \\ s_{12} &= 2Q_1 e_{xy} \end{aligned} \tag{2.1}$$

For the soft material we write

$$\begin{aligned} s_{11} - s &= 2N_2 e_{xx} \\ s_{22} - s &= 2N_2 e_{yy} \\ s_{12} &= 2Q_2 e_{xy} \end{aligned} \tag{2.2}$$

When a uniform shear stress s_{12} is applied, the total average shear strain is

$$e_{xy} = \frac{1}{2} \left(\frac{\alpha_1}{Q_1} + \frac{\alpha_2}{Q_2} \right) s_{12} \tag{2.3}$$

Since α_1 and α_2 represent fractions of the unit thickness of the composite medium, we note the property

$$\alpha_1 + \alpha_2 = 1 \tag{2.4}$$

Next we consider a uniform strain without shear. The components e_{xx} and e_{yy} and the normal stress s_{22} are the same in both media. However, the stress component s_{11} takes values $s_{11}^{(1)}$ and $s_{11}^{(2)}$ in the hard and soft materials, respectively. Taking into account the condition of incompressibility

$$e_{xx} + e_{yy} = 0 \tag{2.5}$$

and subtracting from each other the first two of equations 2.1 and 2.2, we find

$$\begin{aligned} s_{11}^{(1)} - s_{22} &= 4N_1 e_{xx} \\ s_{11}^{(2)} - s_{22} &= 4N_2 e_{xx} \end{aligned} \tag{2.6}$$

The total stress s_{11} for the composite material is

$$s_{11} = s_{11}^{(1)}\alpha_1 + s_{11}^{(2)}\alpha_2 \tag{2.7}$$

From equations 2.4, 2.6, and 2.7 we derive

$$s_{11} - s_{22} = 4(N_1\alpha_1 + N_2\alpha_2)e_{xx} \tag{2.8}$$

Relations (2.3) and (2.8) involve the two coefficients

$$\begin{aligned} N &= N_1\alpha_1 + N_2\alpha_2 \\ Q &= \frac{1}{\frac{\alpha_1}{Q_1} + \frac{\alpha_2}{Q_2}} \end{aligned} \quad (2.9)$$

These expressions yield the two elastic moduli of the laminated medium. We may also write

$$\begin{aligned} s_{11} - s_{22} &= 4Ne_{xx} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (2.10)$$

As already pointed out in section 8 of Chapter 2, if we assume that condition (2.5) corresponding to incompressibility is satisfied, then equations 2.10 are equivalent to

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} \\ s_{22} - s &= 2Ne_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (2.11)$$

If the laminations are comprised of isotropic materials, we write

$$\begin{aligned} N_1 &= Q_1 = \mu_1 \\ N_2 &= Q_2 = \mu_2 \end{aligned} \quad (2.12)$$

where μ_1 and μ_2 are the characteristic shear moduli of each material. The composite moduli become

$$\begin{aligned} N &= \mu_1\alpha_1 + \mu_2\alpha_2 \\ Q &= \frac{1}{\frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2}} \end{aligned} \quad (2.13)$$

Using relation (2.4), we derive

$$N - Q = \frac{\alpha_1\alpha_2(\mu_1 - \mu_2)^2}{\alpha_1\mu_2 + \alpha_2\mu_1} \quad (2.14)$$

Therefore if $\mu_1 \neq \mu_2$ the coefficients satisfy the inequality

$$N > Q \quad (2.15)$$

and the composite medium is anisotropic. An important property of the elastic coefficients is brought out by rotating the coordinate

axes through an angle of 45 degrees. The new stress components are denoted by s'_{11} , s'_{22} , s'_{12} . Applying relations (4.4) of Chapter 1 (putting $\alpha = \pi/4$), we find

$$\begin{aligned} s_{22} - s_{11} &= 2s'_{12} \\ s'_{11} - s'_{22} &= 2s_{12} \end{aligned} \quad (2.16)$$

The strain components are interchanged in the same way, as can readily be shown by transforming the coordinates in the quadratic form of equation 2.3 in Chapter 1. The new strain components are

$$\begin{aligned} e_{yy} - e_{xx} &= -2e_{xx} = 2e'_{xy} \\ e'_{xx} - e'_{yy} &= 2e'_{xx} = 2e_{xy} \end{aligned} \quad (2.17)$$

Equations 2.16 and 2.17 are, of course, also a consequence of the well-known properties of Mohr's circle.*

Relations (2.10) are transformed into

$$\begin{aligned} s'_{11} - s'_{22} &= 4Qe'_{xx} \\ s'_{12} &= 2Ne'_{xy} \end{aligned} \quad (2.18)$$

Therefore a rotation of 45 degrees *interchanges the elastic coefficients*.

This is illustrated in Figure 2.1. Imagine a pile of cards oriented along x (Fig. 2.1a). This medium is easily deformed in shear by sliding the cards in a direction parallel to x while it is relatively rigid when stretched along the x and y directions. This motion is represented by a coefficient Q which is small in comparison with N . If we now orient these cards at an angle of 45 degrees with x (Fig. 2.1b), the medium will be easily deformed by stretching in the x and y directions since a sliding motion of the cards is involved again. On the other hand, the shearing rigidity along x is now high. This corresponds to an interchange of coefficients N and Q . Note that the orthotropic symmetry of the medium about the x and y axes is retained.

Until now we have assumed that there is no initial stress. Let us now consider a laminated medium under initial stress, with principal directions oriented along x and y . Along the laminations the principal stresses in the hard and soft materials are designated by $S_{11}^{(1)}$

* See S. Timoshenko, *Theory of Elasticity*, p. 17, McGraw-Hill Book Company, Inc., New York, 1934.

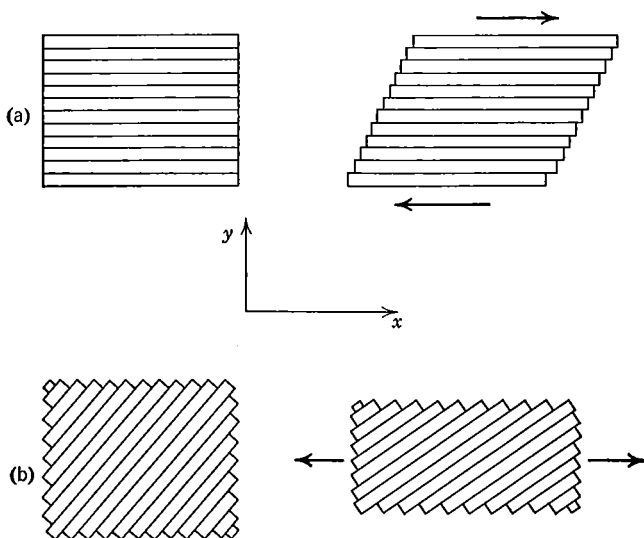


Figure 2.1 Illustration of the interchange of normal and shear rigidities by a 45-degree rotation of the medium.

and $S_{11}^{(2)}$, respectively. The average initial stress in the x direction is therefore

$$S_{11} = S_{11}^{(1)}\alpha_1 + S_{11}^{(2)}\alpha_2 \quad (2.19)$$

In the y direction normal to the laminations the initial stress component S_{22} is constant throughout. We may define quantities

$$\begin{aligned} P &= S_{22} - S_{11} \\ P_1 &= S_{22} - S_{11}^{(1)} \\ P_2 &= S_{22} - S_{11}^{(2)} \end{aligned} \quad (2.20)$$

In the particular case where $S_{22} = 0$ these quantities represent compressive stresses in a direction parallel to the layers. From equations 2.4, 2.19, and 2.20 we derive

$$P = \alpha_1 P_1 + \alpha_2 P_2 \quad (2.21)$$

In the present case the incremental stress-strain relations under initial stress are formally the same as equations 2.1 and 2.2 for the medium initially free of stress. In evaluating the composite coefficient N we may follow exactly the procedure used above; then it is given by the same expression as the first of equations 2.9.

$$N = N_1\alpha_1 + N_2\alpha_2 \quad (2.22)$$

However, in doing so we must make use implicitly of an important property of the incompressible medium, namely, that the strain component e_{yy} is the same in both materials. Hence after deformation the hard and soft layers occupy the same fractions α_1 and α_2 of the total thickness. Otherwise equation 2.7 will not remain valid if an initial stress is present.

Evaluation of the composite shear coefficient under initial stress requires a modification of the procedure. The reason is that the stress component s_{12} is not the same in each layer. We must replace s_{12} by the tangential stress acting on the surface of a deformed layer. This tangential stress as expressed by equation 2.5 of Chapter 3 reads

$$t'_{12} = 2Le_{xy} \quad (2.23)$$

where

$$L = Q + \frac{1}{2}P \quad (2.24)$$

The tangential stress t'_{12} was also analyzed in detail in section 6 of Chapter 2, where it is designated by A_{xy} . It was shown that the slide modulus L is given by the expression (2.24), where $P = S_{22} - S_{11}$. Since t'_{12} is the same throughout the composite medium, the following relations are valid in the hard and soft materials, respectively:

$$\begin{aligned} t'_{12} &= 2L_1 e_{xy}^{(1)} \\ t'_{12} &= 2L_2 e_{xy}^{(2)} \end{aligned} \quad (2.25)$$

The slide moduli of these materials are

$$\begin{aligned} L_1 &= Q_1 + \frac{1}{2}P_1 \\ L_2 &= Q_2 + \frac{1}{2}P_2 \end{aligned} \quad (2.26)$$

The average shear strain of the laminated medium is

$$e_{xy} = \alpha_1 e_{xy}^{(1)} + \alpha_2 e_{xy}^{(2)} \quad (2.27)$$

or

$$e_{xy} = \frac{1}{2} \left(\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2} \right) t'_{12} \quad (2.28)$$

This defines a composite slide modulus

$$L = \frac{1}{\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2}} \quad (2.29)$$

The value of Q for the laminated medium is then given by

$$Q = L - \frac{1}{2}P \quad (2.30)$$

Obvious generalizations of expressions (2.22) and (2.29) for laminated media composed of more than two types of materials are

$$\begin{aligned} N &= \sum_i^i N_i \alpha_i \\ \frac{1}{L} &= \sum_i^i \frac{\alpha_i}{L_i} \end{aligned} \quad (2.31)$$

The elastic coefficients of each material are N_i and L_i , and the layer occupies a fraction α_i of a given thickness. The choice of this averaging thickness is left somewhat arbitrary, and it will depend on the particular features which are to be emphasized by the smoothing-out process.

In the more general case of a compressible material we write the stress-strain relations in the form of equations 2.5 of Chapter 3.

$$\begin{aligned} t_{11} &= C_{11}e_{xx} + C_{12}e_{yy} \\ t_{22} &= C_{12}e_{xx} + C_{22}e_{yy} \\ t'_{12} &= 2Le_{xy} \end{aligned} \quad (2.32)$$

The composite slide modulus L is given by the same expression (2.29) as before.

Derivation of the composite coefficients C_{ij} requires a special procedure because the strain component e_{yy} is not the same in each layer. However, e_{xx} and t_{22} remain the same in both materials. The normal stresses in the hard material are written

$$\begin{aligned} t_{11}^{(1)} &= a_1 e_{xx} + b_1 e_{yy}^{(1)} \\ t_{22} &= b_1 e_{xx} + c_1 e_{yy}^{(1)} \end{aligned} \quad (2.33)$$

For the soft material we write

$$\begin{aligned} t_{11}^{(2)} &= a_2 e_{xx} + b_2 e_{yy}^{(2)} \\ t_{22} &= b_2 e_{xx} + c_2 e_{yy}^{(2)} \end{aligned} \quad (2.34)$$

For convenience the coefficients C_{ij} of equations 2.32 have been written as a, b, c with subscripts 1 and 2 referring to the hard and soft materials, respectively. The average strain e_{yy} is

$$e_{yy} = \alpha_1 e_{yy}^{(1)} + \alpha_2 e_{yy}^{(2)} \quad (2.35)$$

and the average stress is

$$t_{11} = \alpha_1 t_{11}^{(1)} + \alpha_2 t_{11}^{(2)} \quad (2.36)$$

Eliminating the four variables $t_{11}^{(1)}$, $t_{11}^{(2)}$, $e_{yy}^{(1)}$, and $e_{yy}^{(2)}$ from the six equations 2.33, 2.34, 2.35, and 2.36, we find two equations of the form (2.32) with the composite coefficients

$$\begin{aligned} C_{11} &= \alpha_1 a_1 + \alpha_2 a_2 - \frac{\alpha_1 \alpha_2 (b_1 - b_2)^2}{\alpha_1 c_2 + \alpha_2 c_1} \\ C_{12} &= \frac{\alpha_1 b_1 c_2 + \alpha_2 b_2 c_1}{\alpha_1 c_2 + \alpha_2 c_1} \\ C_{22} &= \frac{c_1 c_2}{\alpha_1 c_2 + \alpha_2 c_1} \end{aligned} \quad (2.37)$$

The incremental stresses in the laminated medium are obtained by inserting these values of the coefficients in equations 2.32.

The derivation of the coefficients (2.37) is entirely similar to that used by Postma* and later by Helbig† in the theory of acoustic propagation in a laminated medium which is composed of layers of isotropic materials and is initially stress-free. For this case the coefficients were originally derived by Bruggemann.‡

Validity of Representation by a Continuous Medium. Attention should be called to the limitations of this approximate representation of a laminated medium by a continuous medium of anisotropic properties. The assumptions required for the averaging process to be valid imply that the rigidity contrast of the layers is not too large, and that the layer thickness remains sufficiently small with respect to the wavelength of the deformation field. The validity of the approximation will depend on the type of problem considered and will have to be examined for each one. However, where the assumption is only partially valid and requires additional refinements, it will generally provide useful insight into some of the basic features of the problem.

* G. W. Postma, Wave Propagation in a Stratified Medium, *Geophysics*, Vol. 20, No. 4, pp. 780–806, 1955.

† K. Helbig, Elastische Wellen in anisotropen Medien, *Gerlands Beiträge zur Geophysik*, Vol. 67, pp. 256–288, Leipzig, 1958.

‡ D. A. G. Bruggemann, Berechnung der verschiedenen physikalischen Konstanten von heterogenen Substanzen (part 3), Die elastische Konstanten der quasiisotropen Mischkörper aus isotropen Substanzen, *Annalen der Physik*, Vol. 29, pp. 160–178, 1937.

3. INTERNAL INSTABILITY

In this section we consider a type of instability which may occur in a medium of infinite extent or in a finite region confined between rigid boundaries. The existence of such *internal buckling* is implicit in the theory of acoustic propagation under initial stress derived by the author in 1940.* The existence and the nature of this phenomenon were brought out more clearly in a later detailed analysis and discussion.†

The elastic medium is assumed to be incompressible, homogeneous, and of orthotropic symmetry. The coordinate axes are oriented along the directions of elastic symmetry. The principal initial stresses S_{11}, S_{22}, S_{33} are also oriented along the same directions. Incremental stresses corresponding to plane strain in the x, y plane satisfy the equilibrium equations

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0\end{aligned}\tag{3.1}$$

where

$$P = S_{22} - S_{11}\tag{3.2}$$

These equations are derived by putting $S_{12} = 0$ in equations 6.17 of Chapter 1. The stress-strain relations are

$$\begin{aligned}s_{11} - s &= 2N e_{xx} \\ s_{22} - s &= 2N e_{yy} \\ s_{12} &= 2Q e_{xy}\end{aligned}\tag{3.3}$$

To these equations we must add the condition of incompressibility

$$e_{xx} + e_{yy} = 0\tag{3.4}$$

These equations were derived and discussed in section 8 of Chapter 2. Equation 3.4 is satisfied by writing the displacement as

$$\begin{aligned}u &= -\frac{\partial \phi}{\partial y} \\ v &= \frac{\partial \phi}{\partial x}\end{aligned}\tag{3.5}$$

where ϕ is a function of x and y .

* See reference 7 in the Preface.

† M. A. Biot, Internal Buckling under Initial Stress in Finite Elasticity, *Proceedings of the Royal Society, A*, Vol. 273, pp. 306-328, 1963.

Expressing the strain components in terms of ϕ and substituting the values (3.3) of the stresses into the equilibrium conditions (3.1), we obtain

$$\begin{aligned} \frac{\partial s}{\partial x} - \frac{\partial}{\partial y} \left[\left(2N - Q + \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(Q + \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial y^2} \right] &= 0 \\ \frac{\partial s}{\partial y} + \frac{\partial}{\partial x} \left[\left(2N - Q - \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial y^2} + \left(Q - \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial x^2} \right] &= 0 \end{aligned} \quad (3.6)$$

Elimination of s in these two equations leads to a single equation for ϕ ,

$$\left(Q - \frac{P}{2} \right) \frac{\partial^4 \phi}{\partial x^4} + 2(2N - Q) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \left(Q + \frac{P}{2} \right) \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (3.7)$$

For an isotropic medium ($N = Q$) free of initial stress ($P = 0$) we obtain the well-known biharmonic equation

$$\nabla^4 \phi = 0 \quad (3.8)$$

The occurrence of internal buckling is closely related to the existence of hyperbolic solutions of equation 3.7. Let us put

$$\phi = \varphi(x - \xi y) \quad (3.9)$$

Substitution in equation 3.7 yields

$$\xi^4 + 2m\xi^2 + k^2 = 0 \quad (3.10)$$

with

$$\begin{aligned} m &= \frac{2N - Q}{Q + \frac{1}{2}P} \\ k^2 &= \frac{Q - \frac{1}{2}P}{Q + \frac{1}{2}P} \end{aligned} \quad (3.11)$$

The roots ξ^2 of equation 3.10 are

$$\begin{aligned} \xi_1^2 &= -m + \sqrt{m^2 - k^2} \\ \xi_2^2 &= -m - \sqrt{m^2 - k^2} \end{aligned} \quad (3.12)$$

A solution of the hyperbolic type (3.9) is possible if there exists a real root ξ ; that is, if either ξ_1^2 or ξ_2^2 or both are positive. This occurs in the following cases:

Case 1. $m > 0, k^2 < 0$. The root ξ_1^2 is positive and ξ_1 is real.

Case 2. $m < 0, m^2 > k^2 > 0$. Both ξ_1^2 and ξ_2^2 are positive. Hence ξ_1 and ξ_2 are real.

These two cases correspond to two types of phenomena. We shall call them *instability of the first and second kind*. In the discussion it is advantageous to use the coefficients

$$\begin{aligned} L &= Q + \frac{1}{2}P \\ M &= N + \frac{1}{4}P \end{aligned} \quad (3.13)$$

The significance of the slide modulus L and the coefficient M has been discussed previously (sections 6 and 10 of Chapter 2 and section 4 of Chapter 3). With these coefficients we write

$$\begin{aligned} m &= \frac{2M - L}{L} \\ k^2 &= \frac{L - P}{L} \end{aligned} \quad (3.14)$$

and the characteristic equation 3.10 becomes

$$L\xi^4 + 2(2M - L)\xi^2 + L - P = 0 \quad (3.15)$$

The two cases of instability are then determined by the equivalent conditions discussed hereafter.

It should be noted that there is a *third case*, $m < 0$, $k^2 < 0$, for which the root ξ_1 is real. By definition we shall also consider this case to represent an internal instability of the first kind because there is only one real root. However, physically the phenomenon will be overshadowed by case 2 which will generally occur first as an incipient instability of the second kind. We shall therefore omit this third case in the present discussion.

Internal Instability of the First Kind. This type of instability occurs for

$$m > 0 \quad k^2 < 0 \quad (3.16)$$

(For the reason stated above we exclude the case $m < 0$, $k^2 < 0$.) In terms of the coefficients L and M the condition for instability of the first kind is written

$$2M > L \quad P > L \quad (3.17)$$

Let us denote by ξ_1 the positive value of the root ξ . With positive determination for the radicals its value is

$$\xi_1 = \sqrt{-m + \sqrt{m^2 - k^2}} \quad (3.18)$$

The two real roots are then $\pm \xi_1$. Note that the two other roots $\pm \xi_2$ are pure imaginary. We may write a general solution of equation 3.7 in the form

$$\phi = \varphi_1(x - \xi_1 y) + \varphi_2(x + \xi_1 y) + \varphi_3(x - \xi_2 y) + \varphi_3(x + \xi_2 y) \tag{3.19}$$

where φ_3 is an analytic function of the complex argument. Such a solution is of the mixed elliptic-hyperbolic type.

Another physical interpretation of this case is obtained by considering a particular type of solution:

$$\phi = -\frac{1}{2}C[\cos l(x - \xi y) + \cos l(x + \xi y)] \tag{3.20}$$

or

$$\phi = -C \cos lx \cos \xi ly \tag{3.21}$$

with an arbitrary constant C . For

$$\xi = \xi_1 \tag{3.22}$$

this expression is a solution of equation 3.7. Another way of stating this is to substitute solution (3.21) into equation 3.7. This yields the characteristic equation (3.15). It may be written in the form

$$P = L\xi^4 + 2(2M - L)\xi^2 + L \tag{3.23}$$

When this equation is satisfied, a solution ϕ exists with an arbitrary amplitude factor C . In the discussion which follows we shall denote by ξ a positive real variable. Hence, if it satisfies equation 3.23, it is identical with the positive root ξ_1 .

The displacements corresponding to solution (3.21) are

$$\begin{aligned} u &= -\frac{\partial \phi}{\partial y} = -Cl\xi \cos lx \sin l\xi y \\ v &= \frac{\partial \phi}{\partial x} = Cl \sin lx \cos \xi ly \end{aligned} \tag{3.24}$$

They result from an interference pattern of sinusoidal solutions along the two characteristic directions. This pattern is formally analogous to a pattern of standing acoustic waves in a rectangular domain. The wavelengths along these two coordinate axes are

$$\begin{aligned} \mathcal{L}_x &= \frac{2\pi}{l} \\ \mathcal{L}_y &= \frac{2\pi}{l\xi} \end{aligned} \tag{3.25}$$

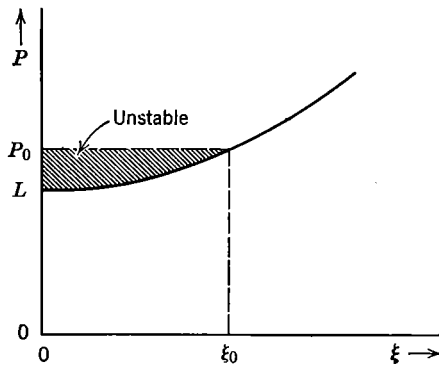


Figure 3.1 Diagram for internal instability of the first kind.

Hence the parameter ξ may also be interpreted as the ratio of these two wavelengths:

$$\xi = \frac{\mathcal{L}_x}{\mathcal{L}_y} \quad (3.26)$$

In the general case the coefficients L and M are functions of the initial stress P . Hence the characteristic equation 3.23 is really an

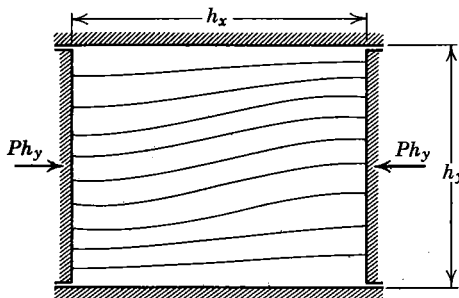


Figure 3.2 Metastable configuration with $n_x = 1$, $n_y = 1$.

implicit relation between ξ and P . In order to simplify the discussion let us assume for the time being that the coefficients L and M are independent of P . Then equation 3.23 and the inequalities (3.17) show that P is an increasing function of ξ . This is illustrated schematically in Figure 3.1.

Consider now a medium confined rigidly and without friction between rectangular boundaries of sides h_x and h_y (Fig. 3.2). The solution (3.21) fits these boundaries provided that we put

$$\begin{aligned}\mathcal{L}_x &= \frac{2h_x}{n_x} \\ \mathcal{L}_y &= \frac{2h_y}{n_y}\end{aligned}\tag{3.27}$$

where n_x and n_y are integers. The value of ξ may be written

$$\xi = \frac{n_y h_x}{n_x h_y}\tag{3.28}$$

The normal displacements of the solution (3.24), as well as the tangential stress, vanish at the boundaries. This corresponds to rigid and perfectly lubricated boundaries.

Let us choose two particular integers,

$$\begin{aligned}n_x &= N_x \\ n_y &= N_y\end{aligned}\tag{3.29}$$

The value of ξ is then

$$\xi_0 = \frac{N_y h_x}{N_x h_y}\tag{3.30}$$

From the diagram of Figure 3.1 we derive the stress P_0 for which the solution (3.24) is possible. It represents an *internal buckling of a rigidly confined medium*. The displacement field of arbitrary amplitude is an equilibrium configuration under the initial stress P_0 . However, it is a metastable solution. This is readily seen by choosing values of n_x and n_y such that

$$\frac{n_y h_x}{n_x h_y} < \xi_0\tag{3.31}$$

There are an infinite number of solutions which satisfy this inequality, all of them corresponding to values of P smaller than P_0 . Hence in the range of values

$$0 < \xi < \xi_0\tag{3.32}$$

there are an infinite number of configurations which are hypercritically unstable under the initial compression P_0 . Three of these

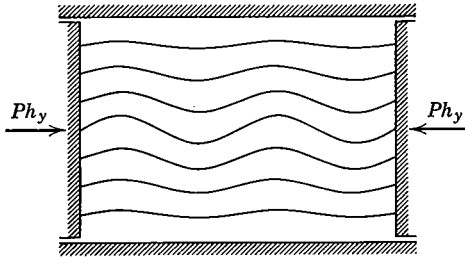


Figure 3.3 Metastable configuration with $n_x = 5$, $n_y = 1$.

configurations are illustrated in Figures 3.2, 3.3, and 3.4. They correspond to

$$\begin{aligned}
 n_x &= 1 & n_y &= 1 \\
 n_x &= 5 & n_y &= 1 \\
 n_x &= 5 & n_y &= 2
 \end{aligned}
 \tag{3.33}$$

Of all these possible modes of internal buckling under compression P_0 , which one is the most unstable? As shown by the diagram of

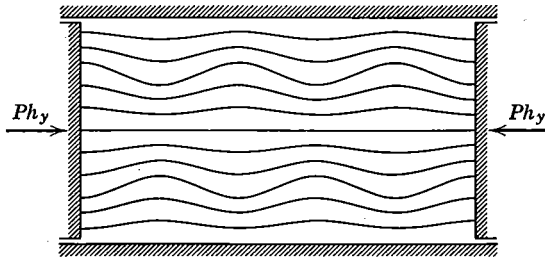


Figure 3.4 Metastable configuration with $n_x = 5$, $n_y = 2$.

Figure 3.1, the most unstable mode corresponds to a vanishingly small value of ξ , because at $\xi = 0$ the difference between the critical load $P = L$ and the actual compression P_0 is maximum. This corresponds to the configuration

$$n_x = \infty \quad n_y = 1
 \tag{3.34}$$

It has a vanishing wavelength. Therefore, theoretically, such a mode of vanishing wavelength will appear as soon as the compression P reaches the value L .

This conclusion may seem paradoxical, but there are inherent limitations in the validity of the theory for very small wavelengths. It is, of course, not valid in the atomic scale. Then non-linearity also enters into play. The result therefore indicates that the buckling wavelength will tend to be the smallest compatible with the small-scale physics of the medium.

The last remark is of particular significance when the theory is applied to a thinly laminated medium. The results indicate that the buckling tends to occur with the shortest possible wavelength compatible with the small scale geometry of the layers and their rigidity contrast. The buckling wavelength is then governed by additional factors such as the layer thickness, which cause a departure from the behavior of the continuous model derived in section 2. An evaluation of this effect in the absence of any confinement ($h_y = \infty$) is given in the last part of this chapter (see Fig. 7.5). The continuous model is useful because it provides a foundation for more elaborate theories and brings out some of the most significant features. An exact solution may, of course, be obtained by applying the general theory of stability of section 7 for multilayered media. The problem has been further investigated in a recent paper by the author as mentioned at the end of section 6 in Chapter 6.

The general conclusions of the theory remain valid when the coefficients $L(P)$ and $M(P)$ are functions of P . Starting from $P = 0$, if we increase the compression gradually there comes a moment when $P = L(P)$. At this point internal instability makes its appearance.

A characteristic property of internal instability is brought out by considering a medium which is isotropic in finite strain. For this case the coefficient Q is determined by equation 7.15 of Chapter 2. Its value is

$$Q = \frac{1}{2}P \frac{\lambda_2^2 + \lambda_1^2}{\lambda_2^2 - \lambda_1^2} \quad (3.35)$$

where λ_1 and λ_2 are the extension ratios of the finite initial strain in the x and y directions. Hence

$$L = P \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2} \quad (3.36)$$

This expression shows that it is not possible in this case to satisfy the condition

$$P \geq L \quad (3.37)$$

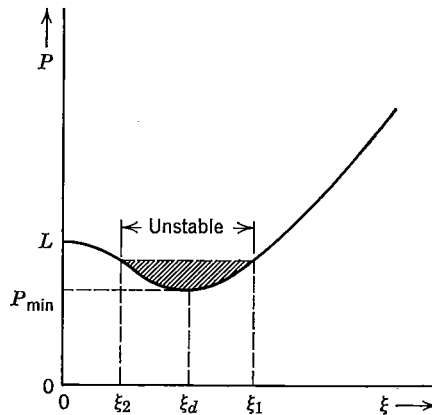


Figure 3.5 Diagram for internal instability of the second kind.

Hence *internal instability of the first kind is not possible in a medium which is isotropic for finite deformations.*

We now analyze the second case of internal instability.

Internal Instability of the Second Kind. This case occurs when

$$m < 0 \quad m^2 > k^2 > 0 \quad (3.38)$$

These conditions may be written in the equivalent form

$$2M < L \quad L > P > \frac{4M}{L}(L - M) \quad (3.39)$$

If the coefficients M and L are constant, the value of P as a function of ξ is given by equation 3.23. The plot is shown schematically in Figure 3.5. Because $2M - L < 0$, the value of P goes through a minimum. This minimum value is

$$P_{\min} = \frac{4M}{L}(L - M) \quad (3.40)$$

and the corresponding abscissa ξ_d is

$$\xi_d = \sqrt{\frac{L - 2M}{L}} \quad (3.41)$$

If $P > P_{\min}$, there is a region of instability in the range

$$\xi_2 < \xi < \xi_1 \quad (3.42)$$

as indicated in the diagram of Figure 3.5.

Incipient instability occurs for

$$P = P_{\min} \quad (3.43)$$

and the corresponding slope of the two characteristic directions is given by

$$\xi = \pm \xi_a \quad (3.44)$$

At this point the characteristic equation (3.23) has a double root. Hence incipient instability is determined by the equation

$$m^2 - k^2 = 0 \quad (3.45)$$

or

$$P = \frac{4M}{L} (L - M) \quad (3.46)$$

In terms of the coefficients N and Q this equation is written

$$4N(N - Q) + \frac{1}{4}P^2 = 0 \quad (3.47)$$

In order to illustrate the significance of this result let us consider the particular case

$$M = 0 \quad (3.48)$$

Here

$$P_{\min} = 0 \quad \xi_a = 1 \quad (3.49)$$

Instability occurs as soon as a small compression is applied and the characteristic directions are oriented at an angle of 45 degrees with the direction of P . A medium with these properties is exemplified by a stack of thin layers, as studied in section 2. The layers are oriented at an angle of 45 degrees with the x direction and slide easily over each other. As shown in Figure 3.6a a direction of slip under the compression obviously exists along the directions of the layers. The other direction of slip is perpendicular to the layers as shown in Figure 3.6b; it will occur if the layers have "perfect flexibility." This type of instability is obviously closely related to the phenomenon of "shear failure" and the appearance of "slip lines" in plasticity. Application of the theory to plasticity will be discussed in section 6 of Chapter 6.

It should be noted that P is defined by equation 3.2 as the difference of the principal stresses and may represent a *tension* in the y direction as well as a compression along x .

If the coefficients L and M are functions of P , condition (3.47) for incipient instability remains valid, but it is now an intrinsic equation for P . An example of such a case is a medium which is isotropic in finite strain. Assume the value of P to be

$$P = 2\mu_0 \frac{1 - \lambda^4}{1 + \lambda^4} \tag{3.50}$$

where λ is the extension ratio in the direction of P and in finite plane

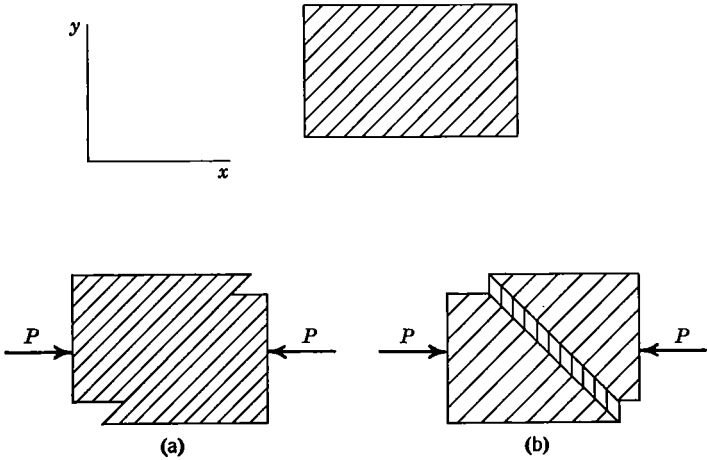


Figure 3.6 Illustration of internal instability of the second kind.

strain. Applying equations 8.38 and 8.39 of Chapter 2, we find the coefficients

$$N = \frac{4\lambda^4}{(\lambda^4 + 1)^2} \mu_0 \tag{3.51}$$

$$Q = \mu_0$$

The finite stress-strain relation (3.50) is shown schematically in Figure 3.7. The incipient buckling condition (3.47) becomes

$$\lambda^2 + \frac{1}{\lambda^2} = 4 \tag{3.52}$$

One root

$$\lambda_1 = \sqrt{2 + \sqrt{3}} = 1.93 \tag{3.53}$$

corresponds to an extension. The other

$$\lambda_2 = \frac{1}{\lambda_1} = 0.517 \tag{3.54}$$

corresponds to a compression. These two roots represent the same phenomenon, and they correspond to an interchange of the x and y directions. Because of isotropy the compression P may be applied

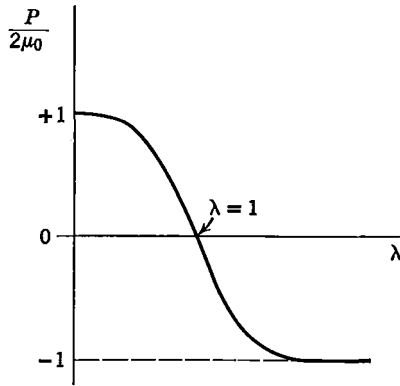


Figure 3.7 Finite stress-strain relation (3.50).

in the direction of x or y . The inclination of the characteristics is obtained by noting that $\xi_d = \pm \lambda$. Their direction lies at an angle of about 27 degrees with the extension axis.

Variational Analysis of Internal Instability. A variational formulation of this phenomenon is particularly illuminating and brings out more clearly the physical mechanism involved. This is readily accomplished by applying expression (4.23) of Chapter 3, namely,

$$\Delta V = 2Me_{xx}^2 + 2Le_{xy}^2 - \frac{1}{2}P \left(\frac{\partial v}{\partial x} \right)^2 \tag{3.55}$$

When $S_{22} = 0$ the initial stress is reduced to a uniaxial compression P , and this expression of ΔV represents the incremental strain energy per unit area inside the rectangular contour of Figures 3.2, 3.3, and 3.4.

However, we may use equation 3.55 for the more general case, where P denotes the stress difference (3.2), in order to evaluate the total incremental potential energy of the elastic medium inside the contour. This can be seen by going back to the discussion of section 4 in Chapter 3. The rigid frictionless and rectangular boundaries illustrated in Figure 4.1 of that chapter provide exactly the same boundary condition as that in the present discussion of internal instability. It was shown that in that case the total incremental strain energy potential \mathcal{P} is obtained by substituting the stress difference (3.2) for the value of P and integrating expression (3.55) for ΔV over the rectangular area. This integral is

$$\mathcal{P} = \int_{-h_y/2}^{h_y/2} dy \int_{-h_x/2}^{h_x/2} \Delta V dx \quad (3.56)$$

Substituting the sinusoidal displacement fields (3.24) into ΔV , we derive

$$\mathcal{P} = \frac{1}{8} h_x h_y C^2 l^4 [L\xi^4 + 2(2M - L)\xi^2 + L - P] \quad (3.57)$$

By putting $\mathcal{P} = 0$ we obtain the characteristic equation (3.23) for internal buckling. The incremental potential energy \mathcal{P} is negative when

$$P > L\xi^4 + 2(2M - L)\xi^2 + L \quad (3.58)$$

When this inequality is satisfied, more strain energy is available in the uniform initial compression than is required to initiate an internal mode of buckling. This is the explanation for the occurrence of internal instability.

4. SURFACE INSTABILITY OF THE ANISOTROPIC HALF-SPACE

As in the preceding section, the medium is assumed to be incompressible and homogeneous of orthotropic symmetry. The coordinate axes are directed along the planes of symmetry. The solid occupies the region $y < 0$ and has a free surface at $y = 0$ (Fig. 4.1). The finite initial strain is homogeneous, and the initial stress in the x, y plane is represented by the constant values

$$\begin{aligned} S_{11} &= -P \\ S_{22} &= 0 \end{aligned} \quad (4.1)$$

This corresponds to a uniform compression P in the direction parallel to the free surface. We shall investigate the stability of this free surface in the state of initial stress. A stress S_{33} may act in a direction perpendicular to the x, y plane, but it does not appear explicitly in the theory. The material presented in this section was originally developed in a recent paper by the author.*

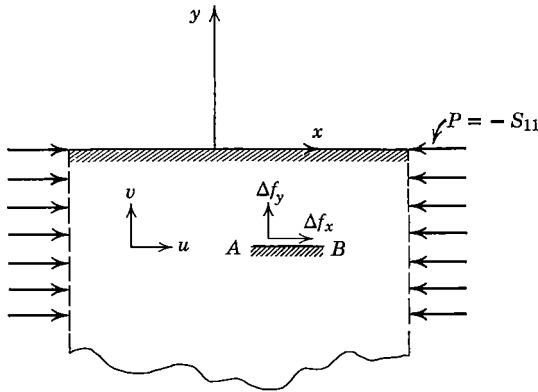


Figure 4.1 Anisotropic half-space under initial compressive stress P .

The equations to be solved are the same as in the previous section. There are two unknowns s and ϕ which must satisfy equations 3.6. Hence

$$\begin{aligned} \frac{\partial s}{\partial x} - \frac{\partial}{\partial y} \left[\left(2N - Q + \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(Q + \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial y^2} \right] &= 0 \\ \frac{\partial s}{\partial y} + \frac{\partial}{\partial x} \left[\left(2N - Q - \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial y^2} + \left(Q - \frac{P}{2} \right) \frac{\partial^2 \phi}{\partial x^2} \right] &= 0 \end{aligned} \tag{4.2}$$

Elimination of s yields equation 3.7. With the use of coefficients (3.11) it is written

$$\frac{\partial^4 \phi}{\partial y^4} + 2m \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2 \frac{\partial^4 \phi}{\partial x^4} = 0 \tag{4.3}$$

* M. A. Biot, Surface Instability in Finite Anisotropic Elasticity under Initial Stress, *Proceedings of the Royal Society, A*, Vol. 273, pp. 329–339, 1963.

We shall use a solution of the type

$$\begin{aligned}\phi l^2 &= f(l y) \sin l x \\ s &= F(l y) \cos l x\end{aligned}\quad (4.4)$$

Substitution of ϕ into equation 4.3 leads to the ordinary differential equation

$$f'''' - 2mf'' + k^2f = 0 \quad (4.5)$$

The primes denote differentiation with respect to the argument ly . Substitution of s and ϕ into equations 4.2 yields F in terms of f .

$$F(l y) = \left(2N - Q + \frac{P}{2}\right) f' - \left(Q + \frac{P}{2}\right) f'' \quad (4.6)$$

All other variables may now be expressed by means of f . The displacement field is

$$\begin{aligned}u &= -\frac{\partial \phi}{\partial y} = -\frac{1}{l} f' \sin l x \\ v &= \frac{\partial \phi}{\partial x} = \frac{1}{l} f \cos l x\end{aligned}\quad (4.7)$$

Similarly, from equations 3.3, the stresses are also expressed in terms of f .

Actually in order to introduce the surface boundary condition we need to evaluate the forces acting on a deformed boundary. Consider the half-space lying below the plane AB of ordinate y (Fig. 4.2a). Before deformation the plane AB is free of stress since the initial stress P is parallel to the x direction. After deformation the surface AB is deformed into a corrugated shape with sinusoidal amplitude and forces are acting on this surface (Fig. 4.2b). The x and y components of these forces per unit initial area are given by equations 6.27 of Chapter 1. They are

$$\begin{aligned}\Delta f_x &= s_{12} + P e_{xy} \\ \Delta f_y &= s_{22}\end{aligned}\quad (4.8)$$

We may express these quantities by means of f and write

$$\begin{aligned}\frac{1}{L} \Delta f_x &= -(f + f'') \sin l x \\ \frac{1}{L} \Delta f_y &= [(2m + 1)f' - f'''] \cos l x\end{aligned}\quad (4.9)$$

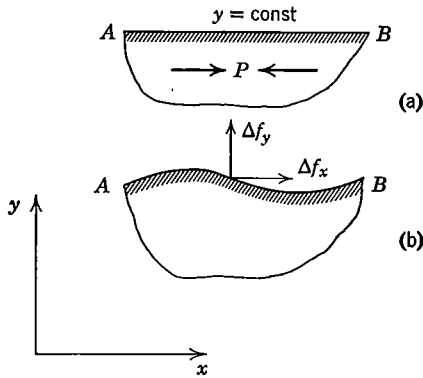


Figure 4.2 Forces acting internally on a deformed surface AB of the medium.

Let us represent the displacements as

$$u = U(l y) \sin l x \tag{4.10}$$

$$v = V(l y) \cos l x$$

and the forces (4.8) by

$$\Delta f_x = \tau(l y) \sin l x \tag{4.11}$$

$$\Delta f_y = q(l y) \cos l x$$

Comparing equations 4.10 and 4.11 with equations 4.7 and 4.9, we derive

$$lU(l y) = -f' \tag{4.12}$$

$$lV(l y) = f$$

and

$$\frac{1}{L} \tau(l y) = -f - f'' \tag{4.13}$$

$$\frac{1}{L} q(l y) = (2m + 1)f' - f'''$$

Using these general results, we shall first solve the problem of finding the surface deformation of the half-space under normal and tangential forces applied to the surface. Let us go back to the differential equation 4.5. Its solutions are of the type $f = \exp(\beta l y)$ where β satisfies the characteristic equation

$$\beta^4 - 2m\beta^2 + k^2 = 0 \tag{4.14}$$

With $\xi = i\beta$, this is identical with equation 3.10 used in the discussion of internal instability. The general solution of equation 4.5 is

$$f = \sum_{i=1}^4 C_i e^{\beta_i y} \quad (4.15)$$

where β_i are the four roots of equation 4.14.

In order to solve the problem of the half-space we must take into account the boundary conditions. Since we are interested in the surface instability, we must consider only the solutions which vanish at $y = -\infty$. Such solutions will decay exponentially with depth. In order to satisfy the boundary condition at the surface ($y = 0$) there must be two independent solutions of this type. Therefore we must exclude all cases where at least one of the roots β_i is pure imaginary. There are two such cases to be excluded, namely,

$$\begin{aligned} \text{Case 1:} & \quad m > 0 \quad \text{with } k^2 < 0 \\ \text{Case 2:} & \quad m < 0 \quad \text{with } m^2 - k^2 > 0 \end{aligned} \quad (4.16)$$

From results obtained in the preceding section it can be seen that these two cases entail precisely the conditions required for the occurrence of internal instability. If we restrict ourselves to pure surface instability, the parameters must lie outside the range of internal instability. Therefore we must assume the conditions

$$m > 0 \quad \text{with } k^2 > 0 \quad (4.17)$$

or

$$m < 0 \quad \text{with } m^2 - k^2 < 0$$

This is equivalent to either of the following conditions.

$$2M > L \quad \text{with } P < L$$

or

$$2M < L \quad \text{with } P < \frac{4M}{L} (L - M) \quad (4.18)$$

Under these conditions the roots β_i of equation 4.14 are either real or complex conjugates. Their real part is different from zero, and it is always possible to choose two roots so that their real part is positive. We designate these two roots by

$$\begin{aligned} \beta_1 &= \sqrt{m + \sqrt{m^2 - k^2}} \\ \beta_2 &= \sqrt{m - \sqrt{m^2 - k^2}} \end{aligned} \quad (4.19)$$

The solution (4.15) becomes

$$f = C_1 e^{\beta_1 ly} + C_2 e^{\beta_2 ly} \quad (4.20)$$

The next step is to introduce the boundary condition at the surface. The values of the displacements and forces at the surface are denoted by

$$\begin{aligned} U(0) &= U & \tau(0) &= \tau \\ V(0) &= V & q(0) &= q \end{aligned} \quad (4.21)$$

Substituting solution (4.20) into equations 4.12 and 4.13 and putting $y = 0$, we derive

$$\begin{aligned} lU &= -C_1\beta_1 - C_2\beta_2 \\ lV &= C_1 + C_2 \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} \frac{\tau}{L} &= -C_1(1 + \beta_1^2) - C_2(1 + \beta_2^2) \\ \frac{q}{L} &= C_1\beta_1(1 + \beta_2^2) + C_2\beta_2(1 + \beta_1^2) \end{aligned} \quad (4.23)$$

In deriving the last equation account is taken of the relation

$$2m = \beta_1^2 + \beta_2^2 \quad (4.24)$$

Solving equations 4.22 for C_1 and C_2 and substituting these values into equations 4.23, we derive

$$\begin{aligned} \frac{\tau}{lL} &= (\beta_1 + \beta_2)U + (\beta_1\beta_2 - 1)V \\ \frac{q}{lL} &= (\beta_1\beta_2 - 1)U + \beta_1\beta_2(\beta_1 + \beta_2)V \end{aligned} \quad (4.25)$$

We may express the product $\beta_1\beta_2$ and the sum $\beta_1 + \beta_2$ in terms of m and k . Properties of the roots of the characteristic equation (4.14) produce the relations

$$\begin{aligned} \beta_1^2\beta_2^2 &= k^2 \\ \beta_1^2 + \beta_2^2 &= 2m \end{aligned} \quad (4.26)$$

In the absence of internal instability, k^2 is positive. We may therefore write the real quantity

$$k = \sqrt{\frac{Q - \frac{1}{2}P}{Q + \frac{1}{2}P}} = \sqrt{\frac{L - P}{L}} \quad (4.27)$$

and choose for k the positive value of the square root. The roots β_1 and β_2 have been selected in such a way that they are either real and positive or complex conjugate with a positive real part. Hence without ambiguity in sign the first of equations 4.26 becomes

$$\beta_1\beta_2 = k \quad (4.28)$$

This result combined with the second of equations 4.26 leads to

$$(\beta_1 + \beta_2)^2 = 2(m + k) \quad (4.29)$$

Since $k^2 > 0$ and $m^2 - k^2 < 0$, we conclude that $|m| < k$. Therefore $m + k$ is positive. Since $\beta_1 + \beta_2$ is also positive, we may write

$$\beta_1 + \beta_2 = \sqrt{2(m + k)} \quad (4.30)$$

where the square root is chosen as a positive quantity.

Substituting from relations (4.28) and (4.30) into equations 4.25, we obtain

$$\frac{\tau}{lL} = \sqrt{2(m + k)} U + (k - 1)V \quad (4.31)$$

$$\frac{q}{lL} = (k - 1)U + k\sqrt{2(m + k)} V$$

These equations yield the normal and tangential displacements of the surface under the action of arbitrary surface forces.

Consider first the case when there is no initial stress, i.e., $P = 0$. Here

$$k = 1 \quad L = Q \quad m + k = \frac{2N}{Q} \quad (4.32)$$

and equations 4.31 become

$$\tau = 2l\sqrt{NQ} U \quad (4.33)$$

$$q = 2l\sqrt{NQ} V$$

and there is *no coupling between normal and tangential displacements*. Moreover for an isotropic medium where

$$\mu = N = Q \quad (4.34)$$

we find

$$\tau = 2l\mu U \quad (4.35)$$

$$q = 2l\mu V$$

This result shows that the surface of the anisotropic half-space free of initial stress behaves as if it were isotropic with an effective modulus

$$\mu = \sqrt{NQ} \tag{4.36}$$

In the presence of initial stress this is no longer true and the two displacements must then be derived by equations 4.31, which contain a coupling term.

Instability of the free surface will arise when a non-vanishing solution of equations 4.31 exists for

$$q = \tau = 0 \tag{4.37}$$

Hence the characteristic equation for surface instability is obtained by putting equal to zero the determinant of equations 4.31. This yields

$$\begin{vmatrix} \sqrt{2(m+k)} & k-1 \\ k-1 & k\sqrt{2(m+k)} \end{vmatrix} = 0 \tag{4.38}$$

or

$$2k(m+1) + k^2 - 1 = 0 \tag{4.39}$$

By putting

$$\zeta = \frac{P}{2Q} \quad k = \sqrt{\frac{1-\zeta}{1+\zeta}} \quad m = \frac{2(N/Q) - 1}{1 + \zeta} \tag{4.40}$$

equation 4.39 takes the form

$$\frac{N}{Q} = \frac{1}{2}\zeta \left(\sqrt{\frac{1+\zeta}{1-\zeta}} - 1 \right) \tag{4.41}$$

The numerical relations between N/Q and ζ required for surface instability are shown in Table 1. The roots β_1 and β_2 of the unstable solution may be either real or complex. They will be complex if

$$m^2 - k^2 < 0 \tag{4.42}$$

or

$$\frac{N}{Q} < \frac{1}{2}(1 + \sqrt{1 - \zeta^2}) \tag{4.43}$$

When combined with equation 4.41, this inequality shows that the roots will be complex in the range

$$\begin{aligned} 0 < \frac{N}{Q} < 0.8 \\ 0 < \zeta < 0.8 \end{aligned} \tag{4.44}$$

Table 1

Critical value of $\zeta = P/(2Q)$
for surface instability according
to equation 4.41

N/Q	ζ
0	0
0.20	0.02
0.40	0.10
0.60	0.30
0.70	0.48
0.80	0.80
1.00	0.84
1.51	0.90
2.45	0.95
4.37	0.98
6.50	0.99
∞	1.00

In this range the solution is oscillatory with an exponential decay of amplitude with depth. In the limiting case, where $N/Q = \zeta = 0.8$, the roots β_1 and β_2 are equal and real.

How to determine the finite strain for which surface instability occurs is illustrated by an example of an incompressible medium isotropic in finite strain. We assume a finite plane strain of extension ratio λ . The compression P and the incremental elastic coefficients are expressed by equations 8.35, 8.38, and 8.39 of Chapter 2.

$$\begin{aligned}
 P &= -S_{11} = -F(\lambda) \\
 \frac{N}{Q} &= \frac{\lambda}{2F} \frac{dF}{d\lambda} \frac{\lambda^4 - 1}{\lambda^4 + 1} \\
 \zeta &= \frac{1 - \lambda^4}{1 + \lambda^4}
 \end{aligned}
 \tag{4.45}$$

We plot a curve of ordinate N/Q and abscissa ζ as a parametric function of λ . It intersects the curve represented in Table 1 at a point corresponding to surface instability. This case will be discussed further in the last paragraph of this section.

The isotropic case $N = Q$ was analyzed in Chapter 3. Putting $N/Q = 1$ in equation 4.41 yields after rationalization

$$\zeta^3 + 2\zeta^2 - 2 = 0 \tag{4.45a}$$

This equation has the real root $\zeta = 0.84$. It is identical with equation 6.20 of Chapter 3. An alternative form of the characteristic equation for this case is obtained by putting $m = 1/(1 + \zeta)$ into equation 4.39, which becomes

$$k(2 + \zeta) - \zeta = 0 \quad (4.45b)$$

Multiplying by $(1 - k)$, we derive

$$(1 + \zeta)^2 k - 1 = 0 \quad (4.45c)$$

which is the same as equation 6.19 of Chapter 3. Note that the roots β_1 and β_2 for this case become

$$\beta_1 = 1 \quad \beta_2 = k \quad (4.45d)$$

Deflection under Surface Loads and Gravity Forces. Let us derive the surface deflection V under the action of a normal load, while the half-space is initially stressed by a horizontal compression P parallel to the surface. The tangential force is assumed to be zero; hence we must solve equations 4.31 after putting $\tau = 0$. We obtain

$$V = \frac{q}{2l\varphi\sqrt{NQ}} \quad (4.46)$$

with

$$\varphi = \frac{1 + \zeta}{2\sqrt{\frac{N}{Q}}} \frac{2k(m + 1) + k^2 - 1}{\sqrt{2(m + k)}} \quad (4.47)$$

In the absence of initial stress we find $\varphi = 1$, and the deflection is the same as that given by equations 4.33 for the initially stress-free medium. The effect of the initial stress is represented by an amplification factor $1/\varphi$. When instability is reached, we see from equation 4.39 that $\varphi = 0$ and the surface deflection V becomes infinite. We note that the amplification factor is independent of the wavelength. Using a Fourier representation of the surface load distribution, we conclude that the shape of the surface deflection is the same as for a medium initially stress-free and isotropic. Only the magnitude of the deflection is affected by the initial stress.

Again we examine the isotropic case $N = Q$, for which $m = 1/(1 + \zeta)$ and equation 4.47 becomes

$$\varphi = \frac{k(2 + \zeta) - \zeta}{\sqrt{2\left(\frac{1}{1 + \zeta} + k\right)}} \quad (4.47a)$$

We also verify the identity

$$\sqrt{2 \left(\frac{1}{1 + \zeta} + k \right)} = 1 + k \quad (4.47b)$$

by squaring both sides of the equation. Substituting expression (4.47b) into equation 4.47a and multiplying numerator and denominator by $1 - k$, we obtain

$$\varphi = \frac{1}{\zeta} [k(1 + \zeta)^2 - 1] \quad (4.47c)$$

This is the value of φ which was previously derived for an isotropic medium. It coincides with equation 6.34 of Chapter 3.

Additional significant features are brought into this problem by introducing gravity forces. We assume the half-space to lie below a horizontal surface, under the action of a uniform gravity field of acceleration g . The medium is of uniform mass density ρ . As already shown in section 8 of Chapter 3, the problem is readily solved by using the solution for the gravity-free problem and adding vertical forces at the surfaces proportional to the normal displacements. In the present case this amounts to replacing q by $q - \rho g V$ in equation 4.46,

$$V = \frac{q - \rho g V}{2l\varphi\sqrt{NQ}} \quad (4.48)$$

Solving for V yields

$$V = \frac{q}{2l\varphi\sqrt{NQ} + \rho g} \quad (4.49)$$

Instability of the surface will occur when

$$2l\varphi\sqrt{NQ} + \rho g = 0 \quad (4.50)$$

This requires that φ be negative. Since $\varphi = 0$ corresponds to instability when gravity is absent, the presence of gravity increases the critical value of the initial compression. As expected, gravity has a stabilizing effect.

When the initial stress is such that $\varphi = 0$, the surface deflection becomes

$$V = \frac{q}{\rho g} \quad (4.51)$$

This shows that the apparent elastic rigidity of the solid vanishes at this point and the surface behaves like a fluid of the same density as the solid. The load applied to the surface will be supported entirely

by the buoyancy of the medium as if it were floating. We must assume, of course, that the deformation of the surface is of sufficiently gentle slope that the linearized theory remains applicable.

Another interesting feature of the instability is the dependence of equation 4.50 on the wavelength. Suppose the initial stress P is higher than the value P_0 required for instability in the gravity-free case. Then φ is negative and we write

$$\varphi = -\varphi'$$

For small values of l , hence for large wavelengths, the left side of equation 4.50 is positive and the surface is stable. Consider the wavelength \mathcal{L} for which equation 4.50 is satisfied

$$\mathcal{L} = \frac{2\pi}{l} = \frac{4\pi\varphi'\sqrt{NQ}}{\rho g} \quad (4.52)$$

For all wavelengths larger than this value the surface is stable. On the other hand, for a given initial compression $P > P_0$ the surface is unstable in the range of wavelengths between zero and a cut-off value given by equation 4.52. This cut-off wavelength depends on the magnitude of the initial stress.

Simplified Criterion for Surface Instability in the Case of Finite Isotropy. Equation 4.41 for surface instability is applicable to an incompressible medium which is isotropic in finite strain. This has already been pointed out for the case where the initial deformation is a state of plane strain. For this condition the values of N/Q and ζ are given by equations 4.45. It is of interest to point out that for a medium of finite isotropy the stability criterion (4.41) may be expressed in a simpler form which is applicable to the general case of triaxial initial strain. We choose the x, z axes to be parallel to the surface of the half-space. Stresses $S_{11} = -P$ and S_{33} are applied, respectively, in the x and z directions, and the corresponding extension ratios are denoted by λ_1 and λ_3 . No stress is applied in the y direction, which is normal to the surface. In this direction the extension ratio λ_2 is determined by the condition of incompressibility; hence

$$\lambda_2 = \frac{1}{\lambda_1\lambda_3} \quad (4.53)$$

For finite isotropy the value of Q is given by equation 3.35 in terms of the extension ratios λ_1 and λ_2 . Hence we may write

$$\zeta = \frac{P}{2Q} = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2} \quad (4.54)$$

With this value of ζ , equation 4.41 becomes

$$4N = P \left(\frac{\lambda_2}{\lambda_1} - 1 \right) \quad (4.55)$$

We may introduce explicitly the finite stress-strain relation by writing

$$-P = S_{11}(\lambda_1, \lambda_3) \quad (4.56)$$

We consider S_{11} to be a function of λ_1 and λ_3 . The coefficient N is given by

$$4N = \lambda_1 \frac{\partial S_{11}}{\partial \lambda_1} \quad (4.57)$$

The derivation of this expression is the same as for equation 8.38 of Chapter 2. With this value of N , equation 4.55 becomes

$$\lambda_1 \frac{\partial S_{11}}{\partial \lambda_1} = \left(1 - \frac{\lambda_2}{\lambda_1} \right) S_{11} \quad (4.58)$$

This form of the stability criterion brings in explicitly the finite stress-strain law $S_{11}(\lambda_1, \lambda_3)$ plotted for $\lambda_3 = \text{const.}$ and the "tangent modulus" represented by the slope $\partial S_{11}/\partial \lambda_1$.

Another form of the stability criterion (4.55) is

$$4M = P \frac{\lambda_2}{\lambda_1} \quad (4.59)$$

where

$$M = N + \frac{1}{4}P \quad (4.60)$$

is an elastic coefficient previously discussed and defined by equation 4.24 of Chapter 3.

5. GENERAL EQUATIONS FOR A PLATE UNDER INITIAL STRESS

In this section we develop the plane strain analysis for a plate of uniform thickness h subject to a compression P parallel to the plane

faces (Fig. 5.1). The deformation is in the x, y plane, with the initial stress acting in the x direction and the boundaries located at $y = \pm h/2$.

The elastic medium is incompressible and orthotropic with axes of elastic symmetry along x and y . It obeys the stress-strain relations (3.3).

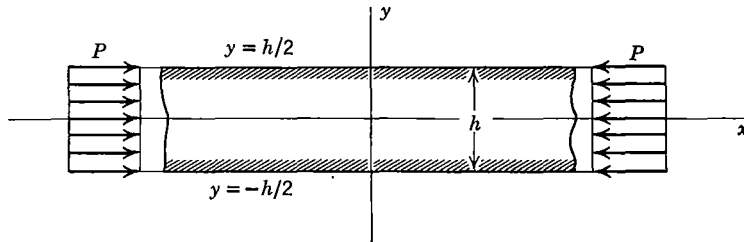


Figure 5.1 Single plate under an initial compression P viewed across the thickness h .

We shall evaluate the forces which must be applied to the boundaries in order to produce two types of deformations. One type, represented in Figure 5.2a, is antisymmetric and corresponds to a bending. The other, represented in Figure 5.2b, is symmetric. In each case we shall consider that the deformation is sinusoidal in the x direction. The material presented in this section is taken from a recent paper by the author.*

Displacements and incremental forces associated with these two types of deformation are easily derived by applying equations 4.12 and 4.13. The displacements are given by

$$\begin{aligned} lU(l y) &= -f' \\ lV(l y) &= f \end{aligned} \tag{5.1}$$

and the incremental forces by

$$\begin{aligned} \frac{1}{L} \tau(l y) &= -f - f'' \\ \frac{1}{L} q(l y) &= (2m + 1)f' - f''' \end{aligned} \tag{5.2}$$

* M. A. Biot, Theory of Stability of Multilayered Continua in Finite Anisotropic Elasticity, *Journal of the Franklin Institute*, Vol. 276, No. 2, pp. 128-153, 1963.

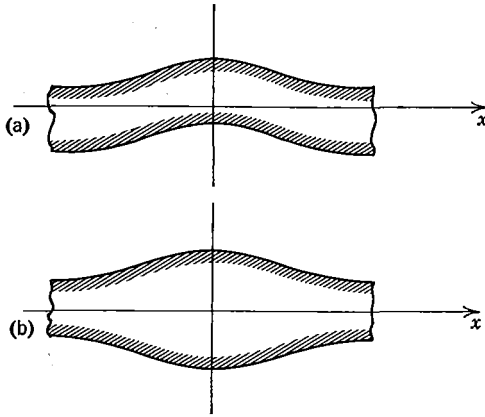


Figure 5.2 Antisymmetric (a) and symmetric (b) deformation of a plate.

The function f is the general solution (4.15)

$$f = \sum_{i=1}^4 C_i e^{\beta_i y} \quad (5.3)$$

We shall assume that the initial stress lies outside the range of internal instability. Then, as shown in the preceding section, if internal instability is excluded, the roots β_i of the characteristic equation (4.5) are either real or complex conjugates and are given by equations 4.19.

The solution f corresponding to the antisymmetric deformation of Figure 5.2a is

$$f = C_1 \cosh \beta_1 l y + C_2 \cosh \beta_2 l y \quad (5.4)$$

When β_1 and β_2 are real, we choose their positive value. When they are complex conjugate, we choose values with positive real parts and also complex conjugate values for the constants C_1 and C_2 . We put

$$\begin{aligned} U(\tfrac{1}{2}lh) &= U_a \\ V(\tfrac{1}{2}lh) &= V_a \\ \tau(\tfrac{1}{2}lh) &= \tau_a \\ q(\tfrac{1}{2}lh) &= q_a \end{aligned} \quad (5.5)$$

These are the displacements and incremental forces at the top surface $y = \frac{1}{2}h$. At the bottom surface $y = -\frac{1}{2}h$ they are

$$\begin{aligned} U(-\tfrac{1}{2}lh) &= -U_a \\ V(-\tfrac{1}{2}lh) &= V_a \\ \tau(-\tfrac{1}{2}lh) &= \tau_a \\ q(-\tfrac{1}{2}lh) &= -q_a \end{aligned} \tag{5.6}$$

Attention should be called to the significance of τ and q at the lower boundary. According to expression (4.11) and Figure 4.2, they represent forces acting on the medium lying *below* the boundary. Hence they must be changed in sign in order to represent forces *acting on the plate* at the bottom surface.

We substitute the value (5.4) of f into equations 5.1 and 5.2 and put $y = \frac{1}{2}h$. This yields the values (5.5) of U_a , V_a , τ_a , and q_a in terms of the two unknown constants C_1 and C_2 . Elimination of these two constants from the four equations 5.5 yields

$$\begin{aligned} \frac{\tau_a}{lL} &= a_{11}U_a + a_{12}V_a \\ \frac{q_a}{lL} &= a_{12}U_a + a_{22}V_a \end{aligned} \tag{5.7}$$

In the evaluation of the coefficients we use the value of m given by equation 4.24. The coefficients of equations 5.7 are

$$\begin{aligned} a_{11} &= \frac{\beta_1^2 - \beta_2^2}{z_1 - z_2} \\ a_{12} &= \frac{\beta_1^2 z_2 - \beta_2^2 z_1}{z_1 - z_2} - 1 \\ a_{22} &= a_{11}z_1z_2 \end{aligned} \tag{5.8}$$

with

$$\begin{aligned} z_1 &= \beta_1 \tanh \beta_1 \gamma \\ z_2 &= \beta_2 \tanh \beta_2 \gamma \end{aligned} \tag{5.9}$$

The parameter

$$\gamma = \frac{1}{2}lh \tag{5.10}$$

(introduced in section 7 of Chapter 3) plays an important role in the applications. Its value in terms of the wavelength \mathcal{L} of the deformation is

$$\gamma = \pi \frac{h}{\mathcal{L}} \tag{5.11}$$

The same analysis can be carried out for the symmetric deformation in Figure 5.2b. In this case we choose for f the value

$$f = C_1 \sinh \beta_1 ly + C_2 \sinh \beta_2 ly \quad (5.12)$$

We write the displacements and forces at the upper face as

$$\begin{aligned} U(\tfrac{1}{2}lh) &= U_s \\ V(\tfrac{1}{2}lh) &= V_s \\ \tau(\tfrac{1}{2}lh) &= \tau_s \\ q(\tfrac{1}{2}lh) &= q_s \end{aligned} \quad (5.13)$$

At the bottom face they are

$$\begin{aligned} U(-\tfrac{1}{2}lh) &= U_s \\ V(-\tfrac{1}{2}lh) &= -V_s \\ \tau(-\tfrac{1}{2}lh) &= -\tau_s \\ q(-\tfrac{1}{2}lh) &= q_s \end{aligned} \quad (5.14)$$

Proceeding as above, we find

$$\begin{aligned} \frac{\tau_s}{lL} &= b_{11}U_s + b_{12}V_s \\ \frac{q_s}{lL} &= b_{12}U_s + b_{22}V_s \end{aligned} \quad (5.15)$$

with the coefficients

$$\begin{aligned} b_{11} &= \frac{\beta_1^2 - \beta_2^2}{\beta_1^2 z_2 - \beta_2^2 z_1} z_1 z_2 \\ b_{12} &= \beta_1^2 \beta_2^2 \frac{z_1 - z_2}{\beta_1^2 z_2 - \beta_2^2 z_1} - 1 \\ b_{22} &= b_{11} \frac{\beta_1^2 \beta_2^2}{z_1 z_2} \end{aligned} \quad (5.16)$$

General plane strain deformation is easily obtained by superposition of the symmetric and antisymmetric solutions. We denote by τ_1, q_1, U_1, V_1 the forces and displacements at the upper face of the plate, and by τ_2, q_2, U_2, V_2 the values of the same quantities at the lower face (Fig. 5.3). Adding equations 5.5 and 5.13, we may write the values at the top side

$$\begin{aligned} \tau_1 &= \tau_a + \tau_s & U_1 &= U_a + U_s \\ q_1 &= q_a + q_s & V_1 &= V_a + V_s \end{aligned} \quad (5.17)$$

Similarly adding equations 5.6 and 5.14, we obtain

$$\begin{aligned} \tau_2 &= \tau_a - \tau_s & U_2 &= -U_a + U_s \\ q_2 &= -q_a + q_s & V_2 &= V_a - V_s \end{aligned} \quad (5.18)$$

We derive

$$\begin{aligned} U_a &= \frac{1}{2}(U_1 - U_2) & U_s &= \frac{1}{2}(U_1 + U_2) \\ V_a &= \frac{1}{2}(V_1 + V_2) & V_s &= \frac{1}{2}(V_1 - V_2) \end{aligned} \quad (5.19)$$

These displacements are substituted into equations 5.7 and 5.15, and

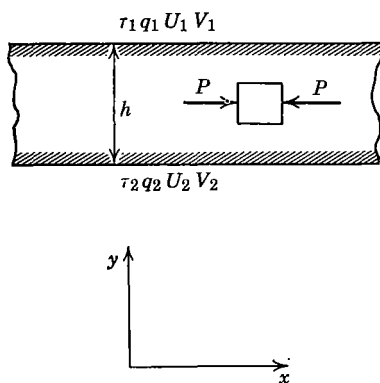


Figure 5.3 Forces and displacements at the top and bottom faces of a plate in the general case.

the resulting expressions for τ_a , q_a , τ_s , q_s are in turn substituted into equations 5.17 and 5.18. We finally obtain

$$\begin{bmatrix} \tau_1 \\ q_1 \\ \tau_2 \\ q_2 \end{bmatrix} = L \begin{bmatrix} A_1 & A_2 & -A_4 & A_5 \\ A_2 & A_3 & -A_5 & A_6 \\ A_4 & A_5 & -A_1 & A_2 \\ -A_5 & -A_6 & A_2 & -A_3 \end{bmatrix} \begin{bmatrix} lU_1 \\ lV_1 \\ lU_2 \\ lV_2 \end{bmatrix} \quad (5.20)$$

The six distinct elements in this matrix are

$$\begin{aligned} A_1 &= \frac{1}{2}(a_{11} + b_{11}) & A_4 &= \frac{1}{2}(a_{11} - b_{11}) \\ A_2 &= \frac{1}{2}(a_{12} + b_{12}) & A_5 &= \frac{1}{2}(a_{12} - b_{12}) \\ A_3 &= \frac{1}{2}(a_{22} + b_{22}) & A_6 &= \frac{1}{2}(a_{22} - b_{22}) \end{aligned} \quad (5.21)$$

It is possible to write the matrix equation (5.20) in more compact

form by introducing a quadratic expression which is proportional to the incremental strain energy. It may be written

$$I = a_{11}U_a^2 + 2a_{12}U_aV_a + a_{22}V_a^2 + b_{11}U_s^2 + 2b_{12}U_sV_s + b_{22}V_s^2 \quad (5.22)$$

When we introduce the values (5.19), this becomes

$$I = \frac{1}{2}A_1(U_1^2 + U_2^2) - A_4U_1U_2 + \frac{1}{2}A_3(V_1^2 + V_2^2) + A_6V_1V_2 + A_2(U_1V_1 - U_2V_2) + A_5(U_1V_2 - U_2V_1) \quad (5.23)$$

With this definition the matrix equation 5.20 is identical with the relations

$$\begin{aligned} \tau_1 &= lL \frac{\partial I}{\partial U_1} & \tau_2 &= -lL \frac{\partial I}{\partial U_2} \\ q_1 &= lL \frac{\partial I}{\partial V_1} & q_2 &= -lL \frac{\partial I}{\partial V_2} \end{aligned} \quad (5.24)$$

Limiting Cases. There are a number of limiting cases of practical interest in which the matrix elements of equation 5.20 are considerably simplified. One such case is found by putting

$$\gamma = \infty \quad (5.25)$$

According to the definition (5.11) of γ , this amounts to either a vanishing wavelength or an infinite thickness. Hence the result must coincide with the equations already derived for the half-space in section 4. Since the initial stress is assumed to be outside the range of internal instability, the real parts of β_1 and β_2 are not zero and have been chosen positive. Hence for $\gamma = \infty$ we may write the limiting values

$$\begin{aligned} z_1 &= \beta_1 \lim_{\gamma \rightarrow \infty} \tanh \beta_1 \gamma = \beta_1 \\ z_2 &= \beta_2 \lim_{\gamma \rightarrow \infty} \tanh \beta_2 \gamma = \beta_2 \end{aligned} \quad (5.26)$$

Introducing these values in coefficients (5.8) and (5.16) and taking into account expressions (4.28) and (4.30) for $\beta_1\beta_2$ and $\beta_1 + \beta_2$, we derive

$$\begin{aligned} a_{11} &= b_{11} = \beta_1 + \beta_2 = \sqrt{2(m+k)} \\ a_{12} &= b_{12} = \beta_1\beta_2 - 1 = k - 1 \\ a_{22} &= b_{22} = \beta_1\beta_2(\beta_1 + \beta_2) = k\sqrt{2(m+k)} \end{aligned} \quad (5.27)$$

Hence

$$\begin{aligned} A_1 &= a_{11} & A_2 &= a_{12} & A_3 &= a_{22} \\ A_4 &= A_5 = A_6 = 0 \end{aligned} \tag{5.28}$$

In this case the coupling terms between the upper and lower surfaces disappear. Equations 5.20 become

$$\begin{bmatrix} \tau_1 \\ q_1 \\ \tau_2 \\ q_2 \end{bmatrix} = L \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & -a_{11} & a_{12} \\ 0 & 0 & a_{12} & -a_{22} \end{bmatrix} \begin{bmatrix} lU_1 \\ lV_1 \\ lU_2 \\ lV_2 \end{bmatrix} \tag{5.29}$$

The matrix elements a_{ij} have the values (5.27). The first two of equations 5.29 coincide with relations (4.25) obtained earlier for the lower half-space. The last two of equations 5.29 correspond to the upper half-space where the medium lies above the free surface.

By returning to the derivation in section 4, it is easy to verify that for the upper half-space there is a change of sign of a_{11} and a_{22} . When the medium of the half-space occupies the region $y > 0$, the condition that the solution vanish at $y = \infty$ requires the roots β_1, β_2 to have negative real parts. This amounts to a change in sign of β_1 and β_2 in equations 4.25. Hence we must reverse the sign of a_{11} and a_{22} while that of a_{12} remains unchanged.

Another limiting case occurs when the characteristic equation 4.14 has double roots. Then

$$m^2 - k^2 = \frac{1}{L} [4N(N - Q) + \frac{1}{4}P^2] = 0 \tag{5.30}$$

The roots become

$$\beta_1 = \beta_2 = \sqrt{-m} = \beta \tag{5.31}$$

According to the inequalities (4.17) the absence of internal instability requires $m > 0$; hence β is a real quantity.

According to equation 5.30 double roots cannot occur if $N > Q$. Hence, referring to the inequality (2.15), we conclude that double roots generally will not occur in a laminated medium for a compression parallel to the layers. When $\beta_1 = \beta_2$, coefficients (5.8) and (5.16) become indeterminate of the form 0/0. Their limiting value is obtained by putting

$$\begin{aligned} \beta_1 &= \beta + \epsilon \\ \beta_2 &= \beta - \epsilon \end{aligned} \tag{5.32}$$

and making ε vanishingly small. We expand the hyperbolic functions to the first power of ε . After we cancel the common factors and put $\varepsilon = 0$, coefficients (5.8) become

$$\begin{aligned} a_{11} &= \frac{4\beta \cosh^2 \beta\gamma}{\sinh 2\beta\gamma + 2\beta\gamma} \\ a_{12} &= \beta^2 \frac{\sinh 2\beta\gamma - 2\beta\gamma}{\sinh 2\beta\gamma + 2\beta\gamma} - 1 \\ a_{22} &= \frac{4\beta^3 \sinh^2 \beta\gamma}{\sinh 2\beta\gamma + 2\beta\gamma} \end{aligned} \quad (5.33)$$

Coefficients (5.16) for the symmetric case become

$$\begin{aligned} b_{11} &= \frac{4\beta \sinh^2 \beta\gamma}{\sinh 2\beta\gamma - 2\beta\gamma} \\ b_{12} &= \beta^2 \frac{\sinh 2\beta\gamma + 2\beta\gamma}{\sinh 2\beta\gamma - 2\beta\gamma} - 1 \\ b_{22} &= \frac{4\beta^3 \cosh^2 \beta\gamma}{\sinh 2\beta\gamma - 2\beta\gamma} \end{aligned} \quad (5.34)$$

For an isotropic medium $N = Q$, and expression (5.30) will vanish only if $P = 0$. The existence of double roots in this case requires the medium to be initially stress-free. The roots become

$$\beta_1 = \beta_2 = \beta = 1 \quad (5.35)$$

By substituting $\beta = 1$, expressions (5.33) are further simplified to

$$\begin{aligned} a_{11} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma + 2\gamma} \\ a_{12} &= - \frac{4\gamma}{\sinh 2\gamma + 2\gamma} \\ a_{22} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma + 2\gamma} \end{aligned} \quad (5.36)$$

and expressions (5.34) become

$$\begin{aligned} b_{11} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma - 2\gamma} \\ b_{12} &= \frac{4\gamma}{\sinh 2\gamma - 2\gamma} \\ b_{22} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma - 2\gamma} \end{aligned} \quad (5.37)$$

Coefficients (5.36) and (5.37) could have been derived directly from the classical theory of elasticity for an incompressible isotropic medium free of initial stress. This requires the solution of the biharmonic equation 3.8.

We note that, for the isotropic medium under initial stress, putting $N = Q$ in equations 4.40 yields

$$m = \frac{1}{1 + \zeta} \tag{5.38}$$

and the roots (4.19) take the values

$$\begin{aligned} \beta_1 &= 1 \\ \beta_2 &= k \end{aligned} \tag{5.39}$$

They check with the general solution (6.13) derived in Chapter 3 for the isotropic case.

Triaxial Initial Stress. Until now in this section and the previous one we have restricted ourselves to a state of initial stress defined by the components $S_{11} = -P$ and $S_{22} = 0$ in the x, y plane.

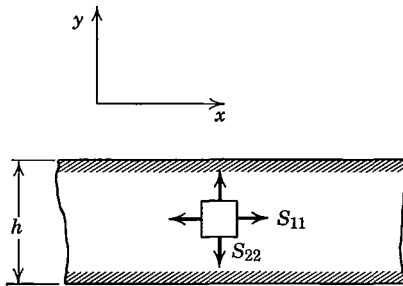


Figure 5.4 Triaxial initial stress in a plate of thickness h . A third component S_{33} may be present in a direction perpendicular to the figure.

