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# Periodic averaging with a second order integral error\*

Thierry Dargent<sup>a</sup>, Florentina Nicolau<sup>b</sup> and Jean-Baptiste Pomet<sup>b</sup>

*Abstract*—In this paper, we study a variation of second order periodic averaging that allows an asymptotic reconstruction of the “fast” variable.

Our goal is to present a simpler method to obtain a second order estimate based on changing the initial condition of the averaged system and approaching a “moving” average of the solutions instead of the solutions themselves. This idea was already present in a publication by the first author, in the context of solving an optimal control problem with averaging techniques and application to low thrust orbit transfer. Namely, adding well suited second order terms to the average system, properly choosing the initial condition for the average system and measuring errors “in the mean” instead of pointwise yields an errors of order 2 with respect to  $\epsilon$  on the slow variable and of order 1 for the fast variable.

## I. INTRODUCTION AND MOTIVATION

In this paper, we study high order periodic averaging for the class of systems of the form

$$(S_\epsilon) : \begin{aligned} \dot{I} &= \epsilon f(I, \varphi) \\ \dot{\varphi} &= \omega(I) + \epsilon g(I, \varphi) \end{aligned}$$

where  $(I, \varphi)$  are local coordinates,  $I \in O \subset \mathbb{R}^n$ ,  $\varphi \in S^1$ ,  $f$ ,  $g$  and  $\omega$  are defined on  $O \times S^1$ , where  $O$  is an open subset of  $\mathbb{R}^n$  and  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . The systems of the form  $(S_\epsilon)$  are called one-frequency “integrable” systems in [8]. Indeed,  $(S_\epsilon)$  is a perturbation of the integrable Hamiltonian system  $\dot{I} = 0$ ,  $\dot{\varphi} = \omega(I)$  called “one-frequency” because  $\varphi$  has dimension 1, while  $\varphi$  usually has the same dimension as  $I$  in the reduction of a generic Hamiltonian system to a system represented in action-angle variables (Liouville’s theorem, see, e.g., [1]). In  $(S_\epsilon)$ , the variables  $I$  are slow, while  $\varphi$  is the fast variable.

The (first order) averaging principle for the system  $(S_\epsilon)$  consists in eliminating the fast variable  $\varphi$  and determining the approximate behavior of the slow variables  $I$  by constructing macroscopic evolution equations (called the averaged system) which handle only certain average characteristics of the small-scale motion, and which give a good approximation to the true evolution of the slow variables on a certain time interval, see, for instance, [1], [3], [8], [10]. These methods

have attracted a lot of attention because of their multiple applications for many interesting physical problems (work and heat in thermodynamics, hydrodynamical and molecular dynamical quantities in fluid mechanics, electronics, celestial mechanics, quantum mechanics, etc.). They are also used in optimal control, see the pioneering work [6] and the more recent [4], [7]. They yield approximation results that are by nature asymptotic (as  $\epsilon$  tends towards 0): typically, these results provide error estimates of order  $\epsilon$  on an interval of time whose length has order  $1/\epsilon$ .

Although the results are asymptotic, averaging methods are very useful in practice when  $\epsilon$  is finite “reasonably small”. The more recent paper [9] gives some theoretical ground to this observation by presenting a fully quantitative estimate of both the time interval and the error for fixed  $\epsilon > 0$ . These results are practically interesting, but we follow here the path of asymptotic estimates.

High order results are also available, e.g. in [8], [10]: averaging of order  $k$  has two purposes: either to obtain  $\mathcal{O}(\epsilon^k)$ -error estimates valid for time  $\mathcal{O}(\frac{1}{\epsilon})$ , with  $k > 1$ , or weaker  $\mathcal{O}(\epsilon^{k-j})$ -error estimates but for longer time  $\mathcal{O}(\frac{1}{\epsilon^{j+1}})$ . These techniques involve sophisticated transformations (depending on the fast variable); they are calculated a recursive way which requires some choice at each step, so that only the first order approximation is uniquely defined.

The results of the present paper stem out of ideas, remarks and numerical experiments of the first author, published in [5]<sup>1</sup>, based on the idea that the solution of the averaged system should not approach the (oscillating) solution of the original system  $(S_\epsilon)$  but rather an average of this solution on the approximate duration of one oscillation. This leads to choosing an initial condition for the averaged system that is not the same as that of the real system, but is rather such that its average on the first oscillation is the same as the average of the solution of the real system on the same oscillation. It is showed numerically in [5] that, on the one hand, this choice of initial condition results in a solution of the averaged system that approaches the average of the true solution with a much better

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<sup>1</sup>The paper [5] deals with averaging applied to a two point boundary value problem arising from optimal control and the Pontryagin maximum principle rather than a Cauchy problem. Here, we develop the theory for Cauchy problems because it is simpler and interesting in its own right. When we talk about initial conditions, it should be translated into initial and final conditions in the context of [5].

accuracy than the one for approaching the solution itself and that, on the other hand, one also obtains an accurate reconstruction of the fast variable that was previously impossible.

This lead us to the following conjecture, in terms of asymptotic results as  $\epsilon$  goes to 0. Firstly, such a choice of the initial value of the slow variable in the averaged system leads to an error between the moving average of the slow part of the real solution and the same moving average for the averaged system that is of order  $\epsilon^2$  instead of an error of order  $\epsilon$  between the instantaneous value at each time of the slow parts of the real solution and the solution of the averaged system obtained in classical first order averaging. Secondly, it allows a prediction of the fast variable through averaging with an error of order  $\epsilon$ , while classical averaging only predicts that the error does not go to infinity as  $\epsilon$  goes to 0.

The goal of this paper is to prove this conjecture. In order to obtain this result, we have to add to the averaged dynamics some corrective terms of order  $\epsilon^2$ , that were not explicitly present in [5]. These terms are of course reminiscent of averaging at order 2 in the sense of [8], [10]. We checked that our corrective terms are not the same as the (non unique) ones obtained in high order averaging. This makes sense since we do not reconstruct the solutions of  $(S_\epsilon)$  up to order 2 with respect to  $\epsilon$ ; the two approaches are however certainly not unrelated. Also, obtaining an asymptotic reconstruction of the fast variable via averaging was not yet proposed in the literature, to the best of our knowledge, and this has a clear practical interest.

