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ON THE LINK BETWEEN FINITE DIFFERENCES AND DERIVATIVES OF POLYNOMIALS

KOLOSOV PETRO

ABSTRACT. The main aim of this paper to establish the relations between forward, backward and central finite and divided differences (that is discrete analog of the derivative) and partial and ordinary high-order derivatives of the polynomials.

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1. INTRODUCTION

Let introduce the basic definition of finite difference. Finite difference is difference between function values with constant increment. There are three types of finite differences: forward, backward and central. Generally, the first order forward difference could be noted as: $\Delta_h f(x) = f(x+h) - f(x)$ backward, respectively, is $\nabla_h f(x) = f(x) - f(x-h)$ and central $\delta_h f(x) = f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)$, where $h = const$, (see [1], [2], [3]). If increment is enough small, but constant, we can say that finite difference divided by increment tends to derivative, but not equals. The error of this approximation could be counted next: $\frac{\Delta_h f(x)}{h} - f'(x) = \mathcal{O}(h) \rightarrow 0$, where h - increment, such that, $h \rightarrow 0$. By means of induction as well right for backward difference. More exact approximation we have using central difference, that is: $\frac{\delta_h f(x)}{h} - f'(x) = \mathcal{O}(h^2)$, note that function should be twice differentiable. The finite difference is the discrete analog of the derivative (see [4]), the main distinction is constant increment of the function's argument, while difference to be taken. Backward and forward differences are opposite each other. More generally, high order finite differences (forward, backward and central, respectively) could be denoted as (see [7]):

$$(1.1) \quad \Delta_h^k f(x) = \Delta^{k-1} f(x+h) - \Delta^{k-1} f(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f(x+(n-k)h)$$

$$(1.2) \quad \delta_h^n f(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} \cdot f\left(x + \left(\frac{n}{2} - k\right)h\right)$$

$$(1.3) \quad \nabla_h^k f(x) = \nabla^{k-1} f(x) - \nabla^{k-1} f(x-h) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f(x-kh)$$

Let describe the main properties of finite difference operator, they are next (see [5])

- (1) Linearity rules $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$
 $\delta(f(x) + g(x)) = \delta f(x) + \delta g(x)$
 $\nabla(f(x) + g(x)) = \nabla f(x) + \nabla g(x)$
- (2) $\Delta(C \cdot f(x)) = C \cdot \Delta f(x)$, $\nabla(C \cdot f(x)) = C \cdot \nabla f(x)$,
 $\delta(C \cdot f(x)) = C \cdot \delta f(x)$
- (3) Constant rule $\Delta C = \nabla C = \delta C = 0$

Strictly speaking, divided difference (see [6]) with constant increment is discrete analog of derivative, when finite difference is discrete analogue of function's differential. They are close related to each other. To show thus, let define the divided difference.

Definition 1.4. Divided difference of fixed increment definition (forward, central, backward respectively)

$$f^+[x_i, x_j] := \frac{f(x_j) - f(x_i)}{x_j - x_i}, \quad j > i, \quad \Delta x \geq 1$$

$$f^-[x_i, x_j] := \frac{f(x_i) - f(x_j)}{x_i - x_j}, \quad j < i, \quad \nabla x \geq 1$$

$$f^c[x_i] := \frac{f(x_{i+m}) - f(x_{i-m})}{2m}$$

Hereby, divided difference could be represented from the finite difference, let be $j = i \pm const$

$$f^\pm[x_i, x_j] \equiv \frac{\Delta f(x_j)}{\Delta x} \equiv \frac{\nabla f(x_i)}{\nabla x}$$

$$f_c[x_i] \equiv \frac{\delta f(x_{i\pm m})}{2m} \equiv \frac{\delta f(x_{i\pm m})}{\delta x}$$

The n -order

$$f^\pm[x_i, x_j]^n \equiv \frac{\Delta^n f(x_j)}{\Delta x^n} \equiv \frac{\nabla^n f(x_i)}{\nabla x^n}$$

$$f_c[x_i]^n \equiv \frac{\delta^n f(x_{i\pm m})}{(2m)^n} \equiv \frac{\delta^n f(x_{i\pm m})}{\delta x^n}$$

Each properties, which holds for finite differences holds for divided differences as well.

