



HAL
open science

Hydrodynamic limit for an active exclusion process

Clément Erignoux

► **To cite this version:**

| Clément Erignoux. Hydrodynamic limit for an active exclusion process. 2016. hal-01350532v2

HAL Id: hal-01350532

<https://hal.science/hal-01350532v2>

Preprint submitted on 5 Feb 2019 (v2), last revised 6 Feb 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HYDRODYNAMIC LIMIT FOR AN ACTIVE EXCLUSION PROCESS

by

Clément Erignoux

Abstract. — Collective dynamics can be observed among many animal species, and have given rise in the last decades to an active and interdisciplinary field of study. Such behaviors are often modeled by active matter, in which each individual is self-driven and tends to update its velocity depending on the one of its neighbors.

In a classical model introduced by Vicsek & al., as well as in numerous related active matter models, a phase transition between chaotic behavior at high temperature and global order at low temperature can be observed. Even though ample evidence of these phase transitions has been obtained for collective dynamics, from a mathematical standpoint, such active systems are not fully understood yet. Significant progress has been achieved in the recent years under an assumption of mean-field interactions, however to this day, few rigorous results have been obtained for models involving purely local interactions.

In this paper, as a first step towards the mathematical understanding of active microscopic dynamics, we describe a lattice active particle system, in which particles interact locally to align their velocities. We obtain rigorously, using the formalism developed for hydrodynamic limits of lattice gases, the scaling limit of this out-of-equilibrium system. This article builds on the multi-type exclusion model introduced by Quastel [35] by detailing his proof and incorporating several generalizations, adding significant technical and phenomenological difficulties.

Résumé (Limite hydrodynamique pour un processus d'exclusion actif). — L'étude des dynamiques collectives, observables chez de nombreuses espèces animales, a motivé dans les dernières décennies un champ de recherche actif et transdisciplinaire. De tels comportements sont souvent modélisés par de la matière active, c'est-à-dire par des modèles dans lesquels chaque individu est caractérisé par une vitesse propre qui tend à s'ajuster selon celle de ses voisins.

De nombreux modèles de matière active sont liés à un modèle fondateur proposé en 1995 par Vicsek & al.. Ce dernier, ainsi que de nombreux modèles proches, présentent une transition de phase entre un comportement chaotique à haute température, et un comportement global et cohérent à faible température. De nombreuses preuves numériques de telles transitions de phase ont été obtenues dans le cadre des dynamiques collectives. D'un point de vue mathématique, toutefois, ces systèmes actifs sont encore mal compris. Plusieurs résultats ont été obtenus récemment sous une approximation de champ moyen, mais il n'y a encore à ce jour que peu d'études mathématiques de modèles actifs faisant intervenir des interactions purement microscopiques.

Dans cet article, nous décrivons un système de particules actives sur réseau interagissant localement pour aligner leurs vitesses. Comme première étape afin d'atteindre une meilleure compréhension des modèles microscopiques de matière active, nous obtenons rigoureusement, à l'aide du formalisme des limites hydrodynamiques pour les gaz sur réseau, la limite macroscopique de ce système hors-équilibre. Nous développons le travail réalisé par Quastel [35], en apportant une preuve plus détaillée et en incorporant plusieurs généralisations posant de nombreuses difficultés techniques et phénoménologiques.

Key words and phrases. — Statistical physics, Hydrodynamic Limits, Lattice gases, Out-of-equilibrium systems, Non-gradient systems, Exclusion process.

Thanks. - I would first like to warmly thank Thierry Bodineau, my PhD advisor, for his unwavering support and help throughout writing this proof, as well as Julien Tailleur for his insight on active matter and MIPS. I would also like to thank Jeremy Quastel for his help navigating his original article and solving the spectral gap issue, as well as Claudio Landim for our numerous discussions. Finally, I would like to thank the anonymous referee, for both the attention given to my work, and for many insightful comments that significantly improved this article.

Contents

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55

1. Introduction.....	3
1.1. Active matter and active exclusion process.....	3
1.2. Hydrodynamics limits for non-gradients systems.....	4
1.3. Multi-type lattice gases, and contributions of this article.....	4
1.4. Active exclusion process and main result.....	5
Description of a simplified process with two particle types.....	5
Hydrodynamic limit.....	6
Description of the active exclusion process.....	7
Model extensions.....	8
1.5. Structure of the article.....	9
2. Notations and Main theorem.....	10
2.1. Main notations and introduction of the Markov generator.....	10
2.2. Measures associated with a smooth profile and definition of the Markov process.....	11
2.3. Hydrodynamic limit.....	13
2.4. Instantaneous currents.....	15
3. Canonical measures, entropy and irreducibility.....	17
3.1. Definition of the canonical measures.....	17
3.2. Entropy production and local equilibrium.....	19
3.3. Irreducibility and control on full clusters.....	23
4. Law of large number for the exclusion process with angles.....	33
4.1. Replacement Lemma.....	33
4.2. Proof of the replacement Lemma.....	34
4.3. Proof of Lemma 4.3 : The one-block estimate.....	36
4.4. Proof of Lemma 4.4 : The two-block estimate.....	40
5. Preliminaries to the non-gradient method.....	43
5.1. Comparison with an equilibrium measure.....	43
5.2. Relative compactness of the sequence of measures.....	45
5.3. Regularity of the density and energy estimate.....	50
6. Non-gradient estimates.....	53
6.1. Replacement of the symmetric current by a macroscopic gradient.....	53
6.2. Replacement of the currents and $\mathcal{L}f$ by their local average.....	55
6.3. Estimation of the gradients on full clusters.....	56
6.4. Replacement of the macroscopic gradients by their local counterparts.....	61
6.5. Projection on non-full sets and reduction to a variance problem.....	66
6.6. Limiting variance and diffusion coefficients.....	70
6.7. Drift part of the hydrodynamic limit.....	80
7. Proof of the hydrodynamic limit.....	83
8. Limiting space-time covariance.....	88
8.1. Spectral gap for the symmetric exclusion process with angles.....	89
8.2. Discrete differential forms in the context of particles systems.....	95
8.3. An integration by parts formula.....	109
8.4. Heuristics on $\ll \cdot \gg_{\hat{\alpha}}$ and Theorem 6.11.....	112
8.5. Proof of Theorem 6.11.....	113
Appendix A. Possible application : Coarsening and global order in active Matter.....	120
A.1. Collective motion among biological organisms.....	120
A.2. Microscopic active matter models.....	121
Vicsek model and phase transition in alignment models.....	121
Motility-Induced Phase Transition (MIPS).....	123
Appendix B. General tools.....	123
B.1. Topological setup.....	123
B.2. Self-diffusion coefficient.....	124
B.3. Entropy.....	126
B.4. Bound on the largest eigenvalue of a perturbed Markov generator.....	126
Appendix C. Space of grand-canonical parameters.....	126

56	C.1. Equivalence of ensembles.....	126
57	C.2. Regularity of the grand-canonical measures in their parameter.....	128
58	C.3. Compactness of the set of grand-canonical parameters.....	129
59	References.....	130

60

1. Introduction

61 **1.1. Active matter and active exclusion process.** — *Active matter* systems, i.e. microscopic
 62 interacting particles models in which each particle consumes energy to self-propel, have been the subject
 63 of intense scrutiny in physics in the recent years. As explained thoroughly in Appendix A, active matter
 64 exhibits a rich phenomenology. Its two most studied features are the emergence of global polarization,
 65 first discovered with Vicsek’s seminal model [50], and the so-called Motility Induced Phase Separation
 66 (MIPS, cf. [11]), which can be roughly described as the particle’s tendency to cluster where they move
 67 more slowly. As detailed in Appendix A, these two phenomena have been extensively studied by the
 68 physics community in the last decade (e.g. [41] [42] [43] for alignment phase transition, [10] [11] for
 69 MIPS).

70 By essence, active matter models are driven out-of-equilibrium at a microscopic level, and although
 71 many are now well-understood from a physics standpoint, their mathematical understanding to this day
 72 remains partial. Inspired by Vicsek’s original model [50], significant mathematical progress has been
 73 achieved using analytical tools for active alignment models submitted to *mean-field* or *local-field* inter-
 74 actions, i.e. for which the particle’s interactions are locally averaged out over a large number of their
 75 neighbors (e.g. [5], [15], [18]). However, in some cases, the local-field approximation is not mathemat-
 76 ically justified, and deriving exact results on models with purely microscopic interactions can provide
 77 welcome insight for their phenomenological study [30].

78 Let us start by briefly describing a simplified version of the active exclusion process studied in this
 79 article before giving some mathematical context. On a two-dimensional periodic lattice, consider two-
 80 types of particles, denoted “+” and “−”, which move and update their type according to their neighbors.

81 — Each particle’s type is randomly updated by a Glauber dynamics depending on its nearest
 82 neighbors.

83 — The motion of any particle is a random walk, weakly biased in one direction depending on its
 84 type : the “+” particles will tend to move to the right, whereas the “−” particles will tend to move
 85 to the left.

86 — The vertical displacement is symmetric regardless of the particle’s type.

87 To model hard-core interactions, an *exclusion rule* is imposed, i.e. two particles cannot be present on the
 88 same site : a particle jump towards an occupied site will be canceled. This induces the congestion effects
 89 which can lead to MIPS, and one can therefore hope that this model encompasses both the alignment
 90 phase transition and MIPS which are characteristic of many of the active models described in Appendix
 91 A. However, mathematically proving such phenomenology for our microscopic active model is still out of
 92 reach.

93 In this article, as a first step towards this goal, we derive the hydrodynamic limit for an extension of the
 94 model briefly described above. From a mathematical standpoint, a first microscopic dynamics combining
 95 alignment and stirring was introduced in [13], where De Masi et al. considered a lattice gas with two
 96 types of particles, in which two neighboring particles can swap their positions, and can change type
 97 according to the neighboring particles. They derived the hydrodynamic limit, as well as the fluctuations,
 98 when the stirring dynamics is accelerated by a diffusive scaling, w.r.t. the alignment dynamics. This scale
 99 separation is crucial to have both alignment and stirring present in the hydrodynamic limit. Generally,
 100 the strategy to obtain the hydrodynamic limit for a lattice gas depends significantly on the microscopic
 101 features of the model, and must be adapted on a case-by-case basis to the considered dynamics. For
 102 example, the exclusion rule in the active exclusion process makes it non-gradient, thus the proof of its
 103 hydrodynamic limit is significantly more elaborate. The end of this introduction is dedicated to putting

104 in context the mathematical contributions of this article and describing the difficulties occurring in the
 105 derivation of the hydrodynamic limit of our model.

106 **1.2. Hydrodynamics limits for non-gradients systems.** — The active exclusion process presented
 107 above belongs to a broad class of microscopic lattice dynamics for which the instantaneous particle
 108 currents along any edge cannot be written as a discrete gradient. This difficulty appears naturally in
 109 exclusion systems, in particular for systems with multiple particle types, or for generalized exclusion
 110 processes where only a fixed number κ ($\kappa \geq 2$) of particles can be present at the same site. Such systems
 111 are called *non-gradients*. A considerable part of this article is dedicated to solving the difficulties posed
 112 by the non-gradient nature the active exclusion process.

113 The first proof for a non-gradient hydrodynamic limit was obtained by Varadhan in [48], and Quastel
 114 [35] (cf. below). To illustrate the difficulty let us consider a general diffusive particle system of size N
 115 in 1 dimension, evolving according to a Markov generator \mathcal{L}_N . Such a diffusive system must be rescaled
 116 in time by a factor N^2 , therefore each jump in \mathcal{L}_N should occur at rate N^2 . Denoting by η_x the state of
 117 the system at the site x (e.g. number of particles, energy of the site), $\mathcal{L}_N \eta_x$ is a microscopic gradient,

$$\mathcal{L}_N \eta_x = N^2(j_{x-1,x} - j_{x,x+1}),$$

118 where $j_{x,x+1}$ is the instantaneous current along the edge $(x, x+1)$, and the N^2 comes from the time-
 119 rescaling. This microscopic gradient balances out a first factor N , and acts as a spatial derivative on
 120 a macroscopic level. In order to obtain a diffusive equation similar to the heat equation, one needs to
 121 absorb the second factor N in a second spatial derivative. This is the main difficulty for non-gradient
 122 systems, for which the instantaneous current $j_{x,x+1}$ does not take the form of a microscopic gradient.
 123 The purpose of the non-gradient method developed by Varadhan is to establish a so-called *microscopic*
 124 *fluctuation-dissipation relation*

$$j_{x,x+1} \simeq -D(\eta_{x+1} - \eta_x) + \mathcal{L}_N g_x,$$

125 where $\mathcal{L}_N g_x$ is a small fluctuation which usually disappears in the macroscopic limit according to Fick's
 126 law for diffusive systems. Although the link to the macroscopic fluctuation-dissipation relation (cf. Section
 127 8.8, p140-141 in [45] for more detail on this relation) is not apparent, the latter is indeed a consequence
 128 of the microscopic identification above.

129 **1.3. Multi-type lattice gases, and contributions of this article.** — The difficulties to derive the
 130 hydrodynamic limit of multi-type particle models vary significantly depending on the specificities of each
 131 microscopic dynamics. Active matter provides natural examples of multi-type particle systems, since
 132 each possible velocity can be interpreted as a different type. When the particles evolve in a continuous
 133 space domains, (e.g. [15], [16]) and in the absence of hard-core interactions, the density of each type
 134 of particles can essentially be considered independently regarding displacement, and the scaling limit
 135 usually decouples the velocity variable and the space variable.

136 In the case of lattice gases, however, it becomes necessary to specify the way particles interact when
 137 they are on the same site. Dynamically speaking, multi-type models often allow either

- 138 — swapping particles with different types, as in [37] for a totally asymmetric system with velocity
 139 flips.
- 140 — The coexistence on a same site of particles with different velocities, as in [14] or [39] for a
 141 model closely related to the one investigated in this article with weak driving forces, or in [20] for
 142 a zero-range model exhibiting MIPS-like behavior.

143 These simplifications allow to bypass the specific issues arising for diffusive systems with complete exclu-
 144 sion between particles, since the latter often require the non-gradient tools mentioned previously.

145 The first hydrodynamic limits for non-gradient microscopic systems were studied by Varadhan and
 146 Quastel. They developed in [48] and [35] a general method to derive the hydrodynamic limit for non-
 147 gradient systems with main requirement a sharp estimate for the Markov generator's spectral gap. Quastel
 148 also notably obtained in [35] an explicit expression for the diffusion and conductivity matrices for the
 149 multi-type exclusion process, as a function of the various particle densities and of the self-diffusion

150 coefficient $d_s(\rho)$ of a tagged particle for the equilibrium symmetric simple exclusion process with density
 151 ρ . This result was then partially extended to the weakly asymmetric case (in [36] as a step to obtain a
 152 large deviation principle for the empirical measure of the symmetric simple exclusion process, and where
 153 the asymmetry does not depend on the configuration, and in [24] for a weak asymmetry with a mean-field
 154 dependency in the configuration), as well as a more elaborate dynamics with creation and annihilation
 155 of particles [38].

156 In this article, we derive the hydrodynamic limit for an active matter lattice gas with purely microscopic
 157 interactions. To do so, we generalize the results obtained by Quastel [35] by incorporating many natural
 158 extensions, and apply in great detail the non-gradient method for multi-type exclusion with a weak drift.

159 There are several reasons behind our choice to detail this difficult proof. First, Quastel’s original
 160 article suffers from typos which are fixed in this paper, in particular the spectral gap for the multi-type
 161 exclusion process is not uniform with respect to the density and this required an adaptation of the original
 162 proof. Second, Quastel’s proof relied significantly on the structure of the microscopic dynamics which
 163 could be controlled by the symmetric exclusion. This played a crucial role in [35] to ensure that the
 164 particle density does not reach 1, because when this is the case, the system loses its mixing properties as
 165 represented by the decay of the spectral gap. When the considered dynamics is a multi-type symmetric
 166 exclusion (identical for any particle type, as in [35]), the macroscopic density for the total number of
 167 particles evolves according to the heat equation, and density control at any given time is ensured by
 168 the maximum principle. In our case, the limiting equation is not diffusive and a priori estimates on the
 169 density are much harder to derive. Finally, [35] was one of the first examples of hydrodynamic limit
 170 for non-gradient systems, and to make the proof more accessible, we used the more recent formalism
 171 developed in [27], in which an important upside is the clear identification of the orders of the estimates
 172 in the scaling parameter N .

173 We extend the proof of the hydrodynamic limit for the multi-type exclusion process [35] to the weakly
 174 asymmetric case when the particle types depend on a continuous parameter. The hydrodynamic limit for
 175 lattice gases with K particle types takes the form of K coupled partial differential equations. Extending
 176 it to a continuum of particle types therefore poses the issue of the well-posedness of the system. To
 177 solve this issue, we therefore introduce an angular variable joint to the space variable. Although the
 178 global outline of the proof remains similar, this induced numerous technical difficulties. In particular, as
 179 opposed to the previous examples, local equilibrium is not characterized by a finite number of real-valued
 180 parameters (e.g. density, local magnetization), which required significant adaptation of the proof of the
 181 hydrodynamic limit.

182 **1.4. Active exclusion process and main result.** — The remainder of this section is dedicated to a
 183 short description of our model and its hydrodynamic limit. For clarity’s sake, we first describe in more
 184 details the simplified model with only two types of particles briefly presented above, and then introduce
 185 the more general active exclusion process studied in this article. Precisely describing the complete model,
 186 and rigorously stating its hydrodynamic limit, will be the purpose of Section 2.

187 *Description of a simplified process with two particle types.* — For the clarity of notations, we describe
 188 and study our model in dimension $d = 2$. The simplified version of the model can be considered as an
 189 active Ising model [43] with an *exclusion rule* : each site x of the periodic lattice \mathbb{T}_N^2 of size N is either

- 190 — occupied by a particle of type “+” ($\eta_x^+ = 1$),
- 191 — occupied by a particle of type “−” ($\eta_x^- = 1$),
- 192 — empty if $\eta_x^+ = \eta_x^- = 0$.

193 Each site contains at most one particle, thus the pair (η_x^+, η_x^-) entirely determines the state of any site
 194 x , and is either $(1, 0)$, $(0, 1)$ or $(0, 0)$. The initial configuration for our particle system is chosen at
 195 local equilibrium and close to a smooth macroscopic profile $\zeta_0 = \zeta_0^+ + \zeta_0^- : \mathbb{T}^2 \rightarrow [0, 1]$, where \mathbb{T}^2
 196 is the continuous domain $[0, 1]^2$ with periodic boundary conditions, and $\zeta_0^+(x/N)$ (resp. $\zeta_0^-(x/N)$) is the
 197 initial probability that the site x contains a “+” particle (resp. “−”). We denote by $\hat{\eta}$ the collection
 198 $((\eta_x^+, \eta_x^-))_{x \in \mathbb{T}_N^2}$.

Each particle performs a random walk, which is symmetric in the direction $i = 2$, and weakly asymmetric in the direction $i = 1$. The asymmetry is tuned via a positive parameter λ , thus a “+” (resp. “−”) particle at site x jumps towards $x + e_1$ at rate $1 + \lambda/N$ (resp. $1 - \lambda/N$) and towards $x - e_1$ at rate $1 - \lambda/N$ (resp. $1 + \lambda/N$). If a particle tries to jump to an occupied site, the jump is canceled. In order to obtain a macroscopic contribution of this displacement dynamics, it must be accelerated by a factor N^2 .

Moreover, the type of the particle at site x is updated at random times, depending on its nearest neighbors. Typically, to model collective motion, a “−” particle surrounded by “+” particles will change type quickly, whereas a “−” particle surrounded by “−” particles will change type slowly, to model the tendency of each individual to mimic the behavior of its neighbors. Although they determine the shape of the last term of the hydrodynamic limit, the microscopic details of this update dynamics are technically not crucial to the proof of the hydrodynamic limit (in the scaling considered here), we therefore choose general, bounded flip rates $c_{x,\beta}(\hat{\eta})$ parametrized by an inverse temperature $\beta \geq 0$ and depending only on the local configuration around x . We further assume that these jump rates depend continuously on the θ_y 's around x .

The complete dynamics can be split into three parts, namely the symmetric and asymmetric contributions of the exclusion process, and the Glauber dynamics, evolving on different time scales. For this reason, each corresponding part in the Markov generator has a different scaling in the parameter N : the two-type process is driven by the generator

$$L_N = N^2 \left[\mathcal{L} + \frac{1}{N} \mathcal{L}^{\text{WA}} \right] + \mathcal{L}^{\text{G}},$$

whose three elements we now define. Fix a function f of the configuration, we denote by

$$\eta_x = \eta_x^+ + \eta_x^- \in \{0, 1\}$$

the total occupation state of the site x . The nearest-neighbor simple symmetric exclusion process generator \mathcal{L} is

$$\mathcal{L}f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \sum_{|z|=1} \eta_x (1 - \eta_{x+z}) (f(\hat{\eta}^{x,x+z}) - f(\hat{\eta})),$$

\mathcal{L}^{WA} encompasses the weakly asymmetric part of the displacement process,

$$\mathcal{L}^{\text{WA}}f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \sum_{\delta=\pm 1} \delta \lambda (\eta_x^+ - \eta_x^-) (1 - \eta_{x+\delta e_1}) (f(\hat{\eta}^{x,x+\delta e_1}) - f(\hat{\eta})),$$

which is not a Markov generator because of its negative jump rates, but is well-defined once added to the symmetric part of the exclusion process. Finally, \mathcal{L}^{G} is the generator which rules the local alignment of the angles

$$\mathcal{L}^{\text{G}}f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \eta_x c_{x,\beta}(\hat{\eta}) (f(\hat{\eta}^x) - f(\hat{\eta})).$$

In the identities above, $\hat{\eta}^{x,x+z}$ is the configuration where the states of x and $x+z$ have been swapped in $\hat{\eta}$, and $\hat{\eta}^x$ is the configuration where the type of the particle at site x has been changed.

Hydrodynamic limit. — Let us denote by $\rho_t^+(u)$ (resp. $\rho_t^-(u)$) the macroscopic density of “+” (resp. “−”) particles, and by $\rho_t(u) = \rho_t^+(u) + \rho_t^-(u)$ the total density at any point u in \mathbb{T}^2 . Let us also denote by $m_t(u) = \rho_t^+(u) - \rho_t^-(u)$ the local average asymmetry.

Then, as a special case of our main result the pair (ρ_t^+, ρ_t^-) is solution, in a weak sense, to the partial differential system

$$(1.1) \quad \begin{cases} \partial_t \rho_t^+ = \nabla \cdot [\mathfrak{d}(\rho_t, \rho_t^+) \nabla \rho_t + d_s(\rho_t) \nabla \rho_t^+] - 2\lambda \partial_{u_1} [m_t \mathfrak{s}(\rho_t, \rho_t^+) + d_s(\rho_t) \rho_t^+] + \Gamma_t, \\ \partial_t \rho_t^- = \nabla \cdot [\mathfrak{d}(\rho_t, \rho_t^-) \nabla \rho_t + d_s(\rho_t) \nabla \rho_t^-] + 2\lambda \partial_{u_1} [m_t \mathfrak{s}(\rho_t, \rho_t^-) - d_s(\rho_t) \rho_t^-] - \Gamma_t \end{cases}$$

with initial profile

$$(1.2) \quad \rho_0^\pm(u) = \zeta^\pm(u).$$

227 In the PDE (1.4), ∂_{u_1} denotes the partial derivative in the first space variable, d_s is the self-diffusion
 228 coefficient for the SSEP in dimension 2 mentioned in the introduction, the coefficients \mathfrak{d} and \mathfrak{s} are given
 229 by

$$(1.3) \quad \mathfrak{d}(\rho, \rho^*) = \frac{\rho^*}{\rho}(1 - d_s(\rho)) \quad \text{and} \quad \mathfrak{s}(\rho, \rho^*) = \frac{\rho^*}{\rho}(1 - \rho - d_s(\rho)),$$

and Γ_t is the local creation rate of particles with type “+”, which can be written as the expectation under a product measure of the microscopic creation rate. Although it is not apparent, the coefficients \mathfrak{d} , \mathfrak{s} , and d_s satisfy a Stokes-Einstein relation in a matrix form when the differential equation is written for the vector (ρ_t^+, ρ_t^-) , in the sense that

$$\begin{pmatrix} \mathfrak{d}(\rho, \rho^+) + d_s(\rho) & \mathfrak{d}(\rho, \rho^+) \\ \mathfrak{d}(\rho, \rho^-) & \mathfrak{d}(\rho, \rho^-) + d_s(\rho) \end{pmatrix} \begin{pmatrix} \rho^+(1 - \rho^+) & -\rho^+\rho^- \\ -\rho^+\rho^- & \rho^-(1 - \rho^-) \end{pmatrix} \\ = \begin{pmatrix} \rho^+[\mathfrak{s}(\rho, \rho^+) + d_s(\rho)] & \rho^-\mathfrak{s}(\rho, \rho^+) \\ \rho^+\mathfrak{s}(\rho, \rho^-) & \rho^+[\mathfrak{s}(\rho, \rho^-) + d_s(\rho)] \end{pmatrix}.$$

230 The second matrix above is the compressibility matrix, whose components are $Cov_{\rho^+, \rho^-}(\eta_0^{s_1}, \eta_0^{s_2})$, where
 231 both s_1 and s_2 take value in $\{+, -\}$.

232 This simplified model is very close to the active Ising model (cf. Appendix A, and [43]) with a weak
 233 driving force. The main difference is the exclusion rule : in the active Ising model, there is no limit to the
 234 number of particles per site, and each particle’s type is updated depending on the other particles present
 235 at the same site. In our two-type model, the exclusion rule creates a strong constraint on the displacement
 236 and therefore changes the form of the hydrodynamic limit, which is no longer the one derived in [43].

237 *Description of the active exclusion process.* — We now describe the *active exclusion process* considered
 238 in this article, which is in some form a generalization of the model presented above. Indeed, although for
 239 technical reasons the proof of our main result cannot be applied verbatim to a finite number of particle
 240 types, the overwhole scheme is extremely similar, and under suitable assumptions on the initial profile,
 241 one can state an analogous result in the case of a finite number of particle types as well. Since the
 242 active exclusion process is thoroughly introduced in Section 2, we briefly describe it here, and only give
 243 a heuristic formulation for our main result. Denoting

$$\mathbb{S} := [0, 2\pi[,$$

244 the *periodic* set of possible angles, the type of any particle is now a parameter $\theta \in \mathbb{S}$ representing the
 245 angular direction of its weak driving force. To compare with the simplified model, the “+” particles
 246 correspond to the angle $\theta = 0$, whereas the “-” particles correspond to the angular direction $\theta = \pi$.

247 Any site is now either occupied by a particle with angle θ ($\eta_x = 1, \theta_x = \theta$), or empty ($\eta_x = 0, \theta_x = 0$
 248 by default). The initial configuration $\widehat{\eta}(0)$ of the system is chosen at local equilibrium, close to a smooth
 249 macroscopic profile $\widehat{\zeta} : \mathbb{T}^2 \times \mathbb{S} \rightarrow \mathbb{R}_+$, where each site x is occupied by a particle with angle $\theta_x \in [\theta, \theta + d\theta[$
 250 with probability $\widehat{\zeta}(x/N, \theta)d\theta$, and the site remains empty w.p. $1 - \int_{\mathbb{S}} \widehat{\zeta}(x/N, \theta)d\theta$.

251 Our active exclusion process is driven by the Markov generator

$$L_N = N^2 \left[\mathcal{L} + \frac{1}{N} \mathcal{L}^{\text{WA}} \right] + \mathcal{L}^{\text{G}},$$

with three parts described below. Fix a function f of the configuration. The nearest-neighbor simple symmetric exclusion process generator \mathcal{L} is unchanged with respect to the two-type case, whereas \mathcal{L}^{WA} is now given by

$$\mathcal{L}^{\text{WA}} f(\widehat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \sum_{\substack{|z|=1 \\ z = \delta e_i}} \delta \lambda_i(\theta_x) \eta_x (1 - \eta_{x+\delta e_i}) (f(\widehat{\eta}^{x, x+\delta e_i}) - f(\widehat{\eta})),$$

252 where the asymmetry in the direction i for a particle with angle θ is encoded by the functions $\lambda_i(\theta)$,

$$\lambda_1(\theta) = \lambda \cos(\theta) \quad \text{and} \quad \lambda_2(\theta) = \lambda \sin(\theta).$$

To fix ideas, The Glauber generator will be taken of the form

$$\mathcal{L}^G f(\widehat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} c_{x,\beta}(\theta, \widehat{\eta}) (f(\widehat{\eta}^{x,\theta}) - f(\widehat{\eta})) d\theta,$$

253 where $\widehat{\eta}^{x,\theta}$ is the configuration where θ_x has been set to θ , and we choose alignment rates similar to the
254 Glauber dynamics of the XY model (cf. Appendix A). More precisely, we consider

$$c_{x,\beta}(\theta, \widehat{\eta}) = \frac{\exp\left(\beta \sum_{y \sim x} \eta_y \cos(\theta_y - \theta)\right)}{\int_{\mathbb{S}} \exp\left(\beta \sum_{y \sim x} \eta_y \cos(\theta_y - \theta')\right) d\theta'},$$

255 which tends to align θ_x with the θ_y 's, for y a neighbor site of x . In the jump rates above, we take the
256 value in $[-\pi, \pi]$ of the angle $\theta_y - \theta$. The intensity λ and the inverse temperature $\beta \geq 0$ still tune the
257 strength of the drift and the alignment.

258 As mentioned before, we settle for now for a heuristic formulation of the hydrodynamic limit. Let us
259 denote by $\rho_t^\theta(u)$ the macroscopic density of particles with angle θ , and by $\rho_t(u) = \int_{\mathbb{S}} \rho_t^\theta(u) d\theta$ the total
260 density at any point u in the *periodic domain* $\mathbb{T}^2 := [0, 1]^2$. Let us also denote by $\vec{\Omega}_t$ the direction of the
261 local average asymmetry

$$\vec{\Omega}_t(u) = \int_{\mathbb{S}} \rho_t^\theta(u) \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} d\theta.$$

262 As expected from (1.1), the main result (cf. Theorem 2.6) of this article is that ρ_t^θ is solution, in a weak
263 sense, to the partial differential equation

$$(1.4) \quad \partial_t \rho_t^\theta = \nabla \cdot [\mathfrak{d}(\rho_t, \rho_t^\theta) \nabla \rho_t + d_s(\rho_t) \nabla \rho_t^\theta] - 2 \nabla \cdot \left[\mathfrak{s}(\rho_t, \rho_t^\theta) \lambda \vec{\Omega}_t + d_s(\rho_t) \rho_t^\theta \begin{pmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{pmatrix} \right] + \Gamma_t,$$

264 with initial profile

$$\rho_0^\theta(u) = \widehat{\zeta}(u, \theta).$$

265 In the PDE (1.4), d_s is the self-diffusion coefficient for the SSEP in dimension 2 mentioned previously, the
266 coefficients \mathfrak{d} and \mathfrak{s} are given by (1.3) as in the two-type case, and Γ_t is the local creation rate of particles
267 with angles θ , which can be written as the expectation under a product measure of the microscopic
268 creation rate.

269 Before properly stating the hydrodynamic limit, let us recall the major difficulties of the proof. The
270 main challenge is the non-gradient nature of the model : the instantaneous current of particles with angle
271 θ between two neighboring sites x and $x + e_i$ can be written

$$j_{x, x+e_i}^\theta = \mathbb{1}_{\{\theta_x = \theta\}} \eta_x (1 - \eta_{x+e_i}) - \mathbb{1}_{\{\theta_{x+e_i} = \theta\}} \eta_{x+e_i} (1 - \eta_x),$$

272 which is not a discrete gradient. One also has to deal with the loss of ergodicity at high densities, and with
273 the asymmetry affecting the displacement of each particle, which drives the system out-of-equilibrium,
274 and complicates the non-gradient method. Finally, the non-linearity of the limiting equation also induces
275 several difficulties throughout the proof.

276 *Model extensions.* — Several design choices for the model have been made either to simplify the notations,
277 or to be coherent with the collective dynamics motivations (cf. Appendix A). However, we present now
278 some of the possible changes for which our proof still holds with minimal adaptations.

279 — The model can easily be adapted to dimensions $d \geq 2$. The dimension 1, however, exhibits very
280 different behavior, since neighboring particles with opposite drifts have pathological behavior and
281 freeze the system due to the exclusion rule.

282 — The nearest neighbor jumps dynamics can be replaced by one with local and irreducible transition
283 function $p(\cdot)$. This involves minor adjustments of the limiting equation, as solved by Quastel [35].

284 In this case, the total jump generator must be split between a symmetric part scaled as N^2 , and
285 an asymmetric part scaled as N whose jumps can be decomposed as a succession of jumps from
286 the symmetric part. However, providing exact criteria for the validity of the extension to a more
287 general jump kernel would be rather difficult, and such extensions are best checked on a case-by-
288 case basis. In the case of nearest-neighbor exclusion, the drift functions can be replaced by any

289 bounded function, and can also involve a spatial dependency, as soon as $\lambda_i(u, \theta)$ is a smooth $C^{1,1}$
 290 function of its two variables u and θ .

291 — We chose for our alignment dynamics a jump process, however analogous results would hold for
 292 diffusive alignment. The jump rates can also be changed to any local and bounded rates, provided
 293 they are smooth in the θ_x 's. The smoothness assumption in the last two comments is there to make
 294 sure that the expectation of their microscopic contribution under the grand-canonical measures is
 295 a Lipschitz-continuous function in the grand-canonical parameter.

296 **1.5. Structure of the article.** — Section 2 is dedicated to the full description of the model, to
 297 introducing the main notations, and the proper formulation of the hydrodynamic limit for the active
 298 exclusion process.

299 Section 3 is composed of three distinct parts. In Subsection 3.1 we characterize local equilibrium for
 300 our process by introducing the set $\mathcal{M}_1(\mathbb{S})$ of parameters for the grand-canonical measures of our process.
 301 We also give a topological setup for $\mathcal{M}_1(\mathbb{S})$, for which some elementary properties are given in Appendix
 302 C. In Subsection 3.2, we prove using classical tools that the entropy of the measure of our process with
 303 respect to a reference product measure is of order N^2 . The last Subsection 3.3 tackles the problem of
 304 irreducibility, which is specific to our model and is one of its major difficulties. Its main result, Proposition
 305 3.12, relies on a-priori density estimates, and states that on a microscopic scale, large local clusters are
 306 seldom completely full, which is necessary to ensure irreducibility on a microscopic level.

307 Section 4 proves a law of large numbers for our process. The so-called Replacement Lemma stated in
 308 Subsection 4.1 relies on the usual one block (Subsection 4.3) and two blocks (Subsection 4.4) estimates.
 309 However, even though we use the classical strategy to prove both estimates, some technical adaptations
 310 are necessary to account for the specificities of our model.

311 Section 5 acts as a preliminary to the non-gradient method. The first result of this section is the
 312 comparison of the active exclusion process's measure to that of an equilibrium process without drift nor
 313 alignment (Subsection 5.1). We also prove, adapting the classical methods, a compactness result for
 314 the sequence of measures of our process, (Subsection 5.2) as well as an energy estimate (Subsection 5.3)
 315 necessary to prove our main result.

316 The non-gradient estimates are obtained in Section 6. It is composed of a large number of intermediate
 317 results which we do not describe in this introduction. The application of the non-gradient method to the
 318 active exclusion process, however, requires to overcome several issues which are specific to our model. One
 319 such difficulty is solved in Subsection 6.3, where we estimate the contributions of microscopic full clusters.
 320 In Subsections 6.6 and 6.7, we prove that for our well chosen diffusion and conductivity coefficients, the
 321 total displacement currents can be replaced by the sum of a gradient quantity and the drift term. For the
 322 sake of clarity, we use to do so the modern formalism for hydrodynamic limits as presented in [27] rather
 323 than the one used in [35]. We state in this section a convergence result at the core of the non-gradient
 324 method (Theorem 6.11) whose proof is intricate and is postponed to the last section.

325 All these results come together in Section 7, where we conclude the proof of the hydrodynamic limit
 326 for our process. Some more specific work is necessary in order to perform the second integration by parts,
 327 due to the delicate shape of the diffusive part of our limiting differential equation.

328 Finally, Section 8 is dedicated to proving Theorem 6.11, following similar steps as in [27]. To do so,
 329 we estimate in Subsection 8.1 the spectral gap of the active exclusion process on a subclass of functions.
 330 We then describe in Subsection 8.2 the notion of germs of closed forms for the active exclusion process,
 331 and prove using the spectral gap estimate a decomposition theorem for the set of germs of closed forms.
 332 A difficulty of this model is that the spectral gap is not uniform in the density, and decays faster as the
 333 density goes to 1. This issue is solved by cutting off large densities (cf. equation (8.2) and Lemma 8.15).
 334 Using the decomposition of closed forms, Theorem 6.11 is derived in Subsection 8.5.

2. Notations and Main theorem

335

336 We describe an interacting particle system, where a particle follows an exclusion dynamics with a
 337 weak bias depending on an angle associated with this particle. At the same time, each particle updates
 338 its angle according to the angles of the neighboring particle. We study the macroscopic behavior of the
 339 corresponding 2-dimensional system with a periodic boundary condition.

340 **2.1. Main notations and introduction of the Markov generator.** — On the two dimensional
 341 discrete set

$$\mathbb{T}_N^2 = \{1, \dots, N\}^2$$

342 with *periodic boundary conditions*, we define the occupation configuration $\eta = (\eta_x)_{x \in \mathbb{T}_N^2} \in \{0, 1\}^{\mathbb{T}_N^2}$ where
 343 $\eta_x \in \{0, 1\}$ is the number of particles at site x . With any *occupied* site $x \in \mathbb{T}_N^2$, we associate an angle
 344 $\theta_x \in \mathbb{S}$ representing the mean direction of the velocity in the plane of the particle occupying the site.
 345 When the site x is empty, we set the angle of the site to $\theta_x = 0$ by default.

346 **Definition 2.1 (Configurations, cylinder & angle-blind functions)**

347 For any site $x \in \mathbb{T}_N^2$, we denote by $\hat{\eta}_x$ the pair (η_x, θ_x) , and by $\hat{\eta} = (\hat{\eta}_x)_{x \in \mathbb{T}_N^2}$ the complete configuration.
 348 The set of all configurations will be denoted by

$$\Sigma_N = \left\{ (\eta_x, \theta_x)_{x \in \mathbb{T}_N^2} \in (\{0, 1\} \times \mathbb{S})^{\mathbb{T}_N^2} \mid \theta_x = 0 \text{ if } \eta_x = 0 \right\}.$$

349 Denote by Σ_∞ the set of infinite configurations above, where \mathbb{T}_N^2 is replaced by \mathbb{Z}^2 . We will call *cylinder*
 350 *function* any function f depending on the configuration only through a finite set of vertices $B_f \subset \mathbb{Z}^2$,
 351 and C^1 w.r.t. each θ_x , for any $x \in B_f$. The set of cylinder functions on \mathbb{Z}^2 will be denoted by \mathcal{C} . Note
 352 that a cylinder function is always bounded, and that any function $f \in \mathcal{C}$ admits a natural image as a
 353 function on Σ_N for any N large enough. This is always the latter that we will consider, and we therefore
 354 abuse the notation and denote in the same way both f and its counterpart on Σ_N .

355 We will call *angle-blind function* any function depending on $\hat{\eta}$ only through the occupation variables
 356 $\eta = (\eta_x)_{x \in \mathbb{T}_N^2}$. In other words, an angle-blind function depends on the position of particles, but not on
 357 their angles. We denote by \mathcal{S} the set of angle-blind functions.

358 We will use on the discrete torus the notations $|\cdot|$ for the norm $|x| = \sum_{i=1}^2 |x_i|$.

359 Let T be a fixed time, we now introduce the process $(\hat{\eta}(t))_{t \in [0, T]}$ on Σ_N which is central to our work.
 360 Our goal is to combine the two dynamics present in Viscek's model [50] : The first part of the process
 361 is the *displacement dynamics*, which rules the *motion of each particle*. The moves occur at rates biased
 362 by the angle of the particle, and follows the exclusion rule. Thus, for $\delta = \pm 1$ the rate $p_x(\delta e_i, \hat{\eta})$ at which
 363 the particle at site x moves to an *empty site* $x + \delta e_i$, letting $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the canonical basis
 364 in \mathbb{Z}^2 , is given by

$$p_x(\delta e_i, \hat{\eta}) = \begin{cases} 1 + \lambda \delta \cos(\theta_x)/N & \text{if } i = 1 \\ 1 + \lambda \delta \sin(\theta_x)/N & \text{if } i = 2 \end{cases},$$

365 where $\lambda \in \mathbb{R}$ is a positive parameter which characterizes the strength of the asymmetry. For convenience,
 366 we will denote throughout the proof

$$(2.1) \quad \lambda_1(\theta) = \lambda \cos(\theta) \quad \text{and} \quad \lambda_2(\theta) = \lambda \sin(\theta).$$

367 The previous rates indicate that the motion of each particle is biased in a direction given by its angle.
 368 The motion follows an exclusion rule, which means that if the target site is already occupied, the jump is
 369 canceled. Note that in order to see the symmetric and asymmetric contributions in the diffusive scaling
 370 limit, we must indeed choose an asymmetry scaling as $1/N$. Furthermore, in order for the system to
 371 exhibit a macroscopic behavior in the limit $N \rightarrow \infty$, we need to accelerate the whole exclusion process
 372 by N^2 , as discussed further later on.

373 The second part of the dynamic is the angle update process, which will be from now on referred to
 374 as the *Glauber part of the dynamics*. A wide variety of choices is available among discontinuous angle

375 dynamics (jump process) and continuous angle dynamics (diffusion). We choose here a Glauber jump
 376 process with inverse temperature $\beta \geq 0$ described more precisely below.

377 The generator of the complete Markov process is given by

$$(2.2) \quad L_N = N^2 \mathcal{L}^D + \mathcal{L}^G,$$

378 where

$$(2.3) \quad \mathcal{L}^D = \mathcal{L} + \frac{1}{N} \mathcal{L}^{\text{WA}}$$

379 is the generator for the displacement process (which two parts are defined below) and \mathcal{L}^G is the generator
 380 of the Glauber dynamics. The process can therefore be decomposed into three distinct parts, with
 381 different scalings in N , namely the symmetric part of the motion, with generator $N^2 \mathcal{L}$, the asymmetric
 382 contribution to the displacement generator $N \mathcal{L}^{\text{WA}}$ with parameter $\lambda \geq 0$, and finally the angle-alignment
 383 with generator \mathcal{L}^G and inverse temperature $\beta \geq 0$, which are defined for any cylinder (and therefore C^1
 384 in the angular variables, cf. Definition 2.1) function $f : \Sigma_N \rightarrow \mathbb{R}$, by

$$(2.4) \quad \mathcal{L}f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \sum_{|z|=1} \eta_x (1 - \eta_{x+z}) (f(\hat{\eta}^{x,x+z}) - f(\hat{\eta})),$$

$$\mathcal{L}^{\text{WA}} f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \sum_{\substack{\delta=\pm 1 \\ i=1,2}} \delta \lambda_i(\theta_x) \eta_x (1 - \eta_{x+\delta e_i}) (f(\hat{\eta}^{x,x+\delta e_i}) - f(\hat{\eta})),$$

385

$$(2.5) \quad \mathcal{L}^G f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} c_{x,\beta}(\theta, \hat{\eta}) (f(\hat{\eta}^{x,\theta}) - f(\hat{\eta})) d\theta.$$

386 Note that \mathcal{L}^{WA} alone is not a Markov generator due to the negative jump rates, but considering the
 387 complete displacement generator $\mathcal{L} + N^{-1} \mathcal{L}^{\text{WA}}$ solves this issue for any N large enough. In the expressions
 388 above, we denoted $\hat{\eta}^{x,x+z}$ the configuration where the occupation variables $\hat{\eta}_x$ and $\hat{\eta}_{x+z}$ at sites x and
 389 $x+z$ have been exchanged in $\hat{\eta}$

$$\hat{\eta}_y^{x,x+z} = \begin{cases} \hat{\eta}_{x+z} & \text{if } y = x, \\ \hat{\eta}_x & \text{if } y = x+z, \\ \hat{\eta}_y & \text{otherwise,} \end{cases}$$

390 and $\hat{\eta}^{x,\theta}$ the configuration where the angle θ_x in $\hat{\eta}$ has been updated to θ

$$\hat{\eta}_y^{x,\theta} = \begin{cases} (\eta_y, \theta) & \text{if } y = x, \\ \hat{\eta}_y & \text{otherwise.} \end{cases}$$

391 For $x, y \in \mathbb{T}_N^2$, we write $x \sim y$ iff $|x - y| = 1$. We choose for $c_{x,\beta}$ the jump rates

$$c_{x,\beta}(\theta, \hat{\eta}) = \frac{\exp\left(\beta \sum_{y \sim x} \eta_y \cos(\theta_y - \theta)\right)}{\int_{\mathbb{S}} \exp\left(\beta \sum_{y \sim x} \eta_y \cos(\theta_y - \theta')\right) d\theta'},$$

392 which tend to align the angle in x with the neighboring particles according to XY-like jump rates (cf.
 393 Appendix A) with inverse temperature $\beta \geq 0$. Note that by construction, for any non-negative β ,
 394 $\int_{\mathbb{S}} c_{x,\beta}(\theta, \hat{\eta}) d\theta = 1$ and that the jump rates $c_{x,\beta}(\theta, \hat{\eta})$ can be uniformly bounded from above and below
 395 by two positive constants depending only on β .

396 The process defined above will be referred to as active exclusion process.

397 **2.2. Measures associated with a smooth profile and definition of the Markov process.** —
 398 We now introduce the important measures and macroscopic quantities appearing in the expression of the
 399 hydrodynamic limit. Let us denote by \mathbb{T}^2 the continuous periodic domain in dimension 2,

$$\mathbb{T}^2 = [0, 1]^2.$$

400 **Definition 2.2 (Density profile on \mathbb{T}^2).** — We denote by $\mathcal{M}_1(\mathbb{S})$ the set of non-negative measures $\hat{\alpha}$
 401 on \mathbb{S} with total mass $\hat{\alpha}(\mathbb{S})$ in $[0, 1]$. We call *density profile on the torus* any function

$$\hat{\rho} : (u, d\theta) \mapsto \hat{\rho}(u, d\theta)$$

402 such that $\hat{\rho}(u, \cdot) \in \mathcal{M}_1(\mathbb{S}) \forall u \in \mathbb{T}_N^2$. For any density profile $\hat{\rho}$ on the torus, $\hat{\rho}(u, d\theta)$ represents the local
 403 density in u of particles with angle in $d\theta$, and $\rho(u)$ represents the total density of particles in u .

404 **Definition 2.3 (Measure associated with a density profile on the torus)**

405 To any density profile on the torus $\hat{\rho}$, we associate $\mu_{\hat{\rho}}^N$, the product measure on Σ_N such that the
 406 distribution of $\hat{\eta}_x$ is given for any $x \in \mathbb{T}_N^2$ by

$$(2.6) \quad \begin{cases} \mu_{\hat{\rho}}^N(\eta_x = 0) = 1 - \rho(x/N), \\ \mu_{\hat{\rho}}^N(\eta_x = 1) = \rho(x/N), \\ \mu_{\hat{\rho}}^N(\theta_x \in d\theta \mid \eta_x = 1) = \hat{\rho}(x/N, d\theta) / \rho(x/N), \end{cases}$$

407 and such that $\hat{\eta}_x, \hat{\eta}_y$ are independent as soon as $x \neq y$.

408 In other words, under $\mu_{\hat{\rho}}^N$, the probability that a site $x \in \mathbb{T}_N^2$ is occupied is $\rho(x/N) = \int_{\mathbb{S}} \hat{\rho}(x/N, \theta) d\theta \in$
 409 $[0, 1]$. Furthermore, the angle of an empty site is set to 0 by default, and the angle of an occupied site x
 410 is distributed according to the probability distribution $\hat{\rho}(x/N, \cdot) / \rho(x/N)$.

411 **Definition of the process.** — Let $\Sigma_N^{[0, T]} := D([0, T], \Sigma_N)$ denote the space of right-continuous and left-
 412 limited (càdlàg) trajectories $\hat{\eta} : t \rightarrow \hat{\eta}(t)$. We will denote by $\hat{\eta}^{[0, T]}$ the elements of $\Sigma_N^{[0, T]}$. For any initial
 413 measure ν on Σ_N , any non-negative drift $\lambda \leq N$ (to make the displacement operator $\mathcal{L} + N^{-1}\mathcal{L}^{\text{wa}}$ a
 414 Markov generator), and any $\beta \geq 0$, we write $\mathbb{P}_\nu^{\lambda, \beta}$ for the measure on $\Sigma_N^{[0, T]}$ starting from the measure
 415 $\hat{\eta}(0) \sim \nu$, and driven by the Markov generator $L_N = L_N(\lambda, \beta)$ described earlier. We denote by $\mathbb{E}_\nu^{\lambda, \beta}$ the
 416 expectation w.r.t. $\mathbb{P}_\nu^{\lambda, \beta}$. In the case $\lambda = \beta = 0$, there is no drift and the angle of the particles are chosen
 417 uniformly in \mathbb{S} . In this case, we will omit λ and β in the previous notation and write \mathbb{P}_ν for the measure
 418 and \mathbb{E}_ν for the corresponding expectation. Let us now define the initial measure from which we start our
 419 process. Let $\hat{\zeta} \in C(\mathbb{T}^2 \times \mathbb{S})$ be a continuous non-negative function on $\mathbb{T}^2 \times \mathbb{S}$, which will define the initial
 420 macroscopic state of our particle system. We assume that for any $u \in \mathbb{T}^2$,

$$(2.7) \quad \zeta(u) := \int_{\mathbb{S}} \hat{\zeta}(u, \theta) d\theta < 1,$$

421 i.e. that the initial density is less than one initially everywhere on \mathbb{T}^2 . This assumption is crucial,
 422 because when the local density hits one, because of the exclusion rule, the system loses most of its mixing
 423 properties. At density 1, mixing only comes from the (slow, because of the scaling) Glauber dynamics,
 424 which is not sufficient to ensure that local equilibrium is preserved.