## II. PROBLEM STATEMENT AND ASSUMPTIONS

We consider the case of one frequency “integrable” systems (see, for instance, [8]), i.e.,  $\varphi \in S^1$  is a phase variable and  $\omega$  is a function of the slow variables  $I$ . In local coordinates, we have:

$$(S_\epsilon): \begin{cases} \dot{I} = \epsilon f(I, \varphi), & I \in O \subset \mathbb{R}^n \\ \dot{\varphi} = \omega(I) + \epsilon g(I, \varphi), & \varphi \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \end{cases} \quad (1)$$

where  $f$ ,  $g$  and  $\omega$  are defined on  $O \times S^1$ , with  $O$  an open subset of  $\mathbb{R}^n$ . Notice that  $\varphi \in S^1$  amounts saying that  $f$  and  $g$  are  $2\pi$ -periodic with respect to  $\varphi$ .

Throughout this paper, we will make the following assumptions (that will be needed in Section III, when recalling the classical results, and in Section IV, when stating our main results):

- (A1)  $f$  and  $\omega$  are  $C^1$ -smooth in all their arguments and  $\omega(I) > \omega_{min} > 0$ , for all  $I \in O$ ;
- (A2)  $g$  is Lipschitz continuous;
- (A3)  $O$  is such that  $f$ ,  $\frac{\partial f}{\partial I}$ ,  $g$ ,  $\omega$ ,  $\frac{\partial \omega}{\partial I}$  are uniformly bounded on  $O \times S^1$ ;
- (A4)  $\frac{\partial f}{\partial I}$  has a global Lipschitz constant on  $O$ .

We define  $\bar{f}$  to be the average of  $f$  with respect to  $\varphi$ :

$$\bar{f}(I) = \frac{1}{2\pi} \int_0^{2\pi} f(I, \xi) d\xi. \quad (2)$$

In addition to assumptions (A1)-(A4), we work around an initial condition  $I_0$  which has to satisfy the following assumption:

- (ICA)  $I_0 \in O$  is such that the trajectory of  $\dot{x} = \bar{f}(x)$ , with  $x(0) = I_0$ , remains in  $O$ , for  $0 \leq t \leq T$ , for some  $T$  such that  $0 < T < +\infty$ .

Notice that the nature of assumptions (A1)-(A4) is very different from that of assumption (ICA). Indeed, (A1)-(A4) are regularity assumptions, while (ICA) is a non explosion assumption.

The aim of averaging is to approach in some way the solutions of  $(S_\epsilon)$  by the solutions of on “averaged system” whose right-hand side depends only on slow variables.

We recall in Section III the classical first order periodic averaging principle yielding an averaged system whose solutions approach those of  $(S_\epsilon)$  with an error of order  $\epsilon$ , as  $\epsilon$  goes to zero. Section IV constructs a modified averaged system that provides, with the right initialization, an average error over one oscillation that is of order  $\epsilon^2$  and also reconstructs the fast variable, with an error of order  $\epsilon$ ; a less general version of the result when the fast variable is time, i.e., when  $\dot{\varphi} = 1$  in  $(S_\epsilon)$ , is also provided. Section V is devoted to the proof of this less general result. The general proof is not given due to space limitation.

## III. FIRST ORDER PERIODIC AVERAGING

We define the averaged systems by:

$$(AvS_\epsilon): \quad \dot{J} = \epsilon \bar{f}(J),$$

where  $\bar{f}$  is given by (2). The basic result is that the solutions of  $(S_\epsilon)$  and  $(AvS_\epsilon)$  remain close (of order  $\epsilon$ ) for a time interval of order  $\frac{1}{\epsilon}$ :

*Theorem 1: Assume that assumptions (A1)-(A3) and (ICA) are satisfied. Consider a family of solutions  $(I_\epsilon(t), \varphi_\epsilon(t))$  of  $(S_\epsilon)$  (resp. a family of solutions  $J_\epsilon(t)$  of  $(AvS_\epsilon)$ ) such that  $\|I_\epsilon(0) - I_0\| \leq k\epsilon$  (resp.  $\|J_\epsilon(0) - I_0\| \leq k\epsilon$ ), for all  $\epsilon$  small enough, where  $k$  is a positive constant. Then there exist two positive constants  $\epsilon_0$  and  $c$  such that for all  $0 \leq \epsilon \leq \epsilon_0$ , the solution of the system  $(S_\epsilon)$  exists for  $0 \leq t \leq \frac{T}{\epsilon}$  and*

$$\|I_\epsilon(t) - J_\epsilon(t)\| \leq c\epsilon,$$

where  $c$  depends on  $k$  and on the bounds in assumption (A3) only.

The main idea of the proof is to reduce the original system to the averaged one up to the order  $\epsilon^2$  by a near identity change of variables and then apply the Gronwall lemma (which will be recalled in Section V) in order to bound the separation between the new variables and  $J_\epsilon$ , see for instance [8]. In [10], a more

recent and shorter proof, that relies on an inequality due to Besjes [2], is proposed. Observe that assumption (A4) is not needed in the above theorem, but it is needed for Theorems 1 and 3 stating our main results.

Notice that the above  $\mathcal{O}(\epsilon)$ -estimate is usually stated with  $I_\epsilon(0) = J_\epsilon(0) = I_0$ . However, allowing a  $\mathcal{O}(\epsilon)$ -discrepancy in the initial conditions does not change the proof.

Another interesting remark is that at first order approximations, one does not keep track of the phase:  $\varphi$  is reconstructed at order  $\mathcal{O}(1)$ .

In the more recent paper [9], the  $\mathcal{O}(\epsilon)$  approximation was replaced with a fully quantitative estimate for a time interval of order  $\frac{1}{\epsilon}$ . In certain cases, their approach also gives a reliable error estimate on time scales larger than  $\mathcal{O}(\frac{1}{\epsilon})$ . Although the estimate of [9] is proved theoretically, its computation requires the numerical solution of a system of differential equations that can be difficult to solve.

High order results are also available (see [8], [10]): it is possible to obtain  $\mathcal{O}(\epsilon^k)$ -error estimates valid for time  $\mathcal{O}(\frac{1}{\epsilon})$ , with  $k > 1$ , and (under more restrictive assumptions) weaker  $\mathcal{O}(\epsilon^{k-j})$ -error estimates but for longer time  $\mathcal{O}(\frac{1}{\epsilon^{j+1}})$ , see [10]. In the latter case, one has actually traded off  $j$  orders of accuracy for longer time of validity. These techniques are more involved: they make use of complicated transformations (that includes the fast oscillations) and imply tedious calculations made in a recursive way.