2. DEFINITIONS FOR x_g DISTRIBUTION

Let be variable $x_g : x_g = g \cdot C$, $C = x_{g+1} - x_g = \text{const}$, $C \in \mathbb{R} \rightarrow x_g \in \mathbb{R}$, $g \in \mathbb{Z}$. Define the finite difference of function of such argument, we take $C = h$ and rewrite forward, backward and central differences of some analytically defined function $f(x_i)$ next way: $\Delta f(x_{i+1}) = f(x_{i+1}) - f(x_i)$, $\nabla f(x_{i-1}) = f(x_i) - f(x_{i-1})$, $\delta f(x_i) = f\left(x_{i+\frac{1}{2}}\right) - f\left(x_{i-\frac{1}{2}}\right)$. The n -th differences of such a function could be written as

$$(2.1) \quad \Delta^n f(x_{i+1}) = \Delta^{n-1} f(x_{i+1}) - \Delta^{n-1} f(x_i) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f(x_{i+n-k})$$

$$(2.2) \quad \delta^n f(x_i) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f\left(x_{i+\frac{n}{2}-k}\right)$$

$$(2.3) \quad \nabla^n f(x_{i-1}) = \nabla^{n-1} f(x_i) - \nabla^{n-1} f(x_{i-1}) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f(x_{i-n+k})$$

Let be differences $\Delta f(x_{i+1})$, $\delta f(x_i)$, $\nabla f(x_{i-1})$, such that $i \in \mathbb{Z}$ and differences is taken starting from point i , which divides the space $\mathbb{Z} = \mathbb{Z}^- \cup \mathbb{Z}^+$ symmetrically (note that $+/-$ symbols mean the left and right sides of start point i , i.e backward and forward direction), this way we have $(i+1) \in \mathbb{Z}^+$, $(i-1) \in \mathbb{Z}^-$, $i \in (\mathbb{Z}^+, \mathbb{Z}^-)$. Let derive some properties of that distribution:

- (1) $\max(\mathbb{Z}^-) = \min(\mathbb{Z}^+) = i$
- (2) Forward difference is taken starting from $\min(\mathbb{Z}^+)$, while backward from $\max(\mathbb{Z}^-)$
- (3) $\text{card}(\mathbb{Z}^+) = \text{card}(\mathbb{Z}^-)$
- (4) Maximal order of forward difference in which it is not equal to zero is $\max(\mathbb{Z}^+)$
- (5) Maximal order of backward difference in which it is not equal to zero is $\min(\mathbb{Z}^-)$
- (6) Maximal order of central difference in which it is not equal to zero is $\max(\mathbb{Z}^+)$
- (7) Forward and backward difference equal each other by absolute value, while to be taken from $i = 0$

3. DIFFERENCE AND DERIVATIVE OF POWER FUNCTION

Since the n -order polynomial defined as summation of argument to power multiplied by coefficient, with higher power n , let describe a few properties of finite (divided) difference of power function.

Lemma 3.1. *For any power function with natural number as exponent holds the equality between forward, backward and central divided differences, and derivative with order respectively to exponent and equals to exponent under factorial sign.*

Proof. Let be function $f(x) = x^n$, $n \in \mathbb{N}$. The derivative of power function, $f'(x) = nx^{n-1}$, so k -th derivative $f^{(k)}(x) = n \cdot (n-1) \cdot \dots \cdot (n-k+1) \cdot x^{n-k}$, $n > k$. Using limit notation, we have: $\lim_{m \rightarrow n^-} f^{(m)}(x) = f^{(n)}(x) = n!$. Let rewrite expressions (2.1, 2.2, 2.3) according to definition $x_i = i \cdot \Delta x$, note that $\Delta x \equiv \nabla x \equiv \delta x$. By

means of power function multiplication property $(i \cdot \Delta x)^n = i^n \cdot \Delta x^n$, we can rewrite the n -th finite difference equations (2.1, 2.2, 2.3) as follows

$$(3.2) \quad \Delta^m(x_{i+1}^n) = \sum_{k=0}^m \binom{m}{k} (-1)^k \cdot (i+m-k)^m \cdot \Delta x^m, \quad m < n \in \mathbb{N}$$