425 We can now define the initial density profile on the torus $\hat{\rho}_0$ by

$$(2.8) \quad \hat{\rho}_0(u, d\theta) = \hat{\zeta}(u, \theta) d\theta.$$

426 We start our process from a random configuration

$$(2.9) \quad \hat{\eta}(0) \sim \mu^N := \mu_{\hat{\rho}_0}^N$$

427 fitting the profile $\hat{\rho}_0$, according to Definition 2.3. Given this initial configuration, we define the Markov
 428 process $\hat{\eta}^{[0, T]} \in \Sigma_N^{[0, T]} \sim \mathbb{P}_{\mu^N}^{\lambda, \beta}$ driven by the generator L_N introduced in (2.2), starting from μ^N .

429 *Topological setup.* — Let us denote by $\mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$ the space of non-negative measures on the continuous
 430 configuration space endowed with the weak topology, and

$$(2.10) \quad \mathcal{M}^{[0,T]} = D([0, T], \mathcal{M}(\mathbb{T}^2 \times \mathbb{S}))$$

the space of right-continuous and left-limited trajectories of measures on $\mathbb{T}^2 \times \mathbb{S}$. Each trajectory $\hat{\eta}^{[0,T]}$
 of the process admits a natural image in $\mathcal{M}^{[0,T]}$ through its empirical measure

$$\pi_t^N \left(\hat{\eta}^{[0,T]} \right) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x(t) \delta_{x/N, \theta_x(t)}.$$

431 We further define the projection π^N , which associates to $\hat{\eta}^{[0,T]}$ the trajectory $t \mapsto \pi_t^N \left(\hat{\eta}^{[0,T]} \right)$. We endow
 432 $\mathcal{M}^{[0,T]}$ with Skorohod's metric defined in Appendix B.1, and the set $\mathcal{P}(\mathcal{M}^{[0,T]})$ of probability measures
 433 on $\mathcal{M}^{[0,T]}$ with the weak topology. We now define $Q^N \in \mathcal{P}(\mathcal{M}^{[0,T]})$ the distribution of the trajectory of
 434 the empirical measure $\pi^N \left(\hat{\eta}^{[0,T]} \right)$ of our process $\hat{\eta}^{[0,T]} \sim \mathbb{P}_{\mu^N}^{\lambda, \beta}$.

435 2.3. Hydrodynamic limit. —

436 *Self-diffusion coefficient.* — The hydrodynamic limit for our system involves the diffusion coefficient of
 437 a tagged particle for symmetric simple exclusion process (SSEP) in dimension 2. Let us briefly remind
 438 here its definition. On \mathbb{Z}^2 , consider an infinite equilibrium SSEP with density ρ and a tagged particle
 439 placed at time 0 at the origin. We keep track of the position $X(t) = (X_1(t), X_2(t)) \in \mathbb{Z}^2$ of the tracer
 440 particle at time t and denote by Q_ρ^* the measure of the process starting with measure μ_ρ on $\mathbb{Z}^2 \setminus \{0\}$ and
 441 a particle at the origin.

442 **Definition 2.4 (Self-Diffusion coefficient).** — The self-diffusion coefficient $d_s(\rho)$ is defined as the
 443 limiting variance of the tagged particle

$$d_s(\rho) := \lim_{t \rightarrow \infty} \frac{\mathbb{E}_{Q_\rho^*}(X_1(t)^2)}{t}.$$

444 The existence of this limit is a consequence of [28]. A variational formula for d_s has been obtained
 445 later by Spohn [44]. The regularity of the self-diffusion coefficient was first investigated in [49], where
 446 Varadhan shows that the self-diffusion matrix is Lipschitz-continuous in any dimension $d \geq 3$. Landim,
 447 Olla and Varadhan since then proved in [31] that the self-diffusion coefficient is in fact of class C^∞ in
 448 any dimension. The matter of self-diffusion being treated in full detail in Section 6, p199-240 of [29], we
 449 do not develop it further here. We summarize in appendix B.2 some useful results on the matter.

Diffusion, conductivity and alignment coefficients. — Given a density profile on the torus $\hat{\rho}(u, d\theta)$, recall
 from Definition 2.2 that $\rho(u) = \int_{\mathbb{S}} \hat{\rho}(u, d\theta)$ is the local density. We introduce the coefficients

$$\widehat{d}(\rho, \hat{\rho})(u, d\theta) = \frac{\hat{\rho}(u, d\theta)}{\rho(u)} (1 - d_s(\rho(u))) \mathbf{1}_{\{\rho(u) > 0\}}, \quad \widehat{s}(\rho, \hat{\rho})(u, d\theta) = (1 - \rho(u) - d_s(\rho(u))) \frac{\hat{\rho}(u, d\theta)}{\rho(u)} \mathbf{1}_{\{\rho(u) > 0\}},$$

450 where d_s is the self-diffusion coefficient described in the previous paragraph. We also define $\vec{\Omega}(\hat{\rho})$, the
 451 vector representing the mean direction of the asymmetry under $\hat{\rho}$,

$$\vec{\Omega}(\hat{\rho})(u) = \int_{\mathbb{S}} \hat{\rho}(u, d\theta') \begin{pmatrix} \cos(\theta') \\ \sin(\theta') \end{pmatrix}.$$

452 as well as $\Gamma(\hat{\rho})$ the local creation and annihilation rate of particles with angle θ

$$\Gamma(\hat{\rho})(u, d\theta) = \rho(u) \mathbb{E}_{\hat{\rho}(u, \cdot)} [c_{0, \beta}(\theta, \hat{\eta})] d\theta - \hat{\rho}(u, d\theta),$$

453 where under $\mathbb{E}_{\hat{\rho}(u, \cdot)}$, each site is occupied independently w.p. $\rho(u)$, and the angle of each particle is chosen
 454 according to the probability distribution $\hat{\rho}(u, \cdot) / \rho(u)$. The precise definition of $\mathbb{E}_{\hat{\rho}(u, \cdot)}$ is given just below
 455 in Definition 3.4.

456 *Weak solutions of the PDE.* — In order to state the hydrodynamic limit of our system, we need to
 457 describe the notion of weak solutions in our case, which is quite delicate because of the angles. For any
 458 measure $\pi \in \mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$ and any function $H : \mathbb{T}^2 \times \mathbb{S} \rightarrow \mathbb{R}$ integrable w.r.t. π , we shorten $\langle \pi, H \rangle =$
 459 $\int_{\mathbb{T}^2 \times \mathbb{S}} H(u, \theta) d\pi(du, d\theta)$.

460 **Definition 2.5 (Weak solution of the differential equation).** — Any trajectory of measures
 461 $(\pi_t)_{t \in [0, T]} \in \mathcal{M}^{[0, T]}$ will be called a *weak solution of the differential system*
 (2.11)

$$\left\{ \begin{array}{l} \partial_t \widehat{\rho}_t = \nabla \cdot [\widehat{\mathfrak{d}}(\rho_t, \widehat{\rho}_t) \nabla \rho_t + d_s(\rho_t) \nabla \widehat{\rho}_t] - 2\lambda \nabla \cdot \left[\widehat{\mathfrak{s}}(\rho_t, \widehat{\rho}_t) \vec{\Omega}_t + \widehat{\rho}_t d_s(\rho_t) \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \right] + \Gamma(\widehat{\rho}_t) \\ \widehat{\rho}_0(u, d\theta) = \widehat{\zeta}(u, \theta) d\theta \end{array} \right. ,$$

462 if the following four conditions are satisfied :

- 463 i) $\pi_0(du, d\theta) = \widehat{\zeta}(u, \theta) dud\theta$
 464 ii) for any fixed time $t \in [0, T]$, the measure π_t is absolutely continuous in space w.r.t. the Lebesgue
 465 measure on \mathbb{T}^2 , i.e. there exists a density profile on the torus (in the sense of Definition 2.2) $\widehat{\rho}_t$, such
 466 that

$$\pi_t(du, d\theta) = \widehat{\rho}_t(u, d\theta) du.$$

- 467 iii) Letting $\rho_t(u) = \int_{\mathbb{S}} \widehat{\rho}_t(u, d\theta)$, ρ is in $H^1([0, T] \times \mathbb{T}^2)$, i.e. there exists a family of functions $\partial_{u_i} \rho_t$
 468 in $L^2([0, T] \times \mathbb{T}^2)$ such that for any smooth function $G \in C^{0,1}([0, T] \times \mathbb{T}^2)$,

$$\int_{[0, T] \times \mathbb{T}^2} \rho_t(u) \partial_{u_i} G_t(u) dt du = - \int_{[0, T] \times \mathbb{T}^2} G_t(u) \partial_{u_i} \rho_t(u) dt du$$

- iv) For any function $H \in C^{1,2,1}([0, T] \times \mathbb{T}^2 \times \mathbb{S})$,

$$\begin{aligned} \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle &= \int_0^T \langle \pi_t, \partial_t H_t \rangle dt \\ &+ \int_0^T \int_{\mathbb{T}^2 \times \mathbb{S}} \left[\sum_{i=1}^2 \left(- \partial_{u_i} H_t(u, \theta) [\widehat{\mathfrak{d}}(\rho_t, \widehat{\rho}_t) - d'_s(\rho_t) \widehat{\rho}_t](u, d\theta) \partial_{u_i} \rho_t(u) + \partial_{u_i}^2 H_t(u, \theta) d_s(\rho_t) \widehat{\rho}_t(u, d\theta) \right. \right. \\ &\left. \left. + \partial_{u_i} H_t(u, \theta) [2\lambda \widehat{\mathfrak{s}}(\rho_t, \widehat{\rho}_t) \Omega_i(\widehat{\rho}_t) + 2\lambda_i(\theta) d_s(\rho_t) \widehat{\rho}_t](u, d\theta) \right) + H_t(u, \theta) \Gamma(\widehat{\rho}_t)(u, d\theta) \right] dudt, \end{aligned}$$

469 where the various coefficients are those defined just before, and the functions λ_i are defined in (2.1).

470 Note that in this Definition, the only quantity required to be in H^1 is the total density ρ : indeed, the
 471 term $d_s(\rho_t) \nabla \widehat{\rho}_t$ is rewritten as

$$d_s(\rho_t) \nabla \widehat{\rho}_t = \nabla(d_s(\rho_t) \widehat{\rho}_t) - d'_s(\rho_t) \widehat{\rho}_t \nabla \rho_t,$$

472 and the first term in the right-hand side above allows another derivative to be applied to the test function
 473 H , whereas the second term only involves the derivative of ρ as wanted.

474 We are now ready to state our main theorem :

475 **Theorem 2.6.** — *The sequence $(Q^N)_{N \in \mathbb{N}}$ defined at the end of Section 2.2 is weakly relatively compact,*
 476 *and any of its limit points Q^* is concentrated on trajectories $(\pi_t)_{t \in [0, T]}$ which are solution of (2.11) in*
 477 *the sense of Definition 2.5.*

478 **Remark 2.7 (Uniqueness of the weak solutions of equation (2.11))**

479 One of the reasons for our weak formulation of the scaling limit of the active exclusion process is
 480 the lack of proof for the uniqueness of weak solutions of equation (2.11). Several features of equation
 481 (2.11) make the uniqueness difficult to obtain : First, our differential equation does not take the form
 482 of an autonomous differential equation : the variation of $\widehat{\rho}_t(u, \theta)$ involves the total density ρ , therefore
 483 the differential equation is in fact a differential system operating on the vector $(\widehat{\rho}_t(u, \theta), \rho_t(u))$. Cross-
 484 diffusive systems can exhibit pathological behavior when the diffusion matrix has negative eigenvalues,
 485 but in our case, both eigenvalues are non-negative and this issue does not appear.

486 However, although cross-diffusive systems are quite well understood (cf. for example [1]), our equation
 487 involves a drift term which factors in via the vector $\vec{\Omega}(\hat{\rho}_t)$ the whole profile $(\hat{\rho}_t(u, \theta))_{\theta \in \mathbb{S}}$. One of the
 488 consequences of this drift term, which is the main obstacle to prove uniqueness, is that even the uniqueness
 489 of the total density $\rho_t(u)$ is not well established. Indeed, contrary to [35], in which the total density evolves
 490 according to the heat equation, the total density in our case is driven by the Burgers-like equation

$$\partial_t \rho_t(u) = \Delta \rho_t(u) - \lambda \nabla \cdot (m_t(u)(1 - \rho_t(u)))$$

491 where m is a quantity which depends on the whole profile $(\hat{\rho}_t(u, \theta))_{\theta \in \mathbb{S}}$, and for which uniqueness is hard
 492 to obtain.

493 **2.4. Instantaneous currents.** — In order to get a grasp on the delicate points of the proof, and to
 494 introduce the particle currents on which rely the proof of Theorem 2.6, we need a few more notations.

495 Throughout the proof, for any function $\varphi : \Sigma_N \rightarrow \mathbb{R}$ and $x \in \mathbb{T}_N^2$, we will denote by $\tau_x \varphi : \Sigma_N \rightarrow \mathbb{R}$ the
 496 function which associates to a configuration $\hat{\eta}$ the value $\varphi(\tau_x \hat{\eta})$, where $\tau_x \hat{\eta} \in \Sigma_N$ is the translation of the
 497 configuration $\hat{\eta}$ by a vector x :

$$(\tau_x \hat{\eta})_y = \hat{\eta}_{x+y}, \quad \forall y \in \mathbb{T}_N^2.$$

498

499 For any function

$$H : [0, T] \times \mathbb{T}^2 \times \mathbb{S} \rightarrow \mathbb{R} \\ (t, u, \theta) \mapsto H_t(u, \theta) ,$$

500 in $C^{1,2,1}([0, T] \times \mathbb{T}^2 \times \mathbb{S})$, and any measure π on $\mathbb{T}^2 \times \mathbb{S}$, let us denote

$$\langle \pi, H_t \rangle = \int_{\mathbb{T}^2 \times \mathbb{S}} H_t(u, \theta) d\pi(u, \theta)$$

501 the integral of H with respect to the measure π . We consider the martingale $M_t^{H,N}$

$$(2.12) \quad M_t^{H,N} = \langle \pi_t^N, H_t \rangle - \langle \pi_0^N, H_0 \rangle - \int_0^t (\partial_s + L_N) \langle \pi_s^N, H_s \rangle ds,$$

502 where π_s^N is the *empirical measure of the process*

$$\pi_s^N = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x(s) \delta_{x/N, \theta_x(s)}.$$

The quadratic variation of this martingale can be explicitly computed, and is equal to (cf. Appendix 1.5 of [27])

$$[M^{H,N}]_t = \int_0^t L_N (\langle \pi_s^N, H_s \rangle^2) - 2 \langle \pi_s^N, H_s \rangle L_N \langle \pi_s^N, H_s \rangle ds \\ = \frac{2}{N^4} \sum_{x \in \mathbb{T}_N^2} \left[\int_0^t L_N [\eta_x(s) \eta_{x+1}(s) H_s(x/N, \theta_x(s)) H_s((x+1)/N, \theta_{x+1}(s))] \right. \\ \left. - \eta_{x+1}(s) H_s((x+1)/N, \theta_{x+1}(s)) L_N [\eta_x(s) H_s(x/N, \theta_x(s))] \right. \\ \left. - \eta_x(s) H_s(x/N, \theta_x(s)) L_N [\eta_{x+1}(s) H_s((x+1)/N, \theta_{x+1}(s))] ds \right].$$

Because of the initial factor N^{-4} , the contributions of the asymmetric and Glauber parts of the dynamic can be crudely bounded respectively by CN^{-1} and CN^{-2} . By computing the symmetric part, we finally obtain

$$[M_t^{H,N}]_t = O(1/N) + \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \left[\int_0^t \eta_x(s) \left[H_s^2(x+1/N, \theta_x(s)) + H_s^2(x-1/N, \theta_x(s)) - 2H_s^2(x/N, \theta_x(s)) \right. \right. \\ \left. \left. + 2\eta_x(s)(1 - \eta_{x+1}(s)) H_s(x/N, \theta_x(s)) [H_s((x+1)/N, \theta_x(s)) - H_s(x/N, \theta_x(s))] \right. \right. \\ \left. \left. + 2\eta_{x+1}(s)(1 - \eta_x(s)) H_s((x+1)/N, \theta_{x+1}(s)) [H_s((x+1)/N, \theta_{x+1}(s)) - H_s(x/N, \theta_{x+1}(s))] \right] ds \right].$$

503 Because we assumed that H is a smooth function, the three lines above are of order at most N^{-1} , and
 504 therefore $[M_t^{H,N}]_t$ vanishes as N goes to infinity. The martingale thus vanishes uniformly in time, in
 505 probability under $\mathbb{P}_{\mu_N}^{\lambda,\beta}$.

506 Assume now that the function H takes the form

$$(2.13) \quad H_s(u, \theta) = G_s(u)\omega(\theta),$$

507 where G and ω are respectively functions on $[0, T] \times \mathbb{T}^2$ and \mathbb{S} . From now on, for any function $\Phi : \mathbb{S} \rightarrow \mathbb{R}$,
 508 any configuration $\hat{\eta}$ and any $x \in \mathbb{T}_N^2$ we will shorten

$$\eta_x^\Phi = \Phi(\theta_x)\eta_x.$$

509 With these notations, recalling that

$$L_N = N^2 (\mathcal{L} + N^{-1} \mathcal{L}^{\text{WA}}) + \mathcal{L}^{\text{G}},$$

510 we can write the generator part of the integral term of (2.12) as

$$(2.14) \quad \int_0^T L_N \langle \pi_s^N, H_s \rangle ds = \frac{1}{N^2} \int_0^T \sum_{x \in \mathbb{T}_N^2} G_s(x/N) (N^2 [\mathcal{L}\eta_x^\omega(s) + N^{-1} \mathcal{L}^{\text{WA}}\eta_x^\omega(s)] + \mathcal{L}^{\text{G}}\eta_x^\omega(s)) ds.$$

511 Let us introduce accordingly the three instantaneous currents in our active exclusion process. Recall that
 512 τ_x represents the translation of a function by x .

513 **Definition 2.8.** — Given a site $x \in \mathbb{T}_N^2$, each part of the generator L_N 's action over η_x^ω can be written

$$(2.15) \quad \mathcal{L}\eta_x^\omega = \sum_{i=1}^2 (\tau_{x-e_i} j_i^\omega - \tau_x j_i^\omega) \quad \text{with} \quad j_i^\omega(\hat{\eta}) = \eta_0^\omega (1 - \eta_{e_i}) - \eta_{e_i}^\omega (1 - \eta_0),$$

514

$$(2.16) \quad \mathcal{L}^{\text{WA}}\eta_x^\omega = \sum_{i=1}^2 (\tau_{x-e_i} r_i^\omega - \tau_x r_i^\omega) \quad \text{with} \quad r_i^\omega(\hat{\eta}) = \eta_0^{\omega\lambda_i} (1 - \eta_{e_1}) + \eta_{e_i}^{\omega\lambda_i} (1 - \eta_0),$$

515 and

$$(2.17) \quad \mathcal{L}^{\text{G}}\eta_x^\omega = \tau_x \gamma^\omega \quad \text{with} \quad \gamma^\omega(\hat{\eta}) = \eta_0 \int_{\mathbb{S}} c_{0,\beta}(\theta, \hat{\eta}) (\omega(\theta) - \omega(\theta_0)) d\theta.$$

516 For $i \in \{1, 2\}$ we will at times write $j_{x,x+e_i}^\omega = \tau_x j_i^\omega$ (resp. $r_{x,x+e_i}^\omega = \tau_x r_i^\omega$), which is interpreted as the
 517 *instantaneous current with intensity ω in the direction i* along the edge $(x, x+e_i)$ of the symmetric (resp.
 518 weakly asymmetric) part of the process. The last quantity $\tau_x \gamma^\omega$ is the *local alignment rate*.

519 When considering the time process $(\hat{\eta}(t))_{t \in [0, T]}$ we will, for the sake of concision, write $j_i^\omega(t)$ for
 520 $j_i^\omega(\hat{\eta}(t))$, and in the same fashion $r_i^\omega(t)$ instead of $r_i^\omega(\hat{\eta}(t))$, and $\gamma^\omega(t)$ instead of $\gamma^\omega(\hat{\eta}(t))$. Finally, in the
 521 case where $\omega \equiv 1$, we will denote by

$$j_i := j_i^1 = \eta_0 - \eta_{e_i}.$$

Performing a first integration by parts on the exclusion part of the right-hand side of (2.14), we obtain
 thanks to equations (2.15), (2.16) and (2.17)

$$(2.18) \quad \int_0^T L_N \langle \pi_s^N, H_s \rangle ds = \frac{1}{N^2} \int_0^T \sum_{x \in \mathbb{T}_N^2} \tau_x \left[\sum_{i=1}^2 (N j_i^\omega(s) + r_i^\omega(s)) \partial_{u_i, N} G_s(x/N) + G_s(x/N) \gamma^\omega(s) \right] ds,$$

522 where $\partial_{u_i, N}$ is the discrete partial derivative

$$(\partial_{u_i, N} G)(x/N) = N [G((x+e_i)/N) - G(x/N)].$$

523

524 The spatial averaging is of great importance throughout the proof of the hydrodynamic limit, we need
 525 some convenient notation to represent this operation. For any site $x \in \mathbb{T}_N^2$ and any integer l , we denote
 526 by

$$B_l(x) = \{ y \in \mathbb{T}_N^2 \mid \|y - x\|_\infty \leq l \}$$

527 the box of side length $2l + 1$ around x . In the case where $x = 0$ is the origin, we will simply write
 528 $B_l := B_l(0)$. For any *finite* subset $B \subset \mathbb{T}_N^2$, $|B|$ denotes the number of sites in B . Given φ a function on
 529 Σ_N , we denote by

$$(2.19) \quad \langle \varphi \rangle_x^l = \frac{1}{|B_l(x)|} \sum_{y \in B_l(x)} \tau_y \varphi$$

530 the average of the function φ over $B_l(x)$. In the case where $\varphi(\hat{\eta}) = \eta_0^\omega$, (resp. $\varphi(\hat{\eta}) = \eta_0$), we will write
 531 $\tau_x \rho_l^\omega = \langle \varphi \rangle_x^l$ (resp. $\tau_x \rho_l$) for the empirical average of η^ω (resp. η) over the box centered in x of side
 532 length $2l + 1$.

533 We will also denote for any integer l by $\hat{\rho}_l$ the empirical angular density defined by

$$(2.20) \quad \hat{\rho}_l = \frac{1}{|B_l|} \sum_{x \in B_l} \eta_x \delta_{\theta_x} \in \mathcal{M}_1(\mathbb{S}),$$

534 where $\mathcal{M}_1(\mathbb{S})$ is the set of non-negative measures on \mathbb{S} with total mass in $[0, 1]$ (cf. Definition 3.1 below).
 535 Finally, to simplify notations throughout the proof, we will write εN instead of the integer part $\lfloor \varepsilon N \rfloor$.

536 3. Canonical measures, entropy and irreducibility

537 **3.1. Definition of the canonical measures.** — Due to the presence of angles, the canonical product
 538 measures for the active exclusion process are not parameterized by the local density $\alpha \in [0, 1]$ like the
 539 SSEP, but rather by a measure $\hat{\alpha}$ on $[0, 2\pi]$ whose total mass $\int_{\mathbb{S}} \hat{\alpha}(d\theta)$ is the local density.

540 **Definition 3.1 (Grand-canonical parameters).** — Recall that \mathbb{T}^2 is the 2-dimensional continuous
 541 torus $(\mathbb{R}/\mathbb{Z})^2$, and let $\mathcal{M}(\mathbb{S})$ be the set of non-negative measures on \mathbb{S} . We will call *grand-canonical*
 542 *parameter* any measure $\hat{\alpha} \in \mathcal{M}(\mathbb{S})$ with total mass $\alpha := \int_{\mathbb{S}} \hat{\alpha}(d\theta) \leq 1$. We denote by

$$(3.1) \quad \mathcal{M}_1(\mathbb{S}) = \{ \hat{\alpha} \in \mathcal{M}(\mathbb{S}) \mid \alpha \in [0, 1] \},$$

543 the set of grand-canonical parameters.

544 We now define a topological setup on $\mathcal{M}_1(\mathbb{S})$. Let us consider on $C^1(\mathbb{S})$, the set of continuously
 545 differentiable functions, the norm $\|g\|^* = \max(\|g\|_\infty, \|g'\|_\infty)$, and let B^* be the unit ball in $(C^1(\mathbb{S}), \|\cdot\|^*)$.

546 **Definition 3.2.** — We endow $M(\mathbb{S})$, the vector space of finite mass signed measures on \mathbb{S} , with the
 547 norm

$$\| \hat{\alpha} \| = \sup_{g \in B^*} \left\{ \int_{\mathbb{S}} g(\theta) d\hat{\alpha}(\theta) \right\},$$

548 and with the corresponding distance

$$d(\hat{\alpha}, \hat{\alpha}') := \sup_{g \in B^*} \left\{ \int_{\mathbb{S}} g(\theta) d\hat{\alpha}(\theta) - \int_{\mathbb{S}} g(\theta) d\hat{\alpha}'(\theta) \right\}.$$

549 We then endow $\mathcal{M}_1(\mathbb{S})$ with the topology induced by $\| \cdot \|$. This distance is a generalization of the
 550 Wasserstein distance to measures which are not probability measures.

551 **Remark 3.3.** — As checked in Appendix C, this topology satisfies

552 — for any cylinder function ψ , the application $\hat{\alpha} \mapsto \mathbb{E}_{\hat{\alpha}}(\psi)$ is Lipschitz-continuous (cf. Proposition
 553 C.2).

554 — any $\hat{\alpha} \in M(\mathbb{S})$ is the limit of combinations of Dirac measures.

555 — if $\theta_k \rightarrow \theta$, then $\| \delta_{\theta_k} - \delta_\theta \| \rightarrow 0$.

556 It is therefore the natural choice for our problem.

557 We now introduce the canonical measures of our process, which are translation-invariant particular
 558 cases of measures associated with a density profile, introduced in Definition 2.3.

559 **Definition 3.4 (Grand canonical measures).** — Consider a *translation invariant* density profile on
 560 the torus $\widehat{\rho}$, i.e. such that for any $u \in \mathbb{T}^2$,

$$\widehat{\rho}(u, d\theta) = \widehat{\alpha}(d\theta)$$

561 for some grand-canonical parameter $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$ independent of u . We will write $\mu_{\widehat{\alpha}}$ for the product
 562 measure $\mu_{\widehat{\rho}}^N$, and $\mathbb{E}_{\widehat{\alpha}}$ will denote the corresponding expectation. This class of measures will be referred
 563 to as *grand-canonical measures*. Furthermore, for any $\alpha \in [0, 1]$, the measure $\mu_{\widehat{\alpha}}$ associated with the
 564 uniform density profile on the torus

$$\widehat{\rho}(u, d\theta) \equiv \alpha d\theta / 2\pi,$$

565 where the angle of each particle is chosen uniformly in \mathbb{S} , will be denoted by μ_{α}^* , and the corresponding
 566 expectation will be denoted by \mathbb{E}_{α}^* .

567 Note that these measures are dependent on N , but due to their translation invariant nature, we will
 568 omit this in our notation.

569 **Remark 3.5.** — For any density $\alpha \in [0, 1]$, the measure μ_{α}^* on Σ_N is not invariant for our dynamic,
 570 because although it is invariant for the symmetric part of the exclusion, the weakly asymmetric part (as
 571 well as the Glauber part as soon as $\beta \neq 0$) breaks this property. We will however prove in Section 3.2
 572 that due to the scaling in N , the stationary distribution of our dynamics is locally close to μ_{α}^* .

573 **Definition 3.6 (Canonical measures).** — Fix a positive integer l , an integer $K \leq (2l + 1)^2$ and
 574 $\Theta_K = (\theta_1, \dots, \theta_K)$ a family of K angles, taken up to reordering of its coordinates, we shorten by \widehat{K} the
 575 pairs (K, Θ_K) , which we will refer to as *canonical states* on B_l . We will denote by \mathbb{K}_l the set of canonical
 576 states \widehat{K} on B_l ,

$$\mathbb{K}_l = \{\widehat{K} = (K, \Theta_K) \mid K \leq (2l + 1)^2\}.$$

577 Since our process loses its fast mixing properties when there is only one or less empty site (In which case
 578 mixing mainly comes from the Glauber dynamics, which is very slow w.r.t. the displacement dynamics,
 579 cf. Section 3.3 below), we also introduce

$$(3.2) \quad \widetilde{\mathbb{K}}_l = \{\widehat{K} \in \mathbb{K}_l \mid K \leq (2l + 1)^2 - 2\},$$

580 the set of \widehat{K} for which the exclusion process on B_l is irreducible. Furthermore, for any fixed $\widehat{K} \in \mathbb{K}_l$, we
 581 denote by

$$(3.3) \quad \Sigma_l^{\widehat{K}} = \left\{ \widehat{\eta} \text{ config. on } B_l \mid \sum_{x \in B_l} \eta_x \delta_{\theta_x} = \sum_{k=1}^K \delta_{\theta_k} \right\},$$

582 the set of configurations on B_l with canonical state \widehat{K} in B_l .

583 Let $\mu_{\alpha, l}^*$ denote the measure μ_{α}^* on B_l , for any density $\alpha \in]0, 1[$, we will denote by $\mu_{l, \widehat{K}}$ the conditioning
 584 of $\mu_{\alpha, l}^*$ to $\Sigma_l^{\widehat{K}}$ (which is therefore a measure on the set of local configurations $\widehat{\eta} \in (\{0, 1\} \times \mathbb{S})^{B_l}$), and by
 585 $\mathbb{E}_{l, \widehat{K}}$ the corresponding expectation

$$\mathbb{E}_{l, \widehat{K}}(g) = \mathbb{E}_{\alpha, l}^* \left(g \mid \widehat{\eta} \in \Sigma_l^{\widehat{K}} \right).$$

586 These measures will be referred to as *canonical measures of the process*.

587 **Definition 3.7.** — Fix $l \in \mathbb{N}$, we associate to any $\widehat{K} \in \mathbb{K}_l$ the grand-canonical parameter

$$\widehat{\alpha}_{\widehat{K}, l} = \frac{1}{(2l + 1)^2} \sum_{k=1}^K \delta_{\theta_k}.$$

588 When there is no ambiguity, we will drop the dependency in l and simply write $\widehat{\alpha}_{\widehat{K}} = \widehat{\alpha}_{\widehat{K}, l}$.

589 The pertinent results regarding the metric space $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$ are regrouped in Appendix C : The
 590 equivalence of ensembles is proved in Section C.1, the Lipschitz-continuity of the expectation w.r.t. $\mu_{\widehat{\alpha}}$
 591 in the parameter $\widehat{\alpha}$ is proved in Section C.2, and finally, the compactness of the set $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$ is
 592 proved in Section C.3.

593 **3.2. Entropy production and local equilibrium.** — *The proof of the replacement Lemma is based*
 594 *on the control of the entropy production of the process. The difficulty here is that the invariant measures of*
 595 *the process are not known, and the decay of the relative entropy w.r.t. these measures cannot be computed*
 596 *directly. Thus we consider approximations of these measures, and for a fixed non-trivial density $\alpha \in]0, 1[$,*
 597 *our goal is to get an estimate of the entropy of the process's time average with respect to the reference*
 598 *measure μ_α^* introduced in Definition 3.4.*

599 Let us fix $\alpha \in]0, 1[$, we are going to prove that regardless of the initial density profile, the entropy of
 600 the active exclusion process w.r.t the measure of a process started from μ_α^* and following a symmetric
 601 simple exclusion process can be controlled by CN^2 for some constant C .

602 The choice of μ_α^* among the $\mu_{\alpha'}^*$, $\alpha' \in]0, 1[$ is not important, since for any different angle density
 603 $\alpha' \in]0, 1[$, the relative entropy between the two product measures μ_α^* and $\mu_{\alpha'}^*$ is of order N^2 as well.

604 For some cylinder function $h \in \mathcal{C}$, and some edge $a = (a_1, a_2)$ in \mathbb{T}_N^2 or \mathbb{Z}^2 , we denote by ∇_a the
 605 gradient representing the transfer of a particle from site a_1 to site a_2 under the exclusion process

$$(3.4) \quad \nabla_a f(\hat{\eta}) = \eta_{a_1} (1 - \eta_{a_2}) (f(\hat{\eta}^{a_1, a_2}) - f(\hat{\eta})).$$

606 We will shorten this notation in the case where $a = (0, e_j)$ by writing $\nabla_j := \nabla_{(0, e_j)}$. Before turning
 607 to the control of the entropy itself, we introduce an important quantity in the context of hydrodynamic
 608 limits.

609 **Definition 3.8 (Dirichlet form of the symmetric dynamics).** — Let h be a cylinder function, we
 610 introduce the Dirichlet form of the process

$$(3.5) \quad \mathcal{D}_{\hat{\alpha}}(h) = -\mathbb{E}_{\hat{\alpha}}(h\mathcal{L}h),$$

611 where \mathcal{L} is the symmetric exclusion generator defined in equation (2.4). It can be rewritten thanks to
 612 the invariance of $\mu_{\hat{\alpha}}$ w.r.t the symmetric exclusion process as

$$\mathcal{D}_{\hat{\alpha}}(h) = \frac{1}{2} \mathbb{E}_{\hat{\alpha}} \left(\sum_{x \in \mathbb{T}_N^2} \sum_{|z|=1} (\nabla_{x, x+z} h)^2 \right).$$

613 If there is no ambiguity, we will omit the dependency in $\hat{\alpha}$ of the Dirichlet form, and simply denote it by
 614 \mathcal{D} . The Dirichlet form is convex and non-negative. Furthermore, any function f in its kernel is such that
 615 $f(\hat{\eta}) = f(\hat{\eta}')$ for any pair $(\hat{\eta}, \hat{\eta}')$ of configurations with the same number of particles $K \leq N^2 - 2$ and the
 616 same family of angles. For any non-negative function h , we also introduce the Dirichlet form

$$(3.6) \quad D(h) = \mathcal{D}(\sqrt{h}),$$

617 which has the same properties as \mathcal{D} .

618 We now investigate the entropy production of the active exclusion process. Let $P_t^{N, \lambda, \beta}$ be the semi-
 619 group of the active exclusion process associated with the complete generator L_N introduced in equation
 620 (2.2), and $\mu_t^N = \mu^N P_t^{N, \lambda, \beta}$ the measure of the configuration at time t . Because we assume the initial
 621 profile to be continuous (and therefore bounded), μ^N is absolutely continuous with respect to the product
 622 measure μ_α^* , with density

$$(3.7) \quad \frac{d\mu^N}{d\mu_\alpha^*}(\hat{\eta}) = \prod_{x \in \mathbb{T}_N^2} \left[(1 - \eta_x) \frac{1 - \zeta(x/N)}{1 - \alpha} + \eta_x \frac{2\pi \hat{\zeta}(x/N, \theta_x)}{\alpha} \right].$$

623 This, and the fact that the alignment rates $c_{x, \beta}$ are bounded from above and below uniformly in θ ,
 624 guarantee that for any time t , μ_t^N is also absolutely continuous w.r.t. μ_α^* . We therefore denote by
 625 $f_t^N = d\mu_t^N / \mu_\alpha^*$ the density of the measure at time t w.r.t. the reference measure μ_α^* . We now prove the
 626 following estimate on the entropy of the function f_t^N .

627 **Proposition 3.9 (Control on the entropy and the Dirichlet form of f_t^N)**

For any density f w.r.t. μ_α^* , we denote by $H(f) = \mathbb{E}_\alpha^*(f \log f)$ the entropy of the density f . Then, for any time $t > 0$, there exists a constant $K_0 = K_0(t, \lambda, \beta, \hat{\zeta})$ such that

$$H\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq K_0 N^2 \quad \text{and} \quad D\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq K_0.$$

628 *Proof of Proposition 3.9.* — The density f_t^N is solution to

$$(3.8) \quad \begin{cases} \partial_t f_t^N = L_N^* f_t^N \\ f_0^N = d\mu^N / d\mu_\alpha^*, \end{cases}$$

629 where L_N^* is the adjoint of L_N in $L^2(\mu_\alpha^*)$. To clarify the proof, we divide it in a series of steps.

630 *Expression of the entropy production of the system.* — The relative entropy of μ_t^N with respect to the
631 reference measure μ_α^* is given by

$$H(\mu_t^N | \mu_\alpha^*) = H(f_t^N) = \mathbb{E}_\alpha^*(f_t^N \log f_t^N),$$

632 which is non-negative due to the convexity on $[0, +\infty[$ of $x \mapsto x \log x$. According to equation (3.8), its
633 time derivative is

$$(3.9) \quad \partial_t H(f_t^N) = \mathbb{E}_\alpha^*(\log f_t^N L_N^* f_t^N) + \mathbb{E}_\alpha^*(L_N^* f_t^N).$$

634 The second term on the right-hand side is equal to

$$\mathbb{E}_\alpha^*(L_N^* f_t^N) = \mathbb{E}_\alpha^*(f_t^N L_N \tilde{1}) = 0,$$

635 since all constant functions are in the kernel of L_N . Equation (3.9) can be rewritten, since L_N^* is the
636 adjoint of L_N in $L^2(\mu_\alpha^*)$, as

$$\partial_t H(f_t^N) = \mathbb{E}_\alpha^*(f_t^N L_N \log f_t^N).$$

637 Now thanks to the elementary inequality

$$\log b - \log a \leq \frac{2}{\sqrt{a}}(\sqrt{b} - \sqrt{a}),$$

638 we can control $L_N \log f_t^N$ by

$$\frac{2}{\sqrt{f_t^N}} L_N \sqrt{f_t^N},$$

639 therefore, the definition of L_N yields

$$\partial_t H(f_t^N) \leq -2N^2 D(f_t^N) + 2N \mathbb{E}_\alpha^*\left(\sqrt{f_t^N} \mathcal{L}^{\text{WA}} \sqrt{f_t^N}\right) + 2 \mathbb{E}_\alpha^*\left(\sqrt{f_t^N} \mathcal{L}^{\text{G}} \sqrt{f_t^N}\right),$$

640 where D is the Dirichlet form defined in Definition 3.8.

641 Integrating between the times 0 and t , we get

$$(3.10) \quad H(\mu_t^N | \mu_\alpha^*) + 2N^2 \int_0^t D(f_s^N) \leq H(\mu^N | \mu_\alpha^*) + 2 \int_0^t \mathbb{E}_\alpha^*\left(\sqrt{f_s^N} (N \mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{G}}) \sqrt{f_s^N}\right) ds$$

642 Since the Dirichlet form of the symmetric exclusion process is non-negative, we now focus on showing that
643 the part of the entropy due to the weakly asymmetric part and Glauber part do not grow too much in N ,
644 in order to get an upper bound on the Dirichlet form $D(f)$ and on the entropy $H(\mu_t^N | \mu_\alpha^*)$. From here,
645 control over the initial relative entropy should suffice to ensure that the measure of the active exclusion
646 process remains close to a product measure.

647 *Bound on the entropy production of the asymmetric part of the dynamics.* — by definition of the asym-
 648 metric dynamic,

$$\mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \mathcal{L}^{\text{WA}} \sqrt{f_s^N} \right) = \mathbb{E}_\alpha^* \left(\sum_{x,i,\delta=\pm 1} \lambda_i(\theta_x) \delta \eta_x (1 - \eta_{\delta e_i}) \sqrt{f_s^N}(\widehat{\eta}) \left(\sqrt{f_s^N}(\widehat{\eta}^{x,x+\delta e_i}) - \sqrt{f_s^N}(\widehat{\eta}) \right) \right).$$

649 Despite the extra factor N , the jump rates of the weakly asymmetric dynamics are not very different
 650 from symmetric exclusion process jump rates, which allows us to estimate the quantity above in terms of
 651 the Dirichlet form. More precisely, thanks to the elementary inequality

$$\mathbb{E}(\varphi\psi) \leq \gamma \mathbb{E}(\varphi^2)/2 + \mathbb{E}(\psi^2)/2\gamma$$

652 which holds for any positive constant γ , we can write with

$$\varphi = \eta_x (1 - \eta_{\delta e_i}) \left(\sqrt{f_s^N}(\widehat{\eta}^{x,x+\delta e_i}) - \sqrt{f_s^N}(\widehat{\eta}) \right),$$

653 and

$$\psi = \lambda_i(\theta_x) \delta \sqrt{f_s^N}(\widehat{\eta})$$

that

$$\begin{aligned} \mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \mathcal{L}^{\text{WA}} \sqrt{f_s^N} \right) \\ \leq \sum_{x,i,\delta=\pm 1} \left[\frac{\mathbb{E}_\alpha^* (\lambda_i(\theta_x)^2 f_s^N)}{2\gamma} + \frac{\gamma}{2} \mathbb{E}_\alpha^* \left(\eta_x (1 - \eta_{\delta e_i}) \left(\sqrt{f_s^N}(\widehat{\eta}^{x,x+\delta e_i}) - \sqrt{f_s^N}(\widehat{\eta}) \right)^2 \right) \right]. \end{aligned}$$

654 In right-hand side above, letting $C_\lambda = 4\lambda^2$ the first term can be bounded by $C_\lambda N^2/2\gamma$, since the number
 655 of terms in the sum is $4N^2$, whereas the second sum of terms is $\gamma D(f_s^N)$. We then let $\gamma = N$ to obtain
 656 the upper bound

$$(3.11) \quad 2N \mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \mathcal{L}^{\text{WA}} \sqrt{f_s^N} \right) \leq C_\lambda N^2 + N^2 D(f_s^N).$$

Bound on the entropy production of the Glauber part of the dynamics. — thanks to the elementary
 inequality $ab \leq (a^2 + b^2)/2$, and since the jump rates $c_{x,\beta}$ are less than $e^{8\beta}/2\pi$, and η_x by 1

$$\begin{aligned} \mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \mathcal{L}^{\text{G}} \sqrt{f_s^N} \right) &= \mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} c_{x,\beta}(\theta, \widehat{\eta}) \left(\sqrt{f_s^N}(\widehat{\eta}^{x,\theta}) - \sqrt{f_s^N}(\widehat{\eta}) \right) d\theta \right) \\ &\leq \frac{e^{8\beta}}{2\pi} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\frac{1}{2} \int_{\mathbb{S}} f_s^N(\widehat{\eta}^{x,\theta}) d\theta + \frac{3}{2} f_s^N(\widehat{\eta}) \right). \end{aligned}$$

657 Since $\mathbb{E}_\alpha^* \left(\frac{1}{2\pi} \int_{\mathbb{S}} f_s^N(\widehat{\eta}^{x,\theta}) d\theta \right) = \mathbb{E}_\alpha^* (f_s^N)$, the expectation can be bounded from above by 2, and we can
 658 therefore write, letting $C_\beta = 2e^{8\beta}/\pi$

$$(3.12) \quad 2\mathbb{E}_\alpha^* \left(\sqrt{f_s^N} \mathcal{L}^{\text{G}} \sqrt{f_s^N} \right) \leq C_\beta N^2.$$

659 *Bound on the Dirichlet form and on the entropy production.* — at this point, we obtain from (3.10),
 660 (3.11) and (3.12)

$$H(\mu_t^N | \mu_\alpha^*) + N^2 \int_0^t D(f_s^N) ds \leq H(\mu^N | \mu_\alpha^*) + t(C_\lambda + C_\beta)N^2$$

661 By (3.7), there exists a constant $K = K(\widehat{\zeta}, \alpha)$, such that for any $N \in \mathbb{N}$, $\|\log d\mu^N/d\mu_\alpha^*\|_\infty \leq KN^2$, and
 662 we can therefore estimate the relative entropy of the initial measure μ^N w.r.t. μ_α^* by

$$(3.13) \quad H(\mu^N | \mu_\alpha^*) \leq KN^2.$$

663 We can therefore write

$$(3.14) \quad H(\mu_t^N | \mu_\alpha^*) + \int_0^t N^2 D(f_s^N) ds \leq K(t)N^2.$$

664 where $K(t) = K + t(C_\lambda + C_\beta)$ is a positive constant. Since $H(f) = \mathbb{E}_\alpha^*(f \log f)$ and $D(f)$ are both non-
 665 negative and convex, we can deduce from (3.14), that for some time-dependent constant $K_0 = \int_0^t K(s)ds$,
 666 we have

$$(3.15) \quad H\left(\frac{1}{t} \int_0^t f_s^N\right) \leq K_0 N^2 \quad \text{and} \quad D\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq K_0.$$

667 This upper bound proves proposition 3.9, and will be necessary in the next Section to apply the replace-
 668 ment Lemma 4.1 to the active exclusion process. \square

669 Before taking on the problem of irreducibility, we give a result that will be needed several times
 670 throughout the proof, and comes from the entropy inequality. Let us denote by $\mathcal{L}^{G,\beta=0}$ the modified
 671 Glauber generator with uniform update of the angle in \mathbb{S} , (i.e. $\beta = 0$)

$$\mathcal{L}^{G,\beta=0} f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \eta_x \frac{1}{2\pi} \int_{\mathbb{S}} (f(\hat{\eta}^{x,\theta}) - f(\hat{\eta})) d\theta$$

672 and denote in a similar fashion

$$(3.16) \quad L_N^{\beta=0} = N^2 \mathcal{L}^D + \mathcal{L}^{G,\beta=0},$$

673 which is the complete generator of the active exclusion process with random update of the angles. Then,
 674 accordingly to our previous notations, $\mathbb{P}_{\mu_\alpha^*}^{\lambda,0}$ is the measure on the trajectories started from μ_α^* and driven
 675 by the generator $L_N^{\beta=0}$. We can now state the following result.

676 **Proposition 3.10 (Comparison of $\mathbb{P}_{\mu_N}^{\lambda,\beta}$ and $\mathbb{P}_{\mu_\alpha^*}^{\lambda,0}$).** — We endow Σ_N with the topology associated
 677 with the metric

$$(3.17) \quad \mathbf{d}(\hat{\eta}, \hat{\eta}') = \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\eta_x = \eta'_x\}} \frac{|\theta_x - \theta'_x|}{\pi} + \mathbb{1}_{\{\eta_x \neq \eta'_x\}},$$

678 where $|\theta_x - \theta'_x| \in [0, \pi]$, and $\Sigma_N^{[0,T]}$ with the corresponding Skorohod's metric. There exists a constant
 679 $K_0 = K_0(T, \beta, \hat{\zeta}) > 0$ such that for any bounded and measurable function $X : \Sigma_N^{[0,T]} \rightarrow \mathbb{R}$ and any $A > 0$,

$$\mathbb{E}_{\mu_N}^{\lambda,\beta} \left[X \left(\hat{\eta}^{[0,T]} \right) \right] \leq \frac{1}{A} \left(K_0 N^2 + \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(AX \left(\hat{\eta}^{[0,T]} \right) \right) \right] \right),$$

680 where $\hat{\eta}^{[0,T]}$ is the notation already introduced at the end of Section 2.2 for a trajectory $(\hat{\eta}(t))_{t \in [0,T]}$.

681 *Proof of Proposition 3.10.* — The proof of this Proposition is rather straightforward thanks to the en-
 682 tropy inequality. In a first step, we compare the same process starting from μ_α^* . First note that for any
 683 function $X : \Sigma_N^{[0,T]} \rightarrow \mathbb{R}$, we can write

$$\mathbb{E}_{\mu_N}^{\lambda,\beta} \left[X \left(\hat{\eta}^{[0,T]} \right) \right] = \mathbb{E}_{\mu_\alpha^*}^{\lambda,\beta} \left(\frac{d\mu^N}{d\mu_\alpha^*}(\hat{\eta}(0)) X \left(\hat{\eta}^{[0,T]} \right) \right).$$

This yields that

$$(3.18) \quad \mathbb{E}_{\mu_N}^{\lambda,\beta} \left[X \left(\hat{\eta}^{[0,T]} \right) \right] \leq \frac{1}{A} \left(H(\mu^N | \mu_\alpha^*) + \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,\beta} \left[\exp \left(AX \left(\hat{\eta}^{[0,T]} \right) \right) \right] \right).$$

684 In the entropy inequality above, $\mathbb{E}_{\mu_N}^{\lambda,\beta}$ is the expectation under the measure of the process started from
 685 μ^N , whereas $\mathbb{E}_{\mu_\alpha^*}^{\lambda,\beta}$ is that of the process started from the stationary measure μ_α^* .