Therefore, our goal is firstly, to construct a simpler method giving a high order estimate, secondly, to obtain a solution for the averaged system which is (or remain very close to) the average of that of the original system and, finally, to make possible an asymptotic reconstruction of the fast variable. We will define the averaged system in a slightly different way than the classical one (by adding a corrective term in the averaged dynamics) and show that the average error between the exact trajectory and the averaged one is of order  $\mathcal{O}(\epsilon^2)$ .

#### IV. MAIN RESULTS

Consider the system  $(S_\epsilon)$ , given by (1). We suppose that the assumptions (A1)-(A4) and (ICA), introduced in Section II, hold for the system  $(S_\epsilon)$ .

In order to give our main result, we have to add to the averaged dynamics a corrective term of order  $\epsilon^2$ . This requires the introduction of a map  $B$  defined by the following proposition.

*Proposition 1: There is a unique map  $B : \mathcal{O} \times S^1 \mapsto \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  such that*

- (i)  $\frac{\partial B}{\partial \varphi}(I, \varphi) = \frac{\partial f}{\partial I}(I, \varphi) - \frac{\partial \bar{f}}{\partial I}(I)$ ,
- (ii)  $B(I, \varphi + 2\pi) = B(I, \varphi)$ ,
- (iii)  $\bar{B}(I) = \frac{1}{2\pi} \int_0^{2\pi} B(I, \xi) d\xi = 0$ .

and it is given by

$$B(I, \varphi) = \int_0^\varphi \frac{\partial f}{\partial I}(I, \xi) d\xi - \frac{1}{2\pi} \int_0^{2\pi} (\pi + \varphi - \xi) \frac{\partial f}{\partial I}(I, \xi) d\xi.$$

*Proof:* The map  $B(I, \varphi)$  is  $2\pi$ -periodic in  $\varphi$  because  $\varphi \mapsto \frac{\partial f}{\partial I}(I, \varphi) - \frac{\partial \bar{f}}{\partial I}(I)$  is  $2\pi$ -periodic and has zero mean. The other properties are clear and the proof of the above proposition reduces to elementary computations. In [8], the map  $B$  is denoted by  $B = \phi[\frac{\partial f}{\partial I}]$  and called the integral with zero mean value of the fluctuating part of  $\frac{\partial f}{\partial I}$ . ■

As we will see below,  $B$  plays an important role in our study since it allows us to define the corrective term  $E$  (needed for the definition of the corrected averaged system  $(CAvS_\epsilon)$ , see relation (5) below) which is crucial for obtaining an  $\mathcal{O}(\epsilon^2)$ -estimate.

##### A. Simplified case

To simplify the understanding of the paper, we will first state our main result for the particular and simpler case where  $\dot{\varphi}$  is constant and, specifically,  $\omega(I) = 1$  and  $g = 0$  in (1), i.e., the dynamics of the fast variable is given by  $\dot{\varphi} = 1$ . With the additional assumption that  $\varphi(0) = 0$ , this yields  $\varphi(t) = t[2\pi]$  and the system  $(S_\epsilon)$  reduces to:

$$(S_\epsilon) : \quad \dot{I} = \epsilon f(I, t). \quad (3)$$

Since the proofs should fit the length of a conference paper, we will show our results only for that particular case. The proof of the general case, for which  $\dot{\varphi} = \omega(I) + \epsilon g(I, \varphi)$ , uses similar arguments, but it is more involved and will be presented in a future paper.

An important remark is that contrary to the classical averaging results, instead of estimating  $\|I_\epsilon(t) - J_\epsilon(t)\|$ , we will estimate an integral error on one period of oscillations (here,  $2\pi$ ), namely we evaluate

$$\| \int_0^{2\pi} I_\epsilon(t + \tau) - J_\epsilon(t + \tau) d\tau \| \quad (4)$$

(in the general case this period will not be constant). Note that this is coordinate dependent, but one may check that the main result does not depend on the coordinates or on the norm choice.

We define the following "corrected averaged system"

$$(CAvS_\epsilon) : \quad \dot{J} = \epsilon \bar{f}(J) - \epsilon^2 E(J). \quad (5)$$

where the corrective term  $E$  is given by

$$E(I) = \frac{1}{2\pi} \int_0^{2\pi} B(I, \tau) f(I, \tau) d\tau.$$

We do not make any assumption on the initial conditions of the averaged system.

Our main result is given by the following theorem that gives a second order estimate based on changing

the initial condition of the averaged system and approaching the averages of the solutions instead of the solutions themselves.

*Theorem 2: Let  $(S_\epsilon)$  and  $(CAvS_\epsilon)$  be given by (3) and (5), respectively. Assume that assumptions (A1)-(A4) and (ICA) are satisfied. Consider a family  $(I_\epsilon(t), \varphi_\epsilon(t))$  of solutions of  $(S_\epsilon)$  such that  $\|I_\epsilon(0) - I_0\| \leq k\epsilon^2$  and a family  $J_\epsilon(t)$  of solutions of  $(CAvS_\epsilon)$  such that*

$$\|J_\epsilon(0) - I_0 - \frac{\epsilon}{2\pi} \int_0^{2\pi} (\pi - \tau) f(I_0, \tau) d\tau\| \leq k\epsilon^2, \quad (6)$$

for all  $\epsilon$  small enough, where  $k$  is a positive constant. Then, there exist two positive constants  $\epsilon_0$  and  $c$  such that for all  $0 \leq \epsilon \leq \epsilon_0$ , the solution of the system  $(S_\epsilon)$  exists for  $0 \leq t \leq \frac{T}{\epsilon}$ , remains in  $O \times S^1$  and

$$\left\| \int_0^{2\pi} I_\epsilon(t + \tau) - J_\epsilon(t + \tau) d\tau \right\| \leq c\epsilon^2, \quad (7)$$

where  $c$  depends on  $k$  and on the bounds in assumptions (A3)-(A4) only.

Notice that we no longer require  $J_\epsilon(0) = I_\epsilon(0)$  or  $\|J_\epsilon(0) - I_\epsilon(0)\| = \mathcal{O}(\epsilon)$ . The goal is to make the expression (4) small which leads to taking initial conditions that are not the same for  $I_\epsilon$  and  $J_\epsilon$ , but rather are such that (4) is small for  $t = 0$ . Indeed, we follow [5] and take for the averaged system an initial condition different from that of the original one. Condition (6) is, in fact, equivalent to

$$\left\| \int_0^{2\pi} I_\epsilon(\tau) - J_\epsilon(\tau) d\tau \right\| \leq k\epsilon^2.$$

So what is now important is not that the initial solutions of the original and averaged systems are close, but the fact that at  $t = 0$ , the average, over one period, of  $I_\epsilon(\tau) - J_\epsilon(\tau)$  is of order  $\epsilon^2$ . We do not claim that  $\|I_\epsilon(t) - J_\epsilon(t)\| \leq c\epsilon^2$  (we no longer compare the trajectories  $I_\epsilon(t)$  and  $J_\epsilon(t)$  at each time, but their averages).