Using limit notation on (3.2), we obtain

$$(3.3) \quad \begin{aligned} \lim_{m \rightarrow n^-} \frac{\Delta^m(x_{i+1}^n)}{\Delta x^m} &= \lim_{m \rightarrow n^-} \sum_{k=0}^m \binom{m}{k} (-1)^k \cdot (i+m-k)^m \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot (i+n-k)^{n-0} = n! \end{aligned}$$

Similarly, going from (2.3), backward n -th difference equals:

$$(3.4) \quad \begin{aligned} \lim_{m \rightarrow n^-} \frac{\nabla^m(x_{i-1}^n)}{\nabla x^m} &= \lim_{m \rightarrow n^-} \sum_{k=0}^m \binom{m}{k} (-1)^k \cdot (i-m+k)^m \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot (i-n+k)^{n-0} = n! \end{aligned}$$

And n -th central (2.2)

$$(3.5) \quad \begin{aligned} \lim_{m \rightarrow n^-} \frac{\delta^m f(x_i)}{\delta x^m} &= \lim_{m \rightarrow n^-} \sum_{k=0}^m \binom{m}{k} (-1)^k \cdot \left(i + \frac{m}{2} - k\right)^m \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot \left(i + \frac{n}{2} - k\right)^{n-0} = n! \end{aligned}$$

As we can see the next conformities hold

$$(3.6) \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\Delta x \rightarrow C} \frac{\Delta^n f}{\Delta x^n} \equiv n!$$

$$(3.7) \quad \lim_{\delta x \rightarrow 0} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\delta x \rightarrow C} \frac{\delta^n f}{\delta x^n} \equiv n!$$

$$(3.8) \quad \lim_{\nabla x \rightarrow 0} \frac{\nabla^n f}{\nabla x^n} \equiv \lim_{\nabla x \rightarrow C} \frac{\nabla^n f}{\nabla x^n} \equiv n!$$

$$(3.9) \quad \lim_{\Delta x \rightarrow C} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\delta x \rightarrow C} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x \rightarrow C} \frac{\nabla^n f}{\nabla x^n} \quad \forall (C \in \mathbb{R}^+)$$

In partial case when $C = 0$

$$(3.10) \quad \frac{d^n f}{dx^n} \equiv \lim_{\delta x \rightarrow 0} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x \rightarrow 0} \frac{\nabla^n f}{\nabla x^n}$$

As well holds

$$(3.11) \quad \frac{df}{dx}(x_0) = \left| \lim_{\nabla x \rightarrow 0} \frac{\nabla^n f}{\nabla x^n}(x_0) \right|$$

$$(3.12) \quad \frac{d^n f}{dx^n} \equiv \lim_{\Delta x \rightarrow C} \frac{\Delta^n f}{\Delta x^n} \equiv \lim_{\delta x \rightarrow C} \frac{\delta^n f}{\delta x^n} \equiv \lim_{\nabla x \rightarrow C} \frac{\nabla^n f}{\nabla x^n}, \quad \forall (C \in \mathbb{R}^+)$$

where $f = x^n$. And there is exist the continuous derivative and difference of order $k \leq n$ since $f \in C^n$ class of smoothness. Thus, from (3.6, 3.7, 3.8), we can conclude

$$(3.13) \quad \frac{d^n x^n}{dx^n} = \frac{\Delta^n(x_{i+1}^n)}{\Delta x^n} = \frac{\delta^n(x_i^n)}{\delta x^n} = \frac{\nabla^n(x_{i-1}^n)}{\nabla x^n} = n!, \quad (\Delta x, \delta x, \nabla x) \not\rightarrow dx$$

This completes the proof. \square

Definition 3.14. We introduce the difference equality operator $E(f)$, such that

$$(3.15) \quad E(f) \stackrel{\text{def}}{=} \left(\frac{\Delta^n f}{\Delta x^n} = \frac{\delta^n f}{\delta x^n} = \frac{\nabla^n f}{\nabla x^n} \right)$$

Property 3.16. Let be central difference written as $\delta_m f(x_i) = f(x_{i+m}) - f(x_{i-m})$ the n -th central difference of n -th power is $\delta_m^n(x_i^n) = n! \cdot 2m \cdot \delta x^n$, where $\delta x = x_{i+m} - x_{i-m} = \text{const}$