686 By (3.13), the first term in the right-hand side above is less than KN^2/A for some fixed constant
 687 $K = K(\hat{\zeta})$. Furthermore, the Radon-Nikodym derivative of the process with alignment ($\beta > 0$) w.r.t the
 688 one without alignment ($\beta = 0$) can be explicitly computed. Given a càdlàg trajectory $\hat{\eta}^{[0,T]} \in \Sigma_N^{[0,T]}$,
 689 consider τ_1, \dots, τ_R the set of angle jumps between times 0 and T , let us denote by x_i the site at which
 690 the angle changed at time τ_i , and by $\theta_i = \theta_{x_i}(\tau_i)$ the new angle at site x_i . Then, the density between the
 691 measures with and without alignment is given by

$$\frac{d\mathbb{P}_\nu^{\lambda,\beta}}{d\mathbb{P}_\nu^{\lambda,0}}(\hat{\eta}^{[0,T]}) = \prod_{i=1}^R \frac{c_{x_i,\beta}(\theta_i, \hat{\eta}(\tau_i))}{c_{x_i,0}(\theta_i, \hat{\eta}(\tau_i))} \leq e^{8\beta R},$$

692 where R is the number of angle updates between times 0 and T . To establish the estimate above, we
 693 used that $c_{x,\beta}(\theta, \hat{\eta})$ can be uniformly bounded from above by $e^{8\beta}/2\pi$, that $c_{x,0}(\theta, \hat{\eta}) = 1/2\pi$, and that
 694 regardless of the configuration and the inverse temperature β , each site updates its angle at rate 1 (i.e.
 695 $\int_{\theta} c_{x,\beta}(\theta, \hat{\eta}) = 1$). We can now estimate the second term in the right-hand side of equation (3.18) by

$$\frac{1}{A} \log \mathbb{E}_{\mu_{\alpha}^{\lambda,0}} \left[e^{8\beta R} \exp \left(AX \left(\hat{\eta}^{[0,T]} \right) \right) \right].$$

696 Applying the Cauchy-Schwarz inequality yields that the quantity above is less than

$$\frac{1}{2A} \left(\log \mathbb{E}_{\mu_{\alpha}^{\lambda,0}} \left[e^{16\beta R} \right] + \log \mathbb{E}_{\mu_{\alpha}^{\lambda,0}} \left[\exp \left(2AX \left(\hat{\eta}^{[0,T]} \right) \right) \right] \right).$$

697 Since the angle updates happen in each site at rate 1 except when the site is empty, we can define on the
 698 same probability space as our process a family P_x of i.i.d. Poisson variable with mean T , and such that
 699 $R \leq \sum_{x \in \mathbb{T}_N^2} P_x$. Thanks to the elementary inequality

$$\log \mathbb{E} \left[e^{16\beta \sum_{x \in \mathbb{T}_N^2} P_x} \right] = T(e^{16\beta} - 1)N^2,$$

700 we now only have to let

$$K_0(T, \beta, \hat{\zeta}) = 2K(\hat{\zeta}) + T(e^{16\beta} - 1)$$

701 and replace A by $2A$ to conclude the proof of Proposition 3.10. \square

702 **3.3. Irreducibility and control on full clusters.** — *Unlike the exclusion process with one type of*
 703 *particles, the multi-type exclusion process is not irreducible when the number of particles is too large,*
 704 *namely when the domain has less than one empty site. When all the sites are occupied for example, the*
 705 *process is stuck in its current configuration, up to realignment, due to the exclusion rule. At high density,*
 706 *we therefore lose the mixing properties we need to reach local equilibrium. To illustrate this statement,*
 707 *consider a square macroscopic domain of size εN , and on it a configuration with the bottom half filled*
 708 *with particles with angle θ , and the top half filled with particles with angle $\theta' \neq \theta$, and letting a finite*
 709 *number of sites be empty, the mixing time of this setup is of order larger than N^2 due to the rigidity of*
 710 *the configuration. In order to reach equilibrium, an empty site needs to "fetch" a particle and transport it*
 711 *in the other cluster, and so on, until the density is homogeneous for both types of particles. The scaling*
 712 *of our alignment dynamics, is, furthermore, not sufficient to ensure sufficiently frequent realignment of*
 713 *the particles to solve this issue.*

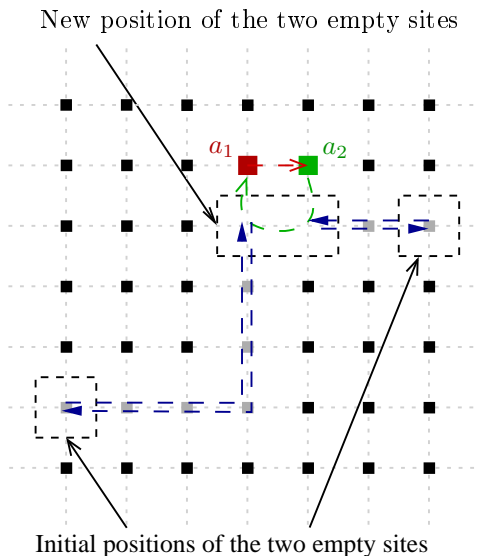
714 *In order to prove the scaling limit of a multi-type exclusion process, it is therefore critical to bound*
 715 *the particle density away from 1. This issue was solved in [35] by using the fact that the total density of*
 716 *the multi-type SSEP (the angle blind model) follows the standard SSEP dynamics (with one specie). Thus*
 717 *the total density could be controlled by the classical argument on the hydrodynamic limit for SSEP. In*
 718 *our case, however, the total density does not follow the SSEP dynamics. In fact, it is not even a Markov*
 719 *chain due to the asymmetric parts which depend on the angles. A different argument is required to control*
 720 *the evolution of the total density, which is the purpose of the subsection.*

721 In the general setup where the number of types of particles in a domain B can reach $|B|$ (which will
 722 often be the case when particles take their angles in \mathbb{S}), it is known that the exclusion process with $|B| - 1$
 723 particles is no longer irreducible, as a consequence of a generalization of the n -puzzle (cf. Johnson &
 724 Story, 1879, see [26]). We therefore need to consider only the local configurations with two empty sites,
 725 on which the exclusion process is irreducible regardless of the number of types of particles, as stated
 726 in the following Lemma. For any integers $a, b \in \mathbb{Z}$, $[[a, b]] = \{a, \dots, b\}$ denotes the segment of integers
 727 between a and b .

728 **Lemma 3.11 (Irreducibility of the displacement process with two empty sites)**

729 *Consider a square domain $B = B_p(x)$, and two configurations $\hat{\eta}, \hat{\eta}'$ two configurations with the same*
 730 *types and number of particles in B , i.e. such that*

$$\sum_{x \in B} \eta_x \delta_{\theta_x} = \sum_{x \in B} \eta'_x \delta_{\theta'_x}.$$

FIGURE 1. Reaching $\hat{\eta}^{a_1, a_2}$ from η .

731 Further assume that the number of empty sites in η and η' is at least 2. Then, there exists a sequence of
 732 configurations $\hat{\eta}^0, \dots, \hat{\eta}^n$, such that $\hat{\eta}^0 = \hat{\eta}$, $\hat{\eta}^n = \hat{\eta}'$, and such that for any $k \in \llbracket 0, n-1 \rrbracket$, $\hat{\eta}^{k+1}$ is reached
 733 from $\hat{\eta}^k$ by one allowed particle jump, i.e.

$$\hat{\eta}^{k+1} = (\hat{\eta}^k)^{x_k, x_k+z_k}, \quad \text{and} \quad \eta_{x_k+z_k}^k = 1 - \eta_{x_k}^k = 0 \quad \text{and} \quad |z_k| = 1.$$

734 Furthermore, there exists a constant C such that $n \leq Cp^4$.

735 *Proof of Lemma 3.11.* — The proof of this statement is quite elementary. Fix a configuration $\hat{\eta} \in \Sigma_N$ on
 736 a rectangular domain B with two empty sites, and let $a = (a_1, a_2)$ be an edge in \mathbb{T}_N^2 . We are first going
 737 to prove that $\hat{\eta}^{a_1, a_2}$ can be reached from $\hat{\eta}$ using allowed particles jumps. Notice that according to the
 738 exclusion rule, we can consider that any empty site is allowed to move freely by exchanging their place
 739 with any site next to it.

740 We first bring ourselves back to a configuration described in Fig. 1, where the two closest empty sites
 741 are brought next to the edge a . More precisely, we reach a configuration where the two empty sites and
 742 the two sites a_1 and a_2 are at the vertices of a side-1 square. From here, we are able to invert the two
 743 particles in a_1 and a_2 by a circular motion of the four empty sites along the edges of the square, and then
 744 bring back the empty sites along the paths that brought them next to a to their original location. Doing
 745 so, one reaches exactly the configuration $\hat{\eta}^{a_1, a_2}$ from $\hat{\eta}$ with allowed particle jumps in B .

746 We deduce from this last statement that for any pair of configurations $\hat{\eta}, \hat{\eta}'$ with the same particles
 747 in B , $\hat{\eta}'$ can be reached from $\hat{\eta}$ with jumps in B since the transition can be decomposed along switches
 748 of nearest neighbor sites. The process is thus irreducible on the sets with fixed numbers \hat{K} of particles,
 749 as soon as K is smaller than $|B| - 2$. Furthermore, this construction ensures that any two neighboring
 750 particles can be switched with a number of particle exchanges of order p where we denoted by p the size
 751 of the box. Since one needs to invert $2p$ pairs of particles at most to move one particle to its final position
 752 in $\hat{\eta}'$, this proves the last statement. \square

753 We now prove that large microscopic boxes are rarely fully occupied under the dynamics. Let us
 754 denote by $E_{p,x}$ the event

$$(3.19) \quad E_{p,x} = \left\{ \sum_{y \in B_p(x)} \eta_y \leq |B_p(x)| - 2 \right\},$$

755 on which the box of size p around x contains at least two empty sites. When the site x is the origin,
 756 we will simply write E_p instead of $E_{p,0}$. In order to ensure that full clusters very rarely appear in the
 757 dynamics, we need the following Lemma.

758 **Proposition 3.12 (Control on full clusters).** — *For any positive time T ,*

$$(3.20) \quad \lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N^{\lambda, \beta}} \left(\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c}(t) dt \right) = 0.$$

759 **Remark 3.13 (Scheme of the proof).** — We first sketch the proof in a continuous idealized setup to
 760 explain the general ideas before giving the rigorous proof. To prove that the box of microscopic size p is
 761 not full, setting $p' = (2p + 1)^2$ the cardinal of B_p , it is enough to prove thanks to the microscopic setting
 762 that

$$\iint_{[0, T] \times \mathbb{T}^2} \rho_t^{p'}(u) du dt \xrightarrow{p' \rightarrow \infty} 0,$$

763 where $\rho_t(u)$ denotes the macroscopic density in u at time t .

764 We expect the total density ρ to follow the partial differential equation

$$(3.21) \quad \partial_t \rho = \Delta \rho - \nabla \cdot (m(1 - \rho)),$$

765 where m is an *a priori* random quantity representing the local direction of the asymmetry, which can be
 766 represented as the vector field which would satisfy at any time t and for any smooth function $H : \mathbb{T}^2 \rightarrow \mathbb{R}$

$$\int_{\mathbb{T}^2} H(u) m_t(u) du = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x) \eta_x(t) \begin{pmatrix} \cos(\theta_x(t)) \\ \sin(\theta_x(t)) \end{pmatrix}.$$

Naturally, making sense of this quantity is not obvious, and it is not our purpose in this paragraph. For
 now, we carry on with our heuristic presentation, and therefore assume that (3.21) holds true. We can
 therefore formally write, letting $\phi(\rho) = 1/(1 - \rho)$

$$(3.22) \quad \begin{aligned} \partial_t \int_{\mathbb{T}^2} \phi(\rho_t) du &= \int_{\mathbb{T}^2} \phi'(\rho_t) [\Delta \rho_t - \nabla \cdot (m_t(1 - \rho_t))] du \\ &= \int_{\mathbb{T}^2} \phi''(\rho_t) [-(\nabla \rho_t)^2 + m_t(1 - \rho_t) \nabla \rho_t] du \\ &\leq \int_{\mathbb{T}^2} \phi''(\rho_t) \left[-(\nabla \rho_t)^2 + \frac{(\nabla \rho_t)^2}{2} + \|m_t\|_\infty^2 (1 - \rho_t)^2 \right] du \\ &\leq \int_{\mathbb{T}^2} \phi''(\rho_t) \|m_t\|_\infty^2 (1 - \rho_t)^2 du = 2 \|m_t\|_\infty^2 \int_{\mathbb{T}^2} \phi(\rho_t) du \end{aligned}$$

767 One could then apply Gronwall's Lemma to obtain that for any time t ,

$$\int_{\mathbb{T}^2} \phi(\rho_t) du \leq e^{2\|m\|_\infty^2 t} \int_{\mathbb{T}^2} \phi(\rho_0) du.$$

768 Furthermore, for any time t ,

$$\int_{\mathbb{T}^2} \phi(\rho_t) du \geq \frac{1}{\delta} \int_{\mathbb{T}^2} \mathbb{1}_{\{\rho_t \geq 1 - \delta\}} + \int_{\mathbb{T}^2} \mathbb{1}_{\{\rho_t \leq 1 - \delta\}} = \frac{1 - \delta}{\delta} \int_{\mathbb{T}^2} \mathbb{1}_{\{\rho_t \geq 1 - \delta\}} + 1,$$

769 therefore, for any time t ,

$$(3.23) \quad \int_{\mathbb{T}^2} \mathbb{1}_{\{\rho_t \geq 1 - \delta\}} \leq \frac{\delta}{1 - \delta} \left[e^{2\|m\|_\infty^2 t} \int_{\mathbb{T}^2} \phi(\rho_0) du - 1 \right] \xrightarrow{\delta \rightarrow 0} 0.$$

770 As a consequence, for any time t , we could therefore write

$$(3.24) \quad \iint_{[0, T] \times \mathbb{T}^2} \rho_t^{p'}(u) du dt \leq T(1 - \delta)^{p'} + \iint_{[0, T] \times \mathbb{T}^2} \mathbb{1}_{\{\rho_t \geq 1 - \delta\}}.$$

771 The first term in the right-hand side vanishes for any fixed δ as $p' \rightarrow \infty$, whereas the second becomes as
 772 small as needed letting $\delta \rightarrow 0$.

773 Since our macroscopic density does not verify equation (3.21), however, the operations above need to
 774 be performed in a microscopic setup. The derivation of equation (3.23) is the purpose of Proposition
 775 3.14. Two intermediate Lemmas 3.15 and 3.16 prove the microscopic equivalent of equation (3.22).

776 Before giving the proof of Proposition 3.12, which is postponed to the end of the subsection, we give
 777 first the following estimate.

778 **Proposition 3.14 (High density estimate).** — Denote

$$\rho_{\varepsilon N} = \frac{1}{2\varepsilon N + 1} \sum_{|y| \leq \varepsilon N} \eta_y$$

779 the average density in a small mesoscopic box centered at 0. For any positive $0 < \delta' < 1/2$, and any time
 780 $t > 0$, we have the bound

$$(3.25) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\tau_x \rho_{\varepsilon N}(t) > 1 - \delta'/2\}} \right) \leq \delta' C,$$

781 where C is a finite constant depending continuously on t , and also depending on the asymmetry λ , and
 782 the initial profile $\widehat{\zeta}$.

783 *Proof of Proposition 3.14.* — For any small $\delta > 0$, let us denote by ϕ_δ the application

$$\phi_\delta : \begin{array}{ccc} [0, 1 + \delta/2] & \longrightarrow & \mathbb{R}_+ \\ \rho & \longmapsto & \frac{1}{1 + \delta - \rho} \end{array} .$$

784 Note that all successive derivatives of order less than k of ϕ_δ are positive (and increasing) functions, and
 785 all are bounded by C_k/δ^{k+1} for some family of universal constants $(C_k)_{k>0}$.

786 We now fix a C^1 function $H : \mathbb{T}^2 \rightarrow \mathbb{R}_+$, and assume that $\int_{\mathbb{T}^2} H(u) du = 1$. For any $u \in \mathbb{T}^2$, we denote
 787 by H_u the function

$$H_u : v \mapsto H(u - v).$$

788 In order to simplify the notations, for any configuration $\widehat{\eta} \in \Sigma_N$, and given its empirical measure π^N , we
 789 shorten

$$(3.26) \quad \rho_x^{N, H}(\widehat{\eta}) := \langle \pi^N, H_{x/N} \rangle = \frac{1}{N^2} \sum_{y \in \mathbb{T}_N^2} H\left(\frac{x-y}{N}\right) \eta_y.$$

790 In some cases, this quantity could be larger than 1, so that we need to take further precautions. For any
 791 fixed δ we will therefore assume that N is large enough for the condition

$$\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x/N) \leq 1 + \frac{\delta}{2},$$

792 to hold, which is possible because we assumed that H is smooth and $\int_{\mathbb{T}^2} H(u) du = 1$. Note that this
 793 restriction to N large enough is not an issue, because in all what follows, H will be fixed and N will go
 794 to ∞ .

795 For N large enough, the density $\rho_x^{N, H}(\widehat{\eta})$ is now in the domain of ϕ_δ , we now write

$$(3.27) \quad \partial_t \mathbb{E}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta(\rho_x^{N, H}(\widehat{\eta})) \right) = \mathbb{E}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} L_N \phi_\delta(\rho_x^{N, H}(\widehat{\eta})) \right),$$

796 where L_N is the generator of the complete process $L_N = N^2 \mathcal{L} + N \mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{G}}$. Our goal is to apply
 797 Gronwall's Lemma to the expectation in the left-hand side, therefore we now need to estimate the right-
 798 hand side.

799 Since $\rho_x^{N, H}$ does not depend on the angles of the particles, neither does $\phi_\delta(\rho_x^{N, H})$, and the contribution
 800 of the Glauber part \mathcal{L}^{G} of the generator L_N in the right-hand side above vanishes. The two other parts
 801 of the generator together yield the wanted bound, and are treated in separate lemmas for the sake of
 802 clarity. As mentioned earlier, these two lemmas are the microscopic equivalent of equation (3.22).

803 **Lemma 3.15.** — [Contribution of the symmetric part] There exists a sequence $(c_N(\delta, H))_{N \in \mathbb{N}}$ depending
 804 only on δ and H , vanishing as $N \rightarrow \infty$, and such that for any configuration $\hat{\eta} \in \Sigma_N$

$$(3.28) \quad \sum_{x \in \mathbb{T}_N^2} \mathcal{L} \phi_\delta(\rho_x^{N,H})(\hat{\eta}) \leq - \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \frac{\phi_\delta''(\rho_{x+e_i}^{N,H}) + \phi_\delta''(\rho_x^{N,H})}{2} (\rho_{x+e_i}^{N,H} - \rho_x^{N,H})^2(\hat{\eta}) + c_N(\delta, H).$$

Lemma 3.16. — [Contribution of the asymmetric part] There exists a sequence $(\tilde{c}_N(\delta, H))_{N \in \mathbb{N}}$ depending
 only on δ and H , vanishing as $N \rightarrow \infty$, and such that for any configuration $\hat{\eta} \in \Sigma_N$

$$(3.29) \quad \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathcal{L}^{\text{va}} \phi_\delta(\rho_x^{N,H})(\hat{\eta}) \leq \sum_{x \in \mathbb{T}_N^2} \left[\sum_{i=1}^2 \frac{\phi_\delta''(\rho_{x+e_i}^{N,H}) + \phi_\delta''(\rho_x^{N,H})}{2} (\rho_{x+e_i}^{N,H} - \rho_x^{N,H})^2 + \frac{4\lambda^2 \phi_\delta(\rho_x^{N,H})}{N^2} \right] (\hat{\eta}) + \tilde{c}_N(\delta, H).$$

805 *Proof of Lemma 3.15.* — By definition of the symmetric part of the generator \mathcal{L} ,

$$\sum_{x \in \mathbb{T}_N^2} \mathcal{L} \phi_\delta(\rho_x^{N,H})(\hat{\eta}) = \sum_{x,y \in \mathbb{T}_N^2} \sum_{i=1}^2 \mathbb{1}_{\{\eta_y \eta_{y+e_i} = 0\}} [\phi_\delta(\rho_x^{N,H}(\hat{\eta}^{y,y+e_i})) - \phi_\delta(\rho_x^{N,H}(\hat{\eta}))].$$

We now develop the gradient of ϕ_δ to the second order, to obtain that the right-hand side above is equal to

$$\sum_{x,y \in \mathbb{T}_N^2} \sum_{i=1}^2 \mathbb{1}_{\{\eta_y \eta_{y+e_i} = 0\}} \left[\phi_\delta'(\rho_x^{N,H}(\hat{\eta})) (\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta})) + \frac{\phi_\delta''(\rho_x^{N,H}(\hat{\eta}))}{2} (\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}))^2 + o\left((\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}))^2\right) \right].$$

806 Note that since the successive derivatives of order less than k of ϕ_δ are uniformly bounded on $[0, 1]$ by
 807 C_k/δ^k , the vanishing quantity $o\left((\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}))^2\right)$ can be bounded uniformly in $\hat{\eta}$, x , y and
 808 i (but not uniformly in δ). Since H is a smooth function,

$$\left| \rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}) \right| = \frac{1}{N^2} \left| H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right|$$

809 is of order N^{-3} , the contributions of the second line above are therefore at most of order N^{-2} and vanish
 810 in the limit $N \rightarrow \infty$. This yields

$$(3.30) \quad \sum_{x \in \mathbb{T}_N^2} \mathcal{L} \phi_\delta(\rho_x^{N,H}) = \sum_{x \in \mathbb{T}_N^2} \phi_\delta'(\rho_x^{N,H}(\hat{\eta})) \sum_{y \in \mathbb{T}_N^2} \sum_{i=1}^2 \mathbb{1}_{\{\eta_y \eta_{y+e_i} = 0\}} (\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta})) + o_N(1),$$

811 where $o_N(1)$ is less than a vanishing sequence $(c_N^1)_{N \in \mathbb{N}}$ depending on δ and H only.

Since for any $z \in \mathbb{T}^2$, $H_u(v+z) = H_{u-z}(v)$, the definition of $\rho_x^{N,H}$ yields

$$\begin{aligned} \mathbb{1}_{\{\eta_y \eta_{y+e_i} = 0\}} (\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta})) &= \frac{1}{N^2} (\eta_y - \eta_{y+e_i}) \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \\ &= \frac{1}{N^2} \eta_y \left(H_{x-e_i/N} \left(\frac{y}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \\ &\quad - \frac{1}{N^2} \eta_{y+e_i} \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \right). \end{aligned}$$

Summing the quantity above over y , one obtains exactly $\rho_{x-e_i}^{N,H} + \rho_{x+e_i}^{N,H} - 2\rho_x^{N,H}$. This is the discrete Laplacian in the variable x of $\rho_x^{N,H}$, and a discrete integration by parts allows us to rewrite the first term on the right-hand side of equation (3.30) as

$$- \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left(\phi_\delta'(\rho_{x+e_i}^{N,H}) - \phi_\delta'(\rho_x^{N,H}) \right) (\rho_{x+e_i}^{N,H} - \rho_x^{N,H}).$$

812 We now write

$$\left(\phi'_\delta \left(\rho_{x+e_i}^{N,H} \right) - \phi'_\delta \left(\rho_x^{N,H} \right) \right) = \frac{\left(\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right) + \phi''_\delta \left(\rho_x^{N,H} \right) \right)}{2} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right) + o \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right),$$

813 in which $\rho_{x+e_i}^{N,H} - \rho_x^{N,H}$ is of order $1/N$ because H is a smooth function, to finally obtain that

$$(3.31) \quad \sum_{x \in \mathbb{T}_N^2} \mathcal{L} \phi_\delta \left(\rho_x^{N,H} \right) = - \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \frac{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right) + \phi''_\delta \left(\rho_x^{N,H} \right)}{2} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)^2 + o_N(1),$$

814 where once again, the o_N can be bounded by a vanishing sequence $(c_N)_N$ depending only on δ , which
815 completes the proof of Lemma 3.15 \square

816 *Proof of Lemma 3.16.* — This proof follows the exact same steps as for the previous one. We first obtain
817 by definition of \mathcal{L}^{wa} and developing the discrete gradient of ϕ that

$$(3.32) \quad \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathcal{L}^{\text{wa}} \phi_\delta \left(\rho_x^{N,H} \right) = o_N(1) + \frac{1}{N} \sum_{x,y \in \mathbb{T}_N^2} \sum_{i=1}^2 (\tau_y j_i^{\lambda_i}) \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right) \left(\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}) \right),$$

818 where $j_i^{\lambda_i}$ is defined according to equation (2.15) as

$$j_i^{\lambda_i}(\hat{\eta}) = \lambda_i(\theta_0)\eta_0(1 - \eta_{e_i}) - \lambda_i(\theta_{e_i})\eta_{e_i}(1 - \eta_0),$$

and $o_N(1)$ is less than a vanishing sequence depending only on δ and H . Once again, similar steps as in the previous case allow us to rewrite

$$\begin{aligned} & (\tau_y j_i^{\lambda_i}) \left(\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}) \right) \\ &= \frac{1}{N^2} \left[\lambda_i(\theta_y)\eta_y(1 - \eta_{y+e_i}) + \lambda_i(\theta_{y+e_i})\eta_{y+e_i}(1 - \eta_y) \right] \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \\ &= \frac{1}{N^2} \lambda_i(\theta_y)\eta_y(1 - \eta_{y+e_i}) \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \\ &\quad + \frac{1}{N^2} \lambda_i(\theta_{y+e_i})\eta_{y+e_i}(1 - \eta_y) \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \\ &= \frac{1}{N^2} \lambda_i(\theta_y)\eta_y(1 - \eta_{y+e_i}) \left(H_{x/N} \left(\frac{y+e_i}{N} \right) - H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \right) \\ &\quad + \frac{1}{N^2} \lambda_i(\theta_{y+e_i})\eta_{y+e_i}(1 - \eta_y) \left(H_{x-e_i/N} \left(\frac{y}{N} \right) - H_{x/N} \left(\frac{y}{N} \right) \right) \end{aligned}$$

Summing once again by parts in x , we obtain that the second term in the right-hand side of equation (3.32) is

$$\begin{aligned} & \frac{1}{N} \sum_{x,y \in \mathbb{T}_N^2} \sum_{i=1}^2 (\tau_y j_i^{\lambda_i}) \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right) \left(\rho_x^{N,H}(\hat{\eta}^{y,y+e_i}) - \rho_x^{N,H}(\hat{\eta}) \right) \\ &= \frac{1}{N^3} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left[\phi'_\delta \left(\rho_{x+e_i}^{N,H}(\hat{\eta}) \right) - \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right) \right] \times \\ &\quad \sum_{y \in \mathbb{T}_N^2} \left[\lambda_i(\theta_y)\eta_y(1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) + \lambda_i(\theta_{y+e_i})\eta_{y+e_i}(1 - \eta_y) H_{x/N} \left(\frac{y}{N} \right) \right] \\ (3.33) \quad & := S_1 + S_2, \end{aligned}$$

819 where

$$S_1 = \frac{1}{N^3} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left[\phi'_\delta \left(\rho_{x+e_i}^{N,H}(\hat{\eta}) \right) - \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right) \right] \sum_{y \in \mathbb{T}_N^2} \left[\lambda_i(\theta_y)\eta_y(1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \right]$$

820 and

$$S_2 = \frac{1}{N^3} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left[\phi'_\delta \left(\rho_{x+e_i}^{N,H}(\hat{\eta}) \right) - \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right) \right] \sum_{y \in \mathbb{T}_N^2} \left[\lambda_i(\theta_{y+e_i}) \eta_{y+e_i} (1 - \eta_y) H_{x/N} \left(\frac{y}{N} \right) \right].$$

821 These two terms are treated in the exact same fashion, we therefore only treat in full detail the case of
 822 S_1 , S_2 will follow straightforwardly. First, we develop the difference $\phi'_\delta \left(\rho_{x+e_i}^{N,H}(\hat{\eta}) \right) - \phi'_\delta \left(\rho_x^{N,H}(\hat{\eta}) \right)$ to the
 823 first order,

$$\phi'_\delta \left(\rho_{x+e_i}^{N,H} \right) - \phi'_\delta \left(\rho_x^{N,H} \right) = \phi''_\delta \left(\rho_{x+e_i}^{N,H} \right) \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right) + o \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right).$$

824 Once again, H being a smooth function, $\rho_{x+e_i}^{N,H} - \rho_x^{N,H}$ is of order $1/N$, therefore the $o \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)$ is
 825 also a $o_N(1/N)$, and the corresponding contribution in S_1 vanishes in the limit $N \rightarrow \infty$. Recall that ϕ''_δ
 826 is a positive function, we now apply in S_1 the elementary inequality $ab \leq a^2/2 + b^2/2$ to

$$a = \sqrt{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)$$

827 and

$$b = \frac{1}{N^3} \sqrt{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)} \sum_{y \in \mathbb{T}_N^2} \left[\lambda_i(\theta_{y+e_i}) \eta_{y+e_i} (1 - \eta_y) H_{x/N} \left(\frac{y}{N} \right) \right].$$

This yields

$$|S_1| \leq o_N(1) + \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \left[\frac{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)}{2} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)^2 + \frac{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)}{2N^6} \left(\sum_{y \in \mathbb{T}_N^2} \lambda_i(\theta_y) \eta_y (1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \right)^2 \right].$$

828 The function H being non-negative, for any y , we can write

$$\lambda_i(\theta_y) \eta_y (1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \leq \lambda (1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right).$$

829 Furthermore, since we assumed that $\int_{\mathbb{T}^2} H = 1$, and since H is smooth, we get that

$$\frac{1}{N^2} \sum_{y \in \mathbb{T}_N^2} H_{x/N}(y/N) = 1 + o_N(1),$$

830 which yields

$$\left| \frac{1}{N^2} \sum_{y \in \mathbb{T}_N^2} \lambda_i(\theta_y) \eta_y (1 - \eta_{y+e_i}) H_{x+e_i/N} \left(\frac{y+e_i}{N} \right) \right| \leq \lambda (1 - \rho_{x+e_i}^{N,H}) + o_N(1)$$

831 This, combined with the previous bound, yields that

$$|S_1| \leq o_N(1) + \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \left[\frac{\phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)}{2} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)^2 + \frac{\lambda^2 \phi''_\delta \left(\rho_{x+e_i}^{N,H} \right)}{2N^2} (1 - \rho_{x+e_i}^{N,H})^2 \right].$$

832 A similar bound can be achieved for S_2 , this time developing the difference $\phi'_\delta \left(\rho_{x+e_i}^{N,H} \right) - \phi'_\delta \left(\rho_x^{N,H} \right)$ in
 833 $\rho_x^{N,H}$ instead of $\rho_{x+e_i}^{N,H}$,

$$|S_2| \leq o_N(1) + \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \left[\frac{\phi''_\delta \left(\rho_x^{N,H} \right)}{2} \left(\rho_{x+e_i}^{N,H} - \rho_x^{N,H} \right)^2 + \frac{\lambda^2 \phi''_\delta \left(\rho_x^{N,H} \right)}{2N^2} (1 - \rho_x^{N,H})^2 \right].$$

Combining these two bounds with identities (3.32) and (3.33), we obtain that

$$\begin{aligned} & \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathcal{L}^{\text{VA}} \phi_\delta (\rho_x^{N,H}) \\ & \leq \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \left[\frac{\phi_\delta'' (\rho_{x+e_i}^{N,H}) + \phi_\delta'' (\rho_x^{N,H})}{2} (\rho_{x+e_i}^{N,H} - \rho_x^{N,H})^2 + \frac{\lambda^2 \phi_\delta'' (\rho_x^{N,H})}{N^2} (1 - \rho_x^{N,H})^2 \right] + o_N(1), \end{aligned}$$

834 where the $o_N(1)$ can be bounded by a vanishing sequence $(\tilde{c}_N)_N$ depending only on H and δ . One easily
835 obtains that for any non-negative δ and any $\rho \in [0, 1 + \delta/2]$,

$$(1 - \rho)^2 \phi_\delta''(\rho) \leq 2\phi_\delta(\rho),$$

836 thus concluding the proof of Lemma 3.16. \square

837 We are now ready to apply Gronwall's Lemma and complete the proof of Proposition 3.14. For that
838 purpose, let us define

$$\Phi(t) = \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta (\rho_x^{N,H}(t)) \right).$$

839 according to the previous Lemmas 3.15, 3.16 and to equation (3.27), there exists a sequence $k_N = c_N + \tilde{c}_N$
840 depending only on δ and H , verifying

$$k_N \xrightarrow{N \rightarrow \infty} 0,$$

841 and such that

$$\partial_t \Phi(t) \leq 4\lambda^2 \Phi(t) + k_N.$$

842 Since ϕ_δ is bounded from below by $1/1 + \delta$, $\Phi(t)$ also is, and therefore

$$\partial_t \Phi(t) \leq (4\lambda^2 + k_N(1 + \delta))\Phi(t).$$

843 Gronwall's Lemma therefore yields that for any non-negative t ,

$$\mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta (\rho_x^{N,H}(t)) \right) \leq \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta (\rho_x^{N,H}(0)) \right) e^{(4\lambda^2 + k_N(1 + \delta))t},$$

844 where this time the right-hand side depends on the trajectory only through its initial state $\hat{\eta}(0)$.

Fix a small $\delta' > 0$. ϕ_δ being a non-decreasing function bounded from below by $1/1 + \delta$, one can write for any $\rho \in [0, 1 + \delta/2]$

$$\phi_\delta(\rho) \geq \frac{1}{\delta + \delta'} \mathbb{1}_{\{\rho > 1 - \delta'\}} + \mathbb{1}_{\{\rho \leq 1 - \delta'\}} \frac{1}{1 + \delta} = \frac{1 - \delta'}{(1 + \delta)(\delta + \delta')} \mathbb{1}_{\{\rho > 1 - \delta'\}} + \frac{1}{1 + \delta}$$

We apply this decomposition to the left-hand side of the inequality above, to obtain that

$$\begin{aligned} (3.34) \quad & \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\rho_x^{N,H}(t) > 1 - \delta'\}} \right) \\ & \leq \frac{(1 + \delta)(\delta + \delta')}{1 - \delta'} \left[\mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta (\rho_x^{N,H}(0)) \right) e^{(4\lambda^2 + k_N(1 + \delta))t} - \frac{1}{1 + \delta} \right]. \end{aligned}$$

Coming back to the definition (3.26) of $\rho_x^{N,H}$, for any smooth non-negative function H with integral equal to 1, taking the $\limsup N \rightarrow \infty$, we thus obtain from equation (3.34)

$$\begin{aligned} (3.35) \quad & \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\rho_x^{N,H}(t) > 1 - \delta'\}} \right) \\ & \leq \limsup_{N \rightarrow \infty} \frac{(1 + \delta)(\delta + \delta')}{1 - \delta'} \left[\mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta (\rho_x^{N,H}(0)) \right) e^{4\lambda^2 t} - \frac{1}{1 + \delta} \right]. \end{aligned}$$

845 Fix a small $\varepsilon > 0$, and let us denote for any $u, v \in \mathbb{T}^2$

$$H^\varepsilon(v) = \frac{1}{(2\varepsilon)^2} \mathbb{1}_{[-\varepsilon, +\varepsilon]^2}(v) \quad \text{and} \quad H_u^\varepsilon(v) = \frac{1}{(2\varepsilon)^2} \mathbb{1}_{[-\varepsilon, +\varepsilon]^2}(v - u).$$

846 Recalling that $\rho_{\varepsilon N}(t)$ is the empirical density in a box of size εN around the origin at time t , we can
847 then write

$$\tau_x \rho_{\varepsilon N}(t) = \frac{(2\varepsilon N)^2}{(2\varepsilon N + 1)^2} \rho_x^{N, H^\varepsilon} = \rho_x^{N, H^\varepsilon} + o_N(1).$$

848 At this point, we want to apply equation (3.35) to $H = H^\varepsilon$, which is an indicator function, and thus
849 need to be smoothed out. For that purpose, consider a sequence $(H_l^\varepsilon)_{l \in \mathbb{N}}$ of functions such that

850 — $\forall l \in \mathbb{N}, \forall u \in \mathbb{T}^2, H_l^\varepsilon(u) \geq 0$ and $\sup_{\mathbb{T}^2} H_l^\varepsilon = \sup_{\mathbb{T}^2} H^\varepsilon = 1/(2\varepsilon)^2$.

851 — $\forall l \in \mathbb{N}, H_l^\varepsilon \in C^1(\mathbb{T}^2)$ and $\int_{\mathbb{T}^2} H_l^\varepsilon(u) du = 1$.

852 — $H_l^\varepsilon(u) \neq H^\varepsilon(u) \Rightarrow \varepsilon - 1/l < \|u\|_\infty < \varepsilon + 1/l$.

853 The existence of such a sequence of functions is quite clear and is left to the reader. In particular, the
854 last condition imposes that

$$I_l := \int_{\mathbb{T}^2} \mathbb{1}_{H_l^\varepsilon(u) \neq H^\varepsilon(u)} du \leq \frac{16\varepsilon}{l},$$

which is the area of the crown on which the two functions may differ. The sequence H_l^ε converges for any fixed ε towards H^ε in $L^1(\mathbb{T}^2)$. Furthermore, notice that for any $x \in \mathbb{T}_N^2$, since both the H_l^ε 's and H^ε are bounded by $1/(2\varepsilon)^2$,

$$\begin{aligned} \left| \rho_x^{N, H_l^\varepsilon} - \rho_x^{N, H^\varepsilon} \right| &\leq \frac{1}{N^2} \sum_{y \in \mathbb{T}_N^2} \eta_y \left| H_{l, x/N}^\varepsilon \left(\frac{y}{N} \right) - H_{x/N}^\varepsilon \left(\frac{y}{N} \right) \right| \\ &\leq \left(\frac{16\varepsilon}{l} + o_N(1) \right) (\|H_l^\varepsilon\|_\infty + \|H^\varepsilon\|_\infty) = \frac{8}{\varepsilon l} + o_N(1), \end{aligned}$$

855 where the last line represents the proportion of sites of the discrete torus in the crown around $u = x/N$
856 on which $H_{l, x/N}^\varepsilon$ and $H_{x/N}^\varepsilon$ can be different. The last observation yields that for any $x \in \mathbb{T}_N^2$, we can
857 write

$$\left| \tau_x \rho_{\varepsilon N}(t) - \rho_x^{N, H_l^\varepsilon}(t) \right| \leq \frac{8}{\varepsilon l} + o_N(1),$$

858 where the $o_N(1)$ can be chosen independent of $\hat{\eta}$ and x . Fix $\varepsilon > 0$ and consider N_0 and l_0 such that for
859 any $N \geq N_0$ and any $l \geq l_0$,

$$\left| \tau_x \rho_{\varepsilon N}(t) - \rho_x^{N, H_l^\varepsilon}(t) \right| \leq \frac{\delta'}{2}.$$

860 For any such pair l, N , we therefore also have

$$\mathbb{1}_{\{\tau_x \rho_{\varepsilon N}(t) > 1 - \delta'/2\}} \leq \mathbb{1}_{\{\rho_x^{N, H_l^\varepsilon}(t) > 1 - \delta'\}}.$$

For any l , by our assumptions, equation (3.35) holds for $H = H_l^\varepsilon$ for any positive δ and δ' . For any $l \geq l_0$, we can therefore write

$$(3.36) \quad \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\tau_x \rho_{\varepsilon N}(t) > 1 - \delta'/2\}} \right) \leq \limsup_{N \rightarrow \infty} \frac{(1 + \delta)(\delta + \delta')}{1 - \delta'} \left[\mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta \left(\rho_x^{N, H_l^\varepsilon}(0) \right) \right) e^{4\lambda^2 t} - \frac{1}{1 + \delta} \right].$$

861 Recall that under $\mathbb{P}_{\mu^N}^{\lambda, \beta}$, the initial configuration $\hat{\eta}(0)$ is distributed according to a product measure fitting
862 the initial profile ζ defined before (2.7). By law of large number, and since ϕ_δ is smooth on $[0, 1 + \delta/2]$,
863 we therefore obtain for any $v \in \mathbb{T}^2$

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\phi_\delta \left(\rho_{[Nv]}^{N, H_l^\varepsilon}(0) \right) \right) = \phi_\delta(\zeta * H_l^\varepsilon(v)),$$

864 where $[Nv] = ([Nv_1], [Nv_2]) \in \mathbb{T}_N^2$ and " $*$ " denotes the convolution operator on \mathbb{T}^2 . By dominated
865 convergence theorem, we thus obtain

$$\mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \phi_\delta \left(\rho_x^{N, H_l^\varepsilon}(0) \right) \right) \xrightarrow{N \rightarrow \infty} \int_{\mathbb{T}^2} \phi_\delta (\zeta * H_l^\varepsilon(v)) dv.$$

866 Since ζ and satisfies (2.7), it is bounded away from 1 uniformly on \mathbb{T}^2 , $\zeta * H_l^\varepsilon$ is also bounded away
867 from 1 uniformly in ε , and therefore

$$\phi_\delta (\zeta * H_l^\varepsilon(v)) \leq C^*,$$

868 where $C^* = C^*(\hat{\zeta})$ is a constant which does not depend on l , ε , v or δ . Letting now δ go to 0, we obtain
869 from (3.36) and the limit above that for any $\varepsilon > 0$ and any time t ,

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\tau_x \rho_{\varepsilon N}(t) > 1 - \delta'/2\}} \right) \leq \frac{\delta'}{1 - \delta'} (e^{4\lambda^2 t} C^* - 1),$$

870 which concludes the proof of Proposition 3.14 since we assumed $\delta' < 1/2$. \square

871 With the estimate stated in Proposition 3.14, we are ready to prove Proposition 3.12.

872 *Proof of Proposition 3.12.* — First notice that in order to prove (3.20), it is sufficient to prove it both
873 for $F_{p,x}$ and $F'_{p,x}$ instead of $E_{p,x}^c$, where

$$F_{p,x} = \left\{ \sum_{y \in B_p(x)} \eta_y = |B_p(x)| \right\} \quad \text{and} \quad F'_{p,x} = \left\{ \sum_{y \in B_p(x)} \eta_y = |B_p(x)| - 1 \right\}.$$

874 We focus on the first case, the second is derived in the exact same fashion.

875 Unlike in [35], the angle blind process's macroscopic density does not evolve according to the heat
876 equation because of the weak drift. However, thanks to the bound (3.15) on the entropy of the measure
877 μ_t^N w.r.t. the reference measure μ_α^* and on the Dirichlet form of the density f_t^N , local equilibrium holds for
878 the angle-blind process. As a consequence, the replacement Lemma 4.1 holds for functions independent
879 of the angles (cf. for example [27], p77). One therefore obtains that to prove

$$(3.37) \quad \lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{F_{p,x}}(s) ds \right) = 0,$$

880 one can replace $\mathbb{1}_{F_{p,x}(s)}$ by its expectation under the product measure with parameter $\tau_x \rho_{\varepsilon N}(s)$, namely

$$\mathbb{E}_{\tau_x \rho_{\varepsilon N}(s)}(\mathbb{1}_{F_{p,x}}) = [\tau_x \rho_{\varepsilon N}(s)]^{p'},$$

881 where $p' = (2p+1)^2$ is the number of sites in B_p .

882 To prove equation (3.37), it is therefore sufficient to prove that $\forall t \in [0, T]$,

$$(3.38) \quad \lim_{p' \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} [\tau_x \rho_{\varepsilon N}(t)]^{p'} \right) = 0.$$

883 To prove the latter, since $\rho_{\varepsilon N}(t)$ is at most 1, one only has to write, as outlined in equation (3.24),

$$\mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} [\tau_x \rho_{\varepsilon N}(t)]^{p'} \right) \leq (1 - \delta)^{p'} + \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{\{\tau_x \rho_{\varepsilon N}(t) > 1 - \delta\}} \right),$$

884 which holds for any positive δ .

885 For any fixed $\delta > 0$, the first term on the right-hand side vanishes as $p \rightarrow \infty$, whereas the second does
886 not depend on p and we can therefore let $\delta \rightarrow 0$ after $N \rightarrow \infty$, then $\varepsilon \rightarrow 0$, then $p' \rightarrow \infty$. Since the
887 right-hand side of equation (3.25) vanishes as $\delta' = 2\delta$ goes to 0, the left-hand side also does, and (3.38)
888 holds for any t thanks to Proposition 3.14. This proves equation (3.37), and the equivalent proposition
889 with $F'_{p,x}$ instead of $F_{p,x}$ is proved in the exact same fashion, thus concluding the proof of Proposition
890 3.12. \square

4. Law of large number for the exclusion process with angles

891

892 **4.1. Replacement Lemma.** — *Our goal in this section is to close the microscopic equations and*
 893 *to replace in the definition of the martingale $M^{H,N}$ introduced in (2.12) any cylinder (in the sense of*
 894 *Definition 2.1) function $g(\hat{\eta})$ by its spatial average $\mathbb{E}_{\hat{\rho}_{\varepsilon N}}(g)$, where $\hat{\rho}_{\varepsilon N}$ is the empirical angular density*
 895 *over a small macroscopic box of size εN . We use this Section to introduce new useful notations. The*
 896 *proof of the main result of this section, the Replacement Lemma 4.1, follows closely the usual strategy*
 897 *(c.f. Lemma 1.10 p.77 of [27]), however it requires several technical adaptations due to the nature of our*
 898 *canonical and grand-canonical measure. In particular, we will need the topological setup and the various*
 899 *results obtained in Section 3.*

900 Consider a cylinder function $g \in \mathcal{C}$, and l a positive integer. Recall from (2.19) that $\langle g \rangle_0^l$ is the average
 901 of the translations of g over a box of side $2l + 1$ centered at the origin. Recall from equation (2.20) and
 902 Definition 3.1 that the empirical angular density $\hat{\rho}_l$ over the box B_l of side $2l + 1$ is the measure on $[0, 2\pi[$

$$\hat{\rho}_l = \frac{1}{|B_l|} \sum_{x \in B_l} \eta_x \delta_{\theta_x}.$$

903 Define

$$(4.1) \quad \mathcal{V}^l(\hat{\eta}) = \langle g(\hat{\eta}) \rangle_0^l - \mathbb{E}_{\hat{\rho}_l}(g) \quad \text{and} \quad \mathcal{W}^l(\hat{\eta}) = g(\hat{\eta}) - \mathbb{E}_{\hat{\rho}_l}(g),$$

 904 and for any smooth function $G \in C(\mathbb{T}^2)$, let

$$(4.2) \quad X^{l,N}(G, \hat{\eta}) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}^l.$$

905 We first state that under the measure of active exclusion process, one can replace the average of g
 906 over a small macroscopic box by its expectation w.r.t. the grand-canonical measure with grand-canonical
 907 parameter $\hat{\rho}_{\varepsilon N}$.

908 **Lemma 4.1 (Replacement Lemma).** — *For every $\delta > 0$, we have with the notation (4.1)*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left[\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x |\mathcal{V}^{\varepsilon N}(\hat{\eta}(t))| dt > \delta \right] = 0.$$

909 The proof is postponed to Subsection 4.2, and requires the control of the full clusters stated in Propo-
 910 sition 3.12. For now, we can deduce from this lemma the following result, which will allow us to replace
 911 in (2.18) the currents by their spatial averages.

912 **Corollary 4.2.** — *For every $\delta > 0$, and any continuous function*

$$G : [0, T] \times \mathbb{T}^2 \longrightarrow \mathbb{R} \\ (t, u) \longmapsto G_t(u) ,$$

913 *we get with the notation (4.2)*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left[\left| \int_0^T X^{\varepsilon N, N}(G_t, \hat{\eta}(t)) dt \right| > \delta \right] = 0.$$

914 *Proof of Corollary 4.2.* — Recall that $\varepsilon \rightarrow 0$ after $N \rightarrow \infty$, which means that the smoothness of G allows
 915 us to replace in the limit $G(x/N)$ by its spatial average on a box of size ε , which is denoted by

$$G^{\varepsilon N}(x/N) := \frac{1}{(2\varepsilon N + 1)^2} \sum_{y \in B_{\varepsilon N}(x)} G(y/N).$$

More precisely, we can write, using notation (2.19) for the local averaging, and since g is a cylinder, hence
 bounded, function,

$$\limsup_{N \rightarrow \infty} \int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} G_t(x/N) \tau_x g dt = \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} G_t^{\varepsilon N}(x/N) \tau_x g dt$$

$$(4.3) \quad = \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \int_0^T \frac{1}{N^2} \sum_{y \in \mathbb{T}_N^2} G_t(y/N) \langle g \rangle_y^{\varepsilon N} dt,$$

916 where the average $\langle g \rangle_y^{\varepsilon N}$ is defined in equation (2.19).

917 As a consequence, $\tau_y g$ can be replaced by its average $\langle g \rangle_y^{\varepsilon N}$. Note that

$$\mathcal{V}^{\varepsilon N}(\hat{\eta}) = \mathcal{W}^{\varepsilon N}(\hat{\eta}) + \langle g \rangle_y^{\varepsilon N} - g,$$

918 and that the replacement Lemma 4.1 implies in particular that for any bounded function $G \in C([0, T] \times \mathbb{T}^2)$

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_N^{\lambda, \beta}} \left[\left| \int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} G_t(x/N) \tau_x \mathcal{V}^{\varepsilon N}(\hat{\eta}(t)) dt \right| > \delta \right] = 0.$$

919 Therefore, thanks to equality (4.3), Corollary 4.2 follows directly from Lemma 4.1. \square

920 **4.2. Proof of the replacement Lemma.** — In order to prove the replacement Lemma 4.1, we will
 921 need the two lemmas below. The first one states that the average of any cylinder function $\langle g(\hat{\eta}) \rangle_0^l$ over
 922 a large microscopic box (a box of size l which tends to infinity after N) can be replaced by its expected
 923 value w.r.t. the grand-canonical measure whose parameter is the empirical density $\mathbb{E}_{\hat{\rho}_l}(g)$.

924 The second states that the empirical angular density does not vary much between a large microscopic
 925 box and a small macroscopic box. We state these two results, namely the one and two-blocks estimates,
 926 in a quite general setup, because they are necessary in several steps of the proof of the hydrodynamic
 927 limit.

928 **Lemma 4.3 (one-block estimate).** — Consider $\alpha \in]0, 1[$ and a density f w.r.t the translation invari-
 929 ant measure μ_α^* (cf. Definition 3.4) satisfying

930 i) There exists a constant K_0 such that for any N

$$H(f) \leq K_0 N^2 \quad \text{and} \quad D(f) \leq K_0.$$

ii)

$$(4.4) \quad \lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_\alpha^* \left(f \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c} \right) = 0.$$

931 Then, for any cylinder function g ,

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_\alpha^* \left(f \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{V}^l \right) = 0,$$

932 where \mathcal{V}^l was defined in (4.1).

933 **Lemma 4.4 (two-block estimate).** — For any $\alpha \in]0, 1[$ and any density f satisfying conditions i)
 934 and ii) of Lemma 4.3,

$$\limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{y \in B_{\varepsilon N}} \mathbb{E}_\alpha^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \left\| \tau_{x+y} \hat{\rho}_l - \tau_x \hat{\rho}_{\varepsilon N} \right\| f \right) = 0,$$

935 where $\tau_z \hat{\rho}_k$ is the local empirical angular density in the box of size k centered in z introduced in (2.20).

936 The proofs of these two lemmas will be presented resp. in Section 4.3 and 4.4. For now, let us show
 937 that they are sufficient to prove the replacement Lemma 4.1.

938 **Proof of Lemma 4.1.** — Lemma 4.1 follows from applying the two previous lemmas to the density

$$\bar{f}_T^N = \frac{1}{T} \int_0^T f_t^N dt,$$

939 where $f_t^N = d\mu_t^N/d\mu_\alpha^*$, defined in Section 3.2, is the density of the active exclusion process at time t
 940 started from μ^N , and prove that Lemma 4.1 follows. Proposition (3.9) proved that \bar{f}_T^N satisfies condition
 941 i) of Lemma 4.3. Furthermore, \bar{f}_T^N also satisfies condition ii)

$$\lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c} \right) = 0$$

942 thanks to Proposition 3.12, thus the one-block and two-blocks estimates apply to $f = \bar{f}_T^N$.