Observe that the averaged system is now defined in a slightly different way than the classical one, since we added the corrective term  $E$  in the averaged dynamics. That term was not explicitly present in [5]. In order to apply Theorem 2, one has to construct the corrected averaged system  $(CAvS_\epsilon)$ . To this end, the corrective term  $E(J)$  has to be computed (for its computation, as well as for the computation of  $\bar{f}$ , one has to perform a numerical integration).

In  $(S_\epsilon)$ , we could also have taken

$$\dot{I} = \epsilon f_1(I, t) + \epsilon^2 f_2(I, t) + \epsilon^3 r(I, t, \epsilon).$$

Adding the terms of higher order in  $\epsilon$  does not change the above result, only the definition of the corrected averaged system changes very slightly: in that case, the equation of  $\dot{J}$  becomes  $\dot{J} = \epsilon \bar{f}_1(J) + \epsilon^2 \bar{f}_2(J) - \epsilon^2 E(J)$ .

Theorem 2 can be reformulated in terms of ‘‘moving averages’’. To a slow solution  $t \mapsto I_\epsilon(t)$  of  $(S_\epsilon)$ , we

associate its ‘‘moving average’’ denoted by  $\tilde{I}_\epsilon$ , and defined as follows:

$$\tilde{I}_\epsilon(t) = \frac{1}{2\pi} \int_0^{2\pi} I_\epsilon(t + \tau) d\tau. \quad (8)$$

Since the average is taken over one period of the fast variable, we may foresee that  $\tilde{I}_\epsilon$  is a low pass filtered version of  $I_\epsilon$ , i.e., it oscillates less. Similarly, we denote by  $\tilde{J}_\epsilon$  the moving average over one period of the solution  $t \mapsto J_\epsilon(t)$  of  $(CAvS_\epsilon)$ :

$$\tilde{J}_\epsilon(t) = \frac{1}{2\pi} \int_0^{2\pi} J_\epsilon(t + \tau) d\tau. \quad (9)$$

It is immediate that the estimate (7) of Theorem 2 is actually equivalent to

$$\|\tilde{I}_\epsilon(t) - \tilde{J}_\epsilon(t)\| \leq c\epsilon^2,$$

and it can be shown that (6) is, in fact, equivalent to

$$\|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| \leq k\epsilon^2,$$

where  $\tilde{I}_\epsilon(t)$  and  $\tilde{J}_\epsilon(t)$  correspond to the moving averages of  $I_\epsilon(t)$  and  $J_\epsilon(t)$ . So the difference between  $I_\epsilon(t)$  and  $J_\epsilon(t)$  remains of order  $\epsilon$ , what is now of order  $\epsilon^2$  is the average error between  $I_\epsilon(t)$  and  $J_\epsilon(t)$ .

To sum up, the interest of our method is three-fold: firstly, it allows us to give high order approximation without using involved transformations including the fast oscillations or reconstructing the slow variables. Secondly, it enables us to conclude that the solution of the averaged system is very close (of order  $\mathcal{O}(\epsilon^2)$ ) to the average of the solution of the original problem. Finally, as we will see when presenting the general case in Section IV-B, it makes possible the reconstruction of the phase up to order  $\epsilon$  (contrary to the classical results, where  $\varphi$  can be reconstructed at order  $\mathcal{O}(1)$ ). This can be very important for the applications. In Theorem 2, there is no statement on the fast variable, since  $\varphi(t) = t[2\pi]$  and this is not altered by averaging.

## B. General case

Let us now turn to the general case and consider the system  $(S_\epsilon)$ , given by (1). The situation is more complex since, on the one hand, we have to handle the fast variable  $\varphi$ , and on the other hand, the period of oscillations, when  $\epsilon = 0$ , is no longer fixed, but given by

$$T(I) = \frac{2\pi}{\omega(I)}. \quad (10)$$

The corrective term  $E$  has to be adapted and it becomes:

$$E(I) = \frac{\pi}{\omega^2(I)} (\omega'(I) \cdot \bar{f}(I)) \bar{f}(I) + \frac{1}{2\pi\omega(I)} \int_0^{2\pi} [B(I, \xi) f(I, \xi) + (g(I, \xi) - \bar{g}(I))(f(I, \xi) - \bar{f}(I))] d\xi$$

The corrected averaged system is defined by

$$(CAvS_\epsilon) : \begin{cases} \dot{J} &= \epsilon \bar{f}(J) - \epsilon^2 E(J) \\ \dot{\psi} &= \omega(J) + \epsilon(\bar{g}(J) - \frac{\pi}{\omega(J)} \omega'(J) \cdot \bar{f}(J)). \end{cases} \quad (11)$$

Notice that contrary to the system  $(AvS_\epsilon)$  for the classical first order averaging (where only the dynamics corresponding to the slow variables appear), in  $(CAvS_\epsilon)$ , given by (11), we have an additional equation corresponding to the fast variable. This will allow us to reconstruct the phase up to order  $\epsilon$ .

The following theorem is the counterpart of Theorem 2 for the general case. In addition to Theorem 2 (where there is no statement on the fast variable, since  $\varphi(t) = t[2\pi]$  and this is not altered by averaging), Theorem 3 also allows an asymptotic reconstruction of the fast variable.

*Theorem 3: Assume that assumptions (A1)-(A4) and (ICA) are satisfied. Consider a family  $(I_\epsilon(t), \varphi_\epsilon(t))$  of solutions of  $(S_\epsilon)$  such that  $\|I_\epsilon(0) - I_0\| \leq k\epsilon^2$  and  $\|\varphi_\epsilon(0) - \varphi_0\| \leq k\epsilon$  and a family  $(J_\epsilon(t), \psi_\epsilon(t))$  of solutions of  $(CAvS_\epsilon)$  such that*

$$\|J_\epsilon(0) - I_0 - \frac{\epsilon}{2\pi\omega(I_0)} \int_0^{2\pi} (\pi - \xi)f(I_0, \varphi_0 + \xi) d\xi\| \leq k\epsilon^2 \quad (12)$$

and  $\|\psi_\epsilon(0) - \varphi_0\| \leq k\epsilon$ , for all  $\epsilon$  small enough, where  $k$  is a positive constant. Then, there exist two positive constants  $\epsilon_0$  and  $c$  such that for all  $0 \leq \epsilon \leq \epsilon_0$ , the solution of the system  $(S_\epsilon)$  exists for  $0 \leq t \leq \frac{T}{\epsilon}$ , remains in  $O \times S^1$  and

$$\left\| \int_0^{T(I_\epsilon(t))} I_\epsilon(t + \tau) - J_\epsilon(t + \tau) d\tau \right\| \leq c\epsilon^2 \quad (13)$$

and

$$\left\| \int_0^{T(I_\epsilon(t))} \varphi_\epsilon(t + \tau) - \psi_\epsilon(t + \tau) d\tau \right\| \leq c\epsilon, \quad (14)$$

where  $c$  depends on  $k$  and on the bounds in assumptions (A3)-(A4) only.