A third component S_{33} perpendicular to the x, y plane may or may not be present and does not appear explicitly in the problem.

Let us examine the case where S_{22} is not zero. The initial stress may be triaxial. In the x, y plane there are two components S_{11} and S_{22} of the initial stress (Fig. 5.4). Hence there is an initial stress S_{22} acting on the top and lower faces of the plate.

Obviously such an initial state of stress may be obtained by superposing upon the preceding example a hydrostatic stress S_{22} .

Physically this is obtained by immersion of the whole system in a fluid under a pressure $p_f = -S_{22}$. Since the medium is incompressible, nothing is substantially changed. A similar situation was discussed in section 6 of Chapter 2 in connection with the definition of certain elastic coefficients. This property must, of course, appear in the formulation. That it does so may be shown as follows.

We consider first the equilibrium equations (3.1). As shown by equation 3.2, they are not affected if we define P as

$$P = S_{22} - S_{11} \quad (5.40)$$

The only other formal difference must appear in the boundary condition. We consider again the incremental forces Δf_x and Δf_y acting on the deformed surface AB of Figure 4.2b. The present case differs by the fact that a stress S_{22} acts initially on the surface AB . When we apply equations 6.27 of Chapter 1 the incremental forces become

$$\begin{aligned} \Delta f_x &= s_{12} - S_{22}\omega - S_{11}e_{xy} \\ \Delta f_y &= s_{22} + S_{22}e_{xx} \end{aligned} \quad (5.41)$$

These expressions are the x and y components of the incremental force acting on the deformed surface AB per unit initial area. With the value (5.40) for P they are written

$$\begin{aligned} \Delta f_x &= \Delta' f_x - S_{22} \frac{\partial v}{\partial x} \\ \Delta f_y &= \Delta' f_y + S_{22} e_{xx} \end{aligned} \quad (5.42)$$

where

$$\begin{aligned} \Delta' f_x &= s_{12} + P e_{xy} \\ \Delta' f_y &= s_{22} \end{aligned} \quad (5.43)$$

The significance of this result is brought out by noting that the term $S_{22}e_{xx}$ is the incremental force due to the change of area, while $-S_{22}(\partial v/\partial x)$ represents the incremental x component due to the change of slope of the surface. Hence it is easy to see that $\Delta' f_x$ and $\Delta' f_y$ represent the force *per unit area after deformation* in directions which are *tangent and normal* to the deformed surface AB (Fig. 5.5).

The right sides of equations 4.8 and 5.43 are formally identical. We put

$$\begin{aligned} \Delta' f_x &= \tau \sin lx \\ \Delta' f_y &= q \cos lx \end{aligned} \quad (5.44)$$

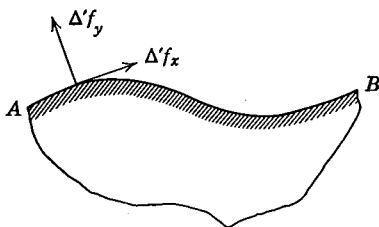


Figure 5.5 Interpretation of the incremental forces $\Delta'f_x$ and $\Delta'f_y$ as normal and tangential stresses on the deformed surface AB (see Fig. 4.2).

Hence all equations in section 4 and in the earlier part of this section are applicable if we replace P by the stress difference (5.40), provided we interpret τ and q as representing the stress increments in directions tangential and normal to the deformed surface.

6. BUCKLING OF A FREE AND EMBEDDED PLATE; INTERFACIAL INSTABILITY

The results of the preceding section will now be discussed for a plate under initial compressive stress P . Consider equations 5.7

$$\begin{aligned}\frac{\tau}{lL} &= a_{11}U + a_{12}V \\ \frac{q}{lL} &= a_{12}U + a_{22}V\end{aligned}\tag{6.1}$$

for the antisymmetric deformation. The coefficients a_{ij} in these equations are those of equations 5.8. We have omitted the subscript a . It is a flexure-type deformation due to the action of normal and tangential forces acting at the upper and lower faces and distributed sinusoidally (Fig. 6.1). The tangential and normal forces acting at the top are $\tau \sin lx$ and $q \cos lx$. On the bottom surface they are reversed. At top and bottom the tangential displacements are, respectively, $U \sin lx$ and $-U \sin lx$, the vertical displacements are the same and are equal to $V \cos lx$. We shall make use of solution (6.1) to derive the buckling condition of a plate stress-free on both faces, or embedded in a medium of infinite extent. The solutions presented here were derived in a recent paper by the author.*

* M. A. Biot, Stability of Multilayered Continua Including the Effect of Gravity and Viscoelasticity, *Journal of the Franklin Institute*, Vol. 276, No. 3, pp. 231-252, 1963.

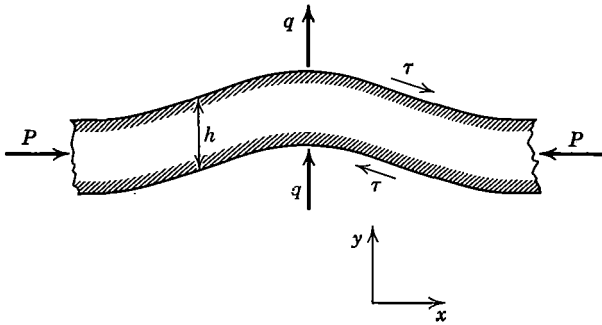


Figure 6.1 Antisymmetric bending deformation of a plate under an initial compressive stress P .

Buckling of a Free Plate. Let us assume that only a normal force is applied to the faces of the plate. Putting $\tau = 0$ in equations 6.1 and eliminating U , we derive

$$\frac{q}{lL} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} V \quad (6.2)$$

This equation yields the deflection of a plate under initial stress due to the application of a normal load.

The coefficient in this equation is evaluated by substituting the expressions (5.8) for a_{ij} . In this evaluation an important simplification results from the cancellation of the common factor $z_1 - z_2$ in numerator and denominator. We find

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{(\beta_1^2 + 1)^2 z_2 - (\beta_2^2 + 1)^2 z_1}{\beta_1^2 - \beta_2^2} \quad (6.3)$$

If the plate is free of surface forces, we must put $q = 0$. Then we obtain the buckling condition

$$(\beta_1^2 + 1)^2 z_2 - (\beta_2^2 + 1)^2 z_1 = 0 \quad (6.4)$$

We recall that

$$\begin{aligned} z_1 &= \beta_1 \tanh \beta_1 \gamma \\ z_2 &= \beta_2 \tanh \beta_2 \gamma \\ \gamma &= \frac{1}{2}lh \end{aligned} \quad (6.5)$$

where h is the plate thickness and $\mathcal{L} = 2\pi/l$ is the buckling wavelength. For a given value of N/Q equation 6.4 is a relation between the variables γ and

$$\zeta = \frac{P}{2Q} \quad (6.6)$$

The solution of equation 6.4 is therefore represented by a family of curves in the γ, ζ plane with N/Q as parameter. These curves are plotted in Figure 6.2 for the three values

$$\frac{N}{Q} = 1, 3, 10 \quad (6.7)$$

The solution for $N/Q = 1$ was derived in Chapter 3 for isotropic media, and it coincides with the curve in Figure 7.3 of that chapter. These plots bring out some basic properties of buckling phenomena in plates. Any one curve may be divided into three portions.

(a) In a range of small values of γ , that is, for large wavelengths the curve is a parabola. This corresponds to buckling by pure bending, and we shall show that here the critical load is given by the classical Euler formula.

(b) There is an intermediate range of wavelengths represented by the rising and approximately straight portion of the curve where the buckling is influenced by the shear rigidity of the plate.

(c) Finally for large values of γ , that is, for small wavelength the curve tends to a horizontal asymptote. This means that the buckling degenerates into a surface instability.

The range of wavelengths for which these three types of instability occur depends on the value of N/Q , and hence on the degree of

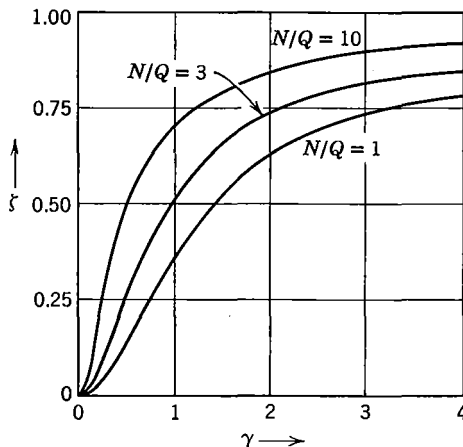


Figure 6.2 Stability parameter ζ as a function of γ for the buckling of an anisotropic plate plotted for three values of N/Q .

anisotropy. It can be seen that for increasing values of N/Q the transition from bending to shear buckling occurs at increasingly larger wavelengths.

The behavior of the buckling deformation as a function of the wavelength is entirely analogous to that of flexural vibrations of a plate which degenerate into surface Rayleigh waves for decreasing wavelengths.

Let us examine more closely the nature of the solution in the extreme ranges of wavelengths. For small values of $\beta_1\gamma$ and $\beta_2\gamma$ we may replace the hyperbolic functions in equations 6.5 by their power series. Retaining only the terms in γ and γ^3 , we write

$$\begin{aligned} z_1 &= \beta_1^2\gamma - \frac{1}{3}\beta_1^4\gamma^3 \\ z_2 &= \beta_2^2\gamma - \frac{1}{3}\beta_2^4\gamma^3 \end{aligned} \quad (6.8)$$

Substituting these values in equation 6.4, we derive

$$k^2 - 1 + \frac{2}{3}(k^2 + m)\gamma^2 = 0 \quad (6.9)$$

With the values (4.40) for k and m , this becomes

$$\zeta = \frac{2}{3} \frac{N}{Q} \frac{\gamma^2}{1 + \frac{1}{3}\gamma^2} \quad (6.10)$$

Again using the assumption that γ is small, we finally write

$$\zeta = \frac{2}{3} \frac{N}{Q} \gamma^2 \quad (6.11)$$

This equation represents an approximation for the curves of Figure 6.2 near the origin. The portion near the origin is replotted on an enlarged scale in Figure 6.3 along with the parabolic approximation (6.11). When we substitute the value (4.40) for ζ , equation 6.11 takes the form

$$P = 4N \frac{l^2 h^2}{12} \quad (6.12)$$

This is the critical load obtained from the classical Euler theory of buckling, as formulated by the equation for the deflection w of a thin plate under the compressive axial stress P ; that is,

$$4N \frac{h^3}{12} \frac{d^2 w}{dx^2} + Phw = 0 \quad (6.13)$$

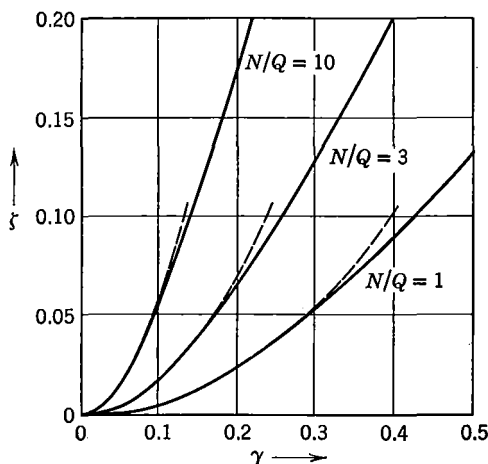


Figure 6.3 Stability parameter ζ of Figure 6.2 in the range of larger wavelengths (γ small). The approximate value obtained from the Euler Theory (equation 6.11) is shown by dotted lines.

The elastic modulus used in the equation is $4N$. When P is small, it is the same as the modulus $4M$ whose significance is discussed in Chapter 3 and illustrated by equation 4.25 of that chapter.

Comparison of the exact curves with the approximate value (6.11), also plotted in Figure 6.3, shows that the Euler theory is valid above a certain limiting wavelength which increases with the value of N/Q . The range of validity of the Euler theory is determined by the inequality

$$\gamma \sqrt{\frac{N}{Q}} < 0.3 \tag{6.14}$$

For the isotropic case it coincides with condition (7.24) of Chapter 3.

For large values of γ the hyperbolic tangents in expressions (6.5) are replaced by unity. We may write

$$\begin{aligned} z_1 &= \beta_1 \\ z_2 &= \beta_2 \end{aligned} \tag{6.15}$$

We insert these values in equation 6.4, cancelling the factor $(\beta_1 - \beta_2)$ and taking into account the values (4.26) and (4.28) for m and k . We derive the relation

$$2k(m + 1) + k^2 - 1 = 0 \tag{6.16}$$

It is identical with equation 4.39 which expresses the condition for surface instability of the half-space. This result shows that for very small wavelengths the value of ζ is represented by a horizontal asymptote which corresponds to surface instability.

A remark is in order here regarding the case where the initial stress lies in the range of internal instability. For simplicity we consider internal instability of the first kind. In this case the inequalities (3.16) must be satisfied. They are

$$m > 0 \quad k^2 = \frac{1 - \zeta}{1 + \zeta} < 0 \quad (6.17)$$

Hence the absolute value of ζ is larger than unity. The root β_1 here is real while β_2 is pure imaginary. We put

$$\beta_2 = i\xi \quad (6.18)$$

Equation 6.4 becomes

$$(1 - \xi^2)^2 \beta_1 \tanh \beta_1 \gamma + (\beta_1^2 + 1)^2 \xi \tan \xi \gamma = 0 \quad (6.19)$$

The solution of this equation has an infinite number of branches which are all located above the value $\zeta = 1$ and below the value $\zeta = -1$. These branches correspond to the phenomenon of internal instability discussed in section 3 for a rigidly confined medium. Equation 6.19 represents the same phenomenon for the slightly more complicated case of a plate with free faces. Such branches have also been plotted in a paper dealing with the closely related problem of an embedded layer.*

Multiple Branch Solution; Instability under Axial Tension. Let us replace equation 6.19 by the simpler one

$$\tan \xi \gamma = 0 \quad (6.19a)$$

The branch solutions of this equation are

$$\xi \gamma = n\pi \quad (6.19b)$$

where n is an integer. They are analogous to the solution of equation 6.19. Since ξ is real, they must correspond to an internal buckling. The value of ξ also satisfies condition (3.23). With the variable $\zeta = P/(2Q)$, an equivalent form of this condition is

$$\zeta = \frac{1 + 2(2N/Q - 1)\xi^2 + \xi^4}{1 - \xi^4} \quad (6.19c)$$

* See M. A. Biot and H. Odé, On the Folding of a Viscoelastic Medium with Adhering Layer under Compressive Initial Stress, *Quarterly of Applied Mathematics*, Vol. XIX, No. 4, pp. 351-355, 1962.

Substitution of $\xi = n\pi/\gamma$ in this equation yields a plot of ζ versus γ with an infinite number of branches represented schematically in Fig. 6.3a. An interesting aspect of this solution is the possibility of instability for negative values of ζ . This corresponds to a plate under axial tension. As illustrated in Fig. 6.3b a shear instability is possible in this case. The occurrence of instability under tension may also be inferred from the discussion of internal buckling in section 3. This can be seen by superposing an isotropic tension equal to the compressive stress P . The instability is unaffected, but the initial compression is replaced by a tension in the perpendicular direction.

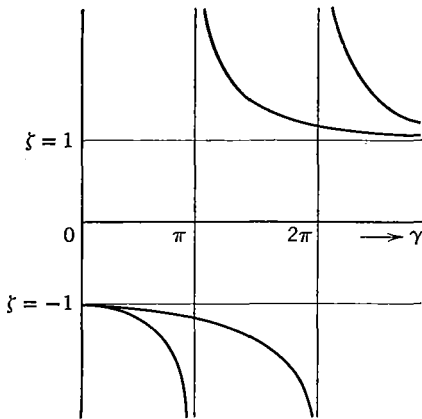


Figure 6.3a Branch solutions for internal instability.

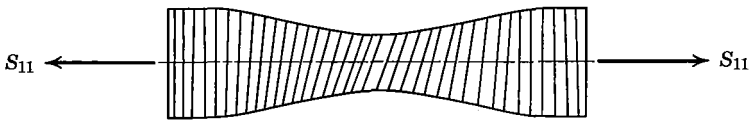


Figure 6.3b Shear instability of an anisotropic plate under axial tension.

Buckling of an Embedded Plate. We now consider the plate to be embedded in an infinite medium as shown in Figure 6.4. As before, we assume incompressibility and incremental orthotropy with planes of symmetry parallel to the plate and to the plane of the figure. The incremental elastic coefficients of the plate and the embedding medium are designated by N, Q and N_1, Q_1 , respectively. The state of initial stress may be triaxial. There may be a stress S_{22} acting uniformly throughout the medium in a direction normal to the plate. The principal stress components parallel to the plate are

designated by S_{11} in the plate, and by $S_{11}^{(1)}$ in the embedding medium. The corresponding stress differences are

$$\begin{aligned} P_1 &= S_{22} - S_{11}^{(1)} \\ P &= S_{22} - S_{11} \end{aligned} \tag{6.20}$$

A third component S_{33} of initial stress may also be present in the direction normal to the figure; it does not appear explicitly.

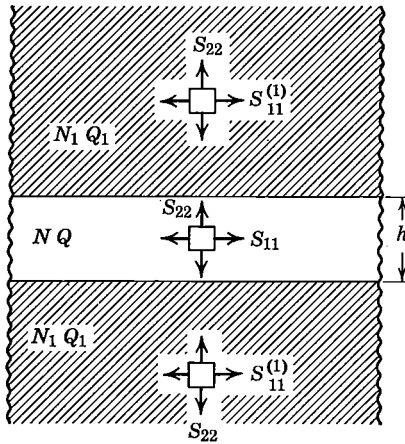


Figure 6.4 Plate and embedding medium under initial stress.

In order to establish the stability equations of this system let us consider the stresses in the upper half-space at the interface. They are given by the last two of equations 5.29. Omitting the subscript 2 and replacing a_{ij} by a'_{ij} , we write

$$\begin{aligned} \tau &= L_1 l(-a'_{11} U + a'_{12} V) \\ q &= L_1 l(a'_{12} U - a'_{22} V) \end{aligned} \tag{6.21}$$

The slide modulus L_1 is

$$L_1 = Q_1 + \frac{1}{2} P_1 \tag{6.22}$$

and the coefficients a'_{ij} are given by equations 5.27 for the half-space. They are

$$\begin{aligned} a'_{11} &= \sqrt{2(m_1 + k_1)} \\ a'_{12} &= k_1 - 1 \\ a'_{22} &= k_1 \sqrt{2(m_1 + k_1)} \end{aligned} \tag{6.23}$$

The subscript 1 refers to values for the embedding medium, where m_1 and k_1 are functions of N_1 , Q_1 , and P_1 . For the upper face of the plate the stress is determined by equations 6.1. Coefficients a_{ij} are given by equations 5.8 and are functions of N , Q , and P .

In the case of perfect adherence between the two media the stresses τ , q and the displacements U , V at the interface are the same. Equating the value of τ and q of relations 6.1 and 6.21 yields the two equations

$$\begin{aligned} (La_{11} + L_1a'_{11})U + (La_{12} - L_1a'_{12})V &= 0 \\ (La_{12} - L_1a'_{12})U + (La_{22} + L_1a'_{22})V &= 0 \end{aligned} \tag{6.24}$$

The buckling condition is obtained by putting equal to zero the determinant

$$(La_{11} + L_1a'_{11})(La_{22} + L_1a'_{22}) - (La_{12} - L_1a'_{12})^2 = 0 \tag{6.25}$$

The numerical solution of the problem is considerably simplified if we introduce two assumptions. We put

$$P_1 = 0 \tag{6.26}$$

This means either that the initial stress is zero in the embedding medium or that in the plane of the figure the initial stress is isotropic. If this is not so, the assumption $P_1 = 0$ is an adequate approximation when the initial stress of the embedding medium is small. The other simplification is to assume perfect slip at the interface. As shown elsewhere,* the influence of interfacial adherence for the single embedded layer is not significant. (A case of perfect adherence is analyzed in the next paragraph.)

With these two assumptions the normal stress for the upper half-space is obtained by applying equation 4.33.

$$q = -2lQ_{\text{eff}}V \tag{6.27}$$

with

$$Q_{\text{eff}} = \sqrt{N_1Q_1}$$

The "effective" modulus Q_{eff} is that of an isotropic medium with the same apparent surface rigidity as given by equation 4.36. The sign

* See M. A. Biot and H. Odé, On the Folding of a Viscoelastic Medium with Adhering Layer under Compressive Initial Stress, *Quarterly of Applied Mathematics*, Vol. XIX, No. 4, pp. 351-355, 1962.

of q has been changed because equation 4.33 refers to the lower half-space. On the other hand, for $\tau = 0$ the stress q in the layer is given by equation 6.2. We must remember that the two values of q represent the stress at the displaced point. If the surfaces at the interface are allowed to slip, the values of q in equations 6.2 and 6.27 are not exactly equal because they represent the normal interfacial stress at different points. However, the relative displacement is of the first order and therefore the values of q differ only by a second order quantity. Hence, neglecting the second order differences, we equate the values (6.2) and (6.27) and obtain

$$\frac{2Q_{\text{eff}}}{L} + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = 0 \quad (6.28)$$

When we introduce expression (6.3) and write the slide modulus in the form

$$L = Q + \frac{1}{2}P = Q(1 + \zeta) \quad (6.29)$$

the buckling condition (6.28) becomes

$$\frac{Q_{\text{eff}}}{Q} = \frac{1}{2}(1 + \zeta) \frac{(\beta_2^2 + 1)^2 z_1 - (\beta_1^2 + 1)^2 z_2}{\beta_1^2 - \beta_2^2} \quad (6.30)$$

On the right side is a function of the parameters N , Q , P of the plate and the variable γ . For given values of Q_{eff}/Q and N/Q equation 6.30 constitutes a relation between $\zeta = P/(2Q)$ and γ . This is plotted in Figure 6.5 for the values

$$\begin{aligned} \frac{Q_{\text{eff}}}{Q} &= \frac{1}{100} \\ \frac{N}{Q} &= 1, 3, 10 \end{aligned} \quad (6.31)$$

The curves show that ζ goes through a minimum value which depends on the two parameters Q_{eff}/Q and N/Q . We may write this minimum value

$$\zeta_{\text{min}} = F(Q_{\text{eff}}/Q, N/Q) \quad (6.32)$$

In order to find the buckling condition under finite strain we must evaluate ζ , Q_{eff}/Q , and N/Q as functions of the finite extension ratios. Buckling will occur as soon as the horizontal drawn through the value of ζ intersects the corresponding curve in Figure 6.5. This will

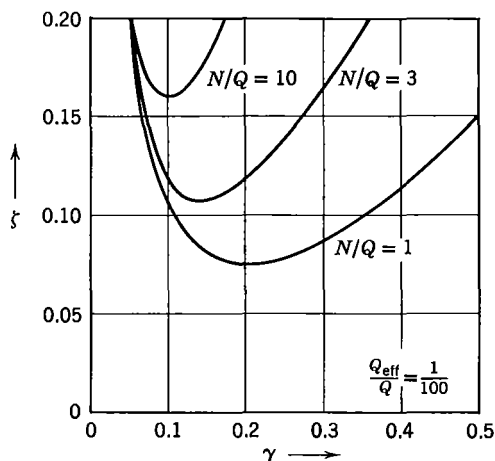


Figure 6.5 Stability parameter ζ as a function of γ for the embedded anisotropic plate.

happen first when ζ_{\min} as given by equation 6.32 is equal to ζ . Therefore the buckling condition is

$$\frac{P}{2Q} = F(Q_{\text{eff}}/Q, N/Q) \quad (6.33)$$

The wavelength of the buckling is given by the value of γ which corresponds to ζ_{\min} .

Buckling of an Embedded Layer in Rubber-like Materials.

The stability problem of an embedded layer becomes particularly simple for isotropic materials obeying the finite stress-strain relations (8.45) of Chapter 2. This problem was solved in a recent paper.* Perfect adherence is assumed at the interface between the layer and the embedding medium. The system is in a state of finite triaxial homogeneous strain, and two of the principal directions of strain are parallel to the layer. The initial stresses are the same as in Figure 6.4. The finite extension ratios λ_1 and λ_2 are in the plane of the figure, and λ_1 represents the extension in the direction of the layer. The third extension λ_3 is normal to the figure and is determined by

* M. A. Biot, Continuum Theory of Stability of an Embedded Layer in Finite Elasticity under Initial Stress, *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 27, Part 1, pp. 19-22, 1964.

the condition of incompressibility. The finite stress-strain relations yield

$$\begin{aligned} P &= S_{22} - S_{11} = \mu_0(\lambda_2^2 - \lambda_1^2) \\ P_1 &= S_{22} - S_{11}^{(1)} = \mu_{01}(\lambda_2^2 - \lambda_1^2) \end{aligned} \quad (6.34)$$

The coefficient μ_0 is the shear modulus of the layer in the original unstressed state and μ_{01} is its value for the embedding medium.

In the plane of Figure 6.4 the material is isotropic for incremental deformations. The incremental elastic coefficients are given by equation 8.51 of Chapter 2; they are

$$\begin{aligned} N &= Q = \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2) \\ N_1 &= Q_1 = \frac{1}{2}\mu_{01}(\lambda_1^2 + \lambda_2^2) \end{aligned} \quad (6.35)$$

Considerable simplification of this problem results from the relation

$$\frac{L_1}{L} = \frac{Q_1 + \frac{1}{2}P_1}{Q + \frac{1}{2}P} = \frac{\mu_{01}}{\mu_0} \quad (6.36)$$

By introducing the rigidity ratio

$$n = \frac{\mu_{01}}{\mu_0} \quad (6.37)$$

we may write the general buckling condition (6.25) as:

$$\begin{aligned} (a_{11}a_{22} - a_{12}^2) + n(a_{22}a'_{11} + a_{11}a'_{22} + 2a_{12}a'_{12}) \\ + n^2(a'_{11}a'_{22} - a_{12}'^2) = 0 \end{aligned} \quad (6.38)$$

Another simplification is due to the fact that the values of ζ , k , and m are the same in both media, namely,

$$\begin{aligned} \zeta &= \frac{P}{2Q} = \frac{P_1}{2Q_1} = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2} \\ k &= \sqrt{\frac{1 - \zeta}{1 + \zeta}} = \frac{\lambda_1}{\lambda_2} \\ m &= \frac{1}{1 + \zeta} \end{aligned} \quad (6.39)$$

Hence the values of a'_{ij} are obtained from equations 6.23 by substituting k and m for k_1 and m_1 , respectively. We write

$$\begin{aligned} a'_{11} &= 1 + k \\ a'_{12} &= k - 1 \\ a'_{22} &= k(1 + k) \end{aligned} \quad (6.40)$$

The coefficients a_{ij} for the layer are obtained from expressions (5.8). For this case we must introduce the values $\beta_1 = 1$ and $\beta_2 = k$ given by equations 5.39. Hence

$$\begin{aligned} a_{11} &= \frac{1 - k^2}{z_1 - z_2} \\ a_{12} &= \frac{2z_2 - (1 + k^2)z_1}{z_1 - z_2} \\ a_{22} &= a_{11}z_1z_2 \end{aligned} \quad (6.41)$$

with

$$\begin{aligned} z_1 &= \tanh \gamma \\ z_2 &= k \tanh k\gamma \\ \gamma &= \frac{1}{2}lh = \frac{\pi h}{\mathcal{L}} \quad (\mathcal{L} = \text{wavelength}) \end{aligned} \quad (6.42)$$

Values of the expressions in parentheses in the buckling condition (6.38) may also be written.

$$\begin{aligned} a_{11}a_{22} - a_{12}^2 &= \frac{1}{z_1 - z_2} [4z_2 - (1 + k^2)^2z_1] \\ a'_{11}a_{22} + a_{11}a'_{22} + 2a_{12}a'_{12} &= \\ \frac{1 - k}{z_1 - z_2} [(1 + k)^2z_1z_2 - 4z_2 + 2(1 + k^2)z_1 + k(1 + k)^2] \\ a'_{11}a'_{22} - a_{12}'^2 &= k(1 + k)^2 - (1 - k)^2 \end{aligned} \quad (6.43)$$

The buckling condition (6.38) may be considered a relation between the variables ζ , γ , and n . The stability parameter ζ as a function of γ is plotted in Figure 6.6 for five values of the rigidity ratio n . It is recalled that γ is defined by equations 6.5 and is proportional to the layer thickness h divided by the buckling wavelength \mathcal{L} .

The following limiting cases of equation 6.38 are of interest.

Case $n = 0$. This case, where the rigidity of the embedding medium vanishes, coincides with the problem of buckling of a plate with free surfaces. By putting $n = 0$ in equation 6.38 it becomes

$$a_{11}a_{22} - a_{12}^2 = 0 \quad (6.44)$$

Reference to expressions (6.43) shows that this condition is equivalent to

$$4z_2 - (1 + k^2)^2z_1 = 0 \quad (6.45)$$

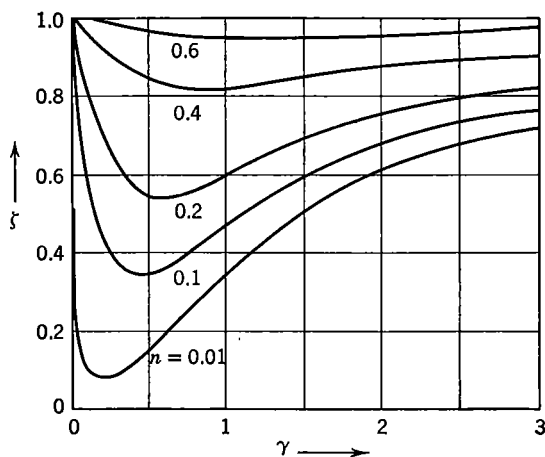


Figure 6.6 Stability parameter ζ as a function of γ for five values of the rigidity ratio $n = \mu_{01}/\mu_0$ for an embedded layer of rubber-like material.

This is the same as the buckling condition (7.12) obtained in Chapter 3 for a thick slab. The solution is plotted in Figure 7.3 of that chapter.

Case $n = \infty$. In this case the rigidity of the layer vanishes, and we are left with the problem of surface instability of a half-space represented by one side of the embedding medium. Equation 6.38 becomes

$$a'_{11}a'_{22} - a'_{12}{}^2 = k(1 + k)^2 - (1 - k)^2 = 0 \quad (6.46)$$

or

$$k^3 + k^2 + 3k - 1 = 0 \quad (6.47)$$

We multiply this equation by $k - 1$ and obtain

$$(k^2 + 1)^2 - 4k = 0 \quad (6.48)$$

It may be written

$$(1 + \zeta)^2 k - 1 = 0 \quad (6.49)$$

This equation coincides with the condition (4.45c) for the surface instability of isotropic materials. It was also derived as equation 6.19 of Chapter 3.

Another limiting case is shown by the plot in Figure 6.6 where the

curves tend toward horizontal asymptotes for $\gamma = \infty$. This will now be discussed in more detail.

Interfacial Instability. The case $\gamma = \infty$ corresponds to that of a vanishing wavelength. Since γ is proportional to the ratio of layer thickness to wavelength \mathcal{L} , this case also corresponds to that of a layer of infinite thickness. Hence it represents the problem of stability at an interface of two adhering half-spaces. For $\gamma = \infty$ the values (6.40) and (6.41) become

$$\begin{aligned} a'_{11} &= a_{11} = 1 + k \\ a'_{12} &= a_{12} = k - 1 \\ a'_{22} &= a_{22} = k(k + 1) \end{aligned} \tag{6.50}$$

With these values the buckling condition (6.38) assumes the simple form

$$k \left(\frac{1+k}{1-k} \right)^2 = \left(\frac{1-n}{1+n} \right)^2 \tag{6.51}$$

From this equation the critical value of ζ required for interfacial instability has been evaluated as a function of the rigidity ratio n of the two media. Numerical values are shown in Table 2. These values of ζ represent horizontal asymptotes for the curves in Figure 6.6 when γ tends to infinity.

Table 2

Critical value of ζ for interfacial instability as a function of the rigidity ratio n for rubber-like media

$n = \mu_{01}/\mu_0$	ζ
0	0.839
0.2	0.924
0.4	0.973
0.6	0.994
0.8	0.999
1.0	1.000

For $n = 0$ Table 2 yields the numerical value (6.21) of Chapter 3 derived as the critical value for surface instability. For $n = 1$ the discontinuity vanishes and no interfacial instability is possible. This

is in accordance with the corresponding value $\zeta = 1$ in the table, since it implies $\lambda_2 = \infty$ and can never be attained.

Interfacial Instability in the General Case. The phenomenon of elastic instability at an interface is quite general and is mathematically analogous to Stoneley wave propagation at the interface of two elastic solids. The condition of instability for the general case, including anisotropic media, has exactly the same form as equation 6.25 except that now the coefficients a_{ij} and a'_{ij} are the expressions (5.27) and (6.23) for the lower and upper half-spaces. Interfacial instability was discussed in more detail by the author in a recent paper.* Values in Table 2 are taken from that paper.

Bending of a Beam Resting on an Elastic Half-Space. The problems treated in this section are closely related to the theory, originally developed by the author,† of bending of a beam resting on an elastic half-space. For the two-dimensional problem of a plate, with initial axial stress, resting on an elastic half-space, the beam theory yields a simple solution. Except in the range of very short wavelengths the beam theory should provide close agreement with an exact solution derived by treating the plate as a continuous medium. This point is illustrated in the discussion of the similar problem of the embedded viscoelastic plate in Chapter 6 (section 8). The solution of the three-dimensional problem of bending of a beam, with initial axial stress, resting on an elastic half-space, requires a treatment which differs from that for the two-dimensional analysis. The equation for the deflection w of the beam is written

$$E_0 I \frac{d^4 w}{dx^4} + PA \frac{d^2 w}{dx^2} + kw = q(x) \quad (6.51a)$$

The x coordinate is measured along the axis of the beam (Fig. 6.7). The parameters for the beam are: Young's modulus E_0 , the compressive initial stress P , the area A , and moment of inertia I of the cross section. It is assumed that the beam is pressed against the half-space by its own weight or the weight it carries. The load per unit length carried by the beam is $q(x)$. The reaction of the elastic half-space is represented by a modulus k whose value is

$$k = \frac{E}{C(1 - \nu^2)} lb\psi(lb) \quad (6.51b)$$

where E and ν are Young's modulus and Poisson's ratio for the half-space,

* M. A. Biot, Interfacial Instability in Finite Elasticity under Initial Stress, *Proceedings of the Royal Society, A*, Vol. 273, pp. 340-344, 1963.

† M. A. Biot, Bending of an Infinite Beam on an Elastic Foundation, *Journal of Applied Mechanics (Transactions ASME)*, Vol. 59, pp. A1-A7, 1937.

respectively. The width of the beam at the surface of contact with the half-space is $2b$. The value of k is valid for a sinusoidal load and deflection

$$q = q_0 \sin lx \quad w = w_0 \sin lx \quad (6.51c)$$

which define the wavelength parameter l . The coefficient C depends slightly on the transverse rigidity of the beam over the width $2b$ at the area of contact

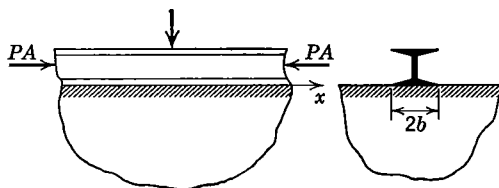


Figure 6.7 Beam resting on an elastic half-space.

with the half-space. The value of C lies between the limits $C = 1$ and $C = 1.13$. Tabulated values of the function $\psi(lb)$ are given in the author's paper.* Additional numerical results have also been derived in a recent paper by Lekkerkerker.† Deflection of the beam under a concentrated load may be evaluated by the Fourier analysis procedure used in the cited references.

7. STABILITY THEORY OF MULTILAYERED MEDIA INCLUDING THE EFFECT OF GRAVITY

We consider here a system of n layers of different materials and retain the assumption of incompressibility.‡ The orthotropic elastic properties of each layer are such that the directions of elastic symmetry are the same for all layers and parallel to the principal directions of the initial stress. The layers are also parallel to one of the planes of elastic symmetry. Under these conditions the equations for the stability of the system of layers are derived immediately from the results obtained in section 5. The layers represented in Figure 7.1 are numbered from 1 to n starting at the top.

* M. A. Biot, Bending of an Infinite Beam on an Elastic Foundation, *Journal of Applied Mechanics (Transactions ASME)*, Vol. 59, pp. A1-A7, 1937.

† J. G. Lekkerkerker, Bending of an Infinite Beam Resting on an Elastic Half-Space, *Proceedings Koninklijke Akademie van Wetenschappen (Amsterdam)*, Ser. B, Vol. 63, pp. 484-497, 1960.

‡ The results presented in this section were obtained in a recent paper (M. A. Biot, Stability of Multilayered Continua Including the Effect of Gravity and Viscoelasticity, *Journal of the Franklin Institute*, Vol. 276, No. 3, pp. 231-252, 1963).

We consider first the case where the stack of layers is free at the top and bottom surfaces and assume that there is perfect adherence at the interfaces. In the j th layer we denote the elastic coefficients by N_j, Q_j and the initial compressive stress by P_j . We denote by U_j, V_j the displacements at the top of the j th layer and by U_{j+1}, V_{j+1} the displacements at the bottom of the same layer. The quadratic form (5.23) for this layer is written (no summation)

$$\begin{aligned} I_j = & \frac{1}{2} {}_j A_1 (U_j^2 + U_{j+1}^2) - {}_j A_4 U_j U_{j+1} \\ & + \frac{1}{2} {}_j A_3 (V_j^2 + V_{j+1}^2) + {}_j A_6 V_j V_{j+1} \\ & + {}_j A_2 (U_j V_j - U_{j+1} V_{j+1}) + {}_j A_5 (U_j V_{j+1} - U_{j+1} V_j) \end{aligned} \quad (7.1)$$

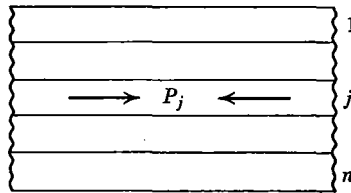


Figure 7.1 Multilayered medium with free end faces.

The coefficients ${}_j A_1, {}_j A_2$, etc., are functions of the following parameters:

- (a) The two elastic coefficients N_j and Q_j of the j th layer.
- (b) The compressive stress P_j in the j th layer.
- (c) The variable $\gamma_j = \frac{1}{2} h_j l$, where h_j is the thickness of the j th layer.

The slide modulus of the j th layer is

$$L_j = Q_j + \frac{1}{2} P_j \quad (7.2)$$

By equations 5.24 the stress at the bottom of this layer is

$$\begin{aligned} \tau_{j+1} &= -l L_j \frac{\partial I_j}{\partial U_{j+1}} \\ q_{j+1} &= -l L_j \frac{\partial I_j}{\partial V_{j+1}} \end{aligned} \quad (7.3)$$

while the stress at the top of the $(j + 1)$ th layer is

$$\begin{aligned} \tau_{j+1} &= lL_{j+1} \frac{\partial I_{j+1}}{\partial U_{j+1}} \\ q_{j+1} &= lL_{j+1} \frac{\partial I_{j+1}}{\partial V_{j+1}} \end{aligned} \tag{7.4}$$

Equating the stresses (7.3) and (7.4) yields the two recurrence equations

$$\begin{aligned} \frac{\partial}{\partial U_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) &= 0 \\ \frac{\partial}{\partial V_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) &= 0 \end{aligned} \tag{7.5}$$

These equations relate the six displacements at three successive interfaces. The stress-free boundary condition at the top and bottom surfaces are written

$$\begin{aligned} \frac{\partial I_1}{\partial U_1} &= 0 & \frac{\partial I_1}{\partial V_1} &= 0 \\ \frac{\partial I_n}{\partial U_{n+1}} &= 0 & \frac{\partial I_n}{\partial V_{n+1}} &= 0 \end{aligned} \tag{7.6}$$

There are $2(n - 1)$ equations (7.5), i.e., two for each interface. The addition of the boundary conditions (7.6) yields a system of $2n + 2$ equations for the $2n + 2$ unknown U_j and V_j .

Equating to zero the determinant of this system yields the characteristic equation for instability. The system (7.5) does not contain more than six variables in each equation. This feature makes the formulation very suitable for numerical solutions with automatic computers.

If the system of layers is embedded in two semi-infinite media (Fig. 7.2a), the initial stress may include a component S_{22} normal to the layers. The initial stress component parallel to the layer may be different in each layer. Its value in the j th layer is denoted by $S_{11}^{(j)}$. In this case the equations for each layer remain the same provided P_j is put equal to the stress difference:

$$P_j = S_{22} - S_{11}^{(j)} \tag{7.7}$$

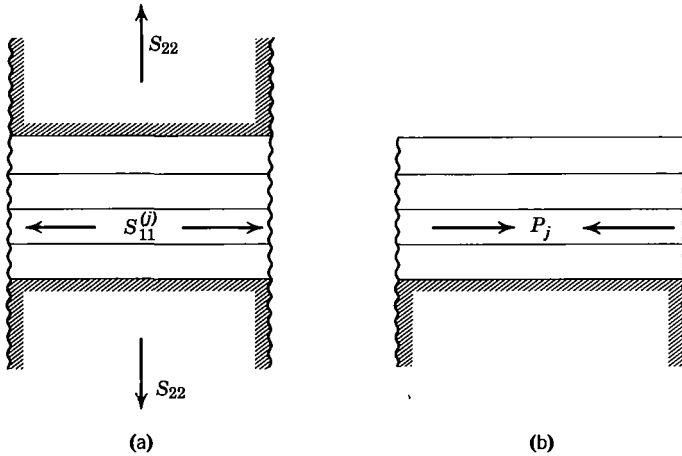


Figure 7.2 Multilayered medium: (a) between two semi-infinite media, (b) lying on top of a semi-infinite medium.

The boundary condition at the top interface becomes

$$L_1 \frac{\partial I_1}{\partial U_1} = L'(-a'_{11}U_1 + a'_{12}V_1)$$

$$L_1 \frac{\partial I_1}{\partial V_1} = L'(a'_{12}U_1 - a'_{22}V_1)$$
(7.8)

and at the bottom interface

$$-L_n \frac{\partial I_n}{\partial U_{n+1}} = L(a_{11}U_{n+1} + a_{12}V_{n+1})$$

$$-L_n \frac{\partial I_n}{\partial V_{n+1}} = L(a_{12}U_{n+1} + a_{22}V_{n+1})$$
(7.9)

In these equations the a_{ij} represent the coefficients (5.27) for the lower half-space, while the a'_{ij} are the values of the same expressions for the upper half-space. The slide moduli in upper and lower half-spaces are, respectively, L' and L .

If the top surface is free (Fig. 7.2b), the boundary conditions are obtained by putting $L' = 0$ in equations 7.8.

The recurrence equations and the boundary conditions for any one

of the three cases which we have just discussed may be written in very compact form by introducing the composite quadratic form

$$\mathcal{J} = LI_l + \sum_{j=1}^n L_j I_j + L' I_u \quad (7.10)$$

In this expression I_u and I_l are quadratic forms representing the upper and lower half-spaces. They are defined as follows

$$\begin{aligned} I_u &= \frac{1}{2} a'_{11} U_1^2 - a'_{12} U_1 V_1 + \frac{1}{2} a'_{22} V_1^2 \\ I_l &= \frac{1}{2} a_{11} U_{n+1}^2 + a_{12} U_{n+1} V_{n+1} + \frac{1}{2} a_{22} V_{n+1}^2 \end{aligned} \quad (7.11)$$

Equations 7.5 and the boundary conditions (7.8) and (7.9) are then written very simply as

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \quad \frac{\partial \mathcal{J}}{\partial V_j} = 0 \quad (7.12)$$

with $j = 1, 2, \dots, n + 1$.

The particular cases of a free surface at the top or at the top and bottom of the layers are obtained by putting $L' = 0$ or $L' = L = 0$ in the expression for \mathcal{J} .

The generality and simplicity of equations 7.12 are a consequence of the fact that they embody a variational principle for the total incremental strain energy of the layered system under initial stress. The variational principle is

$$\delta \mathcal{J} = 0 \quad (7.13)$$

The particular case of buckling of a sandwich plate composed of a soft core between two sheets has been discussed by Goodier* as an application of the stability theory of continuous media.

Solution by Matrix Multiplication. Instead of using the recurrence equations 7.5, it is possible to follow a procedure similar to that proposed by Thomson† and further developed by Haskell‡ for

* J. N. Goodier, Cylindrical Buckling of Sandwich Plates, *Journal of Applied Mechanics*, Vol. 13, pp. A253–A260, 1946.

† W. T. Thomson, Transmission of Elastic Waves through a Stratified Solid Medium, *Journal of Applied Physics*, Vol. 21, pp. 89–93, 1950.

‡ N. A. Haskell, Dispersion of Surface Waves in Multilayered Media, *Bulletin of the Seismological Society of America*, Vol. 43, pp. 17–34, 1953.

the numerical evaluation of wave dispersion in layered media. This is done by rewriting equations 5.20 in the form

$$\begin{bmatrix} \tau_1 \\ q_1 \\ lU_1 \\ lV_1 \end{bmatrix} = \mathcal{M} \begin{bmatrix} \tau_2 \\ q_2 \\ lU_2 \\ lV_2 \end{bmatrix} \quad (7.14)$$

The matrix \mathcal{M} is

$$\mathcal{M} = \begin{bmatrix} B_1 & B_2 & LB_5 & LB_6 \\ B_3 & B_4 & -LB_6 & LB_7 \\ \frac{1}{L} B_8 & \frac{1}{L} B_9 & B_1 & -B_3 \\ -\frac{1}{L} B_9 & \frac{1}{L} B_{10} & -B_2 & B_4 \end{bmatrix} \quad (7.15)$$

The components B_i of this matrix may be written in terms of the coefficients a_{ij} and b_{ij} given by expressions (5.8) and (5.16) for the particular layer considered. They are

$$\begin{aligned} \Delta &= (a_{12} - b_{12})^2 - (a_{11} - b_{11})(a_{22} - b_{22}) \\ B_1 &= \frac{1}{\Delta} [(a_{12}^2 - b_{12}^2) - (a_{11} + b_{11})(a_{22} - b_{22})] \\ B_2 &= \frac{2}{\Delta} (a_{11}b_{12} - a_{12}b_{11}) \\ B_3 &= \frac{2}{\Delta} (a_{12}b_{22} - a_{22}b_{12}) \\ B_4 &= \frac{1}{\Delta} [(a_{22} + b_{22})(a_{11} - b_{11}) - (a_{12}^2 - b_{12}^2)] \\ B_5 &= \frac{2}{\Delta} [a_{12}^2b_{11} - a_{11}b_{12}^2 - a_{11}b_{11}(a_{22} - b_{22})] \\ B_6 &= \frac{2}{\Delta} [-a_{12}b_{12}(a_{12} - b_{12}) + a_{11}a_{22}b_{12} - a_{12}b_{11}b_{22}] \\ B_7 &= \frac{2}{\Delta} [a_{22}b_{22}(a_{11} - b_{11}) + a_{22}b_{12}^2 - a_{12}^2b_{22}] \\ B_8 &= -\frac{2}{\Delta} (a_{22} - b_{22}) \\ B_9 &= -\frac{2}{\Delta} (a_{12} - b_{12}) \\ B_{10} &= \frac{2}{\Delta} (a_{11} - b_{11}) \end{aligned} \quad (7.16)$$

Equation 7.14 relates values of stresses and displacements at two successive interfaces. For the j th layer we write

$$\begin{bmatrix} \tau_j \\ q_j \\ lU_j \\ lV_j \end{bmatrix} = \mathcal{M}_j \begin{bmatrix} \tau_{j+1} \\ q_{j+1} \\ lU_{j+1} \\ lV_{j+1} \end{bmatrix} \quad (7.17)$$

For adhering layers the stresses and displacements are continuous at the interfaces. Hence by a process of matrix multiplications it is possible to express a relation between the variables at the lower and upper boundaries of the layered system. We derive

$$\begin{bmatrix} \tau_1 \\ q_1 \\ lU_1 \\ lV_1 \end{bmatrix} = \prod_{j=1}^n \mathcal{M}_j \begin{bmatrix} \tau_{n+1} \\ q_{n+1} \\ lU_{n+1} \\ lV_{n+1} \end{bmatrix} \quad (7.18)$$

The product $\prod_{j=1}^n \mathcal{M}_j$ of the n matrices \mathcal{M}_j is a 4×4 matrix. If the system of layers is free at top and bottom, we put $\tau_1 = q_1 = \tau_{n+1} = q_{n+1} = 0$, and equations 7.18 reduce to a homogeneous system with two unknowns U_{n+1} and V_{n+1} . A 2×2 system also results if the layers lie on top of a half-space. Then we express τ_{n+1} and q_{n+1} in terms of the displacement U_{n+1} and V_{n+1} at the top of the half-space. If the layers are embedded on top and bottom, we express τ_1 and q_1 in terms of U_1 and V_1 . In this case equations 7.18 constitute a 4×4 system which is immediately reducible to a 2×2 system. In any event the characteristic equation is obtained by equating to zero the determinant of the resulting 2×2 system of equations.

In the numerical procedure the final 2×2 determinant may be plotted against the value of any chosen parameter as, for example, the wavelength. Points where the curve crosses the abscissa are solutions of the characteristic equation. These roots may be evaluated by standard programming techniques.

When using this procedure, it is important to keep in mind that for wavelengths which are sufficiently small relative to the thickness of a particular layer the top and bottom faces become decoupled. In this case the coefficients a_{ij} and b_{ij} tend to become equal, and

singular properties of the \mathcal{M} matrix result. The difficulty is avoided by considering the particular layer to behave as a half-space.

Either the recurrence equations 7.5 or the matrix multiplication process may conveniently be used for the programming with automatic computers when a large number of layers is involved.

A numerical procedure for solving the recurrence equations will be discussed in the last paragraph of this section.

Inclusion of Gravity Forces. We now introduce into the analysis the effect of gravity. It is assumed that the layers lie horizontally in a uniform gravity field of acceleration g and that the density ρ is uniform in each layer.

From the general considerations of Chapter 3 regarding the stability problem in the presence of a gravity field it is readily concluded that its effect is obtained from the gravity-free case by adding elastic forces at the interface which are proportional to the vertical displacement and the density difference between the layers. This may be verified independently as follows. The initial stress in a particular layer may be written

$$\begin{aligned} S_{11} &= -P + \rho gy + C \\ S_{22} &= \rho gy + C \end{aligned} \quad (7.19)$$

where C is a constant, y is the vertical coordinate, ρ is the constant density of the layer, and

$$P = S_{22} - S_{11} \quad (7.20)$$

has a constant value in the layer.

The force components $\Delta'f_x, \Delta'f_y$ defined in section 5 may be evaluated for this case. Applying equation 5.43, we write

$$\begin{aligned} \Delta'f_x &= s_{12} + Pe_{xy} \\ \Delta'f_y &= s_{22} \end{aligned} \quad (7.21)$$

As we have seen, $\Delta'f_x$ and $\Delta'f_y$ represent the normal and tangential incremental stresses on a deformed surface originally horizontal. On the other hand, we have shown in section 8 of Chapter 3 that the stress increments s_{ij} in the heavy medium are immediately derived

when we know the stresses s'_{ij} for the gravity-free case. We may write

$$\begin{aligned} s_{11} &= s'_{11} + \rho g v \\ s_{22} &= s'_{22} + \rho g v \\ s_{12} &= s'_{12} \end{aligned} \quad (7.22)$$

These relations are the same as equations 8.13 of Chapter 3 except that we have changed the sign of the term $\rho g v$ because the orientation of the y axis has been reversed. Substitution of these values of s_{ij} into relations 7.21 yields

$$\begin{aligned} \Delta' f_x &= s'_{12} + P e_{xy} \\ \Delta' f_y &= s'_{22} + \rho g v \end{aligned} \quad (7.23)$$

Therefore, introducing the stresses τ and q evaluated for the gravity-free case, we may write

$$\begin{aligned} \Delta' f_x &= \tau \sin lx \\ \Delta' f_y &= (q + \rho g V) \cos lx \end{aligned} \quad (7.24)$$

If we compare this result with equations 5.44, we see that the effect of gravity is taken into account by introducing the normal stress

$$q = q + \rho g V \quad (7.25)$$

Hence the stresses at the top and bottom of a layer corresponding to equations 5.24 are now

$$\begin{aligned} \tau_1 &= lL \frac{\partial I}{\partial U_1} & \tau_2 &= -lL \frac{\partial I}{\partial U_2} \\ q_1 &= lL \frac{\partial I}{\partial V_1} + \rho g V_1 & q_2 &= -lL \frac{\partial I}{\partial V_2} + \rho g V_2 \end{aligned} \quad (7.26)$$

In applying these results to a multilayered system we denote by ρ_j the density of the j th layer. In each layer the initial stress is represented by the stress difference

$$P_j = S_{22} - S_{11}^{(j)} \quad (7.27)$$

It may vary from one layer to the other but is constant in each layer, whereas the vertical stress component S_{22} varies with depth in piecewise linear segments. Proceeding as for equations 7.3 and 7.4 and

expressing the continuity of the stress τ, q at an interface, we find the recurrence equations

$$\begin{aligned} \frac{\partial}{\partial U_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) &= 0 \\ \frac{\partial}{\partial V_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) + \frac{1}{l} (\rho_{j+1} - \rho_j) g V_{j+1} &= 0 \end{aligned} \quad (7.28)$$

This result shows that the effect of gravity is to add a normal elastic force per unit area at the interface equal to $(\rho_{j+1} - \rho_j)gV_{j+1}$. It is consistent with the general procedure discussed in section 5 of Chapter 3. Note that the boundary condition at a free boundary is written

$$q = q + \rho g V = 0 \quad (7.29)$$

Equations 7.12 may be extended to include gravity by writing

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \quad \frac{\partial}{\partial V_j} (\mathcal{J} + \mathcal{G}) = 0 \quad (7.30)$$

where

$$\mathcal{G} = \frac{1}{2l} \sum_0^n (\rho_{j+1} - \rho_j) g V_{j+1}^2$$

These equations apply to layers embedded between two half-spaces. The densities of the upper and lower half-spaces are denoted by ρ_0 and ρ_{n+1} , respectively.

Again we note that equations 7.30 imply the variational principle

$$\delta(\mathcal{J} + \mathcal{G}) = 0 \quad (7.31)$$

where \mathcal{G} is a quantity proportional to the incremental potential of the gravity forces.

Finally it is possible to extend to this case the process of matrix multiplication represented by equation 7.18. This is accomplished by substituting $q_1 - \rho g V_1$ and $q_2 - \rho g V_2$ for q_1 and q_2 in equation 7.14. The result is

$$\begin{bmatrix} \tau_1 \\ q_1 \\ lU_1 \\ lV_1 \end{bmatrix} = \mathcal{N} \begin{bmatrix} \tau_2 \\ q_2 \\ lU_2 \\ lV_2 \end{bmatrix} \quad (7.32)$$

with the matrix

$$\mathcal{N} = \begin{bmatrix} B_1 & B_2 & LB_5 & LC'_6 \\ C_3 & C_4 & -LC_6 & LC_7 \\ \frac{1}{L} B_8 & \frac{1}{L} B_9 & B_1 & -C'_3 \\ -\frac{1}{L} B_9 & \frac{1}{L} B_{10} & -B_2 & C'_4 \end{bmatrix} \quad (7.33)$$

and the additional coefficients

$$\begin{aligned} C_3 &= B_3 - \frac{\rho g}{lL} B_9 & C'_3 &= B_3 + \frac{\rho g}{lL} B_9 \\ C_4 &= B_4 + \frac{\rho g}{lL} B_{10} & C'_4 &= B_4 - \frac{\rho g}{lL} B_{10} \\ C_6 &= B_6 + \frac{\rho g}{lL} B_2 & C'_6 &= B_6 - \frac{\rho g}{lL} B_2 \\ C_7 &= B_7 - \left(\frac{\rho g}{lL}\right)^2 B_{10} \end{aligned} \quad (7.34)$$

These coefficients are the only ones which contain the gravity parameter $\rho g/lL$.

As before the stresses τ_i and q_i are continuous at the interfaces, and the characteristic equation is obtained by a process of matrix multiplication. We write

$$\begin{bmatrix} \tau_1 \\ q_1 \\ lU_1 \\ lV_1 \end{bmatrix} = \prod_{j=1}^n \mathcal{N}_j \begin{bmatrix} \tau_{n+1} \\ q_{n+1} \\ lU_{n+1} \\ lV_{n+1} \end{bmatrix} \quad (7.35)$$

The matrices \mathcal{N}_j play the same role as \mathcal{M}_j in equation 7.18, and the numerical programming may be carried out in identical fashion.

Perfect adherence between layers was assumed in the preceding analysis. If the possibility of perfect lubrication is introduced, the condition $\tau = 0$ must be verified at the interface. This additional condition makes it possible to eliminate all tangential displacement components U_j from the problem. The procedure was illustrated in section 6 in the example of the single layer. In this case the

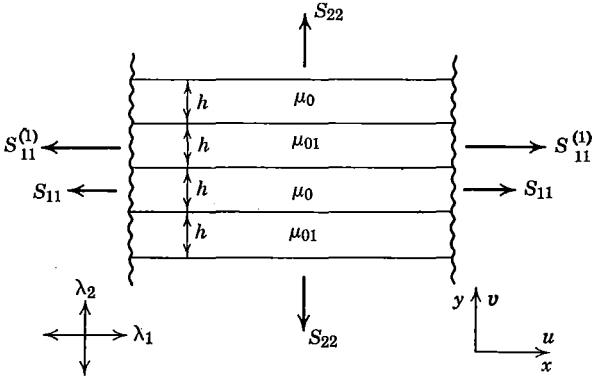


Figure 7.3 Elastic layers of equal thickness h and alternate rigidities μ_0 and μ_{01} under initial stress. The number of layers is infinite.

recurrence equations are reduced to relations between values of the normal displacements V_j at three successive interfaces. Equations 7.14 and 7.32 are replaced by relations which involve only two variables q_j and V_j or q_j and V_j . The matrices corresponding to \mathcal{M} and \mathcal{N} become 2×2 matrices. We shall not spell out the details of the procedure in this case. It is a straightforward application of the methods outlined in the preceding analysis.

Buckling of a Multilayered Periodic Medium. As an example we shall consider a multilayered medium constituted of an infinite number of layers of equal thickness h as shown in Figure 7.3. The effect of gravity is not included. The layers are constituted of two rubber-like materials which alternate and are repeated periodically. The initial stresses in these materials are given by the stress-strain relations (6.34):

$$\begin{aligned} P &= S_{22} - S_{11} = \mu_0(\lambda_2^2 - \lambda_1^2) \\ P_1 &= S_{22} - S_{11}^{(1)} = \mu_{01}(\lambda_2^2 - \lambda_1^2) \end{aligned} \quad (7.36)$$

The initial extension ratios in the plane of the figure are λ_1 and λ_2 . The value λ_1 is measured in the direction of the layers. The materials are assumed to be incompressible, μ_0 and μ_{01} being the elastic coefficients of the two materials.

The buckling is assumed to occur in a mode such that all interfaces

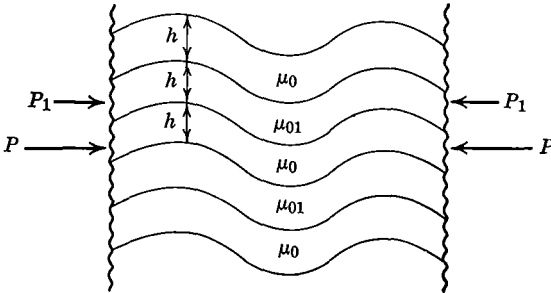


Figure 7.4 Buckling mode of the infinite multilayered system. The “effective” compressions in each layer at buckling are $P = S_{22} - S_{11}$ and $P_1 = S_{22} - S_{11}^{\prime}$.

are deformed with the same normal amplitude as shown in Figure 7.4. The problem has been solved in a recent paper by the author.*

The buckling condition is obtained by expressing the fact that the stresses are continuous at any interface. It is expressed by equations identical in form with equations 6.24. The coefficients a_{ij} represent the layer of rigidity μ_0 . However, the coefficients a'_{ij} in this case are not given by expressions (6.23) but are those of another layer of the same thickness h and rigidity μ_{01} . The values of ζ and γ are the same for both layers, and therefore the coefficients a_{ij} and a'_{ij} are the same. Their values are (see equation 6.41)

$$\begin{aligned}
 a_{11} &= a'_{11} = \frac{1 - k^2}{z_1 - z_2} \\
 a_{12} &= a'_{12} = \frac{2z_2 - (1 + k^2)z_1}{z_1 - z_2} \\
 a_{22} &= a'_{22} = a_{11}z_1z_2
 \end{aligned}
 \tag{7.37}$$

where the variables are defined by equations 6.39 and 6.42.

In addition, relation (6.36) is also valid for this case. Hence

$$\frac{L_1}{L} = \frac{\mu_{01}}{\mu_0} = n
 \tag{7.38}$$

* M. A. Biot, Theory of Stability of Periodic Multilayered Continua, *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 17, Part II, pp. 217-224, 1964.

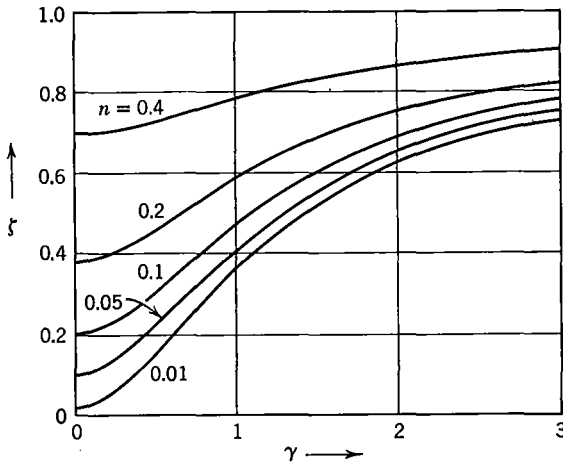


Figure 7.5 Stability diagram for the multilayered medium of Figure 7.4. Values of $\zeta = (\lambda_2^2 - \lambda_1^2)/(\lambda_2^2 + \lambda_1^2)$ as a function of $\gamma = \pi h/\mathcal{L}$ (\mathcal{L} = wavelength) and the rigidity ratio $n = \mu_{01}/\mu_0$.

With the values (7.37) and (7.38) the buckling condition (6.25) takes the remarkably simple form

$$\left(\frac{1-n}{1+n}\right)^2 = \frac{a_{11}a_{22}}{a_{12}^2} \quad (7.39)$$

This equation has been solved numerically, and the stability parameter ζ is plotted in Figure 7.5 as a function of the wavelength variable γ for five values of the rigidity ratio n . We recall that by equations 6.39 and 6.42 the variables in this plot are defined as $\zeta = P/(2Q) = P_1/(2Q_1) = (\lambda_2^2 - \lambda_1^2)/(\lambda_2^2 + \lambda_1^2)$ and $\gamma = \pi h/\mathcal{L}$, where \mathcal{L} is the wavelength.

Two limiting cases are of interest. For $\gamma = \infty$, corresponding to a small wavelength, we put $z_1 = 1$ and $z_2 = k$. Equation 7.39 becomes

$$\left(\frac{1-n}{1+n}\right)^2 = \left(\frac{1+k}{1-k}\right)^2 k \quad (7.40)$$

which is the same as equation 6.51. Hence, at vanishing wavelengths, ζ tends toward asymptotic values given by Table 2 and the buckling degenerates into an interfacial instability.

The other limiting case, obtained by putting $\gamma = 0$, is of particular interest. Here the wavelength is large compared with the thickness of the layers. For this case we put $z_1 = \gamma$ and $z_2 = k^2\gamma$, and equation 7.39 becomes

$$\left(\frac{1-n}{1+n}\right)^2 = k^2 \quad (7.41)$$

or

$$\frac{2}{\zeta} = n + \frac{1}{n} \quad (7.42)$$

This gives the values of ζ on the vertical axis in Figure 7.5. Now we should expect the medium to behave like an anisotropic continuum equivalent to the laminated medium. The buckling in this case should coincide with the internal instability analyzed in section 3. That this is effectively so is easily shown as follows.

The slide moduli of the materials composing the layers are

$$\begin{aligned} L &= Q(1 + \zeta) \\ L_1 &= Q_1(1 + \zeta) \end{aligned} \quad (7.43)$$

where Q and Q_1 have the values (6.35). Applying equation 2.29 for the average slide modulus L_{av} of the composite material, we obtain

$$L_{av} = \frac{2}{\frac{1}{L} + \frac{1}{L_1}} = 2(1 + \zeta) \frac{QQ_1}{Q + Q_1} \quad (7.44)$$

The average value of the effective compressive stress is

$$P_{av} = \frac{1}{2}(P + P_1) = \zeta(Q + Q_1) \quad (7.45)$$

According to the result of section 3 internal instability appears when

$$L_{av} = P_{av} \quad (7.46)$$

Hence

$$2(1 + \zeta)QQ_1 = \zeta(Q + Q_1)^2 \quad (7.47)$$

From the values (6.35) for Q and Q_1 we derive

$$\frac{Q_1}{Q} = \frac{\mu_{01}}{\mu_0} = n \quad (7.48)$$

and equation 7.47 becomes

$$\frac{2}{\zeta} = n + \frac{1}{n} \quad (7.49)$$

This condition is the same as that expressed by the limiting equation 7.42. It shows that the buckling of a multilayered medium is identical with the internal instability of a continuum if the layers are sufficiently thin relative to the wavelength.

For wavelengths gradually decreasing in magnitude the value of ζ increases. This is due to the finite thickness of the laminations and the influence of the bending stiffness of the more rigid layers.

Iteration Process for the Numerical Solution of the Recurrence Equations. The pair of equations (7.5) are linear relations between six variables. These variables are the three tangential displacements U_i, U_{i+1}, U_{i+2} and the three normal displacements V_i, V_{i+1}, V_{i+2} at three successive interfaces. Relations of the same type are also obtained when gravity forces are included as shown by the pair of equations (7.28). In general such recurrence equations are typical of the mechanics of multilayered media and are obtained in a large variety of problems of dynamics and stability treated here and in the following chapters. The recurrence equations provide an alternative method of numerical solution which may be more convenient than the matrix multiplication method outlined above. The numerical procedure is straightforward and elementary and does not need to be spelled out in detail.

Consider for example the pair of recurrence equations (7.5). They may be solved for two of the six variables and written

$$\begin{aligned} U_{i+2} &= \varphi(U_i, V_i, U_{i+1}, V_{i+1}) \\ V_{i+2} &= \psi(U_i, V_i, U_{i+1}, V_{i+1}) \end{aligned} \quad (7.50)$$

The right side represents linear functions of the four displacements at the top face of layers i and $i + 1$. The coefficients in these functions depend on the parameters of these two layers.

We start with the top layer. If the top surface is free the displacements U_1, V_1, U_2, V_2 satisfy the first two of equations (7.6) and we derive U_2, V_2 in terms of U_1, V_1 . If the top surface adheres to a half space we proceed similarly by using the two equations (7.8). Substituting U_1, V_1, U_2, V_2 into equations (7.50) we derive U_3 and V_3 as linear functions of U_1 and V_1 . By repeating this procedure we obtain the displacements $U_n, V_n, U_{n+1}, V_{n+1}$ of the bottom layer as functions of U_1, V_1 . We then substitute these values $U_n, V_n, U_{n+1}, V_{n+1}$ either in equations (7.9) or in the last two of equations (7.6)

depending on the boundary conditions at the lower face. Hence we finally obtain two linear equations for U_1 and V_1 whose determinant must vanish. The value of this determinant and characteristic roots for any parameter may be evaluated numerically by standard programming.

The method is of course quite general and is applicable to equations (7.28) and other recurrence equations of the same type.

CHAPTER FIVE

Dynamics of Elastic Media under Initial Stress

1. INTRODUCTION

Until now we have neglected all dynamical effects, thus restricting the foregoing results to deformations where the inertia forces are not significant.

In this chapter we develop the dynamics of continuous elastic media under initial stress. We deal with vibrations and wave propagation in such media and with problems of dynamic instability.

This general dynamical theory was developed by the author in 1940* and applied to problems of propagation of elastic waves in the presence of initial stress.

The property of elasticity refers only to incremental deformations. The state of initial stress may or may not be associated with any finite strain elasticity and may be due to any other physical cause.

The presence of an initial stress tends to modify the effective rigidity of the medium. In general, a tension tends to increase the rigidity, whereas a compression tends to decrease it. This is reflected in the dynamics of the medium and is associated with a corresponding increase or decrease of frequency of oscillation and wave velocity.

The general equations governing the dynamics of the elastic medium under initial stress and the corresponding variational

* See reference 7 at the end of the Preface.

principles are derived in section 2. In the Lagrangian form with generalized coordinates they are the same as the classical equations for a conservative system perturbed from equilibrium with the difference, however, that beyond a certain range for the initial stress the potential energy is not positive definite. In this case the system becomes unstable and exhibits characteristic values with exponentially increasing amplitudes.

In section 3 the theory is applied to the evaluation of the effect of gravity on Rayleigh waves. Some of the results are discussed in the context of geophysics.

The influence of initial stress on the propagation of body waves is examined in section 4. Some fundamental properties due specifically to the initial stress are brought out. In particular it is shown that the effect of the initial stress cannot be represented by a change in elastic coefficients. Internal instability is shown to correspond to the vanishing of the velocity of propagation of a transverse wave. Complete equations for acoustic propagation in an isotropic elastic medium with first order corrections for the effect of initial stress are readily obtained by combining the dynamical equations with values of the elastic coefficients derived in section 9 of Chapter 2. Special attention is given to the distinction between adiabatic and isothermal deformations and the relation between the corresponding elastic coefficients in the presence of initial stress.

The general theory is applicable to an elastic medium with vanishing rigidity. This leads to the dynamics of a fluid under initial stress or the theory of acoustic-gravity waves developed in section 5. The equations of motion are derived in two different forms referred to as unmodified and modified. The former are obtained immediately by introducing hydrostatic stresses in the equations of the general theory. The latter are obtained by a simple transformation, leading to a form where the terms incorporating the effect of the initial stress may be interpreted as buoyancy forces. For an incompressible fluid, surface and internal gravity waves may be analyzed by using an analog model which is stress-free in the initial state. This provides a conceptually very useful model for the understanding of the properties of gravity waves.

Variational principles for acoustic-gravity waves may also be formulated in corresponding dual form denoted as modified and unmodified principles. They are discussed in section 6. The dual

form corresponds to two different ways of expressing the potential energy of the fluid in terms of the volume change or in terms of buoyancy forces. The formulation by means of generalized coordinates and Lagrangian equations follows immediately from the variational principle. An important feature of the theory which distinguishes it from the classical dynamics of a fluid free of initial stress is the presence of a surface integral in addition to the volume integral in the expression for the total potential energy.

The important problem of wave propagation in elastic plates under initial stress is treated in section 7. In the stability analysis of plates in Chapter 4 we assumed for simplicity that the material is incompressible. In wave propagation problems such an assumption is inadequate, and the problem must be treated for the completely general case of a compressible material. However, it is sufficient to derive the six coefficients a_{ij} and b_{ij} describing the antisymmetric and symmetric deformation of a plate in response to exciting forces applied to the surface. Once these key coefficients are obtained, a large variety of problems are immediately solved by applying exactly the same formulas as in the corresponding stability problem treated in Chapter 4 for incompressible media. The complete solution of the problem of wave propagation in multilayered elastic media under initial stress is included. The solutions apply to phenomena in several categories. By putting the frequency equal to zero they generalize to compressible media theories of internal, surface, and interfacial instability, and the theory of buckling of thick plates, and multilayers. The solutions also represent dynamic instability when the characteristic exponent is real and the amplitudes are proportional to an increasing exponential of time.

2. DYNAMICAL EQUATIONS FOR AN ELASTIC MEDIUM UNDER INITIAL STRESS

From d'Alembert's principle it is well known that static equations may be formally generalized to dynamics by including the inertia forces as part of the body forces. By this procedure the static equations derived in the preceding chapters may be readily extended to dynamics.

Let us go back to the static equilibrium equations 7.29 of Chapter 1. They are

$$\frac{\partial}{\partial x_j} (S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}) + \rho X_i(\xi_i) = 0 \quad (2.1)$$

We should remember the significance of the body force $X_i(\xi_i)$. A particle of the solid initially at point x, y, z is displaced to a point of coordinates

$$\begin{aligned} \xi &= x + u \\ \eta &= y + v \\ \zeta &= z + w \end{aligned} \quad (2.2)$$

In abbreviated notation,

$$\xi_i = x_i + u_i \quad (2.3)$$

The body force per unit mass at the initial point x_i is denoted by $X_i(x_i)$. At the displaced point it becomes $X_i(\xi_i)$.

Assume the medium to be in static equilibrium in the initial state of stress. The following equilibrium equations are satisfied:

$$\frac{\partial S_{ij}}{\partial x_j} + \rho X_i(x_i) = 0 \quad (2.4)$$

If the particle has acquired an acceleration in its displaced position, this acceleration is

$$a_i = \frac{\partial^2 \xi_i}{\partial t^2} = \frac{\partial^2 u_i}{\partial t^2} \quad (2.5)$$

This introduces the time variable t . To apply d'Alembert's principle we must replace $X_i(\xi_i)$ by $X_i(\xi_i) - a_i$ in the static equations (2.1). We derive the dynamical equations

$$\frac{\partial}{\partial x_j} (S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}) + \rho X_i(\xi_i) = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (2.6)$$

By taking into account the initial equilibrium condition (2.4) and putting

$$\Delta X_i = X_i(\xi_i) - X_i(x_i) \quad (2.7)$$

equations 2.6 become

$$\frac{\partial}{\partial x_j} (s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}) + \rho \Delta X_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (2.8)$$

The left side of this equation may be transformed by following exactly the same procedure as in Chapter 1, writing equations 2.8 in the alternative form

$$\begin{aligned} \frac{\partial s_{ij}}{\partial x_j} + \rho \Delta X_i - \rho \omega_{ik} X_k(x_i) - \rho e X_i(x_i) \\ + S_{kj} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S_{ik}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \end{aligned} \quad (2.9)$$

Except for the inertial term on the right side, they are the same as equations 7.42 of Chapter 1.

Boundary conditions for dynamical problems are, of course, not affected and are the same as for static equilibrium.

In two dimensions the dynamical equations (2.9) are written

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \Delta X + \rho \omega Y(x, y) - \rho e X(x, y) \\ - 2S_{12} \frac{\partial \omega}{\partial x} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} \\ - \frac{\partial S_{11}}{\partial x} e_{xx} - \frac{\partial S_{12}}{\partial y} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{12}}{\partial x} \right) e_{xy} = \rho \frac{\partial^2 u}{\partial t^2} \end{aligned} \quad (2.10)$$

$$\begin{aligned} \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho \Delta Y - \rho \omega X(x, y) - \rho e Y(x, y) \\ + 2S_{12} \frac{\partial \omega}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} \\ - \frac{\partial S_{22}}{\partial y} e_{yy} - \frac{\partial S_{12}}{\partial x} e_{xx} - \left(\frac{\partial S_{22}}{\partial x} + \frac{\partial S_{12}}{\partial y} \right) e_{xy} = \rho \frac{\partial^2 v}{\partial t^2} \end{aligned}$$

Variational Principle. A variational principle for the dynamical equations 2.9 is readily obtained as follows.

The previously derived principle for the static case is expressed by equation 5.37 of Chapter 2 and is written

$$\delta \iiint_{\tau} \Delta V \, d\tau = \iiint_{\tau} \Delta X_i \rho \delta u_i \, d\tau + \iint_A \Delta f_i \delta u_i \, dA \quad (2.11)$$

with

$$\Delta V = \frac{1}{2} t_{ij} e_{ij} + \frac{1}{2} S_{ij} (e_{i\mu} \omega_{\mu j} + e_{j\mu} \omega_{\mu i} + \omega_{i\mu} \omega_{j\mu}) \quad (2.12)$$

The triple integrals are extended to the volume τ of the elastic medium, and the surface integral is extended to its boundary A . The incremental boundary force per unit initial area is Δf_i .

By d'Alembert's principle we replace $X_i(\xi_i)$ by $X_i(\xi_i) - a_i$. This is equivalent to replacing ΔX_i by $\Delta X_i - a_i$ in equation 2.11. Hence we write

$$\begin{aligned} \iiint_{\tau} (\delta\Delta V + \rho a_i \delta u_i) d\tau \\ = \iiint_{\tau} \Delta X_{i\rho} \delta u_i d\tau + \iint_A \Delta f_i \delta u_i dA \end{aligned} \tag{2.13}$$

This is the variational principle corresponding to the dynamical equations (2.9). In the variational process the acceleration is considered to be a fixed field whereas the virtual displacements are arbitrary.*

Equation 2.13 expresses the variational principle in its most general form. It may be simplified by considering the particular case where the *externally applied forces are conservative*. These forces include the body force field and the forces applied at the boundary.