Going from lemma (3.1), we have next properties

- (1) $\Delta^k(x_{i+1}^k) = \text{const}$, $(i+1) \in \mathbb{Z}+$: $\max(\mathbb{Z}+) > k \rightarrow \Delta^k(x_{i+1}^k) \equiv \Delta^k(x_i^k)$
- (2) $\nabla^k(x_{i-1}^k) = \text{const}$, $(i-1) \in \mathbb{Z}-$: $\min(\mathbb{Z}-) \ll k \rightarrow \nabla^k(x_{i-1}^k) \equiv \nabla^k(x_i^k)$
- (3) $\delta^k(x_i^k) = \text{const}$, $i \in \mathbb{Z}+$: $\max(\mathbb{Z}+) > k \rightarrow \delta^k(x_i^k) \equiv \delta^k(x_{i+j}^k)$
- (4) $\forall([i+1] \in \mathbb{Z}+, [i-1] \in \mathbb{Z}-) : \Delta^{k+j}(x_{i+1}^k) = \nabla^{k+j}(x_{i-1}^k) = 0$,
 $j > 1$, since $\Delta C = \delta C = \nabla C = 0$
- (5) $\forall(f = x^n, n \in \mathbb{N}, k \leq n) : \Delta^k f = (-1)^{n-1} \cdot \nabla^k f$.
- (6) $\Delta f(x_{i+1}) = |\nabla f(x_{i-1})|$

Hereby, according to above properties, we can write the lemma (3.1) as

$$(3.17) \quad \frac{d^n x^n}{dx^n} = \frac{\Delta^n(x_i^n)}{\Delta x^n} = \frac{\delta^n(x_i^n)}{\delta x^n} = \frac{\nabla^n(x_i^n)}{\nabla x^n} = n!$$

Or

$$(3.18) \quad \left(\frac{d}{dx} \right)^n x^n = E(x^n) = n!$$

4. DIFFERENCE OF POLYNOMIALS

Let be polynomial $P_n(x_g)$ defined as

$$(4.1) \quad P_n(x_g) = \sum_{i=0}^n a_i x_g^i$$

Finite differences of such kind polynomial, are $\Delta P_n(x_i) = P_n(x_{i+1}) - P_n(x_i)$, $\nabla P_n(x_{i-1}) = P_n(x_i) - P_n(x_{i-1})$, $\delta P_n(x_i) = P_n\left(x_{i+\frac{1}{2}}\right) - P_n\left(x_{i-\frac{1}{2}}\right)$. Such way, according to the properties (1, 2, 3) from section 1, high order finite differences of polynomials could be written as:

$$(4.2) \quad \begin{aligned} \Delta^k P_n(x_{i+1}) &= \Delta^k(a_0 \cdot x_{i+1}^0 + \dots + a_n \cdot x_{i+1}^n) = \Delta^k(a_0 \cdot x_{i+1}^0) + \dots + \Delta^k(a_n \cdot x_{i+1}^n) \\ &= a_0 \cdot \Delta^k(x_{i+1}^0) + \dots + a_n \cdot \Delta^k(x_{i+1}^n) \end{aligned}$$

Backward difference, respectively, is

$$(4.3) \quad \begin{aligned} \nabla^k P_n(x_{i-1}) &= \nabla^k(a_0 \cdot x_{i-1}^0 + \dots + a_n \cdot x_{i-1}^n) = \nabla^k(a_0 \cdot x_{i-1}^0) + \dots + \nabla^k(a_n \cdot x_{i-1}^n) \\ &= a_0 \cdot \nabla^k(x_{i-1}^0) + \dots + a_n \cdot \nabla^k(x_{i-1}^n) \end{aligned}$$

And central

$$\begin{aligned}
 \delta^k P_n(x_i) &= \delta^k(a_0 \cdot x_i^0 + \cdots + a_n \cdot x_i^n) = \delta^k(a_0 \cdot x_i^0) + \cdots + \delta^k(a_n \cdot x_i^n) \\
 (4.4) \qquad &= a_0 \cdot \delta^k(x_i^0) + \cdots + a_n \cdot \delta^k(x_i^n)
 \end{aligned}$$

Above expressions hold for any build natural n -order polynomial.