As for the simplified case, Theorem 3 can be stated in terms of moving averages. Now, the moving average of a solution  $t \mapsto (I_\epsilon(t), \varphi_\epsilon(t))$  of  $(S_\epsilon)$  is defined as follows:

$$\begin{aligned} \tilde{I}_\epsilon(t) &= \frac{1}{T(I_\epsilon(t))} \int_0^{T(I_\epsilon(t))} I_\epsilon(t + \tau) d\tau \\ \tilde{\varphi}_\epsilon(t) &= \frac{1}{T(I_\epsilon(t))} \int_0^{T(I_\epsilon(t))} \varphi_\epsilon(t + \tau) d\tau, \end{aligned} \quad (15)$$

and that of a solution  $t \mapsto (J_\epsilon(t), \psi(t))$  of  $(CAvS_\epsilon)$  by :

$$\begin{aligned} \tilde{J}_\epsilon(t) &= \frac{1}{T(J_\epsilon(t))} \int_0^{T(J_\epsilon(t))} J_\epsilon(t + \tau) d\tau \\ \tilde{\psi}_\epsilon(t) &= \frac{1}{T(J_\epsilon(t))} \int_0^{T(J_\epsilon(t))} \psi_\epsilon(t + \tau) d\tau, \end{aligned} \quad (16)$$

where  $T(I)$  is given by (10). Similarly to the simplified case presented in the previous section, it can be shown that the estimates (13) and (14) of Theorem 1 are actually equivalent to

$$\|\tilde{I}_\epsilon(t) - \tilde{J}_\epsilon(t)\| \leq c\epsilon^2 \text{ and } \|\tilde{\varphi}_\epsilon(t) - \tilde{\psi}_\epsilon(t)\| \leq c\epsilon,$$

and (6) is equivalent to

$$\|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| \leq k\epsilon^2.$$

One of the principal interests of our method is that it makes possible the reconstruction of the phase up to order  $\epsilon$  (contrary to the classical results, where  $\varphi$  can be reconstructed at order  $\mathcal{O}(1)$  only). This can be very important for the applications.

## V. PROOF OF THE MAIN RESULT (SIMPLIFIED CASE)

As announced, we will prove only Theorem 2, but not the more general result Theorem 3. The fact that the period depends on the slow variables  $I$  instead of being constant (like in Theorem 2) makes the proof of Theorem 3 longer and more involved. Proving its first estimate (13) follows however the same line as that of the proof of Theorem 2. In order to obtain the second estimate (14), one has to notice that  $\varphi$  is, roughly speaking, the integral of a function of  $I$ , on a time interval of length  $\frac{T}{\epsilon}$ , and that  $I$  is estimated up to an integral error of order  $\epsilon^2$ .

From now on (until the end of this paper), we assume  $\dot{\varphi} = 1$  and  $\varphi(0) = 0$  in (1).

### A. Notations and useful results

Throughout, we will denote by  $\text{Lip } f$  (resp.  $\text{Sup } f$ ) the Lipschitz constant (resp. the supremum) of  $f$ . When we write  $x(t) = y(t') + \mathcal{O}(\epsilon^j)$ , we mean that there exists a positive constant  $k$  such that  $\|\mathcal{O}(\epsilon^j)\| \leq k\epsilon^j$ . In our study,  $k$  depends on the constants associated to  $f$  and  $\bar{f}$  only ( $\text{Lip } f$ ,  $\text{Sup } f$ ,  $\text{Lip } \frac{\partial f}{\partial I}$ , etc.). It is left to the reader to verify it.

In the proof of our main theorem, we will need the Gronwall lemma that will allow us to bound the separation between  $\tilde{I}_\epsilon$  and  $\tilde{J}_\epsilon$ . Gronwall lemma appears frequently in this kind of situation and plays a fundamental role in finding estimates regarding differential equations. We will state it in an integral form. See, for instance, [8] for the proof.

*Lemma 1: (Gronwall lemma [8]) Let  $x(t)$ ,  $a(t)$  and  $\gamma(t)$  be positive continuous functions defined on an interval  $[0, t]$ . Assume that  $\gamma$  is of class  $\mathcal{C}^1$  and the following inequality holds:*

$$x(t) \leq \gamma(t) + \int_0^t a(s)x(s) ds.$$

Then we have:

$$x(t) \leq \gamma(0)e^{\int_0^t a(s) ds} + \int_0^t \gamma'(s)e^{\int_s^t a(u) du} ds.$$

In the proof of Theorem 2, we will also need the following technical results for which we suppose that assumptions (A1)-(A4) and (ICA) hold.

*Proposition 2: Consider the system  $(S_\epsilon)$ , given by (3), the moving average  $\tilde{I}_\epsilon$ , given by (8), of the solution  $I_\epsilon$  of  $(S_\epsilon)$  and  $\tau$  such that  $0 \leq \tau \leq 2\pi$ . Then  $I_\epsilon(t + \tau) = \tilde{I}_\epsilon(t) + \mathcal{O}(\epsilon)$ .*

*Proof:* We have:

$$I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t) = \frac{1}{2\pi} \int_0^{2\pi} I_\epsilon(t + \tau) - I_\epsilon(t + \tau') d\tau'.$$

Applying the mean value theorem, for  $1 \leq i \leq n$  (where  $n$  is the dimension of  $I$ ), there exists  $\tau_i$  between  $\tau$  and  $\tau'$  such that  $I_\epsilon^i(t + \tau) - I_\epsilon^i(t + \tau') = f^i(I_\epsilon(t + \tau_i))(\tau - \tau')$  and we immediately deduce  $I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t) = \mathcal{O}(\epsilon)$  and

$$\|I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t)\| \leq 2\pi\epsilon(\text{Sup } f)$$

thus  $\|\mathcal{O}(\epsilon)\| \leq k\epsilon$ , with  $k$  depending on  $\text{Sup } f$  only. ■

*Proposition 3:* Consider the corrected averaged system  $(CAvS_\epsilon)$ , given by (5), the moving average  $\tilde{J}_\epsilon$ , given by (9), of the solution  $J_\epsilon$  of  $(CAvS_\epsilon)$  and  $\tau$  such that  $0 \leq \tau \leq 2\pi$ . Then  $J_\epsilon(t + \tau) = \tilde{J}_\epsilon(t) + \epsilon f(J_\epsilon(t + \pi))(\tau - \pi) + \mathcal{O}(\epsilon^2)$ .