Let the body force be derived from a potential U ; that is,

$$X_i = - \frac{\partial U}{\partial x_i} \tag{2.14}$$

By linearizing the incremental body force ΔX_i we write

$$\Delta X_i = \frac{\partial X_i}{\partial x_j} u_j = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \tag{2.15}$$

Hence

$$\Delta X_i \delta u_i = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \delta u_i \tag{2.16}$$

If we put

$$\Delta U = \frac{1}{2} \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \tag{2.17}$$

the variational principle (2.13) becomes

$$\delta \iiint_{\tau} (\Delta V + \rho \Delta U) d\tau + \iint_A \rho a_i \delta u_i d\tau = \iint_A \Delta f_i \delta u_i dA \tag{2.18}$$

* Applications of the variational principle (2.13) to the dynamics of plates have been discussed by G. Herrmann, The Influence of Initial Stress on the Dynamic Behaviour of Elastic and Viscoelastic Plates, *Publications of the International Association for Bridge and Structural Engineering*, Vol. 16, pp. 275-294, Zürich, 1956.

On the left side we recognize the quantity already defined as \mathcal{P}_τ by equation 5.47 of Chapter 2:

$$\mathcal{P}_\tau = \iiint_{\tau} (\Delta V + \rho \Delta U) d\tau \quad (2.19)$$

With this definition the variational equation 2.18 is written

$$\delta\mathcal{P}_\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau = \iint_A \Delta f_i \delta u_i dA \quad (2.20)$$

A further simplification is obtained by considering conservative boundary forces, already discussed in section 4 of Chapter 3. A trivial case is obtained if the boundary is composed of free surfaces and surfaces on which the displacement u_i is zero. Then either Δf_i or δu_i vanishes on the boundary and equation 2.20 is simplified to

$$\delta\mathcal{P}_\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau = 0 \quad (2.21)$$

Another example of conservative boundary forces corresponds to the case where the non-free boundaries are in contact with rigid frictionless surfaces. It was shown in section 4 of Chapter 3 that in this case the surface integral on the right side of equation 2.20 becomes

$$\iint_A \Delta f_i \delta u_i dA = -\delta\mathcal{P}_B \quad (2.22)$$

where \mathcal{P}_B is

$$\mathcal{P}_B = -\frac{1}{2} \iint_B S\phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j dA \quad (2.23)$$

The surface integral is extended to the rigid boundary B . In this expression S is the normal initial stress at the boundary at the initial point x_i . The other factors in the integrand have been defined by equations 4.58 and 4.59 of Chapter 3. The value of \mathcal{P}_B depends essentially on the curvature of the boundary. With the definition

$$\mathcal{P} = \mathcal{P}_\tau + \mathcal{P}_B \quad (2.24)$$

the variational principle (2.20) becomes

$$\delta\mathcal{P} + \iiint_{\tau} \rho a_i \delta u_i d\tau = 0 \quad (2.25)$$

In general, when the boundary forces are conservative it is possible

to write the surface integral (2.22) as an exact differential of a function \mathcal{P}_B of the boundary displacements. The same variational principle (2.25) is therefore applicable to this more general case with a suitable expression for \mathcal{P}_B representing the potential of the boundary forces.

Analog Model for an Incompressible Medium. In section 5 of Chapter 3 it was shown that for an incompressible medium the influence of the body force may usually be taken into account by considering an analog model in which the initial body force field has been replaced by "buoyancy forces" depending on the local particle displacements. This analog model is valid for dynamical problems provided that we add the inertia forces. For example, in equations 5.32 of Chapter 3 we simply add the inertia term $\rho(\partial^2 u_i)/(\partial t^2)$ to the right side of the equations. All equations are readily extended to dynamics by this procedure, and there is no need to rewrite them here. A detailed application of the analog model concept including the corresponding variational principle will be developed in the context of acoustic-gravity waves in a fluid (sections 5 and 6 of this chapter). It is also applied to the problem of Rayleigh waves in the next section.

Generalized Coordinates and Lagrangian Equations. From the results above it is possible to derive equations which are identical in form with those of the classical theory of conservative systems in Lagrangian mechanics. This can be shown by expressing the displacement field in terms of generalized coordinates q_i . We write

$$u_i = u_{ij}(x, y, z)q_j \quad (2.26)$$

where $u_{ij}(x, y, z)$ are fixed configuration fields. The actual field may always be approximated to any desired accuracy by an arbitrarily large number of generalized coordinates. With the notation

$$\frac{d^2 q}{dt^2} = \ddot{q} \quad (2.27)$$

the acceleration field is

$$a_{ik} = u_{ikj}(x, y, z)\ddot{q}_j \quad (2.28)$$

The displacement is varied by applying variations to the generalized coordinates q_j ; that is,

$$\delta u_k = u_{kj}(x, y, z)\delta q_j \quad (2.29)$$

Hence we may write

$$\iiint_{\tau} \rho a_i \delta u_i d\tau = m_{ij} \ddot{q}_j \delta q_i \quad (2.30)$$

where

$$m_{ij} = m_{ji} = \iiint_{\tau} \rho u_{ki} u_{kj} d\tau \quad (2.31)$$

On the other hand, the potential energy \mathcal{P} of equation 2.24 is quadratic and homogeneous in the displacements. Expressed by means of generalized coordinates, it takes the form

$$\mathcal{P} = \frac{1}{2} a_{ij} q_i q_j \quad (2.32)$$

with

$$a_{ij} = a_{ji} \quad (2.33)$$

Substitution of expressions (2.30) and (2.32) into the variational principle (2.25) yields

$$a_{ij} q_j \delta q_i + m_{ij} \ddot{q}_j \delta q_i = 0 \quad (2.34)$$

Since the variations δq_i are arbitrary, equation 2.34 implies

$$a_{ij} q_j + m_{ij} \ddot{q}_j = 0 \quad (2.35)$$

These differential equations govern the time dependence of the generalized coordinates and completely describe the dynamics of the system. They may be formulated in a number of equivalent ways. By introducing the kinetic energy

$$\mathcal{T} = \frac{1}{2} \iiint_{\tau} \rho \dot{u}_i \dot{u}_i d\tau = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \quad (2.36)$$

the differential equations 2.35 may be written in the classical Lagrangian form,

$$\frac{\partial \mathcal{P}}{\partial q_i} + \frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) = 0 \quad (2.37)$$

Another useful formalism is obtained by writing the time derivative as

$$p = \frac{d}{dt} \quad (2.38)$$

We also introduce the quadratic form,

$$T = \frac{1}{2} \iiint_{\tau} \rho u_i u_i d\tau = \frac{1}{2} m_{ij} q_i q_j \quad (2.39)$$

and a corresponding operational expression,

$$\hat{\mathcal{T}} = p^2 T \quad (2.40)$$

Considering p as an algebraic quantity, we write the variational principle

$$\delta(\mathcal{P} + \mathcal{F}) = 0 \quad (2.41)$$

This yields the equations

$$\frac{\partial \mathcal{P}}{\partial q_i} + p^2 \frac{\partial T}{\partial q_i} = 0 \quad (2.42)$$

Since $p^2 = d^2/dt^2$, they are identical with the dynamical equations 2.35.

From the Lagrangian equations 2.37, it also follows that the dynamical system obeys Hamilton's principle expressed in the form

$$\delta \int^t (\mathcal{P} - \mathcal{T}) dt = 0 \quad (2.43)$$

with vanishing variations of q_i at the origin and terminal point of the dynamical path.

Normal Modes of Oscillation and Instability. The variational principle and the corresponding Lagrangian equations lead to a very general formulation of the dynamics of instability and oscillations of elastic systems under initial stress.

Consider the differential equations 2.35 for the generalized coordinates q_i . They admit solutions of the type

$$q_i = q'_i e^{pt} \quad (2.44)$$

where p is an algebraic coefficient to be determined. We substitute these values into equations 2.35. For simplicity we write q_i instead of q'_i , and we obtain

$$a_{ij}q_j + p^2 m_{ij}q_j = 0 \quad (2.45)$$

The characteristic values p^2 are the roots of the equation obtained by putting equal to zero the determinant of the linear homogeneous system (2.45). A fundamental property of equations 2.45 results from the fact that the quadratic form (2.39) is always positive definite,

$$T > 0 \quad (2.46)$$

It is a well-known result that in such a case all the *characteristic roots* p^2 are real. The existence of only real characteristic roots depends essentially on two properties: that the matrices $[a_{ij}]$ and $[m_{ij}]$ are symmetric and that at least one of the associated quadratic forms (in this case T) is positive definite.

Proof that the Characteristic Values p^2 are real. Assume a complex solution

$$q_j = \alpha_j + i\beta_j \quad (2.46a)$$

satisfying the equations

$$a_{kj}q_j + \lambda m_{kj}q_j = 0 \quad (2.46b)$$

We write λ instead of p^2 . These equations are also satisfied by the complex conjugate solution

$$q_j^* = \alpha_j - i\beta_j \quad (2.46c)$$

Hence we have the equations

$$a_{kj}q_j^* + \lambda^* m_{kj}q_j^* = 0 \quad (2.46d)$$

where λ^* is the complex conjugate of λ . Multiply equations 2.46b by q_k^* and 2.46d by q_k and subtract the two results. Taking into account that $a_{kj} = a_{jk}$ and $m_{kj} = m_{jk}$ we obtain

$$(\lambda - \lambda^*)m_{kj}q_kq_j^* = 0 \quad (2.46e)$$

Since $m_{kj} = m_{jk}$, the quadratic form in this expression is

$$m_{kj}q_kq_j^* = m_{kj}(\alpha_k\alpha_j + \beta_k\beta_j) \quad (2.46f)$$

Because $T > 0$ it follows that

$$m_{kj}(\alpha_k\alpha_j + \beta_k\beta_j) > 0 \quad (2.46g)$$

Hence equation 2.46e implies

$$\lambda = \lambda^* \quad (2.46h)$$

and λ must be real.

Consider a characteristic root p^2 and the corresponding amplitudes q_j . Because equations 2.45 are homogeneous, these amplitudes contain a common factor which may be chosen arbitrarily. Multiplying equations 2.45 by q_i , we obtain

$$a_{ij}q_jq_i + p^2m_{ij}q_jq_i = 0 \quad (2.47)$$

Hence

$$p^2 = -\frac{\mathcal{P}}{T} \quad (2.48)$$

The root p^2 is positive or negative depending on the sign of \mathcal{P} . Positive roots p^2 yield a real positive value for p . This requires

$$\mathcal{P} < 0 \quad (2.49)$$

As already pointed out, this inequality corresponds to elastic instability. The solution is a *buckling mode*, and all amplitudes are proportional to a real and increasing exponential of time $\exp(pt)$. For a root p^2 which is negative we put

$$p = i\alpha \quad (2.50)$$

The solution yields a *natural mode of oscillation* where all amplitudes are in phase and proportional to the factor $\exp(iat)$. For such a solution, equation 2.48 requires the condition

$$\mathcal{P} > 0 \quad (2.51)$$

Hence for an amplitude distribution represented by an oscillatory mode the potential \mathcal{P} is positive.

In general, the elastic system will contain an infinite sequence of natural modes, some of which are unstable and some oscillatory.*

The properties of these modes which have been derived in terms of generalized coordinates may of course be expressed in terms of the displacement field of the continuum. Consider for instance a particular mode of displacement field u_i and its characteristic root p . All amplitudes are proportional to a factor $\exp(pt)$ and satisfy the boundary conditions. The acceleration is

$$a_i = p^2 u_i \quad (2.52)$$

In the variational principle (2.25) the exponential time factor cancels out. We may write it in terms of a time-independent displacement field u_i which represents the amplitude distribution of the medium in a particular mode:

$$\delta\mathcal{P} + p^2 \iiint_{\tau} \rho u_i \delta u_i d\tau = 0 \quad (2.53)$$

Hence, by defining T as

$$T = \frac{1}{2} \iiint_{\tau} \rho u_i u_i d\tau \quad (2.54)$$

the variational principle is written

$$\delta\mathcal{P} + p^2 \delta T = 0 \quad (2.55)$$

The variational principle (2.55) also can be formulated in an equivalent form as follows. Consider the bound extremum condition that the variation of \mathcal{P} vanishes; that is,

$$\delta\mathcal{P} = 0 \quad (2.56)$$

* For a discussion of the mathematical validity of the stability criterion (2.51) the reader is referred to the remarks in the last paragraph of section 4 in Chapter 3.

for amplitudes u_i which in addition to obeying the boundary condition also satisfy the constraint

$$T = \text{const.} \quad (2.57)$$

It is well known that the solution of this variational problem is obtained by writing the variational equation

$$\delta\mathcal{P} + \lambda\delta T = 0 \quad (2.58)$$

with a Lagrangian multiplier λ and with variations δu_i which are free of the constraint (2.57). This result is identical with the variational principle (2.55) where p^2 plays the role of a Lagrangian multiplier.

Equivalence of Group Velocity and Energy Flux. The variational principle in the form (2.56) has been used by the author to prove a fundamental theorem in wave-guide propagation. This refers to a sinusoidal wave train propagating in an elastic medium. The properties of the medium are independent of x , which represents the direction of propagation of a particular mode of the wave-guide. The state of initial stress is also independent of x . The medium may be non-homogeneous along the y and z directions. The amplitude field of the wave may be written in the form $V_i(y, z) \exp(ikx - i\omega t)$, and the group velocity is defined as da/dl . It was shown by the author* that this group velocity is equal to the energy flux across a plane perpendicular to the direction x of propagation. This property was derived for an elastic system in the most general case of anisotropy whether or not in an initial state of stress. It applies to a large variety of phenomena such as Rayleigh and Stoneley waves, waves in plates, gravity waves, and waves in layered and non-homogeneous media.

3. THE INFLUENCE OF GRAVITY ON RAYLEIGH WAVES

The dynamical equations of the preceding section will now be applied to surface waves in an elastic medium. It was shown by Rayleigh that elastic waves may propagate along the surface of elastic bodies. The amplitude of such waves decreases exponentially with depth. The simplest example is given by waves propagating along the surface of an elastic half-space with isotropic elasticity. It is clear that the weight of the solid must influence the propagation.

* M. A. Biot, General Theorems on the Equivalence of Group Velocity and Energy Transport, *The Physical Review*, Vol. 105, No. 4, pp. 1129–1137, 1957. (The paper also discusses electromagnetic waves.)

The effect must depend on the relative magnitude of the gravity force and the rigidity. In fact, we can imagine cases where the materials considered have decreasing rigidity. In the limit we reach the case of a liquid in which the rigidity is zero and the surface waves such as ocean waves are due entirely to gravity. Rayleigh did not consider the influence of gravity on surface waves. The theory of this effect was first developed by Bromwich.*

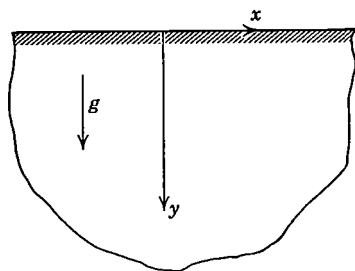


Figure 3.1 Elastic half-space subject to a gravity field.

Application of the general dynamical equations 2.9 to this problem was described by the author in a paper published in 1940.† The resulting equations were shown to be equivalent to those of Bromwich. The problem is treated in two dimensions. The surface of the half-space is located at $y = 0$, and the y axis is directed vertically downward. The components of the body force are therefore (Fig. 3.1)

$$\begin{aligned} X &= 0 \\ Y &= g \end{aligned} \quad (3.1)$$

where g is the acceleration of gravity. We shall assume that the initial stress due to gravity is *hydrostatic*. In practice this is a good assumption. The main interest of this problem lies in its geophysical applications. The initial stress in this case is produced by a slow process of creep where all shear stresses tend to become small or vanish after long periods of time. The state of initial stress is

$$\begin{aligned} S_{11} &= S_{22} = S \\ S_{12} &= 0 \end{aligned} \quad (3.2)$$

* T. J. I'A. Bromwich, On the Influence of Gravity on Elastic Waves and in Particular on the Vibrations of an Elastic Globe, *Proceedings of the London Mathematical Society*, Vol. 30, pp. 98–120, 1898.

† See reference 7 at the end of the Preface.

The stress S is a function of the depth.

The material is also assumed to be homogeneous of mass density ρ . We shall furthermore restrict the analysis by assuming that the material is incompressible. The latter assumption simplifies the treatment, but it does not alter the nature of the conclusions.

Equations 2.4 expressing the equilibrium conditions of the initial stress become

$$\begin{aligned}\frac{\partial S}{\partial x} &= 0 \\ \frac{\partial S}{\partial y} + \rho g &= 0\end{aligned}\tag{3.3}$$

Taking into account relations (3.2) and (3.3) as well as the condition of incompressibility ($e = 0$), we find that the dynamical equations 2.10 for the two-dimensional problem reduce to

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho \omega g - \frac{\partial S}{\partial y} e_{xy} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \frac{\partial S}{\partial y} e_{yy} &= \rho \frac{\partial^2 v}{\partial t^2}\end{aligned}\tag{3.4}$$

From equations 3.3 we may substitute

$$\rho g = - \frac{\partial S}{\partial y}\tag{3.5}$$

Furthermore we have the identities

$$\begin{aligned}\omega + e_{xy} &= \frac{\partial v}{\partial x} \\ e_{yy} &= \frac{\partial v}{\partial y}\end{aligned}\tag{3.6}$$

Hence equations 3.4 become

$$\begin{aligned}\frac{\partial}{\partial x} (s_{11} + \rho g v) + \frac{\partial s_{12}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial}{\partial y} (s_{22} + \rho g v) &= \rho \frac{\partial^2 v}{\partial t^2}\end{aligned}\tag{3.7}$$

Additional equations are furnished by the stress-strain relations. As pointed out in section 4 of Chapter 2, when the initial stress is hydrostatic, the incremental stress-strain relations are formally the same as in a medium initially stress-free. In the present case the

medium is also assumed to be isotropic and incompressible. Hence the stress-strain relations are

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx} \\ s_{22} - s &= 2\mu e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned} \tag{3.8}$$

where μ is the shear modulus and $s = \frac{1}{2}(s_{11} + s_{22})$.

Finally we must consider the boundary conditions and express the fact that the surface at $y = 0$ is unstressed. We go back to equations 6.26 of Chapter 1 for the force components per unit initial area at the boundary. Expressing that these boundary forces vanish at $y = 0$, we derive the two conditions

$$\begin{aligned} S_{12} + s_{12} - S_{22}\omega - S_{11}e_{xy} + S_{12}e_{xx} &= 0 \\ S_{22} + s_{22} + S_{12}\omega - S_{12}e_{xy} + S_{22}e_{xx} &= 0 \end{aligned} \tag{3.9}$$

Since the initial stress due to gravity also vanishes at the boundary, we may put

$$S_{11} = S_{22} = S_{12} = 0 \tag{3.10}$$

Then the boundary conditions at $y = 0$ reduce to

$$s_{12} = s_{22} = 0 \tag{3.11}$$

We must solve equations 3.7 after substituting the values of the stresses (3.8). The two equations thus obtained contain three unknowns u, v, s , and the additional equation is furnished by the condition of incompressibility $e = 0$.

The problem may be transformed into an equivalent one by introducing the fictitious stresses

$$\begin{aligned} s_{11} + \rho gv &= s'_{11} \\ s_{22} + \rho gv &= s'_{22} \\ s_{12} &= s'_{12} \end{aligned} \tag{3.12}$$

The dynamical equations 3.7 become

$$\begin{aligned} \frac{\partial s'_{11}}{\partial x} + \frac{\partial s'_{12}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s'_{12}}{\partial x} + \frac{\partial s'_{22}}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \tag{3.13}$$

and the stress-strain relations (3.8) are now written

$$\begin{aligned} s'_{11} - s' &= 2\mu e_{xx} \\ s'_{22} - s' &= 2\mu e_{yy} \\ s'_{12} &= 2\mu e_{xy} \end{aligned} \tag{3.14}$$

with

$$s' = \frac{1}{2}(s'_{11} + s'_{22}) = s + \rho g v \tag{3.15}$$

The boundary conditions (3.11) become

$$\begin{aligned} s'_{22} &= \rho g v \\ s'_{12} &= 0 \end{aligned} \tag{3.16}$$

The interesting point about this transformation is that the alternative equations 3.13 and 3.14 are the same as the classical equations for an initially unstressed medium. The only difference lies in the boundary condition (3.16). In the transformed problem the boundary is no longer stress-free, and we must introduce a normal force at the surface proportional to the deflection.

The reader will recognize here the same fictitious stress introduced in section 5 of Chapter 3 and discussed in the general context of stability problems. For an incompressible medium we have shown that it leads to an analog model. The same transformation was also applied to the problem of stability of a non-homogeneous elastic half-space treated in section 8 of Chapter 3. The general validity of the analog model for dynamical phenomena was discussed in the preceding section.

This alternative formulation with the fictitious stresses possesses an intuitive character. The boundary force represents a buoyancy effect. It is interesting to note that this intuitive form results rigorously from the general theory. The stress s' may be interpreted physically as the hydrostatic stress increment due to the waves at a fixed point x, y , while s is the hydrostatic stress increment at a point originally of coordinates x, y but displaced with the material to the point of coordinates $x + u, y + v$.

The dynamical equations 3.13 and the boundary conditions (3.16) coincide with those used by Bromwich. In order to solve the

dynamical equations we substitute the values (3.14) of the stress into equations 3.13, and we obtain

$$\begin{aligned}\mu \nabla^2 u + \frac{\partial s'}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \mu \nabla^2 v + \frac{\partial s'}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}\end{aligned}\tag{3.17}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\tag{3.18}$$

In deriving these equations we have taken into account the condition of incompressibility, namely,

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\tag{3.19}$$

Since we are interested in surface waves, we must consider solutions which vanish exponentially with depth. A solution of this type which also satisfies the condition (3.19) for incompressibility is

$$\begin{aligned}u &= (A l e^{-ly} + C r e^{-ry}) \sin (lx - \alpha t) \\ v &= (A l e^{-ly} + C l e^{-ry}) \cos (lx - \alpha t) \\ s' &= A \alpha^2 \rho e^{-ly} \cos (lx - \alpha t)\end{aligned}\tag{3.20}$$

with

$$\begin{aligned}r &= l \sqrt{1 - \beta^2} \\ \beta^2 &= \frac{\alpha^2 \rho}{l^2 \mu}\end{aligned}\tag{3.21}$$

The constants A and C remain to be determined from the boundary conditions (3.16). These boundary conditions may also be written

$$\begin{aligned}s' + 2\mu \frac{\partial v}{\partial y} &= \rho g v \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0\end{aligned}\tag{3.22}$$

Introducing the solution (3.20) into these conditions, we obtain

$$\begin{aligned}(2\mu l^2 - \alpha^2 \rho + \rho g l)A + (2\mu l r + \rho g l)C &= 0 \\ 2l^2 A + (l^2 + r^2)C &= 0\end{aligned}\tag{3.23}$$

Elimination of A and C yields the characteristic equation

$$(2 - \xi^2)^2 - \frac{\rho g \xi^2}{\mu l} = 4\sqrt{1 - \xi^2} \quad (3.24)$$

When equations 3.23 and 3.24 are satisfied, the solution (3.20) represents a surface wave. The phase velocity v_R of the surface wave is obtained from the condition

$$lx - \alpha t = \text{const.} \quad (3.25)$$

or

$$v_R = \frac{dx}{dt} = \frac{\alpha}{l}$$

On the other hand, the velocity of the "shear wave" is known to be

$$v_s = \sqrt{\frac{\mu}{\rho}} \quad (3.26)$$

This is the velocity of propagation of pure transverse waves, also called rotational waves. The dimensionless parameter ξ turns out to be the ratio of the surface wave velocity to that of the shear wave:

$$\xi = \frac{v_R}{v_s} \quad (3.27)$$

Equation 3.24 yields the velocity of the surface wave as a function of the dimensionless parameter $\rho g/\mu l$. This parameter represents the influence of gravity.

The condition that the wavelength be vanishingly small is $l = \infty$. It is identical with the case of zero gravity, $g = 0$, when equation 3.24 becomes

$$(2 - \xi^2)^2 = 4\sqrt{1 - \xi^2} \quad (3.28)$$

It has a real root

$$\xi = 0.955 \quad (3.29)$$

The corresponding surface wave velocity is

$$v_R = 0.955v_s \quad (3.30)$$

which is the value found by Rayleigh for an incompressible material. Let us now go to the other extreme and consider a material of zero rigidity; then we are dealing with a fluid. This corresponds to

putting $\mu = 0$ in equation 3.24. To evaluate this limiting case we first multiply the equation by μ^2 . The value of v_R derived from this equation becomes

$$v_f = \frac{\alpha}{l} = \sqrt{\frac{g}{l}} \tag{3.31}$$

In terms of the wavelength

$$\mathcal{L} = \frac{2\pi}{l} \tag{3.32}$$

we may write

$$v_f = \sqrt{\frac{g\mathcal{L}}{2\pi}} \tag{3.33}$$

This is the well-known velocity of pure gravity surface waves in an incompressible fluid. We derive a useful physical interpretation of the gravity parameter from the relation

$$\frac{\rho g}{\mu l} = \left(\frac{v_f}{v_s}\right)^2 \tag{3.34}$$

It turns out to be the squared ratio of pure gravity to shear wave velocity of the same wavelength \mathcal{L} .

We may solve equation 3.24 for \mathfrak{z} as a function of $\rho g/\mu l$. The result is plotted in Figure 3.2. Starting with $\mathcal{L} = 0$ and the corresponding Rayleigh wave velocity (equation 3.30), we see that the velocity increases with the wavelength until we reach the point of abscissa

$$\frac{\rho g}{\mu l} = \left(\frac{v_f}{v_s}\right)^2 = 1 \tag{3.35}$$

and ordinate

$$\mathfrak{z} = \frac{v}{v_s} = 1 \tag{3.36}$$

Beyond this point the root \mathfrak{z} becomes complex.

The physical significance of this result is easily understood. The effect of gravity is to increase the surface wave velocity to the point where it exceeds the shear wave velocity. When this occurs, the surface wave cannot propagate unattenuated because it will generate shear waves at the surface which will radiate downward and dissipate the energy of the surface wave. We may visualize such an attenuated

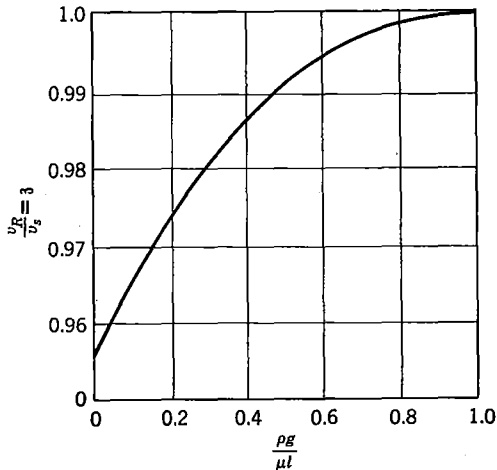


Figure 3.2 The influence of gravity on Rayleigh wave velocity.

surface wave in a jelly-type material. In such a material the shear wave velocity is very low. It is easily exceeded by the gravity wave velocity, and attenuation will take place. On the other hand, if we decrease the rigidity to the limiting value $\mu = 0$ we are dealing with a liquid. Then the surface wave becomes a pure gravity wave with no attenuation. A more detailed analytical discussion of these phenomena appears in the paper cited.*

It is of interest to evaluate the order of magnitude of the gravity correction for seismic waves. The correction depends entirely on the ratio v_f/v_s and becomes significant when its value approaches unity. Usual seismic shear wave velocities are of the order of $v_s = 4$ kilometers per second. Applying equation 3.33, we find that gravity wave velocities v_f of this magnitude require wavelengths of about 10,000 kilometers. For such wavelengths the gravity correction is important. However, at such large wavelengths the waves correspond to modes of oscillations of the earth, and a correct theory should take into account the spherical geometry and the variation of the gravity field due to the deformation itself.

For usual seismic waves of shorter wavelengths the gravity correction will be small. An exception to this must be made for propagation in materials of very low rigidity such as mud where the

* See reference 7 at the end of the Preface.

gravity wave and shear wave velocities may be of comparable magnitudes for observed seismic frequencies.

4. SOME FUNDAMENTAL PROPERTIES OF ACOUSTIC PROPAGATION UNDER INITIAL STRESS

We shall consider a homogeneous medium in a state of initial stress. The material is either isotropic in finite strain or anisotropic with orthotropic symmetry. The principal directions of initial stress are chosen to coincide with the directions of elastic symmetry and the coordinate axes. The state of initial stress is therefore defined by principal components S_{11} , S_{22} , S_{33} .

Let us restrict the analysis to plane strain disturbances with displacements in the x, y plane. In this case the third principal stress S_{33} perpendicular to the x, y plane does not enter explicitly into the propagation equations. Its influence is included indirectly in the density and in the values of the incremental elastic coefficients which appear in the two-dimensional stress-strain relations.

We shall also assume the initial stress to be homogeneous. Hence there is no body force and the dynamical equations 2.10 for plane strain propagation become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} &= \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \quad (4.1)$$

The two-dimensional stress-strain relations for orthotropic symmetry are given by equations 2.4 of Chapter 3:

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (4.2)$$

Existence of an elastic strain energy requires that these coefficients satisfy relations (6.2) of Chapter 2. In the present case these relations reduce to

$$B_{12} - B_{21} = S_{22} - S_{11} \quad (4.3)$$

In order to simplify the writing, we put

$$S_{22} - S_{11} = P \quad (4.4)$$

Substituting the stresses (4.2) into the dynamical equations 4.1, we derive

$$\begin{aligned} B_{11} \frac{\partial^2 u}{\partial x^2} + (B_{12} + Q - \frac{1}{2}P) \frac{\partial^2 v}{\partial x \partial y} + (Q + \frac{1}{2}P) \frac{\partial^2 u}{\partial y^2} &= \rho \frac{\partial^2 u}{\partial t^2} \\ B_{22} \frac{\partial^2 v}{\partial y^2} + (B_{21} + Q + \frac{1}{2}P) \frac{\partial^2 u}{\partial x \partial y} + (Q - \frac{1}{2}P) \frac{\partial^2 v}{\partial x^2} &= \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \quad (4.5)$$

Let us now consider plane waves propagating in the medium. The analysis of a plane longitudinal wave propagating in the x or y direction does not disclose any behavior which is essentially different from that of an unstressed medium. For example, let us consider a longitudinal wave propagating in the x direction,

$$u = u_0 \cos(lx - at) \quad (4.6)$$

Equations 4.5 become

$$B_{11} \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \quad (4.7)$$

Similarly a longitudinal wave propagating in the y direction is governed by the equation

$$B_{22} \frac{\partial^2 v}{\partial y^2} = \rho \frac{\partial^2 v}{\partial t^2} \quad (4.8)$$

These equations have the same form as those for the unstressed medium. The initial stress influences the propagation only through its effect on the magnitudes of the elastic coefficients B_{11} , B_{22} and the density ρ .

Quite different conclusions are obtained when we consider transverse waves. Such a wave propagating in the x direction is represented by

$$\begin{aligned} u &= 0 \\ v &= v_0 \cos(lx - at) \end{aligned} \quad (4.9)$$

This satisfies identically the first of equations 4.5, while the second equation reduces to

$$(Q - \frac{1}{2}P) \frac{\partial^2 v}{\partial x^2} = \rho \frac{\partial^2 v}{\partial t^2} \quad (4.10)$$

The velocity of this wave is obtained by substituting expression (4.9) into equation 4.10 and deriving the value of α/l . The velocity $V_x = \alpha/l$ is given by

$$V_x^2 = \frac{Q - \frac{1}{2}P}{\rho} \quad (4.11)$$

The same calculation may be carried out for the transverse wave

$$\begin{aligned} u &= u_0 \cos (ly - \alpha t) \\ v &= 0 \end{aligned} \quad (4.12)$$

propagating in the y direction. The velocity V_y of this wave is given by

$$V_y^2 = \frac{Q + \frac{1}{2}P}{\rho} \quad (4.13)$$

Comparing equations 4.11 and 4.13, we note that the velocities V_x and V_y are different although the elastic coefficient involved is the same in both expressions. We derive the relation

$$V_y^2 - V_x^2 = \frac{P}{\rho} = \frac{S_{22} - S_{11}}{\rho} \quad (4.14)$$

This equation expresses a property of the wave propagation which is *independent of the elastic coefficients and depends only on the initial stress and the density*. Our results also bring out the important fact that acoustic propagation under initial stress is fundamentally different from the stress-free case and cannot be represented by simply introducing into the classical theory stress-dependent elastic coefficients. These properties were first derived and discussed by the author in a paper published in 1940.*

When $S_{22} = 0$, positive values of P represent a compression in the x direction. As P increases, the velocity V_x given by equation 4.11 will usually decrease and will vanish for

$$Q - \frac{1}{2}P = L - P = 0 \quad (4.15)$$

This is the condition for internal buckling discussed in Chapter 4. When the compression P approaches the value of the slide modulus L , the effective transverse rigidity of the medium tends to vanish, the velocity drops to zero when $P = L$, and internal buckling is incipient.

For a medium which is isotropic in finite strain we have shown that the coefficient Q may be expressed in terms of the initial stress and the extension ratios along the principal directions of initial strain. We denote by λ_1 and λ_2 the finite extension ratios in the x and y

* See reference 7 at the end of the Preface.

directions in the state of initial stress; then application of equation 7.15 of Chapter 2 yields

$$Q = \frac{1}{2}P \frac{\lambda_2^2 + \lambda_1^2}{\lambda_2^2 - \lambda_1^2} \quad (4.16)$$

With this value of Q the velocities V_x and V_y are given by

$$\begin{aligned} V_x^2 &= \frac{P}{\rho} \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} \\ V_y^2 &= \frac{P}{\rho} \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2} \end{aligned} \quad (4.17)$$

In this case the velocities along the x and y directions cannot be made to vanish. This statement is in accordance with the property, discussed in Chapter 4, that the condition $P = L$ which corresponds to an internal instability of the first kind cannot occur in an isotropic medium.

Another point of interest is brought out as a consequence of equations 4.17. We derive

$$\frac{V_x}{\lambda_1} = \frac{V_y}{\lambda_2} \quad (4.18)$$

This result is interpreted as follows. Consider a pair of points on the x axis and another pair on the y axis. Assume these points to be equidistant in the stress-free medium. Equation 4.18 means that after deformation the transit time of the shear waves between the two points on the x axis remains equal to the transit time between the two points on the y axis.

A remark should be made here with respect to internal instability of the second kind. This type of instability may occur in an isotropic medium of a special nature, as already discussed in Chapter 4. In this case we have shown that unstable characteristic solutions appear as slip lines which do not lie along principal directions. Hence vanishing of wave velocity in directions other than those of the principal stresses may be expected. However, we shall not pursue the analysis any further.

Acoustic Propagation in Second Order Elasticity. We shall include here some brief remarks regarding the theory of acoustic waves

in second order elasticity. The extension ratios in the principal directions of initial strain are written

$$\begin{aligned}\lambda_1 &= 1 + \varepsilon_{11} \\ \lambda_2 &= 1 + \varepsilon_{22} \\ \lambda_3 &= 1 + \varepsilon_{33}\end{aligned}\tag{4.19}$$

where ε_{11} , ε_{22} , and ε_{33} are small quantities. In an isotropic medium the corresponding principal stresses including only first and second order terms in ε_{11} , ε_{22} , and ε_{33} are expressed by equations 9.6 in Chapter 2. The relationship involves five elastic coefficients λ , μ , D , F , G . For a medium initially stress-free the elastic coefficients are reduced to λ and μ . Under initial stress the medium becomes orthotropic. The modified elastic coefficients were derived in section 9 of Chapter 2 by adding to λ and μ correction terms of the first order in the initial strain. For example, Q is given by

$$\begin{aligned}2Q &= 2\mu + (\mu + \lambda + D - F)(\varepsilon_{11} + \varepsilon_{22}) \\ &\quad + (\lambda - 2\mu + 2F - G)\varepsilon_{33}\end{aligned}\tag{4.20}$$

If ρ_0 is the density in the stress-free state, its value in the state of initial stress is

$$\rho = \frac{\rho_0}{1 + \varepsilon}\tag{4.21}$$

with

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\tag{4.22}$$

Finally from equations 9.6 of Chapter 2 we derive to the first order

$$P = S_{22} - S_{11} = 2\mu(\varepsilon_{22} - \varepsilon_{11})\tag{4.23}$$

Substitution of the values (4.20), (4.21), and (4.23) into equation 4.11 yields the first order correction of the transverse wave velocity due to the state of initial stress.

The other elastic coefficients B_{11} , B_{12} , etc., are given by equations 9.12 and 9.13 of Chapter 2. For example,

$$\begin{aligned}B_{11} &= (2\mu + \lambda)(1 + \varepsilon_{11} - \varepsilon_{22} - \varepsilon_{33}) \\ &\quad + 2D\varepsilon_{11} + 2F(\varepsilon_{22} + \varepsilon_{33})\end{aligned}\tag{4.24}$$

$$B_{12} = \lambda(1 - \varepsilon_{33}) + 2F(\varepsilon_{11} + \varepsilon_{22}) + G\varepsilon_{33} - S_{11}$$

etc.

General equations for acoustic propagation, with the first order correction due to the initial stress in an isotropic elastic medium, are obtained by substituting the values (4.20), (4.21), (4.23), and (4.24) into the dynamical equations 4.5. Although the procedure is described here for the particular case of plane strain in order to simplify the writing, the three-dimensional equations are similarly obtained without difficulty.

Because the state of initial stress usually corresponds to an isothermal deformation, the incremental coefficients expressed by equations 4.24 will also correspond to isothermal deformations. However, acoustic propagation is generally associated with adiabatic deformations, and a more complete treatment requires the introduction of adiabatic coefficients. They are obtained by adding to expressions (4.24) small correction terms which will now be evaluated.

Relation between Adiabatic and Isothermal Coefficients.

The procedure is best explained by starting with the discussion of a non-isotropic material and without the restrictive assumption that the initial strain is small.

Consider an orthotropic elastic medium in a state of initial stress at an absolute temperature T_r . The principal initial stresses are assumed to be oriented along the axes of elastic symmetry which are also chosen as coordinate axes x, y, z . For simplicity and without loss of generality, we assume incremental deformation in the x, y plane only. The incremental stresses for isothermal deformations are given by equations 4.2. If we increase the absolute temperature by a small amount θ , it becomes $T = T_r + \theta$ and the incremental stresses may be written

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} - \beta_1\theta \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} - \beta_2\theta \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (4.25)$$

Because of the elastic symmetry the temperature does not affect the shear stress component s_{12} . A fourth equation relates the entropy to the strain and the temperature. The increment of entropy per unit initial volume is written in linearized form

$$\bar{s} = \beta'_1 e_{xx} + \beta'_2 e_{yy} + \beta\theta \quad (4.26)$$

The coefficients in this expression are derived from thermodynamic

considerations. We denote by c the heat capacity per unit volume for zero incremental strain. Hence when we put

$$e_{xx} = e_{yy} = 0 \quad (4.27)$$

equation 4.26 becomes

$$\bar{s} = \frac{c\theta}{T_r} = \beta\theta \quad (4.28)$$

Therefore

$$\beta = \frac{c}{T_r} \quad (4.29)$$

The other coefficients are obtained by using the following relations derived from classical thermodynamics

$$\begin{aligned} \frac{\partial s_{11}}{\partial \theta} &= - \frac{\partial \bar{s}}{\partial e_{xx}} \\ \frac{\partial s_{22}}{\partial \theta} &= - \frac{\partial \bar{s}}{\partial e_{yy}} \end{aligned} \quad (4.30)$$

Hence

$$\beta'_1 = \beta_1 \quad \beta'_2 = \beta_2 \quad (4.31)$$

and the increment of entropy (4.26) becomes

$$\bar{s} = \beta_1 e_{xx} + \beta_2 e_{yy} + \frac{c}{T_r} \theta \quad (4.32)$$

Relations (4.30) may be derived as follows. We assume a deformation without shear ($e_{xy} = 0$). For an element of unit volume in the initial state of stress, conservation of energy is expressed by the equation

$$dU = t_{11} de_{xx} + t_{22} de_{yy} + dh \quad (4.32a)$$

In this equation U is the internal energy and h the heat absorbed, and t_{11} and t_{22} are written

$$\begin{aligned} t_{11} &= s_{11} + S_{11}e_{yy} \\ t_{22} &= s_{22} + S_{22}e_{xx} \end{aligned} \quad (4.32b)$$

Equations 4.32b represent the forces per unit initial area defined previously by equations 2.14 of Chapter 2. The entropy \bar{s} is also written:

$$d\bar{s} = \frac{dh}{T} \quad (4.32c)$$

Hence

$$dU = t_{11} de_{xx} + t_{22} de_{yy} + T d\bar{s} \quad (4.32d)$$

The differential of the free energy is

$$d(U - \bar{s}T) = t_{11} de_{xx} + t_{22} de_{yy} - \bar{s} dT \quad (4.32e)$$

Since this is an exact differential, it implies the conditions

$$\begin{aligned}\frac{\partial t_{11}}{\partial T} &= -\frac{\partial \bar{s}}{\partial e_{xx}} \\ \frac{\partial t_{22}}{\partial T} &= -\frac{\partial \bar{s}}{\partial e_{yy}}\end{aligned}\tag{4.32f}$$

By taking into account relations (4.32b) and the definition $T = T_r + \theta$ we derive equations 4.30.

For adiabatic deformations the increment of entropy is zero. Putting $\bar{s} = 0$ in equation 4.32, we obtain the temperature as a function of the strain

$$\theta = -\beta_1 \frac{T_r}{c} e_{xx} - \beta_2 \frac{T_r}{c} e_{yy}\tag{4.33}$$

Substitution of this temperature into equations 4.25 yields the adiabatic stress-strain relations

$$\begin{aligned}s_{11} &= \bar{B}_{11} e_{xx} + \bar{B}_{12} e_{yy} \\ s_{22} &= \bar{B}_{21} e_{xx} + \bar{B}_{22} e_{yy}\end{aligned}\tag{4.34}$$

The incremental adiabatic coefficients are

$$\begin{aligned}\bar{B}_{11} &= B_{11} + \beta_1^2 \frac{T_r}{c} \\ \bar{B}_{12} &= B_{12} + \beta_1 \beta_2 \frac{T_r}{c} \\ \bar{B}_{21} &= B_{21} + \beta_1 \beta_2 \frac{T_r}{c} \\ \bar{B}_{22} &= B_{22} + \beta_2^2 \frac{T_r}{c}\end{aligned}\tag{4.35}$$

The coefficient Q for the shear stress remains the same for isothermal or adiabatic deformations.

Values of β_1 and β_2 may be obtained experimentally by measuring the thermal dilatation under constant stress and using equations 4.25. The value of c is then obtained indirectly from equation 4.32 by measuring the specific heat under the same condition of constant stress.

These values may also be derived analytically if the entropy is known as a function of the deformation and the temperature. This can be shown by considering a cube of unit size in the unstressed

state. The material is assumed to be orthotropic with the faces of the cube oriented along the coordinate axes and the planes of elastic symmetry. Under an initial stress, under forces applied normally to its faces, the cube becomes a parallelepiped of dimensions equal to the extension ratios $\lambda_1, \lambda_2, \lambda_3$. The entropy \mathfrak{S} of this element of the material is expressed as a function of the extension ratios and the temperature. We write

$$\mathfrak{S} = \mathfrak{S}(\lambda_1, \lambda_2, \lambda_3, T) \tag{4.36}$$

Let v_0 denote the volume of the element in a state of initial stress. The entropy of a unit volume in this initial state is then \mathfrak{S}/v_0 , and the incremental entropy is written

$$\delta \mathfrak{S} = \frac{1}{v_0} \left(\frac{\partial \mathfrak{S}}{\partial \lambda_1} d\lambda_1 + \frac{\partial \mathfrak{S}}{\partial \lambda_2} d\lambda_2 + \frac{\partial \mathfrak{S}}{\partial \lambda_3} d\lambda_3 + \frac{\partial \mathfrak{S}}{\partial T} dT \right) \tag{4.37}$$

By taking into account the relations

$$\begin{aligned} e_{xx} &= \frac{d\lambda_1}{\lambda_1} & e_{yy} &= \frac{d\lambda_2}{\lambda_2} \\ \theta &= dT & v_0 &= \lambda_1 \lambda_2 \lambda_3 \end{aligned} \tag{4.38}$$

and comparing equations 4.32 and 4.37, we derive

$$\begin{aligned} \beta_1 &= \frac{1}{\lambda_3 \lambda_2} \frac{\partial \mathfrak{S}}{\partial \lambda_1} \\ \beta_2 &= \frac{1}{\lambda_3 \lambda_1} \frac{\partial \mathfrak{S}}{\partial \lambda_2} \\ \frac{c}{T_r} &= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial \mathfrak{S}}{\partial T} \end{aligned} \tag{4.39}$$

Isotropy and Second Order Elasticity. When the elastic medium is isotropic in finite strain, expressions (4.39) remain applicable. The property of isotropy appears only in the fact that \mathfrak{S} is a function where the variables $\lambda_1, \lambda_2, \lambda_3$ are interchangeable.

Let us consider the case where the initial strain is small and is represented by the quantities $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$ of equations 4.19. To the first order in the initial strain the incremental elastic coefficients are given by equations 4.20 and 4.24. If the initial strain is isothermal, the incremental coefficients derived from these equations are also isothermal. In order to obtain the coefficients for adiabatic incremental deformations we must use equations 4.35. The correction

terms in these equations may also be expressed by linearization with respect to the small initial strains. For example, if we know the theoretical expression for the entropy the coefficients (4.39) may be evaluated to the first order in ε_{11} , ε_{22} , ε_{33} . Because of the isotropy of the medium they are of the form

$$\begin{aligned}\beta_1 &= \beta_0 + a\varepsilon_{11} + b(\varepsilon_{22} + \varepsilon_{33}) \\ \beta_2 &= \beta_0 + a\varepsilon_{22} + b(\varepsilon_{33} + \varepsilon_{11}) \\ c &= c_0 + c_1\varepsilon\end{aligned}\tag{4.40}$$

By substituting these values into expressions (4.35) we obtain the adiabatic coefficients. They may again be simplified by retaining only the linear terms with the initial strain.

We note that β_0 is the common value of β_1 and β_2 in the original stress-free state, while c_0 is the specific heat per unit volume in the same state. In practice the terms containing the initial strain in expressions (4.40) will be very small and will usually be negligible. The adiabatic correction terms will then coincide with those of the stress-free medium.

Adiabatic Coefficients in the General Case. In the most general case of anisotropy the incremental stresses are given by equations 4.15 of Chapter 2. They are written

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu}\tag{4.41}$$

The coefficients $B_{ij}^{\mu\nu}$ may correspond to either isothermal or adiabatic deformation. Let us assume that equation 4.41 corresponds to isothermal deformations. For adiabatic deformations we write

$$s_{ij} = \bar{B}_{ij}^{\mu\nu} e_{\mu\nu}\tag{4.42}$$

The relations between the isothermal and adiabatic coefficients $B_{ij}^{\mu\nu}$ and $\bar{B}_{ij}^{\mu\nu}$ in this general case are derived by following exactly the same procedure as in the foregoing analysis. With an increment θ of the temperature the stresses (4.41) become

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu} - \beta_{ij}\theta\tag{4.43}$$

This defines the coefficients β_{ij} . The increment of entropy per unit volume is found to be

$$\delta s = \beta_{ij} e_{ij} + c \frac{\theta}{T_r}\tag{4.44}$$

Combining equations 4.43 and 4.44 with $\bar{s} = 0$, we derive for the adiabatic coefficients the expressions

$$\bar{B}_{ij}^{\mu\nu} = B_{ij}^{\mu\nu} + \beta_{ij}\beta_{\mu\nu} \frac{T_r}{c} \quad (4.45)$$

where c is again the specific heat per unit volume under zero strain in the initial state of stress.

Application to Stress Measurement and Earthquake Prediction.

The theory of elasticity of a medium under initial stress predicts the influence of the state of stress on the velocity of propagation of elastic waves. This suggests the possible development of new methods of measuring stresses in a solid. In this connection the theory has recently been given a renewed importance by the suggestion that it may be used to predict the occurrence of earthquakes. Variations of stress in the earth should be associated with changes in the velocity of seismic waves. Observation of these changes should give an indication of critical variations of tectonic stresses. Of particular importance in this connection is the variation of velocity of transverse-type waves under horizontal tension or compression, as pointed out in 1940 by the author* and emphasized again in a recent paper.† Detailed prediction of the influence of stress on wave propagation in stratified geological structures can be derived by application of the more elaborate theory developed in section 7 of this chapter.

5. THEORY OF ACOUSTIC-GRAVITY WAVES IN A FLUID

In usual acoustical theory the propagation of elastic waves in a fluid is analyzed by neglecting the initial stress.

In a gas, of course, an initial stress is always present and is represented by the gas pressure in the undisturbed state. When the initial pressure is uniform, the existence of initial stress does not give rise to any difficulty and may be neglected.

If the initial pressure is not uniform the problem is not so simple, as, for example, when the fluid in the undisturbed state is in equilibrium under the action of a body force such as gravity. That the effect of initial stress should produce phenomena which are quite different from those of classical acoustics is immediately evident if we

* See reference 7 at the end of the Preface.

† M. A. Biot, Internal Buckling under Initial Stress in Finite Elasticity, *Proceedings of the Royal Society, A*, Vol. 273, pp. 306-328, 1963.

consider surface gravity waves in an incompressible fluid. In this case the propagation is dispersive. It is due entirely to the initial stress itself, and not to any elastic property of the fluid. More complex behavior arises when the propagation results from the simultaneous action of the initial stress and the compressibility. An example of such behavior is provided by the propagation of pressure waves in the atmosphere. In this case the gas is initially in equilibrium under the action of gravity. The initial pressure and density are functions of the altitude. For large wavelengths the influence of gravity on the propagation becomes very important and must be introduced into the theory.

It is known that in the ocean internal gravity waves also occur because of density differences at various depths.

In the past these problems have been discussed almost exclusively from the standpoint of Euler's equations of fluid dynamics which describe the motion by means of the velocity field.

The classical literature on this subject originated in the late nineteenth century. The theory of gravity waves in heterogeneous liquids is associated with the names of Love, Burnside, Rayleigh, Lamb, and others.* General equations for small motions of a gas about a state of equilibrium in any constant field of force have been derived by Lamb.† Extensive application and development of these theories in the context of oceanography and meteorology are due to Bjerknes‡ and Eckart.§

The present treatment starts from an entirely different viewpoint and, by introducing zero rigidity, considers the fluid as a particular case of an elastic continuum under initial stress. Aside from bringing the theory into the unifying framework of the theory of elasticity it presents many advantages by providing a better understanding of the physical nature of these phenomena and removing many of the obscurities inherent in the traditional treatment by Euler's equations. A detailed discussion based on this viewpoint has been given in some

* Classical references will be found in Lamb's treatise (p. 378), *Hydrodynamics*, Cambridge University Press, 1932 (reprinted by Dover Publications, New York, 1945).

† See Lamb's treatise, p. 554.

‡ V. Bjerknes et al., *Physikalische Hydrodynamik*, J. Springer, Berlin, 1933.

§ C. Eckart, *Hydrodynamics of Oceans and Atmospheres*, Pergamon Press, New York, 1960.

recent papers by the author,* with particular emphasis on some new variational principles. The latter will be presented in the next section.

Unmodified Equations for Acoustic-Gravity Waves. In the particular case of a fluid the initial and incremental stresses are hydrostatic. They are written

$$\begin{aligned} S_{ij} &= S\delta_{ij} \\ s_{ij} &= s\delta_{ij} \end{aligned} \quad (5.1)$$

where S and s are the negative initial and incremental pressures. With these values equations 2.9 are considerably simplified. They become

$$\frac{\partial s}{\partial x_i} + \rho \Delta X_i - \rho e X_i - \rho \omega_{ij} X_j - e_{ij} \frac{\partial S}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.2)$$

These equations are further simplified by taking into account the equilibrium condition for the initial stress,

$$\frac{\partial S}{\partial x_i} + \rho X_i = 0 \quad (5.3)$$

Hence

$$\frac{\partial s}{\partial x_i} + \rho \Delta X_i + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.4)$$

We shall refer to these equations as the *unmodified* equations.

It is of interest to show that equations 5.4 may be derived by a more direct procedure. Consider the particle of fluid initially at point x_i and displaced to the point $\xi_i = x_i + u_i$. The dynamical equations in terms of the coordinates ξ_i are written

$$\frac{\partial}{\partial \xi_i} (S + s) + \rho' X_i(\xi_i) = \rho' \frac{\partial^2 u_i}{\partial t^2} \quad (5.4a)$$

These are exact equations valid for finite deformations. Attention is called to the significance of the variables. At the initial point x_i the stress, the mass

* M. A. Biot, Generalized Theory of Internal Gravity Waves, U.S. Air Force Office of Scientific Research Report, 1962; General Fluid-Displacement Equations for Acoustic-Gravity Waves, *The Physics of Fluids*, Vol. 6, No. 5, pp. 621-626, 1963; Variational Principles for Acoustic-Gravity Waves, *The Physics of Fluids*, Vol. 6, No. 6, pp. 772-778, 1963; Acoustic-Gravity Waves as a Particular Case of the Theory of Elasticity under Initial Stress, *The Physics of Fluids*, Vol. 6, No. 6, pp. 778-780, 1963.

density, and the body force are, respectively, S , ρ , and $X_i(x_i)$. At the displaced point ξ_i they become $S + s$, ρ' , and $X_i(\xi_i)$.

We wish to transform equations 5.4a by introducing x_i as independent variables. An equivalent form of the equations is

$$\frac{\partial x_j}{\partial \xi_i} \frac{\partial}{\partial x_j} (S + s) + \rho' X_i(\xi_i) = \rho' \frac{\partial^2 u_i}{\partial t^2} \quad (5.4b)$$

The derivatives $\partial x_j / \partial \xi_i$ are evaluated by solving the equations

$$d\xi_i = \frac{\partial \xi_i}{\partial x_j} dx_j \quad (5.4c)$$

We find

$$dx_j = \frac{1}{J} M_{ij} d\xi_i \quad (5.4d)$$

The Jacobian J is the determinant of equations 5.4c. The cofactors M_{ij} have been discussed and are given by equations 7.13 of Chapter 1. From equations 5.4d we derive

$$\frac{\partial x_j}{\partial \xi_i} = \frac{1}{J} M_{ij} \quad (5.4e)$$

The density satisfies the relation

$$\rho = J\rho' \quad (5.4f)$$

By using relations (5.4e) and (5.4f), equations 5.4b are transformed into

$$M_{ij} \frac{\partial}{\partial x_j} (S + s) + \rho X_i(\xi_i) = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.4g)$$

These are still exact equations. They may be simplified by keeping only the first order terms. To this approximation M_{ij} is given by expression (7.18) of Chapter 1.

$$M_{ij} = (1 + e)\delta_{ij} - \frac{\partial u_j}{\partial x_i} \quad (5.4h)$$

By substituting this value into equations 5.4g, keeping only first order terms and taking into account the initial equilibrium condition (5.3), we derive

$$\frac{\partial s}{\partial x_i} + \rho \Delta X_i + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.4i)$$

where

$$\Delta X_i = X_i(\xi_i) - X_i(x_i) \quad (5.4j)$$

This result coincides with equations 5.4.

Equations 5.4 simply express Newton's law of motion and do not involve the thermodynamics of the fluid. If we assume that the volume changes are either adiabatic or isothermal, they will depend only on the pressure variation. Linearizing the pressure-volume relation in the vicinity of the initial state, we write

$$s = \lambda e \quad (5.5)$$

This value substituted in equations 5.4 yields the propagation equations

$$\frac{\partial}{\partial x_i} (\lambda e) + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} + \rho \Delta X_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.6)$$

The coefficient λ is either the adiabatic or the isothermal bulk modulus. Both λ and ρ will depend on the local nature of the fluid and the local value of the initial pressure and temperature. Boundary conditions at a free surface are expressed by the equation

$$s = \lambda e = 0 \quad (5.7)$$

If the initial pressure is uniform, the value of S is a constant ($X_i = 0$). In this case equations 5.6 take the classical form for the acoustics of a fluid,

$$\frac{\partial}{\partial x_i} (\lambda e) = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.8)$$

The initial stress does not appear in these equations.

Coriolis Acceleration. In many geophysical problems the Coriolis acceleration must be taken into account. With the additional Coriolis term in the acceleration, equations 5.4 are written

$$\frac{\partial s}{\partial x_i} + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} + \rho \Delta X_i = \rho \left(\frac{\partial^2 u_i}{\partial t^2} + 2\Omega_{ij} \frac{\partial u_j}{\partial t} \right) \quad (5.9)$$

In these equations Ω_{ij} are the elements of the matrix

$$[\Omega_{ij}] = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \quad (5.10)$$

and $\Omega_x, \Omega_y, \Omega_z$ are the components of angular velocity of the frame of reference.

Unmodified Equations for a Fluid in a Constant Gravity Field. Consider a constant gravity field of acceleration g . With a vertical z axis positive upward the body force components are

$$X_i = (0, 0, -g) \quad (5.11)$$

From the equilibrium condition (5.3) the initial stress gradient is found to be

$$\frac{\partial S}{\partial x_i} = -\rho X_i = (0, 0, \rho g) \quad (5.12)$$

Surfaces of constant density are horizontal planes; hence

$$\rho = \rho(z) \quad (5.13)$$

The displacement components are designated by

$$u_i = (u, v, w) \quad (5.14)$$

In this case equations 5.4 become

$$\begin{aligned} \frac{\partial s}{\partial x} - \rho g \frac{\partial w}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s}{\partial y} - \rho g \frac{\partial w}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial s}{\partial z} - \rho g \frac{\partial w}{\partial z} + \rho g e &= \rho \frac{\partial^2 w}{\partial t^2} \end{aligned} \quad (5.15)$$

The incremental hydrostatic stress is

$$s = \lambda e = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (5.16)$$

These equations govern the propagation of acoustic-gravity waves in a compressible fluid with horizontal stratification and a constant gravity field.

In the form (5.15) the equations for a constant gravity field were derived by the author* for a continuum of isotropic properties under initial hydrostatic stress.

An extensive discussion of acoustic-gravity waves based on equations 5.15 has been given in a recent paper by Tolstoy.†

Further Linearization of the Unmodified Equations. Attention should be called to the validity of equations 5.4 for finite displacements. Their validity can be seen from their derivation, which requires that only the gradients of the displacements be small.

For a non-uniform body force field the equations contain a term ΔX_i which is an explicit function of the displacement. It may or may not be linear. For a constant gravity field this term disappears and the equations become linear.

* See reference 7 at the end of the Preface.

† I. Tolstoy, The Theory of Waves in Stratified Fluids Including the Effects of Gravity and Rotation, *Reviews of Modern Physics*, Vol. 35, No. 1, pp. 207-230, 1963.

Equations 5.4 do not contain explicitly the body force potential. A more restricted form of the equations is obtained by introducing explicitly this potential and by linearizing the incremental body force ΔX_i in terms of the components of displacements. With a potential U the body force becomes

$$X_i = - \frac{\partial U}{\partial x_i} \quad (5.17)$$

In addition we assume that ΔX_i may be linearized as follows,

$$\Delta X_i = \frac{\partial X_i}{\partial x_j} u_j = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \quad (5.18)$$

In principle, for a non-uniform field the linearization (5.18) restricts the magnitude of the displacement u_i . However, for the large majority of problems the linearization will be valid. With expression (5.18) for ΔX_i the dynamical equations 5.4 become

$$\frac{\partial s}{\partial x_i} + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.19)$$

Modified Equations for Acoustic-Gravity Waves. The linearized equations 5.19 may be further transformed into an equivalent form which provides additional insight into the physics of the problem. The transformation is a particular case of the more general one discussed in section 5 of Chapter 3 for the stability of a solid in the presence of hydrostatic stress. Using the equilibrium condition (5.3), we write

$$\frac{\partial S}{\partial x_i} = \rho \frac{\partial U}{\partial x_i} \quad (5.20)$$

With this value, equation 5.19 becomes

$$\frac{\partial s}{\partial x_i} + \rho e \frac{\partial U}{\partial x_i} - \rho \frac{\partial u_j}{\partial x_i} \frac{\partial U}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.21)$$

Let us introduce a variable s' defined by the relation

$$s = s' + \rho u_j \frac{\partial U}{\partial x_j} \quad (5.22)$$

Substitution of s into equations 5.21 yields the following result, which we shall refer to as the *modified* equations.

$$\frac{\partial s'}{\partial x_i} - \rho e X_i - u_j X_j \frac{\partial \rho}{\partial x_i} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.23)$$

Comparison with equations 5.23 of Chapter 3 for the more general case of a solid in the presence of hydrostatic stress shows that these equations lead immediately to the same result by putting $\mathcal{F}_i = 0$ and $s_{ij} = s\delta_{ij}$. The physical interpretation of the various terms in equations 5.23 were discussed in section 5 of Chapter 3. The variable s' defined by equation 5.22 may be written

$$s' = s + \rho u_j X_j = s - u_j \frac{\partial S}{\partial x_j} \quad (5.24)$$

This expression shows that s' represents the stress increment *at a fixed point in space*.

Interpretation of the other terms is obtained by taking into account equations 5.20. By evaluating second derivatives of S the following relation is derived:

$$\frac{\partial U}{\partial x_j} \frac{\partial \rho}{\partial x_i} = \frac{\partial U}{\partial x_i} \frac{\partial \rho}{\partial x_j} \quad (5.25)$$

or

$$X_j \frac{\partial \rho}{\partial x_i} = X_i \frac{\partial \rho}{\partial x_j} \quad (5.26)$$

This result expresses the parallelism of the body force and the density gradient. They are both perpendicular to the equipotential surfaces.

We denote by n_i the unit vector normal to the equipotential surface, and by X , $\partial\rho/\partial n$, and u_n the algebraic components of X_i , $\partial\rho/\partial x_i$, and u_i along this normal direction. We may write

$$-\rho e X_i - u_j X_j \frac{\partial \rho}{\partial x_i} = -n_i \left(\rho e X + u_n X \frac{\partial \rho}{\partial n} \right) \quad (5.27)$$

These terms represent the buoyancy force acting on the fluid particle. The first term, $\rho e X$, is the buoyancy force due to the change of volume of the particle. The second term, $u_n X (\partial\rho/\partial n)$, is due to the density gradient and is proportional to the normal displacement.

If we take into account equation 5.26 we may express the buoyancy force (5.27) in the form

$$-\rho e X_i - u_j X_j \frac{\partial \rho}{\partial x_i} = -X_i \frac{\partial}{\partial x_j} (\rho u_j) \quad (5.28)$$

With this result the modified equations 5.23 become

$$\frac{\partial s'}{\partial x_i} - X_i \frac{\partial}{\partial x_j} (\rho u_j) = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5.29}$$

The intuitive significance of equations 5.23 and 5.29 is readily evident because the left-hand side represents the force acting on a fluid particle as the sum of the stress gradient $\partial s'/\partial x_i$ and the buoyancy force.

Modified Equations for a Fluid in a Constant Gravity Field. As an example consider equations 5.15 for the constant gravity field. By substituting

$$s = s' + \rho g w \tag{5.30}$$

they become*

$$\begin{aligned} \frac{\partial s'}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s'}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial s'}{\partial z} + \rho g e + g \frac{\partial \rho}{\partial z} w &= \rho \frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{5.31}$$

They constitute a particular case of the more general equations 5.23.

Relation to Euler's Equations of Fluid Dynamics. In the particular form (5.29) the propagation equations are closely related to the classical result obtained from Euler's equations of motion of a fluid. These equations are

$$-\frac{\partial p_f}{\partial x_i} + \rho X_i = \rho a_i \tag{5.31a}$$

where p_f , ρ , and a_i are the fluid pressure, density, and acceleration at a fixed point. The time derivatives of these equations are

$$-\frac{\partial}{\partial x_i} \left(\frac{\partial p_f}{\partial t} \right) + X_i \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t} a_i + \rho \frac{\partial a_i}{\partial t} \tag{5.31b}$$

Assuming small values of $\partial p_f/\partial t$, $\partial \rho/\partial t$, and a_i , we may write to the first order

$$-\frac{\partial}{\partial x_i} \left(\frac{\partial p_f}{\partial t} \right) + X_i \frac{\partial \rho}{\partial t} = \rho \frac{\partial^2 v_i}{\partial t^2} \tag{5.31c}$$

where v_i denotes the fluid velocity. By introducing the condition of conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) = 0 \tag{5.31d}$$

* A result equivalent to equations 5.31 but slightly different in form was also derived by A. Eliassen and E. Kleinschmitt in *Handbuch der Physik*, Vol. 48 (Geophysics II), p. 52, J. Springer, Berlin, 1957.

equations 5.31c become

$$-\frac{\partial}{\partial x_i} \left(\frac{\partial p_f}{\partial t} \right) - X_i \frac{\partial}{\partial x_j} (\rho v_j) = \rho \frac{\partial^2 v_i}{\partial t^2} \quad (5.31e)$$

They are of the same form as equations 5.29, where s' plays the role of $-\partial p_f/\partial t$ and the displacement u_i becomes the velocity v_i . Equations 5.29 and 5.31e become identical if we divide equation 5.29 by the time increment Δt and consider its limit for infinitesimal values of s' and u_i . The time derivative of the pressure on a fluid particle is

$$\frac{Dp_f}{Dt} = \frac{\partial p_f}{\partial t} + v_i \frac{\partial p_f}{\partial x_i} \quad (5.31f)$$

Comparing with equation 5.24, we see that $-Dp_f/Dt$ plays the role of the incremental stress s on the fluid particle.

Gravity Waves in a Liquid. Analog Model. For a liquid behaving as an incompressible fluid we put $e = 0$. Equations 5.23 are simplified to

$$\frac{\partial s'}{\partial x_i} - X_j \frac{\partial \rho}{\partial x_i} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5.32)$$

To this we must add the condition for incompressibility

$$e = 0 \quad (5.33)$$

The four equations 5.32 and 5.33 are now the propagation equations. They contain four unknowns, s' and u_i .

An important feature brought out by these equations is that they are identical with the propagation equation for a medium *initially stress-free* provided we add a body force proportional to the displacement as represented by the term $-X_j (\partial \rho/\partial x_i) u_j$. We have discussed the physical interpretation of this term as a buoyancy force. It may be looked upon as representing a positive or negative elastic force acting on a fluid particle normally to the equipotential surfaces and proportional to the distance of the displaced particle from this surface. By applying such elastic forces to the fluid particles a fictitious medium is obtained which plays the role of an *analog model* for internal gravity waves. It is obviously a particular case of the analog model discussed previously for the incompressible elastic solid (section 5, Chapter 3, and section 3 of this chapter).

The *stress of a fluid particle in the analog model* is s' . However, it is not the stress of the particle in the actual fluid. The actual stress is $S + s$, and the stress increment s is related to s' by relation (5.24); that is,

$$s = s' - \rho u_j X_j \quad (5.34)$$

The boundary condition $s = 0$ at a free surface is replaced in the analog model by the condition

$$s' = \rho u_j X_j \tag{5.35}$$

The elastic forces in the model are body forces per unit volume. This assumes that the density distribution is continuous. At a surface of discontinuity for the density these volume forces are replaced by forces per unit area. This can be shown by considering the discontinuity as the limiting case of a rapidly varying density and integrating the term $-X_j (\partial\rho/\partial x_i)u_j$ along the normal n_i across the discontinuity. The procedure is illustrated below for the particular case of a constant gravity field. (See also page 475.)

Liquid in a Constant Gravity Field. The significant features of the analog model are well illustrated by considering the particular case of an incompressible fluid in a constant gravity field. The equations for this case are obtained by putting $e = 0$ in equations 5.31. They become

$$\begin{aligned} \frac{\partial s'}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s'}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial s'}{\partial z} + g \frac{\partial \rho}{\partial z} w &= \rho \frac{\partial^2 w}{\partial t^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \tag{5.36}$$

They are a particular case of the more general form (equations 5.32 and 5.33). The boundary condition (5.35) at the free surface becomes

$$s' = -\rho g w \tag{5.37}$$

Equations 5.36 and 5.37 are the same as those for an incompressible fluid free of initial stress with the addition of a vertical body force $g (d\rho/dz)w$ distributed inside the fluid and a pressure $\rho g w$ at the surface. The fluid pressure in this analog model is zero in the initial state of equilibrium, and it becomes $-s'$ when the fluid is in motion.

The distributed vertical body force $g (d\rho/dz)w$ is equivalent to an elastic force proportional to the vertical displacement. The z axis is

positive upward. For a stabilizing density gradient, $d\rho/dz$ is negative, and the force acts in a direction opposite to the displacement and behaves like an elastic restoring force. When $d\rho/dz$ is positive, the body force is destabilizing and acts like an elastic repulsion away from the equilibrium point of the particle.

The boundary condition (5.37) represents a pressure ρgw applied to the boundary of the model and proportional to the vertical displacement. If the fluid lies below the free surface, this surface pressure acts in a direction opposite to the displacement like an elastic restoring force. If the fluid lies above the free surface, the situation is reversed and the surface pressure is destabilizing; in practice this condition, which corresponds to what is known as *Taylor instability*, may be produced by accelerating the fluid.

Of particular interest is the case where the density $\rho(z)$ is a discontinuous function of z . The elastic body force of the model then becomes a force per unit area applied to the surface of discontinuity. Its magnitude is obtained by considering a density ρ which is continuous but changes very rapidly across a horizontal layer of vanishing thickness ε . By integrating the body force across the layer its magnitude per unit area is found to be

$$g \int_z^{z+\varepsilon} w \frac{\partial \rho}{\partial z} dz = gw(\rho_2 - \rho_1) \quad (5.38)$$

where ρ_2 and ρ_1 are the densities above and below the discontinuity. If $\rho_1 > \rho_2$, this force is stabilizing and acts in a direction opposite to w .

Note that the boundary condition (5.37) may be considered as a particular case of equation 5.38 by imagining a fluid of zero density lying on top of the original fluid while the boundary is replaced by a thin layer across which the density drops to zero. Expressions (5.37) and (5.38) become identical by putting $\rho_1 = \rho$ and $\rho_2 = 0$.

The nature of internal gravity waves is further illustrated by assuming a two-dimensional fluid motion parallel to the vertical x, z plane. Hence the last equation (5.36) is satisfied by putting $v = 0$ and

$$u = - \frac{\partial \phi}{\partial z} \quad w = \frac{\partial \phi}{\partial x} \quad (5.39)$$

where ϕ is an unknown function of x and z . Eliminating s' and introducing ϕ into the remaining equations, we obtain

$$\frac{\partial}{\partial z} \left(\rho \frac{\partial^3 \phi}{\partial t^2 \partial z} \right) + \rho \alpha_c^2 \frac{\partial^2 \phi}{\partial x^2} + \rho \frac{\partial^4 \phi}{\partial x^2 \partial t^2} = 0 \tag{5.40}$$

We have put

$$\alpha_c^2 = - \frac{g}{\rho} \frac{d\rho}{dz} \tag{5.41}$$

For a simple harmonic wave propagating in the x direction we put

$$\phi = e^{i(lx - \alpha t)} f(z) \tag{5.42}$$

Substitution of this expression into equation 5.40 leads to a Sturm-Liouville equation for f ,

$$\frac{d}{dz} \left(\rho \frac{df}{dz} \right) + \frac{\rho}{V^2} (\alpha_c^2 - \alpha^2) f = 0 \tag{5.43}$$

We denote by

$$V = \frac{\alpha}{l} \tag{5.44}$$

the phase velocity of the waves in the x direction. The boundary condition (5.37) at the free surface is

$$s' = -\rho g w \tag{5.45}$$

In terms of f it becomes

$$\frac{g}{V^2} f = \frac{df}{dz} \tag{5.46}$$

Equation 5.43 is also written

$$\frac{d^2 f}{dz^2} - \frac{\alpha_c^2}{g} \frac{df}{dz} + \frac{1}{V^2} (\alpha_c^2 - \alpha^2) f = 0 \tag{5.47}$$

For an exponential density distribution

$$\rho = \rho_0 e^{-az} \tag{5.48}$$

we find

$$\alpha_c^2 = ag \tag{5.49}$$

In this case equation 5.47 has constant coefficients and elementary solutions.

Note the physical significance of α_c . When it is a real quantity it represents the angular frequency of a particle of fluid oscillating under the action of the local restoring force wg ($d\rho/dz$). Equations 5.40 and 5.47 coincide with classical results.

Discontinuous Displacements Associated with a Density Discontinuity. It is known that discontinuities of the displacements arise for acoustic propagation in a fluid with density discontinuities. This can be seen by considering the classical equations for the simpler case of a fluid initially stress-free. For waves of circular frequency α they are written

$$\frac{\partial s}{\partial x_i} = -\alpha^2 \rho u_i \quad (5.49a)$$

where u_i is the fluid displacement and s is the negative pressure. From this equation we derive

$$\frac{\partial}{\partial x_j} (\rho u_i) = \frac{\partial}{\partial x_i} (\rho u_j) \quad (5.49b)$$

Hence

$$u_i \frac{\partial \rho}{\partial x_j} - u_j \frac{\partial \rho}{\partial x_i} = \rho \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) \quad (5.49c)$$

The left side of this equation is the vector product of the velocity and the density gradient. On the right side is a quantity proportional to the rotation vector. At a surface of density discontinuity the density gradient becomes infinite. If the velocity is not perpendicular to the surface of discontinuity, equations 5.49c show that the rotation becomes infinite. The rotation vector is tangent to the surface of discontinuity. Hence the tangential component of the displacement is discontinuous as we move across the surface.

The question of the validity of the derivation of the equations of motion arises here since we have assumed that the displacement gradients are small. However, the difficulty may be circumvented by considering the discontinuity surface as a boundary for the fluids lying on each side. By expressing the boundary forces acting on each fluid we find that the discontinuity is equivalent to surface forces proportional to the displacement and normal to the surface of discontinuity. This is exactly the same result as obtained from the dynamical equations by a limiting process and illustrated by equation 5.38.

6. VARIATIONAL PRINCIPLES FOR ACOUSTIC-GRAVITY WAVES

Variational principles for the dynamics of an elastic continuum under initial stress have been discussed in section 2. They are, of course, immediately applicable to the particular case of a fluid.

Instead of proceeding from these general results there is some advantage in deriving variational principles for acoustic-gravity waves in a more direct fashion.

The analysis leads to variational principles of two essentially different types which are distinguished as follows.

(a) *Unmodified Variational Principle.* This principle is the same as the previously derived principle for the theory of elasticity of a continuum under initial stress. It is readily applicable to a fluid by introducing hydrostatic stress components. The potential energy is expressed in terms of stress and strain.

(b) *Modified Variational Principle.* This principle emphasizes the particular properties of the fluid by introducing explicitly the buoyancy force.

It will be shown that these two principles are equivalent. They correspond to the two forms (5.19) and (5.23) of the dynamical equations.

These variational principles were developed and discussed in recent work by the author.* We shall first derive the modified form of the principle.

The Modified Variational Principle. Let us consider the volume integral

$$\mathcal{W}_\tau = \iiint_\tau \left(\frac{1}{2}se + \rho e u_j X_j + \frac{1}{2}X_j \frac{\partial \rho}{\partial x_i} u_i u_j \right) d\tau \quad (6.1)$$

That this expression leads to a variational principle can be inferred if we note the physical significance of the terms appearing in the integrand. We can show that they represent the incremental potential energy of a fluid under initial stress.

The term

$$\frac{1}{2}se = \frac{1}{2}\lambda e^2 \quad (6.2)$$

is the elastic energy due to the change of volume e . The bulk modulus is defined by equation 5.5. Referring to expression (5.27) for the buoyancy force on a fluid particle, we may write the work done against this buoyancy force as

$$\rho e u_j X_j + \frac{1}{2}X_j \frac{\partial \rho}{\partial x_i} u_i u_j = \rho e u_n X + \frac{1}{2}X \frac{\partial \rho}{\partial n} u_n^2 \quad (6.3)$$

In this expression X , u_n , and $\partial \rho / \partial n$ represent, as before, the algebraic

* M. A. Biot, Generalized Theory of Internal Gravity Waves, U.S. Air Force Office of Scientific Research Report, 1962; Variational Principles for Acoustic-Gravity Waves, *The Physics of Fluids*, Vol. 6, No. 6, pp. 772-778, 1963.

components of X_i , u_i , and $\partial\rho/\partial x_i$ along an oriented axis normal to the equipotential surface.

With the values (6.2) and (6.3), expression (6.1) is written

$$\mathcal{W}_\tau = \iiint_\tau \left(\frac{1}{2} \lambda e^2 + \rho e u_n X + \frac{1}{2} X \frac{\partial \rho}{\partial n} u_n^2 \right) d\tau \quad (6.4)$$

It is readily verified that \mathcal{W}_τ leads to a variational principle for acoustic-gravity waves by evaluating its variation for arbitrary displacement u_i . By the usual integration by parts we obtain

$$\begin{aligned} \delta \mathcal{W}_\tau &= \iint_A (s + \rho u_j X_j) n_i \delta u_i dA \\ &\quad - \iiint_\tau \left(\frac{\partial s}{\partial x_i} + \frac{\partial}{\partial x_i} (\rho u_j X_j) - \rho e X_i - X_j \frac{\partial \rho}{\partial x_i} u_j \right) \delta u_i d\tau \end{aligned} \quad (6.5)$$

The bracket in the volume integral is identical with the left side of the dynamical equations 5.23. By writing a_i for the acceleration $\partial^2 u_i / \partial t^2$ of the particle and inserting the value (5.24) for s' , we can write these dynamical equations

$$\frac{\partial s}{\partial x_i} + \frac{\partial}{\partial x_i} (\rho u_j X_j) - X_j \frac{\partial \rho}{\partial x_i} u_j - \rho e X_i = \rho a_i \quad (6.6)$$

If this equation is verified, the variational equation (6.5) becomes

$$\delta \mathcal{W}_\tau + \iiint_\tau \rho a_i \delta u_i d\tau = \iint_A (s + \rho u_j X_j) n_i \delta u_i dA \quad (6.7)$$

This result represents a variational principle for the dynamics of the fluid in the vicinity of equilibrium. Since it leads to the modified equations 5.23, we shall refer to it as the *modified variational principle*.