943 Now let us recall that we want to prove for any $\delta > 0$

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N}^{\lambda, \beta} \left[\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x | \mathcal{V}^{\varepsilon N}(\hat{\eta}(t)) | dt > \delta \right] = 0,$$

944 where

$$\mathcal{V}^{\varepsilon N}(\hat{\eta}) = \langle g(\hat{\eta}) \rangle_0^{\varepsilon N} - \mathbb{E}_{\hat{\rho}_{\varepsilon N}}(g).$$

945 Thanks to the Markov inequality, it is sufficient to prove that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left[\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x | \mathcal{V}^{\varepsilon N}(\hat{\eta}(t)) | dt \right] = 0.$$

946 We can now express the expectation above thanks to the mean density \bar{f}_T^N . Since T is fixed, to obtain
 947 the replacement Lemma it is enough to show that

$$(4.5) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x | \mathcal{V}^{\varepsilon N}(\hat{\eta}) | \right) = 0.$$

948 For any function $\varphi(\cdot)$ on the torus \mathbb{T}_N^2 , recall that we denoted in (2.19) by $\langle \varphi(\cdot) \rangle_x^l$ the average of the
 949 function φ over a box centered in x of size l , and that $\tau_y \hat{\rho}_l$ is the empirical angular density in a box of
 950 size l centered in y defined in (2.20). Let us add and subtract

$$\left\langle \langle g(\hat{\eta}) \rangle_0^l - \mathbb{E}_{\hat{\rho}_l}(g) \right\rangle_0^{\varepsilon N} = \frac{1}{(2\varepsilon N + 1)^2} \sum_{x \in B_{\varepsilon N}} \left[\frac{1}{(2l + 1)^2} \sum_{|y-x| \leq l} \tau_y g - \mathbb{E}_{\tau_x \hat{\rho}_l}(g) \right]$$

951 inside $| \mathcal{V}^{\varepsilon N}(\hat{\eta}) |$. We can then write thanks to the triangular inequality

$$| \mathcal{V}^{\varepsilon N}(\hat{\eta}) | \leq (\mathcal{Z}_1^{l, \varepsilon N} + \mathcal{Z}_2^{l, \varepsilon N} + \mathcal{Z}_3^{l, \varepsilon N})(\hat{\eta}),$$

952 where

$$\mathcal{Z}_1^{l, \varepsilon N} = \left| \frac{1}{(2\varepsilon N + 1)^2} \sum_{x \in B_{\varepsilon N}} \left(\tau_x g - \frac{1}{(2l + 1)^2} \sum_{|y-x| \leq l} \tau_y g \right) \right|,$$

953 is the difference between g and its local average,

$$\mathcal{Z}_2^{l, \varepsilon N} = \frac{1}{(2\varepsilon N + 1)^2} \sum_{x \in B_{\varepsilon N}} \left| \mathbb{E}_{\tau_x \hat{\rho}_l}(g) - \frac{1}{(2l + 1)^2} \sum_{|y-x| \leq l} \tau_y g \right|,$$

954 is the difference between the local average of g and its expectation under the product measure with
 955 parameter the local empirical angular density $\hat{\rho}_l$, and

$$\mathcal{Z}_3^{l, \varepsilon N} = \frac{1}{(2\varepsilon N + 1)^2} \sum_{x \in B_{\varepsilon N}} | \mathbb{E}_{\tau_x \hat{\rho}_l}(g) - \mathbb{E}_{\hat{\rho}_{\varepsilon N}}(g) |$$

956 is the difference between the expectations of g under the empirical microscopic and macroscopic empirical
 957 angular density $\hat{\rho}_l$ and $\hat{\rho}_{\varepsilon N}$.

958 Let us consider the first term, $N^{-2} \sum_x \tau_x \mathcal{Z}_1^{l, \varepsilon N}$. All the terms in $\mathcal{Z}_1^{l, \varepsilon N}$ corresponding to the x 's in
 959 $B_{\varepsilon N-l}$ vanish, since they appear exactly once in both parts of the sum. The number of remaining terms

960 can be crudely bounded by $4\varepsilon Nl$, and each term takes the form $\tau_x g / (2\varepsilon N + 1)^2$. Hence, we have the
 961 upper bound

$$\mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{Z}_1^{l, \varepsilon N} \right) \leq \frac{Kl}{\varepsilon N} \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x |g| \right).$$

962 Since g is a bounded function, this expression can be bounded from above by

$$\frac{Kl \|g\|_\infty}{\varepsilon N} \mathbb{E}_\alpha^* \left(\bar{f}_t^N \right) = C(l, \varepsilon, g) o_N(1),$$

963 which proves that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_\alpha^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{Z}_1^{l, \varepsilon N} \bar{f}_t^N \right) = 0.$$

964 Now since

$$\sum_{x \in \mathbb{T}_N^2} \frac{1}{(2\varepsilon N + 1)^2} \sum_{y \in B_{\varepsilon N}(x)} \tau_y g = \sum_{x \in \mathbb{T}_N^2} \tau_x g,$$

965 the two following terms can respectively be rewritten as

$$(4.6) \quad \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{Z}_2^{l, \varepsilon N} \right) = \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x | \mathbb{E}_{\hat{\rho}_l}(g) - \langle g \rangle_0^l | \right),$$

966 and

$$(4.7) \quad \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{Z}_3^{l, \varepsilon N} \right) = \mathbb{E}_\alpha^* \left(\bar{f}_T^N \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} | \mathbb{E}_{\tau_x \hat{\rho}_l}(g) - \mathbb{E}_{\hat{\rho}_{\varepsilon N}}(g) | \right).$$

967 The quantity (4.6) vanishes in the limit $N \rightarrow \infty$ then $l \rightarrow \infty$ thanks to the one-block estimate stated in
 968 Lemma 4.3.

969 Finally, according to Definition 3.2, (4.7) also vanishes thanks to the two-block estimate of Lemma 4.4
 970 and the Lipschitz-continuity of the application

$$\begin{aligned} \Psi_g & : (\mathcal{M}_1(\mathbb{S}), \|\cdot\|) \longrightarrow \mathbb{R} \\ & \quad \hat{\alpha} \longmapsto \mathbb{E}_{\hat{\alpha}}(g) \end{aligned}$$

971 which was proved in Proposition C.2. The Replacement Lemma 4.1 thus follows from the one and
 972 two-blocks estimates. \square

973 In the next two Sections 4.3 and 4.4, we prove the one-block and two-block estimates. The strategy
 974 for these proofs follows closely these presented in [27], albeit it requires some adjustments due to the
 975 measure-valued nature of the parameter of the product measure $\mu_{\hat{\alpha}}$ and the necessity to control the full
 976 clusters.

977 **4.3. Proof of Lemma 4.3 : The one-block estimate.** — *The usual strategy to prove the one block*
 978 *estimate is to project the estimated quantity on sets with fixed number of particles, on which the density*
 979 *of f should be constant thanks to the bound on the Dirichlet form.*

980 To prove the one-block estimate, thanks to the translation invariance of μ_α^* , it is sufficient to control
 981 the limit as N goes to ∞ , then $l \rightarrow \infty$ of

$$\mathbb{E}_\alpha^* \left(f \cdot \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \mathcal{V}^l \right) = \mathbb{E}_\alpha^* (\mathcal{V}^l \bar{f}),$$

982 where $\bar{f} = N^{-2} \sum_{x \in \mathbb{T}_N^2} \tau_x f$ is the average over the periodic domain of the translations of the density f .
 983 Furthermore, define s_g a fixed integer such that g is measurable w.r.t. $(\hat{\eta}_x)_{x \in B_{s_g}}$. We introduce for l
 984 larger than s_g

$$\tilde{\mathcal{V}}^l = \langle g(\hat{\eta}) \rangle_0^{l-s_g} - \mathbb{E}_{\hat{\rho}_l}(g) = \mathcal{V}^l + o_1(l),$$

985 where the $o_1(l)$ vanishes uniformly in $\hat{\eta}$ as $l \rightarrow \infty$. Proving the one block estimate for $\tilde{\mathcal{V}}^l$ instead of \mathcal{V}^l is
 986 therefore sufficient, and $\tilde{\mathcal{V}}^l$ depends on the configuration only through the sites in B_l .

987 We first eliminate the configurations in which the box B_l is almost full. Notice that the average $\tilde{\mathcal{V}}^l$ is
 988 bounded because g is a cylinder function. We can therefore write

$$\mathbb{E}_\alpha^*(\tilde{\mathcal{V}}^l \bar{f}) \leq \mathbb{E}_\alpha^*(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} \bar{f}) + C(g) \mathbb{E}_\alpha^*(\mathbb{1}_{E_l^c} \bar{f}),$$

989 where E_l is the event on which at least two sites are empty in B_l , defined after equation (3.19), and E_l^c
 990 is its complementary event. The second term in the right-hand side vanishes by definition of \bar{f} , because
 991 f verifies (4.4), and it is therefore sufficient to prove that

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_\alpha^*(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} \bar{f}) = 0.$$

992 Furthermore, the convexity of the Dirichlet form and the entropy yield that condition *i*) of the one-block
 993 estimate is also satisfied by \bar{f} . Since $\tilde{\mathcal{V}}^l \mathbb{1}_{E_l}$ depends on $\hat{\eta}$ only through the $\hat{\eta}_x$'s in the cube B_l we can
 994 replace the density \bar{f} in the formula above by its conditional expectation \bar{f}_l , defined, for any configuration
 995 $\hat{\eta}'$ on B_l by

$$\bar{f}_l(\hat{\eta}') = \mathbb{E}_\alpha^*(\bar{f} \mid \hat{\eta}_x = \hat{\eta}'_x, x \in B_l).$$

996 For any function f depending only on sites in B_l let $\mathbb{E}_{\alpha,l}^*$ be the expectation with respect to the product
 997 measure μ_α^* over B_l . With the previous notations, and in order to prove the one-block estimate, it is
 998 sufficient to prove that

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_{\alpha,l}^*(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} \bar{f}_l) \leq 0.$$

999 In order to proceed, we need to estimate the Dirichlet form and the entropy of \bar{f}_l thanks to that of f ,
 1000 and prove the following Lemma

1001 **Lemma 4.5.** — *We have the following bounds*

$$(4.8) \quad D_l(\bar{f}_l) \leq C(l)N^{-2} \quad \text{and} \quad H(\bar{f}_l) \leq C(l).$$

1002 *Proof of Lemma 4.5.* —

1003 **Estimate on the Dirichlet form of \bar{f}_l** - we denote by $\mathcal{L}_{x,y}$ the symmetric part of the exclusion
 1004 generator corresponding to the transfer of a particle between x and y

$$\mathcal{L}_{x,y} f(\hat{\eta}) = (\eta_x - \eta_y) (f(\hat{\eta}^{y,x}) - f(\hat{\eta})),$$

1005 and by $D^{x,y}$ the part of the Dirichlet form of the exclusion process corresponding to $\mathcal{L}_{x,y}$

$$D^{x,y}(f) = -\mathbb{E}_\alpha^*(\sqrt{f} \mathcal{L}_{x,y} \sqrt{f}).$$

1006 With this notation, we have

$$D(f) = \sum_{|x-y|=1} D^{x,y}(f),$$

1007 where D is the Dirichlet form introduced in equation (3.6). We denote in a similar fashion the Dirichlet
 1008 form restricted to the box of size l for any function h depending only on the sites in B_l by

$$D_l^{x,y}(h) = -\mathbb{E}_{\alpha,l}^*(\sqrt{h} \mathcal{L}_{x,y} \sqrt{h}).$$

1009 Since the conditioning $f \mapsto f_l$ is an expectation, and since the Dirichlet elements $D_l^{x,y}$ are convex, the
 1010 inequality

$$D_l^{x,y}(\bar{f}_l) \leq D^{x,y}(\bar{f})$$

1011 follows from Jensen's inequality. We deduce from the previous inequality, by summing over all edges
 1012 $(x,y) \in B_l$, thanks to the translation invariance of \bar{f} , that

$$D_l(\bar{f}_l) \leq \sum_{(x,y) \in B_l} D^{x,y}(\bar{f}) = 2l(2l+1) \sum_{j=1}^2 D^{0,e_j}(\bar{f}) = \frac{(2l+1)^2}{N^2} D(\bar{f}),$$

1013 where D_l is the Dirichlet form of the process restricted to the particle transfers with both the start and
 1014 end site in B_l . Up to this point, we have proved that for any function f such that $D(\bar{f}) \leq D(f) \leq K_0$,
 1015 we have as wanted

$$(4.9) \quad D_l(\bar{f}_l) \leq C_1(l)N^{-2}.$$

1016 **Estimate on the entropy of \bar{f}_l** - recall that we defined the entropy $H(f) = \mathbb{E}_\alpha^*(f \log f)$ and that
 1017 we already established $H(\bar{f}) \leq K_0 N^2$. Let us partition \mathbb{T}_N^2 in $q := \lfloor N/(2l+1) \rfloor^2$ square boxes $B^1 :=$
 1018 $B_l(x_1), \dots, B^q := B_l(x_q)$, and B^{q+1} , which contains all the site that weren't part of any of the boxes.
 1019 We can thus write

$$\mathbb{T}_N^2 = \bigsqcup_{i=1}^{q+1} B^i.$$

1020 We denote by $\hat{\eta}^i$ the configuration restricted to B^i and by $\hat{\xi}^i$ the complementary configuration to $\hat{\eta}^i$. In
 1021 other words, for any $i \in \llbracket 1, q+1 \rrbracket$, we split any configuration on the torus $\hat{\eta}$ into $\hat{\eta}^i$ and $\hat{\xi}^i$. We define for
 1022 any $i \in \llbracket 1, q \rrbracket$ the densities on the $\hat{\eta}^i$'s

$$\bar{f}_l^i(\hat{\eta}^i) = \mathbb{E}_\alpha^* \left(\bar{f}(\hat{\eta}^i, \hat{\xi}^i) \mid \hat{\eta}^i \right).$$

1023 Let us denote by φ the product density w.r.t. μ_α^* with the same marginals as \bar{f} , defined by

$$\varphi(\hat{\eta}) = \bar{f}_l^1(\hat{\eta}^1) \bar{f}_l^2(\hat{\eta}^2) \dots \bar{f}_l^{q+1}(\hat{\eta}^{q+1}),$$

1024 elementary entropy computations yield that

$$H(\bar{f}) = H_\varphi(\bar{f}/\varphi) + \sum_{i=1}^{q+1} H(\bar{f}_l^i),$$

1025 where $H_\varphi(f) = H(f\mu_\alpha^* \mid \varphi\mu_\alpha^*)$. Since by construction \bar{f} is translation invariant, for any $i = 1, \dots, q$, we
 1026 can write $H(\bar{f}_l^i) = H(\bar{f}_l^1) = H(\bar{f}_l)$, therefore in particular, the previous bound also yields, thanks to
 1027 the non-negativity of the entropy, that

$$H(\bar{f}) \geq qH(\bar{f}_l).$$

1028 Since q is of order N^2/l^2 , this rewrites

$$(4.10) \quad H(\bar{f}_l) \leq \frac{K_0 N^2}{q} \leq C_2(l),$$

1029 and proves equation (4.8). □

1030 Thanks to Lemma (4.5) we now reduced the proof of Lemma 4.3 to

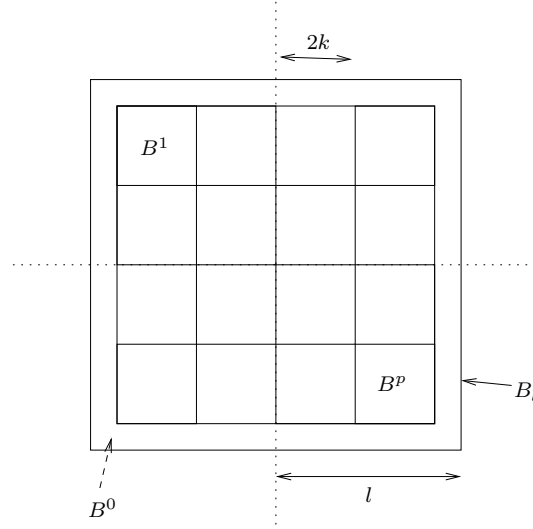
$$(4.11) \quad \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{\substack{D_l(f) \leq C_1(l)N^{-2} \\ H(f) \leq C_2(l)}} \mathbb{E}_{\alpha, l}^* \left(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} f \right) = 0.$$

1031 Since the set of measures with density w.r.t. μ_α^* such that $H(f) \leq C_2(l)$ is weakly compact, to prove the
 1032 one block estimate of Lemma 4.3, it is sufficient to show that

$$\limsup_{l \rightarrow \infty} \sup_{\substack{D_l(f)=0 \\ H(f) \leq C_2(l)}} \mathbb{E}_{\alpha, l}^* \left(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} f \right).$$

1033 Before using the equivalence of ensembles, we need to project the limit above over all sets with fixed
 1034 number of particles $\Sigma_l^{\hat{K}}$ defined in equation (3.3). Recall from Definition 3.6 the projection of the grand-
 1035 canonical measures on the sets with fixed number of particles. For any density f w.r.t. μ_α^* , such that
 1036 $D_l(f) = 0$, thanks to Section 3.3 and the presence of the indicator function, f is constant on $\Sigma_l^{\hat{K}}$ for any
 1037 $\hat{K} \in \tilde{\mathbb{K}}_l$. We therefore denote, for any such f , by $f(\hat{K})$ the value of f on the set $\Sigma_l^{\hat{K}}$. Shortening $\int_{\hat{K} \in \tilde{\mathbb{K}}_l}$
 1038 for the sum $\sum_{K \leq (2l+1)^2} \int_{\theta_1 \in \mathbb{S}} \dots \int_{\theta_K \in \mathbb{S}}$, we can write thanks to the indicator functions $\mathbb{1}_{E_l}$, for any f
 1039 satisfying $D_l(f) = 0$,

$$(4.12) \quad \mathbb{E}_{\alpha, l}^* \left(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} f \right) = \int_{\hat{K} \in \tilde{\mathbb{K}}_l} f(\hat{K}) \mathbb{E}_{l, \hat{K}}(\tilde{\mathcal{V}}^l) d\mu_\alpha^* \left(\Sigma_l^{\hat{K}} \right),$$


 FIGURE 2. Construction of the B^i

1040 where $\tilde{\mathbb{K}}$ was defined in (3.2).

1041 Since $\int_{\hat{K} \in \tilde{\mathbb{K}}_l} f(\hat{K}) d\mu_\alpha^*(\Sigma_l^{\hat{K}}) = 1$ and $\mathbb{E}_{l, \hat{K}}(\tilde{\mathcal{V}}^l) \leq \sup_{\hat{K} \in \tilde{\mathbb{K}}_l} \mathbb{E}_{l, \hat{K}}(\tilde{\mathcal{V}}^l)$, we obtain

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{\substack{D_l(f) \leq C_2(l)N^{-2} \\ H(f) \leq C_2(l)}} \mathbb{E}_{\alpha, l}^*(\tilde{\mathcal{V}}^l \mathbb{1}_{E_l} f) \leq \limsup_{l \rightarrow \infty} \sup_{\hat{K} \in \tilde{\mathbb{K}}_l} \mathbb{E}_{l, \hat{K}}(\tilde{\mathcal{V}}^l).$$

1042 To conclude the proof of equation (4.11) and the one-block estimate, it is therefore sufficient to prove
1043 that the right-hand side above vanishes.

1044 For any $\hat{K} \in \mathbb{K}_l$, recall that $\hat{\alpha}_{\hat{K}} \in \mathcal{M}_1(\mathbb{S})$ is the grand-canonical parameter

$$\hat{\alpha}_{\hat{K}} = \frac{1}{(2l+1)^2} \sum_{k=1}^K \delta_{\theta_k} \in \mathcal{M}_1(\mathbb{S}).$$

1045 Since the expectation $\mathbb{E}_{l, \hat{K}}$ conditions the process to having K particles with angles Θ_K in B_l , by
1046 definition of $\tilde{\mathcal{V}}_l$, letting $l' = l - s_g$ we can write

$$\left| \mathbb{E}_{l, \hat{K}}(\tilde{\mathcal{V}}^l) \right| \leq \mathbb{E}_{l, \hat{K}} \left(\left| \frac{1}{(2l'+1)^2} \sum_{x \in B_{l'}} \tau_x g - \mathbb{E}_{\hat{\alpha}_{\hat{K}}}(g) \right| \right).$$

1047 Let k be an integer that will go to infinity after l , and let us divide $B_{l'}$ according to Figure 2 into q boxes
1048 B^1, \dots, B^q , each of size $(2k+1)^2$, with $q = \lfloor \frac{2l'+1}{2k+1} \rfloor^2$. let $k' = k - s_g$, $B^{i'}$ denotes the box of size $(2k'+1)$
1049 centered inside B^i , and Let $B^0 = B_{l'} - \cup_{i=1}^q B^{i'}$, the number of sites in B^0 is bounded for some constant
1050 $C := C(g)$ by Ckl .

With these notations, the triangular inequality yields

$$\begin{aligned} \mathbb{E}_{l, \hat{K}} \left(\left| \mathbb{E}_{\hat{\alpha}_{\hat{K}}}(g) - \frac{1}{(2l'+1)^2} \sum_{x \in B_{l'}} \tau_x g \right| \right) &\leq \frac{|B^{1'}|}{|B_{l'}|} \sum_{i=0}^q \mathbb{E}_{l, \hat{K}} \left(\left| \mathbb{E}_{\hat{\alpha}_{\hat{K}}}(g) - \frac{1}{|B^{i'}|} \sum_{x \in B^{i'}} \tau_x g \right| \right) \\ &= \frac{(2k'+1)^2}{(2l'+1)^2} \sum_{i=1}^q \mathbb{E}_{l, \hat{K}} \left(\left| \mathbb{E}_{\hat{\alpha}_{\hat{K}}}(g) - \frac{1}{(2k'+1)^2} \sum_{x \in B^{i'}} \tau_x g \right| \right) \\ &\quad + O\left(\frac{k}{l}\right) \end{aligned}$$

1051 Since the distribution of the quantity inside the expectation does not depend on i , the quantity above
1052 can be rewritten

$$\underbrace{q \frac{(2k'+1)^2}{(2l'+1)^2}}_{\rightarrow 1} \mathbb{E}_{l, \widehat{K}} \left(\left| \mathbb{E}_{\widehat{\alpha}_{\widehat{K}}}(g) - \frac{1}{(2k'+1)^2} \sum_{x \in B_{k'}} \tau_x g \right| \right) + O\left(\frac{k}{l}\right).$$

1053 Because g is a cylinder function, and since k goes to ∞ after l , the quantity inside absolute values is a
1054 local function for any fixed k . Letting l go to ∞ , the equivalence of ensembles stated in Proposition C.1
1055 allows us to replace the expectation above, uniformly in \widehat{K} , by

$$\mathbb{E}_{\widehat{\alpha}_{\widehat{K}}} \left(\left| \mathbb{E}_{\widehat{\alpha}_{\widehat{K}}}(g) - \frac{1}{(2k'+1)^2} \sum_{x \in B_{k'}} \tau_x g \right| \right).$$

1056 Finally, since $\cup_{l \in \mathbb{N}} \{\widehat{\alpha}_{\widehat{K}}, \widehat{K} \in \widehat{\mathbb{K}}_l\} \subset \mathcal{M}_1(\mathbb{S})$, where $\mathcal{M}_1(\mathbb{S})$ is the set of angle density profiles introduced
1057 in Definition 3.1,

$$\limsup_{l \rightarrow \infty} \sup_{\widehat{K} \in \widehat{\mathbb{K}}_l} \mathbb{E}_{l, \widehat{K}}(\widehat{\mathcal{V}}^l) \leq \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \mathbb{E}_{\widehat{\alpha}} \left(\left| \mathbb{E}_{\widehat{\alpha}}(g) - \frac{1}{(2k'+1)^2} \sum_{x \in B_{k'}} \tau_x g \right| \right),$$

1058 whose right-hand side vanishes as $k \rightarrow \infty$ by the law of large numbers, thus concluding the proof of the
1059 one-block estimate.

1060 **4.4. Proof of Lemma 4.4 : The two-block estimate.** — *This Section follows the usual strategy*
1061 *for the two-block estimate, with small adaptations to the topological setup on the space of parameters*
1062 *$\mathcal{M}_1(\mathbb{S})$ introduced in Definition 3.2.*

1063 Our goal is to show that for any density f satisfying conditions *i*) and *ii*) in Lemma 4.3,

$$\limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{y \in B_{\varepsilon N}} \mathbb{E}_{\alpha}^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \left\| \tau_{x+y} \widehat{\rho}_l - \tau_x \widehat{\rho}_{\varepsilon N} \right\| \|f\| \right) = 0.$$

1064 The previous expectation can be bounded from above by triangle inequality by

$$\mathbb{E}_{\alpha}^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \frac{1}{(2\varepsilon N + 1)^2} \left\| \sum_{z \in B_{\varepsilon N}} (\tau_{x+y} \widehat{\rho}_l - \tau_{x+z} \widehat{\rho}_l) \right\| \|f\| \right) + o(l/\varepsilon N).$$

1065 In this way, we reduce the proof to comparing average densities in two boxes of size l distant of less than
1066 $2\varepsilon N$. Let us extract in the sum inside the integral the terms in z 's such that $|y - z'| \leq 2l$, the number
1067 of such terms is at most $(4l + 1)^2$, and this quantity is bounded from above by

$$\mathbb{E}_{\alpha}^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \frac{1}{(2\varepsilon N + 1)^2} \left\| \sum_{\substack{z \in B_{\varepsilon N} \\ |y-z| > 2l}} (\tau_{x+y} \widehat{\rho}_l - \tau_{x+z} \widehat{\rho}_l) \right\| \|f\| \right) + o(l/\varepsilon N).$$

1068 This separation was performed in order to obtain independent empirical measures $\tau_{x+y} \widehat{\rho}_l$ and $\tau_{x+z} \widehat{\rho}_l$.
1069 Regarding the expectation above, notice that we now only require to bound each term in the sum in z .
1070 In order to prove the two-block estimate, it is thus sufficient to show that

$$\limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{2l < |y| < 2\varepsilon N} \mathbb{E}_{\alpha}^* \left(\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \left\| \tau_{x+y} \widehat{\rho}_l - \tau_x \widehat{\rho}_l \right\| \|f\| \right) = 0.$$

1071 As in the proof of the one-block estimate, the expectation above can be rewritten

$$\mathbb{E}_{\alpha}^* \left(\left\| \tau_y \widehat{\rho}_l - \widehat{\rho}_l \right\| \|\bar{f}\| \right),$$

1072 where $\bar{f} = N^{-2} \sum_{x \in \mathbb{T}_N^2} \tau_x f$ is the average of the density f . We can also introduce the cutoff functions
1073 $\mathbb{1}_{E_l}$ in the expectation above, thanks to f satisfying (4.4) and $\left\| \tau_y \widehat{\rho}_l - \widehat{\rho}_l \right\|$ being a bounded quantity.

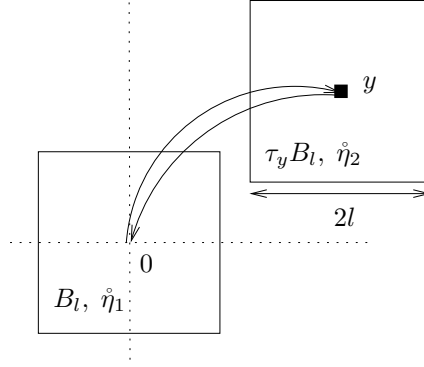


FIGURE 3

1074 Let $B_{y,l}$ be the set $B_l \cup \tau_y B_l$, the quantity under the expectation above is measurable with respect to
 1075 the sites in $B_{y,l}$. Before going further, let us denote, for any configuration $\hat{\eta} \in \Sigma_N$, $\hat{\eta}_1$ the configuration
 1076 restricted to B_l and $\hat{\eta}_2$ the configuration restricted to $y+B_l = \tau_y B_l$. We also denote by $\hat{\eta}$ the configuration
 1077 $(\hat{\eta}_1, \hat{\eta}_2)$ on $B_{y,l}$. Let us finally write $\mu_{y,l}$ for the projection of the product measure μ_α^* on $B_{y,l}$, and $\mathbb{E}_{y,l}$
 1078 the expectation with respect to the latter.

1079 With these notations, the expectation above can be replaced by

$$\mathbb{E}_\alpha^* (\| \tau_y \hat{\rho}_l - \hat{\rho}_l \| \mathbb{1}_{E_l} \bar{f}_{y,l}),$$

1080 where for any density f , $f_{y,l}$ is its conditional expectation with respect to the sigma-field generated by
 1081 $(\hat{\eta}_x)_{x \in B_{y,l}}$,

$$f_{y,l}(\hat{\eta}) = \mathbb{E}_\alpha^* (f \mid \hat{\eta}_{B_{y,l}} = \hat{\eta}),$$

1082 which is well-defined because the two boxes B_l and $\tau_y B_l$ are disjoint, thanks to the condition $|y| > 2l$.

As in the proof of the one-block estimate, we now need to estimate the Dirichlet form of $\bar{f}_{y,l}$ in terms
 of that of f , on which we have some control. For that purpose, let us introduce with the notations of the
 previous Section

$$\begin{aligned}
 D_{l,y}(h) &= -\mathbb{E}_{y,l}(h \mathcal{L}_{0,y} h) - \sum_{\substack{x,z \in B_l \\ |x-z|=1}} \mathbb{E}_{y,l}(h \mathcal{L}_{x,z} h) - \sum_{\substack{x,z \in y+B_l \\ |x-z|=1}} \mathbb{E}_{y,l}(h \mathcal{L}_{x,z} h) \\
 (4.13) \quad &:= D_{l,y}^0 + D_{l,y}^1 + D_{l,y}^2
 \end{aligned}$$

1083 the Dirichlet form corresponding to particle transfers inside the two boxes, and allowing a particle to
 1084 transfer from the center of one box to the center of the other, according to Figure 3. The work of the
 1085 previous section allows us to write that

$$-\mathbb{E}_{y,l}(\bar{f}_{y,l} \mathcal{L}_{x,z} \bar{f}_{y,l}) \leq D^{x,z}(\bar{f}),$$

1086 which implies, if $D(f) \leq C_0$ that

$$(4.14) \quad D_{l,y}^1(\bar{f}_{y,l}) + D_{l,y}^2(\bar{f}_{y,l}) \leq 2C_0 \frac{(2l+1)^2}{N^2},$$

1087 by translation invariance of μ_α and \bar{f} . We now only need to estimate the third term $D_{l,y}^0$. Let us consider
 1088 a path $x_0 = 0, x_1, \dots, x_k = y$ of minimal length, such that $|x_i - x_{i+1}| = 1$ for any $i \in \{0, \dots, k-1\}$.
 1089 For any such path, we have $k \leq 4\epsilon N$, since $|y| \leq 2\epsilon N$, and we can write

$$D_{l,y}^0(\bar{f}) \leq -\mathbb{E}_\alpha^*(\bar{f} \mathcal{L}_{0,y} \bar{f}) = \frac{1}{2} \mathbb{E}_\alpha^* [|\eta_0 - \eta_y| (\bar{f}(\hat{\eta}^{0,y}) - \bar{f}(\hat{\eta}))^2]$$

1090 where $\hat{\eta}^{0,y}$ here is the state where the sites in 0 and y are inverted regardless of the occupation of either
 1091 site. Since $\eta_0 - \eta_y$ vanishes whenever both sites 0 and y are occupied or both are empty, we can for
 1092 example assume that $\eta_0 = 1$ and $\eta_y = 0$. For any configuration $\hat{\eta}^0 = \hat{\eta}$, we let for any $i \in \{1, \dots, k\}$

$$\hat{\eta}^i = (\hat{\eta}^{i-1})^{x_{i-1}, x_i}$$

1093 Thanks to the elementary inequality

$$\left(\sum_{j=1}^k a_j \right)^2 \leq k \sum_{j=1}^k a_j^2,$$

and by definition of the sequence $(\hat{\eta}^i)_{i=0\dots k}$ (which yields in particular $\hat{\eta}^0 = \hat{\eta}$ and $\hat{\eta}^k = \hat{\eta}^{0,y}$), the previous equation yields

$$\begin{aligned} \mathbb{E}_\alpha^* [\eta_0(1-\eta_y)(\bar{f}(\hat{\eta}^{0,y}) - \bar{f}(\hat{\eta}))^2] &\leq k \sum_{i=0}^{k-1} \mathbb{E}_\alpha^* [\eta_0(1-\eta_y)(\bar{f}(\hat{\eta}^{i+1}) - \bar{f}(\hat{\eta}^i))^2] \\ &= k \sum_{i=0}^{k-1} \mathbb{E}_\alpha^* \left[\eta_{x_i}^i (1 - \eta_{x_{i+1}}^i) \left[\bar{f}((\hat{\eta}^i)^{x_i, x_{i+1}}) - \bar{f}(\hat{\eta}^i) \right]^2 \right] \end{aligned}$$

1094 Since μ_α^* is invariant through any change of variable $\hat{\eta} \rightarrow \hat{\eta}^i$, and since we can easily derive the same kind
1095 of inequalities with $\eta_y(1-\eta_0)$ instead of $\eta_0(1-\eta_y)$, we obtain that

$$(4.15) \quad D_l^{0,y}(\bar{f}) \leq k \sum_{i=0}^{k-1} D^{x_{i+1}, x_i}(\bar{f}) = k^2 N^{-2} D(f) \leq 16\varepsilon^2 D(f)$$

1096 thanks to the translation invariance of \bar{f} . Finally, equations (4.13), (4.14) and (4.15) yield

$$(4.16) \quad D_{l,y}(\bar{f}_{y,l}) \leq 2C_0 \frac{(2l+1)^2}{N^2} + 16C_0\varepsilon^2,$$

1097 which vanishes as $N \rightarrow \infty$ then $\varepsilon \rightarrow 0$. A bound on the entropy analogous to (4.8) is straightforward to
1098 obtain. Finally, to prove the two-block estimate, as in the proof of the one-block estimate, we can get
1099 back to proving that

$$(4.17) \quad \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{2l < |y| < 2\varepsilon N} \sup_{D_{l,y}(f) \leq 2C_0 \frac{(2l+1)^2}{N^2} + 16C_0\varepsilon^2} \mathbb{E}_{y,l} (\| \tau_y \hat{\rho}_l - \hat{\rho}_l \| \mathbb{1}_{E_l} f) = 0.$$

1100 Any density satisfying the bound $D_{l,y}(f) \leq 2C_0 \frac{(2l+1)^2}{N^2} + 16C_0\varepsilon^2$ is ultimately constant on any set with
1101 fixed number of particles and angles in the set $B_{y,l}$ with at least two empty sites. More precisely, denote

$$\hat{\alpha}_{y,\ell}(\hat{\eta}) = \frac{1}{2(2l+1)^2} \sum_{x \in B_l \cup \tau_y B_l} \eta_x \delta_{\theta_x}$$

1102 the empirical canonical state of the configuration in $B_l \cup \tau_y B_l$, and denote by $\hat{f}(\cdot)$ the conditional expect-
1103 ation of f w.r.t. the canonical state of the configuration in $B_l \cup \tau_y B_l$, defined for any \hat{K} on $B_l \cup \tau_y B_l$
1104 by

$$\hat{f}(\hat{K}) = \mathbb{E}_\alpha^* (f | \hat{\alpha}_{y,\ell}(\hat{\eta}) = \hat{\alpha}_{\hat{K}}).$$

We can now write for any $|y| > 2l$

$$\begin{aligned} \mathbb{E}_{y,l} (\| \tau_y \hat{\rho}_l - \hat{\rho}_l \| \mathbb{1}_{E_l} f) &\leq \int_{\mathbb{K}_{y,l}} \mathbb{E}_{\hat{K},y,l} (\| \tau_y \hat{\rho}_l - \hat{\rho}_l \|) \hat{f}(\hat{K}) d\hat{K} + \mathbb{E}_{y,l} \left(\mathbb{1}_{E_l} \left| f - \hat{f}(\hat{\alpha}_{y,\ell}(\hat{\eta})) \right| \right) \\ &\leq \sup_{\hat{K} \in \mathbb{K}_{y,l}} \mathbb{E}_{\hat{K},y,l} (\| \tau_y \hat{\rho}_l - \hat{\rho}_l \|) + \mathbb{E}_{y,l} \left(\mathbb{1}_{E_l} \left| f - \hat{f}(\hat{\alpha}_{y,\ell}(\hat{\eta})) \right| \right), \end{aligned}$$

1105 where we shortened $y_l = (2l+1)e_1$, $\mathbb{K}_{y,l}$ denotes the set of canonical parameters on $B_l \cup \tau_y B_l$, and
1106 $\mathbb{E}_{\hat{K},y,l}(\cdot) = \mathbb{E}_\alpha^*(\cdot | \hat{\alpha}_{y,\ell}(\hat{\eta}) = \hat{\alpha}_{\hat{K}})$. By compactness of the set of densities w.r.t. μ_α^* on $B_l \cup \tau_y B_l$, the
1107 supremum over all densities satisfying $D_{l,y}(f) \leq 2C_0 \frac{(2l+1)^2}{N^2} + 16C_0\varepsilon^2$ of the second term above vanishes
1108 uniformly in $|y| > 2l$ as $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, whereas the first term does not depend on y . To prove
1109 (4.17), it is therefore sufficient to prove that

$$\limsup_{l \rightarrow \infty} \sup_{\hat{K} \in \mathbb{K}_{y_l,l}} \mathbb{E}_{\hat{K},y_l,l} (\| \tau_{y_l} \hat{\rho}_l - \hat{\rho}_l \|) = 0,$$

1110 which follows from the equivalence of ensembles.

1111

5. Preliminaries to the non-gradient method

1112 The main focus of Sections 5 and 6 is the symmetric part of the displacement process, whose contri-
 1113 bution to the hydrodynamic limit requires the non-gradient method. Before engaging in the proof of the
 1114 non-gradient estimates, however, we regroup several results which will be needed throughout the proof.

1115 **5.1. Comparison with an equilibrium measure.** — In this section, we prove a result that will
 1116 be used several times throughout the proof, and which allows to control the exponential moments of a
 1117 functional X by a variational formula involving the equilibrium measure μ_α^* . This control is analogous
 1118 to the so called sector condition for asymmetric processes, which ensures that the mixing due to the
 1119 symmetric part of the generator is sufficient to balance out the shocks provoked by the antisymmetric
 1120 part.

1121 **Remark 5.1.** — [Non-stationarity of μ_α^* for the weakly asymmetric process] It has already been pointed
 1122 out that \mathcal{L} is self-adjoint w.r.t any product measure $\mu_{\hat{\alpha}}$, which is not in general the case of $\mathcal{L}^{G,\beta=0}$.
 1123 However, $\mathcal{L}^{G,\beta=0}$ is self-adjoint w.r.t. μ_α^* due to the uniformity in θ of that measure. Asymmetric
 1124 generators are usually "almost" anti-self-adjoint, in the sense that one could expect $\mathcal{L}^{\text{WA}*} = -\mathcal{L}^{\text{WA}}$. This
 1125 identity is for example true for the *TASEP*, for which the asymmetry is constant and does not depend
 1126 on each particle.

1127 It is not true in our case however, due to the exclusion rule and the dependency of the asymmetry
 1128 in the angle of the particle. To clarify this statement, see the adjoint operator as a time-reversal, and
 1129 consider a configuration with two columns of particles wanting to cross each other. This configuration
 1130 would be stuck under \mathcal{L}^{WA} , however, under the time-reversed dynamics $\mathcal{L}^{\text{WA}*}$, it starts to move. This
 1131 illustrates that in our model, the asymmetric generator \mathcal{L}^{WA} is not anti-self-adjoint.

1132 Let us denote accordingly to the previous notation (2.15) and recalling the definition of the λ_i 's (2.1),
 1133 for $i = 1, 2$

$$j_i^{\lambda_i} = \lambda_i(\theta_0)\eta_0(1 - \eta_{e_i}) - \lambda_i(\theta_{e_i})\eta_{e_i}(1 - \eta_0).$$

1134 Elementary computations yield accordingly that the adjoint in $L^2(\mu_\alpha^*)$ of \mathcal{L}^{WA} is in fact given by

$$(5.1) \quad \mathcal{L}^{\text{WA},*} = -\mathcal{L}^{\text{WA}} + 2 \sum_{x \in \mathbb{T}_N^2} \sum_{i=1,2} \tau_x j_i^{\lambda_i}.$$

1135 This identity will be necessary to prove the following result, which compares the measure of the process
 1136 with drift to the measure μ_α^* .

1137 **Lemma 5.2.** — Recall the topology on Σ_N introduced in Proposition 3.10, and fix a bounded measurable
 1138 function

$$X \quad : \quad \Sigma_N \times [0, T] \quad \longrightarrow \quad \mathbb{R} \\ (\hat{\eta}, t) \quad \longmapsto \quad X_t(\hat{\eta}) .$$

1139 For any $\gamma > 0$, we have

$$\frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T X_t(\hat{\eta}(t)) dt \right) \right] \leq \frac{2T\lambda^2}{\gamma} + \frac{1}{\gamma} \int_0^T dt \sup_{\varphi} \left\{ \mathbb{E}_\alpha^* (\varphi \gamma X_t(\hat{\eta})) - \frac{1}{2} D(\varphi) \right\},$$

1140 where the supremum in the right-hand side is taken on the densities w.r.t. μ_α^* .

1141 **Proof of Lemma 5.2.** — Let us denote by $P_t^{\lambda,X}$ the modified semi-group

$$P_t^{\lambda,X} = \exp \left[\int_0^t L_N^{\beta=0} + \gamma N^2 X_s ds \right].$$

where $L_N^{\beta=0}$ is the alignment-free generator introduced in (3.16) and let us denote in this section by
 $\langle \cdot, \cdot \rangle_\alpha$ the inner product in $L^2(\mu_\alpha^*)$. For any $i = 1, 2$, and any H , and $T > 0$, the Feynman-Kac
 formula yields

$$(5.2) \quad \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T X_t(\hat{\eta}(t)) dt \right) \right] = \langle 1, P_T^{\lambda,X} 1 \rangle_\alpha \leq \langle P_T^{\lambda,X} 1, P_T^{\lambda,X} 1 \rangle_\alpha^{1/2} .$$

1142 by definition of $P_t^{\lambda,X}$,

$$(5.3) \quad \frac{d}{dt} \langle P_t^{\lambda,X} 1, P_t^{\lambda,X} 1 \rangle_\alpha = \langle P_t^{\lambda,X} 1, (L_N^{\beta=0} + L_N^{\beta=0,*} + 2\gamma N^2 X_t) P_t^{\lambda,X} 1 \rangle_\alpha,$$

1143 where M^* stands for the adjoint in $L^2(\mu_\alpha^*)$ of M . By definition of $L_N^{\beta=0}$, we have

$$L_N^{\beta=0,*} = N^2 \mathcal{L}^* + N \mathcal{L}^{\text{WA},*} + \mathcal{L}^{\text{G},\beta=0,*}.$$

1144 We now work to control the weakly asymmetric contribution in the right-hand side of equation (5.3),
 1145 which does not vanish in our case, as a consequence of Remark 5.1. For that purpose, consider a function
 1146 $\varphi \in L^2(\mu_\alpha^*)$, identity (5.1) yields

$$\langle \varphi, (\mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{WA},*}) \varphi \rangle_\alpha = 2 \sum_{x \in \mathbb{T}_N^2} \sum_{i=1,2} \mathbb{E}_\alpha^* \left[\varphi^2 \tau_x j_i^{\lambda_i} \right].$$

1147 Recall the definition of $\nabla_a f$ given in equation (3.4). A change of variable $\hat{\eta} \mapsto \hat{\eta}^{x,x+e_i}$ on the second part
 1148 of $\tau_x j_i^{\lambda_i}$ yields that for any x

$$\mathbb{E}_\alpha^* (\varphi^2 \tau_x j_i^{\lambda_i}) = -\mathbb{E}_\alpha^* (\lambda_i(\theta_x) \nabla_{x,x+e_i} \varphi^2) = -\mathbb{E}_\alpha^* [\lambda_i(\theta_x) (\varphi(\hat{\eta}^{x,x+e_i}) + \varphi) \nabla_{x,x+e_i} \varphi],$$

1149 therefore applying the elementary inequality $ab \leq a^2/2 + b^2/2$, to

$$a = \sqrt{N} \nabla_{x,x+e_i} \varphi \quad \text{and} \quad b = -\frac{\lambda_i(\theta_0)}{\sqrt{N}} (\varphi(\hat{\eta}^{x,x+e_i}) + \varphi),$$

we obtain (since $\lambda_i(\theta)$ is either $\lambda \cos(\theta)$ or $\lambda \sin(\theta)$ and is less than λ)

$$\langle \varphi, (\mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{WA},*}) \varphi \rangle_\alpha \leq \frac{N}{2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1,2} \mathbb{E}_\alpha^* \left[(\nabla_{x,x+e_i} \varphi)^2 \right] + \frac{\lambda^2}{2N} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1,2} \mathbb{E}_\alpha^* \left[(\varphi(\hat{\eta}^{x,x+e_i}) + \varphi)^2 \right].$$

1150 Since $(\varphi(\hat{\eta}^{x,x+e_i}) + \varphi)^2$ is less than $2\varphi^2(\hat{\eta}^{x,x+e_i}) + 2\varphi^2$, we finally obtain that,

$$\langle \varphi, N(\mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{WA},*}) \varphi \rangle_\alpha \leq -N^2 \mathbb{E}_\alpha^* [\varphi \mathcal{L} \varphi] + 4\lambda^2 N^2 \mathbb{E}_\alpha^* [\varphi^2].$$

In particular, applying this identity to $\varphi = P_t^{\lambda,X} 1$, we deduce from equation (5.3) that

$$\begin{aligned} \frac{d}{dt} \langle P_t^{\lambda,X} 1, P_t^{\lambda,X} 1 \rangle_\alpha &\leq \langle P_t^{\lambda,X} 1, [2\gamma N^2 X_t + N^2 \mathcal{L} + 2\mathcal{L}^{\text{G},\beta=0} + 4\lambda^2 N^2] P_t^{\lambda,X} 1 \rangle_\alpha \\ &\leq (\nu_\gamma(t) + 4\lambda^2 N^2) \langle P_t^{\lambda,X} 1, P_t^{\lambda,X} 1 \rangle_\alpha + 2 \langle P_t^{\lambda,X} 1, \mathcal{L}^{\text{G},\beta=0} P_t^{\lambda,X} 1 \rangle_\alpha, \end{aligned}$$

where $\nu_\gamma(t)$ is the largest eigenvalue of the self-adjoint operator $N^2 \mathcal{L} + 2\gamma N^2 X_t$. It is not hard to see that the second term above is non-positive. Indeed, for any function φ on Σ_N , by definition of $\mathcal{L}^{\text{G},\beta=0}$ (cf. equation (2.5))

$$\begin{aligned} \langle \varphi, \mathcal{L}^{\text{G},\beta=0} \varphi \rangle_\alpha &= \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\eta_x \varphi(\hat{\eta}) \left[\frac{1}{2\pi} \int_{\mathbb{S}} \varphi(\hat{\eta}^{x,\theta}) d\theta - \varphi(\hat{\eta}) \right] \right) \\ &= -\frac{1}{2} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\eta_x \left[\frac{1}{2\pi} \int_{\mathbb{S}} \varphi(\hat{\eta}^{x,\theta}) d\theta - \varphi(\hat{\eta}) \right]^2 \right) \leq 0. \end{aligned}$$

1151 To establish the last identity, we only used that under μ_α^* , the angles are chosen uniformly, and therefore
 1152 $\mathbb{E}_\alpha^* (\eta_x \varphi(\theta_x)) = \mathbb{E}_\alpha^* (\eta_x) (1/2\pi) \int_{\mathbb{S}} \varphi(\theta') d\theta'$. We thus obtain that

$$\frac{d}{dt} \langle P_t^{\lambda,X} 1, P_t^{\lambda,X} 1 \rangle_\alpha \leq (\nu_\gamma(t) + 4\lambda^2 N^2) \langle P_t^{\lambda,X} 1, P_t^{\lambda,X} 1 \rangle_\alpha,$$

1153 and Grönwall's inequality therefore yields that

$$\langle P_T^{\lambda,X} 1, P_T^{\lambda,X} 1 \rangle_\alpha \leq \exp \left(4T\lambda^2 N^2 + \int_0^T \nu_\gamma(t) dt \right).$$

1154 This, combined with (5.2), allows us to write

$$(5.4) \quad \frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T X_t dt \right) \right] \leq \frac{2T\lambda^2}{\gamma} + \int_0^T \frac{\nu_\gamma(t)}{2\gamma N^2} dt.$$

The variational formula for the largest eigenvalue of the self-adjoint operator $N^2(\mathcal{L} + 2\gamma X_t)$ yields that

$$\nu_\gamma(t) = N^2 \sup_{\psi, \mathbb{E}_\alpha^*(\psi^2)=1} \mathbb{E}_\alpha^*(\psi(\mathcal{L} + 2\gamma X_t)\psi) = 2N^2 \sup_\varphi \left\{ \gamma \mathbb{E}_\alpha^*(X_t \varphi) - \frac{1}{2} D(\varphi) \right\},$$

1155 where the second supremum is taken over all densities φ w.r.t. μ_α^* , which together with (5.4) concludes
 1156 the proof of Lemma 5.2. To prove the last identity, one only has to note that the supremum must be
 1157 achieved by functions ψ of constant sign, so that we can let $\varphi = \sqrt{\psi}$.

1158

□

1159 **5.2. Relative compactness of the sequence of measures.** — *We prove in this section that the*
 1160 *sequence $(Q^N)_{N \in \mathbb{N}}$, defined in equation (B.4), is relatively compact for the weak topology. It follows from*
 1161 *two properties stated in Proposition 5.3 below. The first one ensures that the fixed-time marginals are*
 1162 *controlled, whereas the second ensures that the time-fluctuations of the process's measure are not too wide.*

1163

1164 Given a function $H : \mathbb{T}^2 \times \mathbb{S} \rightarrow \mathbb{R}$, we already introduced in the outline of Section 2.4 the notation

$$\langle \pi, H \rangle = \int_{\mathbb{T}^2 \times \mathbb{S}} H(u, \theta) \pi(du, d\theta).$$

1165 The following result yields sufficient conditions for the weak relative compactness of the sequence $(Q^N)_N$.
 1166 Recall from equation (2.10) the definition of the set of trajectories $\mathcal{M}^{[0, T]}$.