*Remark 1:* In some of the proofs, we will need only the weaker result  $J_\epsilon(t + \tau) = \tilde{J}_\epsilon(t) + \mathcal{O}(\epsilon)$ , with  $\|\mathcal{O}(\epsilon)\| \leq k\epsilon$ , where  $k$  is a positive constant, depending on  $\text{Sup } f$  only. This can be proven following exactly the same line as that of Proposition 2.

*Proof:* We have

$$\begin{aligned} J_\epsilon(t + \tau) &= J_\epsilon(t) + \epsilon \int_t^{t+\tau} \bar{f}(J_\epsilon(\tau')) d\tau' \\ &= J_\epsilon(t) + \epsilon \tau \bar{f}(J_\epsilon(t)) + \mathcal{O}(\epsilon^2) \end{aligned}$$

and

$$\begin{aligned} \tilde{J}_\epsilon(t) &= J_\epsilon(t) + \frac{\epsilon}{2\pi} \int_0^{2\pi} \tau \bar{f}(J_\epsilon(t)) d\tau + \mathcal{O}(\epsilon^2) \\ &= J_\epsilon(t) + \epsilon \pi \bar{f}(J_\epsilon(t)) + \mathcal{O}(\epsilon^2). \end{aligned}$$

From this, it follows immediately  $J_\epsilon(t + \tau) = \tilde{J}_\epsilon(t) + \epsilon \bar{f}(J_\epsilon(t + \pi))(\tau - \pi) + \mathcal{O}(\epsilon^2)$ . ■

*Proposition 4:* Consider the system  $(S_\epsilon)$ , given by (3) and the moving average  $\tilde{I}_\epsilon$ , given by (8), of its solution  $I_\epsilon$ . Then

$$\dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) = \epsilon^2 \dot{A}_\epsilon(t) + \mathcal{O}(\epsilon^3),$$

where  $\|A_\epsilon(t)\|$  is bounded by a positive constant and  $F(\tilde{I}_\epsilon, \epsilon) = \bar{f}(\tilde{I}_\epsilon) - \epsilon E(\tilde{I}_\epsilon)$ .

*Remark 2:* In the proof of our main result we only need  $\|A_\epsilon(t)\|$  to be bounded. There is no assumption on  $\|\dot{A}_\epsilon(t)\|$ .

*Proof:* According to the definitions of  $\tilde{I}_\epsilon$  and  $\bar{f}$ , we have:

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} f(I_\epsilon(t + \tau), t + \tau) d\tau - \epsilon \bar{f}(\tilde{I}_\epsilon) + \epsilon^2 E(\tilde{I}_\epsilon(t)) \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} f(I_\epsilon(t + \tau), t + \tau) - f(\tilde{I}_\epsilon(t), t + \tau) d\tau + \epsilon^2 E(\tilde{I}_\epsilon(t)) \end{aligned}$$

Applying a Taylor expansion of  $f(I_\epsilon(t + \tau), t + \tau)$ , with respect to its first entry, around  $\tilde{I}_\epsilon(t)$  and since  $I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t) = \mathcal{O}(\epsilon)$ , see Proposition 2 above, we deduce:

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} \frac{\partial f}{\partial I}(\tilde{I}_\epsilon(t), t + \tau) (I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t)) d\tau \\ &\quad + \epsilon^2 E(\tilde{I}_\epsilon(t)) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Now recall that  $\frac{\partial B}{\partial t}(I, t) = \frac{\partial f}{\partial t}(I, t) - \frac{\partial \bar{f}}{\partial I}(I)$ , where  $B(I, t)$  is  $2\pi$ -periodic and of zero mean. Replacing in the above expression  $\frac{\partial f}{\partial t}(\tilde{I}_\epsilon(t), t + \tau)$  by  $\frac{\partial \bar{f}}{\partial I}(\tilde{I}_\epsilon(t)) + \frac{\partial B}{\partial t}(\tilde{I}_\epsilon(t), t + \tau)$  and observing that  $\int_0^{2\pi} I_\epsilon(t + \tau) - \tilde{I}_\epsilon(t) d\tau = 0$  and  $\int_0^{2\pi} \frac{\partial B}{\partial t}(\tilde{I}_\epsilon(t), t + \tau) \tilde{I}_\epsilon(t) d\tau = 0$ , we obtain:

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \frac{\epsilon}{2\pi} \int_0^{2\pi} \frac{\partial B}{\partial t}(\tilde{I}_\epsilon(t), t + \tau) I_\epsilon(t + \tau) d\tau \\ &\quad + \epsilon^2 E(\tilde{I}_\epsilon(t)) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Integrating by parts, we have:

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \frac{\epsilon}{2\pi} [B(\tilde{I}_\epsilon(t), t + \tau) I_\epsilon(t + \tau)]_{\tau=0}^{\tau=2\pi} \\ &\quad - \frac{\epsilon^2}{2\pi} \int_0^{2\pi} B(\tilde{I}_\epsilon(t), t + \tau) f(I_\epsilon(t + \tau), t + \tau) d\tau \\ &\quad + \epsilon^2 E(\tilde{I}_\epsilon(t)) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Recall that  $E(\tilde{I}_\epsilon) = \frac{1}{2\pi} \int_0^{2\pi} B(\tilde{I}_\epsilon, \tau) f(\tilde{I}_\epsilon, \tau) d\tau$  and due to the periodicity of  $B$  and  $f$ , we deduce  $E(\tilde{I}_\epsilon(t)) = \frac{1}{2\pi} \int_0^{2\pi} B(\tilde{I}_\epsilon(t), t + \tau) f(\tilde{I}_\epsilon(t), t + \tau) d\tau$ . It can be easily shown that  $f(I_\epsilon(t + \tau), t + \tau) = f(\tilde{I}_\epsilon(t), t + \tau) + \mathcal{O}(\epsilon)$  and by replacing it in the integral of the above relation, we get