Potential Energy of the Free Surface in the Modified Principle. The surface integral of equation 6.7 is extended to the entire fluid boundary A . It may be simplified by introducing boundary conditions and constraints. We shall assume that the condition of zero pressure is verified at the free surface; that is,

$$s = e = 0 \quad (6.8)$$

In addition, the displacement field u_i is so chosen as to satisfy the linear condition that it be tangent to the surface at a rigid boundary; that is,

$$n_i u_i = n_i \delta u_i = 0 \quad (6.9)$$

With these conditions the variational principle (6.7) takes the simpler form

$$\delta \mathcal{W}_\tau + \iiint_\tau \rho a_i \delta u_i d\tau = \iint_F \rho u_j X_j n_i \delta u_i dA \quad (6.10)$$

where the surface integral is extended only to the free surface F of the fluid. This surface integral acquires an interesting interpretation when we note that it represents an exact differential. Note that n_i is the unit normal to the boundary pointing outward. Since the free surface is an equipotential surface, the body force X_i is normal to this surface. Hence we may write

$$\rho u_j X_j n_i \delta u_i = \frac{1}{2} \delta (\rho X u_n^2) \quad (6.11)$$

where X and u_n are the algebraic projection of the body force and the displacement on a direction normal to the free surface and pointing outward. By introducing

$$\mathcal{W}_F = -\frac{1}{2} \iint_F (\rho X u_n^2) dA \quad (6.12)$$

as a "potential energy of the free surface" and

$$\mathcal{P} = \mathcal{W}_\tau + \mathcal{W}_F \quad (6.13)$$

as a total potential energy, the variational principle (6.10) becomes

$$\delta \mathcal{P} + \iiint \rho a_i \delta u_i d\tau = 0 \quad (6.14)$$

With the value (6.13) for \mathcal{P} this result represents another form of the modified variational principle.

Fluid in a Constant Gravity Field. Let us consider a uniform gravity field of acceleration g . The z axis is chosen positive upward, and w denotes the vertical displacement of the fluid. The body force is

$$X_i = (0, 0, -g) \quad (6.15)$$

Expressions (6.1) and (6.12) become

$$\begin{aligned} \mathcal{W}_\tau &= \iiint_\tau \left(\frac{1}{2} \rho w_\tau^2 - \rho g e w - \frac{1}{2} \rho g \frac{d\rho}{dz} w^2 \right) d\tau \\ \mathcal{W}_F &= \frac{1}{2} \rho g \iint_F w_F^2 dA \end{aligned} \quad (6.16)$$

where w_F is the vertical displacement at the free surface. We recognize in the terms $\rho g e w$ and $\frac{1}{2}g(d\rho/dz)w^2$ expressions for the work of the vertical buoyancy force.

Applying the variational principle (6.14) with the particular values (6.16) leads immediately to the dynamical equations 5.15 for a constant gravity field.

The expression for \mathcal{W}_τ in equations 6.16 was also used by Tolstoy* to derive the equations of motion by means of a Lagrangian density for the case of a constant gravity field.

Stability. The static stability of the fluid is determined by the condition that the total potential energy (6.13) be positive definite. This is expressed by the conditions

$$\begin{aligned}\mathcal{W}_F &= -\frac{1}{2} \iint_F \rho X u_n^2 dA > 0 \\ \mathcal{W}_\tau &= \iiint_\tau \left(\frac{1}{2} \lambda e^2 + \rho X e u_n + \frac{1}{2} X \frac{\partial \rho}{\partial n} u_n^2 \right) d\tau > 0\end{aligned}\tag{6.16a}$$

The first inequality requires that

$$X < 0\tag{6.16b}$$

at the free surface. Hence at this surface the body force must be directed inward. The opposite case corresponds to the so-called "Taylor instability" which may occur when the body force is generated by an acceleration field.

The second inequality (6.16a) is satisfied when the quadratic expression in the two variables e and u_n is positive definite. This requires

$$\lambda X \frac{\partial \rho}{\partial n} - \rho^2 X^2 > 0\tag{6.16c}$$

If we assume the axis n , normal to the equipotential surface, and the body force to be oriented in the same direction, X is positive, and the condition becomes

$$\frac{\partial}{\partial n} (\log \rho) > \frac{\rho X}{\lambda}\tag{6.16d}$$

The derivative $(\partial/\partial n) \log \rho$ is the rate of change of $\log \rho$ in the direction of the body force. Hence stability requires that ρ be increasing in that direction. In addition, the rate of increase must be sufficiently large to satisfy the inequality (6.16d). For a constant gravity field this result yields the well-known stability condition of a horizontally stratified gas.

Another interpretation of this result is obtained by considering the condition of neutral equilibrium. We assume that a fluid particle is displaced while the total stress field of the fluid remains undisturbed. This means that the stress

* I. Tolstoy, The Theory of Waves in Stratified Fluids Including the Effect of Gravity and Rotation, *Reviews of Modern Physics*, Vol. 35, No. 1, pp. 207-230, 1963.

increment s' at a fixed point is zero. Using expression (5.24) with components of u_j and X_j along the normal direction and substituting $s = \lambda e$, we write

$$s' = \lambda e + \rho u_n X = 0 \tag{6.16e}$$

The buoyancy force (5.28) on the displaced particle must also vanish; that is

$$\rho e X + u_n X \frac{\partial \rho}{\partial n} = 0 \tag{6.16f}$$

Eliminating e and u_n between equations 6.16e and 6.16f, we obtain

$$\lambda \frac{\partial \rho}{\partial n} - \rho^2 X = 0 \tag{6.16g}$$

This result is also obtained when the inequality (6.16c) is replaced by an equality.

Variational Principle for Gravity Waves in a Liquid. In this case the fluid is incompressible, and we must satisfy the constraint

$$e = 0 \tag{6.17}$$

The expression (6.1) for \mathcal{W}_τ is considerably simplified by putting $e = 0$. It becomes

$$\mathcal{W}_\tau = \frac{1}{2} \iiint_\tau X_j \frac{\partial \rho}{\partial x_i} u_i u_j d\tau \tag{6.18}$$

The surface potential energy \mathcal{W}_F remains the same.

This result could have been derived immediately from the *analog model* discussed in section 5 for a non-homogeneous liquid under initial stress. In this case \mathcal{W}_τ represents the potential energy of the distributed buoyancy forces, and \mathcal{W}_F that of the surface forces acting in the model.

The variational formulation must take care of the constraint (6.17) for incompressibility. This can be accomplished in two ways.

This constraint can be introduced in the displacement field itself by choosing the unknown variables in such a way that the condition $e = 0$ is automatically satisfied. This will be illustrated in the example represented by equations 6.24 discussed below.

Another procedure is to free the variational process of the constraint by introducing a Lagrangian multiplier Λ . The variational principle is then

$$\delta \iiint_\tau \left(\frac{1}{2} X_j \frac{\partial \rho}{\partial x_i} u_i u_j + \Lambda e \right) d\tau + \iiint_\tau \rho a_i \delta u_i d\tau - \frac{1}{2} \delta \iint_F \rho X u_n^2 dA = 0 \tag{6.19}$$

The variations δu_i are now free of the constraint of incompressibility. We derive the differential equations

$$-\frac{\partial A}{\partial x_i} + X_j \frac{\partial \rho}{\partial x_i} u_j + \rho a_i = 0 \quad (6.20)$$

They are identical with the dynamical equations 5.32 provided we put

$$A = s' \quad (6.21)$$

This result provides a physical interpretation of the Lagrangian multiplier as the fictitious stress in the analog model. This interpretation is also in agreement with the boundary condition derived from equations 6.19.

Liquid in a Constant Gravity Field. As an illustration we consider a horizontally stratified liquid in a uniform gravity field. As before, we orient the z axis along the vertical and positively upward. The potential energy of the distributed buoyancy forces is

$$\mathcal{W}_\tau = -\frac{1}{2}g \iiint_\tau \frac{d\rho}{dz} w^2 d\tau \quad (6.22)$$

where w is the vertical displacement. The potential energy of the surface is

$$\mathcal{W}_F = \frac{1}{2}\rho g \iint_F w_F^2 dA \quad (6.23)$$

The problem was analyzed at the end of section 5 for a liquid on top of a horizontal rigid base. A displacement field in the x, z plane satisfying the condition of incompressibility ($e = 0$) is given by equations 5.39 and 5.42. It may be written

$$\begin{aligned} u &= -f' \sin lx e^{iat} \\ w &= lf \cos lx e^{iat} \\ v &= 0 \end{aligned} \quad (6.24)$$

where f is a function of z only. We have put

$$f' = \frac{df}{dz} \quad (6.25)$$

While f is an arbitrary function of z , the displacement field (6.24) always satisfies the condition of incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (6.26)$$

This field represents standing waves. They may be considered to result from two wave trains propagating in opposite directions. In formulating the variational problem the common exponential factor may be dropped. We find

$$\mathcal{W}_\tau = -\frac{1}{2} \int dx \cos^2 lx \int^z gl^2 \frac{d\rho}{dz} f^2 dz \quad (6.27)$$

The integral along x will be evaluated over one wavelength \mathcal{L} . Putting

$$\alpha_c^2 = -\frac{g}{\rho} \frac{d\rho}{dz} \quad (6.28)$$

we find

$$\mathcal{W}_\tau = \frac{1}{4} \mathcal{L} \int^z l^2 \rho \alpha_c^2 f^2 dz \quad (6.29)$$

Similarly the surface potential energy is

$$\mathcal{W}_F = \frac{1}{4} \mathcal{L} \rho gl^2 f_F^2 \quad (6.30)$$

where f_F is the value of f at the free surface. The dynamical term in the variational principle is

$$\iiint_\tau \rho a_i \delta u_i d\tau = \iiint_\tau \rho \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 w}{\partial t^2} \delta w \right) d\tau \quad (6.31)$$

When we drop the exponential time factor, equation 6.31 becomes

$$\iiint_\tau \rho a_i \delta u_i d\tau = -\frac{1}{2} \alpha^2 \delta \iiint_\tau \rho (u^2 + w^2) d\tau \quad (6.32)$$

With the values (6.24) and by integration over one wavelength along x we find

$$\iiint_\tau \rho a_i \delta u_i d\tau = -\frac{1}{4} \mathcal{L} \alpha^2 \delta \int^z \rho (f'^2 + l^2 f^2) dz \quad (6.33)$$

The variational principle (6.14) is

$$\delta \mathcal{W}_\tau + \delta \mathcal{W}_F + \iiint_\tau \rho a_i \delta u_i d\tau = 0 \quad (6.34)$$

Substituting the values (6.29), (6.30), and (6.33), we find

$$\delta \int^z [l^2 \rho (\alpha_c^2 - \alpha^2) f^2 - \alpha^2 \rho f'^2] dz + \delta (\rho gl^2 f_F^2) = 0 \quad (6.35)$$

Except for the condition $f = w = 0$ at the rigid bottom, arbitrary variations are applied to f . This equation is immediately recognized as the variational principle for the Sturm-Liouville equation

$$\alpha^2(\rho f')' + l^2\rho(\alpha_c^2 - \alpha^2)f = 0 \quad (6.36)$$

In addition, it also yields the boundary condition at the surface

$$\alpha^2 f_F' = gl^2 f_F \quad (6.37)$$

These results are identical with the differential equation 5.43 and the boundary condition (5.46) obtained by a different method.

The Unmodified Variational Principle. Other forms of the variational principles may, of course, be obtained by using the general variational equations discussed in section 2 of this chapter for the dynamics of an elastic continuum and applying them to the special case of a fluid.

Let us go back to the variational equation 2.13 for the elastic continuum. It is written

$$\begin{aligned} \delta \iiint \Delta V \, d\tau + \iiint \rho a_i \delta u_i \, d\tau \\ = \iiint \Delta X_{i\rho} \delta u_i \, d\tau + \iint_A \Delta f_i \delta u_i \, dA \end{aligned} \quad (6.38)$$

The term ΔV is defined by equation 2.12. Let us evaluate this term for a fluid. A similar derivation was obtained in section 5 of Chapter 3 in the discussion of the variational principles for the stability of an elastic solid in the presence of hydrostatic stress. The value of t_{ij} is given by equation 5.36 of Chapter 3:

$$t_{ij} = s_{ij} + S_{ij}e - \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (6.39)$$

The stresses being isotropic, we must substitute

$$\begin{aligned} s_{ij} &= s\delta_{ij} \\ S_{ij} &= S\delta_{ij} \end{aligned} \quad (6.40)$$

Hence

$$t_{ij} = (s + Se)\delta_{ij} - Se_{ij} \quad (6.41)$$

With the values (6.40) and (6.41), expression (2.12) for ΔV becomes

$$\Delta V = \frac{1}{2}se + \frac{1}{2}Se^2 + \frac{1}{2}S(2e_{ij}\omega_{ij} + \omega_{ij}\omega_{ij} - e_{ij}e_{ij}) \quad (6.42)$$

The last term is easily simplified. Hence

$$\Delta V = \frac{1}{2}se + \frac{1}{2}S \left(e^2 - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) \quad (6.43)$$

We recognize here the quantity

$$\mathcal{R} = \frac{1}{2}S \left(e^2 - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) \quad (6.44)$$

defined by equation 5.42 of Chapter 3.

We may write

$$\Delta V = \frac{1}{2}se + \mathcal{R} \quad (6.45)$$

With this value of ΔV the variational principle (6.38) takes the form

$$\begin{aligned} \delta \iiint_{\tau} \left(\frac{1}{2}se + \mathcal{R} \right) d\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau \\ = \iiint_{\tau} \Delta X_i \rho \delta u_i d\tau + \iint_A \Delta f_i \delta u_i dA \end{aligned} \quad (6.46)$$

The usual procedure of integration by parts applied to the volume integrals yields the differential equations

$$-\frac{\partial s}{\partial x_i} - \frac{\partial}{\partial x_i} (Se) + \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \right) + \rho a_i = \rho \Delta X_i \quad (6.47)$$

or

$$\frac{\partial s}{\partial x_i} + \rho \Delta X_i + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} = \rho a_i \quad (6.48)$$

This result coincides with the dynamical equations 5.6. This particular form of the equations therefore corresponds to the variational principle (6.46).

A more restricted form of this principle is derived by introducing the body force potential U and by linearizing the body force increment ΔX_i . The procedure is the same as in the derivation of equations 5.19. According to equation 5.18, we write ΔX_i as

$$\Delta X_i = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \quad (6.49)$$

Here we encounter again the incremental body force potential ΔU defined by equation 4.3 of Chapter 3. It is written

$$\Delta U = \frac{1}{2} \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \quad (6.50)$$

Hence

$$\Delta X_i \delta u_i = -\delta \Delta U \quad (6.51)$$

By substituting this value into equation 6.46 it becomes

$$\begin{aligned} \delta \iiint_{\tau} (\tfrac{1}{2}se + \mathcal{R} + \rho \Delta U) d\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau \\ = \iint_A \Delta f_i \delta u_i dA \end{aligned} \quad (6.52)$$

This result will be referred to as the *unmodified variational principle*. It leads directly to the dynamical equations 5.19. Note that it corresponds to the linearized form of the unmodified equations of propagation.

Comparing with equation 2.18, we note that the variational principle (6.52) is in a form corresponding to that for the elastic solid. The term \mathcal{R} in the expression of the elastic energy ΔV represents the product of the initial stress S by the *second order volume change*. This can be shown by expanding the Jacobian of u_i with respect to x_i . The second order term of this Jacobian is the factor multiplying S in the expression (6.44) of \mathcal{R} .

This representation of the energy is quite different from the one appearing for the modified principle (equation 6.7), where \mathcal{W}_{τ} is expressed by means of buoyancy forces.

We must also consider the value of the incremental boundary force Δf_i for this case. This was derived in Chapter 3 and given by equation 4.54 of that chapter:

$$\Delta f_i = (s + Se)n_i - S \frac{\partial u_j}{\partial x_i} n_j \quad (6.53)$$

where n_i is the unit outward normal to the boundary at the initial point.

Rigid Boundary Potential of the Unmodified Principle. Let us go back to equation 2.19 for the general case of an elastic continuum. It defines \mathcal{P}_{τ} as

$$\mathcal{P}_{\tau} = \iiint_{\tau} (\Delta V + \rho \Delta U) d\tau \quad (6.54)$$

For a fluid we have shown that ΔV is reduced to expression (6.45). Hence \mathcal{P}_τ becomes

$$\mathcal{P}_\tau = \iiint_{\tau} (\frac{1}{2}se + \mathcal{R} + \rho \Delta U) d\tau \quad (6.55)$$

The variational principle (6.52) may be written

$$\delta \mathcal{P}_\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau = \iint_A \Delta f_i \delta u_i dA \quad (6.56)$$

This relation is identical in form with equation 2.20.

As already pointed out in section 4 of Chapter 3 and again in section 2 of this chapter, special attention must be paid to the significance of the integral on the right side of equation 6.56.

Let us briefly recall the results previously derived for the elastic continuum. The boundary of the fluid is composed of free surfaces F and rigid boundaries B . At a free surface the initial and incremental stresses are zero; that is,

$$s = S = 0 \quad (6.57)$$

On the other hand, at the rigid boundary we assume that the displacements are tangent to the boundary at the initial point,

$$n_i u_i = n_i \delta u_i \quad (6.58)$$

The initial stress S at this point is, of course, normal to the boundary. The reason for adopting this boundary condition is that it coincides with the linear boundary conditions generally imposed when solving the linear differential equations of the problem. As pointed out in previous discussions, for a curved surface this violates the actual boundary condition by a second order error. It was shown that the surface integral on the right side of equation 6.56 does not vanish and can be expressed by means of the curvature of the rigid boundary. This result was obtained as follows.

By relations (6.53) we write

$$\iint_A \Delta f_i \delta u_i dA = \iint_A \left[(s + Se)n_i - S \frac{\partial u_j}{\partial x_i} n_j \right] \delta u_i dA \quad (6.59)$$

Under the boundary conditions (6.57) and (6.58) this expression is simplified to

$$\iint_A \Delta f_i \delta u_i dA = - \iint_B S \frac{\partial u_j}{\partial x_i} n_j \delta u_i dA \quad (6.60)$$

where the surface integral is now restricted to the rigid boundary B . By equations 4.66 of Chapter 3 we have shown that the surface integral (6.60) is an exact differential

$$\iint_B S \frac{\partial u_j}{\partial x_i} n_j \delta u_i dA = \frac{1}{2} \delta \iint_B S \phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j dA \quad (6.61)$$

The function $F(x_1, x_2, x_3)$ defines the rigid boundary by the equation

$$F(x_1, x_2, x_3) = 0 \quad (6.62)$$

and

$$\phi = \pm \left[\left(\frac{\partial F}{\partial x_1} \right)^2 + \left(\frac{\partial F}{\partial x_2} \right)^2 + \left(\frac{\partial F}{\partial x_3} \right)^2 \right]^{-1/2} \quad (6.63)$$

The \pm sign is so chosen that

$$n_i = \phi \frac{\partial F}{\partial x_i} \quad (6.64)$$

represents a unit normal vector oriented outward from the fluid. Hence by putting

$$\mathcal{P}_B = -\frac{1}{2} \iint_B S \phi \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j dA \quad (6.65)$$

the variational principle (6.56) becomes

$$\delta(\mathcal{P}_\tau + \mathcal{P}_B) + \iiint_\tau \rho a_i \delta u_i d\tau = 0 \quad (6.66)$$

Finally we may define a total potential energy of the fluid as

$$\mathcal{P} = \mathcal{P}_\tau + \mathcal{P}_B \quad (6.67)$$

and write the variational principle

$$\delta \mathcal{P} + \iiint_\tau \rho a_i \delta u_i d\tau = 0 \quad (6.68)$$

It is exactly of the same form as equation 6.14. However, the definition of \mathcal{P} is different.

Equivalence of the Modified and Unmodified Variational Principles. It can be shown that there is a rigorous equivalence between the modified principle (6.7) and the unmodified principle (6.52). They both imply the existence of a body force potential and

the linearization of the incremental body force. A direct verification of this equivalence is obtained by going back to equation 5.48 of Chapter 3. It may be written

$$\frac{\partial}{\partial x_i} \left(S e \delta u_i + \frac{\partial S}{\partial x_j} u_j \delta u_i \right) - \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \delta u_i \right) = \delta(\mathcal{R} + \rho \Delta U - \mathcal{Y}) \quad (6.69)$$

with

$$\mathcal{Y} = \rho X_{,j} u_{,j} e + \frac{1}{2} X_j \frac{\partial \rho}{\partial x_i} u_i u_j \quad (6.70)$$

We substitute the value

$$\frac{\partial S}{\partial x_j} = -\rho X_j \quad (6.71)$$

into equation 6.69 and rearrange the terms. It becomes

$$\delta(\mathcal{R} + \rho \Delta U) - \frac{\partial}{\partial x_i} (S e \delta u_i) + \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \delta u_i \right) = \delta \mathcal{Y} - \frac{\partial}{\partial x_i} (\rho X_{,j} u_j \delta u_i) \quad (6.72)$$

We now add the expression

$$\delta \left(\frac{1}{2} s e \right) + \rho a_i \delta u_i - \frac{\partial}{\partial x_i} (s \delta u_i) \quad (6.73)$$

to both sides of equation 6.72 and integrate over the volume τ of the fluid. Taking into account the value (6.53) for Δf_i , we obtain

$$\begin{aligned} \delta \iiint_{\tau} \left(\frac{1}{2} s e + \mathcal{R} + \rho \Delta U \right) d\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau - \iint_A \Delta f_i \delta u_i dA \\ = \delta \iiint_{\tau} \left(\frac{1}{2} s e + \mathcal{Y} \right) d\tau + \iiint_{\tau} \rho a_i \delta u_i d\tau \\ - \iint_A \left(s + \rho X_{,j} u_j \right) n_i \delta u_i dA \end{aligned} \quad (6.74)$$

Note that expression (6.1) is

$$\mathcal{W}_{\tau} = \iiint_{\tau} \left(\frac{1}{2} s e + \mathcal{Y} \right) d\tau \quad (6.75)$$

By equating to zero the right side of equation 6.74 we obtain the variational principle (6.7). Equating to zero the left side, we obtain

the variational principle (6.52). Hence we have established the equivalence of these two principles.

Lagrangian Equations and Generalized Coordinates. Both the modified variational principle (6.14) and the unmodified principle (6.68) are expressed by the single equation

$$\delta \mathcal{P} + \iiint_{\tau} \rho a_i \delta u_i d\tau = 0 \quad (6.76)$$

However, the value of \mathcal{P} has a dual representation. In the modified principle we put

$$\mathcal{P} = \mathcal{W}_{\tau} + \mathcal{W}_F \quad (6.77)$$

For the unmodified principle we use

$$\mathcal{P} = \mathcal{P}_{\tau} + \mathcal{P}_B \quad (6.78)$$

In both cases the potential energy includes a surface integral. For the modified principle the surface integral is \mathcal{W}_F , and it is extended to the free surface. For the unmodified principle the surface integral is \mathcal{P}_B , and it is extended to the rigid boundary and depends on its curvature.

We have also noted the difference between the volume energies \mathcal{P}_{τ} and \mathcal{W}_{τ} . The first is expressed in terms of the stress and the volume change, whereas the second introduces the energy of the buoyancy force.

The variational principle (6.76) leads immediately to the treatment of acoustic-gravity waves by Lagrangian equations and generalized coordinates. We represent the displacement field as

$$u_i = u_{ij}(x, y, z)q_j \quad (6.79)$$

The generalized coordinates q_j are the unknown amplitudes of fixed configuration fields u_{ij} functions of the space coordinates. The fields u_{ij} are assumed to satisfy the required kinematic constraints of being tangent to the rigid boundaries.

Note that for an incompressible liquid the configurations are chosen to satisfy the additional constraint of being divergence-free.

We proceed exactly as in the discussion at the end of section 2. The value of \mathcal{P} is the quadratic form

$$\mathcal{P} = \frac{1}{2} a_{ij} q_i q_j \quad (6.80)$$

and the kinetic energy is

$$\mathcal{T} = \frac{1}{2} \iiint_{\tau} \rho \dot{u}_i \dot{u}_i d\tau \tag{6.81}$$

or

$$\mathcal{T} = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \tag{6.82}$$

The Lagrangian equations are

$$a_{ij} q_j + m_{ij} \ddot{q}_j = 0 \tag{6.83}$$

For the value (6.80) of \mathcal{P} we may use either of the definitions (6.77) or (6.78). Equations 6.83 are the same as those of the classical theory of oscillations of a conservative system. They lead readily to the application of normal coordinate methods as developed for the analysis of transient wave propagation due to a pulse excitation.*

Hamilton's principle as expressed by equation 2.43 is also applicable in the same form for the particular case of a fluid.

Inclusion of Coriolis Forces in the Equations with Generalized Coordinates. Equations (6.83) for the generalized coordinates q_i may be extended to include the Coriolis forces by applying the variational principle 6.76. We must replace the acceleration a_i by the value which was used in equation 5.9. This value is

$$a_k = \ddot{u}_k + 2\Omega_{k\mu} \dot{u}_\mu \tag{6.83a}$$

The angular velocity of the frame of reference is represented by $\Omega_{k\mu}$. With the value (6.79) for u_k we derive

$$a_k \delta u_k = (u_{kj} u_{ki} \dot{q}_j + 2\Omega_{k\mu} u_{\mu j} u_{ki} \dot{q}_j) \delta q_i \tag{6.83b}$$

Substitution of this expression into the variational principle (6.76) yields the dynamical equations

$$a_{ij} q_j + m_{ij} \ddot{q}_j + c_{ij} \dot{q}_j = 0 \tag{6.83c}$$

The coefficients

$$c_{ij} = 2\Omega_{k\mu} \iiint_{\tau} \rho u_{\mu j} u_{ki} d\tau \tag{6.83d}$$

are due to the Coriolis forces. The definition (5.10) of $\Omega_{k\mu}$ shows that

$$\Omega_{k\mu} = -\Omega_{\mu k} \tag{6.83e}$$

This implies

$$c_{ij} = -c_{ji} \tag{6.83f}$$

* M. A. Biot and I. Tolstoy, Formulation of Wave Propagation in Infinite Media by Normal Coordinates with an Application to Diffraction, *The Journal of the Acoustical Society of America*, Vol. 29, No. 3, pp. 381-391, 1957.

Hence the coefficients c_{ij} are skew-symmetric. The other coefficients, a_{ij} and m_{ij} , are symmetric:

$$a_{ij} = a_{ji} \quad m_{ij} = m_{ji} \quad (6.83g)$$

They are the same as in equations 6.83.

Curvilinear Coordinates. The foregoing equations for acoustic-gravity waves have been derived in cartesian coordinates. However, they are easily obtained in orthogonal curvilinear coordinates. For the modified equations (5.23) the procedure is particularly simple since all the terms in the equations are expressed by scalars, vectors, and the scalar products of vectors. The volume change e is a scalar quantity which appears as such and through its gradient. This volume change in curvilinear coordinates is obtained by using expressions (5.54) of Chapter 2 for the strain components. The unmodified equations in curvilinear coordinates may be derived from the modified form. They may also be obtained directly from the variational principle using expression (6.42) for ΔV after introducing the values (5.54) and (5.55) of Chapter 2 for the strain and the rotation in curvilinear coordinates.

7. DYNAMICS OF ELASTIC PLATES AND MULTILAYERED MEDIA UNDER INITIAL STRESS

In the preceding chapters we have treated static problems of plate mechanics by introducing the simplifying assumption that the material is incompressible. In this section we consider problems of vibrations and wave propagation in elastic plates and multilayered media under initial stress. In dynamical problems it is essential to take into account the compressibility of the material. It will be shown that the general solutions for this case can be derived in relatively simple form by the same procedure as in the analysis of the stability problems in Chapter 4. These solutions were presented in a recent paper.*

Consider the elastic plate illustrated in Figure 5.1 of Chapter 4. The x and y axes are in the plane of the figure. The thickness of the plate is denoted by h , and the faces are located in the planes $y = \pm h/2$.

* M. A. Biot, Continuum Dynamics of Elastic Plates and Multilayered Solids under Initial Stress, *Journal of Mathematics and Mechanics*, Vol. 12, No. 6, pp. 793-810, 1963.

The initial stress is a uniform compression $P = -S_{11}$ acting along the x direction.

An initial stress component S_{33} perpendicular to the plane of the figure may be present. However, it does not appear explicitly in the equations.

We shall solve the problem of vibrations for two-dimensional incremental deformations in the x, y plane.

The dynamical equations 4.1 for two-dimensional deformations applied to the case under discussion become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \quad (7.1)$$

The mass density of the plate is denoted by ρ . It is recalled that u and v are the displacement components in the x, y plane and ω is the rotation:

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (7.2)$$

Equations 7.1 are also derived from equations 2.1 of Chapter 3 by adding the inertia terms on the right side. The stress-strain relations as given by equations 8.31a of Chapter 2 are

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \quad (7.3)$$

The coefficients must satisfy the condition

$$B_{12} = B_{21} + P \quad (7.4)$$

which is required for the existence of a potential energy.

The strain components were defined as

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} & e_{yy} &= \frac{\partial v}{\partial y} \\ e_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \quad (7.5)$$

The stress-strain relations (7.3) correspond to a material orthotropic

under initial stress. The orthotropy may be intrinsic, or it may be induced by the state of initial stress.

It should be kept in mind that the elastic properties represented by the stress-strain relations (7.3) refer only to incremental deformations. The initial stress, on the other hand, does not have to be associated with an elastic finite strain and may result from any other physical cause.

Vibrations of the plate and propagating waves are represented by solutions proportional to the exponential factor $\exp(i\omega t)$. All equations may be written by omitting this factor. The dynamical equations 7.1 then become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} + \alpha^2 \rho u &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} + \alpha^2 \rho v &= 0 \end{aligned} \quad (7.6)$$

By substituting expression (7.2), (7.3), and (7.5) into equations 7.6 we derive

$$\begin{aligned} B_{11} \frac{\partial^2 u}{\partial x^2} + (Q + \frac{1}{2}P) \frac{\partial^2 u}{\partial y^2} \\ + (B_{12} + Q - \frac{1}{2}P) \frac{\partial^2 v}{\partial x \partial y} + \alpha^2 \rho u &= 0 \\ B_{22} \frac{\partial^2 v}{\partial y^2} + (Q - \frac{1}{2}P) \frac{\partial^2 v}{\partial x^2} \\ + (B_{21} + Q + \frac{1}{2}P) \frac{\partial^2 u}{\partial x \partial y} + \alpha^2 \rho v &= 0 \end{aligned} \quad (7.7)$$

These are the two equations which must be satisfied by the displacements u and v .

We shall seek solutions which are sinusoidal along x . They are of the type

$$\begin{aligned} u &= U(y) \sin lx \\ v &= V(y) \cos lx \end{aligned} \quad (7.8)$$

In order to introduce the boundary conditions into the problem we also require an expression for the forces acting on a deformed surface which is initially a plane perpendicular to the y axis. The x and y

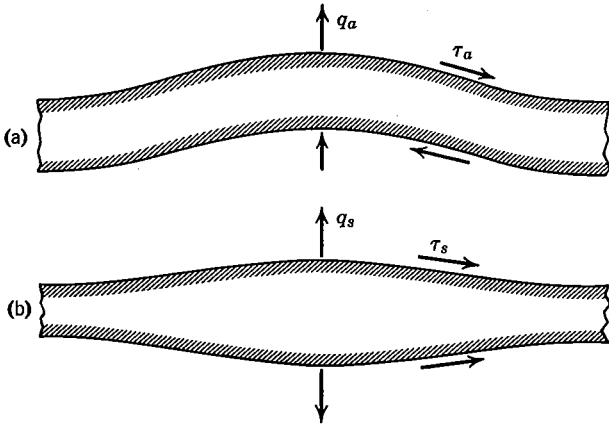


Figure 7.1 Antisymmetric (a) and symmetric (b) oscillations of a plate under normal and tangential surface excitations.

components of these forces are expressed by equations 4.8 of Chapter 4. They are

$$\begin{aligned} \Delta f_x &= s_{12} + P e_{xy} \\ \Delta f_y &= s_{22} \end{aligned} \tag{7.9}$$

The significance of these quantities was discussed in section 4 of Chapter 4 and illustrated in Figure 4.2 of that chapter. The force components (7.9) corresponding to the sinusoidal solution (7.8) are written

$$\begin{aligned} \Delta f_x &= \tau(l_y) \sin lx \\ \Delta f_y &= q(l_y) \cos lx \end{aligned} \tag{7.10}$$

In deriving these solutions we shall follow exactly the same procedure as in section 5 of Chapter 4 by considering separately the symmetric and the antisymmetric case.

Antisymmetric Case. The deformation of the plate in this case is represented in Figure 7.1a. Both faces of the plate remain equidistant, and the deformation is of the flexural type. By substituting expressions (7.8) into equations 7.7 we find two ordinary simultaneous differential equations for $U(l_y)$ and $V(l_y)$. The general solution of these equations for antisymmetric deformations is

$$\begin{aligned} U(l_y) &= C_1 \sinh \beta_1 l_y + C_2 \sinh \beta_2 l_y \\ V(l_y) &= C'_1 \cosh \beta_1 l_y + C'_2 \cosh \beta_2 l_y \end{aligned} \tag{7.11}$$

The values of β_1 and β_2 are obtained from the roots of the characteristic equation

$$\beta^4 - 2m\beta^2 + k^2 = 0 \quad (7.12)$$

with

$$2m = \frac{1}{LB_{22}} \left[\Omega B_{22} - L \left(2B_{21} + P + \frac{\alpha^2 \rho}{l^2} \right) - B_{21}^2 \right] \quad (7.13)$$

$$k^2 = \frac{\Omega}{LB_{22}} \left(L - P - \frac{\alpha^2 \rho}{l^2} \right)$$

We have put

$$\Omega = B_{11} - \frac{\alpha^2 \rho}{l^2} \quad (7.14)$$

$$L = Q + \frac{1}{2}P$$

The quantity L is the slide modulus, also discussed previously. In deriving these results we have also taken into account the value (7.4) for B_{12} .

The values of β_1^2 and β_2^2 are

$$\beta_1^2 = m + \sqrt{m^2 - k^2} \quad (7.15)$$

$$\beta_2^2 = m - \sqrt{m^2 - k^2}$$

The signs of β_1 and β_2 may be chosen arbitrarily in such a way that they possess no negative real part.

The constants of integration C'_i are given in terms of C_i by the relation

$$C'_i = \frac{-\Omega + L\beta_i^2}{(B_{21} + L)\beta_i} C_i \quad (7.16)$$

Hence the solution (7.11) contains only two independent constants C_1 and C_2 . Substitution of the values (7.8) with solution (7.11) into expressions (7.9) yields $\tau(y)$ and $q(y)$ defined by equations 7.10. Their value for $y = h/2$ is

$$\tau_a = \tau(\frac{1}{2}lh) \quad (7.17)$$

$$q_a = q(\frac{1}{2}lh)$$

These quantities represent the tangential and normal forces applied at the top face of the plate. The tangential and normal displacements at the top face are similarly defined as

$$U_a = U(\frac{1}{2}lh) \quad (7.18)$$

$$V_a = V(\frac{1}{2}lh)$$

Equations 7.17 and 7.18 contain the two constants of integration C_1 and C_2 . Their elimination in these two equations yields the relations

$$\begin{aligned} \frac{\tau_a}{lL} &= a_{11}U_a + a_{12}V_a \\ \frac{q_a}{lL} &= a_{12}U_a + a_{22}V_a \end{aligned} \tag{7.19}$$

They represent forced oscillations of the flexural type at a given frequency α under the action of sinusoidally distributed tangential and normal forces τ_a and q_a (Fig. 7.1a).

The expressions for the coefficients a_{ij} are simplified by putting

$$\begin{aligned} \gamma &= \frac{1}{2}lh = \frac{\pi h}{\mathcal{L}} \\ z_1 &= \beta_1 \tanh \beta_1 \gamma \\ z_2 &= \beta_2 \tanh \beta_2 \gamma \end{aligned} \tag{7.20}$$

The parameter γ is the same as defined previously in terms of the wavelength \mathcal{L} measured along the plate. The coefficients in equations 7.19 are written

$$\begin{aligned} a_{11} &= \Omega(\beta_2^2 - \beta_1^2) \frac{1}{A_a} \\ a_{22} &= B_{22}(\beta_2^2 - \beta_1^2)z_1z_2 \frac{1}{A_a} \\ a_{12} &= [(\Omega + B_{21}\beta_2^2)z_1 - (\Omega + B_{21}\beta_1^2)z_2] \frac{1}{A_a} \end{aligned} \tag{7.21}$$

with

$$A_a = (\Omega - L\beta_1^2)z_2 - (\Omega - L\beta_2^2)z_1 \tag{7.22}$$

When deriving these expressions, considerable simplification is obtained by cancellation of common factors in numerators and denominators. In order to bring out these common factors it is necessary to use the relation

$$B_{22} = \frac{\Omega(B_{21} + L)^2}{(\Omega - L\beta_1^2)(\Omega - L\beta_2^2)} \tag{7.23}$$

This relation may be verified by substituting the values $\beta_1^2 + \beta_2^2 = 2m$ and $\beta_1^2\beta_2^2 = k^2$ which are consequences of the characteristic equation 7.12.

When the exciting force is normal to the plate, the deflection V_a is obtained by putting $\tau_a = 0$ in equations 7.19. Elimination of U_a yields

$$\frac{q_a}{lL} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} V_a \quad (7.24)$$

Using the values (7.21) for a_{ij} , we find

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{R_1 z_2 - R_2 z_1}{\Omega(\beta_1^2 - \beta_2^2)} \quad (7.25)$$

with

$$R_1 = \frac{(\Omega + B_{21}\beta_1^2)^2}{\Omega - L\beta_1^2} \quad (7.26)$$

$$R_2 = \frac{(\Omega + B_{21}\beta_2^2)^2}{\Omega - L\beta_2^2}$$

Equation 7.25 is derived after cancellation of the common factor A_a in the numerator and the denominator. This again requires the use of expression (7.23) for B_{22} .

Note that the exciting force q_a in this case represents a normal traction at the top face and a normal push of equal magnitude at the bottom face (see Fig. 7.1a).

Symmetric Case. The solution for the symmetric case is obtained in similar fashion. The type of deformation of the plate is represented in Figure 7.1b. The solution (7.11) is replaced by

$$U(y) = C_1 \cosh \beta_1 ly + C_2 \cosh \beta_2 ly \quad (7.27)$$

$$V(y) = C'_2 \sinh \beta_1 ly + C'_1 \sinh \beta_2 ly$$

The procedure followed is exactly the same as for the antisymmetric case. The applied forces τ_s , q_s and the displacements U_s , V_s at the top surface ($y = h/2$) are related by the equations

$$\frac{\tau_s}{lL} = b_{11}U_s + b_{12}V_s \quad (7.28)$$

$$\frac{q_s}{lL} = b_{12}U_s + b_{11}V_s$$

We put

$$z'_1 = \frac{1}{\beta_1} \tanh \beta_1 \gamma \quad (7.29)$$

$$z'_2 = \frac{1}{\beta_2} \tanh \beta_2 \gamma$$

The coefficients

$$\begin{aligned}
 b_{11} &= \Omega(\beta_2^2 - \beta_1^2)z_1'z_2' \frac{1}{A_s} \\
 b_{22} &= B_{22}(\beta_2^2 - \beta_1^2) \frac{1}{A_s} \\
 b_{12} &= [(\Omega + B_{21}\beta_2^2)z_2' - (\Omega + B_{21}\beta_1^2)z_1'] \frac{1}{A_s}
 \end{aligned}
 \tag{7.30}$$

with

$$A_s = (\Omega - L\beta_1^2)z_1' - (\Omega - L\beta_2^2)z_2'
 \tag{7.31}$$

Comparing the coefficients (7.21) and (7.30), we note that the values of b_{ij} may be derived from those of a_{ij} by simply interchanging the sinh and cosh functions. This amounts to interchanging z_1 and z_2 with $1/z_1'$ and $1/z_2'$ respectively.

The forced oscillation under forces normal to the surface of the plate are obtained by putting $\tau_s = 0$ in equations 7.28. We derive

$$\frac{q_s}{lL} = \frac{b_{11}b_{22} - b_{12}^2}{b_{11}} V_s
 \tag{7.32}$$

Evaluation of the coefficient on the right side yields

$$\frac{b_{11}b_{22} - b_{12}^2}{b_{11}} = \frac{R_1z_1' - R_2z_2'}{\Omega(\beta_1^2 - \beta_2^2)z_1'z_2'}
 \tag{7.33}$$

Free Oscillations and Surface Waves. There are two types of free oscillations corresponding to symmetric and antisymmetric deformations. We consider first antisymmetric waves. In the absence of exciting forces we put $q_a = \tau_a = 0$ in equations 7.19. This yields the characteristic equation

$$a_{11}a_{22} - a_{12}^2 = 0
 \tag{7.34}$$

When we use the value (7.25), this equation becomes

$$R_1z_2 - R_2z_1 = 0
 \tag{7.35}$$

By solving this equation for the frequency α as a function of the wavelength we obtain the velocity dispersion branches for antisymmetric waves in the plate.

For symmetric waves we put expression (7.33) equal to zero, and the characteristic equation becomes

$$R_1 z'_1 - R_2 z'_2 = 0 \quad (7.36)$$

The frequency equation for surface waves is immediately derived from these results. Surface waves are obtained for the limiting case of a plate of infinite thickness. In addition, in order to eliminate body waves, we must assume that the values of β_1 and β_2 are real or complex conjugate. Their sign may be chosen arbitrarily so that the real values and real parts are positive. Infinite thickness corresponds to putting $\gamma = \infty$. Hence in this case

$$\tanh \beta_1 \gamma = \tanh \beta_2 \gamma = 1 \quad (7.37)$$

and

$$z_1 = \beta_1 \quad z_2 = \beta_2 \quad (7.38)$$

Equation 7.35 becomes

$$R_1 \beta_2 - R_2 \beta_1 = 0 \quad (7.39)$$

This equation may be solved for α/l as the unknown. This yields the phase velocity of the surface wave, which is independent of the wavelength. The same equation is obtained by introducing the limiting values (7.37) into equation 7.36 for the symmetric waves.

Surface wave velocities corresponding to solutions of equation 7.39 have been evaluated by Buckens.*

Incompressible Medium. When the material of the plate is incompressible, we may derive the particular values of the coefficients a_{ij} and b_{ij} by writing the elastic coefficients in the form

$$\begin{aligned} B_{11} &= K + N + P \\ B_{12} &= K - N + P \\ B_{21} &= K - N \\ B_{22} &= K + N \end{aligned} \quad (7.40)$$

These expressions are the same as those of equations 8.31e of Chapter

* F. Buckens, The Velocity of Rayleigh Waves along a Prestressed Semi-infinite Medium Assuming Two-Dimensional Anisotropy, *Annali di Geofisica*, Vol. XI, No. 2, pp. 99-112, 1958.

2 where it was shown that incompressibility corresponds to the limiting case obtained by putting

$$K = \infty \quad (7.41)$$

By substituting the limiting values in equation 7.13 we find

$$2m = \frac{1}{L} \left(4M - 2L - \frac{\alpha^2 \rho}{l^2} \right) \quad (7.42)$$

$$k^2 = \frac{1}{L} \left(L - P - \frac{\alpha^2 \rho}{l^2} \right)$$

with

$$M = N + \frac{1}{4}P \quad (7.43)$$

The same substitution and limiting process may be introduced in the coefficients a_{ij} and b_{ij} . For example, for the antisymmetric case expressions (7.21) become

$$a_{11} = \frac{\beta_2^2 - \beta_1^2}{z_2 - z_1}$$

$$a_{22} = \frac{(\beta_2^2 - \beta_1^2)}{z_2 - z_1} z_1 z_2 \quad (7.44)$$

$$a_{12} = \frac{(\beta_2^2 + 1)z_1 - (\beta_1^2 + 1)z_2}{z_2 - z_1}$$

These results are identical with the coefficients (5.8) of Chapter 4, which were derived directly for the static problem in connection with an incompressible material.

The result is formally the same in the present dynamical problem because the frequency appears explicitly only in the values (7.42) for m and k^2 which in turn determine the values of β_1 and β_2 .

By a similar procedure in the limiting case of incompressibility, expressions (7.30) for the coefficients b_{ij} become identical with the values (5.16) of Chapter 4.

Multilayered Media. In the absence of gravity forces the dynamics of multilayered compressible elastic media under initial stress is formally identical with the theory of stability of incompressible multilayered media developed in section 7 of Chapter 4. The basic reason for this formal identity resides in the fact that in both cases the behavior of a single plate may be derived by the superposition of symmetric and antisymmetric solutions which are completely determined by six matrix elements a_{ij} and b_{ij} .

When gravity is present, this procedure is not generally applicable because the symmetry is destroyed. Disturbances in a compressible medium produce changes of density. In the presence of a gravity field a coupling between symmetric and antisymmetric deformations of a layer results. However, for the particular case of an incompressible medium this difficulty disappears when an analog model is introduced. We shall return briefly to this topic later.

Let us assume that there are no gravity forces. The multilayered medium is illustrated in Figure 7.1 of Chapter 4. The principal directions of initial stress are the same in all layers. The initial stress component S_{22} normal to the layers is also the same in all layers. The component S_{11} parallel to the layers may be different in each layer. As shown in Chapter 4 (section 5), all equations derived for the single plate are applicable to each layer provided we replace P by

$$P_j = S_{22} - S_{11}^{(j)} \quad (7.45)$$

where $S_{11}^{(j)}$ denotes the component S_{11} of the initial stress in the j th layer.

The six basic matrix elements attached to a particular layer are

$$\begin{aligned} A_1 &= \frac{1}{2}(a_{11} + b_{11}) & A_4 &= \frac{1}{2}(a_{11} - b_{11}) \\ A_2 &= \frac{1}{2}(a_{12} + b_{12}) & A_5 &= \frac{1}{2}(a_{12} - b_{12}) \\ A_3 &= \frac{1}{2}(a_{22} + b_{22}) & A_6 &= \frac{1}{2}(a_{22} - b_{22}) \end{aligned} \quad (7.46)$$

The quantities a_{ij} and b_{ij} are now defined by expressions (7.21) and (7.30) derived in this section.

With these definitions the formulation of the dynamical problems is obtained by exactly the same formulas as in Chapter 4. Equation 7.1 of that chapter defines a quadratic form I_j for the j th layer. It is a quadratic function of the four displacement components $U_j, V_j, U_{j+1}, V_{j+1}$ at the top and bottom faces of this layer. The coefficients of this quadratic form are given by expressions (7.46). It is also assumed that there is perfect interfacial adherence. We also write

$$\mathcal{J} = \sum_j L_j I_j \quad (7.47)$$

where L_j is the slide modulus of the j th layer. The dynamical

equations are obtained by expressing the continuity of the stresses at the interfaces of the layers. They are written

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \quad \frac{\partial \mathcal{J}}{\partial V_j} = 0 \quad (7.48)$$

They are recurrence equations between the six displacements at three successive interfaces.

It will be noted that equations 7.48 are equivalent to the *variational principle* expressed as $\delta \mathcal{J} = 0$.

The frequency equation for wave propagation in the multilayered system is obtained by equating to zero the determinant of equations 7.48. Solving for the frequency as a function of wavelength yields the various branches of the dispersion curves.

We must add a remark concerning the case where the layers are in contact with a semi-infinite medium. (See Fig. 7.2, Chapter 4.) Since this semi-infinite medium may be considered as a layer of infinite thickness, this case falls within the foregoing formulation. There is a difference, however, in the nature of the matrix elements (7.46). If we assume that the wave is propagated without attenuation in the multilayered system, no radiation, i.e., no body waves, are generated in the adjacent semi-infinite medium. This means that for this adjacent medium neither of the roots β_1 and β_2 is pure imaginary. Their sign may be chosen so that their real part is positive. Then for an infinite thickness (i.e., $\gamma = \infty$) we find

$$\tanh \beta_1 \gamma = \tanh \beta_2 \gamma = 1 \quad (7.49)$$

Hence

$$\begin{aligned} z_1 &= \frac{1}{z_1} = \beta_1 \\ z_2 &= \frac{1}{z_2} = \beta_2 \end{aligned} \quad (7.50)$$

With such values the matrix elements (7.46) become

$$\begin{aligned} A_1 &= a_{11} = b_{22} & A_4 &= 0 \\ A_2 &= a_{12} = b_{22} & A_5 &= 0 \\ A_3 &= a_{22} = b_{22} & A_6 &= 0 \end{aligned} \quad (7.51)$$

The corresponding quadratic form (7.47) for the semi-infinite medium is then simplified accordingly.

Matrix Multiplication Procedure. For numerical computation the same matrix \mathcal{M} given by equation 7.15 of Chapter 4 may be used, the matrix multiplication procedure of Thomson and Haskell being followed.

Iteration Process for the Numerical Solution of the Recurrence Equations. Equations 7.48 are recurrence equations for the six displacements at three successive interfaces. Their numerical solution may be obtained by elementary procedures and standard programming, as already discussed, for the stability problem, at the end of section 7 in Chapter 4.

Multilayered Incompressible Medium in a Gravity Field.

For horizontal layers of incompressible materials the results may be extended to the case where a gravity field is present. This can be seen by applying the concept of an analog model whose validity for dynamical problems was discussed in section 2. In this model the gravity field is replaced by interfacial forces proportional to the vertical displacements. This was shown in section 7 of Chapter 4 for problems of stability of multilayered media in a gravity field. All the results become applicable to the dynamical case by simply introducing the values (7.21) and (7.30) for the coefficients a_{ij} and b_{ij} .

Elastic Material of Finite Isotropy. For a material which is elastic and isotropic in finite strain the incremental stress-strain relations were derived in Chapter 2. Particular attention was given to the derivation of a simple expression for the incremental shear coefficient Q . It should be pointed out that the other elastic coefficients may also be derived quite simply. With principal directions of finite strain oriented along the x, y, z axes and corresponding extension ratios $\lambda_1, \lambda_2, \lambda_3$, the principal stress along x is

$$S_{11} = F(\lambda_1, \lambda_2, \lambda_3) \quad (7.51a)$$

Because of isotropy the function F satisfies the relation

$$F(\lambda_1, \lambda_2, \lambda_3) = F(\lambda_1, \lambda_3, \lambda_2) \quad (7.51b)$$

For the same reason the other stress components are obtained from equation 7.51a by cyclic permutation. Hence

$$\begin{aligned} S_{22} &= F(\lambda_2, \lambda_3, \lambda_1) \\ S_{33} &= F(\lambda_3, \lambda_1, \lambda_2) \end{aligned} \quad (7.51c)$$

For incremental deformations in the x, y plane, λ_3 is constant. The incremental principal stresses are

$$\begin{aligned} s_{11} &= dS_{11} = \frac{\partial S_{11}}{\partial \lambda_1} d\lambda_1 + \frac{\partial S_{11}}{\partial \lambda_2} d\lambda_2 \\ s_{22} &= dS_{22} = \frac{\partial S_{22}}{\partial \lambda_1} d\lambda_1 + \frac{\partial S_{22}}{\partial \lambda_2} d\lambda_2 \end{aligned} \quad (7.51d)$$

With the strain components $e_{xx} = d\lambda_1/\lambda_1$ and $e_{yy} = d\lambda_2/\lambda_2$ the incremental stresses become

$$\begin{aligned} s_{11} &= \lambda_1 \frac{\partial S_{11}}{\partial \lambda_1} e_{xx} + \lambda_2 \frac{\partial S_{11}}{\partial \lambda_2} e_{yy} \\ s_{22} &= \lambda_1 \frac{\partial S_{22}}{\partial \lambda_1} e_{xx} + \lambda_2 \frac{\partial S_{22}}{\partial \lambda_2} e_{yy} \end{aligned} \tag{7.51e}$$

Comparing this result with equations 7.3, we derive

$$\begin{aligned} B_{11} &= \lambda_1 \frac{\partial S_{11}}{\partial \lambda_1} & B_{12} &= \lambda_2 \frac{\partial S_{11}}{\partial \lambda_2} \\ B_{21} &= \lambda_1 \frac{\partial S_{22}}{\partial \lambda_1} & B_{22} &= \lambda_2 \frac{\partial S_{22}}{\partial \lambda_2} \end{aligned} \tag{7.51f}$$

Existence of a potential energy requires that the following condition be verified (see equations 6.2 of Chapter 2).

$$B_{12} + S_{11} = B_{21} + S_{22} \tag{7.51g}$$

This condition may be written

$$\frac{\partial}{\partial \lambda_2} (\lambda_2 S_{11}) = \frac{\partial}{\partial \lambda_1} (\lambda_1 S_{22}) \tag{7.51h}$$

The coefficient Q is given by equation 7.15 of Chapter 2:

$$Q = \frac{1}{2}(S_{11} - S_{22}) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \tag{7.51i}$$

The coefficients for the other coordinate planes are derived similarly. Hence the incremental coefficients are completely defined by a *single function* $F(\lambda_1, \lambda_2, \lambda_3)$ relating directly measurable quantities. When this function corresponds to isothermal deformations, the incremental adiabatic coefficients are obtained from equations 4.35.

Stability. The foregoing analysis has been carried out in the context of wave propagation. However, the results derived in this section are also solutions of the more general problem of static and dynamic stability of elastic plates and multilayered media.

This can be seen by putting

$$p^2 = -\alpha^2 \tag{7.52}$$

Then the equations correspond to solutions proportional to $\exp(pt)$.

We have shown in section 2 that the roots p^2 of the characteristic equation must be real. A positive value of p^2 yields positive and negative values of p . To the positive value corresponds a characteristic solution proportional to an increasing exponential, which represents a dynamic buckling. On the other hand, a negative value of p^2 corresponds to oscillations of a stable system.

If the characteristic equation has a solution for $p = 0$, the equilibrium is neutral. Hence by putting $\alpha = 0$ in the equations of this

section we obtain the generalization to compressible media of the stability theory derived in Chapter 4 for plates and multilayered media in the particular case of incompressibility.

Thus we generalize to compressible media the equations obtained in Chapter 4 for *internal, surface, and interfacial* instability. For example, the condition of surface instability is obtained by putting $\alpha = 0$ in equation 7.39. The equation for internal instability is obtained by expressing the condition that at least one of the roots β_1 and β_2 be pure imaginary. This is done as in Chapter 4 in terms of the quantities m and k^2 which in this case are defined by putting $\alpha = 0$ in equations 7.13. The same distinction may be made between internal instability of the *first and second kind*, depending on whether one or two roots are pure imaginary.

There is an interesting similarity between problems of wave propagation and stability. This was pointed out in our discussion of internal, surface, and interfacial instability (section 1, Chapter 4), which are analogous, respectively, to body waves, Rayleigh waves, and Stoneley waves.

The general character of this analogy is illustrated by considering a single plate. The value of Ω in (7.14) may be written

$$\Omega = B_{11} + P - \left(P + \frac{\alpha^2 \rho}{l^2} \right) \quad (7.53)$$

We put

$$\begin{aligned} \mathcal{A} &= B_{11} + P \\ \mathcal{B} &= P + \frac{\alpha^2 \rho}{l^2} \end{aligned} \quad (7.54)$$

In the basic equations 7.13, 7.21, and 7.30, which govern the plate dynamics, the frequency α appears only in combination with P in the term \mathcal{B} . The other place where P appears explicitly is in the term \mathcal{A} , where it is combined with B_{11} . In general, the initial stress P will represent an insignificant fraction of B_{11} , so that in the first approximation the combined term \mathcal{A} may be considered to be independent of P . Then the characteristic solutions are represented by plotting $\mathcal{B} = P + \alpha^2 \rho / l^2$ as a function of the wavelength.

To this approximation the same curve represents either P with $\alpha = 0$ or $\alpha^2 \rho / l^2$ with $P = 0$. The first case corresponds to static

buckling, and the second corresponds to the phase velocity dispersion of acoustic waves in the absence of initial stress.

Dominant Wavelength in Dynamic Instability. When the layered system is unstable, the time variable is contained in the amplitude factor $\exp(pt)$ which represents an increasing exponential. The characteristic equation of the system may be solved for p as a function of the wavelength. This value of p will generally go through a maximum for a given wavelength \mathcal{L}_d which may be called the dominant wavelength. In a Fourier representation of an initial disturbance the amplitude of the component of dominant wavelength will grow much more rapidly and will represent the significant aspect of the deformation.

This may be illustrated for the simple example of a thin plate of thickness h under an axial compression P . The classical equation for oscillations of the plate is

$$\frac{1}{12} \frac{E}{1 - \nu^2} h^3 \frac{\partial^4 w}{\partial x^4} + Ph \frac{\partial^2 w}{\partial x^2} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{7.55}$$

where

- x = coordinate along the plate
- ρ = mass density
- w = normal deflection
- E = Young's modulus
- ν = Poisson's ratio

We put

$$w = w_0 e^{pt} \cos lx \tag{7.56}$$

Substitution of this value in equation 7.55 yields the characteristic equation

$$P - \frac{\rho p^2}{l^2} = \frac{1}{12} \frac{E}{1 - \nu^2} l^2 h^2 \tag{7.57}$$

We recognize on the left side the combination

$$\mathcal{B} = P - \frac{\rho p^2}{l^2} \tag{7.58}$$

defined by equation 7.54. The value of p^2 is

$$p^2 = \frac{Pl^2}{\rho} - \frac{1}{12} \frac{E}{1 - \nu^2} \frac{h^2}{\rho} l^4 \tag{7.59}$$

It goes through a maximum for

$$l^2 = \frac{6(1 - \nu^2)P}{Eh^2} \quad (7.60)$$

This defines a dominant wavelength

$$\mathcal{L}_d = \frac{2\pi}{l} = \pi h \sqrt{\frac{2E'}{3(1 - \nu^2)P}} \quad (7.61)$$

This result must be interpreted as follows. A thin plate of large size is first loaded by an axial compression P and restrained laterally to prevent buckling. When the lateral restraint is suddenly released, the plate buckles dynamically and the predominant shape is a sinusoidal deflection of wavelength \mathcal{L}_d . The same effect is obtained by a sudden application of a compressive axial load P to a plate which is unrestrained. The theory is applicable to this case provided the speed of propagation of the compressive stress P along the plate is sufficiently high in relation to the time required for the buckling to appear.*

* See, for example, N. J. Hoff, The Dynamics of Buckling of Elastic Columns, *Journal of Applied Mechanics*, Vol. 18, No. 1, pp. 68-74, 1951.

CHAPTER SIX

Mechanics of Viscoelastic Media under Initial Stress

1. INTRODUCTION

As already pointed out, the general equations derived in Chapter 1 for the statics of incremental stresses are applicable to all continuous media independently of the physical properties of the material. The same is true of the dynamical equations 2.9 of Chapter 5. The properties of the material are introduced separately in the relations between incremental stresses and strains. These relations remain arbitrary. In particular, we may apply results of the previous chapters to develop an analysis of viscous and viscoelastic media incorporating the effect of initial stress. Other types of materials with plastic or other non-linear properties may also be analyzed by considering incremental deformations.

The case of a viscoelastic medium initially at rest under initial stress is of considerable interest. Small incremental stresses and strains may be assumed to obey linear relations. A special example in this category is provided by a viscous fluid initially at rest in a gravity field. It is possible to apply the general principles of linear non-equilibrium thermodynamics to such media. The thermodynamic theory does not distinguish between an initially stressed or stress-free system; all that is required is that the initial state be one of thermodynamic equilibrium.

For this reason the first item discussed in section 2 concerns the

application of general thermodynamic principles to dissipative systems. A new feature which does not appear in usual applications of thermodynamics and which is due to the presence of initial stress is the occurrence of instability with a negative value of the free energy near equilibrium.

The linear stress-strain relations for incremental viscoelastic properties are analyzed in section 3. They are expressed in operational form. This formulation is based on the well-known techniques of operational calculus introduced by Heaviside and used extensively in electronics and electrical engineering. These methods are ideally suited for the treatment of viscoelasticity. The particular form of the viscoelastic operators which is imposed by thermodynamics is derived by applying results based on Onsager's relations. This leads to operators which verify with respect to the indices the same symmetry properties as the elastic coefficients. Hence formal solutions of the theory of elasticity are immediately extended to viscoelasticity by replacing the elastic coefficients by operators; thus a "correspondence principle" is established. Although these results of the general theory are based on thermodynamics, there are many particular cases for which these properties remain applicable without recourse to thermodynamics. This point is discussed in more detail at the end of this introduction.

In section 4 the general properties of characteristic solutions are discussed. Sufficient conditions are established for the operators to ensure that the solutions are real and proportional to a real exponential function of time which may be decreasing or increasing, depending on the stability. This criterion is also discussed in relation to thermodynamics.

The correspondence principle is strictly applicable only if the medium is at rest in the initial state. In a medium with fluid properties, an initial stress which is non-hydrostatic is associated with a finite rate of flow and large deformations. In section 5 the case of small deformations superposed on such a state of initial flow is examined. The analysis is carried out in detail for a viscous fluid. The limits of validity of the correspondence principle, in the presence of initial flow, are discussed. A rigorous theory is also derived for viscous buckling of a fluid plate under axial compression with finite strain. The section includes an analysis of fundamental kinematic relations between strain rate and finite strain.

The theory of surface and internal instability which was developed in Chapters 3 and 4 in the context of elasticity theory is readily extended to viscous and viscoelastic media. This is carried out in sections 6 and 7. The problems discussed include the surface instability of a homogeneous and a non-homogeneous half-space and internal instability in anisotropic viscoelasticity and anisotropic fluids. As an approximation the results are also applicable to plastic flow.

Problems of folding due to instability of layered viscous and viscoelastic media are analyzed in section 8. The solutions obtained in the earlier chapters for the elastic medium for single layers and multilayers are extended to viscoelasticity by applying the principle of correspondence. In these problems a complete analysis requires an evaluation of the time history of folding which develops gradually owing to the presence of initial deviations of the layers from perfect flatness. Such a numerical analysis is carried out for the folding under compression of a viscous layer embedded in a viscous medium. This is intended to establish the important result that under usual conditions the dominant wavelength actually emerges as the physically significant feature of the folding. The phenomenon is also discussed in terms of the theory of buckling of a viscous fluid with large deformations established in section 5 as an independent theory. General and rigorous equations are also developed as an independent theory for the folding of an arbitrary number of horizontal layers of viscous fluids undergoing a finite horizontal compression in a gravity field.

Addition of the inertia terms to the equations yields the dynamical theory of viscoelastic media under initial stress developed in section 9. It includes a discussion of the general properties of linear dynamical systems with a potential energy and a dissipation function. This is of interest because the properties of such systems are closely related to those of viscoelastic media which obey thermodynamic principles. Of particular importance is the property that unstable solutions are non-oscillatory and proportional to a real increasing exponential function of time. Criteria which ensure that this property is maintained, independently of thermodynamic considerations, are also derived. General stability criteria and variational principles are established. In particular, the case for which the condition for static stability also ensures dynamic stability is discussed. The analytical

results are used to derive a general theorem for evaluating the power dissipation in periodic motion. The variational method leads to equations with generalized stresses and generalized coordinates applicable to materials of arbitrary physical properties including plasticity. For linear viscoelasticity these results yield a variational principle in operational form and associated reciprocity properties.

As pointed out in previous chapters, the variational principle also provides a simple procedure for deriving the general equations in curvilinear coordinates.

The mechanics of a viscous fluid initially in equilibrium in a gravity field is treated in section 10 as an application of the general theory. Two examples of layered fluids are treated numerically in order to illustrate the procedure for stability problems. The general dynamical equations are also derived with particular attention to variational principles. It is shown that they are obtained from the theory of acoustic-gravity waves of Chapter 5 (section 6) by adding a dissipation function.

Viscoelasticity and Thermodynamics. The results obtained in this chapter are presented partly in the context of the thermodynamics of irreversible processes. This provides a unifying framework for the theory. Many of the results, however, are independent of any thermodynamic theory. In particular, this is the case for properties which depend on the symmetry of the matrix of viscoelastic operators when this symmetry is a consequence of geometric properties. This is immediately evident, for example, for a medium with isotropic properties or cubic symmetry. Another example is provided by the two-dimensional problems for an incompressible medium of orthotropic properties. Many of the problems discussed in this chapter fall in this category, when the viscoelastic properties are defined either by two operators \hat{N} and \hat{Q} or equivalently by \hat{M} and \hat{L} . The same considerations apply to the compressible laminated material analyzed in section 3.

2. THERMODYNAMICS OF VISCOELASTICITY WITH INITIAL STRESS

Let us consider a system in a state of stable thermodynamic equilibrium. When such a system is disturbed, it will tend to revert to its equilibrium state. We shall assume that the disturbances are

“small” and of such a nature that the thermodynamics is expressed by linear equations. In classical thermodynamics it is further assumed that the perturbations of the system occur at an infinitely slow rate through reversible processes.

In order to deal with a viscoelastic medium under initial stress it is necessary to include in the theory the cases for which

- (a) the state of equilibrium may be unstable;
- (b) the transformations may be irreversible.

The occurrence of unstable equilibrium in a system under initial stress was shown, in particular, for the buckling of a purely elastic medium.

In the framework of a linear theory, irreversible processes have recently been incorporated by the author in very general thermodynamics which is applicable to systems under initial stress.

We shall discuss these less familiar aspects of thermodynamics by using a simple example.

Thermodynamics and Stability. In the classical context of reversible isothermal transformations the condition of stable equilibrium of a system is expressed by stating that its free energy is a minimum. It is assumed that the processes are so slow that the system retains a uniform temperature throughout, or that the temperature changes themselves remain small and are not significant parameters of the transformation.

In this classical context, however, thermodynamics need not be confined to stable systems. Equilibrium requires only that the free energy be stationary. In other words, a small variation of the parameters is associated with a variation of the free energy which is of a higher order. Hence the free energy may be a minimum or a maximum. If it is a minimum, the equilibrium is stable; and, if it is a maximum, the equilibrium is unstable. We may also conceive of an equilibrium where the free energy curve plotted as a function of one parameter has an inflection point. At such a point the equilibrium is metastable. This case is characterized by the fact that the free energy, although stationary, is not necessarily a maximum or a minimum.

Such unstable conditions are known in physical chemistry and are represented, for example, by the phenomenon of supercooling.

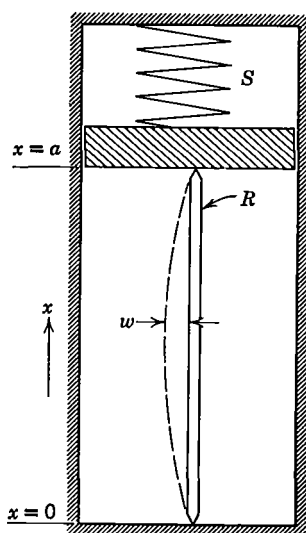


Figure 2.1 Rod R under axial compression due to the action of a spring S .

Some fundamental aspects of this instability will be illustrated by discussing a simple example in the context of mechanics.

Let us consider the system illustrated in Figure 2.1. A thin rod R is mounted in such a way that it is under an axial compression P generated by a compressed spring S . We know that beyond a certain critical value of the compression the rod will buckle. Whether stable or unstable, the system is in equilibrium when the rod is straight. Let us evaluate the free energy of this system when the rod is not straight and acquires a lateral deflection $w(x)$, a function of the coordinate x along the rod.

If we assume that the process is isothermal, the free energy coincides with the mechanical concept of potential strain energy of the elastic system. The length of the rod is denoted by a . The shortening of the rod is

$$\Delta a = \frac{1}{2} \int_0^a \left(\frac{dw}{dx} \right)^2 dx \quad (2.1)$$

The elastic strain energy due to bending of the rod is

$$\mathcal{E} = \frac{1}{2} EI \int_0^a \left(\frac{d^2 w}{dx^2} \right)^2 dx \quad (2.2)$$

where E is Young's modulus and I the moment of inertia of the cross-section about its neutral axis.

We derive the free energy of the system as

$$\mathcal{P} = \mathcal{E} - P \Delta a \tag{2.3}$$

The term $P \Delta a$ represents the decrease of strain energy in the compressed spring S . We are assuming here that the compressive stress is not affected appreciably by the shortening, Δa .

A more convenient form of the free energy (2.3) is obtained by expanding the deflection of the rod in a Fourier series. We write

$$w = \sum_{n=1}^{\infty} q_n \sin \left(n\pi \frac{x}{a} \right) \tag{2.4}$$

Introducing this expression into equation 2.3 yields

$$\mathcal{P} = \frac{1}{4} \frac{\pi^2}{a} \sum_{n=1}^{\infty} [n^2(n^2 P_c - P)q_n^2] \tag{2.5}$$

with

$$P_c = \frac{\pi^2 EI}{a^2} \tag{2.6}$$

The value P_c is the critical buckling load of the rod as derived from the classical Euler Theory.

Expression (2.5) for the free energy \mathcal{P} is a quadratic form which is positive definite for

$$P < P_c \tag{2.7}$$

In this case the free energy is an absolute minimum for zero deflection, and the system is stable.

On the other hand, if

$$(n + 1)^2 P_c > P > n^2 P_c \quad (n = 1, 2, 3, \dots) \tag{2.8}$$

some of the terms in the free energy are negative. Then the quadratic form \mathcal{P} is indefinite, and the system is unstable. If the inequality (2.8) is verified, only the first n terms of the Fourier series correspond to instability and the system may be called metastable.

In any case the system is in equilibrium for zero deflection, that is, for

$$q_1 = q_2 = \dots q_n = 0 \tag{2.9}$$

The equilibrium condition is also expressed by the equations

$$\frac{\partial \mathcal{P}}{\partial q_n} = 0 \quad (2.10)$$

These equations correspond to a stationary value of the free energy.

For a particular value of the compression given by the equation

$$P = n^2 P_c \quad (2.11)$$

the free energy \mathcal{P} becomes independent of q_n . This corresponds to neutral equilibrium with respect to the coordinate q_n . For $n = 1, 2, 3, \dots$, etc., equation 2.11 defines a sequence of values P which are characteristic values of the system. The sinusoidal deflections corresponding to each of these characteristic values are the "modes of instability," in this case the "buckling modes" of the rod.

Although the foregoing example does not introduce explicitly any thermodynamic variables, it is nevertheless quite general and, for reversible phenomena, it contains the essential features of the thermodynamic theory. The variables q_n in this case may represent not only geometric displacements but also other extensive parameters defining the thermodynamic state of the system.

Thermodynamics and Irreversibility. In the foregoing example, only reversible processes have been considered. Within the framework of a linear theory, non-reversible processes may be included in a general thermodynamic theory. This will be illustrated by discussing the same example of a rod under axial compression. Irreversibility may be introduced by the addition of viscous dashpots which restrain the lateral motion of the rod (Fig. 2.2). We shall assume that they are continuously and uniformly distributed lengthwise. The differential equation for the deflection of the rod may now be written:

$$EI \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} = -b \frac{\partial w}{\partial t} \quad (2.12)$$

In this equation b is a coefficient which measures the viscous resistance of the dashpots to the lateral velocity $\partial w / \partial t$. Because the motion is assumed to occur slowly, inertia forces are negligible.

Let us consider a solution w in the form of the Fourier series (2.4). The coefficients $q_n(t)$ of the series are now unknown functions of the

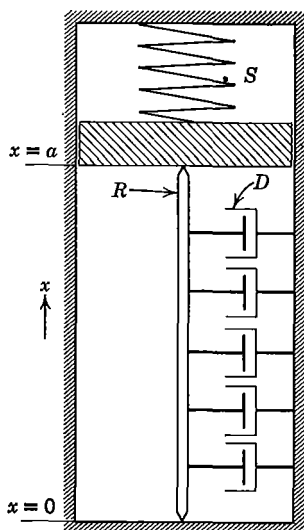


Figure 2.2 Rod R under axial compression with the addition of viscous dashpots D restraining the lateral deflection.

time. Substitution of the Fourier series into equation 2.12 yields

$$\frac{\pi^2 n^2}{a^2} (P - n^2 P_c) q_n = b \dot{q}_n \quad (2.13)$$

where \dot{q}_n denotes the time derivative of q_n . This equation has a solution of the type

$$q_n = C e^{pt} \quad (2.14)$$

with

$$p = \frac{\pi^2 n^2}{b a^2} (P - n^2 P_c) \quad (2.15)$$

If $P < P_c$, that is, if the initial stress is smaller than the buckling load, the values of p are all negative. This means that any component in the Fourier expansion of an initial deflection will decay exponentially. On the other hand, if $P > P_c$, some of the Fourier series components will grow exponentially with time. This we find an infinite sequence of characteristic exponents p some of which may be positive and may correspond to modes of viscoelastic instability.

In order to formulate this result in a more general form let us introduce the so-called *dissipation function*. It is written

$$\mathcal{D} = \frac{1}{2} b \int_0^a \dot{w}^2 dx \quad (2.16)$$

By substituting the Fourier series (2.4) for w we obtain

$$\mathcal{D} = \frac{1}{4} \sum_{n=1}^{\infty} ab\dot{q}_n^2 \quad (2.17)$$

With expressions (2.5) and (2.17) for \mathcal{P} and \mathcal{D} , equations 2.13 may be written

$$\frac{\partial \mathcal{P}}{\partial q_n} + \frac{\partial \mathcal{D}}{\partial \dot{q}_n} = 0 \quad (2.18)$$

These equations coincide with those which govern the linear thermodynamics of irreversible processes in the vicinity of an equilibrium state.

The Linear Thermodynamics of Irreversible Processes.

This phenomenological theory deals with a thermodynamic system initially in an equilibrium state. Small perturbations are superimposed owing to the action of external disturbances. General equations may be written for the thermodynamic variables describing the history of the system. It is essentially a linearized first order theory.

In the general case the free energy \mathcal{P} is written as the quadratic form

$$\mathcal{P} = \frac{1}{2} a_{ij} q_i q_j \quad (2.19)$$

where the q_i 's are generalized thermodynamic coordinates defining the perturbation. They are assumed to be extensive coordinates.

The dissipation function is a quadratic form in the time derivatives \dot{q}_i . It is written

$$\mathcal{D} = \frac{1}{2} b_{ij} \dot{q}_i \dot{q}_j \quad (2.20)$$

It may be defined thermodynamically as a quantity proportional to the rate of entropy production in the system during an irreversible process. Hence \mathcal{D} is a positive definite form.

Strictly speaking, it is only required that \mathcal{D} be non-negative. However, we shall assume that we have excluded all degrees of freedom for which the dissipation vanishes.

It was shown by the author that the differential equations which govern the time history of the thermodynamic system may be written

$$\frac{\partial \mathcal{P}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = Q_i \quad (2.21)$$

The term Q_i on the right side represents the time-dependent forces acting on the system. Equations 2.21 were derived in a series of publications in the years 1954 to 1956.† In this work the author introduced a generalized definition of the free energy \mathcal{P} which includes the case of non-uniform temperature distribution. This made equations 2.21 applicable to thermoelasticity and heat conduction. In this case the extensive coordinates q_i include an “entropy displacement” field. The disturbing forces Q_i are generalized forces in the meaning of Lagrangian mechanics. They may represent mechanical forces or thermodynamic forces. For example, in problems dealing with a non-uniform temperature they include a “thermal force” due to temperature differences. By inserting the values (2.19) and (2.20) for \mathcal{P} and \mathcal{D} into equations 2.21 we obtain

$$a_{ij}q_j + b_{ij}\dot{q}_j = Q_i \quad (2.22)$$

This result represents a system of linear differential equations for the unknown coordinates q_i . They contain the time derivatives \dot{q}_i . The existence of a generalized free energy \mathcal{P} implies the symmetry property

$$a_{ij} = a_{ji} \quad (2.23)$$

The coefficients b_{ij} also satisfy the reciprocity relations

$$b_{ij} = b_{ji} \quad (2.24)$$

The latter property is a consequence of the famous Onsager relations. Their discovery in 1931‡ marked the starting point of a general thermodynamics of irreversible processes. The concepts and methods derived from Onsager’s relations have been applied to many areas of physics and chemistry.§

† M. A. Biot, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Journal of Applied Physics*, Vol. 25, No. 11, pp. 1385–1391, 1954; Variational Principles in Irreversible Thermodynamics with Application to Viscoelasticity, *The Physical Review*, Vol. 97, No. 6, pp. 1463–1469, 1955; Thermoelasticity and Irreversible Thermodynamics, *Journal of Applied Physics*, Vol. 27, No. 3, pp. 240–253, 1956.

‡ L. Onsager, Reciprocal Relations in Irreversible Processes, I, *The Physical Review*, Vol. 37, No. 4, pp. 405–426, 1931; Reciprocal Relations in Irreversible Processes, II, *The Physical Review*, Vol. 38, No. 12, pp. 2265–2279, 1931.

§ See, for example, S. R. De Groot, *Thermodynamics of Irreversible Processes*, North Holland Publishing Company, Amsterdam, and Interscience Division, John Wiley and Sons, New York, 1952; I. Prigogine, *Introduction to Thermodynamics of Irreversible Processes*, 2nd edition, Interscience Division, John Wiley and Sons, New York, 1962.