1167 **Proposition 5.3 (Characterization of the relative compactness on $\mathcal{P}(\mathcal{M}^{[0, T]})$)**

1168 *Let P^N be a sequence of probability measures on the set of trajectories $\mathcal{M}^{[0, T]}$ defined in (2.10), such*
 1169 *that*

1170 (1) *There exists some $A_0 > 0$ such that for any $A > A_0$,*

$$\limsup_{N \rightarrow \infty} P^N \left(\sup_{s \in [0, T]} \langle \pi_s, 1 \rangle \geq A \right) = 0$$

1171 (2) *For any $H \in C^{2,1}(\mathbb{T}^2 \times \mathbb{S})$, $\varepsilon > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} P^N \left(\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} | \langle \pi_{t'}, H \rangle - \langle \pi_t, H \rangle | > \varepsilon \right) = 0.$$

1172 *Then, the sequence $(P^N)_{N \in \mathbb{N}}$ is relatively compact for the weak topology.*

1173 Since this proposition is, with minor adjustments, found in [3] (cf. Theorem 13.2, page 139), we do
 1174 not give its proof, and refer the reader to the latter. For now, our focus is the case of the active exclusion
 1175 process, for which both of these conditions are realized. The strategy of the proof follows closely that
 1176 of Theorem 6.1, page 180 of [27], but requires two adjustments. First, our system is driven out of
 1177 equilibrium by the drift, and we therefore need to use the Lemma 5.2 stated in the previous section to
 1178 carry out the proof. The second adaptation comes from the presence of the angles, and since most of the
 1179 proof is given for a test function $H(u, \theta) = G(u)\omega(\theta)$, we need to extend it in the general case where H
 1180 cannot be decomposed in this fashion.

1181 **Proposition 5.4 (Compactness of $(Q^N)_{N \in \mathbb{N}}$).** — *The sequence $(Q^N)_{N \in \mathbb{N}}$ defined in equation (B.4)*
 1182 *of probabilities on the trajectories of the active exclusion process satisfies conditions (1) and (2) above,*
 1183 *and is therefore relatively compact.*

1184 *Proof of Proposition 5.4.* — The first condition does not require any work since the active exclusion
 1185 process only allows one particle per site and we can thus choose $A_0 = 1$. Regarding the second condition,
 1186 recall that

$$(5.5) \quad \langle \pi_{t'}^N, H \rangle - \langle \pi_t^N, H \rangle = \int_{t'}^t L_N \langle \pi_s^N, H \rangle ds + M_t^H - M_{t'}^H,$$

where M^H is a martingale with quadratic variation of order N^{-2} . For more details, we refer the reader to appendix A of [27]. First, Doob's inequality yields uniformly in δ the crude bound

$$(5.6) \quad \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\sup_{t',t \leq \delta} |M_t^H - M_{t'}^H| \right) \leq 2\mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\sup_{0 \leq t \leq T} |M_t^H| \right) \leq C(H)N^{-1},$$

1187 where $\mathbb{E}_{\mu^N}^{\lambda,\beta}$ is the expectation w.r.t the measure $\mathbb{P}_{\mu^N}^{\lambda,\beta}$ introduced just after Definition 3.4 of the complete
1188 process $\hat{\eta}^{[0,T]}$ started from the initial measure μ^N .

1189 Regarding the integral part of (5.5), we first assume like earlier that H takes the form

$$H(u, \theta) = G(u)\omega(\theta),$$

where G and ω are both C^2 functions. When this is not the case, an application of the periodic Weierstrass Theorem will yield the wanted result. Then, following the same justification as in Section 2.4 we can write

$$\int_{t'}^t L_N < \pi_s^N, H > ds = \frac{1}{N^2} \int_{t'}^t ds \sum_{x \in \mathbb{T}_N^2} \tau_x \left(\sum_{i=1}^2 [Nj_i^\omega + r_i^\omega](s) \partial_{u_i, N} G(x/N) + \tau_x \gamma^\omega(s) G(x/N) \right),$$

1190 where the instantaneous currents j^ω , r^ω and γ^ω were introduced in Definition 2.8.

The weakly asymmetric and Glauber contributions are easy to control, since both jump rates r^ω and γ^ω can be bounded by a same constant K , and we can therefore write

$$\begin{aligned} \int_{t'}^t (N\mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{G}}) < \pi_s^N, H > ds &\leq K \int_{t'}^t ds \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} |G(x/N)| + \sum_{i=1}^2 |\partial_{u_i, N} G(x/N)| \\ &\rightarrow_{N \rightarrow \infty} K(t-t') \int_{\mathbb{T}^2} |G(u)| + \sum_{i=1}^2 |\partial_{u_i} G(u)| du, \end{aligned}$$

which vanishes as soon as $|t' - t| \leq \delta$ in the limit $\delta \rightarrow 0$. Finally,

$$\begin{aligned} &Q^N \left(\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} |< \pi_{t'}, H > - < \pi_t, H >| > \varepsilon \right) \\ &\leq \mathbb{P}_{\mu^N}^{\lambda,\beta} \left[\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} \left| \int_{t'}^t N^2 \mathcal{L} < \pi_s^N, H > ds \right| > \varepsilon/3 \right] \\ &\quad + \mathbb{P}_{\mu^N}^{\lambda,\beta} \left[\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} \left| \int_{t'}^t (N\mathcal{L}^{\text{WA}} + \mathcal{L}^{\text{G}}) < \pi_s^N, H > ds \right| > \varepsilon/3 \right] \\ &\quad + \mathbb{P}_{\mu^N}^{\lambda,\beta} \left[\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} |M_t^H - M_{t'}^H| > \varepsilon/3 \right]. \end{aligned}$$

1191 The second line of the right-hand side vanishes in the limit $N \rightarrow \infty$ then $\delta \rightarrow 0$ thanks to the computation
1192 above, whereas the third line also vanishes thanks to Markov's inequality and equation (5.6). Finally,
1193 the first term vanishes accordingly to Lemma 5.5 below and the Markov inequality, thus completing the
1194 proof in the case where $H(u, \theta) = G(u)\omega(\theta)$. The general case is derived just after the proof of Lemma
1195 5.5.

1196 **Lemma 5.5.** — For any function $H(u, \theta) = G(u)\omega(\theta) \in C^{2,0}(\mathbb{T}^2 \times \mathbb{S})$,

$$(5.7) \quad \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda,\beta} \left(\sup_{\substack{|t-t'| \leq \delta \\ 0 \leq t', t \leq T}} \left| \int_{t'}^t N^2 \mathcal{L} < \pi_s^N, H > ds \right| \right) = 0.$$

1197 *Proof of Lemma 5.5.* — The proof of this Lemma follows, with minor adjustments to account for the
 1198 drift, the proof given in [27]. First, we get rid of the supremum and come back to the reference measure
 1199 with fixed parameter $\alpha \in]0, 1[$ thanks to Lemma 5.2 of Section 5.1. Let us denote

$$(5.8) \quad g(t) = \int_0^t N^2 \mathcal{L} \langle \pi_s^N, H \rangle ds.$$

1200 We now compare the measure of the active exclusion process to that of the process started from equi-
 1201 librium ($\mu^N = \mu_\alpha^*$), and with no alignment ($\beta = 0$), according to Proposition 3.10 with $A = RN^2$
 1202 and

$$X \left(\widehat{\eta}^{[0, T]} \right) = \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} \left| \int_{t'}^t N^2 \mathcal{L} \langle \pi_s^N, H \rangle ds \right| = \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')|.$$

1203 This yields that for some constant $K_0 > 0$, the expectation in equation (5.7) is bounded from above for
 1204 any positive R by

$$(5.9) \quad \frac{1}{RN^2} \left[K_0 N^2 + \log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \exp \left(RN^2 \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right) \right].$$

1205 We therefore reduce the proof of Lemma 5.5 to showing that

$$(5.10) \quad \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{R(\delta)N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \exp \left(R(\delta)N^2 \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right) = 0,$$

1206 where $R(\delta)$ goes to ∞ as δ goes to 0.

1207 Let p and ψ be two strictly increasing functions such that $\psi(0) = p(0) = 0$ and $\psi(+\infty) = +\infty$, with
 1208 ψ continuous, we denote

$$I = \int_{[0, T] \times [0, T]} \psi \left(\frac{|g(t) - g(t')|}{p(|t' - t|)} \right) dt' dt,$$

1209 the Garsia-Rodemich-Rumsey inequality [23] yields that

$$(5.11) \quad \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \leq 8 \int_0^\delta \psi^{-1} \left(\frac{4I}{u^2} \right) p(du).$$

Given any positive a , we choose $p(u) = \sqrt{u}$ and $\psi(u) = \exp(u/a) - 1$, hence $\psi^{-1}(u) = a \log(1 + u)$. An
 integration by parts yields for any $\delta < e^{-2}$ that

$$(5.12) \quad \begin{aligned} \int_0^\delta \psi^{-1} \left(\frac{4I}{u^2} \right) p(du) &= a \int_0^\delta \log \left(1 + \frac{4I}{u^2} \right) \frac{du}{2\sqrt{u}} \\ &= a\sqrt{\delta} \log(1 + 4I\delta^{-2}) + a \int_0^\delta \frac{8I}{u^3 + 4Iu} \sqrt{u} du \\ &\leq a\sqrt{\delta} \log(1 + 4I\delta^{-2}) + a \int_0^\delta \frac{2}{\sqrt{u}} du \\ &= a\sqrt{\delta} [\log(\delta^2 + 4I) - 2 \log \delta + 4] \\ &\leq a\sqrt{\delta} \left[-\frac{\log \delta}{2} \log(\delta^2 + 4I) - 4 \log \delta \right] \\ &\leq a\sqrt{\delta} [-4 \log \delta \log(\delta^2 + 4I) - 4 \log \delta], \end{aligned}$$

since by assumption $-\log(\delta) > 2$. From equations (5.11) and (5.12) we deduce that

$$\log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \exp \left(RN^2 \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right) \leq \log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \exp \left(-32aRN^2 \sqrt{\delta} \log \delta [1 + \log(\delta^2 + 4I + 1)] \right)$$

holds for any $a > 0$. For $\delta < 1$, Let us choose $a = -(32RN^2\sqrt{\delta}\log\delta)^{-1} > 0$, we can write for the second term of (5.9) the upper bound

$$\frac{1}{RN^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \exp \left(RN^2 \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right) \leq \frac{1}{RN^2} [1 + \log(1 + \delta^2 + 4\mathbb{E}_{\hat{\alpha}}(I))].$$

1210 By definition,

$$I = \int_{[0,T] \times [0,T]} \exp \left(\frac{\left| \int_{t'}^t N^2 \mathcal{L} < \pi_u^N, H > du \right|}{a\sqrt{|t-t'|}} \right) dt' dt - T^2.$$

Let us assume, purely for convenience, that $T > 1/2$, for δ sufficiently small, we have $4T^2 - 1 - \delta^2 > 0$, and the quantity inside the limit in equation (5.10) can be estimated by

$$(5.13) \quad \frac{1}{RN^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \exp \left(RN^2 \sup_{\substack{|t'-t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right) \\ \leq \frac{1}{RN^2} \left[1 + \log 4\mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\int_{[0,T] \times [0,T]} \exp \left(\frac{\left| \int_{t'}^t N^2 \mathcal{L} < \pi_s^N, H > ds \right|}{a\sqrt{|t'-t|}} \right) dt' dt \right] \right].$$

1211 If $T \leq 1/2$, we simply carry out a constant term in the log above, which does not alter the proof.

Let us take a look at the two constants a and R . Noting the first bound on the entropy mentioned earlier, in order to keep the first term of (5.9) in check, $R = R(\delta)$ must simply grow to ∞ . Furthermore, we previously obtained that $a = -(RN^2 32\sqrt{\delta}\log\delta)^{-1}$, we can choose $a = N^{-2}$, thus $R = (-1/32\sqrt{\delta}\log\delta)^{-1}$, which is non-negative, and goes to ∞ as $\delta \rightarrow 0^+$. Therefore, the second term above can be rewritten

$$\frac{1}{RN^2} \log \int_{[0,T] \times [0,T]} 4\mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \exp \left[\left| \int_{t'}^t \frac{N}{|t'-t|^{1/2}} \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} j_{x,x+e_i}^\omega(s) \partial_{u_i, N} G(x/N) ds \right| dt' dt \right].$$

1212 In order to estimate the expectation above, we can get rid of the absolute value, since $e^{|x|} \leq e^x + e^{-x}$,
1213 and since the function G is taken in a symmetric class of functions. Furthermore, Lemma 5.2, applied
1214 with $\gamma = 1$ yields that the second term in the right-hand side of (5.13) is less than

$$(5.14) \quad \frac{1}{RN^2} \log \int_{[0,T] \times [0,T]} \exp \left[\frac{(t-t')}{2} [4\lambda^2 N^2 + \nu_N(G, \omega)] \right] dt dt',$$

1215 where $\nu_N(G, \omega)$ is the largest eigenvalue in $L^2(\mu_\alpha^*)$ of the self-adjoint operator

$$N^2 \mathcal{L} + \frac{2N}{|t'-t|^{1/2}} \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} j_{x,x+e_i}^\omega \partial_{u_i, N} G(x/N),$$

1216 which can be rewritten as the variational formula

$$(5.15) \quad \nu_N(G, \omega) = \sup_f \left\{ \frac{2N}{|t'-t|^{1/2}} \sum_{\substack{x \in \mathbb{T}_N^2 \\ i=1,2}} \partial_{u_i, N} G(x/N) \mathbb{E}_\alpha^* (f j_{x,x+e_i}^\omega) - N^2 D(f) \right\},$$

1217 where the supremum is taken on all densities f w.r.t. μ_α^* . In order to prove that the eigenvalue above is
1218 of order N^2 , we now want to transform

$$\frac{N}{|t'-t|^{1/2}} \sum_{x \in \mathbb{T}_N^2} \partial_{u_i, N} G(x/N) \mathbb{E}_\alpha^* (f j_{x,x+e_i}^\omega).$$

For any density f , and $i = 1, 2$, since $j_{x,x+e_i}^\omega(\hat{\eta}^{x,x+e_i}) = -\tau_x j_i^\omega$, we can write

$$\mathbb{E}_\alpha^* (f j_{x,x+e_i}^\omega) \partial_{u_i, N} G(x/N) = -\frac{1}{2} \mathbb{E}_\alpha^* [(f(\hat{\eta}^{x,x+e_i}) - f) j_{x,x+e_i}^\omega] \partial_{u_i, N} G(x/N)$$

$$\begin{aligned} &\leq \frac{1}{4C} \mathbb{E}_\alpha^* \left((j_{x,x+e_i}^\omega)^2 \left(\sqrt{f}(\widehat{\eta}^{x,x+e_i}) - \sqrt{f} \right)^2 \right) \\ &\quad + \frac{C}{4} (\partial_{u_i, N} G(x/N))^2 \mathbb{E}_\alpha^* \left(\left(\sqrt{f}(\widehat{\eta}^{x,x+e_i}) + \sqrt{f} \right)^2 \right). \end{aligned}$$

1219 Since $(j_{x,x+e_i}^\omega)^2 \leq \|\omega\|_\infty^2 \mathbb{1}_{\eta_x \eta_{x+e_i} = 0}$, and since $[\sqrt{f}(\widehat{\eta}^{x,x+e_i}) + \sqrt{f}]^2 \leq 2f(\widehat{\eta}^{x,x+e_i}) + 2f$, we obtain the
1220 upper bound

$$\frac{N}{|t' - t|^{1/2}} \sum_{x \in \mathbb{T}_N^2} \partial_{u_i, N} G(x/N) \mathbb{E}_\alpha^* (f j_{x,x+e_i}^\omega) \leq \frac{N \|\omega\|_\infty^2}{2C |t' - t|^{1/2}} D(f) + \frac{N^3 C}{|t' - t|^{1/2}} \|\partial_{u_i} G\|_\infty^2,$$

which holds for any positive C . We now set $C = |t' - t|^{-1/2} \|\omega\|_\infty^2 / N$ so that the Dirichlet form contributions in the variational formula (5.15) cancel out. We finally obtain that for some positive constant $C_1 = C_1(G, \omega)$, independent of N ,

$$\nu_N(G, \omega) \leq \frac{C_1 N^2}{|t - t'|},$$

1221 which yields that (5.14) vanishes in the limit $N \rightarrow \infty$ and $\delta \rightarrow 0$, since $R = R(\delta)$ goes to ∞ as δ goes to
1222 0. Finally, we have proved thanks to equation (5.13) that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{RN^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \left(\exp \left[RN^2 \sup_{\substack{|t' - t| \leq \delta \\ 0 \leq t', t \leq T}} |g(t) - g(t')| \right] \right) = 0,$$

1223 which concludes the proof of Lemma (5.5). \square

1224 In order to complete the proof of Proposition 5.4, we still have to consider the case when H does not
1225 take a product form $G(u)\omega(\theta)$. In this case, since H is smooth it can be approximated by a trigonometric
1226 polynomial in u_1, u_2 and θ . Each term of the approximation is then of the form $G(u)\omega(\theta)$, and the
1227 previous result can therefore be applied. More precisely, consider a smooth function H , and for any
1228 $\alpha > 0$, there exists a finite family $(p_{ijk}^\alpha)_{0 \leq i, j, k \leq M_\alpha}$ of coefficients such that

$$\sup_{\substack{u \in \mathbb{T}^2, \\ \theta \in \mathbb{S}}} \left| H(u, \theta) - \sum_{i, j, k \in \llbracket 0, M \rrbracket} p_{ijk}^\alpha u_1^i u_2^j \theta^k \right| \leq \alpha.$$

1229 Let us now fix an $\varepsilon > 0$, and let us take $\alpha = \varepsilon/4$. Then, considering the corresponding family $P_{ijk}(u, \theta) =$
1230 $p_{ijk}^\alpha u_1^i u_2^j \theta^k$ we have that

$$\left| \langle \pi_{t'}^N, H \rangle - \langle \pi_t^N, H \rangle \right| \leq \left| \langle \pi_{t'}^N - \pi_t^N, H - \sum_{i, j, k \leq M_\alpha} P_{ijk} \rangle \right| + \sum_{i, j, k \leq M_\alpha} \left| \langle \pi_{t'}^N - \pi_t^N, P_{ijk} \rangle \right|.$$

Since we allow at most 1 particle per site, and since $H - \sum_{i, j, k \leq M_\alpha} P_{ijk}$ is smaller than $\varepsilon/4$, the first term of the right-hand side above is less than $\varepsilon/2$. From this, we deduce that for the left-hand side to be greater than ε , one of the terms $\left| \langle \pi_{t'}^N, P_{ijk} \rangle - \langle \pi_t^N, P_{ijk} \rangle \right|$ must be larger than $\varepsilon/2M_\alpha^3$. This yields that

$$\begin{aligned} &Q^N \left(\sup_{\substack{|s-t| \leq \delta \\ 0 \leq t', t \leq T}} \left| \langle \pi_{t'}^N, H \rangle - \langle \pi_t^N, H \rangle \right| > \varepsilon \right) \\ &\leq \sum_{i, j, k \leq M_\alpha} Q^N \left(\sup_{\substack{|t' - t| \leq \delta \\ 0 \leq t', t \leq T}} \left| \langle \pi_{t'}^N, P_{ijk} \rangle - \langle \pi_t^N, P_{ijk} \rangle \right| > \frac{\varepsilon}{2M_\alpha^3} \right). \end{aligned}$$

1231 Since α is fixed, we can now take the limit $N \rightarrow \infty$ then $\delta \rightarrow 0$, in which the right-hand side vanishes
1232 since all functions are decorrelated in u and θ . The result thus holds for any smooth function H , thus
1233 completing the proof of Proposition 5.4. \square

1234 We now prove that in the limit, the empirical measure of our process admits at any fixed time a density
1235 w.r.t. the Lebesgue measure on \mathbb{T}^2 .

1236 **Lemma 5.6.** — Any limit point Q^* of the sequence Q^N is concentrated on measures $\pi \in \mathcal{M}^{[0,T]}$ with
1237 time marginals absolutely continuous w.r.t the Lebesgue measure on \mathbb{T}^2 ,

$$Q^*(\pi, \pi_t(du, d\theta) = \widehat{\rho}_t(u, d\theta)du, \quad \forall t \in [0, T]) = 1.$$

1238 *Proof of Lemma 5.6.* — For any smooth function $H \in C(\mathbb{T}^2)$ configuration $\widehat{\eta}$ in Σ_N and any correspond-
1239 ing empirical measure π^N , we have

$$|\langle \pi^N, H \rangle| = \left| \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x/N) \eta_x \right| \leq \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} |H(x/N)|.$$

1240 The right-hand side above converges as N goes to ∞ towards $\int_{\mathbb{T}^2} |H(u)| du$. Since for any fixed function
1241 H , the application

$$\pi \mapsto \sup_{0 \leq t \leq T} |\langle \pi_t, H \rangle|$$

1242 is continuous, any limit point Q^* of $(Q^N)_N$ is concentrated on trajectories π such that

$$\sup_{0 \leq t \leq T} |\langle \pi_t, H \rangle| \leq \int_{\mathbb{T}^2} |H(u)| du,$$

1243 for any smooth function H on \mathbb{T}^2 , and therefore is absolutely continuous w.r.t. the Lebesgue measure on
1244 \mathbb{T}^2 . \square

1245 **5.3. Regularity of the density and energy estimate.** — In this section we prove that the macro-
1246 scopic particle density is regular enough for the weak hydrodynamic limit (2.11) to be well defined, i.e.
1247 that criterion iii) of Definition 2.5 is satisfied. The proof follows the same strategy as in [27], we give it
1248 for exhaustivity.

1249 Due to the non-constant diffusion coefficients, the second derivative in equation (2.11) cannot be
1250 applied to the test function, and we need, according to condition iii) of Definition 2.5, to prove that
1251 the macroscopic profiles of our particle system are such that $\nabla \rho$ is well-defined. We can now state the
1252 following result.

1253 **Theorem 5.7.** — Any limit point Q^* of the measure sequence $(Q^N)_N$ is concentrated on trajectories
1254 with $\rho_t(u) \in H_1 = W^{1,2}([0, T] \times \mathbb{T}^2)$ for any $p \geq 1$. In other words, Q^* -a.s., there exists functions
1255 $\partial_{u_i} \rho_t(u)$ in $L^2([0, T] \times \mathbb{T}^2)$ such that for any smooth function $H \in C^{0,1}([0, T] \times \mathbb{T}^2)$

$$(5.16) \quad \iint_{[0, T] \times \mathbb{T}^2} \rho_t(u) \partial_{u_i} H_t(u) du dt = - \iint_{[0, T] \times \mathbb{T}^2} H_t(u) \partial_{u_i} \rho_t(u) du dt$$

1256 Furthermore, there exists a constant $K = K(T, \lambda, \beta, \widehat{\zeta})$ such that for any limit point Q^* of (Q^N) , and for
1257 any i ,

$$(5.17) \quad \mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} [\partial_{u_i} \rho_t(u)]^2 du dt \right) < K.$$

1258 In particular, any such limit point Q^* is concentrated on measures satisfying condition iii) of Definition
1259 2.5.

1260 The proof is postponed to the end of this section. The usual argument to prove this result is Riesz
1261 representation theorem, that yields that if

$$\iint_{[0, T] \times \mathbb{T}^2} \rho_t(u) \partial_{u_i} H_t(u) du dt \leq C \left(\int_{[0, T] \times \mathbb{T}^2} H^2 \right)^{1/2}$$

for any H , there exists a function $\partial_{u_i} \rho \in L^2([0, T] \times \mathbb{T}^2)$ such that (5.16) holds. For that purpose, we need
the estimate given in Lemma 5.8 below. Fix a direction $i \in \{1, 2\}$, for any $x \in \mathbb{T}_N^2$, shorten $x_k = x + ke_i$,

$k \in \{0, \dots, \varepsilon N\}$. Following the strategy of the energy estimate of [27], and recalling that $\tau_x \rho_{\delta N}$ is the empirical particle density in $B_{\delta N}(x)$, we let

$$W_{N,i}(\varepsilon, \delta, H, \eta) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x/N) \left(\frac{1}{\varepsilon} [\tau_{x+\varepsilon N e_i} \rho_{\delta N} - \rho_{\delta N}] - H(x/N) \right).$$

1262 Note that to emphasize that this quantity does not depend on the angles, we denote its third variable as
1263 η instead of $\hat{\eta}$.

1264 **Lemma 5.8.** — Let $\{H^l, l \in \mathbb{N}\}$ be a dense sequence in the separable algebra $C^{0,1}([0, T] \times \mathbb{T}^2)$ endowed
1265 with the norm $\|H\|_\infty + \sum_{i=1}^2 \|\partial_{u_i} H\|_\infty$. For any $i = 1, 2$, there exists a positive constant $K = K(T, \lambda, \beta, \hat{\zeta})$
1266 such that for any $k \geq 1$ and $\varepsilon > 0$,

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N^{\lambda, \beta}}^{\lambda, \beta} \left(\max_{1 \leq l \leq k} \int_0^T W_{N,i}(\varepsilon, \delta, H_t^l, \eta(t)) dt \right) \leq K_0.$$

Proof of Lemma 5.8. — By the replacement Lemma 4.1, it is sufficient to show the result above without the limit in δ , and with $\widetilde{W}_{N,i}(\varepsilon, H, \eta)$ instead of $W_{N,i}$, where

$$\begin{aligned} \widetilde{W}_{N,i}(\varepsilon, H, \eta) &= \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x/N) \left(\frac{1}{\varepsilon} [\eta_{x+\varepsilon N e_i} - \eta_x] - H(x/N) \right) \\ &= \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} H(x/N) \frac{1}{\varepsilon N} \sum_{k=0}^{\varepsilon N-1} [N(\eta_{x_{k+1}} - \eta_{x_k}) - H(x/N)]. \end{aligned}$$

1267 Applying Proposition 3.10 to $A = N^2$ and

$$X(\hat{\eta}^{[0, T]}) = \max_{1 \leq i \leq k} \int_0^T \widetilde{W}_{N,i}(\varepsilon, H_t^l, \eta(t)) dt,$$

the contribution of the Glauber dynamics and the initial measure can be compared to the case $\beta = 0$ started from μ_α^* ,

$$\begin{aligned} \mathbb{E}_{\mu_N^{\lambda, \beta}}^{\lambda, \beta} \left(\max_{1 \leq l \leq k} \int_0^T \widetilde{W}_{N,i}(\varepsilon, H_t^l, \eta(t)) dt \right) \\ \leq K_0(T, \beta, \hat{\zeta}) + \frac{1}{N^2} \left(\log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \left[\exp \left(N^2 \max_{1 \leq l \leq k} \int_0^T \widetilde{W}_{N,i}(\varepsilon, H_t^l, \eta(t)) dt \right) \right] \right). \end{aligned}$$

1268 The max can be taken out of the log in the second term because for any finite family (u_l) ,

$$\exp \left(\max_l u_l \right) \leq \sum \exp u_l \quad \text{and} \quad \limsup_{N \rightarrow \infty} N^{-2} \log \left(\sum_l u_{l, N} \right) \leq \max_l \limsup_{N \rightarrow \infty} N^{-2} \log u_{N, l}.$$

Furthermore, we apply Lemma 5.2 to $\gamma = 1$, and $X_t = \widetilde{W}_{N,i}(\varepsilon, H_t, \eta)$, to obtain that

$$\begin{aligned} \frac{1}{N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda, 0} \left[\exp \left(N^2 \int_0^T \widetilde{W}_{N,i}(\varepsilon, H_t, \eta(t)) dt \right) \right] \\ \leq 2T\lambda^2 + \frac{1}{2} \int_0^T dt \sup_\varphi \left\{ 2\mathbb{E}_\alpha^* \left(\varphi \widetilde{W}_{N,i}(\varepsilon, H_t, \eta) \right) - D(\varphi) \right\}, \end{aligned}$$

1269 where the supremum is taken over all densities w.r.t. μ_α^* . Letting

$$K(T, \lambda, \beta, \hat{\zeta}) = K_0(T, \beta, \hat{\zeta}) + 2T\lambda^2,$$

1270 to prove Lemma 5.8 it is therefore sufficient to show that the second term on the right-hand side of the
1271 inequality above is non-positive in the limit $N \rightarrow \infty$. This will be implied by Lemma 5.9 below, since
1272 the time integral is now only applied to H . \square

1273 **Lemma 5.9.** — For any $H \in C^1(\mathbb{T}^2)$, and $\varepsilon > 0$,

$$\limsup_{N \rightarrow \infty} \sup_{\varphi} \left\{ 2\mathbb{E}_{\alpha}^* \left(\widetilde{W}_{N,i}(\varepsilon, H, \eta)\varphi \right) - D(\varphi) \right\} \leq 0,$$

1274 where the supremum is taken over the densities φ w.r.t the product measure μ_{α}^* .

1275 *Proof of Lemma 5.9.* — The proof of this Lemma follows the exact same steps as the treatment of equa-
 1276 tion (7.3), p.106 in [27], we do not detail it: since $\eta_{x_{k+1}} - \eta_{x_k}$ appearing in the expression of $\widetilde{W}_{N,i}(\varepsilon, H, \eta)$
 1277 can be rewritten $\eta_{x_{k+1}}(1 - \eta_{x_k}) - \eta_{x_k}(1 - \eta_{x_{k+1}})$, the proof of the Lemma is just a matter of performing
 1278 changes of variables $\widehat{\eta} \mapsto \widehat{\eta}^{x_k, x_{k+1}}$, and using the elementary inequality

$$ab(c - d) \leq a^2(c + d) + \frac{b^2}{2}(\sqrt{c} - \sqrt{d})^2,$$

1279 which holds for any positive c, d , to

$$a = H(x/N), \quad b = \eta_{x_{k+1}}(1 - \eta_{x_k}), \quad \eta_{x_k}(1 - \eta_{x_{k+1}}), \quad c = \sqrt{\varphi}(\widehat{\eta}^{x_k, x_{k+1}}), \quad \text{and} \quad d = \sqrt{\varphi}$$

1280 in the first term of $\widetilde{W}_{N,i}(\varepsilon, H, \eta)$. □

1281 Lemma 5.8 allows us to complete the proof of Theorem 5.7.

1282 *Proof of Theorem 5.7.* — Recall that we defined in Section 2.2 $\mathbb{P}_{\mu_N}^{\lambda, \beta}$, the measure on the space
 1283 $D([0, T], \Sigma_N)$ of the active exclusion process $\widehat{\eta}(s)$ started with the measure μ^N , and Q^N is the
 1284 measure on the corresponding measure space $\mathcal{M}^{[0, T]}$. Let us introduce

$$\varphi_{\delta}(u) = (2\delta)^{-2} \mathbb{1}_{[-\delta, \delta]^2}(u).$$

Since $\tau_x \rho_{\delta N} = \frac{(2\delta N)^2}{(2\delta N + 1)^2} < \pi_t$, $\varphi_{\delta}(x/N - \cdot) >$ for any weak limit point Q^* of (Q^N) , Lemma 5.8 yields

$$\limsup_{\delta \rightarrow 0} \mathbb{E}_{Q^*} \left(\max_{1 \leq l \leq k} \iint_{[0, T] \times \mathbb{T}^2} \frac{H_t^l(u)}{\varepsilon} \left(\langle \pi_t, \varphi_{\delta}(u + \varepsilon e_i - \cdot) \rangle - \langle \pi_t, \varphi_{\delta}(u - \cdot) \rangle \right) - H_t^l(u)^2 dudt \right) \leq K.$$

Since thanks to Lemma 5.6 any limit point Q^* of (Q^N) is concentrated on trajectories absolutely contin-
 1285 uous w.r.t. the Lebesgue measure on \mathbb{T}^2 , letting δ then ε go to 0, by dominated convergence, we obtain
 1286 that

$$\mathbb{E}_{Q^*} \left(\max_{1 \leq l \leq k} \iint_{[0, T] \times \mathbb{T}^2} [\partial_{u_i} H_t^l(u) \rho_t(u) - H_t^l(u)^2] dudt \right) \leq K,$$

1285 where $\rho_t(u)$ is the density of the measure $\int_{\mathbb{S}} \pi_t(du, d\theta)$ w.r.t. the Lebesgue measure on \mathbb{T}^2 . By monotone
 1286 convergence, and since the sequence (H_l) is dense in $C^{0,1}([0, T] \times \mathbb{T}^2)$, we therefore obtain

$$(5.18) \quad \mathbb{E}_{Q^*} \left(\sup_H \iint_{[0, T] \times \mathbb{T}^2} [\partial_{u_i} H_t(u) \rho_t(u) - H_t(u)^2] dudt \right) \leq K,$$

1287 where the supremum is taken over all functions $H \in C^{0,1}([0, T] \times \mathbb{T}^2)$. Given a limit point Q^* , let us
 1288 denote by \mathcal{E} the event on which the quantity inside parenthesis above is finite :

$$\mathcal{E} = \left\{ \sup_H \iint_{[0, T] \times \mathbb{T}^2} [\partial_{u_i} H_t(u) \rho_t(u) - H_t(u)^2] dudt < \infty \right\},$$

1289 and denote by ξ the elements of \mathcal{E} . Then, thanks to the L^1 bound we just obtained, we have that
 1290 $Q^*(\mathcal{E}) = 1$.

1291 Define on $C^{0,1}([0, T] \times \mathbb{T}^2)$ the linear operator

$$f_i(H) = \iint_{[0, T] \times \mathbb{T}^2} \partial_{u_i} H_t(u) \rho_t(u) dudt,$$

1292 then equation (5.18) yields that for any $\xi \in \mathcal{E}$, there exists a constant $K(\xi)$ such that for any positive
 1293 constant r , $r f_i(H) - r^2 \iint H^2 \leq K(\xi)$, i.e.

$$f_i(H) \leq \frac{1}{r} K(\xi) + r \iint H^2.$$

1294 Letting $r = \sqrt{K(\xi)/\iint H^2}$, and $C_0 = 2\sqrt{K(\xi)}$, we obtain that for any function $H \in C^{0,1}([0, T] \times \mathbb{T}^2)$,

$$f_i(H) \leq C_0(\xi) \left(\iint_{[0, T] \times \mathbb{T}^2} H^2 \right)^{1/2}.$$

1295 The functional f_i can then be extended to a bounded linear functional in $L^2([0, T] \times \mathbb{T}^2)$. The conclusion
1296 then follows from Riesz's representation theorem. \square

1297

6. Non-gradient estimates

1298 **6.1. Replacement of the symmetric current by a macroscopic gradient.** — *In this section, we*
1299 *focus on the complete exclusion process, and replace the current j_i^ω by a quantity of the form $\tau_{e_i}h - h + \mathcal{L}f$,*
1300 *with f a function of the configuration with infinite support. We then show that the perturbation $\mathcal{L}f$ is*
1301 *of the same order as the weakly asymmetric contribution, and they both contribute to the drift term of*
1302 *equation (2.11). To obtain the non gradient estimates, we use the formalism developed in [27] rather than*
1303 *that of [35]. This changes the proof substantially, with the upside that the orders in N , as well as the*
1304 *studied quantities, are clearly identified at any given point of the proof.*

1305 One of the challenges in proving the non-gradient hydrodynamic limit is to replace the local particle
1306 currents j_i^ω by the gradient of a function of the empirical measure. Recall that we already defined in
1307 equation (2.20) the empirical angular density $\widehat{\rho}_l \in \mathcal{M}_1(\mathbb{S})$,

$$\widehat{\rho}_l = \frac{1}{(2l+1)^2} \sum_{x \in B_l} \eta_x \delta_{\theta_x},$$

1308 and we denote by ρ_l the empirical density

$$\rho_l = \frac{1}{(2l+1)^2} \sum_{x \in B_l} \eta_x = \widehat{\rho}_l(\mathbb{S}).$$

1309 Let

$$\rho_l^\omega = \frac{1}{(2l+1)^2} \sum_{x \in B_l} \eta_x^\omega,$$

1310 be the average of η^ω over a box of side $2l+1$. Finally, for any function φ on Σ_N , recall that δ_i is the
1311 discrete derivative

$$\delta_i \varphi = \tau_{e_i} \varphi - \varphi$$

1312 (for example, $\delta_i \eta_0^\omega = \eta_{e_i}^\omega - \eta_0^\omega$).

1313 The usual strategy in the proof of the non-gradient hydrodynamic limit is to show that for some
1314 coefficients $\mathfrak{d}^\omega, \mathfrak{d} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^+$,

$$j_i^\omega + \mathfrak{d}^\omega(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) \delta_i \rho_{\varepsilon N}^\omega + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) \delta_i \rho_{\varepsilon N}$$

1315 vanishes as $N \rightarrow \infty$. More precisely, the quantity above is in the range of the generator \mathcal{L} , which is
1316 usually sufficient when the functions of the form $\mathcal{L}f$ are negligible. In our case, however, due to the
1317 addition of a weak drift, the usual martingale estimate does not yield that $\mathcal{L}f$ is negligible, but that
1318 $\mathcal{L}^D f = (\mathcal{L} + N^{-1} \mathcal{L}^{\text{WA}})f$ is negligible, therefore this perturbation can be integrated to the drift part, which
1319 is done in Section 6.7.

1320 For this replacement, we will need further notations similar to the ones introduced in Section 4.1. In
1321 our case, the diffusion coefficient $\mathfrak{d}^\omega(\rho, \rho^\omega)$ is in fact the self-diffusion coefficient $d_s(\rho)$, therefore we will
1322 from now on simply write $d_s(\rho)$ for the diffusion coefficient relative to ρ^ω . Note that it depends on the
1323 configuration only through the empirical density, and not on the particle angles. For any positive integer
1324 l , and any cylinder function f , let us thus denote

$$\mathcal{V}_i^{f, \varepsilon N}(\widehat{\eta}) = j_i^\omega + d_s(\rho_{\varepsilon N}) \delta_i \rho_{\varepsilon N}^\omega + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) \delta_i \rho_{\varepsilon N} - \mathcal{L}f,$$

1325 where $\mathfrak{d} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^+$ is the diffusion coefficient given in (1.3).

1326 We introduce for any smooth function $G \in C^2(\mathbb{T}^2)$

$$(6.1) \quad X_{i,N}^{f,\varepsilon N}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{V}_i^{f,\varepsilon N}.$$

1327 Our goal throughout this section is to prove that under the measure of our process, $X_{i,N}^{f,\varepsilon N}(G, \hat{\eta})$ vanishes
1328 for any smooth function G , i.e. that the microscopic currents can be replaced by a macroscopic average
1329 of the gradients up to a perturbation $\mathcal{L}f$ that will be dealt with later on.

1330 The sum contains N^2 terms, and the normalization is only $1/N$, therefore an order N has to be gained,
1331 and this is the major difficulty of the non-gradient dynamics. To prove this statement, we decompose
1332 $X_{i,N}^{f,\varepsilon N}(G, \hat{\eta})$ into distinct vanishing parts. We already introduced in equation (3.19) the set

$$E_{p,x} = \left\{ \sum_{|y-x| \leq p} \eta_y \leq |B_p| - 2 \right\},$$

1333 such that at least two sites are empty in a vicinity of x of size p . The cutoff functions $\mathbb{1}_{E_{p,x}}$ are crucial
1334 in order to control the local variations of the measure of the process with the Dirichlet form.

1335 We set for any integer l

$$(6.2) \quad \rho_l^{\omega,p} = \frac{1}{(2l+1)^2} \sum_{x \in B_l} \eta_x^\omega \mathbb{1}_{E_{p,x}} \quad \text{and} \quad \bar{\rho}_l^{\omega,p} = \rho_l^\omega - \rho_l^{\omega,p} = \frac{1}{(2l+1)^2} \sum_{x \in B_l} \eta_x^\omega \mathbb{1}_{E_{p,x}^c},$$

1336 where $E_{p,x}^c$ is the complementary event of $E_{p,x}$.

1337 We are now ready to split $X_{i,N}^{f,\varepsilon N}$ into 4 vanishing parts. Let us denote by

$$\mathcal{W}_1 = \mathcal{W}_{i,1}^{f,l}(\hat{\eta}) = j_i^\omega - \langle j_i^\omega \rangle_0' - \left(\mathcal{L}f - \langle \mathcal{L}f \rangle_0^{l-s_f} \right),$$

1338 the difference between $j_i^\omega - \mathcal{L}f$ and their local average, and by

$$\mathcal{W}_2 = \mathcal{W}_{i,2}^{\varepsilon N,p}(\hat{\eta}) = d_s(\rho_{\varepsilon N}) \delta_i \bar{\rho}_{\varepsilon N}^{\omega,p}$$

1339 the mesoscopic contributions of full clusters, where $\bar{\rho}_{\varepsilon N}^{\omega,p}$ was defined in equation (6.2) above. Let us also
1340 introduce

$$\mathcal{W}_3 = \mathcal{W}_{i,3}^{l,\varepsilon N,p}(\hat{\eta}) = d_s(\rho_{\varepsilon N}) \delta_i \rho_{\varepsilon N}^{\omega,p} - d_s(\rho_l) \delta_i \rho_{l_p}^{\omega,p} + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) \delta_i \rho_{\varepsilon N} - \mathfrak{d}(\rho_l, \rho_l^\omega) \delta_i \rho_l,$$

1341 where $l_p = l - p - 1$ and $l' = l - 1$, which is the difference between the cutoff microscopic and macroscopic
1342 gradients. Note that the cutoff functions are not needed for the total density ρ , because the gradients
1343 will vanish on full configurations. Finally, we set

$$(6.3) \quad \mathcal{W}_4 = \mathcal{W}_{i,4}^{f,l,p}(\hat{\eta}) = \langle j_i^\omega \rangle_0' + d_s(\rho_l) \delta_i \rho_{l_p}^{\omega,p} + \mathfrak{d}(\rho_l, \rho_l^\omega) \delta_i \rho_l - \langle \mathcal{L}f \rangle_0^{l-s_f},$$

1344 the microscopic difference between currents and gradients, taking into consideration the perturbation $\mathcal{L}f$.

1345 For any smooth function $G \in C^2(\mathbb{T}^2)$, we also introduce

$$Y_1 = Y_{i,1}^{f,l}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}_1, \quad Y_2 = Y_{i,2}^{\varepsilon N,p}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}_2,$$

$$1346 \quad Y_3 = Y_{i,3}^{l,\varepsilon N,p}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}_3 \quad \text{and} \quad Y_4 = Y_{i,4}^{f,l,p}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}_4.$$

1347 By construction,

$$X_{i,N}^{f,\varepsilon N}(G, \hat{\eta}) = \sum_{k=1}^4 Y_k(G, \hat{\eta}).$$

1348 We can now state the main result of this section.

1349 **Theorem 6.1.** — *Let G be a smooth function in $C^{1,2}([0, T] \times \mathbb{T}^2)$, $T > 0$, and $i \in \{1, 2\}$. For any
1350 cylinder function f ,*

$$(6.4) \quad \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N^{\lambda, \beta}} \left(\left| \int_0^T Y_{i,1}^{f,l}(G_t, \hat{\eta}(t)) dt \right| \right) = 0.$$

1351 *Furthermore,*

$$(6.5) \quad \lim_{p \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\left| \int_0^T Y_{i,2}^{\varepsilon N, p}(G_t, \hat{\eta}(t)) dt \right| \right) = 0.$$

1352 *For any integer $p > 1$,*

$$(6.6) \quad \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\left| \int_0^T Y_{i,3}^{l, \varepsilon N, p}(G_t, \hat{\eta}(t)) dt \right| \right) = 0.$$

1353 *Finally,*

$$(6.7) \quad \inf_f \lim_{p \rightarrow \infty} \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\left| \int_0^T Y_{i,4}^{f, l, p}(G_t, \hat{\eta}(t)) dt \right| \right) = 0,$$

1354 *where the infimum in f is taken over the set \mathcal{C} of cylinder functions.*

1355 The core of this section is dedicated to proving these four estimates. The proof of equation (6.4) is
1356 immediate and is sketched in Section 6.2.

1357 Equation (6.5) is quite delicate, and requires both the control on full clusters derived in equation (3.20)
1358 and the energy estimate (5.17). It is proved in Section 6.3, in which the main challenge, as in the control
1359 of full clusters, is to carry out the macroscopic estimate (5.17) in a microscopic setup.

1360 The proof of equation (6.6) is given in Section 6.4. This limit is the non-gradient counterpart of the
1361 two-block estimate stated in Lemma 4.4. It follows closely the replacement of local gradients by their
1362 macroscopic counterparts performed in Lemma 3.1, p.156 of [27], but needs some technical adaptation
1363 due to the presence of the cutoff functions.

1364 The last limit (6.7) requires the tools developed by Varadhan and Quastel [48] [35] for the hydrody-
1365 namic limit for non-gradient systems, and therefore requires more work. It is the non-gradient counterpart
1366 of the one-block estimate of Lemma 4.3. However, if the latter was essentially a consequence of the law
1367 of large numbers, (6.7) is analogous to the central limit theorem, where the gradient term plays the role
1368 of $-\mathbb{E}(j_i^\omega)$. The limit (6.7) is the focus of Sections 6.5-6.6.

1369 Finally, Section 6.7, and in particular Lemma 6.21, is dedicated to the integration of the contribution
1370 $\mathcal{L}f$ to the drift part of the scaling limit.

1371 These four estimates are sufficient to allow the replacement of currents by macroscopic averages of
1372 gradients, up to a perturbation $\mathcal{L}f$.

1373 **Corollary 6.2.** — *Let G be a smooth function in $C^{1,2}([0, T] \times \mathbb{T}^2)$, and $T > 0$, and consider $X_{i,N}^{f, \varepsilon N}$*
1374 *introduced in (6.1). Then for $i \in \{1, 2\}$*

$$(6.8) \quad \inf_f \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left[\left| \int_0^T X_{i,N}^{f, \varepsilon N}(G_t, \hat{\eta}(t)) dt \right| \right] = 0.$$

1375 *Proof of Corollary 6.2.* — Since

$$X_{i,N}^{f, \varepsilon N}(G, \hat{\eta}) = \sum_{k=1}^4 Y_k(G, \hat{\eta}),$$

1376 this Corollary follows immediately from the triangular inequality, and Theorem 6.1 above, taking the
1377 limits $N \rightarrow \infty$, then $\varepsilon \rightarrow 0$ then $l \rightarrow \infty$, then $p \rightarrow \infty$, and finally the infimums over the local functions
1378 f . □

6.2. Replacement of the currents and $\mathcal{L}f$ by their local average. — In this paragraph, we prove
equation (6.4), i.e. that for any $i = 1, 2$, any function $G \in C^{1,2}([0, T] \times \mathbb{T}^2)$, and any cylinder function f ,

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left(\left| \int_0^T Y_1(G_t, \hat{\eta}(t)) dt \right| \right) = 0.$$

1379 We set

$$G^{l,N}(x/N) = \frac{1}{(2l+1)^2} \sum_{y \in \mathbb{T}_N^2, |y-x| \leq l} G(y/N),$$

an integration by parts yields that, shortening $l' = l - 1$

$$\begin{aligned} \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \left(j_{x,x+e_i}^\omega - \frac{1}{(2l'+1)^2} \sum_{|y-x| \leq l'} j_{y,y+e_i}^\omega \right) \\ = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \left(G(x/N) - G^{l',N}(x/N) \right) j_{x,x+e_i}^\omega \leq \frac{C(G)l^2}{N}. \end{aligned}$$

since the difference $G(x/N) - G^{l',N}(x/N)$ is a discrete Laplacian, and is therefore of order l^2/N^2 , and the currents $j_{x,x+e_i}^\omega$ are bounded. By the same reasoning, letting $l_f = l - s_f$, we obtain a similar bound on the difference

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \left(\tau_x \mathcal{L}f - \frac{1}{(2l_f+1)^2} \sum_{|y-x| \leq l_f} \tau_y \mathcal{L}f \right) \leq \frac{C'(G, f)l^2}{N},$$

1380 since $\mathcal{L}f$ is a bounded function (this last statement comes from the fact that f is, and depends only on
1381 a finite number of sites). These two bounds finally yield that for some constant $K = C(G) + C'(G, f)$,

$$|Y_1(G, \hat{\eta})| \leq \frac{Kl^2}{N},$$

1382 which immediately yields equation (6.4) for any cylinder function f .

1383 **6.3. Estimation of the gradients on full clusters.** — We now prove that equation (6.5) holds. Our
1384 goal is to bound $Y_{i,2}^{\varepsilon N, p}(G, \hat{\eta}(s))$ thanks to the control of full clusters functions obtained in (3.20), and to
1385 the energy estimate (5.17). For the sake of clarity, we drop the various dependencies, and simply write

$$Y_2 = Y_{i,2}^{\varepsilon N, p}.$$

By definition of Y_2 and $\bar{\rho}_{\varepsilon N}^{\omega, p}$ (6.2),

$$\begin{aligned} Y_2(G, \hat{\eta}) &= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x (d_s(\rho_{\varepsilon N}) \delta_i \bar{\rho}_{\varepsilon N}^{\omega, p}) \\ &= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \left(d_s(\rho_{\varepsilon N}) \left[\frac{1}{(2\varepsilon N + 1)^2} \sum_{y \in B_{\varepsilon N}(e_i)} \eta_y^\omega \mathbb{1}_{E_{p,y}^c} - \frac{1}{(2\varepsilon N + 1)^2} \sum_{y \in B_{\varepsilon N}} \eta_y^\omega \mathbb{1}_{E_{p,y}^c} \right] \right), \end{aligned}$$

1386 and we can rewrite it by summation by parts as

(6.9)

$$Y_2(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \eta_x^\omega \mathbb{1}_{E_{p,x}^c} \frac{1}{(2\varepsilon N + 1)^2} \left(\sum_{y \in B_{\varepsilon N}(x-e_i)} G(y/N) \tau_y d_s(\rho_{\varepsilon N}) - \sum_{y \in B_{\varepsilon N}(x)} G(y/N) \tau_y d_s(\rho_{\varepsilon N}) \right).$$

1387 Most of the terms in the parenthesis above cancel out, since the boxes $B_{\varepsilon N}(x - e_i)$ and $B_{\varepsilon N}(x)$ overlap
1388 except on the two sides (cf. Figure 4).