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \frac{\epsilon}{2\pi} B(\tilde{I}_\epsilon(t), t) (I_\epsilon(t + 2\pi) - I_\epsilon(t)) \\ &\quad + \mathcal{O}(\epsilon^3). \end{aligned}$$

Let  $C(I, t)$  be a  $2\pi$ -periodic primitive of  $B(I, t)$ . Notice that  $C$  is bounded. We set

$$A_\epsilon(t) = \frac{1}{2\pi\epsilon} C(\tilde{I}_\epsilon(t), t) (I_\epsilon(t + 2\pi) - I_\epsilon(t)).$$

Then

$$\begin{aligned} \dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) &= \\ &= \epsilon^2 [\dot{A}_\epsilon(t) - \frac{1}{2\pi} \frac{\partial C}{\partial I}(\tilde{I}_\epsilon(t), t) \frac{\dot{\tilde{I}}_\epsilon(t)}{\epsilon} (I_\epsilon(t + 2\pi) - I_\epsilon(t)) \\ &\quad - \frac{1}{2\pi} C(\tilde{I}_\epsilon(t), t) (f(I_\epsilon(t + 2\pi), t) - f(I_\epsilon(t), t))] + \mathcal{O}(\epsilon^3), \end{aligned}$$

which gives

$$\dot{\tilde{I}}_\epsilon(t) - \epsilon F(\tilde{I}_\epsilon(t), \epsilon) = \epsilon^2 \dot{A}_\epsilon(t) + \mathcal{O}(\epsilon^3),$$

since  $\frac{\dot{\tilde{I}}_\epsilon(t)}{\epsilon} \leq \text{Sup } f$  and  $I_\epsilon(t + 2\pi) = I_\epsilon(t) + \mathcal{O}(\epsilon)$  (more precisely,  $\|I_\epsilon(t + 2\pi) - I_\epsilon(t)\| \leq 2\pi\epsilon \text{Sup } f$ ). From the last inequality and the definition of  $A_\epsilon(t)$ , we easily deduce that  $\|A_\epsilon(t)\| \leq (\text{Sup } C)(\text{Sup } f)$ . ■

*Proposition 5:* Consider the corrected averaged system  $(CAvS_\epsilon)$ , given by (5), and the moving average  $\tilde{J}_\epsilon$ , given by (9), of its solution  $J_\epsilon$ . Then

$$\epsilon F(\tilde{J}_\epsilon, \epsilon) - \dot{\tilde{J}}_\epsilon = \mathcal{O}(\epsilon^3),$$

where  $F(\tilde{J}_\epsilon, \epsilon) = \bar{f}(\tilde{J}_\epsilon) - \epsilon E(\tilde{J}_\epsilon)$ .

*Proof:* We have

$$\epsilon F(\tilde{J}_\epsilon(t), \epsilon) - \dot{\tilde{J}}_\epsilon(t) =$$

$$\begin{aligned}
&= \bar{\epsilon} \bar{f}(\tilde{J}_\epsilon(t)) - \epsilon^2 E(\tilde{J}_\epsilon(t)) \\
&\quad - \frac{\epsilon}{2\pi} \int_0^{2\pi} \bar{f}(J_\epsilon(t+\tau)) - \epsilon E(J_\epsilon(t+\tau)) d\tau \\
&= \frac{\epsilon}{2\pi} \int_0^{2\pi} \bar{f}(\tilde{J}_\epsilon(t)) - \bar{f}(J_\epsilon(t+\tau)) d\tau \\
&\quad + \frac{\epsilon^2}{2\pi} \int_0^{2\pi} E(J_\epsilon(t+\tau)) - E(\tilde{J}_\epsilon(t)) d\tau.
\end{aligned}$$

Since  $J_\epsilon(t+\tau) = \tilde{J}_\epsilon(t) + \mathcal{O}(\epsilon)$ , we immediately have  $E(J_\epsilon(t+\tau)) = E(\tilde{J}_\epsilon(t)) + \mathcal{O}(\epsilon)$ . It remains to show that  $\frac{1}{2\pi} \int_0^{2\pi} \bar{f}(\tilde{J}_\epsilon(t)) - \bar{f}(J_\epsilon(t+\tau)) d\tau = \mathcal{O}(\epsilon^2)$ . By Proposition 3,  $J_\epsilon(t+\tau) = \tilde{J}_\epsilon(t) + \bar{\epsilon} \bar{f}(J_\epsilon(t+\pi))(\tau - \pi) + \mathcal{O}(\epsilon^2)$ . Thus  $\bar{f}(\tilde{J}_\epsilon(t)) - \bar{f}(J_\epsilon(t+\tau)) = -\epsilon \frac{\partial \bar{f}}{\partial J}(\tilde{J}_\epsilon(t)) \bar{f}(J_\epsilon(t+\pi))(\tau - \pi) + \mathcal{O}(\epsilon^2)$  and it follows that  $\frac{1}{2\pi} \int_0^{2\pi} \bar{f}(\tilde{J}_\epsilon(t)) - \bar{f}(J_\epsilon(t+\tau)) d\tau = \mathcal{O}(\epsilon^2)$ . Finally, we obtain  $\epsilon F(\tilde{J}_\epsilon, \epsilon) - \tilde{J}_\epsilon = \mathcal{O}(\epsilon^3)$ . ■

*Proposition 6:* Consider a family  $(I_\epsilon(t), \varphi_\epsilon(t))$  of solutions of  $(S_\epsilon)$  such that  $\|I_\epsilon(0) - I_0\| \leq k\epsilon^2$ , for all  $\epsilon$  small enough, where  $k$  is a positive constant, and a family  $J_\epsilon(t)$  of solutions of  $(CAvS_\epsilon)$ , given by (5). We have equivalence between

$$\|\tilde{J}_\epsilon(0) - \tilde{I}_\epsilon(0)\| = \mathcal{O}(\epsilon^2)$$

and

$$\|J_\epsilon(0) - I_0 - \frac{\epsilon}{2\pi} \int_0^{2\pi} (\pi - \tau) f(I_0, \tau) d\tau\| = \mathcal{O}(\epsilon^2).$$

*Remark 3:* The above proposition also allows us to express the initial condition  $J_\epsilon(0)$  of the averaged system  $(CAvS_\epsilon)$  as a function of  $I_0$ . Indeed, we can take:

$$J_\epsilon(0) = I_0 + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\pi - \tau) f(I_0, \tau) d\tau.$$

Notice that in order to compute  $J_\epsilon(0)$ , we only have to calculate  $\int_0^{2\pi} (\pi - s) f(I_0, s) ds$ .