The author has shown that they lead to the Lagrangian equations (2.21) for transient phenomena and that these equations are applicable to systems involving non-uniform temperatures by generalizing the concepts of free energy and dissipation function.†

It is important to call attention to one of the essential assumptions of this linear thermodynamic theory, namely, that the departure from a reversible process is measured by forces which depend linearly on the time rates of the thermodynamic variables. Hence the rate of entropy production is a second order quantity. A typical example of this property is provided by viscous behavior. The linear theory is therefore applicable only to those dissipative processes which fall in this general category and which may be called viscoelastic if the word is used in a broad thermodynamic context.

Normal Coordinates. For a system which is not subject to applied forces we must put $Q_i = 0$ in the differential equations 2.22. They become

$$a_{ij}q_j + b_{ij}\dot{q}_j = 0 \tag{2.25}$$

The characteristic solutions of this system may be written $q_i \exp(pt)$, where the q_i 's are time-independent amplitudes. The characteristic exponent p is now determined by the system of algebraic equations

$$a_{ij}q_j + pb_{ij}q_j = 0 \tag{2.26}$$

The roots p of the determinant of this homogeneous system are the characteristic exponents.

It is easily shown that the characteristic roots p are always *real*. This is a direct consequence of two important properties of the coefficients of equations 2.25, namely,

- (a) their symmetry;
- (b) one of the quadratic forms associated with the coefficients (in this case the dissipation function \mathcal{D}) is positive definite.

The proof is exactly the same as given in section 2 of Chapter 5 (p. 270) for the case of dynamics where the equations are written

$$a_{ij}q_j + p^2m_{ij}q_j = 0 \tag{2.27}$$

† For a general discussion of thermodynamics from the viewpoint of equation 2.22 see M. A. Biot, *Linear Thermodynamics and the Mechanics of Solids*, *Proc. Third U.S. National Congress of Applied Mechanics* (Amer. Soc. Mech. Eng.), pp. 1-18, New York, 1958.

In this case the positive definite quadratic form is the kinetic energy $\frac{1}{2}m_{ij}\dot{q}_i\dot{q}_j$. We have shown that the roots p^2 are always real.

We must note the important fact that the symmetry of the coefficients is *not a sufficient condition* for the roots to be real. The additional property must be used that \mathcal{D} is positive definite.

The matrices $[a_{ij}]$ and $[b_{ij}]$ may of course be diagonalized by a linear transformation with real coefficients. The system is then represented by *normal coordinates*. The free energy and dissipation function become a sum of squares, and the differential equations become uncoupled. This happens to be the case in the example of the rod under compression treated in this section; it is illustrated by equations 2.5 and 2.17.

By multiplying equations 2.26 by q_i we derive

$$p = -\frac{\mathcal{P}}{D} \quad (2.28)$$

with

$$D = \frac{1}{2}b_{ij}q_iq_j \quad (2.29)$$

When the free energy \mathcal{P} is positive, the value of p is negative. This corresponds to a *viscoelastic relaxation mode* whose amplitude decays exponentially with time.

For negative values of the free energy, p is positive and it corresponds to a *mode of viscoelastic instability* whose amplitude grows at an exponential rate.

When there are a number of unstable modes, the one which possesses the largest characteristic root p usually represents the *dominant* amplitude which emerges from an initial random disturbance. In the example in Figure 2.2 the value of p is given by equation 2.15. It is maximum for a certain value of n representing the number of half-waves in which the rod buckles. The corresponding wavelength is called the *dominant wavelength* since its amplitude grows at the fastest rate.

In this section we have assumed that inertia forces are negligible. The dynamical theory will be considered in section 9.

3. OPERATIONAL EXPRESSIONS FOR INCREMENTAL STRESSES. CORRESPONDENCE PRINCIPLE

We now come to the problem of expressing the relation between stress and strain in a viscoelastic material under initial stress.

In the previous section we have considered the general phenomenological properties of a finite system of arbitrary complexity under initial stress. In order to determine the stress-strain relations we must restrict ourselves to an element of the material and to homogeneous strain.

For the initially stress-free medium a general thermodynamic theory of stress-strain relations was derived by the author for linear viscoelasticity.† The same theory is immediately applicable to a medium under initial stress under two conditions:

1. The material must be in a state of stable thermodynamic equilibrium under the initial stress.
2. The concept of incremental force per unit initial area must be introduced instead of the true stresses.

What is referred to here as stability is a material property which must be carefully distinguished from the mechanical stability of the structure as a whole. The material must be physically stable, whereas the structure under initial stress may be unstable.

Under these conditions the theory proceeds along exactly the same lines as in the paper cited. This extension of the viscoelastic stress-strain relation to a material under initial stress has been discussed by the author in a subsequent publication.‡

A detailed discussion of the thermodynamic theory lies beyond our purpose in this book. We shall therefore confine ourselves to an illustration of the methods and concepts by treating a simple example.

Let us consider an element of material represented by the system illustrated in Figure 3.1. We shall think of it as a deformable rectangular box oriented along the coordinates. An initial force f is applied to the element on two opposite faces. This force f is balanced by a spring 1 under tension inside the box. This system constitutes a simple example of stable equilibrium under initial stress. Let us assume that the box also contains the additional springs 2 and

† M. A. Biot, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Journal of Applied Physics*, Vol. 25, No. 11, pp. 1385-1391, 1954.

‡ M. A. Biot, On the Instability and Folding Deformation of a Layered Viscoelastic Medium in Compression, *Journal of Applied Mechanics*, Ser. E, Vol. 26, pp. 393-400, 1959.

dashpots 3. These additional springs are initially unstressed. If the force f is increased by an amount Δf , a change q_1 in the length of the box will take place in the same direction. At the same time the internal dashpots and springs 2 and 3 will become active with a certain time lag and relaxation. The time history will be expressed by the differential equations

$$\begin{aligned} k_1 q_1 + k_2(q_1 - q_2) &= \Delta f \\ b\dot{q}_2 + k_2(q_2 - q_1) &= 0 \end{aligned} \quad (3.1)$$

In these equations q_2 represents the displacement of the piston in the dashpot.

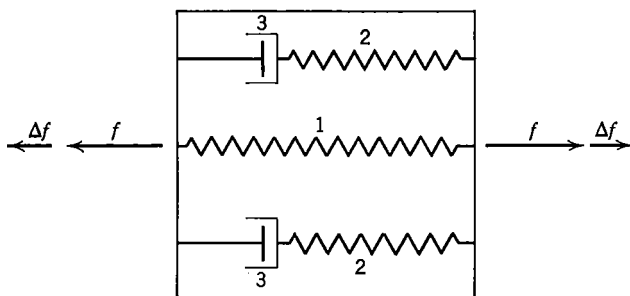


Figure 3.1 Example of a viscoelastic system with initial stress.

This very simple example embodies the essential features of a theory applicable to a very wide range of phenomena.

By introducing the quadratic functions

$$\begin{aligned} \mathcal{P} &= \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_1 - q_2)^2 \\ \mathcal{D} &= \frac{1}{2}b\dot{q}_2^2 \end{aligned} \quad (3.2)$$

we may write the differential equations 3.1 in the general form

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial q_1} &= \Delta f \\ \frac{\partial \mathcal{P}}{\partial q_2} + \frac{\partial \mathcal{D}}{\partial \dot{q}_2} &= 0 \end{aligned} \quad (3.3)$$

This result brings out important generalizations.

Internal Coordinates. We may consider \mathcal{P} as representing the free energy of the system for perturbations near a state of initial

stress. In addition, the coordinate q_2 may be looked upon as an "internal" degree of freedom in contrast to the external coordinate q_1 . Hence we should like to eliminate q_2 from the equations and obtain relations which contain only the incremental force Δf and the external coordinate q_1 .

Since the equations are differential equations, this elimination is most conveniently accomplished by means of the standard operational formalism which was introduced by Heaviside and used extensively by electrical engineers for over a century.† If Δf is a simple harmonic function of time, we may replace it by $\Delta f \exp(i\alpha t)$ where Δf is now a complex amplitude. In a steady state solution the unknowns q_1 and q_2 may be written in the same way. With the notation

$$p = i\alpha \quad (3.4)$$

the differential equations 3.1 become

$$\begin{aligned} k_1 q_1 + k_2(q_1 - q_2) &= \Delta f \\ pbq_2 + k_2(q_2 - q_1) &= 0 \end{aligned} \quad (3.5)$$

Elimination of q_2 yields

$$q_1 \left(k_1 + \frac{pbk_2}{pb + k_2} \right) = \Delta f \quad (3.6)$$

If the incremental force Δf is an arbitrary function of time, it may be expressed by a Fourier integral. By using equation 3.6 the Fourier transform of q_1 is immediately derived and, hence, also the transient response of q_1 to the force. These Fourier transforms are uniquely determined by the condition that $q_1 = \Delta f = 0$ for $t < 0$.

Equation 3.6 may also be looked upon as an operational relation. For example, let us consider the quantity

$$q_1 \frac{pbk_2}{pb + k_2} = k_2 \frac{p}{p + r} q_1 = z \quad (3.7)$$

where $r = k_2/b$. We multiply z by $p + r$ and replace p by the time differential:

$$p = \frac{d}{dt} \quad (3.8)$$

† See, for example, Th. von Kármán and M. A. Biot, *Mathematical Methods in Engineering*, McGraw-Hill Book Co., New York, 1940.

Hence we obtain a first order differential equation for z :

$$k_2 \dot{q}_1 = \dot{z} + rz \quad (3.9)$$

Its solution is

$$z = k_2 e^{-rt} \int_0^t e^{r\tau} \frac{dq_1(\tau)}{d\tau} d\tau \quad (3.10)$$

The constant of integration has been chosen such that $z = 0$ for $t = 0$. The significance of the fractional operator is therefore defined by the equation

$$\frac{p}{p+r} = e^{-rt} \int_0^t e^{r\tau} d \quad (3.11)$$

Hence equation 3.7 may be interpreted in terms of integral operators.

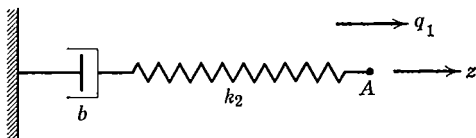


Figure 3.2 Spring and viscous dashpot in series representing a Maxwell element which corresponds to the operator (3.11).

The physical significance of the fractional operator is brought out if we note that z may be considered as a force acting on a spring and dashpot in series as illustrated in Figure 3.2. The displacement q_1 of point A is governed by the differential equation 3.9, and the time history of the force z is determined by the integral (3.10). If point A is suddenly displaced by a unit amount at $t = 0$, we put

$$q_1 = 1(t) \quad (3.12)$$

where $1(t)$ represents the Heaviside step function; that is,

$$\begin{aligned} 1(t) &= 1 & (t > 0) \\ 1(t) &= 0 & (t < 0) \end{aligned} \quad (3.13)$$

The force z is given by equation 3.10, which in this case becomes

$$z = k_2 e^{-rt} \quad (3.14)$$

Operationally this result may be expressed by the equation

$$\frac{p}{p+r} 1(t) = e^{-rt} 1(t) \quad (3.15)$$

The exponential stress relaxation corresponding to this equation is characteristic of the spring-dashpot element of Figure 3.2, which is called a Maxwell element. It is a simplified analog model for a material which is purely elastic for fast deformations and purely viscous for very slow deformations.

The properties of this Maxwell element are easily verified by considering equation 3.7. Since p may be interpreted as $i\alpha$, fast deformations correspond to $p = \infty$. Equation 3.7 then becomes

$$k_2 q_1 = z \quad (3.16)$$

The dashpot is frozen at high frequency, and the system is reduced to a spring of modulus k_2 . At low frequency corresponding to small values of p , equation 3.7 becomes

$$bpq_1 = b\dot{q}_1 = z \quad (3.17)$$

The force in this case is due entirely to the viscous resistance of the dashpot.

In the foregoing example we have considered a system with only a single internal coordinate q_2 . The result may be generalized to any number of such coordinates. Moreover, these internal coordinates may be of a very general thermodynamic nature. We write the generalized free energy of the system as

$$\mathcal{P} = \frac{1}{2} a_{ij} q_i q_j \quad (3.18)$$

and a dissipation function as

$$\mathcal{D} = \frac{1}{2} b_{ij} \dot{q}_i \dot{q}_j \quad (3.19)$$

The coordinate q_1 is the external one. As in Figure 3.1, it represents the change of length of the system. All other coordinates $q_2, q_3, \text{etc.}$, are internal. The system is in equilibrium under an initial force f . The internal coordinates are a measure of the internal thermodynamic perturbation in terms of electrical, physical, chemical, thermal, or any other phenomena involved.

The differential equations of this system under the action of the incremental force Δf are

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial q_1} + \frac{\partial \mathcal{D}}{\partial \dot{q}_1} &= \Delta f \\ \frac{\partial \mathcal{P}}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot{q}_k} &= 0 \quad (k \neq 1) \end{aligned} \quad (3.20)$$

When written operationally, the first equation becomes

$$(a_{1j} + pb_{1j})q_j = \Delta f \quad (3.21)$$

The additional equations (for $k \neq 1$), which may be large in number, are

$$(a_{kj} + pb_{kj})q_j = 0 \quad (3.22)$$

These equations may be solved for q_2, q_3 , etc., in terms of q_1 , and these values are then substituted in equation 3.21. The result is found to be of the form

$$\left(\sum_j^j C_j \frac{p}{p + r_j} + C + C'p \right) q_1 = \Delta f \quad (3.23)$$

From the assumption that the system is stable under the initial stress and from the positive definite nature of the dissipation function, it follows that the quantities C', C, C_j , and r_j are non-negative. The proof of this property will not be given here. It is immediately derived from the author's theory of stress-strain relations in viscoelastic media.† Although the theory was developed in the context of an initially stress-free medium, the proof is directly applicable to the present case.

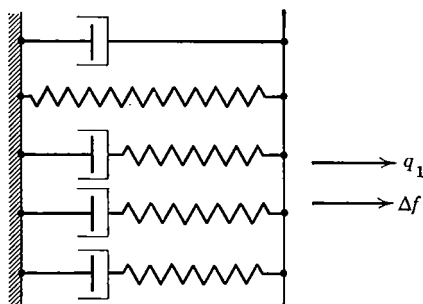


Figure 3.3 Spring-dashpot model for equation 3.23.

Expression (3.23) for Δf is an obvious generalization of equation 3.6. The operational terms on the left side may be interpreted as representing a spring, a dashpot, and a large number of Maxwell elements all in parallel (Fig. 3.3). The mechanical model is also

† M. A. Biot, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Journal of Applied Physics*, Vol. 25, No. 11, pp. 1385-1391, 1954.

an analog for a thermodynamic system. An equivalent electrical analog represented by a resistance-capacitance network may also be derived.

We write the operator in equation 3.23 as

$$\hat{C} = \sum_j C_j \frac{p}{p + r_j} + C + C'p \tag{3.24}$$

Hence

$$\Delta f = \hat{C}q_1 \tag{3.25}$$

Note that the operator \hat{C} is of the same form as that for an initially unstressed medium.

If there is a very large number of relaxation constants r_j , the summation may be replaced by an integral. Hence

$$\hat{C} = \int_0^\infty \frac{p}{p + r} C(r) dr + C + C'p \tag{3.26}$$

The function $C(r)$ is a density function which may be called a relaxation spectrum. If it is assumed that this function may include singular terms of the Dirac function type, it contains the discrete summation of equation 3.24 as a particular case.†

General Stress-Strain Relations. The foregoing considerations apply to the incremental force Δf . For a continuous medium under initial stress an important distinction must be made between the incremental force and the incremental stress. Let us consider an element of the viscoelastic medium represented by a cube of unit size oriented along the coordinate axes. An initial normal stress S_{11} is acting on the faces perpendicular to the x direction. When an incremental force Δf is applied in the x direction, the changes of length of the element along each direction are, respectively, e_{xx} , e_{yy} , e_{zz} and the stress S_{11} becomes $S_{11} + s_{11}$. The system is identical with the previous example provided we put

$$\begin{aligned} f &= S_{11} \\ \Delta f &= s_{11} + S_{11}(e_{yy} + e_{zz}) \\ q_1 &= e_{xx} \end{aligned} \tag{3.27}$$

Equation 3.25 now becomes

$$s_{11} + S_{11}(e_{yy} + e_{zz}) = \hat{C}e_{xx} \tag{3.28}$$

† See M. J. Lighthill, *Fourier Analysis and Generalized Functions*, Cambridge University Press, 1959.

where s_{11} is the incremental stress. We may write

$$t_{11} = \hat{C}e_{xx} \quad (3.29)$$

by putting

$$t_{11} = s_{11} + S_{11}(e_{yy} + e_{zz}) \quad (3.30)$$

This result coincides with the expressions for incremental forces per unit initial area already discussed as alternative stresses in section 2 of Chapter 2.

It is possible to generalize this result to express the incremental stress-strain relations in three dimensions. This possibility becomes evident when we consider that the perturbations of a system in the vicinity of a stable equilibrium state obeys the same thermodynamic equations whether initially stressed or not.

The only difference lies in the definition of the incremental stresses. In the presence of initial stress we must introduce the alternative stress components t_{ij} defined in section 2 of Chapter 2. In this general case there are also a large number of internal coordinates, but there are six observed strain components and six externally applied force components t_{ij} . The operational relations between these external variables are the same as in a medium initially stress-free. Hence the incremental stress-strain relations are written

$$t_{ij} = \hat{C}_{ij}^{\mu\nu} e_{\mu\nu} \quad (3.31)$$

This is the generalized form of equation 3.29.

Thermodynamic principles require the operators to be of the form

$$\hat{C}_{ij}^{\mu\nu} = \int_0^\infty \frac{p}{p+r} C_{ij}^{\mu\nu}(r) dr + C_{ij}^{\mu\nu} + pC_{ij}'^{\mu\nu} \quad (3.32)$$

They must satisfy the symmetry properties

$$\hat{C}_{ij}^{\mu\nu} = \hat{C}_{ji}^{\mu\nu} = \hat{C}_{ij}^{\nu\mu} = \hat{C}_{ji}^{\nu\mu} \quad (3.33)$$

In addition, the quadratic forms $C_{ij}^{\mu\nu}(r)e_{ij}e_{\mu\nu}$, $C_{ij}^{\mu\nu}e_{ij}e_{\mu\nu}$, $C_{ij}'^{\mu\nu}e_{ij}e_{\mu\nu}$ must be non-negative.

Some additional remarks are in order with reference to the thermodynamic properties of the operators (3.32). The author has pointed out† that these operators are the same as those obtained for a material initially stress-free if we use the alternative stress components t_{ij} . That the coefficients $C_{ij}'^{\mu\nu}$ define

† M. A. Biot, On the Instability and Folding Deformation of a Layered Viscoelastic Medium in Compression, *Journal of Applied Mechanics*, Ser. E, Vol. 26, pp. 393-400, 1959.

a non-negative form can be seen by putting $p = 0$. This corresponds to infinitely slow elastic deformations with a non-negative strain energy. For infinite frequency the term $pC_{ij}^{\mu\nu}$ becomes dominant, and the coefficients $C_{ij}^{\mu\nu}$ represent the anisotropic viscosity. Since the dissipation is never negative, the coefficients $C_{ij}^{\mu\nu}$ must define a non-negative quadratic form. These properties also follow from the general expressions of these coefficients derived in the author's thermodynamic theory of relaxation phenomena.† Furthermore it was shown in the same paper that the coefficients $C_{ij}^{\mu\nu}(r)$ may be written in the form

$$C_{ij}^{\mu\nu}(r) = \psi_{ij}(r)\psi_{\mu\nu}(r) \quad (3.33a)$$

The variable r is inversely proportional to the relaxation time of the internal degrees of freedom. Actually there may be a large number of such degrees of freedom with the same relaxation time corresponding to a degeneracy with a multiple characteristic root r . We then write

$$C_{ij}^{\mu\nu}(r) = \sum^s \psi_{ij}^{(s)}(r)\psi_{\mu\nu}^{(s)}(r) \quad (3.33b)$$

There may be an infinite number of terms in this summation leading in the limit to an integration. From expression (3.33b) we derive

$$C_{ij}^{\mu\nu}(r)e_{ij}e_{\mu\nu} = \sum^s [\psi_{ij}^{(s)}(r)e_{ij}]^2 \quad (3.33c)$$

Hence the quadratic form is non-negative. For the same reason all diagonal terms of the matrices $C_{ij}^{\mu\nu}(r)$, $C_{ij}^{\mu\gamma}$, and $C_{ij}^{\mu\nu}$ are non-negative. The non-negative property of the quadratic forms, as a consequence of the thermodynamics, was also discussed by the author in a later paper.‡

The foregoing results may be expressed in terms of the incremental stresses s_{ij} by using equations 2.22 of Chapter 2; that is,

$$t_{ij} = s_{ij} + S_{ij}e - \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (3.34)$$

Hence

$$s_{ij} = \hat{C}_{ij}^{\mu\nu}e_{\mu\nu} - S_{ij}e + \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (3.35)$$

This relation is formally the same as equation 4.17 of Chapter 2. It may be written

$$s_{ij} = \hat{B}_{ij}^{\mu\nu}e_{\mu\nu} \quad (3.36)$$

where

$$\hat{B}_{ij}^{\mu\nu} = \hat{C}_{ij}^{\mu\nu} + D_{ij}^{\mu\nu} - S_{ij}\delta_{\mu\nu} \quad (3.37)$$

and $D_{ij}^{\mu\nu}$ is defined by expression (4.34) of Chapter 2.

As in the analogous elastic case, the operators $\hat{B}_{ij}^{\mu\nu}$ satisfy the relations

$$\hat{B}_{ij}^{\mu\nu} - \hat{B}_{\mu\nu}^{ij} = S_{\mu\nu}\delta_{ij} - S_{ij}\delta_{\mu\nu} \quad (3.38)$$

† M. A. Biot, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Journal of Applied Physics*, Vol. 25, No. 11, pp. 1385-1391, 1954.

‡ M. A. Biot, Linear Thermodynamics and the Mechanics of Solids, *Proc. Third U.S. National Congress of Applied Mechanics* (Amer. Soc. Mech. Eng.), pp. 1-18, New York, 1958.

Hence they are not generally symmetric.

Correspondence Principle. Certain formal analogies between the expressions for the stresses in a viscous fluid and an elastic medium have long been known. It was shown by Alfrey† that the analogy is applicable to problems of static stress analysis for an incompressible and isotropic viscoelastic medium. The complete analogy between the general theory of elasticity and the properties of a viscoelastic medium including compressibility and anisotropy follows immediately from the theory of viscoelastic stress-strain relations derived by the author in 1954‡ in the context of thermodynamics. It was shown that the mechanics of viscoelastic media is governed by equations which may be derived from the theory of elasticity by the simple rule of replacing the elastic coefficients by operators. In order to emphasize its generality and far-reaching consequences the term *correspondence principle* was introduced for this rule in 1955 by the author in a companion paper.§ This paper also discussed applications to wave propagation in viscoelastic media and the dynamics of viscoelastic plates. Similar applications were also discussed in a paper published simultaneously¶ which included a variational form and extensions of the correspondence principle to a viscoelastic medium under initial stress and to a non-linear theory of viscoelastic plates.

The existence of such viscoelastic correspondence is a direct consequence of equations 3.31 and 3.36. These equations are completely analogous to the stress-strain relations of a purely elastic medium under initial stress. The elastic coefficients $C_{ij}^{\mu\nu}$ are replaced by

† T. Alfrey, Non-homogeneous Stress in Viscoelasticity, *Quarterly of Applied Mathematics*, Vol. 2, No. 2, pp. 113–119, 1944.

‡ M. A. Biot, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Journal of Applied Physics*, Vol. 25, No. 11, pp. 1385–1391, 1954.

§ M. A. Biot, Dynamics of Viscoelastic Anisotropic Media, *Proc. Second Midwestern Conference on Solid Mechanics*, Research Series, No. 129, Engineering Experiment Station, Purdue University, Lafayette, Ind., pp. 94–108, 1955.

¶ M. A. Biot, Variational and Lagrangian Methods in Viscoelasticity, *Deformation and Flow of Solids* (IUTAM Colloquium, Madrid 1955), pp. 251–263, Springer, Berlin, 1956. The term “correspondence rule” was also used alternatively by the author instead of “correspondence principle.” The latter term, however, seems to have been generally adopted by other writers. This correspondence was applied extensively by the author in a series of subsequent publications on viscoelastic stability as indicated in sections 7 and 8 and on porous media (see pp. 458 and 490).

operators $\hat{C}_{ij}^{\mu\nu}$. These operators satisfy the symmetry relations (3.33) which are the same as for the elastic coefficients.

It should be pointed out that in the general case of anisotropy these symmetry properties are a consequence of thermodynamic principles. In general, they imply the validity of Onsager's relations. However it is not always necessary to involve thermodynamic principles. For example, in an isotropic medium the symmetry of the operators is a consequence of the isotropy and does not depend on the validity of the thermodynamics. On the other hand, the explicit form of the operators (3.32) in terms of p will still depend on the validity of the thermodynamic principles.

Some examples will now be presented in order to illustrate the application of the correspondence principle.

Viscoelastic Properties of a Laminated Medium. In Chapter 4 (section 2) it was shown how an elastic material composed of thin laminations could be approximately represented by a continuous anisotropic medium. Because of viscoelastic correspondence it is possible to extend these results to viscoelasticity. We consider first an incompressible material. The layers are assumed to be composed of two distinct materials. In material of type one the viscoelastic properties are represented by two operators \hat{L}_1 and \hat{M}_1 . The material occupies a fraction α_1 of the total thickness. The initial stress in this material is denoted by P_1 . Material of type two occupies a fraction α_2 of the total thickness. In this material the initial stress is P_2 , and the viscoelastic properties are defined by the operators \hat{L}_2 and \hat{M}_2 . The average initial stress P in the laminated medium is given by equation 2.21 of Chapter 4. Its value is

$$P = \alpha_1 P_1 + \alpha_2 P_2 \quad (3.39)$$

The operators \hat{L} and \hat{N} for the composite viscoelastic medium are obtained by substituting operators for the elastic coefficients in equations 2.22 and 2.29 of Chapter 4. We obtain

$$\begin{aligned} \hat{L} &= \frac{1}{\frac{\alpha_1}{\hat{L}_1} + \frac{\alpha_2}{\hat{L}_2}} \\ \hat{N} &= \hat{N}_1 \alpha_1 + \hat{N}_2 \alpha_2 \end{aligned} \quad (3.40)$$

The last equation may also be written in terms of the operators \hat{M}_1

and \hat{M}_2 by using the relations

$$\begin{aligned}\hat{M}_1 &= \hat{N}_1 + \frac{1}{4}P_1 \\ \hat{M}_2 &= \hat{N}_2 + \frac{1}{4}P_2\end{aligned}\quad (3.41)$$

By combining equations 3.39, 3.40, and 3.41 and using the property $\alpha_1 + \alpha_2 = 1$, we derive

$$\hat{M} = \hat{M}_1\alpha_1 + \hat{M}_2\alpha_2 \quad (3.42)$$

This is the value of the operator \hat{M} for the "averaged" composite medium. Let us consider two particular cases. We put

$$\begin{aligned}\hat{L}_1 &= L_1 \\ \hat{M}_1 &= M_1 \\ \hat{L}_2 &= \hat{M}_2 = \eta p \\ P_2 &= 0\end{aligned}\quad (3.43)$$

This case corresponds to a medium composed of elastic layers separated by a purely viscous material. The elastic material is defined by the elastic coefficients L_1 and M_1 , the viscous material is defined by a viscosity coefficient η . The average initial stress is

$$P = \alpha_1 P_1 \quad (3.44)$$

It is carried entirely by the elastic layers. The operators representing the properties of the composite material are obtained by substituting the values (3.43) into equations 3.40. They may be written in the form

$$\begin{aligned}\hat{L} &= \frac{p}{p+r} L_r \\ \hat{M} &= M + M'p\end{aligned}\quad (3.45)$$

where

$$\begin{aligned}M &= \alpha_1 M_1 & M' &= \alpha_2 \eta \\ L_r &= \frac{L_1}{\alpha_1} & r &= \frac{\alpha_2 L_1}{\alpha_1 \eta}\end{aligned}\quad (3.46)$$

The physical properties represented by the operators (3.45) will be discussed in section 6.

Another illustration is provided by the case of two materials which are purely viscous. However, the result in this case involves an approximation, because the medium is not at rest but in a state of flow under the initial stress. A discussion of this approximation in

section 5 will establish its general validity. The viscous materials of the two layers are represented approximately by the operators

$$\begin{aligned}\hat{L}_1 &= \hat{M}_1 = \eta_1 p \\ \hat{L}_2 &= \hat{M}_2 = \eta_2 p\end{aligned}\quad (3.47)$$

The coefficients η_1 and η_2 play the role of viscosity coefficients characterizing the two materials. The operators (3.40) for the composite medium may be written

$$\hat{L} = L'p \quad \hat{M} = M'p \quad (3.48)$$

with

$$\begin{aligned}L' &= \frac{1}{\frac{\alpha_1}{\eta_1} + \frac{\alpha_2}{\eta_2}} \\ M' &= \eta_1 \alpha_1 + \eta_2 \alpha_2\end{aligned}\quad (3.49)$$

In the foregoing examples we assumed the materials to be incompressible. Similar procedures may be applied to derive the average operators for a laminated medium of compressible materials. We consider again thin elastic layers separated by a very viscous fluid. The operators giving the incremental stresses in this medium are obtained immediately by using the result derived in Chapter 4 for the purely elastic case. The initial stress is assumed to be carried by the elastic layers which occupy a fraction α_1 of the total thickness. The viscous fluid carries no initial stress and occupies a fraction α_2 of the thickness. We must apply equations 2.33, 2.34, and 2.37 of Chapter 4. For the viscous layers the elastic coefficients α_2 , b_2 , c_2 must be replaced by operators expressing the stress-strain relations in a viscous compressible fluid. For plane-strain deformations the stresses in a compressible fluid of bulk modulus λ and Newtonian viscosity η are expressed operationally by the equations

$$\begin{aligned}\sigma_{xx} &= 2p\eta e_{xx} + (\lambda - \frac{2}{3}p\eta)(e_{xx} + e_{yy}^{(2)}) \\ \sigma_{yy} &= 2p\eta e_{yy} + (\lambda - \frac{2}{3}p\eta)(e_{xx} + e_{yy}^{(2)})\end{aligned}\quad (3.50)$$

The superscript in $e_{yy}^{(2)}$ is used to represent the local strain components in the fluid layer. Since the fluid is not stressed initially, the actual stresses σ_{xx} , σ_{yy} and incremental stresses $t_{11}^{(2)}$ and t_{22} are the same. Hence we may write equations 3.50 in the form

$$\begin{aligned}t_{11}^{(2)} &= \hat{a}_2 e_{xx} + \hat{b}_2 e_{yy}^{(2)} \\ t_{22} &= \hat{b}_2 e_{xx} + \hat{c}_2 e_{yy}^{(2)}\end{aligned}\quad (3.51)$$

with

$$\begin{aligned} \hat{a}_2 &= \hat{c}_2 = \lambda + \frac{4}{3}p\eta \\ \hat{b}_2 &= \lambda - \frac{2}{3}p\eta \end{aligned} \quad (3.52)$$

The incremental stresses in the composite laminated medium are then

$$\begin{aligned} t_{11} &= \hat{C}_{11}e_{xx} + \hat{C}_{12}e_{yy} \\ t_{12} &= \hat{C}_{12}e_{xx} + \hat{C}_{22}e_{yy} \end{aligned} \quad (3.53)$$

where the operators are obtained by using equations 2.37 of Chapter 4 and replacing the coefficients a_2, b_2, c_2 by the operators (3.52). The coefficients a_1, b_1, c_1 are maintained, and they represent the properties of the elastic layers under initial stress. For example, we write

$$\hat{C}_{11} = \alpha_1 a_1 + \alpha_2 \hat{a}_2 - \frac{\alpha_1 \alpha_2 (b_1 - \hat{b}_2)^2}{\alpha_1 \hat{c}_2 + \alpha_2 c_1} \quad (3.54)$$

The tangential stress t'_{12} is written

$$t'_{12} = 2\hat{L}e_{xy} \quad (3.55)$$

The operator \hat{L} for the composite medium is the same as in the case of incompressibility. It is given by the first of equations 3.45. Hence

$$\hat{L} = \frac{1}{\frac{\alpha_1}{L_1} + \frac{\alpha_2}{\eta p}} = \frac{p}{p+r} L_r \quad (3.56)$$

In this expression L_1 represents the slide modulus of the elastic layers.

The same procedure is also applicable if the individual layers themselves are viscoelastic, with properties represented by more complicated operators.

Due caution should be exercised when applying these results. The anisotropic continuous medium may be substituted as an approximation for the laminated material only within certain limits. For the elastic medium these limitations have been discussed in section 2 of Chapter 4. In the case of viscoelasticity similar limitations are imposed. The wavelength of the deformations must be sufficiently large in comparison with the thickness of the laminations. The limiting wavelength depends on the comparative rigidities and viscosities of the layers. For example, for elastic layers separated by a viscous fluid it is obvious that the continuous model will be valid only for a combination of layer thickness, wavelengths, viscosity,

and time intervals such that the fluid will not flow appreciably in the direction of the layering.

In many problems the approximation by a continuous medium will be valid. On the other hand, when not strictly applicable, the approximation will still yield insight into some of the basic features of the problem and provide a foundation for the development of more refined theories by the application of suitable corrections. Further discussion of these points will be found in section 6 of this chapter in connection with the problem of internal instability of a laminated viscoelastic medium.

Torsional Stiffness of a Viscoelastic Bar under Axial Tension. This problem was analyzed for the elastic medium in section 10 of Chapter 2. Let us consider the particular case of a bar of homogeneous material and circular cross section under a uniform axial tension S_{33} . Let us also assume the medium to be transverse-isotropic around the axis of the bar. The torsional strain θ denotes the angle of twist per unit distance along the axis. The torque is obtained by applying the correspondence principle to equation 10.24 of Chapter 2. Replacing the slide modulus L by an operator \hat{L} , we write for the torque

$$T = (\hat{L} + S_{33})I_G\theta \quad (3.57)$$

where

$$I_G = \frac{1}{2}\pi a^4 \quad (3.58)$$

is the polar moment of inertia of the cross section of radius a . The operator \hat{L} is of the form

$$\hat{L} = \int_0^\infty \frac{p}{p+r} L(r) dr + L + pL' \quad (3.59)$$

It may be determined experimentally by applying a tangential stress along the axial direction of the material and measuring the shear displacement as illustrated in Figure 10.2 of Chapter 2.

It is possible to imagine a bar of the nature of a cable made of a composite material where the axial stress is carried by fine steel wires oriented in the axial direction and bonded together by a viscous material. In this case the shear displacement will be approximately a pure viscous flow, and the operator will be reduced to

$$\hat{L} = pL' \quad (3.60)$$

In terms of time derivatives the torque is then

$$T = L'I_G \frac{d\theta}{dt} + S_{33}I_G\theta \quad (3.61)$$

The same method is applicable to the more complicated cases of non-circular cross-sections with orthotropic and non-homogeneous materials. The solution of the corresponding Saint-Venant problem for the elastic medium yields at the same time the solution for the case of viscoelasticity.

4. PROPERTIES OF CHARACTERISTIC SOLUTIONS

When initially displaced from an equilibrium state, a viscoelastic medium will exhibit deformations which are functions of time. They may correspond to relaxation modes or to modes of instability, depending on the stability of the initial state of equilibrium.

These time-dependent deformations are represented by *characteristic solutions* of the homogeneous equations defining the system, with homogeneous boundary conditions.

Slow deformation rates are assumed so that *inertia forces may be neglected*. They will be included later in the more complete theory of section 9.

In a characteristic solution all amplitudes are proportional to the same exponential factor, $\exp(pt)$. The displacement field u_i is then written

$$u_i(x, t) = u_i(x)e^{pt} \quad (4.1)$$

where the amplitudes $u_i(x)$ are functions only of the coordinates x_i , while the time t appears only in the exponential factor.

These characteristic solutions are obtained by substituting expression (4.1) into the field equations and boundary conditions of the viscoelastic medium. Because the equations are homogeneous, the exponential is factored out.

An important advantage of the operational method is that the characteristic equation is obtained immediately by treating the operators as algebraic quantities.

As an example let us consider the viscoelastic system represented in Figure 3.1 and governed by equations 3.1. In the absence of a disturbing force we put $\Delta f = 0$, and the equations become

$$\begin{aligned} k_1q_1 + k_2(q_1 - q_2) &= 0 \\ b\dot{q}_2 + k_2(q_2 - q_1) &= 0 \end{aligned} \quad (4.2)$$

In order to derive a characteristic solution we must replace q_1 and q_2 by $q_1 \exp(pt)$ and $q_2 \exp(pt)$, respectively. Equations 4.2 become

$$\begin{aligned} k_1 q_1 + k_2 (q_1 - q_2) &= 0 \\ pbq_2 + k_2 (q_2 - q_1) &= 0 \end{aligned} \quad (4.3)$$

By eliminating q_1 and q_2 in these equations we find the characteristic equation

$$k_1(pb + k_2) + pbk_2 = 0 \quad (4.4)$$

The same result is obtained immediately by equating to zero the operational coefficient of q_1 in equation 3.6.

The root p of equation 4.4 is the characteristic exponent

$$p = - \frac{k_1 k_2}{b(k_1 + k_2)} \quad (4.5)$$

This value yields a decaying exponential factor, $\exp(pt)$, corresponding to a relaxation mode.

This example points to a general rule of great simplicity whereby equations for the characteristic solutions are immediately derived for any viscoelastic system. By this rule the equations are written by *treating all operators as algebraic quantities*. Then p represents the real or complex coefficient of time in the exponential.

The characteristic equation (4.4) in the previous example is, of course, very simple. In general, the characteristic equation for p may be algebraic or transcendental and may have a finite or an infinite number of roots. The roots may yield decreasing or increasing exponentials, depending on the stability of the system.

An important question which we now examine is whether these roots are real or complex. We shall show that under very broad conditions the roots p are always real.

In order to establish this we shall first derive two lemmas. These lemmas are closely related to the variational principles discussed in Chapter 2.

Lemma 1. Let us write the equilibrium equations in the form (2.24) of Chapter 2. They are

$$\frac{\partial t_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j} (S_{kj} \omega_{ik} - \frac{1}{2} S_{ik} e_{jk} + \frac{1}{2} S_{jk} e_{ik}) + \rho \Delta X_i = 0 \quad (4.6)$$

We assume the body force to be derived from a fixed potential; hence

$$X_i = - \frac{\partial U}{\partial x_i} \quad (4.7)$$

Then the linearized expression for ΔX_i is

$$\Delta X_i = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \quad (4.8)$$

For convenience we put

$$F_{ij} = t_{ij} + S_{kj}\omega_{ik} - \frac{1}{2}S_{ik}e_{jk} + \frac{1}{2}S_{jk}e_{ik} \quad (4.9)$$

The equilibrium equations (4.6) become

$$\frac{\partial F_{ij}}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = 0 \quad (4.10)$$

We now multiply these equations by arbitrary functions \bar{u}_i of the coordinates and integrate the result over an initial volume τ of the solid. We find

$$\iiint_{\tau} \left(\frac{\partial F_{ij}}{\partial x_j} \bar{u}_i - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \bar{u}_i \right) d\tau = 0 \quad (4.11)$$

Integration by parts yields

$$\iiint_{\tau} \left(F_{ij} \frac{\partial \bar{u}_i}{\partial x_j} + \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \bar{u}_i \right) d\tau - \iint_A F_{ij} n_j \bar{u}_i dA = 0 \quad (4.12)$$

The surface integral on the right side is extended to the boundary A of the volume τ . The unit outward normal to this boundary is denoted by n_i . The significance of F_{ij} is brought out by substituting the value (2.22) of Chapter 2 for t_{ij} . We obtain

$$F_{ij} = s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk} \quad (4.13)$$

The incremental boundary force Δf_i expressed by equation 7.56 of Chapter 1 may then be written

$$\Delta f_i = F_{ij} n_j \quad (4.14)$$

We also introduce the notation

$$F(u, \bar{u}) = F_{ij} \frac{\partial \bar{u}_i}{\partial x_j} + \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \bar{u}_i \quad (4.15)$$

Hence we write equation 4.12 in the form

$$\iiint_{\tau} F(u, \bar{u}) d\tau - \iint_A \Delta f_i \bar{u}_i dA = 0 \quad (4.16)$$

As shown below, the value of $F(u, \bar{u})$ may be transformed into the expression

$$F(u, \bar{u}) = t_{ij}\bar{e}_{ij} + S_{ij}(\omega_{ki}\bar{e}_{kj} + e_{ki}\bar{\omega}_{kj} + \omega_{ki}\bar{\omega}_{kj}) + \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \bar{u}_i \tag{4.17}$$

Equation 4.16 with the value (4.17) for $F(u, \bar{u})$ constitutes the first lemma.

In order to derive expression (4.17) we use the identity

$$\frac{\partial \bar{u}_i}{\partial x_j} = \bar{e}_{ij} + \bar{\omega}_{ij} \tag{4.17a}$$

Hence

$$\begin{aligned} F_{ij} \frac{\partial \bar{u}_i}{\partial x_j} &= t_{ij}\bar{e}_{ij} + t_{ij}\bar{\omega}_{ij} \\ &\quad + S_{kj}\omega_{ik}\bar{e}_{ij} + S_{kj}\omega_{ik}\bar{\omega}_{ij} \\ &\quad - \frac{1}{2}S_{ik}e_{jk}\bar{e}_{ij} - \frac{1}{2}S_{ik}e_{jk}\bar{\omega}_{ij} \\ &\quad + \frac{1}{2}S_{jk}e_{ik}\bar{e}_{ij} + \frac{1}{2}S_{jk}e_{ik}\bar{\omega}_{ij} \end{aligned} \tag{4.17b}$$

The quantities in this expression verify the relations

$$\begin{aligned} t_{ij} &= t_{ji} \\ \bar{e}_{ij} &= \bar{e}_{ji} \\ \bar{\omega}_{ij} &= -\bar{\omega}_{ji} \end{aligned} \tag{4.17c}$$

From these relations and the property of dummy indices we find

$$\begin{aligned} t_{ij}\bar{\omega}_{ij} &= 0 \\ \frac{1}{2}S_{ik}e_{jk}\bar{e}_{ij} &= \frac{1}{2}S_{jk}e_{ik}\bar{e}_{ij} \\ -\frac{1}{2}S_{ik}e_{jk}\bar{\omega}_{ij} &= \frac{1}{2}S_{jk}e_{ik}\bar{\omega}_{ij} \end{aligned} \tag{4.17d}$$

Hence

$$F_{ij} \frac{\partial \bar{u}_i}{\partial x_j} = t_{ij}\bar{e}_{ij} + S_{kj}(\omega_{ik}\bar{e}_{ij} + e_{ik}\bar{\omega}_{ij} + \omega_{ik}\bar{\omega}_{ij}) \tag{4.17e}$$

By a change of dummy indices we may also write

$$F_{ij} \frac{\partial \bar{u}_i}{\partial x_j} = t_{ij}\bar{e}_{ij} + S_{ij}(\omega_{ki}\bar{e}_{kj} + e_{ki}\bar{\omega}_{kj} + \omega_{ki}\bar{\omega}_{kj}) \tag{4.17f}$$

Equation 4.17 follows from this result.

Lemma II. Until now the functions \bar{u}_i have remained arbitrary. Let us now assume that they also satisfy the equilibrium equations 4.10. Hence

$$\frac{\partial \bar{F}_{ij}}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} \bar{u}_j = 0 \tag{4.18}$$

We denote by \bar{F}_{ij} the value (4.9) after replacing u_i by \bar{u}_i .

We may repeat the foregoing derivation by multiplying equations 4.18 by u_i and integrating over the same volume. The result amounts to interchanging u and \bar{u} in equation 4.16. Thus

$$\iiint_{\tau} F(\bar{u}, u) d\tau - \iint_A \Delta \bar{f}_i u_i dA = 0 \quad (4.19)$$

Subtracting this equation from equation 4.16, we obtain

$$\iiint_{\tau} [F(u, \bar{u}) - F(\bar{u}, u)] d\tau - \iint_A (\Delta f_i \bar{u}_i - \Delta \bar{f}_i u_i) dA = 0 \quad (4.20)$$

From equation 4.17 and the symmetry property $S_{ij} = S_{ji}$ we also derive

$$F(u, \bar{u}) - F(\bar{u}, u) = t_{ij} \bar{e}_{ij} - \bar{t}_{ij} e_{ij} \quad (4.21)$$

and equation 4.20 finally becomes

$$\iiint_{\tau} (t_{ij} \bar{e}_{ij} - \bar{t}_{ij} e_{ij}) d\tau - \iint_A (\Delta f_i \bar{u}_i - \Delta \bar{f}_i u_i) dA = 0 \quad (4.22)$$

This constitutes the second lemma.

Conditions Leading to Real Values for the Characteristic Exponents. Let us assume that there exists a characteristic solution of amplitude field u_i and complex value of the characteristic exponent p . There must exist also complex conjugate solutions u_i^* and p^* . We may substitute these two solutions in place of u_i and \bar{u}_i in the second lemma (4.22). Hence

$$\iiint_{\tau} (t_{ij} e_{ij}^* - t_{ij}^* e_{ij}) d\tau - \iint_A (\Delta f_i u_i^* - \Delta f_i^* u_i) dA = 0 \quad (4.23)$$

We consider now three types of boundary conditions.

(a) *A free surface where the boundary forces vanish.* For this surface $\Delta f_i = \Delta f_i^* = 0$.

(b) *Vanishing displacement $u_i = u_i^* = 0$ at the boundary.* This will occur at a rigid boundary with perfect adherence or with boundaries at infinity and the condition that the displacement becomes very small with increasing distance, so that boundary integrals of the products $\Delta f_i u_i$ will vanish.

(c) *Perfect slip at a rigid boundary.*

For all three conditions we may write

$$\iint_A (\Delta f_i u_i^* - \Delta f_i^* u_i) dA = 0 \quad (4.24)$$

In cases (a) and (b) this is immediately evident. That it is also true for case (c) can be shown without difficulty by following the procedure used in deriving equations 4.66 and 4.67 of Chapter 3. We may write

$$\Delta f_i u_i^* = -S\phi \frac{\partial^2 F}{\partial x_j \partial x_i} u_j u_i^* \quad (4.25)$$

where ϕ and F are defined by the shape of the rigid surface, and S is the magnitude of the normal initial stress. Equation 4.24 is verified by substituting expression (4.25).

Hence with boundary conditions (a), (b), (c) equation 4.23 is reduced to the vanishing of a volume integral:

$$\iiint_V (t_{ij} e_{ij}^* - t_{ij}^* e_{ij}) d\tau = 0 \quad (4.26)$$

We shall now consider the stress-strain relations between t_{ij} and e_{ij} . We write

$$t_{ij} = \hat{C}_{ij}^{\mu\nu} e_{\mu\nu} \quad (4.27)$$

where the $\hat{C}_{ij}^{\mu\nu}$ represent functions of the complex variable p . Let us also introduce the assumption that these functions satisfy the symmetry condition

$$\hat{C}_{ij}^{\mu\nu} = \hat{C}_{\mu\nu}^{ij} \quad (4.28)$$

With this assumption we derive

$$t_{ij} e_{ij}^* - t_{ij}^* e_{ij} = (\hat{C}_{ij}^{\mu\nu} - \hat{C}_{ij}^{\mu\nu*}) e_{ij}^* e_{\mu\nu} \quad (4.29)$$

We put

$$\hat{C}_{kj}^{\mu\nu} - \hat{C}_{kj}^{\mu\nu*} = 2i \mathcal{I}_{kj}^{\mu\nu} \quad (4.30)$$

where $i \mathcal{I}_{kj}^{\mu\nu}$ is the imaginary part of $\hat{C}_{kj}^{\mu\nu}$. Hence

$$t_{kj} e_{kj}^* - t_{kj}^* e_{kj} = 2i \mathcal{I}_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu} \quad (4.31)$$

Because of our assumption (4.28) the following symmetry condition is also valid:

$$\mathcal{I}_{kj}^{\mu\nu} = \mathcal{I}_{\mu\nu}^{kj} \quad (4.32)$$

Let us write

$$e_{kj} = \alpha_{kj} + i\beta_{kj} \quad (4.33)$$

From equation 4.32 it follows that

$$\mathcal{F}_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu} = \mathcal{F}_{kj}^{\mu\nu} (\alpha_{kj} \alpha_{\mu\nu} + \beta_{kj} \beta_{\mu\nu}) \tag{4.34}$$

Let us assume that, throughout the medium for any arbitrary complex value of p , the quadratic form

$$\mathcal{F}_{kj}^{\mu\nu} \alpha_{kj} \alpha_{\mu\nu} \tag{4.35}$$

is either *non-negative* at all points or *non-positive* at all points. We shall also assume that, for the particular characteristic solution considered, the expression $\mathcal{F}_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu}$ does not vanish everywhere in a finite region of the medium. As a consequence it must be of the same sign at all points where it does not vanish. Hence equation 4.26 cannot be verified unless p is real. Under these conditions and with the implied boundary conditions (a), (b), (c) listed above, *the characteristic exponent p must be real.*

This criterion is generally fulfilled in actual physical situations for characteristic solutions which involve dissipation. As a trivial example in which the criterion is not fulfilled, we note the case of a medium composed of elastic and viscoelastic regions and a characteristic solution where the deformation vanishes in the viscoelastic regions. Under these conditions the expression $\mathcal{F}_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu}$ vanishes throughout the medium for all values of p whether real or complex. However, such a case does not differ from purely elastic buckling.

Relation of Real Characteristic Values to Thermodynamics. If the viscoelastic material obeys the principles of linear thermodynamics, the operators $C_{ij}^{\mu\nu}$ are of the form (3.32). They are written

$$\hat{C}_{ij}^{\mu\nu} = \int_0^\infty \frac{p}{p+r} C_{ij}^{\mu\nu}(r) dr + C_{ij}^{\mu\nu} + p C'_{ij}^{\mu\nu} \tag{4.35a}$$

The operators must satisfy the symmetry condition (3.33). Hence

$$\hat{C}_{ij}^{\mu\nu} = \hat{C}_{\mu\nu}^{ij} \tag{4.35b}$$

For a complex value p the imaginary part of $\hat{C}_{kj}^{\mu\nu}$ is

$$i \mathcal{F}_{kj}^{\mu\nu}(p) = \frac{(p - p^*)}{2} \left[\int_0^\infty \frac{r}{(p+r)(p^*+r)} C_{kj}^{\mu\nu}(r) dr + C'_{kj}^{\mu\nu} \right] \tag{4.35c}$$

As already pointed out in the discussion of expressions (3.32), thermodynamic principles imply that the quadratic forms $C_{kj}^{\mu\nu}(r) e_{kj}^* e_{\mu\nu}$ and $C'_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu}$ be non-negative. Equation 4.35c then shows that $\mathcal{F}_{kj}^{\mu\nu}(p) e_{kj}^* e_{\mu\nu}$ is either non-negative at all points or non-positive at all points, according to the sign of $(p - p^*)/i$. Let us assume that e_{kj} represents the complex deformation field of a characteristic solution. We consider a local deformation $e_{kj} \exp(i\alpha t)$ varying harmonically with time at the frequency $\alpha/2\pi$ with complex strain amplitudes e_{kj} . It

will be shown in a later section (see equation 9.54 below) that the power dissipated per unit volume is $\frac{1}{2}\alpha\mathcal{F}_{kj}^{\mu\nu}(i\alpha)e_{kj}^*e_{\mu\nu}$. If the dissipation thus defined is not everywhere zero, equation 4.35c shows that $\mathcal{F}_{kj}^{\mu\nu}(p)e_{kj}^*e_{\mu\nu}$ does not vanish everywhere unless p is a real quantity.

Hence under these conditions the criterion for real characteristic values p is fulfilled and becomes a consequence of thermodynamic principles.

It is implied, of course, that the system also satisfies the boundary conditions (a), (b), and (c) listed above and associated with equation 4.26.

This conclusion is in agreement with the general properties of characteristic roots of thermodynamic systems already discussed in the more general context of equations 2.26.

Real Characteristic Values for an Incompressible Medium with Horizontal Stratification. These values are of particular interest in the technological problems of creep buckling of laminated plates. They are also applicable to geological problems of folding instability of stratified rock structures under tectonic stresses. We shall show that for such media the criterion ensuring real characteristic values is considerably simplified.

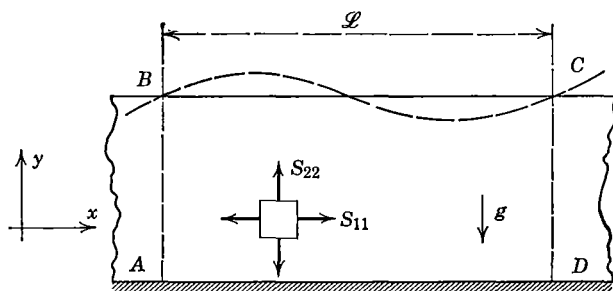


Figure 4.1 Incompressible medium with horizontal stratification deformed sinusoidally along the horizontal direction in a gravity field g .

We assume an incompressible orthotropic medium with vertical and horizontal directions of viscoelastic symmetry and principal initial stresses also oriented along the same directions. We consider incremental deformations in the x, y plane with a vertical y axis. The effect of gravity is taken into account. The two principal components of the initial stress S_{11} and S_{22} in the x, y plane are functions of y (Fig. 4.1).

The incremental stress-strain relations are obtained from equations

8.28 of Chapter 2 by replacing the elastic coefficients N and Q by operators. They are

$$\begin{aligned} s_{11} - s &= 2\hat{N}e_{xx} \\ s_{22} - s &= 2\hat{N}e_{yy} \\ s_{12} &= 2\hat{Q}e_{xy} \end{aligned} \quad (4.36)$$

The two operators \hat{N} and \hat{Q} are functions of y .

Essentially, this case corresponds to a non-homogeneous medium with horizontal stratification either continuous or discontinuous. In stability problems, solutions are considered which are sinusoidal along the horizontal direction. We shall consider such a solution and apply the second lemma (4.23) to the rectangular area $ABCD$ of Figure 4.1.

The vertical sides AB and CD are separated by a distance \mathcal{L} of one wavelength. The surface integral in equation 4.23 is now reduced to a contour integral. It becomes

$$\oint (\Delta f_i u_i^* - \Delta f_i^* u_i) dl = \int_A^B + \int_B^C + \int_C^D + \int_D^A \quad (4.37)$$

The length element along the contour is denoted by dl . Owing to the periodicity of the solution, the line integrals along the vertical sides AB and CD cancel out. If the horizontal side BC is a free surface, we must put $\Delta f_i = 0$ and the integral also vanishes along that side. If the viscoelastic medium rests on a rigid base with perfect adherence, the displacement u_i is zero and the integral along the bottom side AD vanishes. It also vanishes if we assume perfect slip on a rigid base. Then the boundary forces are normal to the displacements, and their scalar product vanishes. Hence

$$\Delta f_i u_i^* = 0 \quad (4.38)$$

For a half-space of infinite depth the side AD is at infinity, and the integral along AD will vanish if the solution tends to zero at large depth. Finally, of course, the top and bottom boundaries may belong to any of the three types considered. For example, they include a free plate or a plate embedded in an infinite medium.

For all these cases the contour integral (4.37) vanishes. Equation 4.23 may then be written

$$\iint_{ABCD} (t_{ij} e_{ij}^* - t_{ij}^* e_{ij}) dx dy = 0 \quad (4.39)$$

This surface integral in the x, y plane is now evaluated over the rectangular domain $ABCD$. Let us write the integrand explicitly:

$$t_{ij}e_{ij}^* - t_{ij}^*e_{ij} = t_{11}e_{xx}^* + t_{22}e_{yy}^* + 2t_{12}e_{xy}^* - t_{11}^*e_{xx} - t_{22}^*e_{yy} - 2t_{12}^*e_{xy} \quad (4.40)$$

The values of t_{ij} in terms of s_{ij} are given by equations 4.33 of Chapter 3. They are

$$\begin{aligned} t_{11} &= s_{11} + S_{11}e_{yy} \\ t_{22} &= s_{22} + S_{22}e_{xx} \\ t_{12} &= s_{12} - \frac{1}{2}(S_{11} + S_{22})e_{xy} \end{aligned} \quad (4.41)$$

When we substitute these values into expression (4.40) and take into account the assumption of incompressibility, the terms containing S_{11} and S_{22} cancel out. For example,

$$t_{11}e_{xx}^* - t_{11}^*e_{xx} = s_{11}e_{xx}^* - s_{11}^*e_{xx} + S_{11}e_{yy}e_{xx}^* - S_{11}e_{yy}^*e_{xx} \quad (4.42)$$

The condition of incompressibility is

$$\begin{aligned} e_{xx} &= -e_{yy} \\ e_{xx}^* &= -e_{yy}^* \end{aligned} \quad (4.43)$$

Hence $e_{yy}e_{xx}^* = e_{yy}^*e_{xx}$, and equation 4.42 is simplified to

$$t_{11}e_{xx}^* - t_{11}^*e_{xx} = s_{11}e_{xx}^* - s_{11}^*e_{xx} \quad (4.44)$$

The same cancellation occurs for the other terms. Therefore the integrand (4.40) may be written

$$t_{ij}e_{ij}^* - t_{ij}^*e_{ij} = s_{11}e_{xx}^* + s_{22}e_{yy}^* + 2s_{12}e_{xy}^* - s_{11}^*e_{xx} - s_{22}^*e_{yy} - 2s_{12}^*e_{xy} \quad (4.45)$$

We now substitute in this equation the values (4.36) for the stress components s_{11} , s_{22} , and s_{12} . Again taking into account the incompressibility conditions (4.43), we derive

$$t_{ij}e_{ij}^* - t_{ij}^*e_{ij} = 4(\hat{N} - \hat{N}^*)e_{xx}e_{xx}^* + 4(\hat{Q} - \hat{Q}^*)e_{xy}e_{xy}^* \quad (4.46)$$

This expression shows that for p complex the integral in equation 4.39 cannot vanish if the following conditions are fulfilled:

1. The coefficients of the imaginary parts of \hat{N} and \hat{Q} are everywhere non-negative or everywhere non-positive.
2. For the particular characteristic solution considered, at least

one of the quantities $(\hat{N} - \hat{N}^*)e_{xx}$ or $(\hat{Q} - \hat{Q}^*)e_{xy}$ is different from zero over a finite region of the medium.

These conditions provide an adequate criterion for ensuring real values for the characteristic roots p . The boundary conditions as stated previously are, of course, implied. A direct proof of this theorem for the same case, that of an incompressible medium with horizontal stratification, was derived in a recent paper by the author.†

The theorem may be expressed in terms of the operators:

$$\begin{aligned}\hat{L} &= \hat{Q} + \frac{1}{2}P \\ \hat{M} &= \hat{N} + \frac{1}{4}P\end{aligned}\tag{4.47}$$

where $P = S_{22} - S_{11}$. Since P is real, the same criterion holds with the operators \hat{L} and \hat{M} instead of \hat{Q} and \hat{N} . Hence the characteristic roots are real if the imaginary parts of \hat{L} and \hat{M} are not zero and always of the same constant sign, and if condition 2 is also satisfied.

For a material whose viscoelastic properties are governed by the principles of linear thermodynamics the operators \hat{L} and \hat{M} are written

$$\begin{aligned}\hat{L} &= \int_0^\infty \frac{p}{p+r} L(r) dr + L + pL' \\ \hat{M} &= \int_0^\infty \frac{p}{p+r} M(r) dr + M + pM'\end{aligned}\tag{4.48}$$

In these expressions the quantities $L(r)$, L' , $M(r)$, and M' are all non-negative. Hence the condition that the imaginary coefficients of \hat{L} and \hat{M} be either non-negative everywhere or non-positive everywhere is always fulfilled.

5. SMALL DEFORMATIONS SUPERPOSED ON AN INITIAL STATE OF FLOW

Until now we have assumed that the medium is initially at rest and in a state of thermodynamic equilibrium.

In many problems of stability this may not be the case. For example, a rod of purely viscous material subject to a constant axial compression will be in an initial state of steady flow. This initial

† M. A. Biot, *Stability of Multilayered Continua Including the Influence of Gravity and Viscoelasticity*, *Journal of the Franklin Institute*, Vol. 276, No. 3, pp. 231-252, 1963.

state is unstable, and a perturbation will induce gradual buckling. We are dealing here with a perturbation in the vicinity of a steady irreversible process. Such perturbations may not obey the principles which govern the linear thermodynamics in the vicinity of an equilibrium state. Onsager's relations, for instance, may not be applicable.

There are cases, however, for which the initial irreversible process does not deviate too much from an equilibrium state. Under these conditions, if the total deformations remain small, the linear thermodynamic theory may still be applicable. The incremental stresses will then be the same as if the medium were initially free of stress.

Certain questions come up in the formulation of such problems in connection with the concept of incremental strain and its relation to strain rates. In particular, it is important to examine the validity of the *correspondence principle* for a medium which is initially in a state of flow. These questions are purely kinematic in nature.

They will now be examined in more detail through an analysis of the perturbations of steady flow in a fluid with Newtonian viscosity. We shall assume slow deformations and neglect inertia forces.

Kinematics of Steady Flow. For an incompressible Newtonian fluid in two-dimensional flow the stresses are given by the equations

$$\begin{aligned}\sigma_{xx} - \sigma &= 2\eta \frac{\partial v_x}{\partial x} \\ \sigma_{yy} - \sigma &= 2\eta \frac{\partial v_y}{\partial y} \\ \sigma_{xy} &= \eta \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)\end{aligned}\tag{5.1}$$

The components of the velocity field at a fixed point x, y are v_x and v_y , and the fluid viscosity is denoted by η .

The condition of incompressibility of the fluid is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\tag{5.2}$$

Let us assume a velocity field defined by the equations

$$\begin{aligned}v_x &= -p_0 x \\ v_y &= p_0 y\end{aligned}\tag{5.3}$$

where p_0 is a constant. The state of stress corresponding to this velocity field is obtained by substituting the components (5.3) into equations 5.1. The shear stress σ_{xy} vanishes. We denote by S_{11} , S_{22} , and S the constant values of σ_{xx} , σ_{yy} , and σ . Equations 5.1 become

$$\begin{aligned} S_{11} - S &= -2\eta p_0 \\ S_{22} - S &= 2\eta p_0 \end{aligned} \quad (5.4)$$

They correspond to a uniform stress with principal components along the x and y axes. Equations 5.4 lead to the result

$$p_0 = \frac{P}{4\eta} \quad (5.5)$$

with

$$P = S_{22} - S_{11} \quad (5.6)$$

For $S_{22} = 0$, this expression represents a compressive stress along x .

Equations 5.3 define the deformation in terms of the velocity field. We shall now determine the corresponding displacements of the fluid particles. We consider a particle whose coordinates are x and y at the time $t = 0$. The coordinates X and Y of the same particle at the time t are functions of t , and they determine the motion of the particle. According to equations 5.3 they satisfy the differential equations

$$\begin{aligned} \frac{dX}{dt} &= -p_0 X \\ \frac{dY}{dt} &= p_0 Y \end{aligned} \quad (5.7)$$

They must be solved with the initial conditions that the coordinates be x and y for $t = 0$. The solution is

$$\begin{aligned} X &= x e^{-p_0 t} \\ Y &= y e^{p_0 t} \end{aligned} \quad (5.8)$$

Since Y increases exponentially, *there is an appearance of instability which is purely kinematic in origin*. A sinusoidal line along the x direction as shown in Figure 5.1a acquires the appearance of Figure 5.1b after a large deformation has taken place.

Kinematics of Strain Rate in Unsteady Flow. We shall first analyze the kinematics of the plane homogeneous deformation for the general case of finite strain and unsteady flow. The homogeneous

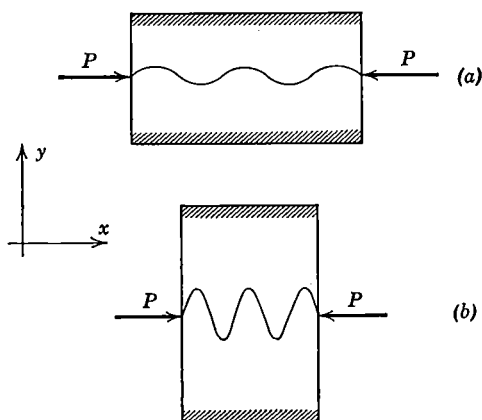


Figure 5.1 Appearance of kinematic instability in a viscous homogeneous fluid in uniform flow.

deformation is represented as in section 1 of Chapter 1. The coordinates X, Y of a particle originally at the point x, y are written:

$$\begin{aligned} X &= \alpha_{11}x + \alpha_{12}y \\ Y &= \alpha_{21}x + \alpha_{22}y \end{aligned} \quad (5.9)$$

It is more convenient here to write α_{11} and α_{22} for the two coefficients $1 + a_{11}$ and $1 + a_{22}$ used in Chapter 1. The four coefficients $\alpha_{11}(t)$, $\alpha_{12}(t)$, $\alpha_{21}(t)$, and $\alpha_{22}(t)$ are now functions of the time t . With the notation

$$\dot{X} = \frac{\partial X}{\partial t} \quad (5.10)$$

the velocity field is

$$\begin{aligned} v_x &= \dot{X} = \dot{\alpha}_{11}x + \dot{\alpha}_{12}y \\ v_y &= \dot{Y} = \dot{\alpha}_{21}x + \dot{\alpha}_{22}y \end{aligned} \quad (5.11)$$

These expressions represent the velocity of a particle at the point X, Y in terms of its initial coordinates x, y .

In order to analyze the stresses in a viscous fluid we need to express the velocity of a particle by means of the coordinates of the particle at the instant considered. Hence we must determine the velocity field (5.11) as a function of the coordinates X, Y instead of the original values x, y . We therefore solve equations 5.9 for x and y .

Since the fluid is incompressible, the determinant of the equation is equal to unity:

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1 \tag{5.12}$$

We derive

$$\begin{aligned} x &= \alpha_{22}X - \alpha_{12}Y \\ y &= -\alpha_{21}X + \alpha_{11}Y \end{aligned} \tag{5.13}$$

Substitution of these values into equations 5.11 yields

$$\begin{aligned} v_x &= \mathcal{A}X + \mathcal{B}Y \\ v_y &= \mathcal{C}X + \mathcal{D}Y \end{aligned} \tag{5.14}$$

with the coefficients

$$\begin{aligned} \mathcal{A} &= \dot{\alpha}_{11}\alpha_{22} - \dot{\alpha}_{12}\alpha_{21} \\ \mathcal{B} &= \dot{\alpha}_{12}\alpha_{11} - \dot{\alpha}_{11}\alpha_{12} \\ \mathcal{C} &= \dot{\alpha}_{21}\alpha_{22} - \dot{\alpha}_{22}\alpha_{21} \\ \mathcal{D} &= \dot{\alpha}_{22}\alpha_{11} - \dot{\alpha}_{21}\alpha_{12} \end{aligned} \tag{5.15}$$

We may, of course, replace X and Y by x and y in equations 5.14, because this is a mere change of notation. These equations then represent the velocity at the point x, y . We write

$$\begin{aligned} v_x &= \mathcal{A}x + \mathcal{B}y \\ v_y &= \mathcal{C}x + \mathcal{D}y \end{aligned} \tag{5.16}$$

The strain-rate components are

$$\begin{aligned} \frac{\partial v_x}{\partial x} &= \mathcal{A} \\ \frac{\partial v_y}{\partial y} &= \mathcal{D} \\ \frac{1}{2} \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) &= \frac{1}{2} (\mathcal{B} + \mathcal{C}) \end{aligned} \tag{5.17}$$

By substitution of these expressions into equations 5.1 we obtain the stress components in the fluid:

$$\begin{aligned} \sigma_{xx} - \sigma &= 2\eta\mathcal{A} \\ \sigma_{yy} - \sigma &= 2\eta\mathcal{D} \\ \sigma_{xy} &= \eta(\mathcal{B} + \mathcal{C}) \end{aligned} \tag{5.18}$$

The condition of incompressibility (5.2) becomes

$$\mathcal{A} + \mathcal{D} = 0 \tag{5.19}$$

The same condition is also derived by taking the time derivative of equation 5.12.

These results are quite general and valid for finite deformations. They will now be used to derive expressions for incremental stresses in the fluid under the assumption that the total deformation is small. This total strain is the combined deformation due to the superposition of a perturbation on the initial steady flow.

Incremental Stresses when the Total Strain Remains Small.

We consider an initial state of stress with principal stresses S_{11} and S_{22} along the x and y directions. In a viscous fluid this constant and uniform stress field is associated with a steady flow. The particle displacement is given by equations 5.8. The corresponding coefficients in the coordinate transformation (5.9) are

$$\begin{aligned}\alpha_{11} &= e^{-p_0 t} & \alpha_{22} &= e^{p_0 t} \\ \alpha_{12} &= a_{21} = 0\end{aligned}\tag{5.20}$$

Let us superpose a small perturbation on this steady state. We express it by writing the coefficients of equations 5.9 in the form

$$\begin{aligned}\alpha_{11} &= e^{-p_0 t} + \gamma_{11} \\ \alpha_{12} &= \gamma_{12} \\ \alpha_{21} &= \gamma_{21} \\ \alpha_{22} &= e^{p_0 t} + \gamma_{22}\end{aligned}\tag{5.21}$$

The perturbations γ_{ij} are assumed to be small quantities of the first order. They are functions of time.

Since we are concerned primarily with stability problems, we shall restrict the analysis to the particular case in which the perturbations are exponential functions of time. Therefore we put

$$\gamma_{ij} = \gamma'_{ij} e^{pt}\tag{5.22}$$

where γ'_{ij} is a constant. Hence the time derivative of γ_{ij} may be written

$$\dot{\gamma}_{ij} = p\gamma_{ij}\tag{5.23}$$

We now substitute expressions (5.21) for the coefficients into equations 5.15 and retain only the first order terms in γ_{ij} . We derive

$$\begin{aligned}\mathcal{A} &= -p_0 + p\gamma_{11}e^{p_0 t} - p_0\gamma_{22}e^{-p_0 t} \\ \mathcal{B} &= (p + p_0)\gamma_{12}e^{-p_0 t} \\ \mathcal{C} &= (p - p_0)\gamma_{21}e^{p_0 t} \\ \mathcal{D} &= p_0 + p\gamma_{22}e^{-p_0 t} + p_0\gamma_{11}e^{p_0 t}\end{aligned}\tag{5.24}$$

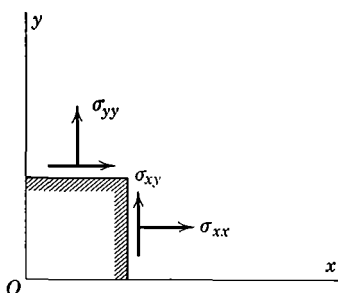


Figure 5.2 Stresses referred to fixed axes x, y .

These expressions are further simplified by introducing the additional assumption

$$\begin{aligned} e^{-p_0 t} - 1 &\ll 1 \\ e^{p_0 t} - 1 &\ll 1 \end{aligned} \tag{5.25}$$

This implies that during the time interval t the deformation associated with the initial state of steady flow is also small. Under this condition equations 5.24 become

$$\begin{aligned} \mathcal{A} &= -p_0 + p\gamma_{11} - p_0\gamma_{22} \\ \mathcal{B} &= (p + p_0)\gamma_{12} \\ \mathcal{C} &= (p - p_0)\gamma_{21} \\ \mathcal{D} &= p_0 + p\gamma_{22} + p_0\gamma_{11} \end{aligned} \tag{5.26}$$

With these values the stresses (5.18) in the fluid are written

$$\begin{aligned} \sigma_{xx} - \sigma &= -2\eta p_0 + 2\eta(p\gamma_{11} - p_0\gamma_{22}) \\ \sigma_{yy} - \sigma &= 2\eta p_0 + 2\eta(p\gamma_{22} + p_0\gamma_{11}) \\ \sigma_{xy} &= \eta p(\gamma_{12} + \gamma_{21}) + \eta p_0(\gamma_{12} - \gamma_{21}) \end{aligned} \tag{5.27}$$

The condition of incompressibility (5.19) becomes

$$\mathcal{A} + \mathcal{D} = (p + p_0)\gamma_{11} + (p - p_0)\gamma_{22} = 0 \tag{5.28}$$

The stress components (5.27) are referred to the x and y directions (Fig. 5.2). Our purpose here is to evaluate the incremental stresses in a form comparable with the stress components s_{ij} which we have used in the general elastic and viscoelastic theory. Hence we must derive expressions for the incremental stresses s_{ij} referred to axes

which have been rotated through the same angle as the material. The angle through which the material has rotated is

$$\omega = \frac{1}{2}(\gamma_{21} - \gamma_{12}) \quad (5.29)$$

If coordinate axes which have undergone the same rotation ω are

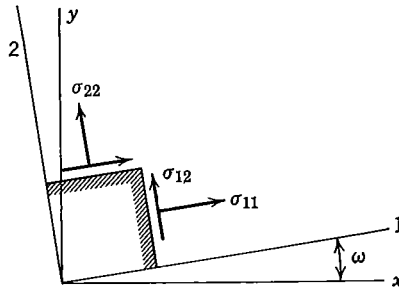


Figure 5.3 Stresses referred to rotated axes 1, 2.

chosen, the coefficients γ_{ij} acquire particular values which satisfy the relation

$$\gamma_{12} = \gamma_{21} \quad (5.30)$$

With stresses σ_{11} , σ_{22} , σ_{12} referred to the same rotated axes (Fig. 5.3), equations 5.27 become

$$\begin{aligned} \sigma_{11} - \sigma &= -2\eta p_0 + 2\eta(p\gamma_{11} - p_0\gamma_{22}) \\ \sigma_{22} - \sigma &= 2\eta p_0 + 2\eta(p\gamma_{22} + p_0\gamma_{11}) \\ \sigma_{12} &= 2\eta p\gamma_{12} \end{aligned} \quad (5.31)$$

By introducing the values (5.4) for the two terms $-2\eta p_0$ and $2\eta p_0$ we derive

$$\begin{aligned} \sigma_{11} - \sigma &= S_{11} - S + 2\eta(p\gamma_{11} - p_0\gamma_{22}) \\ \sigma_{22} - \sigma &= S_{22} - S + 2\eta(p\gamma_{22} + p_0\gamma_{11}) \\ \sigma_{12} &= 2\eta p\gamma_{12} \end{aligned} \quad (5.32)$$

Since $S_{11} - S$ and $S_{22} - S$ represent the initial stress, the remaining terms yield the incremental stresses. They are

$$\begin{aligned} s_{11} - s &= 2\eta(p\gamma_{11} - p_0\gamma_{22}) \\ s_{22} - s &= 2\eta(p\gamma_{22} + p_0\gamma_{11}) \\ s_{12} &= 2\eta p\gamma_{12} \end{aligned} \quad (5.33)$$

These incremental stress-strain relations are correct to the first order. However, their form is quite different from those of incremental stresses in an elastic medium. They contain not only the incremental strain but also the factor p_0 , which is related to the initial strain rate.

The difficulty may be eliminated by introducing a third and important assumption, namely,

$$p_0 \ll p \tag{5.34}$$

This assumption that p_0 is small with respect to p amounts to saying that the exponential factor $\exp(pt)$ which characterizes the unstable perturbation grows much faster than the factor $\exp(p_0t)$, which represents the strain produced by the initial flow. Such an assumption is verified if the unstable solution is physically significant. In practice, it will be so if we have approximately

$$\frac{p_0}{p} < \frac{1}{20} \tag{5.35}$$

Under these conditions we neglect the terms containing p_0 and write the incremental stresses (5.33) as

$$\begin{aligned} s_{11} - s &= 2\eta p \gamma_{11} \\ s_{22} - s &= 2\eta p \gamma_{22} \\ s_{12} &= 2\eta p \gamma_{12} \end{aligned} \tag{5.36}$$

With the same assumption the condition of incompressibility (5.28) becomes

$$\gamma_{11} + \gamma_{22} = 0 \tag{5.37}$$

Validity of the Correspondence Principle for a Viscous Fluid with Initial Flow. On the basis of the preceding results it is possible to show a formal correspondence between the equations of incremental deformations of a solid initially at rest and those of a fluid in a steady state of initial flow. Let us consider the displacements X and Y given by equations 5.8 for the initial state of flow. A perturbation is represented by adding small displacements u_x and u_y to these expressions. The total displacements are therefore

$$\begin{aligned} X + u_x &= xe^{-p_0t} + u_x \\ Y + u_y &= ye^{p_0t} + u_y \end{aligned} \tag{5.38}$$

Since the linear transformation (5.9) is referred to rotated axes, the coefficients are by definition $\alpha_{11} = 1 + \varepsilon_{11}$, $\alpha_{22} = 1 + \varepsilon_{22}$, and $\alpha_{12} = \alpha_{21} = \varepsilon_{12}$. The strain components ε_{ij} referred to the rotated axes 1, 2, are those analyzed in Chapter 1 (section 2), where it was shown that to the first order they are the same as the strain components referred to the fixed axes x, y . For example, $1 + \varepsilon_{11} \cong (\partial/\partial x)(X + u_x)$, etc. Hence, if we assume the total strain to be small, we may write

$$\begin{aligned}\alpha_{11} &= 1 + \varepsilon_{11} \cong \frac{\partial}{\partial x}(X + u_x) = e^{-p_0 t} + \frac{\partial u_x}{\partial x} \\ \alpha_{22} &= 1 + \varepsilon_{22} \cong \frac{\partial}{\partial y}(Y + u_y) = e^{p_0 t} + \frac{\partial u_y}{\partial y} \\ 2\alpha_{12} &= 2\varepsilon_{12} \cong \frac{\partial}{\partial x}(Y + u_y) + \frac{\partial}{\partial y}(X + u_x) = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\end{aligned}\quad (5.39)$$

Comparison of these values with equations 5.21 yields

$$\begin{aligned}\gamma_{11} &= \frac{\partial u_x}{\partial x} & \gamma_{22} &= \frac{\partial u_y}{\partial y} \\ \gamma_{12} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right)\end{aligned}\quad (5.40)$$

Hence with the notation

$$\begin{aligned}e_{xx} &= \frac{\partial u_x}{\partial x} & e_{yy} &= \frac{\partial u_y}{\partial y} \\ e_{xy} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right)\end{aligned}\quad (5.41)$$

the stress-strain relations (5.36) become

$$\begin{aligned}s_{11} - s &= 2\eta p e_{xx} \\ s_{22} - s &= 2\eta p e_{yy} \\ s_{12} &= 2\eta p e_{xy}\end{aligned}\quad (5.42)$$

These equations are also obtained by putting

$$\mu = \eta p \quad (5.43)$$

in the stress-strain relations (8.33) of Chapter 2 for the elastic medium. Although the coefficients (5.39) are derived for homogeneous strain,

equations 5.42 are obviously valid also for a non-homogeneous deformation where the incremental displacements u_x and u_y are arbitrary functions of x and y .