Lemma 4.5. $\forall([i+1] \in \mathbb{Z}^+, [i-1] \in \mathbb{Z}^-) : \Delta^{k+j}(x_{i+1}^k) = \nabla^{k+j}(x_{i-1}^k) = 0, j > 1$

Proof. According to lemma (3.1), the n -th difference of n -th power is constant, consequently, the constant rule (3) holds $\Delta C = \delta C = \nabla C = 0$. \square

According to lemma (4.2) and properties (2, 3), taking the limits from (4.2, 4.3, 4.4), receive:

$$\begin{aligned}
 (4.6) \qquad \Delta^{k \rightarrow n} P_n(x_{i+1}) &= \lim_{k \rightarrow n} \{ \Delta^k(a_0 \cdot x_{i+1}^0) + \cdots + \Delta^k(a_n \cdot x_{i+1}^n) \} \\
 &= \Delta^n(a_n \cdot x_{i+1}^n) = a_n \cdot \Delta^n(x_{i+1}^n)
 \end{aligned}$$

$$\begin{aligned}
 (4.7) \qquad \delta^{k \rightarrow n} P_n(x_i) &= \lim_{k \rightarrow n} \{ \delta^k(a_0 \cdot x_i^0) + \cdots + \delta^k(a_n \cdot x_i^n) \} \\
 &= \delta^n(a_n \cdot x_i^n) = a_n \cdot \delta^n(x_i^n)
 \end{aligned}$$

$$\begin{aligned}
 (4.8) \qquad \nabla^{k \rightarrow n} P_n(x_{i-1}) &= \lim_{k \rightarrow n} \{ \nabla^k(a_0 \cdot x_{i-1}^0) + \cdots + \nabla^k(a_n \cdot x_{i-1}^n) \} \\
 &= \nabla^n(a_n \cdot x_{i-1}^n) = a_n \cdot \nabla^n(x_{i-1}^n), k \leq n
 \end{aligned}$$

Since the n -th difference of n -th power equals to $n!$, we have theorem.

Theorem 4.9. *Each n -order polynomial has the constant n -th finite (divided) difference and derivative, which equals each other and equal constant times $n!$, where n is natural.*

Proof. According to limits (4.6, 4.7, 4.8), we have $\Delta^n P_n(x_{i+1}) = a_n \cdot \Delta^n(x_{i+1}^n)$, $\nabla^n P_n(x_{i-1}) = a_n \cdot \nabla^n(x_{i-1}^n)$, $\delta^n P_n(x_i) = a_n \cdot \delta^n(x_i^n)$, going from lemma (3.1), the n -th difference of n -order polynomial equals to $k_n \cdot n!$, the properties (1, 2, 3, 4) proofs that for enough large sets \mathbb{Z}^+ , \mathbb{Z}^- we have $\Delta^n(x_{i+1}^n) \equiv \Delta^n(x_i^n)$, $\delta^n(x_i^n) \equiv \delta^n(x_{i+j}^n)$, $\nabla^n(x_{i-1}^n) \equiv \nabla^n(x_i^n)$, $\min(\mathbb{Z}^-) < n < \max(\mathbb{Z}^+)$. Therefore, we have equality

$$(4.10) \qquad \frac{d^n P_n(x)}{dx^n} = \frac{\Delta^n P_n(x_i)}{(\Delta x)^n} = \frac{\delta^n P_n(x_i)}{(\delta x)^n} = \frac{\nabla^n P_n(x_i)}{(\nabla x)^n} = a_n \cdot n!$$

Or, by means of definition (3.14)

$$(4.11) \qquad \left(\frac{d}{dx} \right)^n P_n(x) = E(P_n(x)) = a_n \cdot E(x^n)$$

\square

5. RELATION WITH PARTIAL DERIVATIVES

Let be partial finite differences defined as

$$(5.1) \quad \Delta f(u_1, u_2, \dots, u_n)_{u_1} := f(u_1 + h, u_2, \dots, u_n) - f(u_1, u_2, \dots, u_n)$$

$$(5.2) \quad \delta f(u_1, u_2, \dots, u_n)_{u_1} := f(u_1 + h, u_2, \dots, u_n) - f(u_1 - h, u_2, \dots, u_n)$$

$$(5.3) \quad \nabla f(u_1, u_2, \dots, u_n)_{u_1} := f(u_1, u_2, \dots, u_n) - f(u_1 - h, u_2, \dots, u_n)$$

By means of mathematical induction, going from Lemma (3.1), we have equality between n -th partial derivative and n -th partial difference, while be taken of polynomial defined function or power function.