*Proof:* We have:

$$\begin{aligned}
I_\epsilon(t) &= I_0 + \epsilon \int_0^t f(I(s), s) ds + \mathcal{O}(\epsilon^2) \\
&= I_0 + \epsilon \int_0^t f(I_0, s) ds + \mathcal{O}(\epsilon^2)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{I}_\epsilon(0) &= I_0 + \frac{\epsilon}{2\pi} \int_0^{2\pi} \int_0^\tau f(I_0, s) ds d\tau + \mathcal{O}(\epsilon^2) \\
&= I_0 + \frac{\epsilon}{2\pi} \int_0^{2\pi} (2\pi - s) f(I_0, s) ds + \mathcal{O}(\epsilon^2). \tag{17}
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_\epsilon(t) &= J_\epsilon(0) + \epsilon \int_0^t \bar{f}(J(s)) - \epsilon E(J(s)) ds \\
&= J_\epsilon(0) + \epsilon \bar{f}(I_0) t + \mathcal{O}(\epsilon^2)
\end{aligned}$$

and

$$\tilde{J}_\epsilon(0) = J_\epsilon(0) + \frac{\epsilon}{2\pi} \int_0^{2\pi} \pi f(I_0, s) ds + \mathcal{O}(\epsilon^2). \tag{18}$$

It follows from (17) and (18) that  $\|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| = \mathcal{O}(\epsilon^2)$  if and only if  $\|J_\epsilon(0) - I_0 - \frac{\epsilon}{2\pi} \int_0^{2\pi} (\pi - s) f(I_0, s) ds\| = \mathcal{O}(\epsilon^2)$ . ■

## B. Proof of Theorem 2

We have

$$\dot{\tilde{I}}_\epsilon - \dot{\tilde{J}}_\epsilon = \dot{\tilde{I}}_\epsilon - \epsilon F(\tilde{I}_\epsilon, \epsilon) + \epsilon (F(\tilde{I}_\epsilon, \epsilon) - F(\tilde{J}_\epsilon, \epsilon)) + \epsilon F(\tilde{J}_\epsilon, \epsilon) - \dot{\tilde{J}}_\epsilon,$$

where  $F(\tilde{I}_\epsilon, \epsilon) = \bar{f}(\tilde{I}_\epsilon) - \epsilon E(\tilde{I}_\epsilon)$ . By Propositions 4 and 5, we get

$$\dot{\tilde{I}}_\epsilon(t) - \dot{\tilde{J}}_\epsilon(t) = \epsilon^2 \dot{A}_\epsilon(t) + \epsilon (F(\tilde{I}_\epsilon(t), \epsilon) - F(\tilde{J}_\epsilon(t), \epsilon)) + \epsilon^3 b(\epsilon, t),$$

where  $b$  is bounded. Thus

$$\begin{aligned}
\tilde{I}_\epsilon(t) - \tilde{J}_\epsilon(t) &= \tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0) + \epsilon^2 (A_\epsilon(t) - A_\epsilon(0)) \\
&\quad + \int_0^t \epsilon (F(\tilde{I}_\epsilon(\tau), \epsilon) - F(\tilde{J}_\epsilon(\tau), \epsilon)) \\
&\quad + \epsilon^3 b(\epsilon, \tau) d\tau,
\end{aligned}$$

and we have

$$\begin{aligned}
\|\tilde{I}_\epsilon - \tilde{J}_\epsilon\| &\leq \|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| \\
&\quad + \epsilon^2 (\|A_\epsilon\| + \|A_\epsilon(0)\|) + \epsilon^3 (\text{Sup } b) t \\
&\quad + \epsilon \int_0^t (\text{Lip } F) \|\tilde{I}_\epsilon - \tilde{J}_\epsilon\| d\tau.
\end{aligned}$$

Now, recall that according to the theorem's assumptions  $\|J_\epsilon(0) - I_0 - \frac{\epsilon}{2\pi} \int_0^{2\pi} (\pi - \tau) f(I_0, \tau) d\tau\| = \mathcal{O}(\epsilon^2)$ . Therefore, by Proposition 6, we have  $\|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| = \mathcal{O}(\epsilon^2)$ . The result follows immediately by applying Gronwall lemma with  $x(t) = \tilde{I}_\epsilon(t) - \tilde{J}_\epsilon(t)$ ,  $\|\tilde{I}_\epsilon(0) - \tilde{J}_\epsilon(0)\| = \mathcal{O}(\epsilon^2)$  and for  $t$  such that  $0 \leq t \leq \frac{T}{\epsilon}$ .

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## REFERENCES

- [1] V. Arnold. *Mathematical Methods of Classical Mechanics*. Graduate Text in Mathematics 60, Springer-Verlag New York, 1978.
- [2] J. Besjes. On asymptotic methods for non-linear differential equations. *Journal de Mécanique*, 8:357–373, 1969.
- [3] N. N. Bogoliubov and Y. Mitropolsky. *Asymptotic methods in the theory of non-linear oscillations*. Gordon and Breach, New York, 1961.
- [4] F. Chaplais. Averaging and deterministic optimal control. *SIAM J. Control Optim.*, 25(3):767–780, 1987.
- [5] T. Dargent. Initial and final boundaries transformation when solving optimal control problems with averaging techniques and application to low thrust orbit transfer. In *Proceedings of the 66th International Astronautical Congress IAC 2015, Jerusalem, Israel*, 2015.
- [6] T. N. Edelbaum. Optimum power-limited orbit transfer in strong gravity fields. *AIAA J.*, 3:921–925, 1965.
- [7] S. Géffroy. *Généralisation des techniques de moyennation en contrôle optimal – Application aux problèmes de rendez-vous orbitaux en poussée faible*. Thèse de doctorat, Institut National Polytechnique de Toulouse, Toulouse, France, Oct. 1997.
- [8] P. Lochak and C. Meunier. *Multiphase averaging for classical systems. With applications to adiabatic theorems*, volume 72. Applied Mathematical Sciences. Springer-Verlag, 1988.
- [9] C. Morosi and L. Pizzocchero. On the average principle for one-frequency systems. *Journal of Physics A: Mathematical and General*, 39(14):3673–3702, 2006.
- [10] J. Sanders, F. Verhulst, and J. Murdock. *Averaging methods in nonlinear dynamical systems*, volume 59. Springer, 2007.