1389 For any $k \in \llbracket -\varepsilon N, \varepsilon N \rrbracket$, we let according to Figure 4

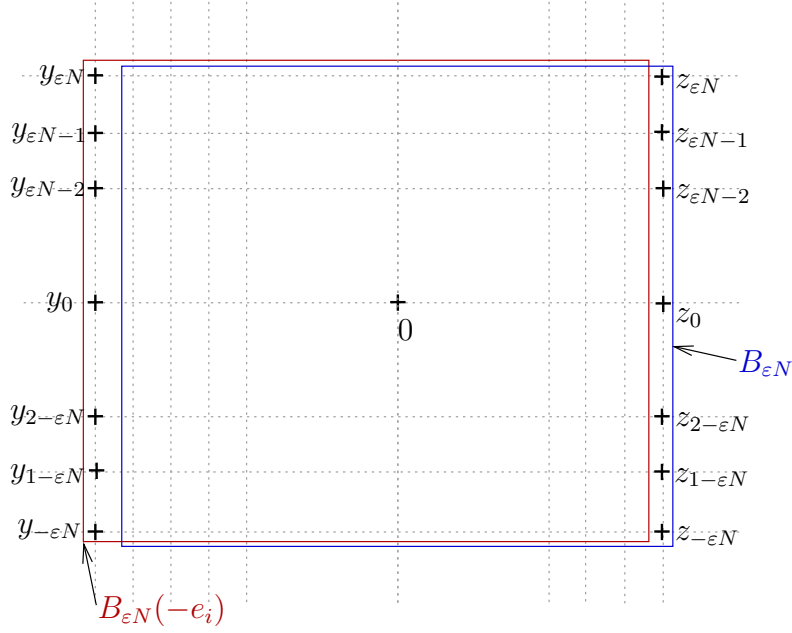
$$y_k = -(\varepsilon N + 1)e_i + ke_{i'} \quad \text{and} \quad z_k = \varepsilon N e_i + ke_{i'},$$

1390 where $i' \neq i$ is the second direction on the torus, which are defined so that $B_{\varepsilon N}(-e_i) \setminus B_{\varepsilon N} =$
1391 $\{y_{-\varepsilon N}, \dots, y_{\varepsilon N}\}$ and $B_{\varepsilon N} \setminus B_{\varepsilon N}(-e_i) = \{z_{-\varepsilon N}, \dots, z_{\varepsilon N}\}$.

We thus obtain from (6.9)

$$(6.10) \quad Y_2(G, \hat{\eta}(s))$$

$$= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \eta_x^\omega \mathbb{1}_{E_{p,x}^c} \frac{1}{(2\varepsilon N + 1)^2} \left(\sum_{k=-\varepsilon N}^{\varepsilon N} G\left(\frac{x+y_k}{N}\right) d_s(\tau_{x+y_k} \rho_{\varepsilon N}) - G\left(\frac{x+z_k}{N}\right) d_s(\tau_{x+z_k} \rho_{\varepsilon N}) \right).$$


 FIGURE 4. Definition of the y_k 's and z_k 's.

1392 We can now rewrite the quantity inside the parenthesis as the sum over k of

$$\left[G\left(\frac{x+y_k}{N}\right) - G\left(\frac{x+z_k}{N}\right) \right] d_s(\tau_{x+y_k}\rho_{\varepsilon N}) - G\left(\frac{x+z_k}{N}\right) [d_s(\tau_{x+z_k}\rho_{\varepsilon N}) - d_s(\tau_{x+y_k}\rho_{\varepsilon N})].$$

1393 Since y_k and z_k are distant of $2\varepsilon N + 1$, the first term in the decomposition above can be bounded in
 1394 absolute value uniformly in x and k by $(2\varepsilon N + 1) \|\partial_{u_i} G\|_{\infty} / N$. Let $C(G, \omega) = \|\partial_{u_i} G\|_{\infty} \|\omega\|_{\infty} \|d_s\|_{\infty}$,
 1395 the corresponding contribution in (6.10) is

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \underbrace{\eta_x^{\omega}}_{\leq \|\omega\|_{\infty}} \mathbb{1}_{E_{p,x}^c} \frac{1}{(2\varepsilon N + 1)^2} \left(\sum_{k=-\varepsilon N}^{\varepsilon N} \underbrace{\left[G\left(\frac{x+y_k}{N}\right) - G\left(\frac{x+z_k}{N}\right) \right]}_{\leq (2\varepsilon N + 1) \|\partial_{u_i} G\|_{\infty} / N} \underbrace{d_s(\tau_{x+y_k}\rho_{\varepsilon N})}_{\leq \|d_s\|_{\infty}} \right),$$

1396 and can therefore be bounded by

$$\frac{C(G, \omega)}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c}.$$

1397 Furthermore, since d_s is C^{∞} on $[0, 1]$, it is Lipschitz-continuous on $[0, 1]$ with Lipschitz constant c , we
 1398 let $C'(G, \omega) = c \|G\|_{\infty} \|\omega\|_{\infty} / 2$. We can now write thanks to the previous considerations that

$$|Y_2| \leq \frac{C(G, \omega)}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c} + \frac{C'(G, \omega)}{N^2} \sum_{x \in \mathbb{T}_N^2} \frac{1}{(2\varepsilon N + 1)} \sum_{k=-\varepsilon N}^{\varepsilon N} \mathbb{1}_{E_{p,x}^c} \frac{|\tau_{x+y_k}\rho_{\varepsilon N} - \tau_{x+z_k}\rho_{\varepsilon N}|}{\varepsilon}.$$

1399 For any positive γ , we have the elementary bound

$$\mathbb{1}_{E_{p,x}^c} \frac{|\tau_{x+y_k}\rho_{\varepsilon N} - \tau_{x+z_k}\rho_{\varepsilon N}|}{\varepsilon} \leq \gamma \mathbb{1}_{E_{p,x}^c} + \frac{1}{\gamma} \frac{(\tau_{x+y_k}\rho_{\varepsilon N} - \tau_{x+z_k}\rho_{\varepsilon N})^2}{\varepsilon^2},$$

and finally, for any positive γ ,

$$\begin{aligned} |Y_2| &\leq \frac{C + \gamma C'}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c} + \frac{C'}{\gamma N^2} \sum_{x \in \mathbb{T}_N^2} \frac{1}{(2\varepsilon N + 1)} \sum_{k=-\varepsilon N}^{\varepsilon N} \frac{(\tau_{x-(\varepsilon N+1)e_i}\rho_{\varepsilon N} - \tau_{x+\varepsilon N e_i}\rho_{\varepsilon N})^2}{\varepsilon^2} \\ (6.11) \quad &= \frac{C + \gamma C'}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{1}_{E_{p,x}^c} + \frac{C'}{\gamma N^2} \sum_{x \in \mathbb{T}_N^2} \frac{(\tau_{x-(\varepsilon N+1)e_i}\rho_{\varepsilon N} - \tau_{x+\varepsilon N e_i}\rho_{\varepsilon N})^2}{\varepsilon^2}. \end{aligned}$$

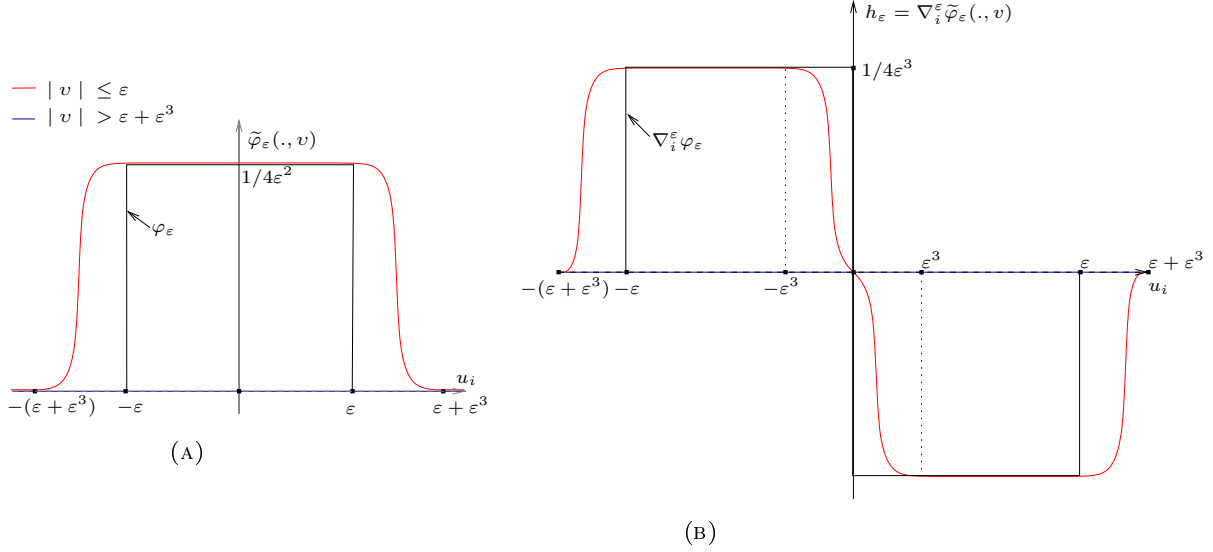


FIGURE 5. (a) Representations of $\tilde{\varphi}_\varepsilon(\cdot, v)$ depending on the value of v .
 (b) Representation of $h_\varepsilon(\cdot, v) = \nabla_i^\varepsilon \tilde{\varphi}_\varepsilon(\cdot, v)$ depending on the value of v .

Recall that we want to prove (6.5), i.e.

$$\lim_{p \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^{\lambda, \beta} \left(\int_0^T |Y_2(G_t, \hat{\eta}(t))| dt \right) = 0.$$

1400 The contribution of the first term in the bound for $|Y_2|$ in equation (6.11) vanishes for any γ as N then
 1401 p goes to ∞ , thanks to Proposition 3.12.

1402 Furthermore, we can replace $\tau_{x - (\varepsilon N + 1)e_i} \rho_{\varepsilon N}$ by $\tau_{x - \varepsilon N e_i} \rho_{\varepsilon N}$ in (6.11) since the difference between these
 1403 two quantities is of order $1/N$ and vanishes in the limit $N \rightarrow \infty$. This replacement allows us to work
 1404 only with quantities that can be expressed in terms of the empirical measure of the process. Equation
 1405 (6.5) therefore holds according to Lemma 6.3 below, letting γ go to ∞ after $N \rightarrow \infty$ then $\varepsilon \rightarrow 0$ then
 1406 $p \rightarrow \infty$. ■

Lemma 6.3. — *There exists a positive constant K such that*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^{\lambda, \beta} \left(\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \frac{(\tau_{x - \varepsilon N e_i} \rho_{\varepsilon N}(t) - \tau_{x + \varepsilon N e_i} \rho_{\varepsilon N}(t))^2}{\varepsilon^2} dt \right) \leq K.$$

1407 *Proof of Lemma 6.3.* — This Lemma states that the difference of macroscopic densities between two
 1408 points distant from 2ε is also of order ε , and is a consequence of the energy estimate (5.17). We are
 1409 going to prove this macroscopic estimate in the topological setup of the space of càdlàg trajectories of
 1410 measures on $\mathbb{T}^2 \times \mathbb{S}$. Recall from Section 5.2 that $\mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$ is the space of non-negative measures on
 1411 the continuous configuration space,

$$\mathcal{M}^{[0, T]} = D([0, T], \mathcal{M}(\mathbb{T}^2 \times \mathbb{S}))$$

1412 is the space of right-continuous, left-limit trajectories on the set of measures on $\mathbb{T}^2 \times \mathbb{S}$, and that Q^N
 1413 is the distribution on $\mathcal{M}^{[0, T]}$ of the process's empirical measure π^N . We have proved in Proposition 5.4
 1414 that the sequence $(Q^N)_{N \in \mathbb{N}}$ is relatively compact for the weak topology. Let $\Lambda_\varepsilon = [\varepsilon, \varepsilon]^2 \subset \mathbb{T}^2$ be the
 1415 cube of size ε , and $(\varphi_\varepsilon)_{\varepsilon > 0}$ be a family of localizing functions on \mathbb{T}^2

$$\varphi_\varepsilon(\cdot) = \frac{1}{(2\varepsilon)^2} \mathbb{1}_{\Lambda_\varepsilon}(\cdot),$$

1416 we then have

$$\tau_x \rho_{\varepsilon N}(t) = \frac{(2\varepsilon N)^2}{(2\varepsilon N + 1)^2} \langle \pi_t^N, \varphi_\varepsilon(\cdot + x/N) \rangle.$$

1417 We define the *mesoscopic gradient*

$$\nabla_i^\varepsilon \varphi(\cdot) = \varepsilon^{-1}(\varphi(\cdot - \varepsilon e_i) - \varphi(\cdot + \varepsilon e_i)),$$

represented in Figure 5b. Note that $\nabla_i^\varepsilon \varphi_\varepsilon$ is at most of order ε^{-3} since φ_ε is of order ε^{-2} . We can rewrite the left-hand side in Lemma 6.3 as

$$(6.12) \quad \mathbb{E}_{Q^N} \left(\int_0^T \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \langle \pi_t, \nabla_i^\varepsilon \varphi_\varepsilon(\cdot + x/N) \rangle^2 dt \right) + o_N(1).$$

Furthermore, since for any two sites $x, x' \in \mathbb{T}^2$ distant from less than $1/N$,

$$| \langle \pi_t, \nabla_i^\varepsilon \varphi_\varepsilon(\cdot + x/N) \rangle - \langle \pi_t, \nabla_i^\varepsilon \varphi_\varepsilon(\cdot + x'/N) \rangle | \leq C(\varepsilon) \frac{1}{N},$$

1418 we can replace the sum above by the integral over the continuous torus.

1419 However, regarding the weak topology on $\mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$, it will be convenient later on to consider smooth
1420 functions instead of φ_ε . We therefore introduce for any ε a function $\tilde{\varphi}_\varepsilon$, represented in Figure 5a verifying

1421 — $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon$ on Λ_ε and on $\mathbb{T}^2 \setminus \Lambda_{\varepsilon+\varepsilon^3}$.

1422 — $\|\tilde{\varphi}_\varepsilon\|_\infty = \|\varphi_\varepsilon\|_\infty$.

1423 — $\tilde{\varphi}_\varepsilon$ is in $C^1(\mathbb{T}^2)$.

Since $\tilde{\varphi}_\varepsilon$ and φ_ε coincide everywhere except on $\Lambda_{\varepsilon+\varepsilon^3} \setminus \Lambda_\varepsilon$, and since $\|\tilde{\varphi}_\varepsilon\|_\infty = (2\varepsilon)^{-2}$ we can write for any $x \in \mathbb{T}_N^2$

$$\begin{aligned} | \langle \pi_t^N, \varphi_\varepsilon(\cdot + x/N) \rangle - \langle \pi_t^N, \tilde{\varphi}_\varepsilon(\cdot + x/N) \rangle | &\leq \frac{1}{(2\varepsilon)^2} \underbrace{\langle \pi_t^N, \mathbb{1}_{\Lambda_{\varepsilon+\varepsilon^3} \setminus \Lambda_\varepsilon}(\cdot + x/N) \rangle}_{\leq 4\varepsilon \times \varepsilon^3} \\ &\leq C\varepsilon^2, \end{aligned}$$

1424 for some positive constant C . This bound immediately yields

$$| \langle \pi_t^N, \nabla_i^\varepsilon \varphi_\varepsilon(\cdot + x/N) \rangle - \langle \pi_t^N, \nabla_i^\varepsilon \tilde{\varphi}_\varepsilon(\cdot + x/N) \rangle | \leq C\varepsilon,$$

1425 which allows us to replace in equation (6.12), in the limit $N \rightarrow \infty$ then $\varepsilon \rightarrow 0$, φ_ε by $\tilde{\varphi}_\varepsilon$.

1426 To prove Lemma 6.3 it is therefore sufficient to prove that

$$(6.13) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{Q^N} \left(\iint_{[0,T] \times \mathbb{T}^2} \langle \pi_t, h_\varepsilon(\cdot + u) \rangle^2 dudt \right) \leq K,$$

1427 where $h_\varepsilon = \nabla_i^\varepsilon \tilde{\varphi}_\varepsilon$, is a continuous bounded function, represented in Figure 5b. Let us denote by Π the
1428 subset of $\mathcal{M}^{[0,T]}$

$$\Pi = \left\{ \pi \in \mathcal{M}^{[0,T]} \mid \sup_{t \in [0,T]} \langle \pi_t, \mathbb{1} \rangle \leq 1 \right\}$$

1429 of trajectories with mass less than one at all times, which is compact w.r.t Skorohod's topology introduced
1430 in Section 5.2.

1431 Consider a weakly convergent subsequence $Q_{N_k} \rightarrow Q^*$, in order to substitute Q^* to Q^N in the limit
1432 above, we want to prove that for any fixed $\varepsilon > 0$, the application

$$I_\varepsilon : \pi \mapsto \iint_{[0,T] \times \mathbb{T}^2} \langle \pi_t, h_\varepsilon(\cdot + u) \rangle^2 dudt$$

1433 is bounded, and continuous on Π w.r.t. Skorohod's topology.

1434 Note that this application is bounded on Π by construction, we now prove the following Lemma.

1435 **Lemma 6.4.** — Fix $\varepsilon > 0$, the application I_ε is continuous on (Π, d) , where d is the Skorohod metric
1436 defined in equation (B.3).

1437 *Proof of Lemma 6.4.* — For any two trajectories π and π' in Π , and some continuous strictly increasing
1438 function κ from $[0, T]$ into itself, such that $\kappa_0 = 0$ and $\kappa_T = T$, we can write

$$I_\varepsilon(\pi) - I_\varepsilon(\pi') = \iint_{[0,T] \times \mathbb{T}^2} du \langle \pi'_t + \pi_t, h_\varepsilon(\cdot + u) \rangle \langle \pi'_t - \pi_{\kappa_t} + \pi_{\kappa_t} - \pi_t, h_\varepsilon(\cdot + u) \rangle dt.$$

The first factor $\langle \pi'_t + \pi_t, h_\varepsilon(\cdot + u) \rangle$ can be crudely controlled by $2\|h_\varepsilon\|_\infty$, which yields

(6.14)

$$|I_\varepsilon(\pi) - I_\varepsilon(\pi')| \leq 2\|h_\varepsilon\|_\infty \iint_{[0,T] \times \mathbb{T}^2} |\langle \pi'_t - \pi_{\kappa_t}, h_\varepsilon(\cdot + u) \rangle + \langle \pi_{\kappa_t} - \pi_t, h_\varepsilon(\cdot + u) \rangle| \, dudt.$$

1439 Note that by definition of $\|\kappa\|$, one easily gets that for any $t \in [0, T]$, $|t - \kappa_t| \leq T(e^{\|\kappa\|} - 1)$,
 1440 therefore, $\kappa_t \rightarrow t$ uniformly on $[0, T]$ as $\|\kappa\| \rightarrow 0$. Let us fix $\pi \in \Pi$, and assume that $d(\pi, \pi^n) \rightarrow 0$
 1441 for some sequence of trajectories $(\pi^n)_n \in \Pi^{\mathbb{N}}$, there exists a sequence $(\kappa^n)_{n \in \mathbb{N}}$ such that $\|\kappa^n\| \rightarrow 0$
 1442 and $\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \delta(\pi_t^n, \pi_{\kappa_t^n}) = 0$. This last statement yields in particular that for any $t \in [0, T]$,
 1443 $\delta(\pi_t^n, \pi_{\kappa_t^n}) \rightarrow 0$, therefore for any $t \in [0, T]$, and for any $u \in \mathbb{T}^2$,

$$\lim_{n \rightarrow \infty} \langle \pi_t^n - \pi_{\kappa_t^n}, h_\varepsilon(\cdot + u) \rangle = 0,$$

1444 since $h_\varepsilon(\cdot + u)$ is a continuous bounded function, and δ is a metric of the weak convergence. Furthermore,
 1445 since κ_t^n converges uniformly towards t on $[0, T]$ and since $t \rightarrow \pi_t$ is weakly continuous almost everywhere
 1446 on $[0, T]$ by definition of $\mathcal{M}^{[0, T]}$, we also have that for almost every $(t, u) \in [0, T] \times \mathbb{T}^2$,

$$\lim_{n \rightarrow \infty} \langle \pi_{\kappa_t^n} - \pi_t, h_\varepsilon(\cdot + u) \rangle = 0.$$

Since π and the π^n 's are in Π , both of these quantities are crudely bounded in absolute value by $2\|h_\varepsilon\|_\infty$, which is naturally integrable on $[0, T] \times \mathbb{T}^2$. One finally obtains by dominated convergence, from (6.14) applied to $\pi' = \pi^n$ and $\kappa = \kappa^n$, that

$$|I_\varepsilon(\pi) - I_\varepsilon(\pi^n)| \xrightarrow{n \rightarrow \infty} 0.$$

1447 Lemma 6.4 is complete. \square

We have now proved that the application I_ε is continuous for any fixed ε , therefore the left-hand side of (6.13) is less than

$$\limsup_{\varepsilon \rightarrow 0} \sup_{Q^*} \mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} du \langle \pi_t, h_\varepsilon(\cdot + u) \rangle^2 dt \right),$$

1448 where the supremum is taken over all limit points Q^* of the sequence Q^N . Since by definition $h_\varepsilon =$
 1449 $\nabla_i^\varepsilon \tilde{\varphi}_\varepsilon$ does not depend on θ , we drop the dependency of π on θ and consider simply for any $u \in \mathbb{T}^2$,
 1450 $\rho(t, u) = \int_{\mathbb{S}} \hat{\rho}_t(u, d\theta)$, where $\hat{\rho}_t(u, d\theta)$ is the density of $\pi_t(\cdot, d\theta)$ w.r.t. the Lebesgue measure \mathbb{T}^2 , which
 1451 exists Q^* -a.s. according to Lemma 5.6. We can write

(6.15)

$$\mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} du \langle \pi_t, h_\varepsilon(\cdot + u) \rangle^2 dt \right) = \mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} \left(\int_{v \in \mathbb{T}^2} \rho(t, v) \nabla_i^\varepsilon \tilde{\varphi}_\varepsilon(v + u) dv \right)^2 dudt \right).$$

1452 We can now express $\nabla_i^\varepsilon \tilde{\varphi}_\varepsilon$ as a gradient, by writing

$$\nabla_i^\varepsilon \tilde{\varphi}_\varepsilon(u) = \partial_{u_i} \int_{-1/2}^{u_i} \nabla_{i'}^\varepsilon \tilde{\varphi}_\varepsilon(v e_i + u_{i'} e_{i'}) dv = \partial_{u_i} \Phi_{\varepsilon, i},$$

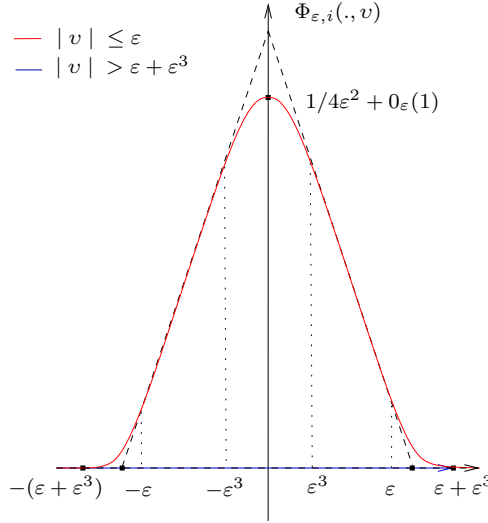
1453 where $i' \neq i$ still denotes the second direction on the torus.

1454 Furthermore, $\Phi_{\varepsilon, i}$, represented in Figure 6, is in $C^2(\mathbb{T}^2)$ because $\tilde{\varphi}_\varepsilon$ is C^1 , and the various integrals
 1455 can be freely swapped since all quantities are bounded at any fixed ε . Since Q^* -a.s. $\rho \in W^{1,2}([0, T] \times \mathbb{T}^2)$
 1456 according to Theorem 5.7, the right-hand side in equation (6.15) is therefore equal to

$$(6.16) \quad \mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} \left(\int_{v \in \mathbb{T}^2} \Phi_{\varepsilon, i}(v + u) \partial_{u_i} \rho(t, v) dv \right)^2 dudt \right).$$

In order to conclude, we adapt the proof of Young's Inequality, and apply Cauchy-Schwarz inequality to $f = (\Phi_{\varepsilon, i}(v + u))^{1/2}$ and $g = (\Phi_{\varepsilon, i}(v + u))^{1/2} \partial_{u_i} \rho(t, v)$, to finally obtain that

$$\mathbb{E}_{Q^*} \left(\iint_{[0, T] \times \mathbb{T}^2} du \langle \pi_t, h_\varepsilon(\cdot + u) \rangle^2 dt \right)$$


 FIGURE 6. Representation of $\Phi_{\varepsilon,i}(\cdot, v)$ depending on v .

$$\begin{aligned} &\leq \mathbb{E}_{Q^*} \left(\iint_{[0,T] \times \mathbb{T}^2} \|\Phi_{\varepsilon,i}\|_1 \left[\int_{v \in \mathbb{T}^2} \Phi_{\varepsilon,i}(v+u) (\partial_{u_i} \rho(t,v))^2 dv \right] dudt \right) \\ &= \|\Phi_{\varepsilon,i}\|_1^2 \mathbb{E}_{Q^*} \left(\iint_{[0,T] \times \mathbb{T}^2} (\partial_{u_i} \rho(t,u))^2 dudt \right), \end{aligned}$$

1457 where the last identity was obtained by integrating first w.r.t. u , then w.r.t. v . Since $\|\Phi_{\varepsilon,i}\|_1 = 1 + o_\varepsilon(1)$,
 1458 Lemma 6.3 follows from equation (5.17). \square

1459 **6.4. Replacement of the macroscopic gradients by their local counterparts.** — We now prove
 1460 equation (6.6), i.e. that the macroscopic average of the gradients can be replaced by a local average. To
 1461 simplify the notations, throughout this section, we drop the various dependencies of $Y_{i,3}^{\gamma,l,\varepsilon N,p}$ and simply
 1462 denote it by Y_3 .

1463 Recall that $\mathcal{L}^{G,\beta=0}$ stands for the modified Glauber generator without alignment of the angles, where
 1464 each angle is updated uniformly in \mathbb{S} ,

$$\mathcal{L}^{G,\beta=0} f(\hat{\eta}) = \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} \frac{(f(\hat{\eta}^{x,\theta}) - f(\hat{\eta}))}{2\pi} d\theta,$$

and

$$L_N^{\beta=0} = N^2 \mathcal{L}^D + \mathcal{L}^{G,\beta=0}.$$

1465 Recall that $\mathbb{P}_{\mu_\alpha^*}^{\lambda,0}$ is the measure on the trajectories starting from the equilibrium measure μ_α^* and driven
 1466 by the generator $L_N^{\beta=0}$, and that the expectation w.r.t the latter is denoted by $\mathbb{E}_{\mu_\alpha^*}^{\lambda,0}$. We first apply
 1467 Proposition 3.10 to the positive functional

$$X(\hat{\eta}^{[0,T]}) = \left| \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right|,$$

1468 letting $A = \gamma N^2$, and obtain that for some constant $K_0 = K_0(T, \beta, \hat{\zeta})$,

$$\mathbb{E}_{\mu_N^{\lambda,\beta}}^{\lambda,\beta} \left(\left| \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right| \right) \leq \frac{K_0}{\gamma} + \frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \left| \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right| \right) \right].$$

1469 Letting γ go to ∞ after N , to prove (6.6) it is therefore enough to show that for any integer $p > 1$

$$(6.17) \quad \lim_{\gamma \rightarrow \infty} \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \left| \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right| \right) \right] = 0.$$

1470 We now get rid of the absolute value by using both of the elementary inequalities

$$e^{|x|} \leq e^x + e^{-x}$$

1471 and

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log(a_N + b_N) \leq \max \left(\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log a_N, \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log b_N \right).$$

1472 Both of these imply that the limit in equation (6.6) is bounded up by the maximum of the limits of

$$\frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right) \right]$$

1473 and

$$\frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(-\gamma N^2 \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right) \right].$$

1474 Since $-Y_3(G, \hat{\eta}) = Y_3(-G, \hat{\eta})$, and since the identity above must be true for any function G , to obtain
1475 the wanted result it is sufficient to show that for any γ and any $G \in C^{1,2}([0, T] \times \mathbb{T}^2)$

$$(6.18) \quad \lim_{\gamma \rightarrow \infty} \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right) \right] \leq 0.$$

1476 We now get back to a variational problem, since Lemma 5.2 yields

$$\frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_\alpha^*}^{\lambda,0} \left[\exp \left(\gamma N^2 \int_0^T Y_3(G_t, \hat{\eta}(t)) dt \right) \right] \leq \frac{2T\lambda^2}{\gamma} + \frac{1}{\gamma} \int_0^T \sup_{\varphi} \left\{ \mathbb{E}_\alpha^* (\varphi \gamma Y_3(G_t, \hat{\eta})) - \frac{1}{2} D(\varphi) \right\}.$$

1477 The first term in the right-hand side above vanishes as γ goes to ∞ . Furthermore, the time integral is
1478 now only applied to the function G_t , therefore to obtain equation (6.6), it is sufficient to prove that for
1479 any γ and any function $G \in C^2(\mathbb{T}^2)$,

$$(6.19) \quad \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\varphi} \{ 2\gamma \mathbb{E}_\alpha^* (\varphi Y_3(G, \hat{\eta})) - D(\varphi) \} \leq 0.$$

1480 Since this must be true for any G and any γ , we can safely assume that $\gamma = 1/2$, and equation (6.19)
1481 follows from Lemma 6.5 below. Thus this completes the proof of (6.6).

1482 In order to avoid repeating a similar proof twice, we forget for the moment that $\mathfrak{d}^\omega(\rho, \rho^\omega) = d_s(\rho)$ only
1483 depends on the total particle density, and present the proof of the following Lemma in the most difficult
1484 case where the gradient is on $\rho^{\omega,p}$ and where the diffusion coefficient depends on both ρ and ρ^ω . We
1485 simply assume throughout this proof that the diffusion coefficient \mathfrak{d}^ω is a uniformly continuous function
1486 of ρ and ρ^ω on the set

$$\left\{ (\alpha, \alpha_\omega) \in [0, 1] \times [-\|\omega\|_\infty, \|\omega\|_\infty] \text{ such that } |\alpha_\omega| \leq \|\omega\|_\infty \alpha \right\}.$$

1487 **Lemma 6.5.** — *Let us fix $1 \leq i, j \leq 2$, we shorten*

$$\mathcal{D}_k = \mathfrak{d}^\omega(\rho_k, \rho_k^\omega) \text{ and } v_k = \delta_i \rho_k^{\omega,p}.$$

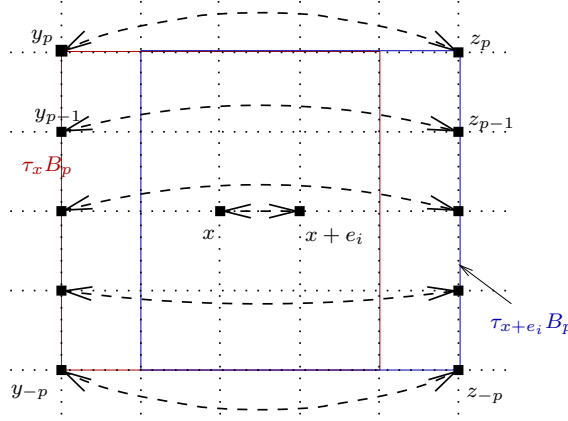
1488 *For any $G \in C^2(\mathbb{T}^2)$*

$$(6.20) \quad \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\varphi} \left\{ \sum_{x \in \mathbb{T}_N^2} \left[\frac{1}{N} G(x/N) \mathbb{E}_\alpha^* (\varphi \tau_x (\mathcal{D}_{\varepsilon N} v_{\varepsilon N} - \mathcal{D}_l v_{l_p})) \right] - D(\varphi) \right\} \leq 0,$$

1489 *where as before $l_p = l - p - 1$, and the supremum is taken over all probability densities with respect to*
1490 *μ_α^* . The same result is true for the gradients $v_k = \delta_i \rho_k$ instead of $\delta_i \rho_k^{\omega,p}$, \mathfrak{d} instead of \mathfrak{d}^ω , and $l' = l - 1$*
1491 *instead of l_p .*

1492 **Proof of Lemma 6.5.** — The difficulty of this Lemma comes from the extra factor N , which prevents us
1493 from using directly the replacement Lemma 4.1. We hence need to get some precise control over each
1494 term to ensure that they are small enough. We start by splitting in two parts the quantity in Lemma 6.5
1495 by noticing that

$$(6.21) \quad \mathcal{D}_{\varepsilon N} v_{\varepsilon N} - \mathcal{D}_l v_{l_p} = \mathcal{D}_{\varepsilon N} (v_{\varepsilon N} - v_{l_p}) + (\mathcal{D}_{\varepsilon N} - \mathcal{D}_l) v_{l_p}.$$


 FIGURE 7. Change of variable $\hat{\eta} \rightarrow T_{i,p}^x \hat{\eta}$.

1496 Both terms are treated in the same fashion due to the continuity of the diffusion coefficients (which follows
 1497 directly from their explicit expression). More precisely, we intend to show that the difference between
 1498 the average over a microscopic and macroscopic box is of order $1/N$, and hence yields the extra factor
 1499 N needed to use the replacement Lemma. Let us thus consider the first term appearing in the Lemma,
 1500 namely

$$\frac{1}{N} \mathbb{E}_\alpha^* \left(\varphi \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{D}_{\varepsilon N} (v_{\varepsilon N} - v_{l_p}) \right).$$

Recall that we denoted $B_l = \{x \in \mathbb{T}_N^2, |x| \leq l\}$, and $|B_l| = (2l+1)^2$. Since both $v_{\varepsilon N}$ and v_{l_p} are merely spatial averages of the gradients $\delta_i(\eta_0^\omega \mathbb{1}_{E_p})$, a first summation by parts yields that the quantity above is equal to

$$\frac{1}{N} \mathbb{E}_\alpha^* \left(\varphi \sum_{x \in \mathbb{T}_N^2} (\eta_{x+e_i}^\omega \mathbb{1}_{E_{p,x+e_i}} - \eta_x^\omega \mathbb{1}_{E_{p,x}}) \left[\frac{1}{|B_{\varepsilon N}|} \sum_{|y-x| \leq \varepsilon N} G(y/N) \tau_y \mathcal{D}_{\varepsilon N} - \frac{1}{|B_{l_p}|} \sum_{|y-x| \leq l_p} G(y/N) \tau_y \mathcal{D}_{\varepsilon N} \right] \right).$$

1501 Now let $S_x(\hat{\eta})$ denote the quantity inside braces, i.e

$$S_x(\hat{\eta}) = \frac{1}{|B_{\varepsilon N}|} \sum_{|y-x| \leq \varepsilon N} G(y/N) \tau_y \mathcal{D}_{\varepsilon N} - \frac{1}{|B_{l_p}|} \sum_{|y-x| \leq l_p} G(y/N) \tau_y \mathcal{D}_{\varepsilon N}.$$

1502 We are now going to prove that

$$(6.22) \quad \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\varphi} \left\{ \frac{1}{N} \mathbb{E}_\alpha^* \left(\varphi \sum_{x \in \mathbb{T}_N^2} S_x(\eta_{x+e_i}^\omega \mathbb{1}_{E_{p,x+e_i}} - \eta_x^\omega \mathbb{1}_{E_{p,x}}) \right) - \frac{1}{2} D(\varphi) \right\} \leq 0.$$

1503 In order to transfer the gradient appearing in the expression above on φ and S_x , we need the specific
 1504 change of variable represented in Figure 7. For any direction $i \in \{1, 2\}$, let $i' \neq i$ be the second direction
 1505 on the torus. Given x in the torus, we denote for any $k \in \llbracket -p, p \rrbracket$ (See Figure 4)

$$y_k = x - p e_i + k e_{i'} \in B_p(x) \quad \text{and} \quad z_k = x + (p+1) e_i + k e_{i'} \in B_p(x + e_i).$$

1506 Given these, we denote, for any configuration $\hat{\eta}$, by

$$T_{i,p}^x(\hat{\eta}) = (((\hat{\eta}^{x,x+e_i})_{y_{-p}, z_{-p}}) \cdots)_{y_p, z_p}$$

1507 the configuration where the sites x and $x + e_i$ have been swapped, as well as the boundary sites y_k and
 1508 z_k .

1509 By construction, we have

$$\eta_x^\omega \mathbb{1}_{E_{p,x}}(T_{i,p}^x \hat{\eta}) = \eta_{x+e_i}^\omega \mathbb{1}_{E_{p,x+e_i}}(\hat{\eta})$$

The first term in the left-hand side of (6.22) can be rewritten as

$$(6.23) \quad \begin{aligned} \frac{1}{N} \mathbb{E}_\alpha^* \left(\varphi \sum_{x \in \mathbb{T}_N^2} S_x (\eta_{x+e_i}^\omega \mathbb{1}_{E_{p,x+e_i}} - \eta_x^\omega \mathbb{1}_{E_{p,x}}) \right) &= \frac{1}{N} \mathbb{E}_\alpha^* \left(\sum_{x \in \mathbb{T}_N^2} \eta_x^\omega \mathbb{1}_{E_{p,x}} ((\varphi S_x)(T_{i,p}^x \hat{\eta}) - \varphi S_x) \right) \\ &= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* (\eta_x^\omega \mathbb{1}_{E_{p,x}} [\varphi(T_{i,p}^x \hat{\eta}) (S_x(T_{i,p}^x \hat{\eta}) - S_x) \\ &\quad + (\varphi(T_{i,p}^x \hat{\eta}) - \varphi) S_x]). \end{aligned}$$

1510 We are going to show that the contribution of the first term of the right-hand side in (6.23) vanishes
1511 in the limit $N \rightarrow \infty$, whereas the second term can be controlled with the Dirichlet form $D(\varphi)$. Recall
1512 that S_x is defined as

$$S_x(\hat{\eta}) = \frac{1}{|B_{\varepsilon N}|} \sum_{|y-x| \leq \varepsilon N} G(y/N) \tau_y \mathcal{D}_{\varepsilon N} - \frac{1}{|B_{l_p}|} \sum_{|y-x| \leq l_p} G(y/N) \tau_y \mathcal{D}_{\varepsilon N}.$$

Since the only dependency of S_x in $\hat{\eta}$ lies in $\mathcal{D}_{\varepsilon N}$, which is the diffusion coefficient evaluated in the macroscopic empirical density $\hat{\rho}_{\varepsilon N}$, in order to control the first term in the right-hand side of (6.23), we can write

$$(6.24) \quad \begin{aligned} S_x(T_{i,p}^x \hat{\eta}) - S_x &= \\ \frac{1}{|B_{\varepsilon N}|} \sum_{|y-x| \leq \varepsilon N} G(y/N) \tau_y [\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})] &- \frac{1}{|B_{l_p}|} \sum_{|y-x| \leq l_p} G(y/N) \tau_y [\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})]. \end{aligned}$$

1513 Recall that $\tau_y \mathcal{D}_{\varepsilon N}(\hat{\eta}) = \mathfrak{d}^\omega(\tau_y \rho_{\varepsilon N}, \tau_y \rho_{\varepsilon N}^\omega)$. Since it depends on the configuration through an average over
1514 $B_{\varepsilon N}(y)$, $\tau_y \mathcal{D}_{\varepsilon N}(\hat{\eta})$ is invariant under any exchange of a pair of sites with both ends in $B_{\varepsilon N}(y)$. We deduce
1515 from this remark that for any $|y-x| \leq l_p$, the quantity

$$\tau_y [\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})]$$

1516 vanishes, since all the exchanges happen between sites at a distance at most p of x , and therefore at a
1517 distance at most $p + l_p$ of y . This yields that the second term in the right-hand side of (6.24) vanishes.

1518 We now consider the first term in the right-hand side of (6.24). For the same reason as before, for
1519 any y in $B_{\varepsilon N-p-1}(x)$, all the exchanges in $T_{i,p}^x$ have both ends in $B_{\varepsilon N}(y)$, and $\tau_y [\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})]$
1520 vanishes. We can finally rewrite (6.24) as

$$(6.25) \quad S_x(T_{i,p}^x \hat{\eta}) - S_x = \frac{1}{|B_{\varepsilon N}|} \sum_{y \in B_{\varepsilon N}(x) \setminus B_{\varepsilon N-p-1}(x)} G(y/N) \tau_y [\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})].$$

1521 We now take a closer look at each of the remaining term. By definition, the configuration $T_{i,p}^x \hat{\eta}$ can be
1522 obtained from $\hat{\eta}$ by inverting $2p+2$ pair of sites in $\hat{\eta}$. Furthermore, fix a y in the sum above, and consider
1523 any inversion $\hat{\eta}^{z_1, z_2}$ with $z_1 \in B_{\varepsilon N}(y)$ and $z_2 \notin B_{\varepsilon N}(y)$, we can write by definition of $\rho_{\varepsilon N}$ and $\rho_{\varepsilon N}^\omega$

$$|\tau_y \rho_{\varepsilon N}(\hat{\eta}^{z_1, z_2}) - \tau_y \rho_{\varepsilon N}(\hat{\eta})| \leq \frac{1}{|B_{\varepsilon N}|} \quad \text{and} \quad |\tau_y \rho_{\varepsilon N}^\omega(\hat{\eta}^{z_1, z_2}) - \tau_y \rho_{\varepsilon N}^\omega(\hat{\eta})| \leq \frac{2\|\omega\|_\infty}{|B_{\varepsilon N}|}.$$

1524 By assumption, $\mathfrak{d}^\omega(\alpha, \alpha_\omega)$ is uniformly continuous on the set

$$\left\{ (\alpha, \alpha_\omega) \in [0, 1] \times [-\|\omega\|_\infty, \|\omega\|_\infty] \text{ such that } |\alpha_\omega| \leq \|\omega\|_\infty \alpha \right\}.$$

1525 We deduce from this that

$$\tau_y (\mathcal{D}_{\varepsilon N}(\hat{\eta}^{z_1, z_2}) - \mathcal{D}_{\varepsilon N}(\hat{\eta})) = o_N(1),$$

1526 therefore

$$|\tau_y (\mathcal{D}_{\varepsilon N}(T_{i,p}^x \hat{\eta}) - \mathcal{D}_{\varepsilon N}(\hat{\eta}))| \leq o_N(1),$$

1527 where this time $o_N(1)$ stands for a constant depending on p which vanishes as $N \rightarrow \infty$. We inject the
1528 latter identity in equation (6.25), to obtain that

$$S_x(T_{i,p}^x \hat{\eta}) - S_x = \frac{|B_{\varepsilon N}(x) \setminus B_{\varepsilon N-p-1}(x)|}{|B_{\varepsilon N}|} o_N(1) = \frac{1}{N} o_N(1),$$

1529 where the last $o_N(1)$ depends on p and ε , but vanishes as $N \rightarrow \infty$. This allows us to get back to equation
 1530 (6.23), in which the first term in the right-hand side can be rewritten

$$\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\eta_x^\omega \mathbb{1}_{E_{p,x}} \varphi(T_{i,p}^x \hat{\eta}) (S_x(T_{i,p}^x \hat{\eta}) - S_x) \right) \right| \leq \frac{\|\omega\|_\infty}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\varphi(T_{i,p}^x \hat{\eta}) \right) o_N(1) = o_N(1),$$

1531 since μ_α^* is invariant under the change of variable $T_{i,p}^x \hat{\eta}$, and therefore $\mathbb{E}_\alpha^* \left(\varphi(T_{i,p}^x \hat{\eta}) \right) = \mathbb{E}_\alpha^* (\varphi) = 1$.

1532 We now work on the contribution of the second part of (6.23), namely

$$(6.26) \quad \mathbb{E}_\alpha^* \left(N^{-1} \sum_{x \in \mathbb{T}_N^2} \eta_x^\omega \mathbb{1}_{E_{p,x}} S_x(\hat{\eta}) [\varphi(T_{i,p}^x \hat{\eta}) - \varphi] \right),$$

that we wish to estimate by the Dirichlet form $D(\varphi)$. The elementary bound

$$cd(a-b) \leq \frac{Ac^2}{2} (\sqrt{a} + \sqrt{b})^2 + \frac{d^2}{2A} (\sqrt{a} - \sqrt{b})^2,$$

1533 which holds for any positive constant A , applied to

$$a = \varphi(T_{i,p}^x \hat{\eta}), \quad b = \varphi, \quad c = \eta_x^\omega S_x \text{ and } d = \mathbb{1}_{E_{p,x}}$$

1534 yields that the quantity above (6.26) can be bounded from above for any positive A by

$$(6.27) \quad \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\frac{A}{2} (\eta_x^\omega S_x)^2 (\sqrt{\varphi}(T_{i,p}^x \hat{\eta}) + \sqrt{\varphi})^2 + \frac{1}{2A} \mathbb{1}_{E_{p,x}} (\sqrt{\varphi}(T_{i,p}^x \hat{\eta}) - \sqrt{\varphi})^2 \right).$$

1535 Since we already established that $S_x(T_{i,p}^x \hat{\eta}) = S_x + (\varepsilon N)^{-1} o_N(1)$, since η_x^ω can be bounded by $C(\omega) > 0$,
 1536 and since $\mathbb{1}_{E_{p,x}} \leq \mathbb{1}_{E_{p+1,x}}$ the sum above is less than

$$(6.28) \quad \frac{AC^2}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* (\varphi S_x^2) + \frac{1}{2AN} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* \left(\mathbb{1}_{E_{p+1,x}} (\sqrt{\varphi}(T_{i,p}^x \hat{\eta}) - \sqrt{\varphi})^2 \right) + o_N(1).$$

1537 According to Section 3.3, on the event $E_{p+1,x}$ on which there are two empty sites in B_{p+1} , there exists
 1538 a sequence of allowed jumps permitting to reach $T_{i,p}^x \hat{\eta}$ from $\hat{\eta}$. However, this sequence is random, which
 1539 we avoid by crudely bounding

$$\mathbb{1}_{E_{p+1,x}} \leq \sum_{z_1, z_2 \in B_{p+1}} (1 - \eta_{z_1})(1 - \eta_{z_2}),$$

1540 since the right-hand side only vanishes when there are less than one empty site in B_{p+1} . Given two fixed
 1541 empty sites z_1 and z_2 there exists an integer $n_p(z_1, z_2)$ bounded by a constant C_p , and a sequence of
 1542 edges $((a_m, b_m))_{m \in [0, n_p]}$ such that

$$\hat{\eta} = \hat{\eta}(0), \quad T_{i,p}^x \hat{\eta} = \hat{\eta}(n_p), \quad \text{and } \hat{\eta}(m+1) = \hat{\eta}(m)^{a_m, b_m} \quad \forall m \in [0, n_p - 1],$$

where a_m and b_m are neighboring sites in $B_{p+1}(x)$ and $\eta_{a_m}(\hat{\eta}(m)) = 1 - \eta_{b_m}(\hat{\eta}(m)) = 1$. We can therefore write

$$\begin{aligned} \mathbb{E}_\alpha^* \left(\mathbb{1}_{E_{p,x}} (\sqrt{\varphi}(T_{i,p}^x \hat{\eta}) - \sqrt{\varphi})^2 \right) &\leq \sum_{z_1, z_2 \in B_{p+1}} \mathbb{E}_\alpha^* \left(n_p \sum_{m=0}^{n_p-1} \mathbb{1}_{E_{p,x}} (\sqrt{\varphi}(\hat{\eta}(m+1)) - \sqrt{\varphi}(\hat{\eta}(m)))^2 \right) \\ &\leq K_p D_{N,p+1}(\varphi), \end{aligned}$$

1543 since $\hat{\eta}(m+1)$ is reached from $\hat{\eta}(m)$ by an allowed particle jump, where $D_{N,p+1}(\varphi)$ is the contribution
 1544 of edges in B_{p+1} in $D(\varphi)$.

1545 The sum in the second term of (6.28) can therefore be bounded by $C_p^* D(\varphi)$, where $C_p^* = (2p+1)^2 K_p$.

1546 Finally, (6.26) can be bounded, for any positive A by

$$\frac{AC^2}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* (\varphi S_x^2) + \frac{C_p^*}{2AN} D(\varphi) + o_N(1).$$

We can now set $A = C_p^*/N$, to obtain that

$$\mathbb{E}_\alpha^* \left(N^{-1} \sum_{x \in \mathbb{T}_N^2} \eta_x^\omega \mathbb{1}_{E_{p,x}} S_x(\hat{\eta}) [\varphi(T_{i,p}^x \hat{\eta}) - \varphi] \right) \leq \frac{C(p, \omega)}{N^2} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^*(\varphi S_x^2) + \frac{1}{2} D(\varphi) + o_N(1).$$

1547 The first term in the right-hand side above vanishes as a consequence of the two-block estimate stated
 1548 in Lemma 4.3, since the diffusion coefficients are continuous according to their explicit expression. This
 1549 concludes the proof of equation (6.22).

1550 The contribution of the second part of equation (6.21) is treated in a similar fashion. Denoting by

$$S'_x(\hat{\eta}) = \frac{1}{|B_{l_p}|} \sum_{|y-x| \leq l_p} G(y/N) (\tau_y \mathcal{D}_{\varepsilon N} - \tau_y \mathcal{D}_l).$$

As before, the corresponding contribution in the left-hand side of (6.20) can be written as

$$-\frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^*(\eta_x^\omega \mathbb{1}_{E_{p,x}} (\varphi(T_{i,p}^x \hat{\eta}) - \varphi) S'_x),$$

1551 since this time, S'_x is invariant under the action of $T_{i,p}^x$ by definition of l_p , whereas the second term can
 1552 be controlled in the limit $N \rightarrow \infty$ as well by $D(\varphi)/2$. This completes the proof of Lemma 6.5 in the case
 1553 where $\mathcal{D}_k = \mathfrak{d}^\omega(\rho_k, \rho_k^\omega)$ and $v_k = \delta_i \rho_k^{\omega, p}$.