Theorem 5.4. *For each n -th natural power of many variables the n -th partial divided differences and n -th partial derivatives equal each other.*

Proof. Let be function $Z = f(u_1, u_2, \dots, u_n) = (u_1, u_2, \dots, u_n)^n$, where dots mean the general relations, i.e multiplication and summation between variables. We denote the equality operator of partial difference as $E(F(u_1, u_2, \dots, u_n))_{u_k}$, where u_k is variable of taken difference. On this basis

$$(5.5) \quad \frac{\partial^n Z}{\partial u_k^n} = \frac{\Delta^n Z_{u_k}}{\Delta u_k^n} = \frac{\delta^n Z_{u_k}}{\delta u_k^n} = \frac{\nabla^n Z_{u_k}}{\nabla u_k^n} = A \cdot n!$$

Or, using equality operator

$$(5.6) \quad \frac{\partial^n Z}{\partial u_k^n} = E(Z)_{u_k} = A \cdot n!$$

where A is free constant, depending of relations between variables and $0 \leq k \leq n$. \square

Property 5.7. *Let be partial differences of the function $f(u_1, \dots, u_k) = u_1^n \pm u_2^n \pm \dots \pm u_k^n$, $n \in \mathbb{N}$, $\Delta f(u_1, \dots, u_k)_M$, $\delta f(u_1, \dots, u_k)_M$, $\nabla f(u_1, \dots, u_k)_M$, where M - complete set of variables, i.e $M = \{u_i\}_i^k$ the n -th partial differences of each variables are*

$$(5.8) \quad \Delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = \pm k \cdot n! \cdot (\Delta u_1)^n \dots (\Delta u_k)^n$$

$$(5.9) \quad \delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = \pm k \cdot n! \cdot (\delta u_1)^n \dots (\delta u_k)^n$$

$$(5.10) \quad \nabla^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = \pm k \cdot n! \cdot (\nabla u_1)^n \dots (\nabla u_k)^n$$

$$\forall k \in \mathbb{Z} : \max(\mathbb{Z}^+) > n > \min(\mathbb{Z}^-), \quad (\delta u_1) \equiv (\delta u_2) \equiv \dots \equiv (\delta u_k), \\ (\nabla u_1) \equiv (\nabla u_2) \equiv \dots \equiv (\nabla u_k), \quad (\Delta u_1) \equiv (\Delta u_2) \equiv \dots \equiv (\Delta u_k)$$

Otherwise

$$(5.11) \quad \Delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = n! \cdot \sum_{i=1}^k (\Delta u_i)^n$$

$$(5.12) \quad \delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = n! \cdot \sum_{i=1}^k (\delta u_i)^n$$

$$(5.13) \quad \nabla^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = n! \cdot \sum_{i=1}^k (\nabla u_i)^n$$

Note that here the partial differences of non-single variable defined as

$$\begin{aligned}\Delta^n f(u_1, \dots, u_k)_M &= \Delta^{n-1} f(u_1 + h, \dots, u_k + h)_M - \Delta^{n-1} f(u_1, \dots, u_k)_M \\ \delta^n f(u_1, \dots, u_k)_M &= \delta^{n-1} f(u_1 + h, \dots, u_k + h)_M - \delta^{n-1} f(u_1 - h, \dots, u_k - h)_M \\ \nabla^n f(u_1, \dots, u_k)_M &= \nabla^{n-1} f(u_1, \dots, u_k)_M - \nabla^{n-1} f(u_1 - h, \dots, u_k - h)_M\end{aligned}$$

Moreover, the n -th partial difference taken over enough large set \mathbb{Z}^+ and $\forall i : \Delta x_i = 1$ has the next connection with single variable n -th derivative of n -th power

$$(5.14) \quad \Delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = \sum_{i=1}^k \left(\frac{d}{du_i} \right)^n f(u_i)$$

With partial derivative we have relation

$$(5.15) \quad \Delta^n f(u_1, u_2, u_3, \dots, u_k)_{u_1, u_2, u_3, \dots, u_k} = \sum_{i=1}^k \left(\frac{\partial}{\partial u_i} \right)^n f(u_1, u_2, u_3, \dots, u_k)$$

Theorem 5.16. For each non-single variable polynomial with order n holds the equality between $k \leq n$ -order partial differences and derivative.