The condition (5.37) for incompressibility becomes

$$e_{xx} + e_{yy} = 0 \tag{5.44}$$

The equilibrium conditions for the incremental stress field are obtained by applying the equations derived in Chapter 1. They are applied by considering the particle coordinates to be x_i at $t = 0$ and to become $x_i + \bar{u}_i + u_i$ at the time t . The term \bar{u}_i represents the displacement due to the initial steady flow, while u_i is the component due to the perturbation. The equations derived in Chapter 1 are valid for the present case if u_i is replaced by $\bar{u}_i + u_i$. Their validity requires the assumption that the incremental stresses, as well as the gradients of the total displacements $\bar{u}_i + u_i$, remain small. The equilibrium conditions are given by equations 6.17 of Chapter 1. They are

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0 \end{aligned} \tag{5.45}$$

Note that only the perturbation u_i contributes to the total rotation ω . Hence

$$\omega = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \tag{5.46}$$

Equations 5.42 and 5.45 are identical with those of an elastic medium. The only difference comes from the interpretation of the elastic coefficient μ which is now given by expression (5.43). Thus the correspondence principle is extended to a medium of Newtonian viscosity under initial stress. All solutions derived from the elastic medium are therefore immediately applicable to the viscous medium provided that we replace μ by $p\eta$.

We must remember that this correspondence is formulated in terms of strain components (5.41) defined by means of displacements u_x and u_y which represent a perturbation superimposed on the initial steady state of flow. Validity of the correspondence requires the assumption that the gradients of the total displacements including both

the perturbed and unperturbed motion are small. In addition, condition (5.34) must be verified; that is, the instability must be significant.

Relation to the Navier-Stokes Equations. Since we are dealing with a viscous fluid, a question naturally arises regarding the relationship of the present results and those obtained from the equations of fluid dynamics.

This relationship is clarified by considering the equilibrium equations 5.45 for the stress field. According to equations 4.13 of Chapter I we may write

$$\begin{aligned}\bar{s}_{\xi\xi} &= s_{11} \\ \bar{s}_{\eta\eta} &= s_{22} \\ \bar{s}_{\xi\eta} &= s_{12} - P\omega\end{aligned}\tag{5.47}$$

These quantities represent the incremental stresses referred to the fixed axes x and y . When we use these expressions, the equilibrium conditions (5.45) become

$$\begin{aligned}\frac{\partial \bar{s}_{\xi\xi}}{\partial x} + \frac{\partial \bar{s}_{\xi\eta}}{\partial y} &= 0 \\ \frac{\partial \bar{s}_{\xi\eta}}{\partial x} + \frac{\partial \bar{s}_{\eta\eta}}{\partial y} &= 0\end{aligned}\tag{5.48}$$

Let us turn our attention to the stress-strain relations for the incremental stresses (5.47). When we compare them with the values given by equations 5.27, we obtain a *paradoxical result* which requires some clarification, as follows.

Substituting into equations 5.47 the values (5.42) and the value $P = 4\eta p_0$ derived from equation 5.5, we obtain

$$\begin{aligned}\bar{s}_{\xi\xi} - s &= 2\eta p e_{xx} \\ \bar{s}_{\eta\eta} - s &= 2\eta p e_{yy} \\ \bar{s}_{\xi\eta} &= 2\eta p e_{xy} - 4\eta p_0 \omega\end{aligned}\tag{5.49}$$

On the other hand, the same incremental stresses referred to fixed axes may be derived from equations 5.27. They are written

$$\begin{aligned}\bar{s}_{\xi\xi} - s &= 2\eta(p\gamma_{11} - p_0\gamma_{22}) \\ \bar{s}_{\eta\eta} - s &= 2\eta(p\gamma_{22} + p_0\gamma_{11}) \\ \bar{s}_{\xi\eta} &= \eta p(\gamma_{12} + \gamma_{21}) + \eta p_0(\gamma_{12} - \gamma_{21})\end{aligned}\tag{5.50}$$

The variables γ_{ij} in this case are referred to the fixed axes x, y . By definition, $e_{xx} = \gamma_{11}$, $e_{yy} = \gamma_{22}$, $2e_{xy} = \gamma_{12} + \gamma_{21}$, and $2\omega = \gamma_{21} - \gamma_{12}$. With these values, equations 5.50 become

$$\begin{aligned}\bar{s}_{\xi\xi} - s &= 2\eta(pe_{xx} - p_0e_{yy}) \\ \bar{s}_{\eta\eta} - s &= 2\eta(pe_{yy} + p_0e_{xx}) \\ \bar{s}_{\xi\eta} &= 2\eta pe_{xy} - 2\eta p_0\omega\end{aligned}\tag{5.51}$$

This result does not coincide with equations 5.49. In addition, equations 5.49 and 5.51 both contain the rotation ω and the parameter p_0 which defines the initial strain rate. However, within the approximation of the present theory these difficulties may be eliminated by the following procedure.

In the first place, according to the assumption (5.34) ($p_0 \ll p$), we may drop the terms with p_0 in the first two of equations 5.51. The same argument cannot always be applied to the third of equations 5.49 and 5.51, because we must consider the possibility that the strain is much smaller than the rotation. If the strain is smaller, we introduce a *fictitious* strain component e'_{xy} defined as

$$\bar{s}_{\xi\eta} = 2\eta \frac{de_{xy}}{dt} - 2K\eta p_0\omega = 2\eta \frac{de'_{xy}}{dt}\tag{5.52}$$

For $K = 1$ this relation expresses the third of equations 5.51, and for $K = 2$ the third of equations 5.49. By integration with respect to time, equation 5.52 yields

$$e_{xy} + Kp_0t\omega_{av} = e'_{xy}\tag{5.53}$$

We denote by ω_{av} an average value of the rotation. The strain due to the initial stress is $\exp(p_0t) - 1 \cong p_0t$ and is assumed to be of the first order. Hence the product $p_0t\omega_{av}$ is of the second order. Therefore to the first order we may write $e_{xy} = e'_{xy}$. Note that the time derivatives of these quantities are not the same to the first order, but this is immaterial since we are interested not in the actual strain rates but in the displacements.

These considerations lead to the conclusion that, within the approximations introduced above, equations 5.49 and 5.51 are equivalent to the form

$$\begin{aligned}\bar{s}_{\xi\xi} - s &= 2\eta pe_{xx} \\ \bar{s}_{\eta\eta} - s &= 2\eta pe_{yy} \\ \bar{s}_{\xi\eta} &= 2\eta pe_{xy}\end{aligned}\tag{5.54}$$

By substituting these stresses in the equilibrium equations (5.48) and taking into account the condition of incompressibility (5.44), we derive the equations

$$\begin{aligned}\eta p \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_x + \frac{\partial s}{\partial x} &= 0 \\ \eta p \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_y + \frac{\partial s}{\partial y} &= 0\end{aligned}\tag{5.55}$$

Let us go back to expressions (5.1) for the stresses σ_{xx} , σ_{yy} , and σ_{xy} in a viscous fluid. These stresses satisfy the equilibrium conditions

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0\end{aligned}\tag{5.56}$$

Substituting in these equations the values (5.1) for the stresses and using the incompressibility condition (5.2), we obtain

$$\begin{aligned}\eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v_x + \frac{\partial \sigma}{\partial x} &= 0 \\ \eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v_y + \frac{\partial \sigma}{\partial y} &= 0\end{aligned}\tag{5.57}$$

These relations are a particular form of the Navier-Stokes equations for the mechanics of an incompressible viscous fluid when inertia forces are negligible. They are identical with equations 5.55 provided we put

$$\begin{aligned}v_x &= pu_x \\ v_y &= pu_y\end{aligned}\tag{5.58}$$

Hence we have shown that under the present assumptions the theory of incremental deformations of a viscous fluid leads to the Navier-Stokes equations. We note that the initial stresses do not appear at all in equations 5.55. However, they still remain in the boundary conditions, which must now be expressed by means of the stress components $\bar{s}_{\xi\xi}$, $\bar{s}_{\eta\eta}$, $\bar{s}_{\xi\eta}$.

As an example, let us consider the boundary conditions corresponding to equations 6.17 of Chapter 3. They are written

$$\begin{aligned}\Delta f_x &= s_{12} + Pe_{xy} \\ \Delta f_y &= s_{22}\end{aligned}\tag{5.59}$$

These conditions correspond to the case of an initial compression P in the x direction. The quantities $\Delta f_x, \Delta f_y$ are the tangential and normal components of stress on a deformed surface initially coincident with a plane perpendicular to the y axis. Substituting in expressions (5.59) the values of s_{12} and s_{22} taken from equations 5.47, we find

$$\begin{aligned} \Delta f_x &= \bar{s}_{zn} + P \frac{\partial u_y}{\partial x} \\ \Delta f_y &= \bar{s}_{nn} \end{aligned} \tag{5.60}$$

These boundary conditions contain the initial stress P . They are the same as those derived in the latter part of this section dealing with incremental stresses in a fluid under conditions of finite strain.

Attention should be called to the fact that equilibrium conditions (5.45) and (5.48) are mathematically equivalent, while the stress-strain relations (5.42) and (5.54) are not. This is because of the approximations which have been introduced in these relations and which involve the assumption $p_0 \ll p$. Hence solutions of equations 5.42 and 5.45 will not be mathematically equivalent to solutions of the Navier-Stokes equations (5.55). However, from the physical viewpoint their differences will not be significant.

The interest in the forms (5.42) and (5.45) of the equations lies, of course, in their formal identity with those of an elastic medium. As a consequence the exact solution for the elastic medium is also valid as an approximation in stability problems for viscous fluids.

Application of this approximation will be discussed in later sections of this chapter.

Anisotropic Fluids and Non-linear Flow Properties. In the foregoing analysis of a medium with initial flow we have restricted ourselves to a viscous fluid of Newtonian viscosity. Application of the correspondence principle may be extended to stability problems of materials with anisotropic and non-linear flow properties. This can be seen by considering the flow properties of the material in a rotating frame of reference. In two dimensions the total strain components referred to rotating axes are $\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$ as defined in section 2 of Chapter 1. In a material with flow properties the stress depends on the time derivatives $\dot{\epsilon}_{11}, \dot{\epsilon}_{22}, \dot{\epsilon}_{12}$, and the incremental stresses s_{11}, s_{22}, s_{12} in the same rotating frame of reference depend on the increments $\Delta \dot{\epsilon}_{11}, \Delta \dot{\epsilon}_{22}$, and $\Delta \dot{\epsilon}_{12}$.

The medium may be an anisotropic fluid. The procedure is also applicable to a material with non-linear flow properties. For such a material a plot of the stress σ as a function of the strain rate $\dot{\epsilon}$ is not a straight line, as shown in Figure 5.3a. The incremental stresses are governed by a *differential viscosity*

coefficient proportional to the local slope $d\sigma/d\dot{\epsilon}$ of the flow curve. This differential viscosity may be considerably smaller than the average viscosity for the over-all deformation as represented by the slope of OA (Fig. 5.3a).

In such cases of anisotropy and non-linearity the *correspondence principle* will be applicable as an approximation provided the incremental stresses s_{ij} are expressed by means of the differential viscosity coefficients. The assumption (5.34), which implies that the instability is significant, is also required.

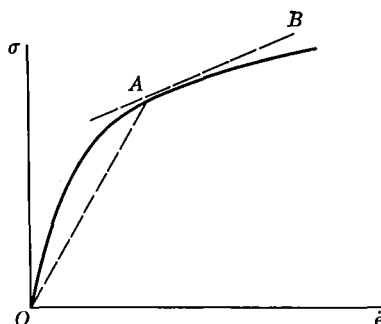


Figure 5.3a Stress σ as a function of the strain rate $\dot{\epsilon}$ for non-linear flow properties. The slope of the tangent AB defines a "differential viscosity" coefficient.

Attention should be called to a fundamental kinematic property of the strain rate. According to equations 2.28 of Chapter 1 we may write the shear strain referred to rotating axes as

$$\epsilon_{12} = e_{xy} + \frac{1}{2}(e_{yy} - e_{xx})\omega \quad (5.60a)$$

The quantities e_{xx} , e_{yy} , and e_{xy} are the strain components referred to the fixed axes, and ω is the rotation of the material. To the first order, $\epsilon_{12} \cong e_{xy}$. This relation does not hold for the time derivatives, however. Let us write

$$\begin{aligned} e_{xx} &= E_{xx} + \Delta e_{xx} \\ e_{yy} &= E_{yy} + \Delta e_{yy} \end{aligned} \quad (5.60b)$$

where the strains E_{xx} and E_{yy} are due to the initial stress alone. The time derivative \dot{E}_{xx} and \dot{E}_{yy} of these quantities is not small but is of the order of the initial stress. Therefore, if only first order terms are retained, the time derivative of equation 5.60a is

$$\dot{\epsilon}_{12} = \dot{e}_{xy} + \frac{1}{2}(\dot{E}_{yy} - \dot{E}_{xx})\omega \quad (5.60c)$$

Hence $\dot{\epsilon}_{12} \neq \dot{e}_{xy}$. This corresponds to the paradox discussed in connection with equation 5.52, in which the same term containing ω appears. The difficulty is resolved in the same way by integrating equation 5.60c with respect to the time and introducing the assumption that the total deformation is small. Hence $\epsilon_{12} \cong e_{xy}$. By considering a perturbation ϵ_{12} proportional

to $\exp(pt)$, we may write $\epsilon_{12} = p\epsilon_{12} \cong pe_{xy}$. With this approximation the correspondence principle is applicable.

Plastic Buckling. The theory is obviously not restricted to cases in which the actual initial state is one of uniform stress. It is applicable also to problems where the incremental deformation is at all times one of small deviation from a uniform, but variable, finite field. Such a case is represented, for example, in the problem of plastic buckling of a column where an eccentric load is applied gradually. The problem has been treated in a fundamental paper by Shanley.†

Incremental Stresses in a Viscous Fluid Undergoing Large Deformations. The foregoing analysis of incremental stresses in the vicinity of steady flow has been restricted to the case in which the total strain remains small. We shall now consider the case where a small perturbation is superposed on a large deformation.

The problem can be treated very simply for an incompressible viscous fluid of Newtonian viscosity in slow motion. The simplicity in this case is due to the linearity of the Navier-Stokes equations (5.57). The method has been developed in a recent paper by the author.‡

Let us consider a sinusoidal surface attached to the fluid particles and illustrated in Figure 5.1 for steady flow under a compressive stress P . We isolate that portion of the fluid lying below the surface as shown in Figure 5.4. The normal and tangential components of the stress acting on the surface at point A are

$$\begin{aligned}\sigma_{12} &= P \frac{\partial u_y}{\partial x} \\ \sigma_{22} &= 0\end{aligned}\tag{5.61}$$

The ordinate of the surface is u_y , and the slope angle $\alpha = \partial u_y / \partial x$ is assumed to be small.

On this steady flow we now superpose a velocity field of components v_x and v_y . Because of this perturbation an additional stress

† F. R. Shanley, Inelastic Column Theory, *Journal of the Aeronautical Sciences*, Vol. 14, No. 5, pp. 261–268, 1947. A later paper by the same author discusses tensile instability (*Aerospace Engineering*, Vol. 20, No. 12, pp. 30–31, 51–56, 58–61, 1961).

‡ M. A. Biot, Theory of Viscous Buckling of Multilayered Fluids Undergoing Finite Strain, *The Physics of Fluids*, Vol. 7, No. 6, pp. 855–859, 1964.

field is generated in the fluid. The normal and tangential stress components at the boundary are now

$$\sigma_{12} = \bar{\sigma}_{12} + P \frac{\partial u_y}{\partial x} \quad (5.62)$$

$$\sigma_{22} = \bar{\sigma}_{22}$$

Because of the linearity of equations 5.1 and 5.57, the additional stresses $\bar{\sigma}_{12}$ and $\bar{\sigma}_{22}$ are *the same as in a fluid initially at rest*, and with the same velocity field as the perturbation. The wavelength of the

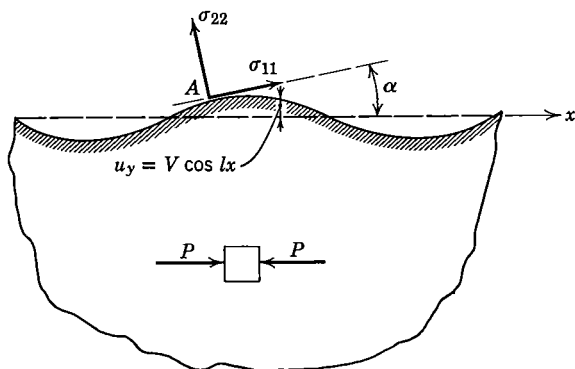


Figure 5.4 Boundary stresses in a viscous fluid with initial stress.

surface varies with time because of the initial steady flow. The x axis in Figure 5.4 represents a plane attached to the fluid particle in the unperturbed steady flow. With respect to this moving plane, the ordinate of a fluid particle lying on the sinusoidal surface may be written

$$u_y = V \cos lx \quad (5.63)$$

Corresponding sinusoidal distributions for the stresses are

$$\begin{aligned} \bar{\sigma}_{12} &= \bar{\tau} \sin lx & \sigma_{12} &= \tau \sin lx \\ \bar{\sigma}_{22} &= \bar{q} \cos lx & \sigma_{22} &= q \cos lx \end{aligned} \quad (5.64)$$

With the values (5.63) and (5.64), equations 5.62 become

$$\begin{aligned} \tau &= \bar{\tau} - P l V \\ q &= \bar{q} \end{aligned} \quad (5.65)$$

Similarly, sinusoidal distributions of the perturbation velocities at the surface are

$$\begin{aligned} v_x &= U' \sin lx \\ v_y &= V' \cos lx \end{aligned} \tag{5.66}$$

An important kinematic relation will now be derived. It relates the perturbation velocity V' to V and its time derivative.

At any given instant the normal velocity of a fluid particle relative to the x axis is

$$\dot{u}_y = \dot{V} \cos lx \tag{5.67}$$

This is not equal to the perturbation velocity v_y , however, because \dot{u}_y does not vanish in the absence of a perturbation. A velocity still remains which is due to the initial strain rate p_0 given by equation 5.5. The value of this velocity is $p_0 u_y$. Hence the total velocity of the fluid is

$$\dot{u}_y = v_y + p_0 u_y \tag{5.68}$$

For a sinusoidal distribution we substitute in this equation the values (5.63), (5.66), and (5.67), and we obtain

$$V' = \dot{V} - p_0 V \tag{5.69}$$

This important result expresses the perturbation velocity in terms of the surface ordinate and its time derivative.

Viscous Buckling of a Fluid Plate with Large Deformations.

The results represented by equations 5.65 and 5.69 provide an exact procedure for the evaluation of stability problems and the complete time history of a perturbation in a viscous fluid undergoing steady flow with large deformations. Let us consider, for example, a fluid plate under a constant compressive stress P . After a time t its original thickness h_0 has become (Fig. 5.5)

$$h = h_0 e^{p_0 t} \tag{5.70}$$

where p_0 is the strain rate of the steady flow. The value of p_0 is given by equation 5.5. We superimpose a small sinusoidal deformation of the fluid plate of initial wavelength \mathcal{L}_0 . At time t this wavelength has been reduced to the value

$$\mathcal{L} = \mathcal{L}_0 e^{-p_0 t} \tag{5.71}$$

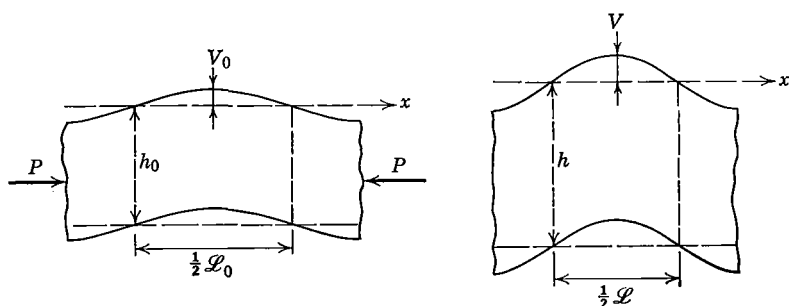


Figure 5.5 Viscous buckling of a fluid plate with large deformation.

We also write

$$l = \frac{2\pi}{\mathcal{L}} = \frac{2\pi}{\mathcal{L}_0} e^{\nu_0 t} \quad (5.72)$$

We assume that the perturbation represents a flexural deformation; hence it is antisymmetric relative to the axis of the plate.

Now we consider the relation between the stresses and the instantaneous velocity field at the time t . Since the boundaries are free of stress, we put

$$\tau = q = 0 \quad (5.73)$$

at the top surface. Hence equations 5.65 become

$$\begin{aligned} \bar{\tau} &= PlV \\ \bar{q} &= 0 \end{aligned} \quad (5.74)$$

The problem is now to find relations between the perturbation velocities U' and V' and their corresponding stresses $\bar{\tau}$ and \bar{q} . This problem has already been solved in section 5 of Chapter 4 in the context of elasticity. The solution for the case of a viscous fluid is governed by the Navier-Stokes equations (5.57). They are the same as the equations of elasticity of an incompressible isotropic medium initially free of stress, provided that we replace the velocity by the displacement and the viscosity coefficient by the shear modulus. Hence we write

$$\begin{aligned} \bar{\tau} &= l\eta(a_{11}U' + a_{12}V') \\ \bar{q} &= l\eta(a_{12}U' + a_{22}V') \end{aligned} \quad (5.75)$$

The viscosity coefficient of the fluid is η . The coefficients a_{ij} are given by equations 5.36 of Chapter 4. They are

$$\begin{aligned} a_{11} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma + 2\gamma} \\ a_{12} &= -\frac{4\gamma}{\sinh 2\gamma + 2\gamma} \\ a_{22} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma + 2\gamma} \end{aligned} \quad (5.76)$$

with

$$\gamma = \frac{1}{2}lh = \frac{\pi h_0}{\mathcal{L}_0} e^{2p_0 t} \quad (5.77)$$

Equating the values (5.74) and (5.75) for the stresses and using equation 5.5, we find

$$\begin{aligned} a_{11}U' + a_{12}V' &= 4p_0V \\ a_{12}U' + a_{22}V' &= 0 \end{aligned} \quad (5.78)$$

Elimination of U' and substitution of the value (5.69) for V' yields the differential equation

$$\frac{\dot{V}}{V} = \frac{4\gamma}{\sinh 2\gamma - 2\gamma} p_0 + p_0 \quad (5.79)$$

Integration by quadrature is immediate. Denoting by V_0 the value of V at $t = 0$, we find

$$\log \frac{V}{V_0} = \int_0^t \frac{4\gamma p_0 dt}{\sinh 2\gamma - 2\gamma} + p_0 t \quad (5.80)$$

If we put

$$\log A(t) = \int_0^t \frac{4\gamma p_0 dt}{\sinh 2\gamma - 2\gamma} \quad (5.81)$$

equation 5.80 becomes

$$\frac{V}{V_0} = A(t)e^{p_0 t} \quad (5.82)$$

The amplification ratio is thus the product of a factor $e^{p_0 t}$, which represents the purely kinematic instability, and a factor $A(t)$, which corresponds to a true mechanical instability. The integral for the

value (5.81) of $\log A(t)$ may be simplified by taking into account equation 5.77. We derive

$$d\gamma = 2\gamma p_0 dt \quad (5.83)$$

Hence

$$\log A(t) = \int_{\kappa}^{\gamma} \frac{2 d\gamma}{\sinh 2\gamma - 2\gamma} \quad (5.84)$$

where $\kappa = \pi h_0 / \mathcal{L}_0$ represents the value of γ at $t = 0$.

Extension to Unsteady and Triaxial Initial Flow. In the foregoing analysis the initial flow field was assumed to be a two-dimensional constant velocity field. The results may be readily extended to the case where the initial stress P is time-dependent. We write

$$p_0(t) = \frac{P(t)}{4\eta} \quad (5.84a)$$

and equations 5.8 are replaced by

$$\begin{aligned} X &= x \exp\left(-\int_0^t p_0 dt\right) \\ Y &= y \exp\left(\int_0^t p_0 dt\right) \end{aligned} \quad (5.84b)$$

The value of γ becomes

$$\gamma = \frac{\pi h_0}{\mathcal{L}_0} \exp\left(2 \int_0^t p_0 dt\right) \quad (5.84c)$$

The differential equation 5.79 remains valid, and its integration is carried out exactly like that for steady flow.

When the initial flow is triaxial, the product XY is not necessarily constant. We write

$$\begin{aligned} X &= x \exp\left(-\int_0^t p'_0 dt\right) \\ Y &= y \exp\left(\int_0^t p_0 dt\right) \end{aligned} \quad (5.84d)$$

For example, if the initial flow is symmetric with respect to the x axis, incompressibility requires that the product XY^2 be constant. Hence in this case $2p'_0 = p_0$, and we must replace $2p_0$ by $p_0 + p'_0$ in the value (5.84c) of γ . Relation (5.84a) between p_0 and P must also be modified accordingly. Except for changes in the numerical factors, the equations for viscous buckling remain formally the same as for plane initial flow.

The criterion (5.35) for the validity of the correspondence principle is obviously applicable to triaxial initial flow, because it expresses the physical fact that the true mechanical instability overshadows the kinematic instability.

6. INTERNAL INSTABILITY IN ANISOTROPIC VISCOELASTICITY

In section 3 of Chapter 4 the phenomenon of internal instability was analyzed for a purely elastic incompressible medium. Because of the general correspondence between elastic and viscoelastic deformations, similar properties must appear for a viscoelastic medium under initial stress.

In the discussion which follows we recognize several distinct aspects of the problem which depend on the physical nature of the material.

We initiate the analysis by considering first a viscoelastic material of infinite extent and at rest in the initial state of stress. The initial stress components are principal stresses S_{11} and S_{22} along the x and y directions. These components correspond to an effective compressive stress

$$P = S_{22} - S_{11} \quad (6.1)$$

acting along x . The medium is assumed to be incompressible and orthotropic with axes of symmetry parallel to the coordinate axes.

The viscoelastic properties for deformations in the x, y plane are characterized by the two operators (4.47). In order to facilitate the discussion, we shall first consider a material defined by the operators

$$\begin{aligned} \hat{L} &= \frac{p}{p+r} L_r \\ \hat{M} &= M + M'p \end{aligned} \quad (6.2)$$

These equations are a particular application of expressions (4.48) for a material obeying the principles of linear thermodynamics. As we shall see, the conclusions derived for this particular case are applicable to more general viscoelastic properties.

The operator \hat{L} represents a viscoelastic model constituted of a spring and dashpot in series, usually called a Maxwell model (Fig. 3.2). The operator \hat{M} represents a spring and dashpot in parallel, sometimes called a Kelvin model.

Comparison with equations 3.45 shows that the operators (6.2) may also represent within suitable limitations a thinly laminated material of elastic and viscous layers. The limitations of this approximation in problems of internal instability are discussed in the latter part of this section.

In order to discuss the instability, we go back to equation 3.15 of Chapter 4, which represents the condition of internal instability in an elastic medium. Applying the correspondence principle, we replace the elastic coefficients L and M by the operators (6.2). We obtain

$$\hat{L}\xi^4 + 2(2\hat{M} - \hat{L})\xi^2 + \hat{L} - P = 0 \quad (6.3)$$

As already stated, we assume a medium of infinite extent. The variable ξ represents the slope of the characteristic lines. The displacement field of the medium is given by

$$u = -\frac{\partial\phi}{\partial y} \quad v = \frac{\partial\phi}{\partial x} \quad (6.4)$$

and

$$\phi = \varphi(x - \xi y) \quad (6.5)$$

is an arbitrary function of the argument $(x - \xi y)$.

In order to discuss the significance of these equations, we write the condition (6.3) in the form

$$\frac{1 - \frac{4\hat{M}}{P}\xi^2}{(1 - \xi^2)^2} = \frac{\hat{L}}{P} \quad (6.6)$$

This may be considered as a relation between the variable defining the slope of the characteristic direction and the algebraic quantity p . As shown in sections 3 and 4, this value of p yields a time-dependent solution

$$\phi e^{pt} = \varphi(x - \xi y) e^{pt} \quad (6.7)$$

proportional to the exponential factor $\exp(pt)$. Hence this is an unstable solution whose degree of instability depends on the slope ξ of the characteristic direction.

Let us determine the functional dependence of p on ξ by examining the properties of equation 6.6. For a given value of ξ the roots p of equation 6.6 are obtained by plotting both sides of the equation as functions of p . The right side of the equation is represented by the hyperbola (a) of Figure 6.1. The left side is represented by the straight line (b) with a negative slope. These curves always intersect at two points, A and B , which represent the only roots because equation 6.6 is of the second degree in p . Hence the roots are always real. This conclusion is in agreement with the general theorems of

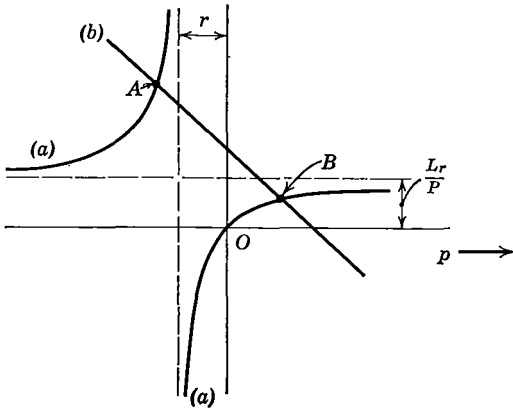


Figure 6.1 Graphical construction for the roots p of equation 6.6.

section 4, since the contour integral (4.37) also vanishes in the present case. Examination of Figure 6.1 shows that there is never more than one positive root.

The limiting case for which the positive root becomes negative is obtained by putting $p = 0$ in equation 6.6. The equation becomes

$$1 - \frac{4M}{P} \xi^2 = 0 \tag{6.8}$$

and yields two values of the slope

$$\xi = \pm \frac{1}{2} \sqrt{\frac{P}{M}} \tag{6.9}$$

This result is interpreted as follows. The two values (6.9) of ξ determine two characteristic lines whose angle of inclination with the direction normal to the compression is given by

$$\theta = \tan^{-1} \left[\frac{1}{2} \sqrt{\frac{P}{M}} \right] \tag{6.10}$$

This is shown in Figure 6.2. For characteristic directions lying within the sector of angle 2θ the solution increases exponentially as the factor $\exp(p t)$.

We now examine the case $\xi = 0$. This value of ξ defines the characteristic direction normal to the compression. The corresponding value of p is determined by putting $\xi = 0$ in equation 6.3.

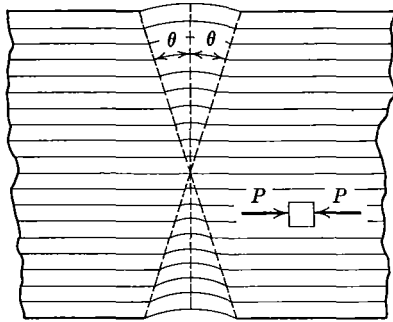


Figure 6.2 Spreading of a disturbance due to internal instability of a viscoelastic medium in compression.

The equation becomes

$$P = \hat{L} \quad (6.11)$$

Substituting the value (6.2) of the operator in equation 6.11 and solving it for p , we find

$$p = \frac{rP}{L_r - P} \quad (6.12)$$

This value of p becomes infinite for $P = L_r$. For $P > L_r$ the solution loses its physical meaning because the material will instantaneously collapse elastically in the direction $\xi = 0$.

We may inquire whether this normal direction is also the one for which p is maximum. Theoretically this is not necessarily the case, and it is possible for p to be maximum for two directions which are oriented symmetrically with respect to the normal to the compression. However, this will happen only in exceptional cases, and then the left side of equation 6.6 must be an increasing function of ξ in the vicinity of $\xi = 0$. This will occur if

$$\frac{4\hat{M}}{P} < 2 \quad (6.13)$$

or

$$2M + 2M'p < P \quad (6.14)$$

In general, this condition will not be fulfilled because the value of P will be smaller than that of the elastic modulus M . Hence under usual conditions the value of p will be maximum for a characteristic direction normal to the compression.

The physical behavior of the medium is illustrated by considering a localized disturbance. The disturbance is unstable and will spread entirely within the sector of angle 2θ as shown in Figure 6.2. Under usual conditions the maximum rate of growth of the disturbance will occur in the direction normal to the compression.

The conclusions for the viscoelastic medium considered in the foregoing analysis remain valid for the general case of viscoelasticity if the operators \hat{L} and \hat{M} obey thermodynamic principles. Then these operators are increasing functions of p , and therefore it can be shown that the qualitative properties derived from equation 6.6 remain unaltered.

Internal Instability of a Viscous Anisotropic Medium. We consider now a medium of infinite extent in a state of steady flow under initial stress. We have shown in section 5 that the general theory is applicable to this medium as an approximation, provided the amplification of a disturbance is sufficiently large when compared with the simultaneous deformation due to the initial steady state of flow. This condition is expressed by the criterion (5.35).

We assume again an initial stress represented by the effective compression (6.1). The viscous medium is incompressible with orthotropic properties along directions parallel and normal to the initial stress. The incremental stresses in this case are derived by introducing the operators

$$\begin{aligned}\hat{L} &= L'p \\ \hat{M} &= M'p\end{aligned}\tag{6.15}$$

These operators represent purely viscous behavior of the incremental deformations. The instability properties are derived by substituting the operators (6.15) into equation 6.3. We obtain

$$\frac{P}{L'p} = 1 + 2\left(\frac{2M'}{L'} - 1\right)\xi^2 + \xi^4\tag{6.16}$$

We assume that the medium exhibits preferential resistance to flow in the direction of the initial compression. Hence

$$\frac{2M'}{L'} > 1\tag{6.17}$$

in analogy with the condition for *internal instability of the first kind*

of the elastic medium analyzed in Chapter 4. The maximum value of p occurs for $\xi = 0$. This maximum value is

$$p = \frac{P}{L'} \quad (6.18)$$

Therefore the characteristic direction ($\xi = 0$) normal to the compression corresponds to the highest instability.

In order to evaluate the magnitude of this instability, we turn our attention to the steady state flow. We assume that the flow under the initial compression P is governed approximately by the same viscosity coefficient M' as for the incremental deformation. The initial flow for plane-strain deformations is represented by a shortening proportional to the exponential function of time, $\exp(-p_0 t)$. The value of p_0 is given by equation 5.5, where the viscosity η is replaced by M' . Hence

$$p_0 = \frac{P}{4M'} \quad (6.19)$$

From equations 6.18 and 6.19 we derive

$$\frac{pt}{p_0 t} = \frac{4M'}{L'} \quad (6.20)$$

For a shortening of about 10% we put $p_0 t = 0.1$. During the time t a perturbation is amplified by a factor $\exp(pt)$. Significant instability will appear if this factor is about 50. A value $pt \cong 4$ is required. Hence from equation 6.20 we derive

$$\frac{M'}{L'} = 10 \quad (6.21)$$

We conclude that internal instability in a viscous medium will be significant only for a material with sufficiently large anisotropy. Note that in this case $p_0/p = 1/40$ and the criterion (5.35) for the validity of the approximate theory is fulfilled.

Filtering Properties Due to Confinement. When the anisotropic viscous medium is confined between two rigid boundaries (see Figure 3.3 of Chapter 4), the variable ξ acquires a different meaning. Then ξ becomes proportional to the wavelength of the unstable deformations (see equation 3.26 of Chapter 4). Equation 6.16 then shows that, for ξ very small, the value of p is almost constant. Hence, in the range of small wavelengths, the unstable disturbances

are evenly amplified. For large wavelengths, p tends to zero and the amplification vanishes. This corresponds to a filtering-out of the large wavelength disturbances.† This property may be approximated by introducing a “cut-off” wavelength to represent the filtering.

Relation of Internal Instability to the Occurrence of Slip Lines in Plasticity. The preceding analysis may be extended to materials for which the viscous properties are restricted to incremental deformations. The stress-strain relations for purely viscous incremental stresses are obtained by introducing the operators (6.15). The material may be strongly non-linear and may exhibit only a small amount of flow until a certain initial value is reached for the compressive stress P . Beyond this value the rate of flow of the material may increase very rapidly. Hence in the vicinity of this yield point we may define a differential viscosity as already discussed in section 5 and illustrated in Figure 5.3a.

In a material which is intrinsically isotropic and exhibits a plastic yield point, a strong incremental anisotropy will generally be induced. An estimate of this effect may be obtained from the analysis leading to equations 7.15 and 7.16 of Chapter 2. It was shown that the superposition of a small shear on the initial finite extension is equivalent to a rotation of the strain axes without change of magnitude. Hence the value of M' will generally be much smaller than that of L' , and we may assume

$$\frac{2M'}{L'} < 1 \quad (6.22)$$

This is analogous to the condition for *internal instability of the second kind* which was analyzed in Chapter 4 for an elastic medium. Equation 6.16 shows that the directions for maximum strain rate lie at an angle with the axis of the initial stress. They correspond to slip lines. Note that the value of P incorporates the combined effect of stresses S_{11} and S_{22} in two perpendicular directions, and P represents either a tension or a compression, as may be seen by carrying the discussion in the context of equation 6.19c of Chapter 4 using coefficients N and Q .

† See M. A. Biot, Internal Instability of Anisotropic Viscous and Viscoelastic Media under Initial Stress (*Journal of the Franklin Institute*, Vol. 279, in press).

The analogous problem for an elastic medium was discussed in the latter part of section 3 of Chapter 4. The properties of the slip lines were brought out in the context of the elastic problem (see Fig. 3.6 of Chapter 4). A specific finite stress-strain relation with an "elastic yielding" was also discussed as an illustration (see Fig. 3.7 of Chapter 4).

Internal Instability of a Laminated Viscoelastic Medium.

The properties of the laminated medium may be expressed by the operators derived in Section 4 for a continuous anisotropic medium which approximates the behavior of the composite material. For example, for a medium composed of elastic and viscous layers, the operators \hat{L} and \hat{M} are given by equations 3.45. They are identical with expressions (6.2) for the continuous anisotropic medium. For laminations of purely viscous materials, the operators are given by expressions (3.48), which are the same as equations 6.15. Hence, within the limits of validity of this approximation, the laminated medium will exhibit the same type of internal instability as found in the preceding analysis for continuous anisotropic media with viscoelastic or purely viscous properties.

If the medium is confined between rigid boundaries, however, the analysis of the analogous case for an elastic material in section 3 of Chapter 4 shows that new features will appear. In the discussion of that case it was shown that the medium tends to buckle in the range of the shortest possible wavelengths. In that range the instability is governed by additional factors, such as the layer thickness, which are not included in the approximate continuous model, and corrections must be introduced. However, the simplified continuous model retains its usefulness by providing a theoretical foundation which brings out some of the essential features.

An exact treatment of the problem of internal instability of laminated media may, of course, be obtained from the general theory of stability of multilayered viscous and viscoelastic media developed in section 8 of this chapter. A very simple approximate solution of this problem which takes into account the thickness of the layers has also been developed by the author in the context of geology.†

† M. A. Biot, Theory of Internal Buckling of a Confined Multilayered Structure, *Geological Society of America Bulletin*, Vol. 75, No. 6, pp. 563-568, 1964.

7. SURFACE INSTABILITY OF VISCOELASTIC MEDIA

The stability of the elastic half-space was analyzed in previous chapters. In Chapter 3 the material was assumed to exhibit incremental isotropy, and the problem was solved for the homogeneous and non-homogeneous medium. In Chapter 4 the problem was treated for the case of anisotropy. In both discussions the medium was assumed to be incompressible.

Analytical solutions for surface instability for a compressible elastic medium with or without anisotropy were developed in section 7 of Chapter 5.

These results may be immediately extended to viscoelasticity. We shall consider only an incompressible material with incremental isotropy.

Viscoelastic Instability of a Non-homogeneous Half-Space Initially at Rest. We shall discuss first a non-homogeneous viscoelastic medium lying below a horizontal surface. The medium is incompressible, and the initial stress is due to a horizontal compression superposed on the action of gravity.

Let us assume that the medium is at rest in the state of initial stress. Application of the principle of viscoelastic correspondence to this medium yields an exact solution of the stability problem. The corresponding problem for the elastic medium was treated in section 8 of Chapter 3.† For the viscoelastic medium the incremental stresses are assumed to obey the equations

$$\begin{aligned} s_{11} - s &= 2\hat{Q}e_{xx} \\ s_{22} - s &= 2\hat{Q}e_{yy} \\ s_{12} &= 2\hat{Q}e_{xy} \end{aligned} \quad (7.1)$$

The operator \hat{Q} is written

$$\hat{Q} = e^{-av}\hat{Q}_0 \quad (7.2)$$

with

$$\hat{Q}_0 = \int_0^\infty \frac{p}{p+r} Q(r) dr + Q + pQ' \quad (7.3)$$

† For a more extensive discussion of this problem in the context of viscoelasticity see M. A. Biot, Instability of a Continuously Inhomogeneous Viscoelastic Half-Space under Initial Stress, *Journal of the Franklin Institute*, Vol. 270, No. 3, pp. 190-201, 1960.

The thermodynamic properties of this expression will be examined below. The same coordinate system is chosen as that shown in Figure 8.1 of Chapter 3, the plane $y = 0$ is horizontal and located at the free surface, and the y axis is directed positively downward.

As for the elastic medium, it is also assumed that the incompressible medium is of uniform density ρ and under the action of a constant gravity field of acceleration g .

The operator (7.2) represents a medium whose rigidity decreases exponentially with depth. It gradually changes into a perfect fluid as the depth increases.

The medium is in a state of initial stress represented by a hydrostatic field due to gravity and an additional horizontal compression P decreasing exponentially with depth with the same exponential factor as the operator (7.2):

$$P = P_0 e^{-ay} \quad (7.4)$$

An important property of the operator (7.2) is indicated by the fact that the factor \hat{Q}_0 which incorporates the operational features is independent of y . Because of this property the solution for the elastic medium may be immediately extended to the viscoelastic medium. In order to transpose the elastic solution to the viscoelastic problem we must replace the elastic coefficient μ by the operator \hat{Q} . Referring to equations 8.20 of Chapter 3, we conclude that in the formal solution for the elastic medium we must replace the value of ζ by

$$\zeta = \frac{P}{2\hat{Q}} = \frac{P_0}{2\hat{Q}_0} \quad (7.5)$$

This quantity is a function of p through expression (7.3) for \hat{Q}_0 .

By the general theorems of section 4 the characteristic values p are all real in the present stability problem. As a consequence the value of ζ is always real, and the numerical solution for the elastic medium is applicable. This solution is plotted in Figure 8.2 of Chapter 3 as the value of ζ versus the dimensionless variable

$$\delta = \frac{l}{a} \quad (7.6)$$

The solution corresponds to unstable modes, sinusoidal along x and of wavelength

$$\mathcal{L} = \frac{2\pi}{l} \quad (7.7)$$

Hence

$$\delta = \frac{2\pi}{\mathcal{L}a} \quad (7.8)$$

is a non-dimensional parameter proportional to the inverse of the wavelength.

There is a family of curves depending on the parameter

$$G = \frac{\rho g}{P_0 a} \quad (7.9)$$

For a given value of G the plot of ζ versus δ yields the value of p as a function of the wavelength. Each mode is associated with a value of p such that its amplitude grows proportionally to $\exp(pt)$. The wavelength for which p is maximum is the most unstable. We shall call it the dominant wavelength \mathcal{L}_a .

At this point a property of the operator \hat{Q} is of particular significance. According to equations 4.47 we may write

$$\hat{L} = \hat{Q} + \frac{1}{2}P \quad (7.10)$$

where the operator \hat{L} is expressed by the first of equations 4.48. Thermodynamic principles require that $L(r)$ and L' be non-negative. Hence equation 7.10 leads to the conclusion that $Q(r)$ and Q' are also non-negative. As a consequence, \hat{Q} is an increasing function of p and the minimum value of ζ corresponds to the maximum value of p . The dominant wavelength is therefore the same as the buckling wavelength in the elastic medium.

This dominant wavelength is

$$\mathcal{L}_a = \frac{2\pi}{a\delta_a} \quad (7.11)$$

where δ_a is the value of δ corresponding to the minimum of ζ . An empirical expression for δ_a is given by equation 8.30 of Chapter 3. The value of p for the dominant wavelength is obtained by writing the minimum value of ζ as

$$\zeta_{\min} = \frac{P_0}{2\hat{Q}_0} \quad (7.12)$$

and solving this equation for p .

In order to bring out some of the characteristic features of the instability let us assume that

$$Q > 0 \quad (7.13)$$

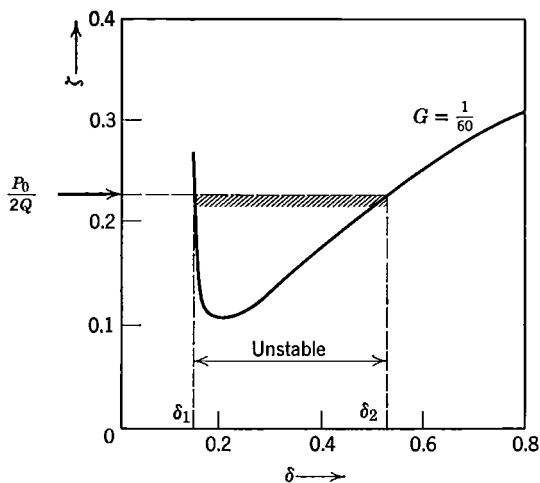


Figure 7.1 Unstable range of wavelengths for a non-homogeneous viscoelastic half-space under the simultaneous action of gravity and a horizontal compression.

For physical reasons this is a natural assumption for a material which is at rest in the state of initial stress. An unstable solution is possible only if

$$P_0 > 2Q\zeta_{\min} \quad (7.14)$$

Hence, for instability to appear, the compressive stress at the surface must lie above a critical value $2Q\zeta_{\min}$.

The case $G = 1/60$ is illustrated in Figure 7.1. If it is assumed that the inequality (7.14) is satisfied, instability will appear for a limited range of wavelengths corresponding to values of δ between the limits δ_1 and δ_2 . The limiting points are determined by the intersection of the curve with the line of ordinate

$$\zeta = \frac{P_0}{2Q} \quad (7.15)$$

At these limiting points, p is zero and the instability vanishes. The existence of a common horizontal asymptote for the value $\zeta = 0.839$ has been derived for the elastic medium (section 8, Chapter 3). Hence, when $P_0/2Q$ approaches this value, the limiting point corresponding to δ_2 tends to infinity. In the treatment of the elastic

medium it was also pointed out that there is a vertical asymptote for the value of ζ . The abscissa δ of this asymptote depends on the parameter G and corresponds to a higher limit for the wavelength beyond which instability vanishes. This is due to the influence of gravity, which is stabilizing and becomes predominant at large wavelengths.

Instability of a Non-homogeneous Viscous Half-Space. We consider now the stability problem of a non-homogeneous half-space for a medium which is not at rest under the initial stress. As an illustration, we shall discuss a viscous incompressible material of constant density and Newtonian viscosity. The initial state of the medium is one of steady flow under a horizontal compression superposed on the action of gravity.

It was shown in section 5 that the results for the elastic medium are applicable to a viscous medium as an approximation, provided that the instability is of significant magnitude. The validity criterion is expressed by the inequality (5.35).

The viscosity of the medium is assumed to be distributed exponentially as in equation 7.2. We write for the viscosity

$$\eta = \eta_0 e^{-ay} \quad (7.16)$$

and the operator \hat{Q}_0 becomes

$$\hat{Q}_0 = \eta_0 p \quad (7.17)$$

The value of ζ is

$$\zeta = \frac{P_0}{2\eta_0 p} \quad (7.18)$$

The problem is formally identical with the preceding one. The dominant wavelength retains the value (7.11), the same as for the elastic and viscoelastic media, and depends only on G . The value of p for this dominant wavelength is obtained from equation 7.18 by substituting the minimum value of ζ on the left side. In order to verify the validity of the approximation implied in this solution, let us assume that the initial state of flow is one of plane strain. According to equations 5.8 the finite extension ratio in the x direction under a compression P is $\exp(-p_0 t)$, where

$$p_0 = \frac{P}{4\eta} = \frac{P_0}{4\eta_0} \quad (7.19)$$

Hence expression (7.18) for ζ becomes

$$\zeta = \frac{2p_0}{p} \quad (7.20)$$

According to the condition (5.35) the ratio p_0/p must be smaller than about $\frac{1}{20}$. Hence the theory is valid provided that

$$\zeta < \frac{1}{10} \quad (7.21)$$

Attention is called to the magnitude of the amplification in this case. A value $p_0 t = \frac{1}{10}$ corresponds to a shortening of about 10%. For $p_0/p = \frac{1}{20}$ the amplification is $\exp(pt) = 7.38$. Below this value the amplification becomes insignificant. Hence regions of the solutions where ζ is larger than about $\frac{1}{10}$ lose meaning and validity for a purely viscous medium.

For significant instability the minimum value of ζ must be smaller than about $\frac{1}{10}$. Figure 7.1 shows that this implies

$$G < \frac{1}{60} \quad (7.22)$$

With reference to the definition (7.9) of G , the condition (7.22) means that the compressive stress P_0 must be sufficiently high or the thickness $1/a$ of the region of high viscosity must be sufficiently small.

Surface Instability of a Homogeneous Viscoelastic Medium Initially at Rest. We consider now the problem of surface instability of a viscoelastic homogeneous half-space without taking into account any gravity forces. The medium is incompressible and at rest under the initial stress. This initial stress is a compression P parallel to the surface.

The corresponding solution was derived in Chapter 3 (section 6) for the elastic half-space. Those results are directly applicable to the present case, and they provide an exact solution for the viscoelastic half-space. The elastic modulus appearing in the elastic solution must be replaced by the operator

$$\hat{Q} = \int_0^\infty \frac{p}{p+r} Q(r) dr + Q + pQ' \quad (7.23)$$

The value of ζ (equation 6.14 of Chapter 3) becomes

$$\zeta = \frac{P}{2\hat{Q}} \quad (7.24)$$

The characteristic equation is (equation 6.20 of Chapter 3)

$$\zeta^3 + 2\zeta^2 - 2 = 0 \quad (7.25)$$

There are three roots, ζ_1 , ζ_2 , ζ_3 . For our purpose it is convenient to write the inverse of their numerical values:

$$\begin{aligned} \frac{1}{\zeta_1} &= 1.192 \\ \frac{1}{\zeta_2} &= -0.596 + 0.255i \\ \frac{1}{\zeta_3} &= -0.596 - 0.255i \end{aligned} \quad (7.26)$$

The real root

$$\zeta_1 = 0.839 \quad (7.27)$$

has already been evaluated for the elastic medium.

The complex roots must be discarded because of the general theorem of section 4. According to this theorem, the characteristic values p must be real, and hence the values of ζ also. It is interesting to verify this directly by going back to the solution (6.13) of Chapter 3. It is written

$$\phi = \frac{1}{\sqrt{2}} (C_1 e^{ly} + C_2 e^{kly}) \sin lx \quad (7.28)$$

The y axis is chosen positive upward in this solution, and the medium is located in the region $y < 0$. Hence the value of k must have a *positive real part* in order that the solution vanish at $y = -\infty$. Actually, the characteristic equation (7.25) is the rationalized form of the original one. This non-rationalized form which includes explicitly the value of k is given by equation 4.45b of Chapter 4; that is,

$$k(2 + \zeta) - \zeta = 0 \quad (7.29)$$

We write it

$$k = \frac{1}{\frac{2}{\zeta} + 1} \quad (7.30)$$

Substituting the complex values (7.26) of $1/\zeta$ into this expression, we find that the corresponding values of k have a negative real part.

Hence the complex roots do not satisfy the condition that the displacement vanish at infinite depth and do not correspond to a solution of the problem.

In the present case of the weightless homogeneous half-space there is no dominant wavelength. The same amplification factor applies to all wavelengths, and surface disturbances are amplified without change of profile.

Surface Instability for a Kelvin Material. As a more specific example we consider a medium of viscoelastic properties defined by the simple operator

$$\hat{Q} = Q + pQ' \quad (7.30a)$$

This represents a so-called Kelvin model illustrated by a spring and dashpot in parallel. The viscous damping is superposed on the elasticity.

The value of p in the amplification factor $\exp(pt)$ is determined by the equation

$$Q + pQ' = \frac{P}{2\zeta_1} = 0.596P \quad (7.30b)$$

Hence p is real, in agreement with the general theorem of section 4. For

$$0.596P > Q \quad (7.30c)$$

the value of p is positive, and the surface is unstable. Similar considerations apply to more general operators.

Surface Stability of a Viscous Fluid. Let us consider a homogeneous half-space constituted of an incompressible viscous fluid of viscosity η and subject to a compressive stress parallel to the surface. Gravity and inertia forces are neglected. Here the initial state is one of steady flow. If we attempt to apply the solution of the elastic theory to this medium, we must introduce the operator

$$\hat{Q} = p\eta \quad (7.31)$$

and according to equation 7.24 we write

$$\zeta = \frac{P}{2\eta p} \quad (7.32)$$

If we assume that the initial state of flow corresponds to a plane strain deformation, the value p_0 is given by equation 5.5. Hence

$$\zeta = \frac{2p_0}{p} \quad (7.33)$$

In this equation we replace ζ by the positive root of equation 7.25.

We must substitute for ζ the numerical value $\zeta_1 = 0.839$ given by equations (7.26). Hence

$$\frac{p_0}{p} = 0.42 \quad (7.34)$$

This value does not satisfy the criterion (5.35), and the theory is not applicable here. In fact, equation 7.34 shows that the amplification factor $\exp(pt)$ is of the same order as the kinematic amplification factor $\exp(p_0t)$. Hence the result does not correspond to an actual instability.

We may verify that true surface instability is rigorously absent for first order perturbations in a viscous fluid. This can be seen by considering the limiting case of infinite thickness ($\gamma = \infty$) in equation 5.79 for the viscous buckling of a fluid plate. When we put $\gamma = \infty$ this equation becomes

$$\frac{\dot{V}}{V} = p_0 \quad (7.35)$$

Hence

$$V = V_0 e^{p_0 t} \quad (7.36)$$

The amplification in this case is reduced to the purely kinematic phenomenon represented by equations 5.8. It is not a true instability.

Surface Wrinkling in Plastic Deformations. The theory is applicable for a material initially at rest and behaving as a viscous fluid only for small incremental stresses. The operator in this case is

$$\hat{Q} = Q'p \quad (7.37)$$

In fact, as already pointed out in section 5, the material need not be initially at rest and the operator (7.37) may govern approximately deviations from a steady strain-rate. Although it is difficult to conceive of a material behaving exactly in this fashion, it seems permissible to assume that certain plastic materials may approach this behavior in the vicinity of the yield point. For example, we have already discussed the medium for which the rate of flow increases very rapidly beyond a critical stress. This was illustrated in Figure 5.3a when the properties of the incremental deformations are expressed by a differential viscosity coefficient. If the anisotropy induced by the yielding process is small, the incremental stresses are

governed by the single operator (7.37) where Q' represents a differential viscosity coefficient. Surface instability will appear as in a Kelvin material defined by the operator (7.30a). The value of p is obtained by putting $Q = 0$ in equation 7.30b. Surface irregularities are amplified proportionally to the factor

$$\exp(pt) = \exp\left(0.596 \frac{Pt}{Q'}\right) \quad (7.38)$$

Since the over-all rate of flow due to the initial stress is assumed to be very small or non-existent, the value of the time t is not limited, as it is for a viscous fluid, and significant amplification may be attained.

If the incremental deformations are not isotropic, their properties may be approximated by the operators

$$\hat{Q} = Q'p \quad \hat{N} = N'p \quad (7.39)$$

In this case we apply the theory of surface instability for the anisotropic elastic material and use the numerical result in Table 1 of Chapter 4 (section 4). By viscoelastic correspondence the value of N/Q is replaced by

$$\frac{\hat{N}}{\hat{Q}} = \frac{N'}{Q'} \quad (7.40)$$

and the table yields the numerical value of

$$\zeta_{cr} = \frac{P}{2Q'p} \quad (7.41)$$

as a function of N'/Q' . We derive the value $p = P/(2Q'\zeta_{cr})$ which must be substituted in the amplification factor $\exp(pt)$. Hence surface wrinkling will appear. The rate of growth of the wrinkles will depend on the induced anisotropy as measured by the ratio N'/Q' .

8. FOLDING INSTABILITY OF LAYERED MEDIA

The solutions obtained in Chapter 4 for the elastic stability of the embedded layer and multilayered media are also applicable to viscoelastic media. In these media the instability is manifested by a gradual folding of the layers:

Embedded Layer. We shall consider a single viscoelastic layer of thickness h under a uniform initial compression P (Fig. 8.1). It

is embedded in an infinite viscoelastic medium free of initial stress. Both materials are incompressible. We shall assume first that the medium is initially at rest and in equilibrium under the initial stress. For simplicity we shall consider first the case where perfect slip takes place between the layer and the medium. It was shown† that for the single layer, if the embedding medium is soft, there is no significant difference between adherence and slip.

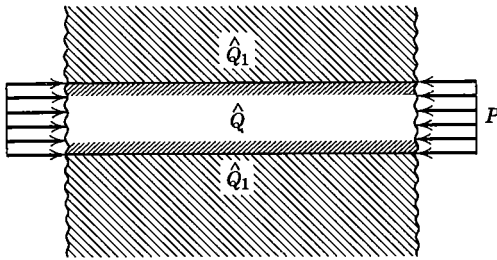


Figure 8.1 Viscoelastic layer under initial compression P embedded in a viscoelastic medium initially stress-free.

The corresponding elastic case is represented by the characteristic equation (6.30) of Chapter 4. This equation is

$$\frac{Q_{\text{eff}}}{Q} = \frac{1}{2} (1 + \zeta) \frac{(\beta_2^2 + 1)^2 z_1 - (\beta_1^2 + 1)^2 z_2}{\beta_1^2 - \beta_2^2} \quad (8.1)$$

where

$$\begin{aligned} z_1 &= \beta_1 \tanh \beta_1 \gamma \\ z_2 &= \beta_2 \tanh \beta_2 \gamma \end{aligned} \quad (8.2)$$

The parameter $\gamma = \frac{1}{2}lh$ is related to the wavelength $\mathcal{L} = 2\pi/l$ of the deformation of the layer. We shall first write this equation for the case where the layer and the embedding material are both isotropic for incremental stresses. The elastic coefficient of the embedding medium is

$$Q_1 = Q_{\text{eff}} \quad (8.3)$$

† M. A. Biot, On the Instability and Folding Deformation of a Layered Viscoelastic Medium in Compression, *Journal of Applied Mechanics*, Ser. E, Vol. 26, pp. 393-400, 1959; M. A. Biot and H. Odé, On the Folding of a Viscoelastic Medium with Adhering Layer under Compressive Initial Stress, *Quarterly of Applied Mathematics*, Vol. XIX, No. 4, pp. 351-355, 1962.

and the roots β_1 and β_2 become

$$\begin{aligned}\beta_1 &= 1 \\ \beta_2 &= k = \sqrt{\frac{1-\zeta}{1+\zeta}}\end{aligned}\quad (8.4)$$

For a viscoelastic material we must replace the elastic coefficients by the corresponding operators. The layer is represented by the operator \hat{Q} , and the embedding medium by the operator \hat{Q}_1 . Hence

$$\zeta = \frac{P}{2\hat{Q}}\quad (8.5)$$

The characteristic equation for the viscoelastic medium becomes

$$\frac{\hat{Q}_1}{\hat{Q}} = \frac{1}{\zeta} [\tanh \gamma - (1 + \zeta)^2 k \tanh k\gamma]\quad (8.6)$$

This equation constitutes a relation between p and the wavelength parameter γ . We solve the equation for p as a function of γ and consider the value γ_a for which p is a maximum. This determines the *dominant wavelength*

$$\mathcal{L}_a = \frac{\pi h}{\gamma_a}\quad (8.7)$$

An initial sinusoidal disturbance of the layer of wavelength \mathcal{L}_a has a maximum rate of growth.

An approximate solution of equation 8.6 is obtained by replacing the hyperbolic functions by their power series representation. Limiting the series to its first two terms, we write

$$\begin{aligned}\tanh \gamma &= \gamma - \frac{1}{3}\gamma^3 \\ k \tanh k\gamma &= k^2\gamma - \frac{1}{3}k^4\gamma^3\end{aligned}\quad (8.8)$$

Equation 8.6 becomes (using 8.4 for k)

$$\frac{\hat{Q}_1}{\hat{Q}} = \zeta\gamma - \frac{1}{3}(2 - \zeta)\gamma^3\quad (8.9)$$

Assuming ζ to be small, we write

$$\frac{\hat{Q}_1}{\hat{Q}} = \zeta\gamma - \frac{2}{3}\gamma^3\quad (8.10)$$

In the particular case of "spectral homogeneity," that is, if the

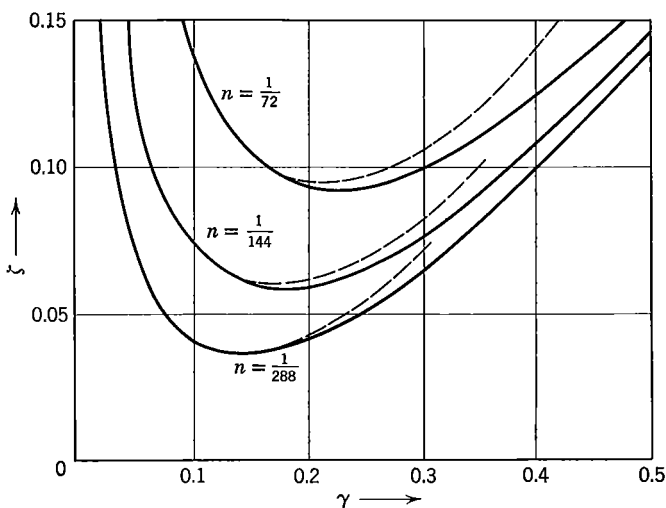


Figure 8.2 Stability parameter ζ versus the wavelength parameter γ for the embedded viscoelastic layer of Figure 8.1. Exact solutions are represented by solid lines. Broken lines correspond to the approximate solution (8.14) ($n = \hat{Q}_1/\hat{Q}$).

layer and the embedding medium have the same relaxation spectrum, we may write

$$\hat{Q}_1 = n\hat{Q} \quad (8.11)$$

where n is a numerical coefficient. Here the characteristic equation (8.6) is considerably simplified because p appears only in the parameter ζ . Equation 8.6 becomes

$$n = \frac{1}{\zeta} [\tanh \gamma - (1 + \zeta)^2 k \tanh k\gamma] \quad (8.12)$$

It has been solved numerically, and the value of ζ is plotted as a function of γ in Figure 8.2. The curves are plotted for three values of n ,

$$n = 1/72, 1/144, 1/288 \quad (8.13)$$

They are reproduced from the author's paper.† Plots for other values of n are also given in the same paper.

† M. A. Biot, Folding of a Layered Viscoelastic Medium Derived from an Exact Stability Theory of a Continuum under Initial Stress, *Quarterly of Applied Mathematics*, Vol. XVII, No. 2, pp. 185-204, 1959; see also Stability Problems of Inhomogeneous Viscoelastic Media, *Non-homogeneity in Elasticity and Plasticity* (IUTAM Symposium, Warsaw, 1958), pp. 311-321, Pergamon Press, New York, 1959.

The value of ζ obtained from the approximate equation (8.10) is

$$\zeta = \frac{n}{\gamma} + \frac{2}{3}\gamma^2 \quad (8.14)$$

This value is also plotted in Fig. 8.2 for comparison.

The dominant wavelength is readily obtained from this result. The operator is of the form

$$\hat{Q} = \int_0^\infty \frac{p}{p+r} Q(r) dr + Q + Q'p \quad (8.15)$$

It is an increasing function of p . Therefore the minimum of ζ maximizes the value of p and determines the dominant wavelength. The value ζ_{\min} of this minimum is a function of n . Instability requires

$$\zeta > \zeta_{\min} \quad (8.16)$$

Hence

$$P > 2Q\zeta_{\min} \quad (8.17)$$

For a compression satisfying this inequality there is a range of unstable wavelengths in the vicinity of the dominant wavelength. This type of behavior was discussed in section 7 for the case of the nonhomogeneous half-space and illustrated in Figure 7.1. The argument is entirely similar and will not be repeated.

Adhering Layer. For the case of perfect adherence between the layer and the embedding medium an exact solution may be derived by viscoelastic correspondence from the elastic solution derived in section 6 of Chapter 4 and plotted in Fig. 6.6 of that chapter. The solution is exactly applicable for a medium initially at rest and if we assume

$$n = \frac{P_1}{P} = \frac{Q_1}{Q} \quad (8.18)$$

where P and P_1 are, respectively, the initial compressive stresses in the layer and the embedding medium.

As already pointed out, the effect of adherence was also discussed in two papers (see references on page 415).

Thin Plate Theory as a Limiting Case. Comparison with exact solutions shows that the classical thin plate theory is quite accurate for most applications. Its relation to the exact theory is

brought out by considering the approximate equations 8.10 and 8.14. The accuracy of the latter has been illustrated in Figure 8.2.

By substituting $\zeta = P/2\hat{Q}$ and $\gamma = \frac{1}{2}lh$, equation 8.10 becomes

$$4l\hat{Q}_1 = l^2Ph - \frac{4\hat{Q}}{12}h^3l^4 \tag{8.19}$$

If the normal deflection w of the layer is proportional to $\cos lx$, equation 8.19 may be written

$$-4l\hat{Q}_1w = Ph \frac{d^2w}{dx^2} + \frac{4\hat{Q}}{12}h^3 \frac{d^4w}{dx^4} \tag{8.20}$$

The right side represents the normal load on a thin plate under an axial compressive stress P . The elastic coefficient of the plate has been replaced by the operator \hat{Q} . The left side represents the viscoelastic reaction of the embedding medium per unit length. This can be shown by applying equation 6.27 of Chapter 4 to the viscoelastic half-space and by introducing a factor 2 in order to add the reactions of both top and lower half-spaces.

Folding of an Elastic Plate in a Viscous Medium. The classical thin plate theory is applicable to an elastic material, and it includes the case of compressibility. For an elastic layer equation 8.20 is replaced by

$$-4l\hat{Q}_1w = Ph \frac{d^2w}{dx^2} + \frac{1}{12} \frac{E}{1 - \nu^2} h^3 \frac{d^4w}{dx^4} \tag{8.21}$$

Young's modulus and Poisson's ratio for the elastic plate are denoted, respectively, by E and ν . The shear modulus is

$$Q = \frac{E}{2(1 + \nu)} \tag{8.22}$$

We note that

$$\frac{E}{1 - \nu^2} = \frac{2Q}{1 - \nu} \tag{8.23}$$

For an incompressible material we put $\nu = \frac{1}{2}$, and the modulus $E/(1 - \nu^2)$ becomes equal to $4Q$. This result yields equation 8.20 for the incompressible medium if we replace Q by \hat{Q} .

We now discuss the stability of the elastic plate embedded in a viscous medium. The operator for the embedding medium of viscosity η_1 is

$$\hat{Q}_1 = \eta_1 p \tag{8.24}$$

and equation 8.21 becomes

$$-4l\eta_1pw = Ph \frac{d^2w}{dx^2} + \frac{1}{12} \frac{E}{1-\nu^2} h^3 \frac{d^4w}{dx^4} \quad (8.25)$$

Substituting a sinusoidal amplitude w proportional to $\cos lx$, we find

$$4\eta_1p = Ph - \frac{1}{12} \frac{E}{1-\nu^2} l^3 h^3 \quad (8.26)$$

The value of p goes through a maximum for a certain value l_d of l which is given by

$$l_d h = 2 \sqrt{\frac{(1-\nu^2)P}{E}} \quad (8.27)$$

This value corresponds to a dominant wavelength \mathcal{L}_d expressed by

$$\mathcal{L}_d = \frac{2\pi}{l_d} = \pi h \sqrt{\frac{E}{(1-\nu^2)P}} \quad (8.28)$$

This result was obtained previously by the author as a particular case of a more general theory.† Disturbances of wavelength \mathcal{L}_d in the layer will exhibit the maximum rate of growth with time. Because of this selective amplification, the dominant wavelength will actually be observed.

Equation 8.28 has been verified experimentally by testing thin layers of cellulose acetate and aluminum embedded in corn syrup.‡ The viscosity of the syrup was of the order of

$$\eta_1 \cong 10^2 \text{ to } 10^4 \text{ poises} \quad (8.29)$$

The compressive load P was also varied. The theoretical result (8.28) shows that the wavelength varies as the inverse of \sqrt{P} . The buckling of an elastic sheet of cellulose acetate under various loads is shown in Figure 8.3. The dominant wavelength was measured under varying conditions of layer thickness, elastic rigidity, viscosity of the embedding medium, and compressive loads. The experimental results are plotted in Figure 8.4 and compared with the

† M. A. Biot, Folding Instability of a Layered Viscoelastic Medium under Compression, *Proc. Roy. Soc., A*, Vol. 242, pp. 444–454, 1957.

‡ M. A. Biot, H. Odé, W. L. Roever, Experimental Verification of the Theory of Folding of Stratified Viscoelastic Media, *Geological Society of America Bulletin*, Vol. 72, No. 11, pp. 1621–1631, 1961.

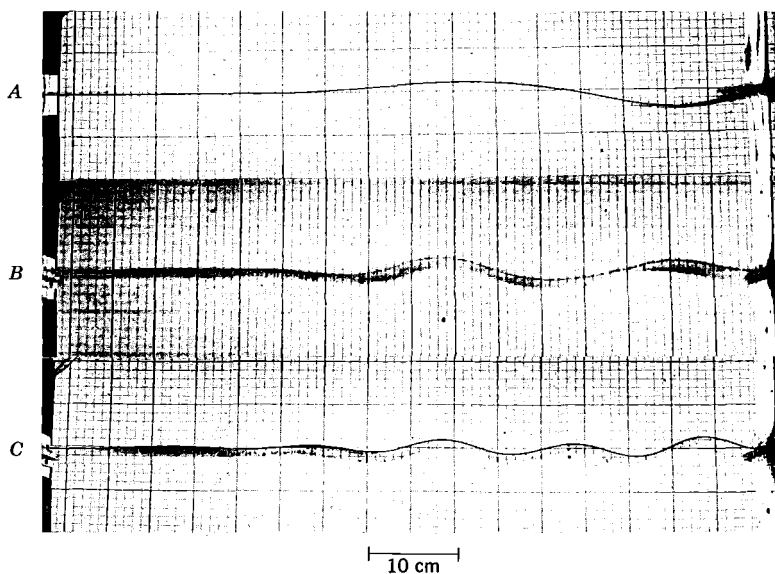


Figure 8.3 Photograph showing buckling of an elastic layer in a viscous medium. The layer is of acetate 1 mm thick and is embedded in corn syrup. The total compressive loads are 1.6 kg, 6.6 kg, and 11.6 kg for cases *A*, *B*, *C*, respectively.

theoretical value (8.28). They show good agreement with the theory. As predicted, the dominant wavelength does not depend on the viscosity of the embedding medium. The viscosity affects only the deformation rate.

Folding of a Viscous Layer in a Viscous Medium. A viscous layer of viscosity η under an axial compression P and embedded in a viscous medium of viscosity η_1 is analyzed by the same procedure. The operators are

$$\hat{Q} = \eta p \quad \hat{Q}_1 = \eta_1 p \quad (8.30)$$

The medium is initially in a state of steady flow, and the applicability of the theory is subject to the restrictions established in section 5. We shall show that they are satisfied for this problem. The thin plate theory yields an equation which coincides with the approximate characteristic equation (8.14) derived from the exact theory. It is written

$$\zeta = \frac{n}{\gamma} + \frac{2}{3}\gamma^2 \quad (8.31)$$

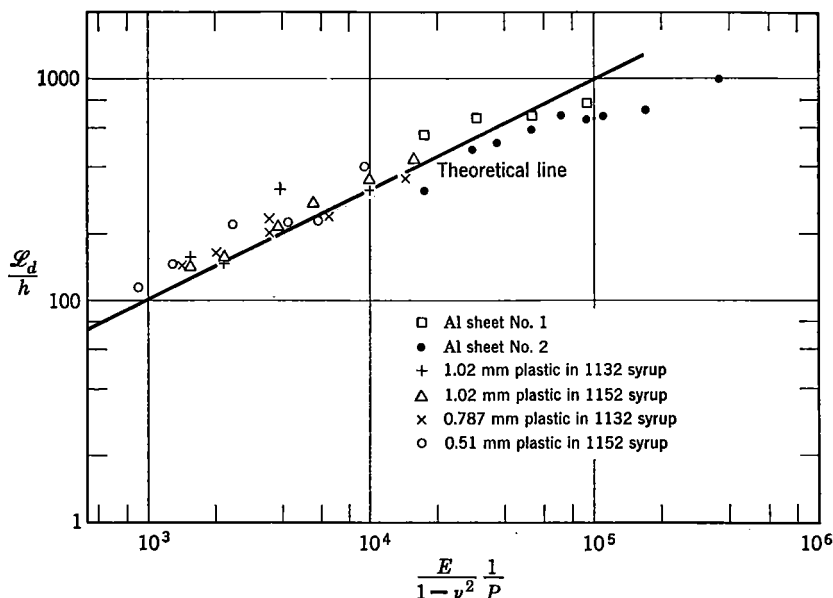


Figure 8.4 Experimental results showing the buckling wavelength \mathcal{L}_d as a function of the compressive stress P for an elastic layer in a viscous medium (plotted non-dimensionally with $h =$ layer thickness, $E =$ Young's modulus of the layer, $\nu =$ Poisson's ratio of the layer). Materials are aluminum and plastic for the layer, and corn syrup for the medium. The theoretical line is given by equation 8.28.

with

$$n = \frac{\eta_1}{\eta} \quad (8.32)$$

$$\zeta = \frac{P}{2\eta p}$$

The dominant wavelength occurs when p is maximum. This maximum value is

$$p_m = \frac{P}{2\eta \zeta_{\min}} \quad (8.33)$$

where ζ_{\min} is the minimum value of ζ obtained when γ is equal to

$$\gamma_d = \frac{1}{2} \sqrt[3]{\frac{6\eta_1}{\eta}} \quad (8.34)$$

The corresponding value of ζ is

$$\zeta_{\min} = \frac{1}{2} \left(\frac{6\eta_1}{\eta} \right)^{\frac{2}{3}} \quad (8.35)$$

The dominant wavelength (8.7) is therefore†

$$\mathcal{L}_d = 2\pi h \sqrt[3]{\frac{\eta}{6\eta_1}} \quad (8.36)$$

In this case it is independent of the initial stress P .

Values of this dominant wavelength (8.36) are given in Table 1 as a function of the viscosity ratio η/η_1 of the two materials.