1554 In the case where $\mathcal{D}_k = \mathfrak{d}(\rho_k, \rho_k^\omega)$ and $v_k = \delta_i \rho_k$, the proof is easier and no longer requires indicator
 1555 functions, since unlike $\delta_i \eta_x^\omega$, $\delta_i \eta_x$ vanishes when there is no empty site. We do not give a detailed proof,
 1556 which would be an easier version of the previous case. We will instead just give a brief outline and the
 1557 equivalent quantities to the previous ones. The same summation by parts allows us to rewrite

$$\frac{1}{N} G(x/N) \mathbb{E}_\alpha^* \left(\varphi \tau_x (\mathcal{D}_{\varepsilon N} v_{\varepsilon N} - \mathcal{D}_l v_{l_p}) \right) = \frac{1}{N} \mathbb{E}_\alpha^* \left(\varphi \sum_{x \in \mathbb{T}_N^2} (S_x + S'_x) (\eta_{x+e_i} - \eta_x) \right),$$

1558 where

$$S_x = \frac{1}{|B_{\varepsilon N}|} \sum_{|y-x| \leq \varepsilon N} G(y/N) \tau_y \mathcal{D}_{\varepsilon N} - \frac{1}{|B_{l'}|} \sum_{|y-x| \leq l'} G(y/N) \tau_y \mathcal{D}_{\varepsilon N},$$

1559 and

$$S'_x(\hat{\eta}) = \frac{1}{|B_{l'}|} \sum_{|y-x| \leq l'} G(y/N) (\tau_y \mathcal{D}_{\varepsilon N} - \tau_y \mathcal{D}_l).$$

1560 We can now rewrite $\eta_{x+e_i} - \eta_x = \eta_{x+e_i}(1 - \eta_x) - \eta_x(1 - \eta_{x+e_i})$, to obtain that the quantity above is

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N^2} \mathbb{E}_\alpha^* (\eta_x (1 - \eta_{x+e_i}) ((S_x + S'_x) \varphi)(\hat{\eta}^{x, x+e_i}) - (S_x + S'_x) \varphi).$$

1561 The gradients of S_x and S'_x still vanish, whereas the average of the gradients $\varphi(\hat{\eta}^{x, x+e_i}) - \varphi$ can be
 1562 controlled by the sum of a vanishing term and the Dirichlet form of φ , since this time the jump rates
 1563 $\eta_x(1 - \eta_{x+e_i})$ are already present. This concludes the proof of Lemma 6.5. \square

1564 **6.5. Projection on non-full sets and reduction to a variance problem.** — We now prove the
 1565 limit (6.7), which states that in a local average, the current can be replaced by gradients, up to a
 1566 perturbation $\mathcal{L}f$. Following the exact same steps as in Section 6.4, up until the statement of Lemma
 1567 6.5, where we reduced the proof of equation (6.6) to (6.19), we reduce the proof of equation (6.7) to the
 1568 variational formula

$$(6.29) \quad \inf_f \lim_{p \rightarrow \infty} \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{\varphi} \{ \mathbb{E}_\alpha^* (\varphi Y_4(G, \hat{\eta})) - D(\varphi) \} \leq 0,$$

1569 where we shortened

$$Y_4(G, \hat{\eta}) = Y_{i,4}^{f,l,p}(G, \hat{\eta}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{W}_{i,4}^{f,l,p},$$

1570 and $\mathcal{W}_{i,4}^{f,l,p}$ was introduced in equation (6.3). Since this step is performed in the exact same way as in
 1571 the beginning of Section 6.4, we do not detail them here and refer the reader to the latter. To simplify
 1572 notations, we shorten

$$\mathcal{W}_i^l = \mathcal{W}_{i,4}^{f,l,p}$$

1573 for the local average of the difference between gradients and currents in the direction i .

1574 We will now work to get an estimate of the largest eigenvalue of the small perturbation $\mathcal{L} + Y_4$ of \mathcal{L} .
 1575 The strategy is close to the one used in the one-block estimate of Section 4.3. To do so, we break down
 1576 the process on finite boxes with a fixed number of particles, where the generator \mathcal{L} has a positive spectral
 1577 gap. In order to introduce this restriction, we adopt once again the notations introduced in Section 4.3,
 1578 which we briefly recall here. Let $B_l = \llbracket -l, l \rrbracket^2$ be the box of size l , $\widehat{K} = (K, \{\theta_1, \dots, \theta_K\})$ be some particle
 1579 number and angles. Recall that \mathbb{K}_l is the set of \widehat{K} 's such that $K \leq (2l+1)^2$, and denote by $\widehat{\alpha}_{\widehat{K}}$ the
 1580 grand-canonical parameter

$$\widehat{\alpha}_{\widehat{K}} = \frac{1}{(2l+1)^2} \sum_{k=1}^K \delta_{\theta_k} \in \mathcal{M}_1(\mathbb{S}).$$

Recall that we already defined in (3.3)

$$\Sigma_l^{\widehat{K}} = \{ \widehat{\eta} \in \Sigma_N \mid \widehat{\rho}_l = \widehat{\alpha}_{\widehat{K}} \}$$

1581 the set of configurations with K particles in B_l with angles θ_k 's. Also recall that $\mu_{l,\widehat{K}}$ is the canonical
 1582 measure $\mu_\alpha^*(\cdot \mid \Sigma_l^{\widehat{K}})$ conditioned to particle configurations of the form \widehat{K} in B_l .

1583 We denote for any site x $\varphi^x = \tau_{-x}\varphi$, and by $\varphi_{l,\widehat{K}}^x$ the density induced by φ^x on $\Sigma_l^{\widehat{K}}$. It can be defined
 1584 for any configuration $\widehat{\zeta}$ on B_l by

$$\varphi_{l,\widehat{K}}^x(\widehat{\zeta}) = \frac{\mathbb{E}_\alpha^*(\varphi^x \mid \widehat{\eta}|_{B_l} = \widehat{\zeta})}{\mathbb{E}_\alpha^*(\varphi^x \mid \Sigma_l^{\widehat{K}})}.$$

1585 Let us now get back to the quantity of interest,

$$(6.30) \quad \mathbb{E}_\alpha^*(\varphi Y_4(G, \widehat{\eta})) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \mathbb{E}_\alpha^*(\varphi \tau_x \mathcal{W}_i^l) = \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \mathbb{E}_\alpha^*(\mathcal{W}_i^l \varphi^x).$$

1586 Because \mathcal{W}_i^l only depends on the vertices in B_l , we can replace the expectation under μ_α^* by the integral
 1587 over \mathbb{K}_l of the expectation under $\mu_{l,\widehat{K}}$. More precisely, let us denote

$$m_x(d\widehat{K}) = \mathbb{E}_\alpha^*(\varphi^x \mathbb{1}_{\Sigma_l^{d\widehat{K}}}),$$

the infinitesimal probability of being on the set $\Sigma_l^{\widehat{K}}$ under the measure with density φ^x w.r.t μ_α^* . Thanks
 to (6.30), letting $\mathbb{E}_{l,\alpha}^*$ be the conditional expectation of \mathbb{E}_α^* w.r.t the sites inside of B_l , we can write

$$(6.31) \quad \begin{aligned} \mathbb{E}_\alpha^*(\varphi Y_4(G, \widehat{\eta})) &= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \mathbb{E}_{l,\alpha}^*(\mathcal{W}_i^l \varphi^x) \\ &= \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \int_{\widehat{K} \in \mathbb{K}_l} \mathbb{E}_{l,\widehat{K}}(\mathcal{W}_i^l \varphi_{l,\widehat{K}}^x) m_x(d\widehat{K}). \end{aligned}$$

1588 Let us now decompose in a similar fashion the Dirichlet form. For φ some density with respect to μ_α ,
 1589 let $D_{l,\widehat{K}}$ be the Dirichlet form on $\Sigma_l^{\widehat{K}}$

$$D_{l,\widehat{K}}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in B_l \\ |x-y|=1}} \mathbb{E}_{l,\widehat{K}} \left[\eta_x (1 - \eta_y) \left(\sqrt{\varphi(\widehat{\eta}^{x,y})} - \sqrt{\varphi} \right)^2 \right].$$

1590 We have with the same tools as in the proof of Lemma 4.3

$$(6.32) \quad \sum_{x \in \mathbb{T}_N^2} \int_{\widehat{K} \in \mathbb{K}_l} D_{l,\widehat{K}}(\varphi_{l,\widehat{K}}^x) m_x(d\widehat{K}) \leq (2l+1)^2 D(\varphi).$$

From the previous considerations, we can localize the quantity inside braces in equation (6.29), which is bounded above thanks to (6.31) and (6.32) by

$$\begin{aligned}
\mathbb{E}_\alpha^*(\varphi Y_4(G, \hat{\eta})) - D(\varphi) &= \sum_{x \in \mathbb{T}_N^2} \int_{\hat{K} \in \mathbb{K}_l} m_x(d\hat{K}) \left(\frac{1}{N} G(x/N) \mathbb{E}_{l, \hat{K}} \left(\mathcal{W}_i^l \varphi_{l, \hat{K}}^x \right) - (2l+1)^{-2} D_{l, \hat{K}} \left(\varphi_{l, \hat{K}}^x \right) \right) \\
&\leq \kappa_1 \sum_{x \in \mathbb{T}_N^2} \sup_{\hat{K} \in \mathbb{K}_l} \left[\frac{\kappa_2}{N} \mathbb{E}_{l, \hat{K}} \left(\mathcal{W}_i^l \varphi_{l, \hat{K}}^x \right) - D_{l, \hat{K}} \left(\varphi_{l, \hat{K}}^x \right) \right] \\
(6.33) \quad &\leq \kappa_1 \sum_{x \in \mathbb{T}_N^2} \sup_{\hat{K} \in \mathbb{K}_l} \sup_{\psi} \left[\frac{\kappa_2}{N} \mathbb{E}_{l, \hat{K}} \left(\mathcal{W}_i^l \psi \right) - D_{l, \hat{K}} \left(\psi \right) \right],
\end{aligned}$$

1591 since $\int_{\hat{K} \in \mathbb{K}_l} m_x(d\hat{K}) = 1$, where

$$\kappa_1 = (2l+1)^{-2} \quad \text{and} \quad \kappa_2 = G(x/N)(2l+1)^2,$$

1592 and the supremum is taken over all densities ψ with respect to $\mu_{l, \hat{K}}$.

1593 We now wish to exclude in the supremum over \hat{K} above the configurations with one or less empty
1594 sites since on the corresponding sets, the exclusion process is not irreducible as investigated in Section
1595 3.3. First note that for any \hat{K} such that $K = |B_l|$, \mathcal{W}_i^l vanishes. Indeed, thanks to our cutoff functions
1596 $\mathbb{1}_{E_p}$, and since l goes to ∞ before p , in that case, the currents, the gradients as well as the $\mathcal{L}f$'s in \mathcal{W}_i^l
1597 all vanish as well as $D_{l, \hat{K}}(\psi)$.

1598 We now consider the case where $K = |B_l| - 1$, i.e. when there is one empty site in B_l . We state the
1599 corresponding estimate as a separate lemma for the sake of clarity.

1600 **Lemma 6.6.** — *There exists a constant $C = C(G, \omega, f)$ such that for any \hat{K} such that $K = |B_l| - 1$,*

$$\frac{\kappa_2}{N} \mathbb{E}_{l, \hat{K}} \left(\mathcal{W}_i^l \psi \right) \leq D_{l, \hat{K}}(\psi) + \frac{C}{N^2}.$$

1601 *Proof of Lemma 6.6.* — First note that all the gradients $\delta_i \eta^{\omega, p}$ vanish in the expression of \mathcal{W}_i^l due to the
1602 cutoff functions. We can therefore write, for any configuration with one or less empty site, that

$$\mathcal{W}_i^l = \frac{1}{(2l'+1)^2} \sum_{x \in B_{l'}} (j_{x, x+e_i}^\omega + \mathfrak{d}_{\hat{K}} j_{x, x+e_i}) - \frac{1}{(2l_f+1)^2} \mathcal{L}_l \bar{f},$$

1603 where we denoted by $\mathfrak{d}_{\hat{K}}$ the value on $\Sigma_{l, \hat{K}}$ of $\mathfrak{d}(\rho_l, \rho_l^\omega)$, which does not depend on the configuration, and
1604 $\bar{f} = \sum_{x \in B_{l'}} \tau_x f$. The quantity we want to estimate can therefore be rewritten

$$\frac{\kappa_2}{N} \mathbb{E}_{l, \hat{K}} \left(\mathcal{W}_i^l \psi \right) = \frac{\kappa_2}{N(2l'+1)^2} \mathbb{E}_{l, \hat{K}} \left(\psi \sum_{x \in B_{l'}} (j_{x, x+e_i}^\omega + \mathfrak{d}_{\hat{K}} j_{x, x+e_i}) \right) - \frac{\kappa_2}{N(2l_f+1)^2} \mathbb{E}_{l, \hat{K}} \left(\psi \mathcal{L}_l \bar{f} \right),$$

where \mathcal{L}_l is the generator of the symmetric exclusion process restricted to jumps with both ends in B_l . Since κ_2 , $(2l'+1)^2$, and $(2l_f+1)^2$ are of order $(2l+1)^2$, and since the sign of f is arbitrary, to prove Lemma 6.6 it is sufficient to prove that for any $A > 0$, we have both

$$\begin{aligned}
(6.34) \quad \frac{1}{N} \mathbb{E}_{l, \hat{K}} \left(\psi \sum_{x \in B_{l'}} (j_{x, x+e_i}^\omega + \mathfrak{d}_{\hat{K}} j_{x, x+e_i}) \right) &\leq \frac{D_{l, \hat{K}}(\psi)}{2A} + \frac{AC(\omega)}{N^2} \\
&\text{and} \quad \frac{1}{N} \mathbb{E}_{l, \hat{K}} \left(\psi \mathcal{L}_l \bar{f} \right) \leq \frac{D_{l, \hat{K}}(\psi)}{2A} + \frac{AC(f)}{N^2}.
\end{aligned}$$

1605 The two inequalities above are proved in the same way. We treat in detail the second, which is the
1606 most delicate, and simply sketch the adaptations to obtain the first. Using the elementary inequality

$$(6.35) \quad ab \leq \frac{\gamma a^2}{2} + \frac{b^2}{2\gamma},$$

which holds for any positive γ , we first write

$$\begin{aligned}
 \mathbb{E}_{l,\widehat{K}}(\psi \mathcal{L}_l \bar{f}) &= \sum_{x,x+z \in B_l} \mathbb{E}_{l,\widehat{K}}(\psi \nabla_{x,x+z} \bar{f}) \\
 &= -\frac{1}{2} \sum_{x,x+z \in B_l} \mathbb{E}_{l,\widehat{K}}(\nabla_{x,x+z} \psi \nabla_{x,x+z} \bar{f}) \\
 &\leq \sum_{x,x+z \in B_l} \frac{\gamma}{4} \mathbb{E}_{l,\widehat{K}}\left((\nabla_{x,x+z} \sqrt{\psi})^2\right) + \frac{1}{4\gamma} \mathbb{E}_{l,\widehat{K}}\left((\nabla_{x,x+z} \bar{f})^2 (\sqrt{\psi} + \sqrt{\psi}(\widehat{\eta}^{x,x+z}))^2\right) \\
 &= \frac{\gamma}{2} D_{l,\widehat{K}}(\psi) + \frac{1}{4\gamma} \mathbb{E}_{l,\widehat{K}}\left(\sum_{x,x+z \in B_l} \eta_x (1 - \eta_{x+z}) (\bar{f} - \bar{f}(\widehat{\eta}^{x,x+z}))^2 (\sqrt{\psi} + \sqrt{\psi}(\widehat{\eta}^{x,x+z}))^2\right).
 \end{aligned}$$

1607 One only has now to carefully account for the order of the different quantities in the second term. Since
 1608 f is a bounded local function, by definition of \bar{f} , it is invariant under particle jumps with both ends
 1609 outside of its domain. There hence exists a constant $C(f)$ such that for any x and $x+z$, $\bar{f} - \bar{f}(\widehat{\eta}^{x,x+z}) \leq$
 1610 $C(f)$. In particular, the constant $C(f)$ does not depend on l . We can also crudely bound η_x by 1 and
 1611 $(\sqrt{\psi} + \sqrt{\psi}(\widehat{\eta}^{x,x+z}))^2$ by $2\psi + \psi(\widehat{\eta}^{x,x+z})$. These bounds and a change of variable $\widehat{\eta} \rightarrow \widehat{\eta}^{x,x+z}$ finally yield
 1612 that for any positive γ ,

$$\mathbb{E}_{l,\widehat{K}}(\psi \mathcal{L}_l \bar{f}) \leq \frac{\gamma}{2} D_{l,\widehat{K}}(\psi) + \frac{C(f)}{2\gamma} \mathbb{E}_{l,\widehat{K}}\left(\sum_{x,x+z \in B_l} (2 - \eta_x - \eta_{x+z}) \psi\right).$$

1613 Furthermore, since there is only one empty site in B_l ,

$$\sum_{|y| \leq l-1} (2 - \eta_y - \eta_{y+e_i}) = \underbrace{|B_{l-1}| - \sum_{y \in B_{l-1}} \eta_y}_{\leq 1} + \underbrace{|\tau_{e_i} B_{l-1}| - \sum_{y \in \tau_{e_i} B_{l-1}} \eta_y}_{\leq 1} \leq 2,$$

1614 therefore, since ψ is a probability density, and setting $\gamma = N/A$ proves the second identity of (6.34).

1615 The second identity is obtained in the same way, since

$$\frac{1}{N} \mathbb{E}_{l,\widehat{K}}\left(\psi \sum_{x \in B_{l'}} (j_{x,x+e_i}^\omega + \mathfrak{d}_{\widehat{K}} j_{x,x+e_i})\right) = \frac{1}{N} \sum_{|y| \leq l-1} \mathbb{E}_{l,\widehat{K}}((\omega(\theta_y) + \mathfrak{d}_{\widehat{K}}) \nabla_{y,y+e_i} \psi),$$

we also obtain

$$\begin{aligned}
 \frac{1}{N} \mathbb{E}_{l,\widehat{K}}\left(\psi \sum_{x \in B_{l'}} (j_{x,x+e_i}^\omega + \mathfrak{d}_{\widehat{K}} j_{x,x+e_i})\right) \\
 \leq \frac{\gamma}{2} D_{l,\widehat{K}}(\psi) + \frac{(\|\omega\|_\infty + \|\mathfrak{d}\|_\infty)^2}{2\gamma} \mathbb{E}_{l,\widehat{K}}\left(\sum_{x,x+e_i \in B_l} (2 - \eta_x - \eta_{x+e_i}) \psi\right).
 \end{aligned}$$

1616 The last estimate, in turn, yields the first inequality in (6.34), which concludes the proof of Lemma
 1617 6.6. \square

1618 In the limit $N \rightarrow \infty$ then $l \rightarrow \infty$, Lemma 6.6 yields, since κ_1 vanishes as $l \rightarrow \infty$, and since all
 1619 quantities vanish when $K = |B_l|$, that

$$\kappa_1 \sum_{x \in \mathbb{T}_N^2} \sup_{\substack{\widehat{K} \in \mathbb{K}_l \\ K \geq |B_l| - 1}} \sup_{\psi} \left[\frac{\kappa_2}{N} \mathbb{E}_{l,\widehat{K}}(\mathcal{W}_i^l \psi) - D_{l,\widehat{K}}(\psi) \right] \rightarrow 0.$$

1620 We can therefore restrict the supremum over \widehat{K} to those satisfying $K \leq |B_l| - 2$. Recall that we
 1621 denoted in equation (3.2) by \mathbb{K}_l the set of such \widehat{K} , the left-hand side of (6.29) is bounded by

$$(6.36) \quad \inf_f \lim_{p \rightarrow \infty} \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \kappa_1 \sum_{x \in \mathbb{T}_N^2} \sup_{\widehat{K} \in \mathbb{K}_l} \sup_{\psi} \left[\frac{\kappa_2}{N} \mathbb{E}_{l,\widehat{K}}(\mathcal{W}_i^l \psi) - D_{l,\widehat{K}}(\psi) \right],$$

where the supremum is taken over all densities ψ w.r.t. $\mu_{l,\widehat{K}}$. On all the sets $\Sigma_l^{\widehat{K}}$ considered, \mathcal{L}_l is invertible and the supremum over ψ is a variational formula for the largest eigenvalue of the operator $\mathcal{L}_l + \kappa_2 \mathcal{W}_i^l / N$. Proposition B.7 then allows us to bound the quantity whose limit is taken in (6.36) by

$$\limsup_{N \rightarrow \infty} \sup_{\widehat{K} \in \widetilde{\mathbb{K}}_l} \frac{\kappa_1 \kappa_2^2}{1 - 2\gamma_l \|\mathcal{W}_i^l\|_\infty} \mathbb{E}_{l,\widehat{K}} (\mathcal{W}_i^l (-\mathcal{L}_l)^{-1} \mathcal{W}_i^l) \leq \|G\|_\infty^2 (2l+1)^2 \sup_{\widehat{K} \in \widetilde{\mathbb{K}}_l} \mathbb{E}_{l,\widehat{K}} (\mathcal{W}_i^l (-\mathcal{L}_l)^{-1} \mathcal{W}_i^l).$$

1622 To obtain the last inequality, we denoted by γ_l the spectral gap of the local generator \mathcal{L}_l , which is positive,
 1623 and used that $\|\mathcal{W}_i^l\|_\infty$ is finite, and $\kappa_1 \kappa_2^2 = \|G\|_\infty^2 (2l+1)^2$. In order to obtain inequality (6.29), and
 1624 conclude the proof of equation (6.7), it is therefore sufficient to prove the following result.

Proposition 6.7 (Estimate of the local covariance). — Recall that \mathcal{W}_i^l is the local average of the difference between currents and gradients up to $\mathcal{L}f$, namely

$$\mathcal{W}_i^l = \langle j_i^\omega \rangle_0^l + d_s(\rho_l) \delta_i \rho_{l_p}^{\omega,p} + \mathfrak{d}(\rho_l, \rho_l^\omega) \delta_i \rho_{l'} - \langle \mathcal{L}f \rangle_0^l,$$

1625 where \mathfrak{d} is given by equation (1.3). Recall that $\widetilde{\mathbb{K}}_l$ only takes into account configurations with two empty
 1626 sites in B_l . Then,

$$(6.37) \quad \inf_f \lim_{p \rightarrow \infty} \limsup_{l \rightarrow \infty} \sup_{\widehat{K} \in \widetilde{\mathbb{K}}_l} (2l+1)^2 \mathbb{E}_{l,\widehat{K}} (\mathcal{W}_i^l (-\mathcal{L}_l)^{-1} \mathcal{W}_i^l) = 0.$$

1627 **6.6. Limiting variance and diffusion coefficients.** — In Section 6.5, we reduced the proof of (6.7),
 1628 and that of Theorem 6.1, to estimating a local variance. In this section, we introduce the limiting variance
 1629 $\ll \cdot \gg_{\widehat{\alpha}}$ and investigate its properties and the structure of a set of functions with mean-0 w.r.t. any
 1630 canonical measures, equipped with $\ll \cdot \gg_{\widehat{\alpha}}$. The presence of indicator functions in $\delta_i \eta_0^{\omega,p}$ and the necessity
 1631 for a uniform estimate in the canonical state $\widehat{K} \in \widetilde{\mathbb{K}}_l$ makes this section fairly technical, however, most
 1632 of the results come from elementary linear algebra. The main results of this section is Proposition 6.14,
 1633 which is the main ingredient to prove Proposition 6.7, and therefore concludes the proof of Theorem 6.1.
 1634

1635 To prove Proposition 6.7, we are now going to investigate the limit as $l \rightarrow \infty$ and $\widehat{\alpha}_{\widehat{K}_l} \rightarrow \widehat{\alpha}$ (cf
 1636 Definition 3.2) of

$$(6.38) \quad \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}_l} \left((-\mathcal{L}_l)^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) := \ll \psi \gg_{\widehat{\alpha}},$$

1637 where ψ is supported by B_{s_ψ} and $l_\psi = l - s_\psi - 1$ is chosen such that $\sum_{x \in B_{l_\psi}} \tau_x \psi$ is measurable w.r.t.
 1638 sites in B_l . There are therefore two important steps to prove (6.37) :

1639 — prove that the limit (6.38) is well-defined for any function ψ in a convenient class of functions
 1640 containing at least the currents, the gradients and $\mathcal{L}C$. This is done in Definitions 6.8, 6.9, and
 1641 Theorem 6.11 below.

1642 — Prove that, shortening $\mathbb{E}_{\widehat{\alpha}}(\omega) = \mathbb{E}_{\widehat{\alpha}}(\omega(\theta_0) | \eta_0 = 1)$ and letting

$$(6.39) \quad \mathfrak{d}(\widehat{\alpha}) = \mathbb{E}_{\widehat{\alpha}}(\omega)(1 - d_s(\alpha)),$$

1643 we have

$$(6.40) \quad \inf_{f \in \mathcal{C}} \lim_{p \rightarrow \infty} \sup_{\widehat{\alpha}} \ll j_i^\omega + d_s(\alpha) \delta_i (\eta_0^\omega \mathbb{1}_{E_p}) + \mathfrak{d}(\widehat{\alpha}) \delta_i \eta_0 - \mathcal{L}f \gg_{\widehat{\alpha}} = 0.$$

1644 which is done below in Proposition 6.16.

1645 We introduce a class of local functions with mean 0 w.r.t. any $\mu_{B,\widehat{K}}$. When there are less than
 1646 one empty site in the domain B , we require these functions to vanish in order to avoid classifying the
 1647 irreducible subsets of Σ_N when there is only one empty site. Recall that we already introduced in
 1648 Definition 3.6 the sets \mathbb{K}_l and $\widetilde{\mathbb{K}}_l$. We now define

$$(6.41) \quad \mathcal{C}_0 = \left\{ \psi \in \mathcal{C} \mid \mathbb{E}_{s_\psi,\widehat{K}}(\psi) = 0 \quad \forall \widehat{K} \in \widetilde{\mathbb{K}}_{s_\psi} \quad \text{and} \quad \psi|_{\Sigma_{\widehat{K}}} \equiv 0 \quad \forall \widehat{K} \in \mathbb{K}_{s_\psi} \setminus \widetilde{\mathbb{K}}_{s_\psi} \right\}.$$

1649 In particular, any function $\psi \in \mathcal{C}_0$ has mean zero w.r.t any canonical measure. Note that $\psi \in \mathcal{C}_0$, and any
 1650 $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, conditioning w.r.t. the canonical state of the configuration in B_{s_ψ} , we obtain in particular
 1651 that $\mathbb{E}_{\widehat{\alpha}}(\psi) = 0$. Further define

$$(6.42) \quad T^\omega = \left\{ f \in \mathcal{C} \mid f(\widehat{\eta}) = \varphi(\eta) + \sum_{x \in \mathbb{Z}^2} \eta_x^\omega \psi_x(\eta), \quad \varphi, \psi_x \in \mathcal{S}, \forall x \in \mathbb{Z}^2 \right\},$$

1652 of functions whose only dependency in the θ_x 's is a linear combination of the $\omega(\theta_x)$. Note that since we
 1653 only consider local functions, this set is well-defined.

1654 Denote

$$(6.43) \quad \mathcal{T}_0^\omega = \mathcal{C}_0 \cap T^\omega.$$

1655 Note that \mathcal{T}_0^ω and \mathcal{C}_0 are stable by the symmetric exclusion generator \mathcal{L} . Further note that by construction,
 1656 $\delta_i(\eta_0^\omega \mathbb{1}_{E_p}) \in \mathcal{T}_0^\omega$.

1657 Recall that for any function Φ on \mathbb{S} , $j_i^\Phi = \Phi(\theta_0)\eta_0(1 - \eta_{e_i}) - \Phi(\theta_{e_i})\eta_{e_i}(1 - \eta_0)$ denotes the symmetric
 1658 current associated with Φ (we also shortened $j_i = j_i^1 = \eta_0 - \eta_{e_i}$). We define J^* the set of linear
 1659 combinations of currents spanning any smooth angular functions,

$$(6.44) \quad J^* = \left\{ j_1^{\Phi_1} + j_2^{\Phi_2}, \quad \text{for } \Phi_1, \Phi_2 \in C^1(\mathbb{S}) \right\},$$

1660 and let

$$(6.45) \quad J^\omega = J^* \cap T^\omega = \left\{ j^{a,b} := \sum_{i=1,2} a_i j_i^\omega + b_i j_i, \quad a, b \in \mathbb{R}^2 \right\}.$$

1661 We now have all the notations needed to introduce the limiting variance $\ll \cdot \gg_{\widehat{\alpha}}$. In order to be able
 1662 to estimate concisely the drift term later on, and to solve a technical issue, we need a rather general
 1663 result. In particular, we give two distinct constructions for $\ll f \gg_{\widehat{\alpha}}$ depending on the nature of the
 1664 function f . Fix $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$.

1665 **Definition 6.8 (Definition of $\ll \cdot \gg_{\widehat{\alpha}}$ on $J^* + \mathcal{L}\mathcal{C}$).** — For any $\Phi_1, \Phi_2 \in C^1(\mathbb{S})$ and for any local
 1666 function $g \in \mathcal{C}$, we define

$$(6.46) \quad \ll j_1^{\Phi_1} + j_2^{\Phi_2} + \mathcal{L}g \gg_{\widehat{\alpha}} = \sum_{i=1,2} \mathbb{E}_{\widehat{\alpha}} \left(\eta_0(1 - \eta_{e_i}) \left[\Phi_i(\theta_0) + \Sigma_g(\widehat{\eta}^{0,e_i}) - \Sigma_g \right]^2 \right),$$

1667 where $\Sigma_g = \sum_{x \in \mathbb{Z}^2} \tau_x g$, which is not a priori well-defined, but whose gradient $\Sigma_g(\widehat{\eta}^{0,e_i}) - \Sigma_g$ is, because
 1668 g is a local function. For any function $\psi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$, define

$$(6.47) \quad \ll \psi, \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \gg_{\widehat{\alpha}} = -\mathbb{E}_{\widehat{\alpha}} \left(\psi \left[\Sigma_g + \sum_{x \in \mathbb{Z}^2} (x_1 \eta_x^{\Phi_1} + x_2 \eta_x^{\Phi_2}) \right] \right)$$

1669 which once again is well-defined because any $\psi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$ is a local function with mean-0 w.r.t. any
 1670 $\mu_{\widehat{\alpha}}$, therefore the expectation above only involves a finite number of non-0 contributions. In particular,
 1671 an elementary computation yields that for any $g \in \mathcal{C}$, and $j \in J^*$

$$\ll \mathcal{L}g + j, \mathcal{L}g + j \gg_{\widehat{\alpha}} = \ll \mathcal{L}g + j \gg_{\widehat{\alpha}}$$

1672 where the left hand-side is given by (6.47) and the right-hand side by (6.46).

1673 **Definition 6.9 (Definition of $\ll \cdot \gg_{\widehat{\alpha}}$ on \mathcal{T}_0^ω).** — For any $\psi \in \mathcal{T}_0^\omega$, define

$$(6.48) \quad \ll \psi \gg_{\widehat{\alpha}} = \sup_{\substack{g \in \mathcal{T}_0^\omega \\ j \in J^*}} \left\{ 2 \ll \psi, \mathcal{L}g + j \gg_{\widehat{\alpha}} - \ll \mathcal{L}g + j \gg_{\widehat{\alpha}} \right\},$$

1674 where \mathcal{T}_0^ω , \mathcal{T}_0^ω and J^ω were defined in (6.43) and (6.45), and the two terms inside braces are respectively
 1675 given by (6.46) and (6.47).

1676 For $\psi \in \mathcal{T}_0^\omega$ and $j_1^{\Phi_1} + j_2^{\Phi_2} + \mathcal{L}g \in J^* + \mathcal{L}\mathcal{C}$, we also define

$$\ll \psi + \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \gg_{\widehat{\alpha}} = \ll \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \gg_{\widehat{\alpha}} + \ll \psi \gg_{\widehat{\alpha}} + 2 \ll \psi, \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \gg_{\widehat{\alpha}},$$

1677 where the three terms in the right-hand side are respectively given by (6.46), (6.48) and (6.47).

1678 These definitions allow us to finally define on $\mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$ a bilinear form $\ll \cdot, \cdot \gg_{\hat{\alpha}}$ by letting
 1679 $\ll \psi, \psi \gg_{\hat{\alpha}} = \ll \psi \gg_{\hat{\alpha}}$ for any $\psi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$, by polarization identity on $\mathcal{T}_0^{\omega^2}$ and $(J^* + \mathcal{L}\mathcal{C})^2$, and
 1680 by (6.47) on $\mathcal{T}_0^\omega \times (J^* + \mathcal{L}\mathcal{C})$.

1681 **Remark 6.10.** — We will see in the proof of Theorem 6.11 below that this definition coincides with
 1682 Definition 6.8 for any $\psi \in \mathcal{T}_0^\omega \cap \{J^* + \mathcal{L}\mathcal{C}\} \subset J^\omega + \mathcal{L}T^\omega$, since in this case the supremum in (6.48) is
 1683 reached for $f = \mathcal{L}g + j^{a,b}$ itself.

1684 For any cylinder function ψ , recall that s_ψ is the smallest fixed integer such that ψ is measurable with
 1685 respect to \mathcal{F}_{s_ψ} , and let $l_\psi = l - s_\psi - 1$ for any integer l large enough. The following result justifies the
 1686 definitions above, and states that $\ll \psi \gg_{\hat{\alpha}}$ defined for any $\psi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$ is the limit of (6.38).

1687 **Theorem 6.11.** — Fix $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, and a sequence $(\hat{K}_l)_{l \in \mathbb{N}}$ such that $\hat{K}_l \in \tilde{\mathbb{K}}_l$ and $\left\| \hat{\alpha}_{\hat{K}_l} - \hat{\alpha} \right\| \rightarrow 0$,
 1688 where $\hat{\alpha}_{\hat{K}_l} \in \mathcal{M}_1(\mathbb{S})$ is the grand-canonical parameter defined in (3.7).

1689 The bilinear form $\ll \cdot, \cdot \gg_{\hat{\alpha}}$ introduced in Definition 6.9 is a semi-inner product on $\mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$,
 1690 and, for any functions $\psi, \varphi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$,

$$(6.49) \quad \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\varphi}} \tau_x \varphi \right) = \ll \psi, \varphi \gg_{\hat{\alpha}}.$$

1691 Furthermore, for any $\psi, \varphi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$, the application $\hat{\alpha} \rightarrow \ll \psi, \varphi \gg_{\hat{\alpha}}$ is continuous in $\hat{\alpha}$, and the
 1692 convergence above is uniform in $\hat{\alpha}$. In particular, for any $\psi \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$,

$$(6.50) \quad \lim_{l \rightarrow \infty} \sup_{\hat{K} \in \tilde{\mathbb{K}}_l} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) = \sup_{\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \ll \psi \gg_{\hat{\alpha}}.$$

1693 The proof of Theorem 6.11 is the purpose of Section 8, and is postponed for now. It requires many
 1694 adaptations because of the angles, but follows the global strategy presented in [27]. Let us explicitly write
 1695 the dependency in p and f of $\mathcal{W}_i^l = \mathcal{W}_{i,p}^{f,l}$ appearing in Proposition 6.7, and define for any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$

$$(6.51) \quad \mathcal{V}_{i,p}^f(\hat{\alpha}) = j_i^\omega + d_s(\alpha) \delta_i \eta_0^{\omega \cdot p} + \mathfrak{d}(\hat{\alpha}) \delta_i \eta_0 + \mathcal{L}f \in \mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}.$$

1696 Recall that $l_f = l - s_f - 1$, where s_f is also the size of the support of $\mathcal{V}_{i,p}^f$ (since we can safely increase
 1697 s_f , in order to have $s_f = s_{\mathcal{V}_{i,p}^f}$) and define

$$Q_1 = (2l+1)^2 \mathcal{W}_{i,p}^{f,l} - \sum_{x \in B_{l_f}} (\tau_x \mathcal{V}_{i,p}^f)(\hat{\rho}_l) \quad \text{and} \quad Q_2 = \sum_{x \in B_{l_f}} \left[(\tau_x \mathcal{V}_{i,p}^f)(\hat{\rho}_l) - (\tau_x \mathcal{V}_{i,p}^f)(\hat{\alpha}) \right].$$

1698 For h a cylinder function measurable w.r.t. sites in B_l , define $\mathcal{D}_{l, \hat{K}}(h) = \mathbb{E}_{l, \hat{K}}(h(-\mathcal{L}_l)h)$. For $\hat{\alpha}_{\hat{K}_l} \rightarrow \hat{\alpha}$,
 1699 the variational formula for the variance yields

$$\begin{aligned} \mathbb{E}_{l, \hat{K}_l} \left(\mathcal{W}_{i,p}^{f,l}(-\mathcal{L}_l^{-1}) \mathcal{W}_{i,p}^{f,l} \right) &= \sup_h \left\{ \mathbb{E}_{l, \hat{K}_l} \left(h \mathcal{W}_{i,p}^{f,l} \right) - \mathcal{D}_{l, \hat{K}_l}(h) \right\} \\ &\leq \sup_h \left\{ \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left(h \sum_{x \in B_{l_f}} (\tau_x \mathcal{V}_{i,p}^f)(\hat{\alpha}) \right) - \frac{1}{3} \mathcal{D}_{l, \hat{K}_l}(h) \right\} \\ &\quad + \sup_h \left\{ \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} (h Q_1) - \frac{1}{3} \mathcal{D}_{l, \hat{K}_l}(h) \right\} + \sup_h \left\{ \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} (h Q_2) - \frac{1}{3} \mathcal{D}_{l, \hat{K}_l}(h) \right\} \\ &\leq \frac{3}{(2l+1)^4} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_f}} (\tau_x \mathcal{V}_{i,p}^f)(\hat{\alpha}) \cdot \sum_{x \in B_{l_f}} (\tau_x \mathcal{V}_{i,p}^f)(\hat{\alpha}) \right) \\ &\quad + \sup_h \left\{ \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} (h Q_1) - \frac{1}{3} \mathcal{D}_{l, \hat{K}_l}(h) \right\} + \sup_h \left\{ \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} (h Q_2) - \frac{1}{3} \mathcal{D}_{l, \hat{K}_l}(h) \right\}. \end{aligned}$$

1700 Since the discrepancies in $Q_1 = (2l+1)^2 \mathcal{W}_{i,p}^{f,l} - \sum_{x \in B_{l_f}} \mathcal{V}_{i,p}^f(\widehat{\rho}_l)$ occur only in $B_{l-1} \setminus B_{l_f}$, letting $\gamma =$
 1701 $1/(2l+1)^2$, Lemma 8.23 below yields that the second term above is less than

$$C_f |B_{l-1} \setminus B_{l_f}| (2l+1)^{-4} = O(l^{-3}).$$

The last term multiplied by $(2l+1)^2$ vanishes as well thanks to Lemma 8.23 and because the diffusion coefficients d_s and \mathfrak{d} are continuous in $\widehat{\alpha}$. Furthermore, as in Lemma 8.23, both of these convergences are uniform in \widehat{K}_l and $\widehat{\alpha}$. We can therefore apply Theorem 6.11 to the first term to obtain that for any $f \in \mathcal{C}$,

$$\limsup_{l \rightarrow \infty} \sup_{\widehat{K}} (2l+1)^2 \mathbb{E}_{l, \widehat{K}} \left(\mathcal{W}_{i,p}^{f,l} (-\mathcal{L}_l)^{-1} \mathcal{W}_{i,p}^{f,l} \right) \leq 3 \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \ll \mathcal{V}_{i,p}^f(\widehat{\alpha}) \gg_{\widehat{\alpha}},$$

1702 therefore to prove Proposition 6.7, and thus Equation (6.7), it is sufficient to prove

$$(6.52) \quad \inf_{f \in \mathcal{C}} \lim_{p \rightarrow \infty} \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \ll \mathcal{V}_{i,p}^f(\widehat{\alpha}) \gg_{\widehat{\alpha}} = 0.$$

1703 This estimate is proved later on in Proposition 6.16, and requires to understand the structure of the space
 1704 $\mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$ equipped with $\ll \cdot \gg_{\widehat{\alpha}}$. It is the main result of this section.

1705 For any $\Phi \in C^1(\mathbb{S})$ and any $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, we shorten

$$\mathbb{E}_{\widehat{\alpha}}(\Phi) := \mathbb{E}_{\widehat{\alpha}}(\Phi(\theta_0) \mid \eta_0 = 1) \quad \text{and} \quad V_{\widehat{\alpha}}(\Phi) := \text{Var}_{\widehat{\alpha}}(\Phi(\theta_0) \mid \eta_0 = 1),$$

1706

$$\text{Cov}_{\widehat{\alpha}}(\omega, \Phi) = \mathbb{E}_{\widehat{\alpha}}(\omega \Phi) - \mathbb{E}_{\widehat{\alpha}}(\omega) \mathbb{E}_{\widehat{\alpha}}(\Phi), \quad \text{and} \quad \widehat{\Phi}(\theta) = \Phi(\theta) - \mathbb{E}_{\widehat{\alpha}}(\Phi).$$

In particular, we denote by $j_i^{\widehat{\Phi}} = j_i^\Phi - \mathbb{E}_{\widehat{\alpha}}(\Phi) j_i = j_i^\Phi + \mathbb{E}_{\widehat{\alpha}}(\Phi) \delta_i \eta$ the associated current. Note that any element $j_1^{\Phi_1} + j_2^{\Phi_2}$ of J^* can be written as a linear combination of the $j_i^{\widehat{\Phi}_i}$ and j_i 's, $i = 1, 2$. For any fixed $\widehat{\alpha}$, we finally define the function h_i^p by

$$\begin{aligned} h_i^p(\widehat{\eta}) &= d_s(\alpha) (\delta_i \eta_0^{\omega, p} + \mathbb{E}_{\widehat{\alpha}}(\omega) j_i) = \delta_i [d_s(\alpha) (\eta_0^\omega \mathbb{1}_{E_p} - \mathbb{E}_{\widehat{\alpha}}(\omega) \eta_0)] \\ &= d_s(\alpha) (\eta_{e_i}^{\widehat{\omega}} - \eta_0^{\widehat{\omega}}) - d_s(\alpha) \left[\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_0^\omega \mathbb{1}_{E_p^c} \right], \end{aligned}$$

1707 where as before $E_p = \left\{ \sum_{x \in B_p} \eta_x \leq |B_p| - 2 \right\}$.

1708 We can now rewrite (6.51) as

$$(6.53) \quad \mathcal{V}_{i,p}^f(\widehat{\alpha}) = j_i^{\widehat{\omega}} + h_i^p + \mathcal{L}f.$$

1709 Note that both $j_i^{\widehat{\omega}}$ and h_i^p depend on $\widehat{\alpha}$ as well as ω , but to simplify notations, we do not write it explicitly.

1710 Throughout this section, we will not indicate the dependencies in ω which is a fixed smooth function.

1711 We now compute the inner product $\ll \cdot, \cdot \gg_{\widehat{\alpha}}$ of h_i^p with elements of $J^* + \mathcal{L}\mathcal{C}$.

1712 **Corollary 6.12.** — For any $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, $g \in \mathcal{C}$, $\Phi \in C^1(\mathbb{S})$ and $i, k = 1, 2$,

$$(6.54) \quad \ll h_i^p, \mathcal{L}g \gg_{\widehat{\alpha}} = 0, \quad \ll h_i^p, j_k^{\widehat{\Phi}} \gg_{\widehat{\alpha}} = \mathbb{1}_{\{i=k\}} q_p^\Phi(\widehat{\alpha}) \quad \text{and} \quad \ll h_i^p, j_k \gg_{\widehat{\alpha}} = \mathbb{1}_{\{i=k\}} r_p(\widehat{\alpha}),$$

1713 where we shortened

$$q_p^\Phi(\widehat{\alpha}) = -\alpha d_s(\alpha) \text{Cov}_{\widehat{\alpha}}(\omega, \Phi) \mu_{\widehat{\alpha}}(E_p \mid \eta_0 = 1)$$

1714 and

$$r_p(\widehat{\alpha}) = d_s(\alpha) \mathbb{E}_{\widehat{\alpha}}(\omega) \mathbb{E}_{\widehat{\alpha}} \left(\eta_0 \mathbb{1}_{E_p^c} [1 - \eta_{e_1} - (2p+1)^2 (\alpha - \eta_{e_1})] \right).$$

1715 Furthermore, shortening $q_p(\widehat{\alpha}) := q_p^\omega(\widehat{\alpha})$,

$$(6.55) \quad \lim_{p \rightarrow \infty} \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} |q_p(\widehat{\alpha}) \mu_{\widehat{\alpha}}(E_p \mid \eta_0 = 1) + \alpha d_s(\alpha) V_{\widehat{\alpha}}(\omega)| = 0 \quad \text{and} \quad \lim_{p \rightarrow \infty} \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \frac{r_p^2(\widehat{\alpha})}{\alpha(1-\alpha)} = 0.$$

1716 In particular, $q_p(\widehat{\alpha}) \rightarrow -\alpha d_s(\alpha) V_{\widehat{\alpha}}(\omega)$ and $r_p(\widehat{\alpha}) \rightarrow 0$ as $p \rightarrow \infty$ uniformly in $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$.

1717 *Proof of Corollary 6.12.* — The three identities in (6.54) are consequences of (6.47). Regarding the first
1718 one,

$$\ll h_i^p, \mathcal{L}g \gg_{\hat{\alpha}} = -\mathbb{E}_{\hat{\alpha}}(h_i^p \Sigma_g) = -d_s(\alpha) \mathbb{E}_{\hat{\alpha}} \left(\Sigma_g \left[\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p} - \eta_0^\omega \mathbb{1}_{E_p} - \mathbb{E}_{\hat{\alpha}}(\omega) \eta_{e_i} + \mathbb{E}_{\hat{\alpha}}(\omega) \eta_0 \right] \right) = 0$$

1719 by translation invariance of $\mu_{\hat{\alpha}}$.

For the second, we write

$$\begin{aligned} \ll h_i^p, j_k^{\hat{\Phi}} \gg_{\hat{\alpha}} &= - \sum_{x \in \mathbb{Z}^2} x_k \mathbb{E}_{\hat{\alpha}}(h_i^p \eta_x^{\hat{\Phi}}) \\ &= -d_s(\alpha) \sum_{x \in \mathbb{Z}^2} x_k \mathbb{E}_{\hat{\alpha}}((\eta_{e_i}^{\hat{\omega}} - \eta_0^{\hat{\omega}}) \eta_x^{\hat{\Phi}}) + d_s(\alpha) \sum_{x \in \mathbb{Z}^2} x_k \mathbb{E}_{\hat{\alpha}}((\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_0^\omega \mathbb{1}_{E_p^c}) \eta_x^{\hat{\Phi}}). \end{aligned}$$

Since by construction $\hat{\Phi}$ has mean 0 w.r.t. the product measure $\mu_{\hat{\alpha}}$, for any function ψ which does not depend on θ_x , $\mathbb{E}_{\hat{\alpha}}(\psi \eta_x^{\hat{\Phi}}) = 0$. In particular, in both sums, any term $x \neq 0, e_i$ vanishes. The terms for $x = 0$ also vanishes because of the factor x_k , and so does the term for $x = e_i$ if $i \neq k$. This yields

$$\begin{aligned} \ll h_i^p, j_k^{\hat{\Phi}} \gg_{\hat{\alpha}} &= -\mathbb{1}_{\{i=k\}} d_s(\alpha) \left\{ \mathbb{E}_{\hat{\alpha}}(\eta_{e_i}^{\hat{\omega}} \eta_{e_i}^{\hat{\Phi}}) - \mathbb{E}_{\hat{\alpha}}(\eta_{e_i}^\omega \eta_{e_i}^{\hat{\Phi}} \mathbb{1}_{\tau_{e_i} E_p^c}) \right\} \\ &= -\mathbb{1}_{\{i=k\}} \alpha d_s(\alpha) \text{Cov}_{\hat{\alpha}}(\omega, \Phi) \mu_{\hat{\alpha}}(E_p | \eta_0 = 1) \end{aligned}$$

1720 as wanted.