Proof. Let be non-single variable polynomial

$$(5.17) \quad \mathcal{P}_n(u_n) = \sum_{i=1}^n M_i \cdot u_i^i$$

Going from above property (5.7), the k -th partial differences of one variable are

$$(5.18) \quad \begin{aligned}\Delta^k \mathcal{P}_n(u_n)_{u_k} &= M_k \cdot k! \cdot (\Delta u_k)^k, & \delta^k \mathcal{P}_n(u_n)_{u_k} &= M_k \cdot k! \cdot (\delta u_k)^k, \\ \nabla^k \mathcal{P}_n(u_n)_{u_k} &= M_k \cdot k! \cdot (\nabla u_k)^k\end{aligned}$$

$0 \leq k \leq n$. The k -th partial derivative:

$$(5.19) \quad \frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = M_k \cdot k!$$

Hereby

$$(5.20) \quad \frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = \frac{\Delta^k \mathcal{P}_n(u_n)_{u_k}}{\Delta u_k^k} = \frac{\delta^k \mathcal{P}_n(u_n)_{u_k}}{\delta u_k^k} = \frac{\nabla^k \mathcal{P}_n(u_n)_{u_k}}{\nabla u_k^k}$$

Also could be noted as

$$(5.21) \quad \frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = E(\mathcal{P}_n(u_n))_{u_k} = M_k \cdot k!, \quad k \leq n$$

□

6. THE ERROR OF APPROXIMATION

The error of derivative approximation done by forward finite difference with respect to order $k \leq n$ could be calculated as follows

$$(6.1) \quad \left(\frac{\Delta}{\Delta x} \right)^k x^n - \left(\frac{d}{dx} \right)^k x^n = \mathcal{O}(x^{n-k})$$

For polynomials is

$$(6.2) \quad \left(\frac{\Delta}{\Delta x} \right)^k P_n(x) - \left(\frac{d}{dx} \right)^k P_n(x) = \mathcal{O}(x^{n-k})$$

The partial, if $m \leq k$

$$(6.3) \quad \left(\frac{\Delta}{\Delta u_k} \right)^m Z - \left(\frac{\partial}{\partial u_k} \right)^m Z = \mathcal{O}(u_k^{k-m})$$

Where \mathcal{O} - Landau-Bachmann symbol (see [8], [9]).

7. SUMMARY

In this section we summarize the obtained results in the previous chapters and establish the relationship between them. According to lemma (3.1), theorems (4.9), (5.4), (5.14) we have concluded

$$(7.1) \quad \frac{d^n x^n}{dx^n} = E(x^n) = n!$$

$$(7.2) \quad \frac{d^n P_n(x)}{dx^n} = E(P_n(x)) = a_n \cdot E(x^n)$$

$$(7.3) \quad \frac{\partial^n Z}{\partial u_k^n} = E(Z)_{u_k} = A \cdot n!$$

$$(7.4) \quad \frac{\partial^k \mathcal{P}_n(u_n)}{\partial u_k^k} = E(\mathcal{P}_n(u_n))_{u_k} = M_k \cdot k!$$

Generalizing these expressions, we can derive the general relations between ordinary, partial derivatives and finite (divided) differences

$$(7.5) \quad \underbrace{E(u^n) = E(P_n(u_g)) = E(Z)_{u_k} = E(\mathcal{P}_{n+j}(u_{n+j}))_{u_n}}_Y$$

$$(7.6) \quad \underbrace{\frac{d^n u^n}{du^n} = \frac{d^n P_n(u)}{du^n} = \frac{\partial^n Z}{\partial u_k^n} = \frac{\partial^n \mathcal{P}_{n+j}(u_{n+j})}{\partial u_n^n}}_U, \quad j \geq 0$$

$$\forall(A, M_n, a_n) = 1$$

I.e the equalities hold with precision to constant. Function Z defined as $Z = f(u_1, u_2, \dots, u_n) = (u_1, u_2, \dots, u_n)^n$. And finally

$$Y = U$$

with same limitations.

8. CONCLUSION

In this paper were established the equalities between ordinary and partial finite (divided) differences and derivatives of power function and polynomials, with order equal between each other.

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