Table 1

Ratio of the dominant wavelength \mathcal{L}_d to the thickness h of the layer as a function of the ratio of the layer viscosity η to the viscosity η_1 of the embedding medium

η/η_1	\mathcal{L}_d/h
12	7.9
36	11.4
72	14.4
100	16.0
144	18.1
288	22.9
500	27.4
1000	34.5
2000	43.5
10000	74.5

Since in this case the system is in a steady state of flow under the initial stress, we must verify the validity of the result by evaluating the magnitude of the instability. As already discussed in section 5, the theory is strictly applicable only for a time interval during which the steady state deformation remains small. On the other hand, during this time the instability, to be significant, must correspond to an amplification factor $\exp(pt)$ of sufficient magnitude. This is

† This result was derived in 1957 by the author as a particular application of a more general treatment (see reference † on page 420).

expressed by condition (5.35). Consider the same time $t = t_0$ already introduced in the problem of the viscoelastic half-space treated in section 7. The value t_0 is defined by putting $p_0 t_0 = 0.1$. It corresponds to a shortening of about 10% of the viscous layer. At the time t_0 the amplification factor $\exp(p_m t_0)$ of the dominant wavelength may be expressed in terms of ζ_{\min} by using equation 7.20. Hence

$$\exp(p_m t_0) = \exp(0.2/\zeta_{\min}) \quad (8.37)$$

When the value (8.35) for ζ_{\min} is substituted, the amplification factor (8.37) becomes a function of the viscosity ratio η/η_1 . Values of this factor are shown in Table 2. It can be seen that the amplification

Table 2

Amplification factor $\exp(p_m t_0)$ of the dominant wavelength at the time t_0 as a function of the viscosity ratio; at the time t_0 the shortening of the layer is about 10%

η/η_1	$\exp(p_m t_0)$
27	2.91
64	24.5
125	148
216	1340

becomes significant only for values of the viscosity ratio which are larger than about 100. Beyond this value the amplification increases very rapidly. The value of ζ_{\min} is given by equation 8.35. For $\eta/\eta_1 = 100$ it is

$$\zeta_{\min} = 0.067 \quad (8.38)$$

From equation 7.20 we derive

$$\frac{p_0}{p_m} = \frac{1}{2}\zeta_{\min} = 0.033 \quad (8.39)$$

This value verifies the criterion (5.35) for the validity of the correspondence principle.

These results are still based on the assumption that the over-all strain is not large. We shall now show that the validity of the theory is not restricted to small deformations.

The Effect of Large Deformations on Viscous Folding. In a viscous medium the initial compression causes a gradual shortening of the layer and an increase in thickness. If we allow sufficient time to elapse for the deformation to become appreciable, these two effects will tend to influence the folding in opposite directions. The shortening will tend to decrease the wavelengths of the folds. On the other hand, the thickening of the layer increases the dominant wavelength, and therefore it will also tend to increase the folding wavelengths. Since these effects are associated with large deformations, their evaluation requires the use of a theory which is also valid for large deformations. Such a theory was developed in section 5 for viscous buckling of a fluid.

In order to bring out the characteristic features due to large deformations, let us consider the buckling of a free plate. The amplification factor $A(t)$ representing the true instability is given by equation 5.81. For the purpose of comparison with approximate theories it is convenient to write the amplification factor in the form

$$A(t) = \exp \left[\int_0^t p(t) dt \right] \quad (8.40)$$

which introduces a function of time $p(t)$ that may be looked on as an "instantaneous value" for the exponential factor and coincides with the previous definitions for the case when p is constant. The value of $p(t)$ derived from equation 5.81 is

$$p = \frac{4\gamma p_0}{\sinh 2\gamma - 2\gamma} \quad (8.41)$$

We introduce expression (5.5) for p_0 and consider the parameter

$$\zeta = \frac{P}{2\eta p} = \frac{\sinh 2\gamma}{2\gamma} - 1 \quad (8.42)$$

In the purely elastic medium a similar parameter was defined as

$$\zeta = \frac{P}{2\mu} \quad (8.43)$$

where μ is the elastic coefficient of the stress-strain relations (8.33) of Chapter 2. The two values (8.42) and (8.43) are interchangeable by applying the principle of correspondence, replacing μ by ηp .

Of considerable interest is the approximate value of ζ obtained by

expanding expression (8.42) in a power series of γ . Reduced to its first term, this approximation is

$$\zeta = \frac{2}{3}\gamma^2 \quad (8.44)$$

This equation coincides with the result derived from the thin plate theory. This can be seen by putting $n = 0$ in equation 8.31. The exact value (8.42) of ζ for the viscous fluid and the approximate value (8.44) are compared in Figure 8.5. It can be seen that the thin

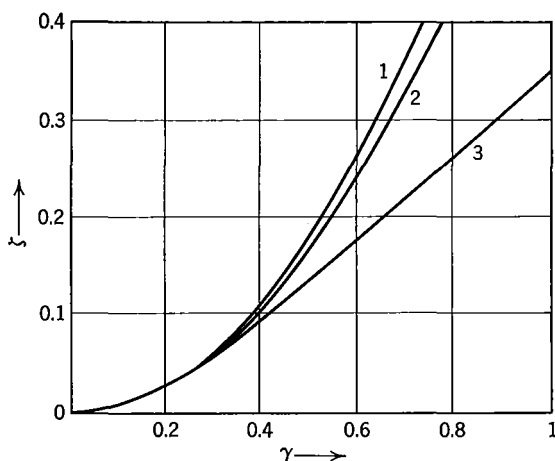


Figure 8.5 Stability parameter ζ for the buckling of a plate as derived from viscous fluid theory (1), thin plate theory (2), and the theory of elasticity (3) plotted as a function of γ .

plate theory provides an excellent approximation. The value of ζ obtained from the exact elastic theory and reproduced from Figure 7.5 of Chapter 3 is also plotted for comparison. All three values are equally satisfactory in the range $\gamma < 0.3$, which corresponds to wavelengths larger than about ten times the thickness.

We conclude that the thin plate theory provides a good approximation for estimating the effect of finite deformations. In the case of an embedded layer we apply equation 8.31. It may be written

$$\frac{P}{2\eta p} = \frac{n}{\gamma} + \frac{2}{3}\gamma^2 \quad (8.45)$$

where $n = \eta_1/\eta$ is the ratio of viscosities of the embedding medium

and the layer. According to equation 8.40 the instantaneous value of p is

$$p = \frac{A}{\mathcal{L}} \quad (8.46)$$

Equation 8.45 is therefore a differential equation for A with a time-dependent variable γ . For plane strain initial flow, γ is expressed by equation 5.77. Its value is

$$\gamma = \frac{\pi h_0}{\mathcal{L}_0} e^{2p_0 t} \quad (8.47)$$

In order to simplify the integration we replace γ by a constant average value during the time interval t . To do so we replace γ by its value γ' at the instant $\frac{1}{2}t$. Hence

$$\gamma' = \frac{\pi h_0}{\mathcal{L}_0} e^{p_0 t} \quad (8.48)$$

Because \mathcal{L}_0 is the initial wavelength, $\mathcal{L} = \mathcal{L}_0 \exp(-p_0 t)$ is the wavelength at the time t . Therefore

$$\gamma' = \frac{\pi h_0}{\mathcal{L}} \quad (8.49)$$

is defined in terms of the initial thickness h_0 and the *final* wavelength \mathcal{L} . With this value γ' , equation 8.45 becomes

$$\frac{P}{2\eta p} = \frac{n}{\gamma'} + \frac{2}{3}\gamma'^2 \quad (8.50)$$

Comparing this result with equation 8.31, we see that the maximum value of p occurs for γ' equal to expression (8.34). Hence, if we measure the wavelength after deformation and substitute the initial thickness h_0 for h , the dominant wavelength \mathcal{L}_d is expressed by equation 8.36. We write

$$\mathcal{L}_d = 2\pi h_0 \sqrt[3]{\frac{\eta}{6\eta_1}} \quad (8.51)$$

According to these definitions the dominant wavelength is not affected, in first approximation, by the finite compression.

As already mentioned, this is due to a compensation resulting from the combined effect of thickening and shortening of the layer. The validity of this conclusion requires that the factor $\exp(2p_0 t)$ remain

below a certain limit; otherwise the averaging process would not be applicable. The total strain therefore should not exceed about 25%.

Time History of Folding Initiated by a Local Disturbance.

In the previous analysis we have considered only the case of folding represented by a pure sine wave of infinite extent. We now derive the time history of folding of a viscous layer embedded in a viscous medium for the case where the initial disturbance is not a pure sine

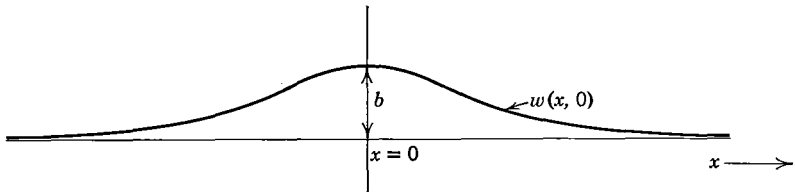


Figure 8.6 Shape of the assumed initial disturbance of an embedded viscous layer represented by equation 8.52.

wave but a localized departure from a plane surface. The analysis which follows and the numerical results were obtained in a recent paper.† The normal deflection w of the initial disturbance is expressed by the equation

$$w = \frac{b}{1 + \left(\frac{x}{a}\right)^2} \quad (8.52)$$

This equation represents a bell-shaped curve illustrated in Figure 8.6. The disturbance may be expressed as a Fourier integral,

$$w(x, 0) = \frac{b}{1 + \left(\frac{x}{a}\right)^2} = ba \int_0^{\infty} e^{-la} \cos lx \, dl \quad (8.53)$$

After a time t each sinusoidal component is multiplied by the amplification factor $\exp(pt)$. Hence at time t the deflection is

$$w(x, t) = ba \int_0^{\infty} e^{pt-la} \cos lx \, dl \quad (8.54)$$

† M. A. Biot, H. Odé, W. L. Roever, Experimental Verification of the Theory of Folding of Stratified Viscoelastic Media, *Geological Society of America Bulletin*, Vol. 72, No. 11, pp. 1621–1631, 1961.

The value of p is a function of l given by equations 8.31 and 8.32. Introducing the variable ζ , we may write

$$pt = \frac{0.2}{\zeta} \frac{t}{t_0} \quad (8.55)$$

The quantity t_0 is the time required for a shortening of about 10% of the medium under the action of the initial compression P (see equation 8.37). The parameter ζ becomes a function of l when $\gamma = \frac{1}{2}lh$ is substituted in equation 8.31.

The integral (8.54) has been evaluated numerically† for a viscosity ratio

$$n = \frac{\eta_1}{\eta} = \frac{1}{1000} \quad (8.56)$$

In plotting the result, the maximum deflection $w(0, t)$ which occurs at the center ($x = 0$) has been normalized to unity. Hence we have plotted the ratio

$$\frac{w(x, t)}{w(0, t)} = f\left(\frac{t}{t_0}, \frac{x}{h}, \frac{a}{h}\right) \quad (8.57)$$

It is a function of the dimensionless time t/t_0 and of the dimensionless distance x/h from the center of the disturbance. There is also a parameter a/h which is a measure of the initial "flatness" of the disturbance. The result is plotted in Figure 8.7 for three types of initial disturbances defined by the values

$$\frac{a}{h} = 2.75, 5.50, 11 \quad (8.58)$$

The shape of the layer is shown at the time $t = 0$ (initial shape) and at the instants corresponding to

$$\frac{t}{t_0} = 0.625, 1.25, 2.5 \quad (8.59)$$

The value $t/t_0 = 2.5$ corresponds to a shortening of the layer of about 25%.

We have shown in the preceding discussion that these results are valid in this range of large compressions, where h represents the initial thickness h_0 of the layer and the wavelengths must be measured

† See reference on page 428.

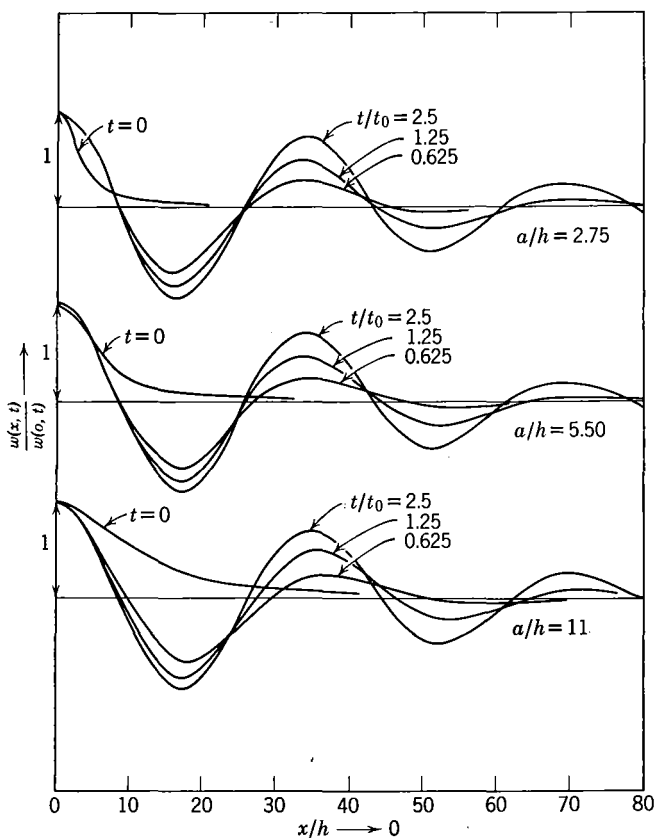


Figure 8.7 Calculated time history of folding under compression of an embedded viscous layer in a viscous medium for a viscosity ratio $\eta_1/\eta = 1/1000$. Three types of initial disturbances (at $t = 0$) are considered. All amplitudes are reduced to unity at the center ($x = 0$). Only the right half of the symmetrical figure is shown.

in the final deformed state. This means that expression (8.52) does not represent the shape of the actual initial disturbance but the shape it would acquire because of the shortening alone. However, as indicated by the plots, this does not affect the folding in any significant way, because the result is quite insensitive to a flattening or sharpening of the initial disturbance. The dominant wavelength for $n = 1/1000$ is (Table 1)

$$\mathcal{L}_a = 34.5h \quad (8.60)$$

It is interesting to note from Figure 8.7 that this wavelength represents very closely the distance between crests of the folds. This dominant wavelength is therefore a distinctive feature of the deformation. It appears quite clearly for $t/t_0 = 0.625$. At that time the shortening of the layer is about 6%. As already pointed out, the result shows that for all practical purposes the time history is very insensitive to the parameter a/h , which represents the flatness of the initial disturbance. Hence, if we exclude extreme cases, it does not matter whether the average width of the initial disturbance is close to the dominant wavelength or not. As for the magnitude of the deflection $w(0, t)$ at the center, it was shown[†] that it follows very closely the exponential amplification factor $\exp(p_m t)$ for the corresponding dominant wavelength. Numerical solutions have also been obtained[†] for the viscosity ratio $n = 1/100$. The folding in this case requires a longer time and is less sharp.

Model tests for the case of purely viscous media have also been carried out using asphalt layers embedded in corn syrup.[†] The results are found to be in good agreement with the theory. In particular, it was verified that the wavelength is independent of the compressive stress.

Folding Instability of Multilayered Viscous and Viscoelastic Media. By application of the correspondence principle, the folding instability of viscous and viscoelastic multilayered media may be treated by using the formal results¹ derived for elastic media in section 7 of Chapter 4 and section 7 of Chapter 5. The equations are applicable to isotropic and anisotropic media, compressible or incompressible. For incompressible materials they include the effect of gravity.[‡] The corresponding equations for viscous and viscoelastic materials are obtained by replacing the elastic coefficients by operators, following the procedure illustrated in this section for the embedded layer. For systems with a large number of layers, numerical solution may be obtained by using the matrix multiplication procedure based on equation 7.18 of Chapter 4. The only

[†] See reference on page 428.

[‡] The general problem of folding of multilayered viscous and viscoelastic media has been discussed in more detail in a paper by the author, Stability of Multilayered Continua Including the Influence of Gravity and Viscoelasticity, *Journal of the Franklin Institute*, Vol. 276, No. 3, pp. 231-252, 1963.

difference in the present case lies in the fact that the initial stress is given. The unknown is now the characteristic exponent p which must be evaluated as a function of the wavelength. Therefore, mathematically, this case resembles the vibration problem of multi-layered elastic media which has been treated in Chapter 5. The previously described iteration procedures also provide methods of numerical solution for media with a large number of dissimilar layers. (See Chapter 5, section 7.)

Property of a Viscous Anisotropic Half-Space. An interesting result is obtained by applying the correspondence principle to a purely viscous incompressible half-space with anisotropic properties and free of initial stress. For the elastic medium it was found that tangential and normal stresses τ and q sinusoidally distributed induce corresponding surface displacements expressed by equations 4.33 of Chapter 4:

$$\begin{aligned}\tau &= 2l\sqrt{N\bar{Q}} U \\ q &= 2l\sqrt{N\bar{Q}} V\end{aligned}\quad (8.60a)$$

For a purely viscous material the corresponding operators are

$$\begin{aligned}\hat{Q} &= p\eta_t \\ \hat{N} &= p\eta_n\end{aligned}\quad (8.60b)$$

The two viscosity coefficients for tangential and normal stresses are η_t and η_n . By substituting the operators (8.60b) in equations 8.60a we obtain

$$\begin{aligned}\tau &= 2l\sqrt{\eta_t\eta_n} pU \\ q &= 2l\sqrt{\eta_t\eta_n} pV\end{aligned}\quad (8.60c)$$

With

$$\eta = \sqrt{\eta_t\eta_n}\quad (8.60d)$$

these last equations become

$$\begin{aligned}\tau &= 2l\eta pU \\ q &= 2l\eta pV\end{aligned}\quad (8.60e)$$

These equations are the same as those expressing the surface stresses in an isotropic viscous fluid of viscosity η . Hence the anisotropic viscous half-space may be replaced by an equivalent isotropic viscous fluid with a viscosity coefficient η given by equation 8.60d.

This result is useful in problems of folding of layers embedded in an anisotropic viscous medium. It was also shown that such a medium may be constituted by thin laminations of purely viscous isotropic fluid with two different viscosities η_1 and η_2 . The two viscosity coefficients η_t and η_n for the equivalent anisotropic fluid may be obtained from equations 3.49. They are

$$\begin{aligned}\eta_t &= \frac{1}{\frac{\alpha_1}{\eta_1} + \frac{\alpha_2}{\eta_2}} \\ \eta_n &= \alpha_1\eta_1 + \alpha_2\eta_2\end{aligned}\quad (8.60f)$$

In these expressions α_1 and α_2 represent the fraction of the total thickness occupied, respectively, by the fluids of viscosity η_1 and η_2 .

If there are more than two different fluids in the laminations, it is possible in some cases to use weighted averages over a suitable averaging thickness. We may write

$$\frac{1}{\eta_t} = \sum \frac{\alpha_t}{\eta_t} \quad (8.60g)$$

$$\eta_n = \sum \alpha_t \eta_t$$

This is analogous to equations 2.31 of Chapter 4 for the elastic medium.

Viscous Buckling of a Multilayered Fluid with Large Deformations. The theory developed in the latter part of section 5 may be used to derive the time history of buckling of a viscous medium composed of an arbitrary number of horizontal fluid layers. The problem of viscous buckling is considered for homogeneous compressive strain of arbitrary magnitude applied to the system in the horizontal direction. The medium is assumed to be incompressible and of high viscosity, so that inertia forces may be neglected.†

Let us consider one of the layers of viscosity η and thickness h . Following the argument in section 5, we must consider the applied stresses τ and q , the perturbation stresses $\bar{\tau}$ and \bar{q} , the corresponding perturbation velocities U' and V' , and the vertical displacement V . We attach a subscript 1 to denote the values of these quantities at the top face of the layer. The subscript 2 denotes the same quantities at the bottom face.

It was pointed out in section 5 that the perturbation stresses $\bar{\tau}_1, \bar{q}_1, \bar{\tau}_2, \bar{q}_2$, due entirely to the perturbation velocities U'_1, V'_1, U'_2, V'_2 are the same as in a viscous fluid of viscosity η initially at rest. The relation between the perturbation stresses and velocities is derived by applying the principle of correspondence to the solution obtained in Chapter 4 for an isotropic incompressible elastic medium *free of initial stress*. The correspondence in this case leads to an *exact solution* for the viscous fluid. We apply the matrix equation (5.20) of Chapter 4, substituting velocities for the displacements and

† The present theory was originally developed more extensively and with specific examples in a recent paper (see M. A. Biot, Theory of Viscous Buckling of Multilayered Fluids Undergoing Finite Strain, *The Physics of Fluids*, Vol. 7, No. 6, pp. 855-859, 1964.

replacing L by η . For the matrix coefficients we must use the values (see Chapter 4, equations 5.36 and 5.37)

$$\begin{aligned} A_1 &= \frac{1}{2}(a_{11} + b_{11}) & A_4 &= \frac{1}{2}(a_{11} - b_{11}) \\ A_2 &= \frac{1}{2}(a_{12} + b_{12}) & A_5 &= \frac{1}{2}(a_{12} - b_{12}) \\ A_3 &= \frac{1}{2}(a_{22} + b_{22}) & A_6 &= \frac{1}{2}(a_{22} - b_{22}) \end{aligned} \quad (8.61)$$

with

$$\begin{aligned} a_{11} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma + 2\gamma} & b_{11} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma - 2\gamma} \\ a_{12} &= -\frac{4\gamma}{\sinh 2\gamma + 2\gamma} & b_{12} &= \frac{4\gamma}{\sinh 2\gamma - 2\gamma} \\ a_{22} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma + 2\gamma} & b_{22} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma - 2\gamma} \end{aligned} \quad (8.62)$$

The variable γ is defined in terms of the thickness h of the layer and the wavelength $\mathcal{L} = 2\pi/l$ of the disturbance,

$$\gamma = \frac{1}{2}lh = \frac{\pi h}{\mathcal{L}} \quad (8.63)$$

It was shown (Chapter 4, section 5) that the matrix equation may be written in compact form by introducing the quadratic function

$$\begin{aligned} I &= \frac{1}{2}A_1(U_1'^2 + U_2'^2) - A_4U_1'U_2' \\ &+ \frac{1}{2}A_3(V_1'^2 + V_2'^2) + A_6V_1'V_2' \\ &+ A_2(U_1'V_1' - U_2'V_2') + A_5(U_1'V_2' + U_2'V_1') \end{aligned} \quad (8.64)$$

The relations between the perturbation stresses and velocities at the top and bottom faces are written

$$\begin{aligned} \bar{\tau}_1 &= l\eta \frac{\partial I}{\partial U_1'} & \bar{q}_1 &= l\eta \frac{\partial I}{\partial V_1'} \\ \bar{\tau}_2 &= -l\eta \frac{\partial I}{\partial U_2'} & \bar{q}_2 &= -l\eta \frac{\partial I}{\partial V_2'} \end{aligned} \quad (8.65)$$

According to equations 5.65, the total stresses are

$$\begin{aligned} \tau_1 &= l\eta \frac{\partial I}{\partial U_1'} - PlV_1 & q_1 &= l\eta \frac{\partial I}{\partial V_1'} \\ \tau_2 &= -l\eta \frac{\partial I}{\partial U_2'} - PlV_2 & q_2 &= -l\eta \frac{\partial I}{\partial V_2'} \end{aligned} \quad (8.66)$$

The horizontal compressive stress in the fluid is denoted by P .

These results may be applied to derive recurrence equations for the multilayered fluid by proceeding exactly as in the corresponding analysis of stability of multilayered solids in Chapter 4. The n layers are numbered from 1 to n (Fig. 8.8). For the j th layer the viscosity, the thickness, and the stress are denoted by η_j , h_j , and P_j . The quadratic form I_j for the j th layer is obtained from expression (8.64)

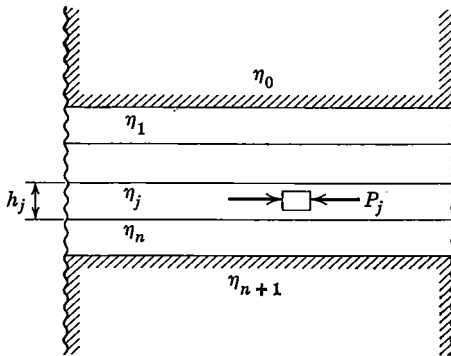


Figure 8.8 Multilayered viscous fluid under initial compression.

by substituting the subscripts j and $j + 1$ for 1 and 2 in the velocity components U'_1 , U'_2 , V'_1 , etc. The variable γ for the j th layer is

$$\gamma_j = \frac{1}{2}kh_j \tag{8.67}$$

We now write the condition that the stresses τ and q are continuous at an interface between the two adjacent layers j and $j + 1$. We derive (no summation)

$$\begin{aligned} \frac{\partial}{\partial U'_{j+1}} (\eta_j I_j + \eta_{j+1} I_{j+1}) &= (P_{j+1} - P_j) V_{j+1} \\ \frac{\partial}{\partial V'_{j+1}} (\eta_j I_j + \eta_{j+1} I_{j+1}) &= 0 \end{aligned} \tag{8.68}$$

If the layers are embedded between two fluid half-spaces, we may include this case in equations 8.68 by adding corresponding layers of infinite thickness denoted by the subscripts 0 and $n + 1$ (Fig. 8.8).

With the total quadratic form

$$\mathcal{J} = \sum_{j=0}^{n+1} \eta_j I_j \tag{8.69}$$

equations 8.68 may be further abbreviated. They are written

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial U'_{j+1}} &= (P_{j+1} - P_j)V_{j+1} \\ \frac{\partial \mathcal{F}}{\partial V'_{j+1}} &= 0\end{aligned}\tag{8.70}$$

Finally we introduce the differential relation (5.69). In this case it becomes

$$V'_j = \dot{V}_j - p_0 V_j\tag{8.71}$$

where \dot{V}_j denotes the time derivative of V_j . Substituting this value of V'_j into equations 8.70, we obtain a system of $2n + 2$ linear homogeneous differential equations for the $2n + 2$ unknowns U'_j and V_j . Each pair of equations 8.70 contains the unknowns for three successive interfaces.

The equations are completely general, and they include boundary conditions. For example, if the top surface is free, we put $\eta_0 = 0$ in the equations. Other cases are handled exactly as in the previous discussion for the elastic medium (see Chapter 4).

The coefficients in the differential equations are functions of the time through the variables γ_j . For an initial flow with plane strain and steady state the time dependence of γ corresponds to equation 5.77. We may write

$$\gamma_j = \kappa_j e^{2p_0 t}\tag{8.72}$$

where κ_j is the initial value of γ_j in the j th layer.

Several generalizations of these equations are readily obtained. In the derivation we have assumed that the initial stress is a compression P_j in the direction of the layers. Actually we may superpose a constant fluid pressure on the whole system. This is equivalent to adding a normal stress S_{22} and considering P_j as representing the effective compression:

$$P_j = S_{22} - S_{11}^{(j)}\tag{8.73}$$

This does not affect the equations, because they depend only on the differences $P_{j+1} - P_j$ and the term S_{22} cancels out.

Equations 8.70 may also be generalized to include the effect of gravity. In this case P_j is still constant within one layer, but the stresses S_{22} and $S_{11}^{(j)}$ are functions of the altitude coordinate. (See

equations 7.19, Chapter 4.) The results obtained for the elastic material are again applicable. It was shown that the problem is solved by introducing an analog model free of gravity stresses with interfacial forces proportional to the density difference and the vertical displacement. We must introduce the quadratic function given by equations 7.30 of Chapter 4:

$$\mathcal{G} = \frac{1}{2l} \sum_0^n (\rho_{j+1} - \rho_j) g V_{j+1}^2 \tag{8.74}$$

where ρ_j is the mass density of the j th layer, and g represents the acceleration of gravity. The differential equations 8.70 are then replaced by

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial U'_{j+1}} &= (P_{j+1} - P_j) V_{j+1} \\ \frac{\partial \mathcal{J}}{\partial V'_{j+1}} + \frac{\partial \mathcal{G}}{\partial V_{j+1}} &= 0 \end{aligned} \tag{8.75}$$

The effect of the gravity forces is embodied in the additional term \mathcal{G} .

A useful simplification of equations 8.75 is obtained by introducing the transformation

$$\begin{aligned} U'_j &= \dot{u}_j e^{p_0 t} \\ V_j &= v_j e^{p_0 t} \end{aligned} \tag{8.76}$$

From equation 8.71 we derive

$$V'_j = \dot{v}_j e^{p_0 t} \tag{8.77}$$

Since equations 8.75 are homogeneous, the factor $e^{p_0 t}$ cancels out. We denote by \mathcal{J}' and \mathcal{G}' the quadratic functions (8.69) and (8.74) written with variables \dot{u}_j , v_j , and \dot{v}_j replacing U'_j , V_j , and V'_j . Equations 8.75 become

$$\begin{aligned} \frac{\partial \mathcal{J}'}{\partial \dot{u}_{j+1}} &= (P_{j+1} - P_j) v_{j+1} \\ \frac{\partial \mathcal{J}'}{\partial \dot{v}_{j+1}} + \frac{\partial \mathcal{G}'}{\partial v_{j+1}} &= 0 \end{aligned} \tag{8.78}$$

These results are also applicable to the general case of unsteady and triaxial initial flow if the slight modification discussed in the latter part of section 5 is introduced. Then p_0 , P_j , and γ become arbitrary functions of time, and $p_0 t$ is replaced by the time integral of p_0 .

Applications to Geology. The problem of folding of stratified geological structures under tectonic stresses has been analyzed by the author as an application of the general theory of stability of multilayered viscous fluids and viscoelastic solids.† This simple quantitative approach to geological folding with its emphasis on flow properties and time dependence has opened a new phase in geodynamics. The basic theoretical and experimental work was performed under a systematic program initiated around 1950 and sponsored by the Shell Development Company. The results were presented in a series of papers published since 1957 and cited in this section. It was shown that the assumption of viscous behavior as an approximate model for rock deformation leads to folding patterns which closely resemble the observed structures. The time history of the deformation was also evaluated and found to be in good agreement with the geological time scale. Problems of orogenesis were also discussed in the same paper‡ in a treatment of the effect of gravity for the non-homogeneous viscous and viscoelastic half-space (see section 7) and for multilayers resting on top of a homogeneous half-space. The latter problem was discussed by using the results derived in an earlier paper by the author.§ The problem of internal folding of a confined multilayered structure has also been solved in a simplified theory¶ which brings out the significant factors and provides an explanation for one of the predominant features of geological structures.

Problems of folding instability of *porous* multilayered media are also of interest in geology and are briefly discussed at the end of section 9 of this chapter.

9. DYNAMICS OF VISCOELASTIC MEDIA UNDER INITIAL STRESS

The dynamical equations for the incremental stresses in a continuous medium were derived in section 2 of Chapter 5. Equations 2.8 of that chapter may be written

$$\frac{\partial F_{ij}}{\partial x_j} + \rho \Delta X_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (9.1)$$

† M. A. Biot, Theory of Folding of Stratified Viscoelastic Media and its Implications in Tectonics and Orogenesis, *Geological Society of America Bulletin*, Vol. 72, No. 11, pp. 1595–1632, 1961.

‡ M. A. Biot, The Influence of Gravity on the Folding of a Layered Viscoelastic Medium under Compression, *Journal of the Franklin Institute*, Vol. 267, No. 3, pp. 211–228, 1959.

§ M. A. Biot, Theory of Internal Buckling of a Confined Multilayered Structure, *Geological Society of America Bulletin*, Vol. 75, No. 6, 563–568, 1964.

where F_{ij} is a short-hand notation for a combination of terms given in two equivalent forms by equations 4.9 and 4.13. They are

$$\begin{aligned} F_{ij} &= t_{ij} + S_{kj}\omega_{ik} - \frac{1}{2}S_{ik}e_{jk} + \frac{1}{2}S_{jk}e_{ik} \\ F_{ij} &= s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk} \end{aligned} \tag{9.2}$$

When the body force is derived from a potential field and the value (4.8) for ΔX is substituted in equations 9.1, we obtain

$$\frac{\partial F_{ij}}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{9.3}$$

These equations generalize to dynamics the previously obtained result (4.10).

To these equations must be added the stress-strain relations in the form (3.31) or (3.36). They are

$$\begin{aligned} t_{ij} &= \hat{C}_{ij}^{\mu\nu} e_{\mu\nu} \\ s_{ij} &= \hat{B}_{ij}^{\mu\nu} e_{\mu\nu} \end{aligned} \tag{9.4}$$

By substituting these values of the stresses into F_{ij} equations 9.3 become the field equations for the three unknown displacements u_i . These field equations may be expressed in operational form by writing p^2 for the differential operator $\partial^2/\partial t^2$; that is,

$$\frac{\partial F_{ij}}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = p^2 \rho u_i \tag{9.5}$$

The operational coefficients in the stress-strain relations (9.4) may be interpreted as integro-differential operators as illustrated in section 3. The field equations are then also integro-differential equations with respect to the time variable.

For harmonic oscillations we must put

$$p = i\alpha \tag{9.6}$$

By this procedure solutions may be obtained which correspond to forced oscillations of given frequency $\alpha/2\pi$.

In problems of natural oscillations and stability the equations are solved by considering p as the unknown with real or complex values.

Vibrations and Dynamic Stability of Viscoelastic Plates and Multilayered Media. In section 7 of Chapter 5 general solutions were obtained for the dynamics of plates and multilayered media in

the elastic case. By viscoelastic correspondence these solutions are immediately applicable to viscoelastic media. As already pointed out, the correspondence yields exact equations if the medium is initially at rest under the initial stress.

Let us consider, for example, the dynamics of an incompressible viscoelastic plate. We replace the elastic coefficients L and M by operators \hat{L} and \hat{M} in the coefficients (7.44) of Chapter 5. These coefficients become operators \hat{a}_{11} , \hat{a}_{12} , \hat{a}_{22} , and equations 7.19 of Chapter 5 become operational relations:

$$\begin{aligned}\frac{\tau_a}{i\hat{L}} &= \hat{a}_{11}U_a + \hat{a}_{12}V_a \\ \frac{q_a}{i\hat{L}} &= \hat{a}_{12}U_a + \hat{a}_{22}V_a\end{aligned}\tag{9.7}$$

If we substitute $p = i\alpha$ in the expression for the operators, equations 9.7 correspond to forced oscillations of the flexural type with given frequency, as shown in Figure 7.1a of Chapter 5. Equations 9.7 relate the amplitude and phase difference of the driving forces and surface displacements of the plate. The tangential and normal driving forces are distributed sinusoidally along the surface and they are equal to $\tau_a \sin lx$ and $q_a \cos lx$.† The corresponding tangential and normal displacements at the surface of the plate are $U_a \sin lx$ and $V_a \cos lx$. The operational coefficients \hat{a}_{ij} are complex functions of the frequency variable $p = i\alpha$ and of the wavelength variable $\gamma = \frac{1}{2}lh$ ($h =$ plate thickness). Hence equations 9.7 determine τ_a and q_a also as complex quantities.

For a plate of compressible isotropic or anisotropic viscoelastic properties the forced flexural oscillations are derived by using the same equations 9.7. For this medium the operational coefficients \hat{a}_{ij} are obtained by substituting operators \hat{B}_{ij} and \hat{L} for the elastic coefficients in the more general expressions of a_{ij} given by equations 7.21 of Chapter 5.

The general problem of forced oscillations of multilayered viscoelastic media is formulated exactly as in the case of pure elasticity. For each layer we must consider six operators. They are the three operators \hat{a}_{ij} already discussed and three additional operators \hat{b}_{ij}

† The quantity q_a in equations 9.7 represents the normal stress and should not be confused with the generalized coordinates q_i which are used below.

obtained by substituting operators \hat{B}_{ij} and \hat{L} in equations 7.30 of Chapter 5.

In problems of free oscillations and dynamic stability of plates and multilayered viscoelastic media, solutions are readily obtained by using the same characteristic equation as that obtained for the corresponding problem in the purely elastic medium. We replace the elastic coefficients by operators. The characteristic equation is then solved for p as the unknown representing a real or complex characteristic exponent.

The numerical methods discussed in Chapter 5 (section 7) are also applicable.

It is of interest to derive fundamental theorems regarding the nature of characteristic exponents which are applicable to the general case of a viscoelastic medium with initial stress. From the thermodynamic viewpoint the properties of a viscoelastic medium are closely related to those of a certain class of dynamical systems with dissipation. We shall therefore begin with a brief analysis of some basic properties of the characteristic solutions for dynamical systems.

Stability Properties of Dynamical Systems with a Potential Energy and a Dissipation Function. Let us consider a dynamical system with a potential energy \mathcal{P} and initially in equilibrium. With the addition of damping forces represented by a dissipation function \mathcal{D} , the small motion of the system near equilibrium is governed by the equations

$$\frac{\partial \mathcal{P}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} + \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) = 0 \quad (9.8)$$

where

$$\begin{aligned} \mathcal{P} &= \frac{1}{2} a_{ij} q_i q_j \\ \mathcal{D} &= \frac{1}{2} b_{ij} \dot{q}_i \dot{q}_j \\ \mathcal{T} &= \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \end{aligned} \quad (9.9)$$

The deviation from the equilibrium position is measured by the generalized coordinates q_i and \mathcal{T} is the kinetic energy of the system. For physical reasons, \mathcal{D} is non-negative† and \mathcal{T} is positive definite. For a system with initial stress, however, the sign of \mathcal{P} is not determined. This has been shown in previous discussions of the stability problem (see section 4 of Chapter 3 and section 2 of Chapter 5).

† If we exclude the degrees of freedom for which the dissipation vanishes, \mathcal{D} becomes also positive definite.

The linear equations of motion (9.8) are written explicitly:

$$a_{ij}q_j + b_{ij}\dot{q}_j + m_{ij}\ddot{q}_j = 0 \quad (9.10)$$

The stability of the system is determined by the nature of the characteristic solutions. These solutions are proportional to an exponential function of time which we write $\exp(pt)$. Hence we replace q_j by $q_j \exp(pt)$, with q_j representing constant amplitudes. With this substitution equations 9.10 become a set of algebraic equations:

$$(a_{ij} + pb_{ij} + p^2m_{ij})q_j = 0 \quad (9.11)$$

By equating to zero the determinant of this homogeneous system of equations a set of characteristic roots for the exponent p is determined. Stability of these characteristic solutions depends on the sign of the real part of p . Some well-known theorems regarding stability properties of dynamical systems governed by equations 9.10 will now be derived.

Non-oscillatory Character of Unstable Motion. We consider a set of values q_j and the associated characteristic exponent p satisfying equations 9.11. The complex conjugate quantities q_j^* and p^* satisfy the same equations. Hence

$$(a_{ij} + p^*b_{ij} + p^{*2}m_{ij})q_j^* = 0 \quad (9.12)$$

We multiply equations 9.11 by q_i^* and equations 9.12 by q_i and obtain

$$\begin{aligned} (a_{ij} + pb_{ij} + p^2m_{ij})q_jq_i^* &= 0 \\ (a_{ij} + p^*b_{ij} + p^{*2}m_{ij})q_j^*q_i &= 0 \end{aligned} \quad (9.13)$$

The coefficients in these expressions are symmetric; that is,

$$a_{ij} = a_{ji} \quad b_{ij} = b_{ji} \quad m_{ij} = m_{ji} \quad (9.14)$$

Therefore equations 9.13 may also be written

$$\begin{aligned} (a_{ij} + pb_{ij} + p^2m_{ij})q_jq_i^* &= 0 \\ (a_{ij} + p^*b_{ij} + p^{*2}m_{ij})q_j^*q_i &= 0 \end{aligned} \quad (9.15)$$

The difference of the two equations 9.15 yields

$$(p - p^*)[b_{ij}q_jq_i^* + (p + p^*)m_{ij}q_jq_i^*] = 0 \quad (9.16)$$

As already stated, the dissipation function \mathcal{D} is non-negative and the

kinetic energy \mathcal{T} is positive definite. Hence (see equation 2.46f of Chapter 5) the following inequalities are verified:

$$b_{ij}q_j q_i^* \geq 0 \quad m_{ij}q_j q_i^* > 0 \quad (9.17)$$

Let us assume that the solution which we have considered is unstable. The singular case of neutral equilibrium $p = 0$ is non-oscillatory. For $p \neq 0$ instability requires

$$p + p^* > 0 \quad (9.18)$$

The inequalities 9.17 and 9.18 show that the bracketed term in equation 9.16 is positive. Therefore equation 9.16 implies

$$p = p^* \quad (9.19)$$

and p is a real quantity.

Thus we have established an important property of a dynamical system with a potential energy and a dissipation function. For such a system an unstable motion is non-oscillatory. Excluding the special case of neutral equilibrium, all displacements in this case are proportional to a real exponential function of time.

Stability Criterion. A sufficient condition for stability of the dynamical system is immediately derived from the preceding results. Let us consider an unstable solution q_i . We have just shown that such a solution is real with a positive or zero value of p . The solution satisfies equations 9.15. With real values of p and q_i these equations are the same, and they become

$$(a_{ij} + pb_{ij} + p^2 m_{ij})q_j q_i = 0 \quad (9.20)$$

An unstable solution requires

$$p \geq 0 \quad (9.21)$$

At the same time the inequalities (9.17) must be satisfied. As a consequence, if

$$\mathcal{P} > 0 \quad (9.22)$$

equation 9.20 cannot be verified and the system cannot be unstable.

We conclude that, for a dynamical system with a potential energy and a dissipation function, stability is ensured if the potential energy \mathcal{P} is positive definite.

From the physical viewpoint this stability condition is also a consequence of the law of conservation of energy.

The stable solutions may be complex or real. The complex solutions represent damped oscillations, and the real solutions correspond to critical or hypercritical damping.

Stability in the Presence of Coriolis Forces. General properties of dynamical systems of the type discussed here have long been known in celestial mechanics. They are important in the problem of stability of equilibrium forms of rotating masses of fluid under the action of self-gravitational forces. The problem was first investigated by Poincaré in 1885. For a rotating system additional terms corresponding to Coriolis forces must be added. These additional terms were included in equations 6.83c of Chapter 5 for acoustic-gravity waves. Similar terms may be added to the general dynamical equations (9.10).

When we examine how the presence of Coriolis forces affects the stability properties, we find that equation 9.16 does not remain valid and unstable characteristic solutions may exist which are complex. Hence, for a rotating system, unstable motion may be oscillatory.

However, the condition $\mathcal{P} > 0$ for the potential energy remains a sufficient criterion for stability. This conclusion follows from the fact that the additional terms representing the Coriolis force constitute a skew-symmetric matrix. Hence, when the operations leading to equations 9.15 are performed, the additional terms cancel out. We then add the two equations 9.15 after dividing them, respectively, by p and p^* . The result is

$$\left[\frac{p + p^*}{pp^*} a_{ij} + b_{ij} + (p + p^*)m_{ij} \right] q_j q_i^* = 0 \quad (9.23)$$

If $\mathcal{P} > 0$, the equations of motion show that they cannot be verified for $p = 0$. Therefore $p \neq 0$. Then equation 9.23 cannot be verified unless $p + p^* \leq 0$. Hence a rotating system with a potential energy and a dissipation function is always stable if its potential energy is positive definite.†

Dynamical Equations Extended to Thermodynamic Systems. The general dynamical equations (9.10) may be extended to thermodynamic systems. For these systems the equations are obtained by including the inertia terms in the thermodynamic

† For a more extensive discussion of problems of this type see, for example, R. A. Lyttleton, *The Stability of Rotating Liquid Masses*, Cambridge University Press, 1953.

equations (2.18). This generalization is justified by applying d'Alembert's principle to the thermodynamic system and treating inertia forces as if they were externally applied forces. The applicability of equations 9.10 is conditioned, of course, by the range of validity of the principles of linear thermodynamics. The generalized coordinates represent a large variety of physical variables including thermodynamic and mechanical quantities. Under these conditions the properties of dynamical systems with a potential energy and a dissipation function are applicable to thermodynamics.

In particular, these properties are applicable to viscoelastic media provided they obey the principles of linear thermodynamics. As pointed out in section 3, the viscoelastic medium may be described thermodynamically by a set of generalized coordinates which are of two different types and which have been referred to as external and internal coordinates. The latter represent the unobserved thermodynamic degrees of freedom which are responsible for the relaxation effects. From the thermodynamic viewpoint, however, these two types of coordinates are not distinct. They are governed by the dynamical equations (9.10). Hence under these assumptions the viscoelastic medium exhibits the same properties as a dynamical system with a potential energy and a dissipation function. As a consequence, unstable characteristic solutions will be non-oscillatory.

It is of interest to examine the stability properties of viscoelastic media independently of thermodynamic principles. This will be done hereafter by deriving some general theorems which are formulated entirely by means of the operators appearing in the stress-strain relations.

Extension to Dynamics of Lemmas I and II of Section 4.

Properties of characteristic solutions for a viscoelastic medium with initial stress under the assumption that the inertia forces are negligible were examined in section 4. In order to generalize the analysis to dynamics we must extend the two lemmas derived in section 4 to dynamical systems. We start by multiplying equations 9.5 by an arbitrary displacement field \bar{u}_i . We then integrate the result over the volume τ . We derive

$$\iiint_{\tau} \left(\frac{\partial F_{ij}}{\partial x_j} \bar{u}_i - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j \bar{u}_i - p^2 \rho u_i \bar{u}_i \right) d\tau = 0 \quad (9.24)$$

Following the procedure used in deriving equation 4.12, we integrate

equation 9.24 by parts. We obtain

$$\iiint_{\tau} [F(u, \bar{u}) + p^2 \rho u_i \bar{u}_i] d\tau = \iint_A \Delta f_i \bar{u}_i dA \quad (9.25)$$

where the Δf_i are the incremental forces applied to the boundary A . This result extends the first lemma (4.16) to dynamics. The expression $F(u, \bar{u})$ is defined by equation 4.17. Let \bar{u}_i also be a solution of the dynamical equations 9.5 with the characteristic value \bar{p} . We may then interchange u_i and \bar{u}_i in equation 9.25 provided that we replace p by \bar{p} . Hence

$$\iiint_{\tau} [F(\bar{u}, u) + \bar{p}^2 \rho u_i \bar{u}_i] d\tau = \iint_A \Delta \bar{f}_i u_i dA \quad (9.26)$$

We now subtract from each other the two equations 9.25 and 9.26. By applying equation 4.21 we obtain

$$\begin{aligned} \iiint_{\tau} [\bar{t}_{ij} \bar{e}_{ij} - \bar{t}_{ij} e_{ij} + (p^2 - \bar{p}^2) \rho u_i \bar{u}_i] d\tau \\ = \iint_A (\Delta f_i \bar{u}_i - \Delta \bar{f}_i u_i) dA \end{aligned} \quad (9.27)$$

This result generalizes to dynamics the second lemma (4.22).

Conditions Ensuring Non-oscillatory Unstable Motion in Dynamic Viscoelasticity. In section 4 we examined some general properties of the mechanics of viscoelastic media under initial stress for the case where inertia forces are negligible, and we established a criterion which ensures that the characteristic solutions are real.

For dynamical systems the criterion obviously cannot remain valid, because damped oscillations represent characteristic solutions of a stable viscoelastic medium. However, if the system is unstable, we shall show that, if we add a simple restriction, a criterion similar to that derived in section 4 will ensure that the unstable solutions are always real.

Let us assume the existence of a complex characteristic solution u_i of equations 9.5 with a characteristic exponent p . The complex conjugate quantities u_i^* and p^* constitute a solution also.

We assume also that these solutions satisfy the conservative boundary conditions listed under (a), (b), and (c) in section 4 (page 369). They correspond to either a free boundary or a rigid boundary with perfect adherence or perfect slip.

We now substitute u_i^* and p^* for \bar{u}_i and \bar{p} in equation 9.27. Because of the assumed boundary conditions the surface integral vanishes as in equation 4.24. Hence we may write

$$\iiint_{\tau} [t_{ij}e_{ij}^* - t_{ij}^*e_{ij} + (p^2 - p^{*2})\rho u_i u_i^*] d\tau = 0 \quad (9.28)$$

At this point we introduce some restrictions regarding the viscoelastic properties. As before (see equation 4.28), we assume that the viscoelastic operators are symmetric. Hence we write

$$\hat{C}_{ij}^{\mu\nu} = \hat{C}_{\mu\nu}^{ij} \quad (9.29)$$

As a consequence equation 4.31 is also valid; that is,

$$t_{kj}e_{kj}^* - t_{kj}^*e_{kj} = 2i\mathcal{I}_{kj}^{\mu\nu}e_{kj}^*e_{\mu\nu} \quad (9.30)$$

with

$$\hat{C}_{kj}^{\mu\nu} - \hat{C}_{kj}^{\mu\nu*} = 2i\mathcal{I}_{kj}^{\mu\nu} \quad (9.31)$$

The dummy index i has been changed to k to avoid confusion with the imaginary symbol. We note that $i\mathcal{I}_{kj}^{\mu\nu}$ represents the imaginary part of $\hat{C}_{kj}^{\mu\nu}$ obtained when we substitute into the operator a complex value of p .

Substituting equation 9.30 into equation 9.28, we obtain

$$\iiint_{\tau} \left[\mathcal{I}_{kj}^{\mu\nu}e_{kj}^*e_{\mu\nu} + (p + p^*)\left(\frac{p - p^*}{2i}\right)\rho u_k u_k^* \right] d\tau = 0 \quad (9.32)$$

The singular case of neutral equilibrium ($p = 0$) is non-oscillatory. Hence we assume $p \neq 0$. The value of p may be either one of the complex conjugate values. Let us choose for p the value with a *positive* coefficient of the imaginary part. Then

$$\frac{p - p^*}{2i} > 0 \quad (9.33)$$

Let us also assume that the characteristic solution which we have considered is unstable. Hence

$$p + p^* > 0 \quad (9.34)$$

Finally let us assume that $\mathcal{I}_{kj}^{\mu\nu}$ defines a *non-negative* quadratic form. Hence, as shown by equation 4.34, we may write the inequality

$$\mathcal{I}_{kj}^{\mu\nu}e_{kj}^*e_{\mu\nu} \geq 0 \quad (9.35)$$

However, equation 9.32 *cannot* be verified under the three assumptions (9.33), (9.34), and (9.35) unless

$$p = p^* \quad (9.36)$$

Hence p must be a real quantity.

We conclude that unstable characteristic solutions for a viscoelastic medium are always non-oscillatory if the following conditions are satisfied:

1. The boundary conditions must be conservative as described previously in conditions (a), (b), and (c) of section 4.
2. The viscoelastic operators must be symmetric; that is, $\hat{C}_{kj}^{\mu\nu} = \hat{C}_{\mu\nu}^{kj}$.
3. We denote by $i\mathcal{F}_{kj}^{\mu\nu}$ the imaginary part of $\hat{C}_{kj}^{\mu\nu}$ obtained when substituting a complex value of p such that $(p - p^*)/i$ is positive. We assume that $\mathcal{F}_{kj}^{\mu\nu}$ must define a non-negative quadratic form.

We see that these conditions differ from the criterion derived in section 4 by the additional requirement that expressions (9.33) and (9.35) be of the same sign.

Conditions Ensuring Stability in Dynamic Viscoelasticity.

In the preceding discussion we have considered an unstable system and derived sufficient conditions for the motion to be non-oscillatory. Let us now examine conditions under which a viscoelastic medium is dynamically stable. In order to arrive at a criterion which is of practical value we assume that conditions 1, 2, and 3 above are fulfilled. These conditions, as we have seen, ensure that any unstable characteristic solution is real. They state that the boundary conditions are conservative, that the viscoelastic operators are symmetric ($\hat{C}_{kj}^{\mu\nu} = \hat{C}_{\mu\nu}^{kj}$), and that their imaginary parts satisfy a certain condition of non-negativeness.

We go back to equation 9.25 and substitute $u_i = \bar{u}_i$. We obtain

$$\iiint_{\tau} [F(u, u) + p^2 \rho u_i u_i] d\tau = \iint_A \Delta f_i u_i dA \quad (9.37)$$

From equation 4.17 we may write

$$\frac{1}{2} F(u, u) = \Delta \hat{V} + \Delta U \quad (9.38)$$

with

$$\begin{aligned} \Delta \hat{V} &= \frac{1}{2} \hat{C}_{ij}^{\mu\nu} e_{ij} e_{\mu\nu} + \frac{1}{2} \mathcal{S}_{ij} (\omega_{ki} e_{kj} + e_{ki} \omega_{kj} + \omega_{ki} \omega_{kj}) \\ \Delta U &= \frac{1}{2} \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \end{aligned} \quad (9.39)$$

These terms are formally identical with the quantities ΔV and ΔU represented by equations 2.12 and 2.17 of Chapter 5. The expression $\Delta \hat{V}$ is derived from ΔV by substituting operators $\hat{C}_{ij}^{\mu\nu}$ for the elastic coefficients $C_{ij}^{\mu\nu}$.

In analogy with equation 2.19 of Chapter 5 we may write

$$\hat{\mathcal{P}}_{\tau} = \iiint_{\tau} (\Delta \hat{V} + \Delta U) d\tau \tag{9.40}$$

We have shown that for conservative boundary forces the surface integral in equation 9.37 may be written

$$\mathcal{P}_B = -\frac{1}{2} \iint_A \Delta f_i u_i dA \tag{9.41}$$

where \mathcal{P}_B represents a potential energy originating at a curved rigid boundary. For perfect slip it is expressed by equation 4.67 of Chapter 3.

We put

$$\hat{\mathcal{P}} = \hat{\mathcal{P}}_{\tau} + \mathcal{P}_B \tag{9.42}$$

Equation 9.37 may then be written

$$\hat{\mathcal{P}} + \frac{1}{2} p^2 \iiint_{\tau} \rho u_i u_i d\tau = 0 \tag{9.43}$$

Under our assumptions an unstable solution must be real, and the value of p must be real and non-negative.

If, in addition, we assume that

$$\hat{\mathcal{P}} > 0 \tag{9.44}$$

for arbitrary displacements and for all real non-negative values of p , equation 9.43 cannot be fulfilled. Thus we have shown that the viscoelastic medium is dynamically stable if $\hat{\mathcal{P}}$ is positive for all non-negative values of p and if, in addition, the conditions for non-oscillatory unstable motion are satisfied.

Stable solutions may be real or complex. They represent an exponential decay or damped oscillations. Undamped oscillations must be considered as a limiting case of stability.

The criterion (9.44) ensures the stability of characteristic solutions. In a physical system, stability of the response to an external excitation is also implied, as can be shown by applying standard Laplace

transform procedures. Under very broad conditions the response is expressed by a contour integral in the complex plane which is evaluated as a sum of residues. The terms in this series are proportional to exponential factors of the type $\exp(pt)$ and are the same as in the characteristic solutions considered above. If the characteristic solutions are stable, real parts of p are never positive, and therefore the response to a perturbation is either damped or non-increasing.

Uniqueness. The stability criterion is closely related to the condition of uniqueness of solutions under external excitation. This can be seen by assuming that two different responses R_1 and R_2 are caused by the same excitation. Because the system is linear, the difference $R_1 - R_2$ represents a solution occurring without external excitation. In a stable system such a spontaneous solution cannot occur. Therefore $R_1 - R_2 = 0$, and the solution is unique.

Thermodynamics and Viscoelastic Stability. We have already pointed out the relationship between thermodynamics and the properties of dynamical systems with a potential energy and a dissipation function. In particular, a viscoelastic medium which is governed by the principle of linear thermodynamics should exhibit the same stability properties as the aforementioned dynamical system. This can be verified by taking into account the special nature of the viscoelastic operators $\hat{C}_{ij}^{\mu\nu}$ derived from thermodynamics and given by equation 4.35a. If in this case the boundary conditions are conservative, any unstable characteristic solution must be real. This conclusion is obtained by referring to equation 4.35c for the expression of $\mathcal{S}_{kj}^{\mu\nu}$. This equation shows that conditions 1, 2, 3, ensuring a non-oscillatory unstable motion, are fulfilled (page 448).

Moreover, in this case the stability condition (9.44) is simplified. The expression for $\hat{\mathcal{P}}$ is defined as a function of p by equations 9.39 and 9.40. The inequality (9.44) must be verified for positive values of p . Writing the value of $\hat{\mathcal{P}}$ using the operators 4.35a, we find that for $p > 0$ it is never smaller than its value for $p = 0$. This result follows from the property that the coefficients $C_{ij}^{\mu\nu}(r)$, $C_{ij}^{\mu\nu}$, and $C'_{ij}{}^{\mu\nu}$ define non-negative quadratic forms. Hence the stability condition (9.44) may be written

$$\mathcal{P} > 0 \tag{9.44a}$$

where \mathcal{P} is the value of $\hat{\mathcal{P}}$ for $p = 0$. This value of \mathcal{P} is also obtained by substituting the elastic coefficients $C_{ij}^{\mu\nu}$ for the operators $\hat{C}_{ij}^{\mu\nu}$.

We conclude that, with conservative boundary conditions and viscoelastic operators derived from thermodynamics, the condition for static stability also ensures dynamic stability. The same property was obtained for a dynamical system with a potential energy and a dissipation function (page 443).

Application to an Incompressible Viscoelastic Medium with Horizontal Stratification. The medium is the same as that considered in the latter part of section 4. It is orthotropic with vertical and horizontal directions of viscoelastic symmetry. The principal directions of the initial stress are also vertical and horizontal, and we consider the incremental deformations in the vertical x, y plane. The effect of the gravity force is included. The incremental viscoelastic properties are defined by the two operators \hat{N} and \hat{Q} of the stress-strain relations (4.36). These operators and the initial stress components may be continuous or discontinuous functions of the vertical coordinate y .

As in section 4, we consider a solution which is periodic in the horizontal direction x . Let us apply equation 9.27 to the rectangular area $ABCD$ illustrated in Figure 4.1 and assume that the vertical sides AB and CD are separated by a distance of one wavelength. We also substitute the complex conjugate quantities $\bar{u}_i = u_i^*$ and $\bar{p} = p^*$. Proceeding as in section 4 and assuming the same boundary conditions, we derive

$$\iint_{ABCD} [t_{ij}e_{ij}^* - t_{ij}^*e_{ij} + (p^2 - p^{*2})\rho u_i u_i^*] dx dy = 0 \quad (9.45)$$

with

$$t_{ij}e_{ij}^* - t_{ij}^*e_{ij} = 4(\hat{N} - \hat{N}^*)e_{xx}e_{xx}^* + 4(\hat{Q} - \hat{Q}^*)e_{xy}e_{xy}^* \quad (9.46)$$

Except for the addition of a dynamic term, this result coincides with equations 4.39 and 4.46. When $p + p^*$ is positive, equation 9.45 cannot be verified if $(\hat{N} - \hat{N}^*)/i$, $(\hat{Q} - \hat{Q}^*)/i$, and $(p - p^*)/i$ are never of different sign. Hence under these conditions we must have $p = p^*$, and the characteristic solution must be real. We conclude that, for the viscoelastic system considered here, an unstable motion will be non-oscillatory if the coefficients of the imaginary parts of \hat{N} , \hat{Q} , and p are never of different sign.

Power Dissipation Theorem. The foregoing results are immediately applicable to the evaluation of the power dissipation in a viscoelastic medium in forced oscillations. Let us consider driving forces $\Delta f_i \exp(i\alpha t)$ applied to the boundary. They are harmonic functions of time with a frequency $\alpha/2\pi$. We apply the second lemma (9.27) and put

$$p = i\alpha \quad \bar{p} = p^* = -i\alpha \quad (9.47)$$

We derive

$$\iiint_{\tau} (t_{kj}e_{kj}^* - t_{kj}^*e_{kj}) d\tau = \iint_A (\Delta f_k u_k^* - \Delta f_k^* u_k) dA \quad (9.48)$$

This result may be interpreted as expressing the power dissipated in the volume τ . In order to show this we consider two real variables z_1 and z_2 which are sinusoidal functions of time. They are represented by the rotating vectors $Z_1 \exp(i\alpha t)$ and $Z_2 \exp(i\alpha t)$, where Z_1 and Z_2 are complex amplitudes. Hence

$$\begin{aligned} z_1 &= \frac{1}{2}(Z_1 e^{i\alpha t} + Z_1^* e^{-i\alpha t}) \\ z_2 &= \frac{1}{2}(Z_2 e^{i\alpha t} + Z_2^* e^{-i\alpha t}) \end{aligned} \quad (9.49)$$

The product of these quantities averaged over one period is

$$\frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} z_1 z_2 dt = \frac{1}{4}(Z_1 Z_2^* + Z_2 Z_1^*) \quad (9.50)$$

Let us apply this result by putting

$$\begin{aligned} Z_{1k} &= \Delta f_k \\ Z_{2k} &= i\alpha u_k \end{aligned} \quad (9.51)$$

The last expression represents the velocity. Hence the power input per unit area at the boundary is

$$\frac{1}{4}(Z_{1k} Z_{2k}^* + Z_{2k} Z_{1k}^*) = -\frac{i\alpha}{4} (\Delta f_k u_k^* - \Delta f_k^* u_k) \quad (9.52)$$

We now multiply equation 9.48 by $-i\alpha/4$. The result is

$$-\frac{i\alpha}{4} \iiint_{\tau} (t_{kj}e_{kj}^* - t_{kj}^*e_{kj}) d\tau = -\frac{i\alpha}{4} \iint_A (\Delta f_k u_k^* - \Delta f_k^* u_k) dA \quad (9.53)$$

The right side of this equation represents the power input through the boundary; therefore the left side is the power dissipated in the volume τ .

If the viscoelastic operators are symmetric ($\hat{C}_{kj}^{\mu\nu} = \hat{C}_{\mu\nu}^{kj}$), we may apply equations 9.30 and 9.31. Then the power dissipation (9.53) becomes

$$-\frac{i\alpha}{4} \iiint_{\tau} (t_{kj}e_{kj}^* - t_{kj}^*e_{kj}) d\tau = \frac{1}{2}\alpha \iiint_{\tau} \mathcal{I}_{kj}^{\mu\nu} e_{kj}^* e_{\mu\nu} d\tau \quad (9.54)$$

The quantity $i\mathcal{I}_{kj}^{\mu\nu}$ represents the imaginary part of $\hat{C}_{kj}^{\mu\nu}$ for $p = i\alpha$.

Variational Principle and Lagrangian Equations.† In Chapters 2, 3, and 5 we have discussed several forms of variational principles for elastic media with initial stress. Actually the variational principles are a consequence of the more fundamental principle of virtual work expressed by equation 5.50 of Chapter 2. It is written

$$\begin{aligned} \iiint_{\tau} (t_{ij}\delta e_{ij} + S_{ij}\delta\eta_{ij}) d\tau \\ = \iiint_{\tau} \rho \Delta X_i \delta u_i d\tau + \iint_A \Delta f_i \delta u_i dA \end{aligned} \quad (9.55)$$

where

$$\eta_{ij} = \frac{1}{2}(e_{i\mu}\omega_{\mu j} + e_{j\mu}\omega_{\mu i} + \omega_{i\mu}\omega_{j\mu}) \quad (9.56)$$

This principle does not involve any material property, and it may be looked upon as an invariant formulation of the equilibrium conditions. As pointed out in Chapter 2, the principle may be used to obtain the equilibrium equations for the incremental stresses in a *curvilinear coordinate system*. It also leads to the formulation of the equilibrium equation in terms of generalized forces and generalized coordinates.

The virtual work principle (9.55) is applicable to dynamics by d'Alembert's principle. Let us consider the body force increment ΔX_i . We may write it as

$$\Delta X_i = - \frac{\partial^2 U}{\partial x_i \partial x_j} u_j - a_i \quad (9.57)$$

where U is a body force potential, and a_i is the particle acceleration. The surface integral in equation 9.55 represents the virtual work of the incremental boundary forces Δf_i . In order to simplify the writing we put

$$\begin{aligned} \Delta U &= \frac{1}{2} \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \\ \mathcal{K} &= \iiint_{\tau} (S_{ij}\eta_{ij} + \rho \Delta U) d\tau \end{aligned} \quad (9.58)$$

† The variational theory and its operational form presented in the latter part of this section are based essentially on the methods developed in the author's paper, *Variational and Lagrangian Methods in Viscoelasticity, Deformation and Flow of Solids* (IUTAM Colloquium, Madrid, 1955), pp. 251-263, Springer, Berlin, 1956.

Equation 9.55 becomes

$$\iiint_{\tau} t_{ij} \delta e_{ij} d\tau + \delta \mathcal{K} + \iiint_{\tau} \rho a_i \delta u_i d\tau = \iint_A \Delta f_i \delta u_i dA \quad (9.59)$$

We now express the displacement field u_i by means of generalized coordinates q_i . We write

$$u_k = u_{kj}(x) q_j \quad (9.60)$$

where the $u_{kj}(x)$ are given vector field functions of the spatial coordinates x_i . In terms of the variables q_j , the value of \mathcal{K} is of the form

$$\mathcal{K} = \frac{1}{2} k_{ij} q_i q_j \quad (9.61)$$

Also

$$\begin{aligned} e_{\mu\nu} &= E_{\mu\nu i} q_i \\ E_{\mu\nu i} &= \frac{1}{2} \left(\frac{\partial u_{\mu i}}{\partial x_\nu} + \frac{\partial u_{\nu i}}{\partial x_\mu} \right) \end{aligned} \quad (9.62)$$

The virtual work principle (9.59) may be formulated in terms of the generalized coordinates q_i and arbitrary variations δq_i . By substituting the field (9.60), the integral terms in equation 9.59 are written:

$$\begin{aligned} \iiint_{\tau} t_{ij} \delta e_{ij} d\tau &= \iiint_{\tau} t_{\mu\nu} E_{\mu\nu i} d\tau \delta q_i \\ \iiint_{\tau} \rho a_i \delta u_i d\tau &= m_{ij} \ddot{q}_j \delta q_i \\ \iint_A \Delta f_i \delta u_i dA &= Q_i \delta q_i \end{aligned} \quad (9.63)$$

Hence the variational principle (9.59) leads to the equations

$$\iiint_{\tau} t_{\mu\nu} E_{\mu\nu i} d\tau + k_{ij} q_j + m_{ij} \ddot{q}_j = Q_i \quad (9.64)$$

Let us examine the significance of these equations. The right side, Q_i , corresponds to the generalized force applied at the boundary. The coefficients m_{ij} are defined by equations 2.31 and 2.36 of Chapter 5. They may be obtained by writing the kinetic energy in the form

$$\mathcal{T} = \frac{1}{2} \iiint_{\tau} \rho \dot{u}_i \dot{u}_i d\tau = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \quad (9.65)$$

Up to this point no mention has been made of the particular physical properties of the material. The volume integral of $t_{\mu\nu}E_{\mu\nu t}$ represents a generalized stress. Equations 9.64 therefore are applicable to a very large category of materials. They may be *linear* or *non-linear*, and they may exhibit *plastic properties*.

For a linear material the stresses may be written

$$t_{ij} = \hat{C}_{ij}^{\mu\nu} e_{\mu\nu} \tag{9.66}$$

where $\hat{C}_{ij}^{\mu\nu}$ represents a linear operator. Substituting this value of t_{ij} into equations 9.64, we derive

$$\hat{\mathcal{L}}_{ij} q_j + k_{ij} q_j + m_{ij} \ddot{q}_j = Q_i \tag{9.67}$$

The operator $\hat{\mathcal{L}}_{ij}$ is defined by the integral:

$$\hat{\mathcal{L}}_{ij} = \iiint_{\tau} E_{kij} E_{\mu\nu t} \hat{C}_{kl}^{\mu\nu} d\tau \tag{9.68}$$

In the most general case this expression will be an integro-differential operator with respect to time. Equations 9.67 may therefore be considered integro-differential equations in Lagrangian form for a viscoelastic medium with initial stress.

The significance of the operators may be illustrated as follows. Let us consider equation 3.10. It provides the following interpretation of the fractional operator:

$$\frac{p}{p+r} z(t) = e^{-rt} \int_{\tau=0}^{\tau=t} e^{r\tau} \frac{dz}{d\tau} d\tau \tag{9.68a}$$

where z is a function of time. In abbreviated form equation 9.68a is written

$$\frac{p}{p+r} = e^{-rt} \int_{\tau=0}^{\tau=t} e^{r\tau} d \tag{9.68b}$$

When the viscoelastic medium obeys thermodynamic principles, the operator $\hat{C}_{ij}^{\mu\nu}$ is given by equation 3.32:

$$\hat{C}_{ij}^{\mu\nu} = \int_0^\infty \frac{p}{p+r} C_{ij}^{\mu\nu}(r) dr + C_{ij}^{\mu\nu} + p C'_{ij}{}^{\mu\nu} \tag{9.68c}$$

Putting

$$F_{ij}^{\mu\nu}(t) = \int_0^\infty e^{-rt} C_{ij}^{\mu\nu}(r) dr \tag{9.68d}$$

and taking into account equation 9.68b, we obtain the integro-differential operator:

$$\hat{C}_{ij}^{\mu\nu} = \int_{\tau=0}^{\tau=t} F_{ij}^{\mu\nu}(t-\tau) d + C_{ij}^{\mu\nu} + C'_{ij}{}^{\mu\nu} \frac{d}{dt} \tag{9.68e}$$

This operator has been derived for a material obeying thermodynamic

principles. In this case the function $F_{ij}^{\mu\nu}(t)$ is restricted to the particular form (9.68d), and the symmetry property $\hat{C}_{ij}^{\mu\nu} = \hat{C}_{\mu\nu}^{ij}$ is verified.

Obviously expression (9.68e) also represents a linear operator of a more general type which may be non-symmetric and for which the nature of the function $F_{ij}^{\mu\nu}(t)$ is not restricted to the form (9.68d). Note that the integral operator in expression (9.68e) must be interpreted as representing a Stieltjes integral.

Operational Form of the Variational Principle. Let us assume that the operators $\hat{C}_{ij}^{\mu\nu}$ are symmetric. Hence

$$\begin{aligned}\hat{C}_{ij}^{\mu\nu} &= \hat{C}_{\mu\nu}^{ij} \\ \hat{\mathcal{L}}_{ij} &= \hat{\mathcal{L}}_{ji}\end{aligned}\tag{9.69}$$

The second equation follows from the first because of the definition (9.68). Equations 9.67 may be obtained by a procedure which is formally identical with the variational theory of elasticity and may be considered an *extension of the principle of viscoelastic correspondence to variational methods*. We put

$$\begin{aligned}\hat{\mathcal{P}}_{\tau} &= \iiint_{\tau} (\Delta \hat{V} + \rho \Delta U) d\tau \\ \Delta \hat{V} &= \frac{1}{2} \hat{C}_{ij}^{\mu\nu} e_{ij} e_{\mu\nu} + S_{ij} \eta_{ij}\end{aligned}\tag{9.70}$$

As can be seen, these expressions are obtained from the elastic potential energy \mathcal{P}_{τ} given by equation 2.19 of Chapter 5 by replacing the elastic coefficients by the operators $\hat{C}_{ij}^{\mu\nu}$. Substituting the values (9.62) for $e_{\mu\nu}$, we obtain

$$\hat{\mathcal{P}}_{\tau} = \frac{1}{2} (\hat{\mathcal{L}}_{ij} + k_{ij}) q_i q_j\tag{9.71}$$

We also introduce the quadratic form (2.39) of Chapter 5, namely,

$$T = \frac{1}{2} \iiint_{\tau} \rho u_i u_i d\tau = \frac{1}{2} m_{ij} q_i q_j\tag{9.72}$$

We then write the variational principle

$$\delta(\hat{\mathcal{P}}_{\tau} + p^2 T) = Q_i \delta q_i\tag{9.73}$$

with the operator $p^2 = \partial^2/\partial t^2$. In performing the variation the operators are treated as constant algebraic quantities. The variational principle (9.73) leads to the equations

$$(\hat{\mathcal{L}}_{ij} + k_{ij} + p^2 m_{ij}) q_j = Q_i\tag{9.74}$$

These Lagrangian equations in operational form are equivalent to the integro-differential equations (9.67).

By this formal correspondence it is possible to extend immediately to viscoelasticity the variational equations derived for the elastic medium. In particular, we may readily extend to viscoelasticity the variational principle (5.52) of Chapter 3 for a medium in the presence of hydrostatic stress. This will be illustrated in the next section for the particular case of a viscous fluid.

Materials with Internal Resonance. It was stated that for a material which obeys the principles of linear thermodynamics the viscoelastic operators are expressed by equations 3.32. However, these expressions are based on the assumption that the inertia effects of the internal coordinates are negligible. When these effects are not negligible, the operators are of a more general type and contain terms which are representative of resonance phenomena. Nevertheless, the reciprocity relations (3.33) will still be verified. Such internal resonance is illustrated by the vibration of dislocation lines in a crystal or by the pulsation of small air bubbles in a liquid. From the thermodynamic viewpoint the complete system of external and internal degrees of freedom will, of course, continue to obey the dynamical equations (9.10) and exhibit the same general properties.

Curvilinear Coordinates. The equations of dynamic viscoelasticity may be expressed in orthogonal curvilinear coordinates by applying the variational principle (9.59), using the general expressions (5.54) and (5.55) derived in section 5 of Chapter 2 for the strain components and the rotation.

Reciprocity Properties. By definition $k_{ij} = k_{ji}$ and $m_{ij} = m_{ji}$. Hence, if the operators obey the condition of symmetry (9.69), the operational matrix in (9.67) is completely symmetric. As a consequence, the system governed by these equations exhibits reciprocity properties. This means that the response at point B due to a forcing function at point A remains the same when we interchange points A and B . A direct proof is also obtained from the second lemma (9.27). We denote by Δf_i and $\Delta f_i^{\bar{}}$ the forces applied, respectively, at points A and B of the surface. The corresponding displacement fields are u_i and \bar{u}_i . The forces are applied at the same frequency $\alpha/2r$; hence $p^2 = \bar{p}^2 = -\alpha^2$. Because of the symmetry assumption (9.69) we have $t_{ij}\bar{e}_{ij} = \bar{t}_{ij}e_{ij}$. Hence the left side of equation 9.27 vanishes, and the remaining surface integral must be zero. Therefore, if we denote by u'_i and \bar{u}'_i the displacements at points A and B ,

respectively, we obtain the relation $\Delta f \bar{u}'_i = \Delta \bar{f}'_i u'_i$, which expresses the reciprocity property. By Fourier representation of the variables we can see that the property is valid for transients.

The theorem obviously applies to the elastic medium under initial stress. Another particular case is that of a viscoelastic medium initially stress-free. The reciprocity property in this case is implicit in the results established by the author in 1955 in a paper† on variational principles in isotropic and anisotropic viscoelasticity.

Mechanics of Porous Media with Initial Stress. General equations for the mechanics of porous media with initial stress were derived by the author.‡ The solid matrix is elastic, and the pores are filled with a viscous fluid. Problems of consolidation and acoustic propagation taking into account the initial state of stress in the earth are important in many fields such as foundation engineering, hydrology, and petroleum geology. On the other hand, the stability theory of multilayered porous media is of interest in geological problems of tectonic folding. In order to bring out the significant physical features involved in folding instability of porous media the buckling of a porous slab, free or embedded, has been the object of a simplified analysis.§

Viscoelastic Correspondence. Application of the author's correspondence principle (see section 3 of this chapter) to porous viscoelastic media was discussed in several papers¶ dealing with problems of consolidation and acoustic propagation. The medium in this case is constituted of a solid matrix of viscoelastic material whose pores are filled with a viscous fluid. The correspondence principle was also extended to a porous viscoelastic medium under initial stress‡ leading to the theory of acoustic propagation and consolidation for this more general case.

† See reference on page 453.

‡ M. A. Biot, Theory of Stability and Consolidation of a Porous Medium under Initial Stress, *Journal of Mathematics and Mechanics*, Vol. 12, No. 4, pp. 521–542, 1963.

§ M. A. Biot, Theory of Buckling of a Porous Slab and its Thermoelastic Analogy, *Journal of Applied Mechanics*, Ser. E, Vol. 31, No. 2, pp. 194–198, 1964.

¶ M. A. Biot, Theory of Deformation of a Porous Viscoelastic Anisotropic Solid, *Journal of Applied Physics*, Vol. 27, No. 5, pp. 459–467, 1956; Mechanics of Deformation and Acoustic Propagation in Porous Media, *ibid.*, Vol. 33, No. 4, pp. 1482–1498, 1962; Generalized Theory of Acoustic Propagation in Porous Dissipative Media, *The Journal of the Acoustical Society of America*, Vol. 34, No. 5, Part I, pp. 1254–1264, 1962.

Thermoelastic Analogy. The mechanics of a thermoelastic medium and that of a porous medium containing a massless fluid were shown by the author to be formally identical.† Solutions derived for porous media are applicable to thermoelasticity by a simple change of notation. The temperature plays the role of the pore pressure. The same analogy is applicable to the theory of acoustic propagation in a thermoelastic medium with initial stress.‡ In stability problems this analogy leads to a distinction between *isothermal* and *adiabatic buckling* as illustrated on the particular problem of a slab.‡ The analogy is more than a formal identity and results from deeper underlying principles of non-equilibrium thermodynamics.

10. INSTABILITY AND SMALL MOTION DYNAMICS OF A VISCOUS FLUID IN A GRAVITY FIELD

In this section we shall consider the particular case of a non-homogeneous viscous fluid in a gravity field. In the initial equilibrium state the density is a function of the coordinates, and the initial stress is purely hydrostatic.

Viscoelastic correspondence is rigorously applicable because the medium is at rest in the initial state. Hence we may apply the general solutions derived for the corresponding case with purely elastic properties. For the same reason the various solutions derived for viscous fluids are applicable to isotropic elastic and viscoelastic media under initial hydrostatic stress. This point is discussed in more detail in the last paragraph of this section.

We begin with problems of stability for layered incompressible fluids, and special attention is given to the case of large viscosity for which inertia forces may be neglected. In the latter part of this section we shall consider the general small motion dynamics of a compressible viscous fluid in a gravity field.

For the incompressible fluid the effect of gravity is conveniently introduced by using the concept of analog model. When the fluid is constituted of a superposition of homogeneous horizontal layers, the gravity field is replaced by buoyancy forces applied at the

† M. A. Biot, Thermoelasticity and Irreversible Thermodynamics, *Journal of Applied Physics*, Vol. 27, No. 3, pp. 240–253, 1956.

‡ See page 458, references † and §.

interfaces and proportional to the vertical displacement. The incremental stress-strain relations for an incompressible viscous fluid of viscosity η are written in operational form as follows:

$$\begin{aligned} s_{11} - s &= 2\eta p e_{xx} \\ s_{22} - s &= 2\eta p e_{yy} \\ s_{12} &= 2\eta p e_{xy} \end{aligned} \quad (10.1)$$

Hence the operators \hat{N} and \hat{Q} become

$$\hat{N} = \hat{Q} = \eta p \quad (10.2)$$

The initial stress being hydrostatic, we put

$$P = S_{11} - S_{22} = 0 \quad (10.3)$$

As a consequence we may write

$$\hat{M} = \hat{N} = \eta p \quad \hat{L} = \hat{Q} = \eta p \quad (10.4)$$

All equations derived previously for incompressible elastic media are immediately applicable to incompressible viscous fluids if we put $P = 0$ and substitute the values (10.2) for the operators. Then, if we assume the inertia forces to be negligible, the six basic coefficients a_{ij} and b_{ij} of the mechanics of layered media are given by the limiting values (5.36) and (5.37) of Chapter 4. They are

$$\begin{aligned} a_{11} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma + 2\gamma} \\ a_{12} &= -\frac{4\gamma}{\sinh 2\gamma + 2\gamma} \\ a_{22} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma + 2\gamma} \end{aligned} \quad (10.5)$$

and

$$\begin{aligned} b_{11} &= \frac{4 \sinh^2 \gamma}{\sinh 2\gamma - 2\gamma} \\ b_{12} &= \frac{4\gamma}{\sinh 2\gamma - 2\gamma} \\ b_{22} &= \frac{4 \cosh^2 \gamma}{\sinh 2\gamma - 2\gamma} \end{aligned} \quad (10.6)$$

They are functions only of

$$\gamma = \frac{1}{2}lh = \frac{\pi h}{\mathcal{L}} \tag{10.7}$$

where h is the layer thickness and \mathcal{L} is the wavelength.

The coefficients (10.5) and (10.6) may also be derived more directly from the Navier-Stokes equations (5.57) for the motion of an incompressible viscous fluid when inertia terms are neglected.

We discuss first two examples for incompressible fluids which involve the stability of a single embedded layer. In these examples we shall assume the medium to be sufficiently viscous so that inertia forces may be neglected. For a layer embedded below a medium of higher density, an instability arises with the appearance of a wavy interface between the two media. Such problems are of interest in geophysics (see geology of salt structures, page 470).

Gravity Instability of an Embedded Fluid Layer.† *Example I.*

In this example we consider a fluid layer of thickness h , viscosity η , and density ρ resting on top of a horizontal rigid base (Fig. 10.1). It is surmounted by a fluid of infinite thickness of viscosity η' and

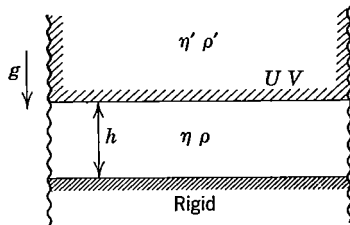


Figure 10.1 Viscous layer on top of a rigid base surmounted by an infinite viscous medium in a gravity field.

density ρ' . Perfect adherence is assumed at the interface. Therefore the displacements vanish on the bottom side of the layer. Hence the problem may be formulated entirely in terms of the two displacement components U and V at the top of the layer.

The stresses τ' and q' in the upper infinite fluid at the interface are given by applying equations (4.35) of Chapter 4. We must replace

† These two examples were treated in a recent paper by the author: Stability of Multilayered Continua Including the Effect of Gravity and Viscoelasticity, *Journal of the Franklin Institute*, Vol. 276, No. 3, pp. 231-252, 1963.

the elastic modulus μ by $\eta'p$ and change the sign of the stresses. They become

$$\begin{aligned}\tau' &= -2l\eta'pU \\ q' &= -2l\eta'pV\end{aligned}\quad (10.8)$$

The change in sign is required because the equations are applied here to the upper half-space instead of the lower half-space.

The stresses τ and q on top of the layer are obtained from the first two of equations 5.20 of Chapter 4. In those equations we put

$$\begin{aligned}U &= U_1 & V &= V_1 \\ U_2 &= V_2 = 0\end{aligned}\quad (10.9)$$

and

$$L = \eta p \quad (10.10)$$

Hence

$$\begin{aligned}\tau &= l\eta p(A_1U + A_2V) \\ q &= l\eta p(A_2U + A_3V)\end{aligned}\quad (10.11)$$

with

$$\begin{aligned}A_1 &= \frac{1}{2}(a_{11} + b_{11}) \\ A_2 &= \frac{1}{2}(a_{12} + b_{12}) \\ A_3 &= \frac{1}{2}(a_{22} + b_{22})\end{aligned}\quad (10.12)$$

Since inertia forces are neglected, these coefficients are obtained by introducing the values (10.5) and (10.6). They are functions of the variable γ which embodies the effect of layer thickness and the wavelength.

We now introduce the effect of gravity by using the analog model. This is done by applying a vertical force $(\rho - \rho')gV$ per unit area at the interface. Hence

$$\begin{aligned}\tau' &= \tau \\ q' &= q + (\rho - \rho')gV\end{aligned}\quad (10.13)$$

The gravity acceleration is denoted by g . These equations are also derived by applying equation 7.25 of Chapter 4.

Substituting the stresses (10.8) and (10.11) into equations 10.13, we obtain

$$\begin{aligned}(2 + \kappa A_1)U + \kappa A_2V &= 0 \\ \kappa A_2U + \left(2 + \kappa A_3 - \frac{\sigma}{2\gamma}\right)V &= 0\end{aligned}\quad (10.14)$$

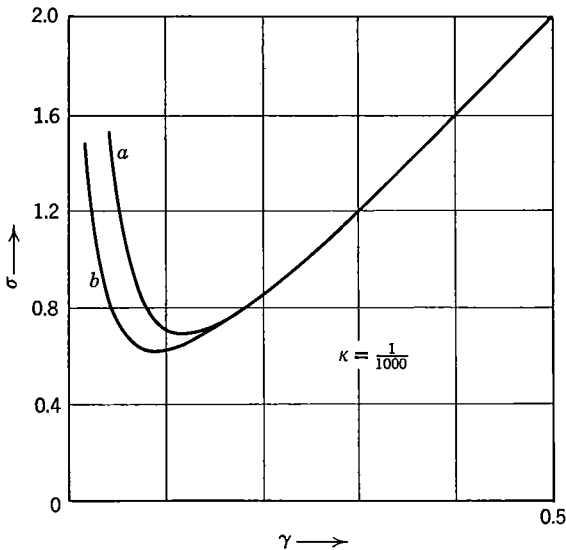


Figure 10.2 Parameter σ representing gravity instability as a function of γ for a viscosity ratio $\kappa = 1/1000$. (Curve a for medium of Figure 10.1; curve b for medium of Figure 10.3.)

The parameters are defined as

$$\kappa = \frac{\eta}{\eta'} \quad \sigma = \frac{(\rho' - \rho)gh}{\eta'p} \quad (10.15)$$

Putting equal to zero the determinant of equations 10.14, we derive the characteristic equation

$$\sigma = 2\gamma \left[2 + \kappa A_3 - \frac{\kappa^2 A_2^2}{2 + \kappa A_1} \right] \quad (10.16)$$

The parameter σ has been evaluated numerically as a function of γ for the following value of the ratio of viscosities:

$$\kappa = \frac{\eta}{\eta'} = \frac{1}{1000} \quad (10.17)$$

The value of σ is plotted in Figure 10.2 (curve a). If the top layer has the larger density ($\rho' > \rho$), a positive value of σ yields a positive value of the characteristic exponent p . Hence the interface is unstable. The minimum σ_{\min} of σ corresponds to a maximum of p

and occurs for $\gamma = \gamma_a$. Hence the dominant wavelength is

$$\mathcal{L}_a = \frac{\pi h}{\gamma_a} \tag{10.18}$$

The values of σ_{\min} , γ_a , and \mathcal{L}_a/h have also been evaluated as functions of the viscosity ratio κ . They are shown in Table 3.

Table 3

Dominant wavelength \mathcal{L}_a as a function of the ratio of viscosities $\kappa = \eta/\eta'$ for the layer represented in Fig. 10.1.

$\kappa = \eta/\eta'$	σ_{\min}	γ_a	\mathcal{L}_a/h
1/1000	0.687	0.114	27.6
1/100	1.49	0.243	12.9
1/10	3.36	0.495	6.35

Example II. As another example we consider a fluid layer of thickness h , density ρ , and viscosity η embedded horizontally in a fluid of infinite extent of density ρ' and viscosity η' . Here the rigid base has been replaced by the same fluid as in the surmounting medium (Fig. 10.3).

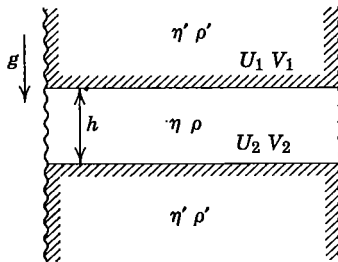


Figure 10.3 Viscous layer embedded in an infinite viscous medium in a gravity field.

There are now two interfaces, and we must consider the displacements U_1, V_1 at the top of the layer and the displacements U_2, V_2 at the bottom interface. The equations are obtained, as previously, by considering the analog model and replacing the effect of gravity by forces applied to the interface. In the present case this is done by

applying equations 5.20 of Chapter 4. The following equations are obtained.

$$\begin{bmatrix} -\frac{2}{\kappa} U_1 \\ \left(-\frac{2}{\kappa} + \frac{\sigma}{2\gamma\kappa}\right) V_1 \\ \frac{2}{\kappa} U_2 \\ \left(\frac{2}{\kappa} + \frac{\sigma}{2\gamma\kappa}\right) V_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & -A_4 & A_5 \\ A_2 & A_3 & -A_5 & A_6 \\ A_4 & A_5 & -A_1 & A_2 \\ -A_5 & -A_6 & A_2 & -A_3 \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix} \quad (10.19)$$

The coefficients A_1, A_2, A_3 are given by equations 10.12. The other three are

$$\begin{aligned} A_4 &= \frac{1}{2}(a_{11} - b_{11}) \\ A_5 &= \frac{1}{2}(a_{12} - b_{12}) \\ A_6 &= \frac{1}{2}(a_{22} - b_{22}) \end{aligned} \quad (10.20)$$

These six matrix elements A_1, A_2, \dots , etc., are functions of the wavelength parameter γ through expressions (10.5) and (10.6).

Examination of equations 10.19 shows that they remain unchanged if U_1, V_1, U_2, V_2 are replaced by $U_2, -V_2, U_1, -V_1$ and if at the same time σ is replaced by $-\sigma$. Hence, when we equate to zero the determinant of the system of equations (10.19), we must obtain a characteristic equation which contains only σ^2 . For $\rho' > \rho$ the positive root σ corresponds to an instability. It is plotted as a function of γ for $\kappa = 1/1000$ and is shown by curve *b* in Figure 10.2. It is seen that the result does not differ greatly from that obtained when the base is rigid.

In this second example the layer is always unstable if the densities are different. The amplitudes of the vertical displacements will be larger at the top or the bottom, depending on which material is the denser.

Gravity Instability of Multilayered Viscous Fluids. The problem of gravity instability of a multilayered incompressible viscous fluid may be handled as a particular application of the general methods derived previously for elastic and viscoelastic multilayered structures. Each layer is homogeneous, of viscosity η_j ,

density ρ_j , and thickness h_j . We assume that there are n layers numbered as shown in Figure 10.4. The layers are resting on a half-space which may be considered a layer of infinite thickness numbered $n + 1$.

Because the fluid is incompressible, we may apply the concept of analog model and replace the effect of the gravity field by interfacial and surface forces proportional to the vertical displacements and density differences.

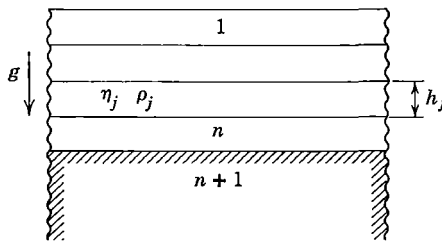


Figure 10.4 Multilayered viscous fluid in a gravity field.

Let us assume first that the inertia forces are negligible. Then the basic coefficients a_{ij} and b_{ij} are given by equations 10.5 and 10.6. For the j th fluid layer we must replace γ by $\gamma_j = \frac{1}{2}l h_j$. Through equations 10.12 and 10.20 we derive the six coefficients ${}_j A_1, {}_j A_2$, etc., of the j th layer. It is convenient to use the quadratic form (7.1) of Chapter 4, which reads (no summation)

$$\begin{aligned}
 I_j = & \frac{1}{2} {}_j A_1 (U_j^2 + U_{j+1}^2) - {}_j A_4 U_j U_{j+1} \\
 & + \frac{1}{2} {}_j A_3 (V_j^2 + V_{j+1}^2) + {}_j A_6 V_j V_{j+1} \\
 & + {}_j A_2 (U_j V_j - U_{j+1} V_{j+1}) + {}_j A_5 (U_j V_{j+1} - U_{j+1} V_j) \quad (10.21)
 \end{aligned}$$

where U_j and V_j are the horizontal and vertical displacements at the top of the j th layer. We also write the composite quadratic form (7.10) of Chapter 4:

$$\mathcal{I} = p \sum_{j=1}^{n+1} n_j I_j \quad (10.22)$$

By the correspondence principle we have replaced the slide modulus L_j by $p\eta_j$, according to equation 10.4. Finally we apply equations 7.30 of Chapter 4. This yields for the interfacial displacements the

recurrence equations

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \quad \frac{\partial}{\partial V_j} (\mathcal{J} + \mathcal{G}) = 0 \quad (10.23)$$

with

$$\mathcal{G} = \frac{1}{2l} \sum_{j=1}^{n+1} (\rho_j - \rho_{j-1}) g V_j^2$$

For $j = 1$ we must put $\rho_0 = 0$. Thus we have obtained $2n + 2$ equations for the $2n + 2$ unknown displacements U_j and V_j . Positive solutions for the characteristic exponent p correspond to modes of instability.

We note that equations 10.23 are equivalent to the variational principle

$$\delta(\mathcal{J} + \mathcal{G}) = 0 \quad (10.24)$$

which is a particular application of the more general variational principles discussed below.

When the layers are resting on a rigid base, the number of unknowns is reduced to $2n$ if we put $U_{n+1} = V_{n+1} = 0$.

Since inertia forces are neglected, the general theorem of section 4 is applicable. Hence we need only look for real characteristic values. The amplitudes are always proportional to real exponential factors. They are decaying or increasing exponentials, depending on the stability of the particular characteristic solution.

We note that equations 10.23 also represent a particular case of the general solution obtained in section 8 for a viscous fluid under initial stress. For a fluid initially at rest, equations 8.75 and 10.23 are identical.

Solution by Matrix Multiplication. For a large number of layers the instability may be evaluated numerically by the matrix multiplication procedure represented by equation 7.35 of Chapter 4. This procedure is well suited for digital programming. The characteristic values of p are obtained as outlined for vibration problems of elastic media (see Chapter 5, section 7).

The matrix \mathcal{N} to be applied is given by equation 7.33 of Chapter 4. In the coefficients we must replace L by the operator ηp . Since we are dealing with a viscous fluid, the coefficients B_1, B_2, \dots, B_{10} may be obtained in a simpler form by substituting the values of a_{ij} and

b_{ij} from equations 10.5 and 10.6 into equations 7.16 of Chapter 4. We find

$$\begin{aligned}
 B_1 &= 2\gamma \sinh 2\gamma + \cosh 2\gamma \\
 B_2 &= 2\gamma \cosh 2\gamma \\
 B_3 &= -2\gamma \cosh 2\gamma \\
 B_4 &= \cosh 2\gamma - 2\gamma \sinh 2\gamma \\
 B_5 &= 2(\sinh 2\gamma + 2\gamma \cosh 2\gamma) \\
 B_6 &= 4\gamma \sinh 2\gamma \\
 B_7 &= 2(\sinh 2\gamma - 2\gamma \cosh 2\gamma) \\
 B_8 &= \frac{1}{2}(\sinh 2\gamma + 2\gamma \cosh 2\gamma) \\
 B_9 &= \gamma \sinh 2\gamma \\
 B_{10} &= \frac{1}{2}(\sinh 2\gamma - 2\gamma \cosh 2\gamma)
 \end{aligned} \tag{10.25}$$

Solution by Iteration. The recurrence equations 10.23 relate the six displacements at three successive interfaces. Their numerical solution may be also obtained by the iteration process already discussed in Chapter 5 (page 332).

Dynamic Instability. The problem of dynamic instability of an incompressible multilayered viscous fluid when inertia forces are taken into account may be solved in a similar way by applying the results obtained for the dynamics of multilayered elastic media. As illustrated by equation 7.44 of Chapter 5, for an incompressible medium the coefficients a_{ij} and b_{ij} expressed in terms of the roots β_1 and β_2 in dynamic instability are formally the same as for the static problem of buckling. When inertia terms are added, they affect only the values of

$$\begin{aligned}
 \beta_1 &= \sqrt{m + \sqrt{m^2 - k^2}} \\
 \beta_2 &= \sqrt{m - \sqrt{m^2 - k^2}}
 \end{aligned} \tag{10.26}$$

In expressions (7.42) of Chapter 5 for m and k^2 we replace α^2 by $-p^2$, and the elastic coefficients by the operators

$$M = N = L = \eta p \tag{10.27}$$

Since the initial stress is hydrostatic, we also put $P = 0$. We derive

$$\begin{aligned}
 \beta_1 &= \sqrt{1 + \frac{p\rho}{\eta l^2}} \\
 \beta_2 &= 1
 \end{aligned} \tag{10.28}$$

The coefficients a_{ij} and b_{ij} are obtained by substituting these values of β_1 and β_2 into equations 5.8 and 5.16 of Chapter 4.

With these values of a_{ij} and b_{ij} , the dynamic stability problem is formulated by the same equations as in the previous non-dynamical problems with anisotropy. We may use either the recurrence equations (10.23) or the matrix multiplication process.

Since we are interested in unstable solutions, we apply the general theorem of section 9 which indicates that such solutions must be real. Hence we need only consider real positive values of the characteristic exponent p . The values (10.28) of β_1 and β_2 are also real and positive in this case.

Gravity Instability of Anisotropic Fluids. The correspondence principle may be applied to problems of gravity instability of anisotropic fluids. For an incompressible fluid the stress-strain relations are

$$\begin{aligned} s_{11} - s &= 2p\eta_n e_{xx} \\ s_{22} - s &= 2p\eta_n e_{yy} \\ s_{12} &= 2p\eta_t e_{xy} \end{aligned} \tag{10.29}$$

where η_t and η_n are the viscosity coefficients for tangential and normal stresses. According to equations 10.29, the operators are

$$\begin{aligned} \hat{Q} &= p\eta_t \\ \hat{N} &= p\eta_n \end{aligned} \tag{10.30}$$

They coincide with expression (8.60b). Since the initial stress is hydrostatic, we put $P = 0$. Hence $\hat{L} = \hat{Q}$ and $\hat{M} = \hat{N}$. With these operators the equations derived in Chapter 4 (section 7) for the elastic medium are immediately applicable to problems of gravity instability of anisotropic fluids.

As an example, let us consider two anisotropic fluids adhering at a horizontal interface. We assume the viscous property to be symmetric with respect to vertical and horizontal directions. For the lower medium the density is denoted by ρ , and the two viscosity coefficients are η_t and η_n . The corresponding quantities for the upper medium are ρ' , η'_t , and η'_n . Applying equation 8.60c, we write for the vertical stresses in the lower and upper medium at the interface

$$\begin{aligned} q &= 2l\sqrt{\eta_t\eta_n} pV \\ q' &= -2l\sqrt{\eta'_t\eta'_n} pV \end{aligned} \tag{10.31}$$

The negative sign is required for the upper half-space, as already pointed out (see equations 5.29 and 6.27 of Chapter 4). In the analog model the effect of gravity is obtained by applying interfacial buoyancy forces. In accordance with equation 7.25 of Chapter 4, this is equivalent to requiring the continuity of the quantity

$$q + \rho g V = q' + \rho' g V \quad (10.32)$$

Combining equations 10.31 and 10.32, we derive

$$p = \frac{(\rho' - \rho)g}{2l(\sqrt{\eta_t \eta_n} + \sqrt{\eta'_t \eta'_n})} \quad (10.33)$$

This value of p gives a measure of the instability as a function of the wavelength. Interfacial disturbances of wavelength $\mathcal{L} = 2\pi/l$ grow proportionally to the factor $\exp(pt)$.

Problems of instability of multilayered viscous media may be conveniently approximated by substituting an equivalent anisotropic fluid. The equivalent viscosities η_t and η_n are given by equations 3.49. The approximation is valid for a thinly laminated medium or, in general, when the wavelength is large compared to the thickness of the layers.

Application to the Geology of Salt Structures and Isostatic Compaction. Problems of gravity instability of stratified fluids are of considerable interest in many geological problems. Because of the geological time scale, rock structures may be assumed to behave approximately like viscous fluids with very high viscosity. Since the rate of deformation is very small, inertia forces do not enter into the picture. Gravity instability occurs when a rock formation of higher density lies over a layer of salt. An instability arises at the interface, and, as shown by the earlier examples treated in this section, the deformation of the interface exhibits a dominant wavelength. Actually, of course, the structure is the result of gradual sedimentation and compaction. The thickness and density of the overlying material vary with time. A solution of this more complex problem has been obtained[†] by applying the variational principle (10.24). This variational principle is also discussed further in the next paragraph in the more general context of dynamics and compressible fluids.

In the foregoing analysis the emphasis has been put on the evaluation of unstable solutions. However, the methods are equally applicable to layered viscous media with stable configurations. When inertia forces are negligible, the characteristic solutions are real and proportional to decaying exponential

[†] M. A. Biot and H. Odé, Theory of Gravity Instability with Variable Overburden and Compaction (*Geophysics*, Vol. 30, in press).

factors. Such solutions represent a gradual return to equilibrium when disturbing loads are removed, and they correspond to solutions of geophysical problems of *isostatic compensation*. For example, a gradual uplift will result from the load removal due to the melting of the ice-cap. Problems of this type have been treated on the basis of viscous flow theory.† The present results given here provide a systematic procedure applicable to a large number of layers. Extension to viscoelastic media is also readily obtained, as indicated in the last paragraph of this section. It should be pointed out that a structure with these stratifications may also be approximated by an anisotropic fluid. In this case problems of gravity instability or isostatic compensation may be treated as illustrated in the preceding paragraph.

Small Motion Dynamics of a Viscous Fluid in a Gravity Field. In section 9 we have considered the general dynamics of a viscoelastic medium under initial stress. In this paragraph we shall discuss the particular case of a non-homogeneous viscous fluid initially in equilibrium in a gravity field; it includes inertia forces and compressibility.

The dynamical equations may be derived by combining the results obtained in Chapters 3 and 5 for the solid and the fluid in the presence of hydrostatic stress.

Inserting the initial hydrostatic stress

$$S_{ij} = S\delta_{ij} \tag{10.34}$$

and the potential field U into the dynamical equations (2.9) of Chapter 5, we obtain

$$\frac{\partial s'_{ij}}{\partial x_j} + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{10.35}$$

Analogous to equations 5.19 of Chapter 5, these equations are in the “unmodified form.” A “modified form” of these equations is obtained by following the procedure used in previous chapters (see equations 5.23 of Chapter 3, and Chapter 5). The modified equations are

$$\frac{\partial s'_{ij}}{\partial x_j} - \rho e X_i - X_j \frac{\partial \rho}{\partial x_i} u_j = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{10.36}$$

with

$$s'_{ij} = s_{ij} + \rho u_k X_k \delta_{ij} \tag{10.37}$$

† N. A. Haskell, Motion of a Viscous Fluid under a Surface Load, *Physics*, Vol. 6, pp. 265–269, 1935, and Vol. 7, pp. 56–61, 1936; W. H. Heiskanen and F. A. Vening-Meinesz, *The Earth and Its Gravity Field*, McGraw-Hill Book Co., New York, 1958.

The incremental stress s_{ij} appearing in these equations is given by the stress-strain relations which govern a compressible fluid of Newtonian viscosity. Hence

$$s_{ij} = 2\eta \frac{\partial}{\partial t} (e_{ij} - \frac{1}{3}e\delta_{ij}) + \lambda e\delta_{ij} \quad (10.38)$$

The viscosity coefficient is denoted by η , and λ represents the incremental bulk modulus.

The equations of motion are obtained by substituting the values (10.38) of s_{ij} into the dynamical equations (10.35) or (10.36). In a constant gravity field of intensity g using a vertical z axis positive upward, equations 10.35 become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} - \rho g \frac{\partial w}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - \rho g \frac{\partial w}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y} + \frac{\partial s_{33}}{\partial z} - \rho g \frac{\partial w}{\partial z} + \rho g e &= \rho \frac{\partial^2 w}{\partial t^2} \end{aligned} \quad (10.39)$$

The cartesian displacements of the fluid are denoted by u , v , w . For vanishing viscosity ($\eta = 0$) the equations coincide with those obtained for acoustic-gravity waves in a perfect fluid (section 5, Chapter 5).

Variational principles may also be formulated as a particular application of the more general theory developed in the preceding section for viscoelasticity.

We consider the fluid to be bounded by surfaces which are either free of stress or rigid with perfect adherence of the fluid. We may write

$$Q_i \delta q_i = \iint_A \Delta f_i \delta u_i dA = 0 \quad (10.40)$$

The variational principle (9.73) becomes

$$\delta(\hat{\mathcal{P}} + p^2 T) = 0 \quad (10.41)$$

We may put $\mathcal{P} = \mathcal{P}_i$ here because the contribution \mathcal{P}_B of the boundary vanishes (see equation 9.42).

Let us introduce explicitly into these equations the properties of the fluid. From equation 2.22 of Chapter 2 the stress t_{ij} is

$$t_{ij} = s_{ij} + S_{ij}e - \frac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}) \quad (10.42)$$

With the initial hydrostatic stress (10.34) this expression becomes

$$t_{ij} = s_{ij} + S e \delta_{ij} - S e_{ij} \quad (10.43)$$

For the elastic medium, equation 2.12 of Chapter 5 yields

$$\Delta V = \frac{1}{2} t_{ij} e_{ij} + S_{ij} \eta_{ij} \quad (10.44)$$

where η_{ij} denotes expression (9.56). Substituting expressions (10.34) and (10.43) for the fluid, we obtain

$$\Delta V = \frac{1}{2} s_{ij} e_{ij} + \mathcal{R} \quad (10.45)$$

with

$$\mathcal{R} = \frac{1}{2} S \left(e^2 - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) \quad (10.46)$$

In operational form the stress-strain relations (10.38) for a viscous fluid are

$$s_{ij} = 2\eta p(e_{ij} - \frac{1}{3}e\delta_{ij}) + \lambda e\delta_{ij} \quad (10.47)$$

With these values for s_{ij} , expression (10.45) for ΔV also becomes operational

$$\Delta \hat{V} = \frac{1}{2} \lambda e^2 + \mathcal{R} + \eta p(e_{ij} e_{ij} - \frac{1}{3}e^2) \quad (10.48)$$

Hence we may write

$$\hat{\mathcal{P}} = \iiint_{\tau} (\Delta \hat{V} + \rho \Delta U) d\tau \quad (10.49)$$

If we put

$$\begin{aligned} \hat{\mathcal{P}} &= \mathcal{P} + pD \\ \mathcal{P} &= \iiint_{\tau} \left(\frac{1}{2} \lambda e^2 + \mathcal{R} + \rho \Delta U \right) d\tau \end{aligned} \quad (10.50)$$

$$D = \iiint_{\tau} \eta (e_{ij} e_{ij} - \frac{1}{3} e^2) d\tau$$

the variational principle (10.41) becomes

$$\delta(\mathcal{P} + pD + p^2 T) = 0 \quad (10.51)$$

The term pD represents a dissipation function in operational form, and \mathcal{P} is the potential energy.

We conclude that the variational principle (10.51) is obtained by adding a dissipation function to the variational equation derived previously for a perfect fluid (section 6, Chapter 5).

As before, it is possible to obtain a modified form of the principle which introduces the buoyancy forces. For this purpose we go back to equation 6.72 of Chapter 5. It is written

$$\begin{aligned} \delta(\mathcal{R} + \rho \Delta U) - \frac{\partial}{\partial x_i} (S e \delta u_i) + \frac{\partial}{\partial x_j} \left(S \frac{\partial u_j}{\partial x_i} \delta u_i \right) \\ = \delta \mathcal{Y} - \frac{\partial}{\partial x_i} (\rho X_j u_j \delta u_i) \end{aligned} \quad (10.52)$$

with

$$\mathcal{Y} = \rho X_j u_j e + \frac{1}{2} X_j \frac{\partial \rho}{\partial x_i} u_i u_j \quad (10.53)$$

After adding $\frac{1}{2} \lambda e^2$ to both sides, we integrate equation 10.52 over the volume τ of the fluid and take into account that $S = 0$ at a free boundary while $\delta u_i = 0$ at a solid boundary. We obtain

$$\delta \mathcal{P} = \delta(\mathcal{W}_\tau + \mathcal{W}_F) \quad (10.54)$$

with

$$\begin{aligned} \mathcal{W}_\tau &= \iiint_\tau \left(\frac{1}{2} \lambda e^2 + \rho X_j u_j e + \frac{1}{2} X_j \frac{\partial \rho}{\partial x_i} u_i u_j \right) d\tau \\ \mathcal{W}_F &= -\frac{1}{2} \iint_F \rho X u_n^2 dA \end{aligned} \quad (10.55)$$

The surface integral \mathcal{W}_F is the potential energy of the free surface F , as already expressed in equation 6.12 of Chapter 5. The quantities X and u_n are the algebraic components of the body force and displacement along the outward normal to the free surface. When the value

$$\mathcal{P} = \mathcal{W}_\tau + \mathcal{W}_F \quad (10.56)$$

is substituted in equation 10.51, the variational principle is expressed in terms of buoyancy forces. This result could also have been derived by viscoelastic correspondence from equation 5.52 of Chapter 3, which expresses a general variational principle for a solid in the presence of hydrostatic stress.

The displacement field may be represented by generalized coordinates q_i in accordance with equation 9.60. The variational principle (10.51) then leads to the following Lagrangian equations in operational form:

$$\frac{\partial \mathcal{P}}{\partial q_i} + p \frac{\partial D}{\partial q_i} + p^2 \frac{\partial T}{\partial q_i} = 0 \quad (10.57)$$

With a dissipation function

$$\mathcal{D} = \iiint_{\tau} \eta(\dot{e}_{ij}\dot{e}_{ij} - \frac{1}{3}\dot{e}^2) d\tau \tag{10.58}$$

and the kinetic energy

$$\mathcal{T} = \frac{1}{2} \iiint_{\tau} \rho \dot{u}_i \dot{u}_i d\tau \tag{10.59}$$

equations 10.57 are equivalent to

$$\frac{\partial \mathcal{P}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} + \frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) = 0 \tag{10.60}$$

They are the same as equations 9.8 for a dynamical system with a potential energy and a dissipation function. The general properties of such systems therefore apply to the viscous fluid in a gravity field. In particular, *unstable characteristic solutions are always real*. We note that the rate of energy dissipation in the fluid is

$$\frac{\partial \mathcal{D}}{\partial \dot{q}_i} \dot{q}_i = 2\mathcal{D} \tag{10.61}$$

For an incompressible fluid ($e = 0$) and a vertical gravity field of z component equal to $-g$, the formulation is considerably simplified. The potential energy (10.56) becomes

$$\mathcal{P} = -\frac{1}{2}g \iiint_{\tau} w^2 \frac{d\rho}{dz} d\tau + \frac{1}{2}g \iint_F \rho w^2 dx dy \tag{10.62}$$

where w is the vertical displacement. The dissipation function is

$$\mathcal{D} = \iiint_{\tau} \eta \dot{e}_{ij}\dot{e}_{ij} d\tau \tag{10.63}$$

For further discussion the same procedure as in section 6 of Chapter 5 for a frictionless fluid may be followed.

It should also be noted that the *analog model* discussed in section 5 of Chapter 5 is directly applicable to dynamics for a viscous incompressible fluid.

Generalized Analog Model for Large Deformations. The analog model for incompressible viscous fluids is valid for large deformations. That this is the case may be derived from the following equations of fluid dynamics:

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} - \rho \frac{\partial U}{\partial x_i} &= \rho a_i \\ \sigma_{ij} - \sigma \delta_{ij} &= \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \end{aligned} \tag{10.63a}$$

At a fixed point in space the variables are the stress σ_{ij} , the pressure $-\sigma$, the acceleration a_i , the velocity v_i , the density ρ , and the viscosity η . The body force potential U is a given function of the coordinates. The fluid is heterogeneous with values of ρ and η depending on the time at a fixed point. In order to solve the dynamical problem we must add to equations 10.63a certain kinematic relations for the velocity field expressing the acceleration, the condition of incompressibility, and the transport of the values of ρ and η attached to the fluid particles. They have not been written because they are not relevant to the argument. We put

$$\begin{aligned}\sigma'_{ij} &= \sigma_{ij} - \rho U \delta_{ij} \\ \sigma' &= \sigma + \rho U\end{aligned}\tag{10.63b}$$

With these variables, equations 10.63a become

$$\begin{aligned}\frac{\partial \sigma'_{ij}}{\partial x_j} + U \frac{\partial \rho}{\partial x_i} &= \rho a_i \\ \sigma'_{ij} - \sigma' \delta_{ij} &= \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)\end{aligned}\tag{10.63c}$$

These equations represent a fluid under the action of a body force $U \partial \rho / \partial x_i$ per unit volume. This result provides an analog model in which the velocity field is the same as in the actual fluid. However, the stress σ'_{ij} in the model is different. Hence boundary conditions involving stresses are not retained and must be modified in accordance with equations 10.63b.

Of particular interest are homogeneous fluids of different densities. We consider two fluids of densities ρ_1 and ρ_2 . At the surface of discontinuity the body force becomes a force per unit area applied normally to the surface and of magnitude $U(\rho_2 - \rho_1)$. It is obtained by integration of $U d\rho/dx_i$ across the discontinuity.

The analog model is not restricted to a fluid. The key property involved here is that the deformation depends only on the "stress deviator" $\sigma_{ij} - \sigma \delta_{ij}$. This property is usually associated with the assumption of incompressibility. Hence the analog model is generally applicable to an incompressible material.

As an illustration let us consider the problem of gravity instability of two fluids. The fluid of higher density ρ_2 lies on top of a fluid of density ρ_1 . With a vertical z coordinate the gravity potential is $U = gz$. An arbitrary constant may be added to this value and may be chosen so that $z = 0$ represents the horizontal interface in the undisturbed initial state. We assume that instability has produced a dome-shaped intrusion into the upper region. In the gravity-free analog model the driving force of this intrusion is represented by normal forces of magnitude $(\rho_2 - \rho_1)gz$ per unit area applied to the interface as shown in Figure 10.4a.

The model is valid for an arbitrary number of layers of incompressible homogeneous fluids of densities $\rho_1, \rho_2 = \rho_1 + \Delta\rho_1, \rho_3 = \rho_2 + \Delta\rho_2$, etc. This can be shown by substituting these values of ρ into equations 10.63a and applying the foregoing procedure to each term $\Delta\rho_i$. For a vertical gravity field the gravity-free model is obtained by applying normal interfacial forces

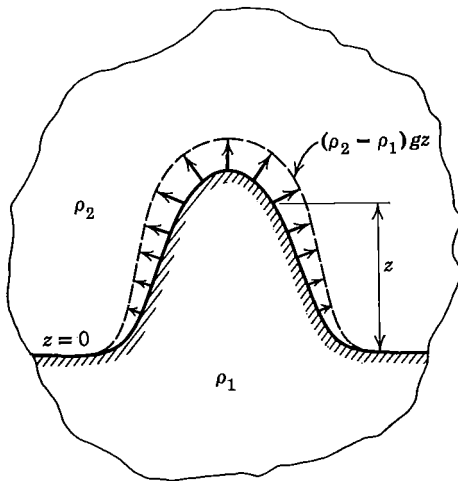


Figure 10.4a Analog model for a dome-shaped intrusion due to gravity instability of two fluids of different densities ρ_1 and ρ_2 ($\rho_2 > \rho_1$).

of magnitude $g \Delta\rho_i z_i$ (no summation), where $\Delta\rho_i$ is the density increment at the i th interface and z_i is the altitude above an arbitrary horizontal plane which may be different for each interface, and may be chosen to coincide with its initial equilibrium position.

Isotropic Viscoelastic Media in the Presence of Hydrostatic Stress. The dynamics of a viscous fluid in a gravity field is a particular case of the corresponding problem for an isotropic viscoelastic medium. It may be a viscoelastic fluid or solid. Because the initial stress is hydrostatic, the property of isotropy is retained for the incremental stresses. They are written

$$s_{ij} = 2\hat{\mu}e_{ij} + \delta_{ij}\hat{\lambda}e \tag{10.64}$$

with two operators $\hat{\mu}$ and $\hat{\lambda}$. The dynamical equations in operational form are obtained by substituting the values (10.64) into equations 10.35 or 10.36.

The particular case of a viscous fluid considered above is obtained by substituting the operators

$$\hat{\mu} = \eta p \quad \hat{\lambda} = \lambda - \frac{2}{3}\eta p \tag{10.65}$$

With the value (10.64) for s_{ij} , expression (10.45) assumes the form

$$\Delta\hat{V} = \hat{\mu}e_{ij}e_{ij} + \frac{1}{2}\hat{\lambda}e^2 + \mathcal{R} \tag{10.66}$$

By introducing this expression into equations 10.41 and 10.49 we obtain the variational principle in operational form. This form of the variational principle assumes perfect adherence at a rigid boundary. For perfect slip the principle is easily modified by addition of the term \mathcal{P}_B as in equation 9.42.

As for a fluid, the variational principle (10.41) may be expressed in a modified form by writing

$$\hat{\mathcal{P}} = \iiint_{\tau} (\frac{1}{2}\hat{\lambda}e^2 + \hat{\mu}e_{ij}e_{ij} + \mathcal{Y}) d\tau + \mathcal{W}_F \quad (10.67)$$

The values of \mathcal{Y} and \mathcal{W}_F are given by equations 10.53 and 10.55. This modified principle may also be considered a consequence of equations 5.52 of Chapter 3.

The stress-strain relations may also be expressed by means of the stresses t_{ij} . Because of isotropy they are written

$$t_{ij} = 2\hat{Q}e_{ij} + \hat{R}e\delta_{ij} \quad (10.68)$$

Using equations 10.43 and 10.64, we derive

$$\begin{aligned} \hat{\lambda} &= \hat{R} - S \\ \hat{\mu} &= \hat{Q} + \frac{1}{2}S \end{aligned} \quad (10.69)$$

Additional properties are derived by assuming that thermodynamic principles are applicable to the operators \hat{Q} and \hat{R} . They correspond to the general expressions $\hat{C}_{ij}^{\mu\nu}$ of equations 3.32. Hence we write

$$\begin{aligned} \hat{Q} &= \int_0^\infty \frac{p}{p+r} Q(r) dr + Q + pQ' \\ \hat{R} &= \int_0^\infty \frac{p}{p+r} R(r) dr + R + pR' \end{aligned} \quad (10.70)$$

The condition of symmetry $\hat{C}_{ij}^{\mu\nu} = \hat{C}_{\mu\nu}^{ij}$ required by thermodynamics is automatically verified in this case as a consequence of isotropic symmetry.

Additional thermodynamic conditions were stated earlier in connection with the more general expressions (4.35a), and they require that $C_{ij}^{\mu\nu}(r)$, $C_{ij}^{\mu\nu}$, and $C_{ij}^{\prime\mu\nu}$ define non-negative quadratic forms. In the present case this requires that the quadratic forms $2Q(r)e_{ij}e_{ij} + R(r)e^2$, $2Qe_{ij}e_{ij} + Re^2$, and $2Q'e_{ij}e_{ij} + R'e^2$ be non-negative. The necessary and sufficient condition for this to be true is that all six

quantities $Q(r)$, Q , Q' , and $R(r) + \frac{2}{3}Q(r)$, $R + \frac{2}{3}Q$, and $R' + \frac{2}{3}Q'$ be non-negative. This can be shown by referring the strain to principal directions x , y , z and writing

$$2Qe_{ij}e_{ij} + Re^2 = (R + \frac{2}{3}Q)e^2 + \frac{2}{3}Q[(e_{xx} - e_{yy})^2 + (e_{yy} - e_{zz})^2 + (e_{zz} - e_{xx})^2] \quad (10.71)$$

The condition is obviously sufficient. That it is also necessary results from substituting successively $e = 0$ and $e_{xx} = e_{yy} = e_{zz}$.

Incompressible Viscoelastic Fluids. For an incompressible medium, equations 10.66 and 10.67 are simplified by putting $e = 0$. The properties of the medium are described by the single operator $\hat{\mu}$. The problems treated in examples I and II at the beginning of this section may be immediately generalized to this case provided we assume that the operators $\hat{\mu}$ and $\hat{\mu}'$ describing the viscoelastic properties of the two media are in a constant ratio:

$$\frac{\hat{\mu}}{\hat{\mu}'} = \kappa \quad (10.72)$$

The equations are the same as for viscous fluids provided we replace ηp and $\eta' p$ by $\hat{\mu}$ and $\hat{\mu}'$. The parameter σ becomes

$$\sigma = \frac{(\rho' - \rho)gh}{\hat{\mu}'} \quad (10.73)$$

The graphs of Figure 10.2 are applicable to this case. When the operator $\hat{\mu}'$ obeys thermodynamic principles, it is derived from equations 10.69 and 10.70. Then $\hat{\mu}'$ is an increasing function of p when p is positive. Hence the minimum of σ yields the same dominant wavelength for all media, whether they are viscous or viscoelastic. The results are also applicable to the stability of incompressible and purely elastic isotropic media with initial hydrostatic stress. In this case κ represents the ratio of the elastic moduli, and σ_{\min} yields the critical density difference and the buckling wavelength.

Medium Initially Stress-Free. The general properties discussed in this chapter for anisotropic media with initial stress are obviously applicable to the simpler particular case of isotropic media initially stress-free. In this case the operators \hat{Q} and \hat{R} become equal to $\hat{\mu}$ and $\hat{\lambda}$.

Curvilinear Coordinates. The general equations for the small motion dynamics of a viscous fluid in a gravity field may be readily expressed in orthogonal curvilinear coordinates by using the variational principle. For a viscous fluid this procedure is a particular application of the general method mentioned at the end of section 9. If we use equations 10.36, a simplification occurs as in the similar case for acoustic-gravity waves discussed in the last paragraph of section 6 in Chapter 5. In equations 10.36 all terms except those containing the viscosity are in vector form, and their formulation in curvilinear coordinates is immediate.

APPENDIX

Non-linear Theories and Finite Strain

Introduction. In the foregoing chapters we have been concerned with the linear theory of small deformations in the vicinity of a state of initial stress. However, the concepts and methods used in this development are equally applicable to the analysis of non-linear theories and finite strain. Many of the results of the linearized theory may readily be extended to include non-linear features. Moreover, it is found that there is a formal analogy between the equations of the non-linear and linearized theories. In fact, in the author's earlier papers* (1934–1940) the problem was considered in the context of non-linear theories, and small deformations of an initially stressed medium were treated as a particular case derived by linearization.

In this book the process has been reversed, and the linear theory has first been developed extensively on an independent basis.

The purpose of this Appendix is to give a brief survey of the methods and results derived by extension of the theory to include non-linearity and finite strain. The material presented here is fundamentally the same as that contained in the author's work already cited.

Non-linear equations of equilibrium for the stress field are conveniently derived by considering incremental deformations which are virtual and by applying the principle of virtual work.

This is followed by a discussion of post-buckling behavior which

* As listed at the end of the Preface.

leads to the possibility of applying the equations to an initially stressed medium with non-linear incremental stress-strain relations.

It is also shown how the underlying concepts lead naturally to a distinction between non-linearity of purely geometrical origin and that arising from material properties. As a consequence it is possible to derive non-linear equations for large deformations of a medium with linear viscoelastic properties.

The same distinction is also applicable under the assumption that the strain remains very small while the rotation may be of larger magnitude, and simplified equations are obtained by introducing explicitly this assumption into the general results.

As indicated in a short paragraph, the non-linear equations are easily expressed in curvilinear coordinates by applying the principle of virtual work.

Finally, it is shown that the linearized theory of a medium under initial stress implies as a particular and trivial limiting case the equations for the velocity field and rate variables which are applicable to finite deformations. By the same token variational principles are obtained for the velocity field.

Incremental Deformations and the Principle of Virtual Work. As already pointed out, the basic equations for incremental deformations were first derived by the author in the context of non-linear theories. By considering virtual incremental deformations it is possible to formulate the principle of virtual work in straightforward fashion. This principle provides the simplest and most general method of derivation of the fundamental equations of the non-linear theory.

The concepts of stress and strain used in deriving non-linear equations are basically the same as those described in detail throughout this book. Let us consider, for example, a two-dimensional deformation. A point P is displaced to P' . On the infinitesimal scale, the deformation in the vicinity of point P may be considered homogeneous. In this homogeneous deformation a square of material originally of unit size and oriented along the fixed directions x and y becomes the parallelogram $P'ABC$ (Fig. 1). The transformation of the square into the parallelogram may be obtained by applying to the square a linear transformation with symmetric coefficients followed by a rotation θ . These coefficients ϵ_{11} , ϵ_{22} and

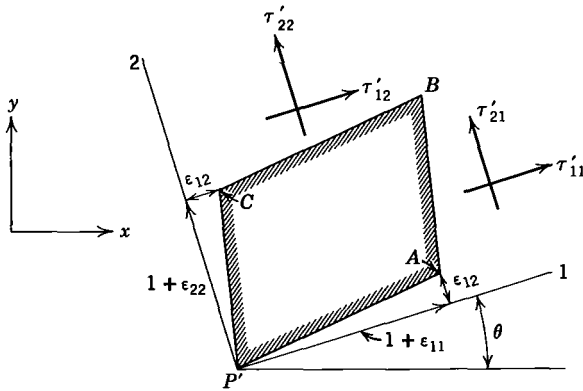


Figure 1 Representation of the finite strain ϵ_{ij} and corresponding forces τ'_{ij} .

$\epsilon_{12} = \epsilon_{21}$ are chosen to represent the finite strain, and their physical significance relative to a locally rotated system of axes 1, 2 is illustrated in Figure 1 (see also page 9).

The stress is defined as follows. In two-dimensional strain the square and the parallelogram are considered to be cut out of a slab of unit thickness parallel to the figure. The forces acting on side AB in directions parallel to the rotated axes 1 and 2 are denoted by τ'_{11} and τ'_{21} . Similarly the forces acting on side CB are τ'_{22} and τ'_{12} .

The foregoing definitions are readily generalized to three dimensions. An infinitesimal region around a point P undergoes a solid rotation and a finite deformation. Local rectangular axes originally parallel to x, y, z become the axes 1, 2, 3 when subject to the same solid rotation as the medium. The finite deformation is defined by a linear transformation with symmetric coefficients $\epsilon_{ij} = \epsilon_{ji}$ relative to the rotated axes 1, 2, 3. Note that the procedure described here defines implicitly both the finite strain and the solid rotation.

This particular form of separation of rotation and finite strain is, of course, arbitrary, and the magnitude of the rotation is determined entirely by the symmetry condition $\epsilon_{ij} = \epsilon_{ji}$ as a matter of definition. Other definitions are possible which will not be discussed at this time. However, the present definition has some theoretical advantage from the standpoint of formal symmetry. There is also a physical reason for this choice since the symmetric coefficients define a transformation with the property that there are three rectangular directions of the material which remain unchanged. This type of deformation has

been discussed in detail in Chapter 1 where it is referred to as a "pure deformation."

The stresses in three dimensions are defined as the forces τ'_{ij} acting on the faces of a parallelepiped which is initially a cube of unit size with sides oriented along x, y, z . The forces τ'_{ij} are the components of these forces in directions parallel to the locally rotated axes 1, 2, 3. Note that the nine force components τ'_{ij} must satisfy three relations expressing that there is no moment acting on the deformed element. Hence only six components are independent.

Let us consider a medium initially enclosed in a volume Ω bounded by the surface A before deformation. Initial coordinates x_i of a material particle become $\xi_i = x_i + u_i$ after deformation. A virtual displacement δu_i defines a virtual deformation $\delta \varepsilon_{ij}$. The virtual work of the forces τ'_{ij} per unit initial volume is

$$\tau'_{ij} \delta \varepsilon_{ij} = \tau_{ij} \delta \varepsilon_{ij} \quad (1)$$

where

$$\tau_{ij} = \frac{1}{2}(\tau'_{ij} + \tau'_{ji}) \quad (2)$$

This result is a consequence of the symmetry relation ($\varepsilon_{ij} = \varepsilon_{ji}$). The forces τ_{ij} may be considered as alternative stress components, in complete analogy with the definition of the alternative incremental stresses t_{ij} in Chapter 2 (section 2).

With these definitions the principle of virtual work is written

$$\iiint_{\Omega} \tau_{ij} \delta \varepsilon_{ij} d\Omega = \iiint_{\Omega} \rho X_i(\xi_i) \delta u_i d\Omega + \iint_A f_i \delta u_i dA \quad (3)$$

In this expression $X_i(\xi_i)$ is the body force field per unit mass at the displaced point ξ_i , and ρ is the initial density at the point x_i before deformation. The principle is applicable to dynamics by including in the body force field the negative acceleration of the particle. In the term expressing the virtual work of the body force we have used the law of conservation of mass,

$$\rho d\Omega = \rho' d\Omega' \quad (4)$$

where ρ' and $d\Omega'$ are the density and the volume element after deformation.

The last term on the right side of equation 3 represents the virtual work of the boundary forces, and f_i is the force per unit initial area acting on the boundary surface A . The exact formulation of equilibrium conditions by the virtual work principle in the

form (3) was first described by the author in two papers published in 1939* and 1940*. More recently, use of the same principle was also suggested in the context of viscoelasticity for the purpose of deriving a non-linear theory.†

Brief reference is made here to some of the more relevant papers in the earlier literature. The first attempt to take into account a state of initial stress is found in a paper by Cauchy‡ (1827) using his assumption that the stress is due to central forces between molecular particles. The case of uniform initial stress was discussed by Southwell§ (1913). The stability theory of Biezeno and Hencky (1928–29) mentioned in Chapter 2 (p. 63) introduces incremental stress-strain relations which are assumed to be those of isotropic elasticity with small deformations. The non-linear theories of Trefftz|| and Kappus¶ (1933–39) involve the use of the metric tensor and the corresponding non-cartesian definition of stress. They also provide a useful physical interpretation. The difficulties inherent in this non-cartesian approach have already been pointed out with reference to more recent developments along this line (see p. 96).

Formal Analogy of Non-linear and Linearized Theories.

Equation 3 provides an exact formulation of the equilibrium equations in finite strain. It is valid for any continuous medium and does not refer to any material property. Hence it is applicable to materials with arbitrary rheological properties.

In order to obtain corresponding differential equations for the stress field we must derive expressions for the finite strain components ϵ_{ij} in terms of the displacement gradients $\partial u_i/\partial x_j$. There are many ways of obtaining such expressions. However, from a practical viewpoint we may use the non-linear expressions derived in Chapter 1. Equations (3.27) of that chapter read

$$\epsilon_{ij} = e_{ij} + \frac{1}{2}(e_{i\mu}\omega_{\mu j} + e_{j\mu}\omega_{\mu i}) + \frac{1}{2}\omega_{i\mu}\omega_{j\mu} \quad (5)$$

* See references 3 and 5 at the end of the Preface.

† M. A. Biot, "Variational and Lagrangian Methods in Viscoelasticity," in *Deformation and Flow of Solids* (IUTAM Colloquium, Madrid, 1955), pp. 251–263, Springer, Berlin, 1956.

‡ A. L. Cauchy, *Exercices de Mathématique*, Vol. 2, Bure Frères, Paris, 1827.

§ R. V. Southwell, On the General Theory of Elastic Stability, *Philosophical Transactions of the Royal Society, A*, Vol. 213, pp. 187–244, 1913.

|| E. Trefftz, Zur Theorie der Stabilität des elastischen Gleichgewichts, *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol 13, No. 2, pp. 160–165, 1933.

¶ R. Kappus, Zur Elastizitätstheorie endlicher Verschiebungen, *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 19, No. 5, pp. 271–285, and No. 6, pp. 344–361, 1939.

The explicit values of these six expressions are given in two and three dimensions by equations 2.28 and 3.18 of Chapter 1. They were derived by the author in two successive publications in 1938–39.* These expressions are correct to the second order and are quite satisfactory for the treatment of a vast category of non-linear problems in physics and technology. The virtual work principle (3) is equivalent to a system of partial differential equations and boundary conditions. By introducing the values (5) for ε_{ij} , the partial differential equations are obtained by following exactly the same procedure as used in section 5 of Chapter 2 for the linearized equations of a continuum under initial stress. By integration by parts we may write the identity

$$\iiint_{\Omega} \tau_{ij} \delta \varepsilon_{ij} d\Omega = - \iiint_{\Omega} \frac{\partial \mathcal{A}_{ij}}{\partial x_j} \delta u_i d\Omega + \iint_A \mathcal{A}_{ij} n_j \delta u_i dA \quad (6)$$

In this expression we have put

$$\mathcal{A}_{ij} = \tau_{ij} + \tau_{kj} \omega_{ik} + \frac{1}{2} \tau_{kj} e_{ki} - \frac{1}{2} \tau_{ki} e_{kj} \quad (7)$$

and n_j denotes the unit vector directed normally outward at the initial boundary A .

When we use the identity (6), the virtual work principle (3) becomes

$$\iiint_{\Omega} \left[\frac{\partial \mathcal{A}_{ij}}{\partial x_j} + \rho X_i(\xi_i) \right] \delta u_i d\Omega + \iint_A (f_i - \mathcal{A}_{ij} n_j) \delta u_i dA = 0 \quad (8)$$

Since the virtual displacements δu_i are arbitrary, equation 8 implies

$$\frac{\partial \mathcal{A}_{ij}}{\partial x_j} + \rho X_i(\xi_i) = 0 \quad (9)$$

These equations must be verified in the volume Ω . Equation 8 also implies the boundary condition

$$f_i = \mathcal{A}_{ij} n_j \quad (10)$$

Equations 9 are essentially equilibrium conditions for the stress field τ_{ij} and do not postulate any material properties. The material properties are introduced by expressing the stresses τ_{ij} by means of the strain components ε_{ij} . These functional relations may be linear or non-linear and express properties of elasticity, fluid viscosity,

* See references 2 and 4 at the end of the Preface.

viscoelasticity, plasticity, etc. We note that the strain components ϵ_{ij} appearing in these relations must in turn be expressed by the non-linear relations (5) in terms of the displacement gradients.

The foregoing results may be compared with the equations derived for the linearized theory of a continuum under initial stress. The two sets of results are formally similar, and they point to a fundamental analogy between the non-linear and linearized theories. If we put

$$\tau_{ij} = t_{ij} + S_{ij} \tag{11}$$

where S_{ij} is the initial stress, the virtual work principle (3) becomes identical with equation 5.48 of Chapter 2. Similarly the equilibrium conditions (9) and boundary conditions (10) are formally identical with equations 5.28 of Chapter 2. By substituting $\tau_{ij} = t_{ij} + S_{ij}$ into the value (7) for \mathcal{A}_{ij} and linearizing the expression with respect to t_{ij} , e_{ij} , and ω_{ij} we obtain equation 5.20 of Chapter 2 for the corresponding value A_{ij} in the linearized theory with initial stress.

Instead of stresses τ_{ij} referred to initial areas we may use the true stresses σ_{ij} referred to actual areas after deformation. Both stress systems τ_{ij} and σ_{ij} are referred to the same locally rotated axes 1, 2, 3. Relations between the two systems of stress components are easily obtained by applying equations 7.6 of Chapter 1, which express the forces per unit initial area. In applying these equations we must use strain components ϵ_{ij} referred to the rotated axes 1, 2, 3. The exact relations involve the 2×2 Jacobians; hence they contain linear and quadratic terms in ϵ_{ij} . In the large majority of applications it is not necessary to retain the quadratic terms, and we may write

$$\tau'_{ij} = \sigma_{ij}(1 + \epsilon) - \sigma_{ik}\epsilon_{kj} \tag{12}$$

with

$$\epsilon = \epsilon_{ij}\delta_{ij} \tag{13}$$

Hence

$$\tau_{ij} = \sigma_{ij}(1 + \epsilon) - \frac{1}{2}\sigma_{ik}\epsilon_{kj} - \frac{1}{2}\sigma_{jk}\epsilon_{ki} \tag{14}$$

Here again we note a formal analogy with the results of the linearized theory for the initially stressed medium. By substituting $\tau_{ij} = t_{ij} + S_{ij}$, $\sigma_{ij} = s_{ij} + S_{ij}$, and linearizing expressions (14) we obtain equations 2.22 of Chapter 2.

Second and Higher Order Theories. The equations presented here include all second order terms. Thus a theory of elasticity is

provided which from the mathematical viewpoint is exact to the second order in the displacement gradients. These equations also contain the particular third order terms which are physically significant in most applications. However, the theory is not restricted to this case, and higher order terms may be evaluated and included if necessary.

Non-linear Theories with Initial Stress. Post-buckling Behavior. In some cases buckling and related problems cannot be treated adequately by considering the linearized theory of incremental deformations in the vicinity of a state of initial stress. However, a non-linear theory for initially stressed media is immediately derived by putting $\tau_{ij} = t_{ij} + S_{ij}$ into equations 6, 9, and 10, this time retaining the non-linear terms. The incremental stresses t_{ij} may also be non-linear functions of the strain ϵ_{ij} . Such equations are, of course, applicable to materials with either elastic or non-elastic properties.

In the case of elastic materials it will generally be sufficient to consider the incremental stresses t_{ij} to be linear functions of the strain ϵ_{ij} while retaining the non-linear expressions (5) for the strain itself. Such a theory will embody the essential features of the so-called "post-buckling" behavior of thin plates and shells.

Distinction between Material and Geometric Non-linearity. The foregoing results lead naturally to a distinction between two fundamentally different types of non-linearity. On the one hand, the non-linearity may be the result of material properties and it is expressed by non-linear relations between stress and strain. On the other hand, we have the non-linearity due to the geometry which is embodied, for example, in expressions (5) for the strain components.

A first attempt to establish a theory which brings out the non-linear terms of purely geometrical origin was made by the author in 1934.* It was shown that this non-linearity is due essentially to the fundamental role played by the rotation ω_{ij} . In a second paper in 1938* the question was examined more rigorously and the foundation was laid for the later developments. To quote from the 1938 paper, with reference to the theory of elasticity the following was stated: "The

* See references 1 and 2 at the end of the Preface.

restriction of the classical theory to small strain justifies in most cases the use of Hooke's law. However, there is no necessity to assume that also the rotation is small since it may become large with respect to the strain while the latter is still a small quantity." Actually the equations derived in this 1938 paper were not restricted to the case where the strain is much smaller than the rotation but are rigorously applicable to the more general case where both are of the same order.

These results opened the way to an exact theory of elasticity with initial stress and provided a method by which non-linearities of physical and geometrical origin could be separated. For example, physical non-linearity may be represented by non-linear relations

$$\sigma_{ij} = \sigma_{ij}(\epsilon_{\mu\nu}) \tag{15}$$

relating the six stresses σ_{ij} to the six strain components $\epsilon_{\mu\nu}$. Substitution of expressions (15) for σ_{ij} into equations 14 also yields non-linear relations for τ_{ij} in terms of $\epsilon_{\mu\nu}$ which we write

$$\tau_{ij} = \tau_{ij}(\epsilon_{\mu\nu}) \tag{16}$$

It can be seen that from a purely mathematical viewpoint the distinction between physical linearity and non-linearity is not necessarily a sharp one, since it depends on the particular definition of the stress. For example, if the stress-strain relations for σ_{ij} are linear, relations (14) show that the stress-strain relations for τ_{ij} will not be linear. However, the difference between these two systems of stresses will often be negligible, and in any case the physical reason for the non-linearity will be clearly brought out.

The stress-strain relations may represent a large variety of linear or non-linear rheological properties, including plasticity and creep. They may also correspond to viscoelasticity where linear stress-strain relations are valid for deformations which are not small in the mathematical sense, as indicated in the following paragraph.

Linear Viscoelasticity and Non-linear Geometry. Within suitable limitations in the magnitude of the strain many viscoelastic materials obey linear stress-strain relations which are expressible in the form

$$\tau_{ij} = \hat{C}_{ij}^{\mu\nu} \epsilon_{\mu\nu} \tag{17}$$

In these equations $\hat{C}_{ij}^{\mu\nu}$ are integro-differential operators given by

equation 9.68e of Chapter 6. They are written

$$\hat{C}_{ij}^{\mu\nu} = \int_{\tau=0}^{\tau=t} F_{ij}^{\mu\nu}(t - \tau) d\tau + C_{ij}^{\mu\nu} + C'_{ij}{}^{\mu\nu} \frac{d}{dt} \quad (18)$$

Substitution of expressions (17) for τ_{ij} into the equilibrium conditions (9) yields non-linear integro-differential field equations for the displacements.

Viscoelastic Correspondence in Non-linear Problems. As can be seen, the non-linear field equations (9) for τ_{ij} are independent of the material properties and are therefore valid for either elastic or viscoelastic materials. The viscoelasticity is embodied only in equations 17 which are obtained from elastic stress-strain relations by substituting operators $\hat{C}_{ij}^{\mu\nu}$ for the elastic coefficients. In this sense and with the obvious restrictions associated with non-linear equations the correspondence principle may be extended to viscoelasticity with non-linear geometry. Application of such correspondence was made by the author in generalizing to viscoelasticity the non-linear Kármán-Föppl equations for an elastic plate with large deflections.*

Strains Small Relative to the Rotations. In a large number of technological problems, in particular those dealing with structural metals, design specifications ensure that the stresses do not exceed a fraction of the elastic limit. In such cases the strain remains of the order of 10^{-3} or smaller. The non-linear properties in such instances are due entirely to the geometry. In particular, this will be true if the rotation ω_{ij} becomes of larger magnitude than the strain, as illustrated

* M. A. Biot, "Variational and Lagrangian Methods in Viscoelasticity," in *Deformation and Flow in Solids* (IUTAM Colloquium, Madrid, 1955), pp. 251-263, Springer, Berlin, 1956. The principle of viscoelastic correspondence was first enunciated and its far-reaching consequences spelled out by the author in colloquium lectures in 1954 at the California Institute of Technology and Brown University. In particular, the extension of elastic solutions to viscoelasticity for applied forces proportional to a function of time is an immediate consequence of the results in the author's 1954 paper (see p. 359). There it is shown that the operators may be manipulated algebraically as if they were elastic coefficients. In generality and substance this principle is quite different from the "viscoelastic analogy" proposed by E. H. Lee (*Stress Analysis in Viscoelastic Bodies, Quarterly of Applied Mathematics*, Vol. 13, No. 2, pp. 183-190, 1955) which is restricted to static problems in isotropic media and is based on a particular differential formulation of the stress-strain relations in terms of the stress deviator.

in many problems of elastic thin shells and plates. With the foregoing assumptions we write expressions (7) by dropping the non-linear terms which do not contain the rotation. We obtain

$$\mathcal{A}_{ij} = \tau_{ij} + \tau_{kj}\omega_{ik} \tag{19}$$

Since the strain is small, the components τ_{ij} may be replaced by the stresses σ_{ij} . Hence the equilibrium equations (9) become

$$\frac{\partial}{\partial x_j} (\sigma_{ij} + \sigma_{kj}\omega_{ik}) + \rho X_i(\xi_i) = 0 \tag{20}$$

In two dimensions and in the absence of a body force these equations take the form

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} - \frac{\partial}{\partial x} (\sigma_{12}\omega) - \frac{\partial}{\partial y} (\sigma_{22}\omega) &= 0 \\ \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial}{\partial x} (\sigma_{11}\omega) + \frac{\partial}{\partial y} (\sigma_{12}\omega) &= 0 \end{aligned} \tag{21}$$

where ω is the rotation in the x, y plane as expressed by equation 2.26 of Chapter 1.

The author obtained equations 21 in 1934,* and their more general form (20) in 1939.*

Equations 20 may, of course, be applied to derive a linearized theory of small deformations and large rotations for a medium under initial stress. By linearization of equations 20 we derive

$$\frac{\partial}{\partial x_j} (s_{ij} + S_{kj}\omega_{ik}) + \rho \Delta X_i(\xi_i) = 0 \tag{22}$$

The similarity of equations 20 and 22 is another example of the formal analogy between non-linear and linearized theories.

For an elastic medium and with the assumption of small strain the values of σ_{ij} may be expressed linearly in terms of ε_{ij} by Hooke's law. However, the strain components ε_{ij} are given by the non-linear expressions (5). For small strain they may be simplified by writing

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2}\omega_{i\mu}\omega_{j\mu} \tag{23}$$

Note that this simplified form may be introduced in the expression of Hooke's law. However, in the variational principle (3) the complete value (5) is required because the assumption of small strain

* See references 1 and 4 at the end of the Preface.

is to be introduced only after the variation $\delta\varepsilon_{ij}$ has been evaluated.

Equations 23, expressing the strain components when they are small in comparison with the rotation, were given by the author in 1938* in the explicit form

$$\begin{aligned} \varepsilon_{11} &= e_{xx} + \frac{1}{2}(\omega_z^2 + \omega_y^2) & \varepsilon_{23} &= e_{yz} - \frac{1}{2}\omega_y\omega_z \\ \varepsilon_{22} &= e_{yy} + \frac{1}{2}(\omega_x^2 + \omega_z^2) & \varepsilon_{31} &= e_{zx} - \frac{1}{2}\omega_z\omega_x \\ \varepsilon_{33} &= e_{zz} + \frac{1}{2}(\omega_y^2 + \omega_x^2) & \varepsilon_{12} &= e_{xy} - \frac{1}{2}\omega_x\omega_y \end{aligned} \quad (24)$$

The quantities appearing in these expressions are defined by equations 3.10 of Chapter 1.

Non-linear Equations in Curvilinear Coordinates. The equilibrium equations (9) are readily obtained in orthogonal curvilinear coordinates by introducing into the strain components (5) the values (5.54) and (5.55) derived in Chapter 2 for e_{ij} and ω_{ij} in curvilinear coordinates. With these values the equilibrium equations are derived by applying the principle of virtual work (3). As already pointed out, this procedure leads also to the non-linear equations for the medium under initial stress and for other particular cases of non-linearity due to the geometry and the rotation described in the foregoing discussion.

The non-linear dynamical equations in curvilinear coordinates are derived from the variational principle by including the negative particle acceleration $-u_i$ in the body force $X_i(\xi_i)$.

Attention is called to the special type of curvilinear coordinates used here to represent the deformation. The strain components are *locally cartesian* and fundamentally different from the classical representation of finite deformation using the metric tensor. A particle initially at a point P moves to a point P' . The vector $\vec{PP'}$ defines the displacement of a particle and is represented by its projections u_i on a local cartesian system of axes. The origin of this local coordinate system coincides with the initial position P of the particle, and the axes are tangent to the curvilinear coordinate lines passing through point P . The displacement components u_i

* See reference 2 at the end of the Preface. The results represented by equations 20 and 23 were later incorporated by a number of authors in discussions of non-linear behavior leading to useful applications in theories of plates and shells (see, e.g., V. V. Novozhilov, *Foundations of the Non-linear Theory of Elasticity*, Graylock Press, Rochester, N.Y., 1953).

are functions of the curvilinear coordinates q_1, q_2, q_3 of the initial point P (see Chapter 2, section 5). In the variational principle (3) the quantities $X_i(\xi_i)$ represent the components of the body force field at the *displaced point* P' projected on the local cartesian axes with its origin at the *initial point* P . The stresses τ_{ij} are again referred to axes which are displaced and locally rotated with the material and are initially tangent to the coordinate lines at point P .

Rate Variables as a Limiting Case of Incremental Deformations. The mechanics of small deformations of a medium under initial stress leads directly to equations which in principle are applicable to finite strain and are expressed in terms of rate variables. These equations are derived very simply by a trivial and elementary limiting process which follows, as a natural consequence of the mechanics of incremental deformations.

An arbitrary time-dependent finite deformation may be looked on as a continuous sequence of incremental deformations. Any instantaneous configuration at the time t is considered to be an initial state. At the instant $t + \Delta t$ the particle displacements are u_i , and the stress on a moving particle referred to axes rotating with the particle has increased by an amount s_{ij} .

Let us go back to equations 5.21 of Chapter 1. They are written

$$\bar{\sigma}_{ij} - S_{ij} = s_{ij} + S_{\mu j} \omega_{i\mu} + S_{i\mu} \omega_{j\mu} \tag{25}$$

The left side represents the stress increment on a moving particle referred to fixed axes. We divide equations 25 by Δt and consider limiting values for Δt vanishingly small. In the limit we find

$$\frac{\bar{\sigma}_{ij} - S_{ij}}{\Delta t} \rightarrow \frac{D\sigma_{ij}}{Dt} = \frac{\partial \sigma_{ij}}{\partial t} + v_k \frac{\partial \sigma_{ij}}{\partial x_k} \tag{26}$$

The instantaneous velocity field v_i is the limiting value of

$$\frac{u_i}{\Delta t} \rightarrow v_i \tag{27}$$

We also introduce the limiting value,

$$\frac{s_{ij}}{\Delta t} \rightarrow \mathcal{S}_{ij} \tag{28}$$

This quantity is the rate of change of the stress at a moving particle

with components referred to axes undergoing the same instantaneous rotation as the particle. This instantaneous rotation Ω_{ij} is expressed by the limiting value

$$\frac{\omega_{ij}}{\Delta t} \rightarrow \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) = \Omega_{ij} \quad (29)$$

The initial stress S_{ij} becomes the instantaneous stress field σ_{ij} . Hence the limiting form of equations 25 is

$$\frac{D\sigma_{ij}}{Dt} = \mathcal{S}_{ij} + \sigma_{\mu j} \Omega_{i\mu} + \sigma_{i\mu} \Omega_{j\mu} \quad (30)$$

We consider also the stress-strain relations (4.15) of Chapter 2. They are

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu} \quad (31)$$

Let us divide this equation by Δt . Its limiting form, for vanishing Δt , is

$$\mathcal{S}_{ij} = B_{ij}^{\mu\nu} \mathcal{E}_{\mu\nu} \quad (32)$$

The strain-rate components $\mathcal{E}_{\mu\nu}$ in these equations are defined as

$$\frac{e_{ij}}{\Delta t} \rightarrow \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \mathcal{E}_{ij} \quad (33)$$

Elimination of \mathcal{S}_{ij} between equations 30 and 32 yields

$$\frac{D\sigma_{ij}}{Dt} = B_{ij}^{\mu\nu} \mathcal{E}_{\mu\nu} + \sigma_{\mu j} \Omega_{i\mu} + \sigma_{i\mu} \Omega_{j\mu} \quad (34)$$

We may write four more equations. They are Newton's equations of motion

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho X_i = \rho \frac{Dv_i}{Dt} \quad (35)$$

where ρ is the mass density and X_i the body force field, and the equation of conservation of mass which is written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \quad (36)$$

The interest in these equations is brought out if the coefficients $B_{ij}^{\mu\nu}$ are functions only of the stresses. This will obviously be the case if the medium is a homogeneous isotropic elastic medium undergoing isothermal deformations. The ten equations 34, 35, and

36 then constitute a differential system for the ten unknown functions of time σ_{ij} , v_i , and ρ . They are valid for *finite deformations*.

Actually, for a homogeneous isotropic elastic medium with isothermal deformations the density may be written as an explicit function of the stress σ_{ij} . Hence in this case equation 36 is superfluous. This becomes evident when we write equation 36 in the form

$$\frac{D}{Dt} (\log \rho) = \mathcal{E}_{ij} \delta_{ij} = \mathcal{E} \tag{37}$$

In this form it is a direct consequence of the definition of the density for a given mass of variable volume.

As can be seen, the general equations which govern the rate variables in finite deformation follow as a natural and immediate consequence of the linearized theory of elasticity with initial stress developed by the author in 1939.*

Equilibrium Equations with Rate Variables. The limiting process of dividing the linear equations for incremental deformations by a vanishingly small time increment Δt is quite general. In particular, it may be applied to equations 7.32 of Chapter 1 which express the equilibrium condition for the incremental stresses s_{ij} . They are written

$$\frac{\partial}{\partial x_j} (s_{ij} + S_{jk} \omega_{ik} + S_{ij} e - S_{ik} e_{jk}) + \rho \Delta X'_i = 0 \tag{38}$$

As a consequence of d'Alembert's principle, we find that they are valid for dynamics by including the inertia forces as part of the body force field, and putting

$$X'_i = X_i - \frac{Dv_i}{Dt} \tag{39}$$

We divide equations 38 by a vanishingly small time increment Δt and derive in the limit

$$\frac{\partial}{\partial x_j} (\mathcal{S}_{ij} + \sigma_{jk} \Omega_{ik} + \sigma_{ij} \mathcal{E} - \sigma_{ik} \mathcal{E}_{jk}) + \rho \frac{DX'_i}{Dt} = 0 \tag{40}$$

These three equations for the rate variables may then be used instead of equations 35.

* See reference 4 at the end of the Preface. More recently these concepts have been used by C. Truesdell and others in connection with mathematical developments sometimes referred to as "hypoelasticity."

It is of interest to show that equations 40 are also a direct consequence of relations 30, 35, and 36. We write equations 35 by introducing the value (39) for X'_i . The time derivatives of equations 35 are

$$\frac{\partial}{\partial x_j} \left(\frac{\partial \sigma_{ij}}{\partial t} \right) + \rho \frac{\partial X'_i}{\partial t} + X'_i \frac{\partial \rho}{\partial t} = 0 \quad (40a)$$

Equation 36 may be written

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x_j} (\rho v_j) \quad (40b)$$

Hence

$$X'_i \frac{\partial \rho}{\partial t} = - X'_i \frac{\partial}{\partial x_j} (\rho v_j) = - \frac{\partial}{\partial x_j} (X'_i \rho v_j) + \rho v_j \frac{\partial X'_i}{\partial x_j} \quad (40c)$$

We substitute this expression into equations 40a. They become

$$\frac{\partial}{\partial x_j} \left(\frac{\partial \sigma_{ij}}{\partial t} - X'_i \rho v_j \right) + \rho \frac{DX'_i}{Dt} = 0 \quad (40d)$$

Equations 26 and 35 are written

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial t} &= \frac{D\sigma_{ij}}{Dt} - v_k \frac{\partial \sigma_{ij}}{\partial x_k} \\ X'_i \rho &= - \frac{\partial \sigma_{ik}}{\partial x_k} \end{aligned} \quad (40e)$$

By introducing these values into equations 40d we obtain

$$\frac{\partial}{\partial x_j} \left(\frac{D\sigma_{ij}}{Dt} - v_k \frac{\partial \sigma_{ij}}{\partial x_k} + v_j \frac{\partial \sigma_{ik}}{\partial x_k} \right) + \rho \frac{DX'_i}{Dt} = 0 \quad (40f)$$

From the identities

$$\frac{\partial}{\partial x_j} \left(-v_k \frac{\partial \sigma_{ij}}{\partial x_k} + v_j \frac{\partial \sigma_{ik}}{\partial x_k} \right) = \frac{\partial}{\partial x_j} \left(-\sigma_{ik} \frac{\partial v_j}{\partial x_k} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \right) \quad (40g)$$

it follows that equations 40f are equivalent to

$$\frac{\partial}{\partial x_j} \left(\frac{D\sigma_{ij}}{Dt} - \sigma_{ik} \frac{\partial v_j}{\partial x_k} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \right) + \rho \frac{DX'_i}{Dt} = 0 \quad (40h)$$

Finally we substitute in these equations the values (30) for $D\sigma_{ij}/Dt$ and the expressions \mathcal{E}_{ij} , \mathcal{E} , and Ω_{ij} defined by relations (29), (33), and (37). The term in parentheses in equations 40h becomes

$$\frac{D\sigma_{ij}}{Dt} - \sigma_{ik} \frac{\partial v_j}{\partial x_k} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} = \mathcal{S}_{ij} + \sigma_{jk} \Omega_{ik} + \sigma_{ij} \mathcal{E} - \sigma_{ik} \mathcal{E}_{jk} \quad (40i)$$

Hence equations 40h are identical with equations 40 derived by the limiting process.

Variational Principle for the Velocity Field. Let us consider an elastic medium undergoing slow finite deformations with negligible inertia forces. In this case $X'_i = X_i$, and the limiting process leads to equations for the velocity field which are derived from the

incremental theory by a simple change of notation. In particular, the displacement u_i is replaced by the velocity v_i , and the incremental boundary force Δf_i is replaced by the time derivative \dot{f}_i of the boundary force itself. Because of this formal identity the variational principles derived in Chapters 2 and 3 for the displacements u_i under conditions of static loading are readily applicable to the velocity field. Hence, for given values of \dot{f}_i and a given state of deformation and stress, the velocity field of the medium is determined by a variational principle. Application of such a principle provides a step-by-step evaluation of the finite deformation under variable loading. Obviously the condition of uniqueness of the solution should be verified at each step. Hence the deformation must be confined within the limits of stability.

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