We now turn to the third identity, for which we can write, applying the same steps as before

$$\ll h_i^p, j_k \gg_{\hat{\alpha}} = -d_s(\alpha) \sum_{x \in \mathbb{Z}^2} x_k \mathbb{E}_{\hat{\alpha}}((\eta_{e_i}^{\hat{\omega}} - \eta_0^{\hat{\omega}}) \eta_x) + d_s(\alpha) \sum_{x \in \mathbb{Z}^2} x_k \mathbb{E}_{\hat{\alpha}}((\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_0^\omega \mathbb{1}_{E_p^c}) \eta_x).$$

By definition of $\hat{\omega}$, each term in the first sum vanishes. Regarding the second term, recall that $B_p(x) = x + B_p$, for any $x \in (B_p \cup B_p(e_i))^c$ and any $x \in B_p \cap B_p(e_i) \setminus \{0, e_i\}$, the corresponding contribution vanishes, because $\eta_{e_i}^\omega \eta_x \mathbb{1}_{\tau_{e_i} E_p^c}$ and $\eta_0^\omega \eta_x \mathbb{1}_{E_p^c}$ have the same distribution. The term for $x = 0$ vanishes once again because of the factor x_k . We can therefore write

$$\begin{aligned} \ll h_i^p, j_k \gg_{\hat{\alpha}} &= \mathbb{1}_{\{i=k\}} d_s(\alpha) \mathbb{E}_{\hat{\alpha}} \left((\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_0^\omega \mathbb{1}_{E_p^c}) \eta_{e_i} \right) \\ &\quad + d_s(\alpha) \sum_{\substack{x \in B_p, x_i = -p \\ \text{or } x \in B_p(e_i), x_i = p+1}} x_k \mathbb{E}_{\hat{\alpha}} \left((\eta_{e_i}^\omega \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_0^\omega \mathbb{1}_{E_p^c}) \eta_x \right). \end{aligned}$$

If $i \neq k$, the sum in the second line vanishes because the contributions for $x_k = q$ cancel out the contributions for $x_k = -q$. If $i = k$, all the contributions for $x_i = -p$ (i.e. $x \in B_p \setminus B_p(e_i)$) are identical and equal to $-pd_s(\alpha) \mathbb{E}_{\hat{\alpha}}(\omega) \mathbb{E}_{\hat{\alpha}}(\alpha \eta_{e_i} \mathbb{1}_{\tau_{e_i} E_p^c} - \eta_x \eta_0 \mathbb{1}_{E_p^c}) = -pd_s(\alpha) \mathbb{E}_{\hat{\alpha}}(\omega) \mathbb{E}_{\hat{\alpha}}((\alpha - \eta_{e_1}) \eta_0 \mathbb{1}_{E_p^c})$ and the contributions for $x_i = p+1$ (i.e. $x \in B_p(e_i) \setminus B_p$) are each equal to $-(p+1)d_s(\alpha) \mathbb{E}_{\hat{\alpha}}(\omega) \mathbb{E}_{\hat{\alpha}}((\alpha - \eta_{e_1}) \eta_0 \mathbb{1}_{E_p^c})$. Since each of those contributions appear $2p+1$ times, we finally obtain as wanted that

$$\ll h_i^p, j_k \gg_{\hat{\alpha}} = \mathbb{1}_{\{i=k\}} d_s(\alpha) \mathbb{E}_{\hat{\alpha}}(\omega) \left[\mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{e_1}) \eta_0 \mathbb{1}_{E_p^c} \right) - (2p+1)^2 \mathbb{E}_{\hat{\alpha}} \left((\alpha - \eta_{e_1}) \eta_0 \mathbb{1}_{E_p^c} \right) \right].$$

1721 According to Proposition B.3, $c(1-\rho) \leq d_s(\rho) \leq C(1-\rho)$ for some positive constants c, C . Using this
1722 fact, the uniform estimates (6.55) follow from elementary computations : for high densities, the factor
1723 $\mu_{\hat{\alpha}}(E_p^c | \eta_0)$ fail to converge uniformly in $\hat{\alpha}$, but then $d_s(\alpha)$ provides the needed control. Regarding r_p the
1724 principle is the same, and the extra factor $(2p+1)^2$ is balanced out as $\alpha \rightarrow 1$ by the factor $\alpha - \eta_1$. We
1725 start with the first estimate. To prove that $q_p(\hat{\alpha}) \mu_{\hat{\alpha}}(E_p | \eta_0 = 1) + \alpha d_s(\alpha) V_{\hat{\alpha}}(\omega)$ vanishes uniformly in $\hat{\alpha}$,
1726 by definition of q_p and since ω is bounded, it is enough to prove that $|1 - \mu_{\hat{\alpha}}(E_p | \eta_0 = 1)| \alpha d_s(\alpha)$ also
1727 does. The probability $\mu_{\hat{\alpha}}(E_p | \eta_0 = 1)$ is explicit, and given by

$$\mu_{\hat{\alpha}}(E_p | \eta_0 = 1) = 1 - \alpha^P - P(1 - \alpha) \alpha^{P-1}$$

1728 where we shortened $P = (2p+1)^2 - 1 = |B_p \setminus \{0\}|$. In particular, since $d_s(\alpha) \leq C(1 - \alpha)$,

$$|1 - \mu_{\hat{\alpha}}(E_p | \eta_0 = 1)| \alpha d_s(\alpha) \leq C \alpha (1 - \alpha) [2\alpha^P + 2P(1 - \alpha) \alpha^{P-1} - [\alpha^P + P(1 - \alpha) \alpha^{P-1}]^2].$$

1729 Thanks to the prefactor $1 - \alpha$, Each of the terms above is bounded by $P^a(1 - \alpha)^{a+1}\alpha^{C_1P}$ for some
 1730 different constants $a \in \{0, 1, 2\}$ and $C_1 > 0$ independent of P . The previous expression is maximal in
 1731 $\alpha_P = C_1P/(a + 1 + C_1P)$, and is therefore, uniformly in $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, less than

$$P^a \left(\frac{a + 1}{a + 1 + C_1P} \right)^{a+1},$$

1732 which vanishes as wanted as $P \rightarrow \infty$.

1733 We now turn to the second estimate. Once again, since $d_s(\alpha) \leq C(1 - \alpha)$, we obtain immediately

$$\frac{r_p(\hat{\alpha})^2}{\alpha(1 - \alpha)} \leq C' \frac{1 - \alpha}{\alpha} \mathbb{E}_{\hat{\alpha}} \left(\eta_0 \mathbb{1}_{E_p^c} [1 - \eta_{e_1} - (2p + 1)^2(\alpha - \eta_{e_1})] \right)^2.$$

1734 The expectation above can be split in two terms, resp. $(1 - (2p + 1)^2) \mathbb{E}_{\hat{\alpha}} \left(\eta_0(1 - \eta_{e_1}) \mathbb{1}_{E_p^c} \right)$ and $(1 -$
 1735 $\alpha)(2p + 1)^2 \mathbb{E}_{\hat{\alpha}} \left(\eta_0 \mathbb{1}_{E_p^c} \right)$. We still shorten $P = (2p + 1)^2 - 1 = |B_p \setminus \{0\}|$, to obtain the bound

$$\left| \mathbb{E}_{\hat{\alpha}} \left(\eta_0 \mathbb{1}_{E_p^c} [1 - \eta_{e_1} - (2p + 1)^2(\alpha - \eta_{e_1})] \right) \right| \leq P(1 - \alpha)\alpha^P + \alpha(1 - \alpha)(P + 1)\mu_{\hat{\alpha}}(E_p^c | \eta_0 = 1).$$

1736 the last probability $\mu_{\hat{\alpha}}(E_p^c | \eta_0 = 1)$ has already been computed for the previous estimate, and one obtains
 1737 straightforwardly that $r_p(\hat{\alpha})^2/\alpha(1 - \alpha)$ is also bounded from above by a (finite) sum of terms of the form
 1738 $C_1P^a(1 - \alpha)^{a+1}\alpha^{C_2P}$ for $a \in \{2, 3, 4\}$ and C_1, C_2 positive constants. As before, each of those vanishes
 1739 uniformly in $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, which concludes the proof. \square

1740 We are ready to investigate the structure of \mathcal{T}_0^ω with respect to the semi-norm $\ll \cdot \gg_{\hat{\alpha}}$ on $\mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C}$.
 1741 Denote by $\mathcal{N}_{\hat{\alpha}} = \text{Ker } \ll \cdot \gg_{\hat{\alpha}}$ and define $\mathcal{H}_{\hat{\alpha}}^\omega$ the completion of $(\mathcal{T}_0^\omega + J^* + \mathcal{L}\mathcal{C})/\mathcal{N}_{\hat{\alpha}}$ with respect to
 1742 $\ll \cdot \gg_{\hat{\alpha}}^{1/2}$. We need to define $\ll \cdot \gg_{\hat{\alpha}}$ on a rather general space, including in particular $J^* + \mathcal{L}\mathcal{C}$, in order
 1743 to be able later on to estimate the drift contribution to the hydrodynamic limit. However for now, we
 1744 focus on the symmetric current, and further define H^ω the closure in $\mathcal{H}_{\hat{\alpha}}^\omega$ of $(\mathcal{T}_0^\omega + J^\omega + \mathcal{L}T^\omega)/\mathcal{N}_{\hat{\alpha}}$.

1745 **Proposition 6.13 (Structure of H^ω).** — For any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, $(\mathcal{H}_{\hat{\alpha}}^\omega, \ll \cdot \gg_{\hat{\alpha}}^{1/2})$ is a Hilbert space, and

$$H^\omega = \frac{\overline{\mathcal{L}T^\omega}}{\mathcal{N}_{\hat{\alpha}}} \oplus J^\omega,$$

1746 where $\overline{\mathcal{L}T^\omega}/\mathcal{N}_{\hat{\alpha}}$ is the closure of $\mathcal{L}T^\omega/\mathcal{N}_{\hat{\alpha}}$ w.r.t. $\ll \cdot \gg_{\hat{\alpha}}$ in $\mathcal{H}_{\hat{\alpha}}^{\omega,0}$.

1747 *Proof of Proposition 6.13.* — First note that if $\alpha = 0$ or 1 , $\ll \cdot \gg_{\hat{\alpha}} \equiv 0$ and therefore $\mathcal{H}_{\hat{\alpha}}^\omega = \{0\}$ is
 1748 trivial. We now assume that $\hat{\alpha}$ is such that $\alpha \in]0, 1[$. Since we took the quotient by $\mathcal{N}_{\hat{\alpha}}$, the fact that
 1749 $(\mathcal{H}_{\hat{\alpha}}^\omega, \ll \cdot \gg_{\hat{\alpha}}^{1/2})$ is a Hilbert space is immediate. By construction H^ω is a closed linear subspace of $\mathcal{H}_{\hat{\alpha}}^\omega$,
 1750 and the inclusion

$$\frac{\overline{\mathcal{L}T^\omega}}{\mathcal{N}_{\hat{\alpha}}} + J^\omega \subset H^\omega$$

1751 is immediate. Since both sets are closed subspaces of $\mathcal{H}_{\hat{\alpha}}^\omega$, we have

$$H^\omega = \left(\frac{\overline{\mathcal{L}T^\omega}}{\mathcal{N}_{\hat{\alpha}}} + J^\omega \right) \oplus \left(\frac{\overline{\mathcal{L}T^\omega}}{\mathcal{N}_{\hat{\alpha}}} + J^\omega \right)^{\perp, H^\omega},$$

1752 where the second set on the right-hand side denotes the orthogonal complement of $\frac{\overline{\mathcal{L}T^\omega}}{\mathcal{N}_{\hat{\alpha}}} + J^\omega$ in H^ω . To
 1753 prove the converse inclusion, it is therefore sufficient to prove that this orthogonal complement is reduced
 1754 to $\{0\}$. This is rather straightforward, although a bit technical because of the different definitions for
 1755 $\ll \cdot \gg_{\hat{\alpha}}$. For that purpose, and to give a proof as clear as possible, let us shorten $M = \overline{\mathcal{L}T^\omega}/\mathcal{N}_{\hat{\alpha}} + J^\omega$,
 1756 and denote by $m = j^{a,b} + \mathcal{L}h$ its elements. Since $\overline{M}^{\perp, H^\omega} \subset H^\omega$, and since H^ω is by definition the closure
 1757 of $\mathcal{T}_0^\omega + M$ any of its element can be written either as $g + m$, where $g \in \mathcal{T}_0^\omega$ and $m \in M$, or as the limit
 1758 of elements of this type. In order to avoid taking convergent sequences, fix

$$g_0 + m_0 \in \overline{M}^{\perp, H^\omega},$$

1759 where $g_0 \in \mathcal{T}_0^\omega$ and $m_0 \in M$, we want to prove that $g_0 + m_0 = 0$. By construction, for any $m \in M$

$$\ll g_0 + m_0, m \gg_{\hat{\alpha}} = 0.$$

and since $g_0 \in \mathcal{T}_0^\omega$, we can rewrite by the definition of $\ll \cdot \gg_{\hat{\alpha}}$ on \mathcal{T}_0^ω (cf. (6.48))

$$(6.56) \quad \ll g_0 \gg_{\hat{\alpha}} = \sup_{m \in M} \{2 \ll g_0, m \gg_{\hat{\alpha}} - \ll m \gg_{\hat{\alpha}}\},$$

1760 therefore there exists a sequence $(m^k)_{k \rightarrow \infty}$ of elements of M such that $\ll g_0 + m^k \gg_{\hat{\alpha}} \rightarrow 0$ as $k \rightarrow \infty$.
1761 We can thus write

$$\ll g_0 + m_0 \gg_{\hat{\alpha}} = \ll g_0 + m^k, g_0 + m_0 \gg_{\hat{\alpha}} + \ll m_0 - m^k, g_0 + m_0 \gg_{\hat{\alpha}}.$$

1762 The second term vanishes because $m_0 - m^k \in M$, whereas the first term in the right-hand side vanishes
1763 as $k \rightarrow \infty$, therefore $\ll g_0 + m_0 \gg_{\hat{\alpha}} = 0$ as wanted. The same proof holds if $g_0 + m_0$ is replaced by a
1764 convergent sequence of elements of $\mathcal{T}_0^\omega + M$, which proves the reverse inclusion.

1765 Only remains to prove that the sum $\frac{\mathcal{L}T^\omega}{\mathcal{N}_{\hat{\alpha}}} + \mathcal{J}^\omega$ is direct. Assume that for some coefficients a_i, b_i , and
1766 for some cylinder function $g \in \mathcal{T}_0^\omega$

$$\ll \sum_{i=1,2} a_i j_i^{\hat{\omega}} + b_i j_i - \mathcal{L}g \gg_{\hat{\alpha}} = 0.$$

1767 (We should really write this identity for a sequence g_n instead of g , with the identity above holding only
1768 as $n \rightarrow \infty$, but this is purely cosmetic and the proof below holds in this case as well). Thanks to equation
1769 (6.54), we can take the inner product of the identity above w.r.t. h_i^p and since we assumed that $0 < \alpha < 1$
1770 let $p \rightarrow \infty$ to obtain that for $i = 1, 2$, $a_i d_s(\alpha) V_{\hat{\alpha}}(\omega) \alpha (1 - \alpha) = 0$, therefore $a_1 V_{\hat{\alpha}}(\omega) = a_2 V_{\hat{\alpha}}(\omega) = 0$. In
1771 both cases, we therefore have $\ll a_1 j_1^{\hat{\omega}} \gg_{\hat{\alpha}} = \ll a_2 j_2^{\hat{\omega}} \gg_{\hat{\alpha}} = 0$. This yields

$$\ll b_1 j_1 + b_2 j_2 - \mathcal{L}g \gg_{\hat{\alpha}} = 0,$$

1772 so that we can now take the inner product with $\delta_i \eta_0 = -j_i$ (which is orthogonal to $\mathcal{L}g$), to obtain that
1773 $b_1 \alpha (1 - \alpha) = b_2 \alpha (1 - \alpha) = 0$, therefore $b_1 = b_2 = 0$ as wanted. This proves that the sum is direct, and
1774 concludes the proof of Proposition 6.13. \square

1775 The next Proposition states that in $\mathcal{H}_{\hat{\alpha}}^\omega$, j_i^ω can be written as a combination of h_i^p and j_i , up to a
1776 function which takes the form $\mathcal{L}g$, and that the coefficients converge as $p \rightarrow \infty$ to those given in (6.53).

1777 **Proposition 6.14 (Decomposition of the currents).** — For any positive integer p , define

$$c_p(\alpha) = \begin{cases} \mu_{\hat{\alpha}}(E_p | \eta_0 = 1)^{-1} & \text{if } \alpha < 1 \\ 1 & \text{else} \end{cases}, \quad \text{and} \quad d_p(\hat{\alpha}) = \begin{cases} -r_p(\hat{\alpha}) c_p(\alpha) / \alpha (1 - \alpha) & \text{if } 0 < \alpha < 1 \\ 0 & \text{else} \end{cases},$$

1778 where r_p was defined in Corollary 6.12. Then, for any $i \in 1, 2$ and $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$.

$$(6.57) \quad \inf_{g \in T^\omega} \ll j_i^{\hat{\omega}} + c_p(\alpha) h_i^p + d_p(\hat{\alpha}) j_i + \mathcal{L}g \gg_{\hat{\alpha}} = 0.$$

1779 Furthermore, any sequence $(g_m)_m$ ultimately realizing (6.57) can be chosen independently of p , and also
1780 ultimately realizes

$$(6.58) \quad \inf_{g \in T^\omega} \ll j_i^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}}.$$

1781 *Proof of Proposition 6.14.* — We start by clearing out the trivial cases when $\alpha = 0$ and $\alpha = 1$. In
1782 those, all quantities vanish and (6.57) is trivially true for any coefficients. Another trivial case is when
1783 $V_{\hat{\alpha}}(\omega) = 0$. In this case, $j_i^{\hat{\omega}} = 0$ in $\mathcal{H}_{\hat{\alpha}}^\omega$, therefore, the h_i^p and j_i being orthogonal (as local gradients) to
1784 $\mathcal{L}T^\omega$, and h_i^p being orthogonal to j_k for $k \neq i$, as a consequence of Proposition 6.13 we can then write
1785 $\ll h_i^p + a_p j_i \gg_{\hat{\alpha}} = 0$ for some constant a_p . This constant can be determined using Lemma 6.12 and taking
1786 the inner product of the previous quantity with j_i , which yields $a_p = -r_p(\hat{\alpha}) / \ll j_i \gg_{\hat{\alpha}} = -r_p(\hat{\alpha}) / \alpha (1 - \alpha)$.
1787 In this case, $\ll c_p(\alpha) h_i^p + d_p(\hat{\alpha}) j_i \gg_{\hat{\alpha}} = 0$ for any p , as wanted.

1788 We now fix $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$ satisfying $\alpha \in]0, 1[$ and $V_{\hat{\alpha}}(\omega) > 0$. Fix $p \in \mathbb{N}$, and define c_p, d_p as in
1789 Proposition 6.14, we now prove that (6.57) holds. According to Proposition 6.13, there exists coefficients
1790 $a_{i,k}^p$ and $b_{i,k}^p$ such that,

$$(6.59) \quad \inf_{g \in T^\omega} \ll h_i^p + \sum_{k=1,2} a_{i,k}^p j_k^{\hat{\omega}} + b_{i,k}^p j_k + \mathcal{L}g \gg_{\hat{\alpha}} = 0.$$

1791 In order not to burden the proof, we will assume that the infimum in g is reached, i.e. that there exists
 1792 a function $g_i^p \in T^\omega$ such that

$$(6.60) \quad \ll h_i^p + \left[\sum_{k=1,2} a_{i,k}^p j_k^{\widehat{\omega}} + b_{i,k}^p j_k \right] + \mathcal{L}g_i^p \gg_{\widehat{\alpha}} = 0.$$

1793 This assumption is purely for convenience, and we can substitute at any point to g_i^p a sequence of functions
 1794 $(g_{i,m}^p)_{m \in \mathbb{N}}$ such that the previous identity holds in the limit $m \rightarrow \infty$.

1795 Using (6.47), one obtains immediately that $\ll j_i^{\widehat{\omega}}, j_k^{\widehat{\omega}} \gg_{\widehat{\alpha}} = \mathbb{1}_{\{i=k\}} V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha)$, $\ll j_i^{\widehat{\omega}}, j_k \gg_{\widehat{\alpha}} = 0$ and
 1796 $\ll j_i, j_k \gg_{\widehat{\alpha}} = \mathbb{1}_{\{i=k\}} \alpha(1-\alpha)$. Using these formulas and Corollary 6.12, we take the inner product of the
 1797 function in (6.60) with $j_l^{\widehat{\omega}}$, j_l , $\mathcal{L}g_l^p$, and h_l^p , to obtain the four identities

$$\mathbb{1}_{\{i=l\}} q_p(\widehat{\alpha}) + a_{i,l}^p V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha) + \ll \mathcal{L}g_i^p, j_l^{\widehat{\omega}} \gg_{\widehat{\alpha}} = 0, \quad \mathbb{1}_{\{i=l\}} r_p(\widehat{\alpha}) + b_{i,l}^p \alpha(1-\alpha) = 0,$$

1798

$$(6.61) \quad \sum_{k=1,2} a_{i,k}^p \ll j_k^{\widehat{\omega}}, \mathcal{L}g_i^p \gg_{\widehat{\alpha}} + \ll \mathcal{L}g_i^p, \mathcal{L}g_i^p \gg_{\widehat{\alpha}} = 0 \quad \text{and} \quad \ll h_i^p, h_i^p \gg_{\widehat{\alpha}} + a_{i,l}^p q_p(\widehat{\alpha}) + b_{i,l}^p r_p(\widehat{\alpha}) = 0.$$

1799 Note that since we assumed $\alpha \in]0, 1[$, $V_{\widehat{\alpha}}(\omega) > 0$ and $p > 0$, we have $q_p(\widehat{\alpha}) < 0$. Define A_p , B_p , H_p ,
 1800 G_p and J_p the matrices whose respective elements are given for $i, k = 1, 2$ by $a_{i,k}^p$, $b_{i,k}^p$, $\ll h_i^p, h_k^p \gg_{\widehat{\alpha}}$,
 1801 $\ll \mathcal{L}g_i^p, \mathcal{L}g_k^p \gg_{\widehat{\alpha}}$ and $\ll \mathcal{L}g_i^p, j_k^{\widehat{\omega}} \gg_{\widehat{\alpha}}$. Note in particular that H_p and G_p are symmetric with non-negative
 1802 eigenvalues. Further denote by I the two-dimensional identity matrix. The four identities above then
 1803 rewrite in matrix form as

$$J_p = -q_p(\widehat{\alpha})I - V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha) A_p, \quad B_p = -\frac{r_p(\widehat{\alpha})}{\alpha(1-\alpha)} I$$

1804

$$-A_p J_p^\dagger = G_p \quad \text{and} \quad -q_p(\widehat{\alpha}) A_p - r_p(\widehat{\alpha}) B_p = H_p,$$

1805 where J_p^\dagger is the transposed matrix of J_p . The second and last identities show that B_p and A_p are
 1806 symmetric, therefore so is J^p , and that

$$A_p = -\frac{1}{q_p(\widehat{\alpha})} \left[H_p - \frac{r_p(\widehat{\alpha})^2}{\alpha(1-\alpha)} I \right].$$

1807 In particular, since H_p is positive in the matrix sense, it is diagonalizable, and thus so is A_p . Finally, the
 1808 first and third identities then yields

$$A_p [q_p(\widehat{\alpha})I + V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha) A_p] = G_p.$$

1809 therefore, since G_p is positive in the matrix sense, any eigenvalue λ of A_p must satisfy

$$\lambda [q_p(\widehat{\alpha}) + V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha) \lambda] \geq 0,$$

1810 and therefore $\lambda > -q_p(\widehat{\alpha})/V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha) > 0$. Let C_p denote the inverse of A_p , which is a positive matrix
 1811 with eigenvalues bounded from above by $-V_{\widehat{\alpha}}(\omega) \alpha(1-\alpha)/q_p(\widehat{\alpha})$. Since A_p is invertible, we can therefore
 1812 rewrite (6.60) as

$$(6.62) \quad \ll j_i^{\widehat{\omega}} + \left[\sum_{k=1,2} c_{i,k}^p h_k^p + d_{i,k}^p j_k \right] + \mathcal{L}\tilde{g}_i^k \gg_{\widehat{\alpha}} = 0.$$

which holds for $i = 1, 2$, where $\tilde{g}_i^k = \sum_{k=1,2} c_{i,k}^p g_k^p$, and the $c_{i,k}^p$ (resp. $d_{i,k}^p$) are the matrix elements
 of C_p (resp. $D_p := C_p B_p$). For $x, y \in \mathbb{R}^2$, shorten $x \cdot y = x_1 y_1 + x_2 y_2$ their usual inner product. Let
 $j^{\widehat{\omega}} = (j_1^{\widehat{\omega}}, j_2^{\widehat{\omega}})$, and define the quadratic form Q as

$$x^\dagger Q x = \inf_{g \in T^\omega} \ll x \cdot j^{\widehat{\omega}} + \mathcal{L}g \gg_{\widehat{\alpha}}.$$

1813 Then, (6.62) yields for any $x \in \mathbb{R}^2$

$$(6.63) \quad \inf_{g \in T^\omega} \ll x \cdot j^{\widehat{\omega}} + \left[\sum_{i,k=1,2} x_i c_{i,k}^p h_k^p + x_i d_{i,k}^p j_k \right] + \mathcal{L}g \gg_{\widehat{\alpha}} = 0.$$

Taking the inner product of the expression above with $x \cdot j^{\widehat{\omega}} + \mathcal{L}g$, and since the terms in the sum are orthogonal to any $\mathcal{L}g$, we obtain

$$\begin{aligned} x^\dagger Qx &= \inf_{g \in T^\omega} \ll x \cdot j^{\widehat{\omega}} + \mathcal{L}g \gg_{\widehat{\alpha}} = - \ll x \cdot j^{\widehat{\omega}}, \sum_{i,k=1,2} x_i c_{i,k}^p h_k^p + x_i d_{i,k}^p j_k \gg_{\widehat{\alpha}} \\ &= - \sum_{i,k=1,2} x_i x_k c_{i,k}^p \ll h_k^p, j_k^{\widehat{\omega}} \gg_{\widehat{\alpha}} + x_i x_k d_{i,k}^p \ll j_k, j_k^{\widehat{\omega}} \gg_{\widehat{\alpha}} \\ &= -q_p(\widehat{\alpha}) x^\dagger C_p x, \end{aligned}$$

1814 thanks to Corollary 6.12 and because j_k and $j_k^{\widehat{\omega}}$ are orthogonal. We prove in Appendix B.2, equation
1815 (B.6), that $Q = \alpha V(\widehat{\alpha}) d_s(\alpha) I$, therefore

$$C_p = -\frac{\alpha V(\widehat{\alpha}) d_s(\alpha)}{q_p} I = \mu_{\widehat{\alpha}}(E_p \mid \eta_0 = 1)^{-1} I = c_p(\alpha) I,$$

1816 and $D_p = [-c_p(\alpha) r_p(\widehat{\alpha}) / \alpha(1 - \alpha)] I = d_p(\widehat{\alpha}) I$, where c_p, d_p were defined in Proposition 6.14. We can now
1817 rewrite (6.63) as wanted as

$$(6.64) \quad \inf_{g \in T^\omega} \ll j_i^{\widehat{\omega}} + c_p(\alpha) h_i^p + d_p(\widehat{\alpha}) j_i + \mathcal{L}g \gg_{\widehat{\alpha}} = 0.$$

1818 Since h_i^p and j_i are both orthogonal to any $\mathcal{L}g$, taking the inner product of the identity above with
1819 $j_i^{\widehat{\omega}} + \mathcal{L}g$, one obtains that any sequence of functions realizing the infimum above also realizes $\inf_{g \in T^\omega} \ll$
1820 $j_i^{\widehat{\omega}} + \mathcal{L}g \gg_{\widehat{\alpha}}$, which proves the last statement and concludes the proof of Proposition (6.14). \square

1821 **Remark 6.15 (Bound on $\ll h_i^p \gg_{\widehat{\alpha}}$).** — We already obtained in (6.61) $\ll h_i^p, h_i^p \gg_{\widehat{\alpha}} + a_{i,l}^p q_p(\widehat{\alpha}) +$
1822 $b_{i,l}^p r_p(\widehat{\alpha}) = 0$. Since we now have an explicit expression for the matrix $A_p = C_p^{-1} = c_p^{-1}(\alpha) I$, and
1823 $B_p = -r_p(\widehat{\alpha}) / \alpha(1 - \alpha) I$, we obtain $\ll h_i^p \gg_{\widehat{\alpha}} = -q_p(\widehat{\alpha}) c_p^{-1}(\alpha) + \frac{r_p(\widehat{\alpha})^2}{\alpha(1 - \alpha)}$. Equation (6.55) then yields the
1824 uniform bound

$$(6.65) \quad \lim_{p \rightarrow \infty} \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} | \ll h_i^p \gg_{\widehat{\alpha}} - \alpha d_s(\alpha) V_{\widehat{\alpha}}(\omega) | = 0.$$

1825 We now prove equation (6.52), and thus concludes the proof of Theorem 6.1. Up until now, we have
1826 only used $\ll \cdot \gg_{\widehat{\alpha}}$ for functions in T^ω , but in (6.52) the function f is a priori no longer in T^ω but rather
1827 in \mathcal{C} , we therefore need the extension of $\ll \cdot \gg_{\widehat{\alpha}}$ to $\mathcal{L}\mathcal{C}$ introduced in Definitions 6.8 and 6.9. Thanks to
1828 (6.53), the result can be stated as follows.

1829 **Proposition 6.16 (Uniform bound on $\ll \mathcal{V}_{i,p}^f \gg_{\widehat{\alpha}}$).** — Identity (6.52) holds, in the sense that there
1830 exists a sequence of local functions $f_n \in \mathcal{C}$ such that

$$(6.66) \quad \limsup_{n \rightarrow \infty} \limsup_{p \rightarrow \infty} \sup_{\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \ll j_i^{\widehat{\omega}} + h_i^p + \mathcal{L}f_n \gg_{\widehat{\alpha}} = 0.$$

1831 Furthermore, for any $\widehat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, $\lim_{n \rightarrow \infty} \ll j_i^{\widehat{\omega}} + \mathcal{L}f_n \gg_{\widehat{\alpha}} = \inf_{g \in T^\omega} \ll j_i^{\widehat{\omega}} + \mathcal{L}g \gg_{\widehat{\alpha}}$

Proof of Proposition 6.16. — In order not to burden with technical estimates, we start by cutting off the
extreme densities for which the convergences as $p \rightarrow \infty$ can be problematic. For any $\widehat{\alpha}$, we can write by
triangular inequality and using (6.65),

$$\begin{aligned} \ll j_i^{\widehat{\omega}} + h_i^p + \mathcal{L}f \gg_{\widehat{\alpha}} &\leq \ll j_i^{\widehat{\omega}} \gg_{\widehat{\alpha}} + \ll h_i^p \gg_{\widehat{\alpha}} + \ll \mathcal{L}f \gg_{\widehat{\alpha}} \\ &\leq V_{\widehat{\alpha}}(\omega) \alpha(1 - \alpha) + \alpha d_s(\alpha) V_{\widehat{\alpha}}(\omega) (1 + o_p(1)) + \sum_{i=1,2} \mathbb{E}_{\widehat{\alpha}}(\eta_0(1 - \eta_{e_i}) [\Sigma_f(\widehat{\eta}^{0,e_i}) - \Sigma_f]^2), \end{aligned}$$

where the $o_p(1)$ does not depend on $\widehat{\alpha}$. As stated in Proposition B.3, $d_s(\alpha) \leq C(1 - \alpha)$, ω is bounded,
and f is a cylinder function and therefore $\Sigma_f(\widehat{\eta}^{0,e_i}) - \Sigma_f$ is bounded as well. Fix $\epsilon > 0$, in particular, the
estimate above yields, for some constant $C_{\omega,f}$, and for any $\widehat{\alpha}$ such that $\alpha \notin [\epsilon, 1 - \epsilon]$

$$\ll j_i^{\widehat{\omega}} + h_i^p + \mathcal{L}f \gg_{\widehat{\alpha}} \leq C_{\omega,f} (1 + o_p(1)) \epsilon.$$

1832 We now fix $\widehat{\alpha}$ such that $\epsilon \leq \alpha \leq 1 - \epsilon$, by triangular inequality,

$$\ll j_i^{\widehat{\omega}} + h_i^p + \mathcal{L}f \gg_{\widehat{\alpha}} \leq \ll j_i^{\widehat{\omega}} + c_p(\alpha) h_i^p + d_p(\widehat{\alpha}) j_i + \mathcal{L}f \gg_{\widehat{\alpha}} + \ll (c_p(\alpha) - 1) h_i^p + d_p(\widehat{\alpha}) j_i \gg_{\widehat{\alpha}}.$$

1833 Since $\hat{\alpha}$ is bounded away from the extreme densities, the second term in the right-hand side is $C_{\epsilon} o_p(1)$,
 1834 and we can therefore write

$$\sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + h_i^p + \mathcal{L}f \gg_{\hat{\alpha}} \leq \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + c_p(\alpha)h_i^p + d_p(\hat{\alpha})j_i + \mathcal{L}f \gg_{\hat{\alpha}} + C_{\omega, f}\epsilon + C_{\epsilon, \omega, f} o_p(1).$$

1835 We then let $p \rightarrow \infty$ and then $\epsilon \rightarrow 0$ to obtain that

$$\limsup_{p \rightarrow \infty} \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + h_i^p + \mathcal{L}f \gg_{\hat{\alpha}} \leq \limsup_{p \rightarrow \infty} \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + c_p(\alpha)h_i^p + d_p(\hat{\alpha})j_i + \mathcal{L}f \gg_{\hat{\alpha}}.$$

1836 Proposition (6.16) is therefore a consequence of Lemma (6.17) below. \square

Lemma 6.17. — *There exists a sequence of local functions $f_n \in \mathcal{C}$ such that*

$$\limsup_{p \rightarrow \infty} \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + c_p(\alpha)h_i^p + d_p(\hat{\alpha})j_i + \mathcal{L}f_n \gg_{\hat{\alpha}} \leq \frac{3}{n},$$

1837 and for any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, $\lim_{n \rightarrow \infty} \ll j_i^{\hat{\omega}} + \mathcal{L}f_n \gg_{\hat{\alpha}} = \inf_{g \in T^\omega} \ll j_i^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}}$

1838 *Proof of Lemma 6.17.* — The proof of this Lemma is analogous to that of Theorem 5.6, p.176 of [27]. We
 1839 now write explicitly the dependency of h_i^p in $\hat{\alpha}$. According to Theorem 6.11 the application $\hat{\alpha} \mapsto \ll \psi \gg_{\hat{\alpha}}$
 1840 is continuous on $\mathcal{M}_1(\mathbb{S})$, and thanks to equation (6.52), for any $\hat{\alpha}_0 \in \mathcal{M}_1(\mathbb{S})$, there exists a function
 1841 $g_{\hat{\alpha}_0} \in T^\omega$ and a neighborhood $\mathcal{N}_{\hat{\alpha}_0}$ of $\hat{\alpha}_0$ such that for any $\hat{\alpha} \in \mathcal{N}_{\hat{\alpha}_0}$,

$$\ll j_i^{\hat{\omega}} + c_p(\alpha_0)h_i^p(\hat{\alpha}_0) + d_p(\hat{\alpha}_0)j_i + \mathcal{L}g_{\hat{\alpha}_0} \gg_{\hat{\alpha}} \leq n^{-1}.$$

1842 Furthermore, thanks to the last statement in Proposition 6.14, this function is an approximation of the
 1843 one realizing $\inf_{g \in T^\omega} \ll j_i^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}_0}$, and can be chosen independently of p .

1844 We prove in Proposition C.3 that $\mathcal{M}_1(\mathbb{S})$ is compact, it therefore admits a finite covering $\mathcal{M}_1(\mathbb{S}) \subset$
 1845 $\cup_{j=1}^m \mathcal{N}_{\hat{\alpha}_j}$. We can build a C^2 interpolation of the $g_{\hat{\alpha}_j}$'s, and therefore obtain a function $(\hat{\alpha}, \eta) \mapsto \psi(\hat{\alpha}, \eta)$
 1846 which coincides in $\hat{\alpha} = \hat{\alpha}_j$ with $g_{\hat{\alpha}_j}$, with the two following properties :

- 1847 — let B be a finite set of edges in \mathbb{Z}^2 containing the support of all the $g_{\hat{\alpha}_j}$'s, $\psi(\hat{\alpha}, \cdot)$ is a cylinder
 1848 function in T^ω with support included in B for any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$.
- 1849 — For any fixed configuration $\hat{\eta}$, $\psi(\cdot, \hat{\eta})$ is in $C^2(\mathcal{M}_1(\mathbb{S}))$.
- 1850 — for any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$

$$(6.67) \quad \ll j_i^{\hat{\omega}} + c_p(\alpha)h_i^p(\hat{\alpha}) + d_p(\hat{\alpha})j_i + \mathcal{L}\psi(\hat{\alpha}, \cdot) \gg_{\hat{\alpha}} \leq 2n^{-1}.$$

1851 Recall that we introduced in (2.20) $\hat{\rho}_r = |B_r|^{-1} \sum_{x \in B_r} \eta_x \delta_{\theta_x}$ the empirical angular density in the box
 1852 of side $(2r+1)$ around the origin. Define

$$f_r(\hat{\eta}) = \psi(\hat{\rho}_r, \hat{\eta}),$$

1853 for any r large enough for the support B of the $\psi(\hat{\alpha}, \eta)$'s to be contained in B_r . Note that f_r is not
 1854 necessarily in T^ω , but it is a local function for r fixed.

1855 By triangle inequality,

$$(6.68) \quad \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + c_p(\alpha)h_i^p(\hat{\alpha}) + d_p(\hat{\alpha})j_i + \mathcal{L}f_r \gg_{\hat{\alpha}} \leq 2n^{-1} + \sup_{\hat{\alpha}} \ll \mathcal{L}(f_r - \psi(\hat{\alpha}, \cdot)) \gg_{\hat{\alpha}}.$$

1856 The second term in the right-hand side is

$$\sum_i \mathbb{E}_{\hat{\alpha}} \left(\left(\nabla_{0, e_i} \sum_{x \in \mathbb{Z}^2} \tau_x [f_r - \psi(\hat{\alpha}, \cdot)] \right)^2 \right) = \sum_i \mathbb{E}_{\hat{\alpha}} \left(\left(\sum_{x \in \mathbb{Z}^2} \nabla_{x, x+e_i} [f_r - \psi(\hat{\alpha}, \cdot)] \right)^2 \right),$$

1857 by translation invariance of $\mu_{\hat{\alpha}}$. We extend B by 1 in such a way that for any edge a outside of B ,
 1858 $\nabla_a \psi(\hat{\alpha}, \cdot)$ vanishes. Therefore, the only contributions outside of B in the sums above are at the boundary
 1859 of B_r , where f_r has a variation in its first argument of order $(2r+1)^{-2}$. Thanks to the regularity of ψ in
 1860 $\hat{\alpha}$, and since the number of corresponding edges is roughly $4(2r+1)$, the contribution of all these jumps
 1861 is of order r^{-1} in the whole sum.

1862 Then, since the number of edges in B depends only on ψ , and since $\mathbb{E}_{\hat{\alpha}}((\nabla_a f)^2) \leq 4\mathbb{E}_{\hat{\alpha}}(f^2)$, we obtain
 1863 by definition of f_r that

$$(6.69) \quad \sup_{\hat{\alpha}} \ll \mathcal{L}(f_r - \psi(\hat{\alpha}, \cdot)) \gg_{\hat{\alpha}} \leq \sup_{\hat{\alpha}} C(\psi) \mathbb{E}_{\hat{\alpha}} \left[(\psi(\hat{\rho}_r, \cdot) - \psi(\hat{\alpha}, \cdot))^2 \right] + O(r^{-2}),$$

1864 whose right-hand side vanishes as r goes to infinity by the law of large numbers.

1865 Let us fix r_n such that the right-hand side of (6.69) is less than $1/n$, and let $f_n = f_{r_n}$, (6.68) finally
 1866 yields

$$(6.70) \quad \sup_{\hat{\alpha}} \ll j_i^{\hat{\omega}} + c_p(\alpha) h_i^p(\hat{\alpha}) + d_p(\hat{\alpha}) j_i + \mathcal{L} f_n \gg_{\hat{\alpha}} \leq 3n^{-1},$$

1867 as wanted. The last statement of the Lemma is a direct consequence of the construction of f_n and of
 1868 Proposition 6.14. This concludes the proof of Lemma 6.17. \square

1869 **6.7. Drift part of the hydrodynamic limit.** — Recall that $L_N = N^2 \mathcal{L} + N \mathcal{L}^{\text{WA}} + \mathcal{L}^G$ is the complete
 1870 generator of our process introduced in (2.2). In the previous section, we proved that the symmetric currents
 1871 can be replaced by a gradient, up to a perturbation $\mathcal{L}f$. In our case, this perturbation is not negligible,
 1872 and must be added to the asymmetric currents induced by the asymmetric generator \mathcal{L}^{WA} to complete the
 1873 drift term in equation (2.11). This is the purpose of this section.

1874 To achieve that goal, we need notations similar to the ones introduced in Section 4.1. For any positive
 1875 integer l , and any smooth function $G \in C([0, T] \times \mathbb{T}^2)$, let us introduce

$$\mathcal{R}_i^{f,l}(\hat{\eta}) = r_i^{\omega} + \mathcal{L}^{\text{WA}} f - \mathbb{E}_{\hat{\rho}_i}(r_i^{\omega} + \mathcal{L}^{\text{WA}} f),$$

1876 and

$$Y_{i,N}^{f,l}(G, \hat{\eta}) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} G(x/N) \tau_x \mathcal{R}_i^{f,l},$$

where r_i^{ω} is the asymmetric current introduced in (2.16). According to Theorem 6.1, for any i , there
 exists a family of cylinder functions $(f_{i,n}^{\omega})_{n \in \mathbb{N}}$ introduced in Proposition 6.16 such that

$$\lim_{\gamma \rightarrow \infty} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{\gamma N^2} \log \mathbb{E}_{\mu_{\alpha}^{\lambda, \beta}} \left[\exp \left(\gamma N^2 \left| \int_0^T X_{i,N}^{f_{i,n}^{\omega}, \varepsilon N}(G_t, \hat{\eta}(t)) dt \right| \right) \right] = 0,$$

1877 where $X_{i,N}^{f, \varepsilon N}$ was defined in equation (6.1). Furthermore, we also established in Proposition 6.16 that
 1878 this sequence satisfies for any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$

$$(6.71) \quad \lim_{n \rightarrow \infty} \ll j_i^{\omega} + \mathcal{L} f_{i,n}^{\omega} \gg_{\hat{\alpha}} = \inf_{f \in T^{\omega}} \ll j_i^{\omega} + \mathcal{L} f \gg_{\hat{\alpha}}.$$

1879 The replacement Lemma 4.1 applied to $g(\hat{\eta}) = r_i^{\omega} + \mathcal{L}^{\text{WA}} f$ yields the following result.

1880 **Lemma 6.18.** — Let G be some smooth function in $C^{1,2}([0, T] \times \mathbb{T}^2)$, and $T \in \mathbb{R}_+^*$, then for $i \in \{1, 2\}$
 1881 we have

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[\left| \int_0^T Y_{i,N}^{f_{i,n}^{\omega}, \varepsilon N}(G, \hat{\eta}) ds \right| \right] = 0.$$

1882 Furthermore, we now prove the following result, which states that any function of the form $N \mathcal{L}^D f$
 1883 vanishes in the hydrodynamic limit, where $\mathcal{L}^D = \mathcal{L} + N^{-1} \mathcal{L}^{\text{WA}}$ is the generator of whole exclusion process.

Lemma 6.19. — For any function $G : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ in $C^{1,2}$, and any cylinder function f ,

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[\left| \int_0^T \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(s, x/N) \tau_x \mathcal{L}^D f(\hat{\eta}(s)) ds \right| \right] = 0.$$

1884 *Proof of Lemma 6.19.* — For any such smooth function H and cylinder function f , let us denote

$$F_G(s, \hat{\eta}(s)) = N^{-2} \sum_{x \in \mathbb{T}_N^2} G(s, x/N) \tau_x f(\hat{\eta}(s)).$$

1885 The process

$$M_G(t) = F_G(t, \hat{\eta}(t)) - F_G(0, \hat{\eta}(0)) - \int_0^t \partial_s F_G(s, \hat{\eta}(s)) ds - \int_0^t L_N F_G(s, \hat{\eta}(s)) ds$$

is a martingale, where L_N is the complete generator of our process, introduced in (2.2). Since f is bounded, the first three terms are of order 1, it remains to control $\int_0^T L_N F_G ds$. The quadratic variation of this martingale is given (cf. Appendix 1.5, Lemma 5.1 in [27]) by

$$\begin{aligned} [M_G(\cdot, \hat{\eta}(\cdot))]_t &= \int_0^T L_N F_G(s, \hat{\eta}(s))^2 - 2F_G(s, \hat{\eta}(s)) L_N F_G(s, \hat{\eta}(s)) ds \\ &= \int_0^T ds N^2 \sum_{\substack{x \in \mathbb{T}_N^2 \\ \delta = \pm 1, i \in \{1,2\}}} \tau_{x,z,i,\delta}^\lambda [F_G(s, \hat{\eta}^{x,x+\delta e_i}(s)) - F_G(s, \hat{\eta}(s))]^2 \\ &\quad + \int_0^T ds \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} c_{x,\beta}(\theta, \hat{\eta}) [F_G(s, \hat{\eta}^{x,\theta}(s)) - F_G(s, \hat{\eta}(s))]^2 d\theta \\ &= \frac{1}{N^2} \int_0^T ds \sum_{\substack{x \in \mathbb{T}_N^2 \\ \delta = \pm 1, i \in \{1,2\}}} \tau_{x,z,i,\delta}^\lambda(\hat{\eta}(s)) \left[\sum_{y \in \mathbb{T}_N^2} G(s, y/N) (\tau_y f(\hat{\eta}^{x,x+z}(s)) - \tau_y f(\hat{\eta}(s))) \right]^2 \\ &\quad + \frac{1}{N^4} \int_0^T ds \sum_{x \in \mathbb{T}_N^2} \eta_x \int_{\mathbb{S}} c_{x,\beta}(\theta, \hat{\eta}) \left[\sum_{y \in \mathbb{T}_N^2} G(s, y/N) (\tau_y f(\hat{\eta}^{x,x+z}(s)) - \tau_y f(\hat{\eta}(s))) \right]^2 d\theta, \end{aligned}$$

1886 where

$$\tau_{x,z,i,\delta}^\lambda(\hat{\eta}) = \left(1 + \frac{\delta \lambda_i(\theta_x)}{N} \right) \eta_x (1 - \eta_{x+z})$$

1887 is the total displacement jump rate.

Since f is a local function, all but a finite number of terms in the y sums vanish, and the quadratic variation is hence of order N^{-2} . We deduce from the estimate of the quadratic variation of M_G and the order of the three first terms in the expression of M_G that

$$\mathbb{E}_{\mu^N} \left(\left| \int_0^T N^{-1} L_N F_G(s, \hat{\eta}(s)) ds \right| \right) \leq N^{-1} \left[\underbrace{\mathbb{E}_{\mu^N} ([M_G(t, \hat{\eta}(t))]^2)}_{O(N^{-1})} + O_N(1) \right] \xrightarrow{N \rightarrow \infty} 0.$$

The previous martingale estimate shows that $\mathbb{E}_{\mu^N} \left(\left| \int_0^T N^{-1} L_N F_G(s, \hat{\eta}(s)) ds \right| \right)$ vanishes in the limit $N \rightarrow \infty$. Furthermore, elementary computations yield a crude bound on the contribution of the Glauber generator of order N^{-1} . Finally, since $L_N = N^2 \mathcal{L}^D + \mathcal{L}^G$, we obtain

$$\mathbb{E}_{\mu^N} \left(\left| \int_0^T N \mathcal{L}^D F_G(s, \hat{\eta}(s)) ds \right| \right) \xrightarrow{N \rightarrow \infty} 0,$$

1888 which completes the proof of Lemma 6.19. □

1889 We now use these two Lemmas to prove that the total displacement current can be replaced by the
1890 wanted averages. More precisely, let

$$\mathcal{U}_i^{f,l}(\hat{\eta}) = j_i^\omega + \frac{1}{N} \gamma_i^\omega + d_s(\rho_l) \delta_i \rho_l^\omega + \mathfrak{d}(\rho_l, \rho_l^\omega) \delta_i \rho_l - \frac{1}{N} \mathbb{E}_{\hat{\rho}_l}(r_i^\omega + \mathcal{L}^{\text{WA}} f),$$

1891 we can state the following result.

1892 **Corollary 6.20.** — For any $G \in C^{1,2}([0, T] \times \mathbb{T}^2)$, $T \in \mathbb{R}_+^*$, and $i \in \{1, 2\}$,

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N}^{\lambda, \beta} \left[\left| \int_0^T \frac{1}{N} \sum_{x \in \mathbb{T}_N^2} G(x/N) \mathcal{U}_i^{f_{i,n}^{\omega, \varepsilon N}}(G, \hat{\eta}) ds \right| \right] = 0.$$

1893 *Proof of Corollary 6.20.* — Adding and subtracting $\mathcal{L}^D f_{i,n}^{\omega}$ to $\mathcal{U}_i^{f_{i,n}^{\omega, \varepsilon N}}$, we can split it into three parts,

$$j_i^{\omega} + d_s(\rho_{\varepsilon N}) \delta_i \rho_{\varepsilon N}^{\omega} + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^{\omega}) \delta_i \rho_{\varepsilon N} + \mathcal{L} f_{i,n}^{\omega},$$

1894

$$\frac{1}{N} (r_i^{\omega} + \mathcal{L}^{\text{WA}} f_{i,n}^{\omega}) - \frac{1}{N} \mathbb{E}_{\hat{\rho}_{\varepsilon N}} (r_i^{\omega} + \mathcal{L}^{\text{WA}} f_{i,n}^{\omega}), \quad \text{and} \quad -\mathcal{L}^D f_{i,n}^{\omega}.$$

1895 The contribution of the first quantity vanishes in the limit of Corollary 6.20, according to Corollary
1896 6.2. The second contribution also does thanks to Lemma 6.18, as well as the third due to Lemma 6.19,
1897 thus completing the proof of the Corollary. \square

1898 We now derive an explicit expression for the limit of $\mathbb{E}_{\hat{\rho}_{\varepsilon N}} (r_i^{\omega} + \mathcal{L}^{\text{WA}} f_{i,n}^{\omega})$, appearing in $\mathcal{U}_i^{f_{i,n}^{\omega, \varepsilon N}}$, as n goes
1899 to ∞ .

1900 **Lemma 6.21.** — For any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$,

$$(6.72) \quad \lim_{n \rightarrow \infty} \mathbb{E}_{\hat{\alpha}} (r_i^{\omega} + \mathcal{L}^{\text{WA}} f_{i,n}^{\omega}) = 2d_s(\alpha) \alpha_{\omega \lambda_i} + 2 \frac{\alpha_{\omega} \alpha_{\lambda_i}}{\alpha} (1 - \alpha - d_s(\alpha)),$$

1901 where for any function $\Phi \in C^1(\mathbb{S})$, we defined $\alpha_{\Phi} = \mathbb{E}_{\hat{\alpha}}(\Phi(\theta_0)\eta_0)$.

1902 *Proof of Lemma 6.21.* — By definition of $r_i^{\omega} = \lambda_i(\theta_0)\omega(\theta_0)\eta_0(1 - \eta_{e_1}) + \lambda_i(\theta_{e_1})\omega(\theta_{e_1})\eta_{e_1}(1 - \eta_0)$, we can
1903 write, shortening as before $\mathbb{E}_{\hat{\alpha}}(\Phi) = \mathbb{E}_{\hat{\alpha}}(\Phi(\theta_0)|\eta_0 = 1)$,

$$(6.73) \quad \mathbb{E}_{\hat{\alpha}}(r_i^{\omega}) = 2\mathbb{E}_{\hat{\alpha}}(\lambda_i \omega) \alpha (1 - \alpha) = 2 \ll j_i^{\lambda_i}, j_i^{\omega} \gg_{\hat{\alpha}}.$$

1904 For any cylinder function f , by translation invariance of $\mu_{\hat{\alpha}}$ and Definition 6.8, one also obtains by
1905 elementary computations that

$$(6.74) \quad \mathbb{E}_{\hat{\alpha}}(\mathcal{L}^{\text{WA}} f) = 2 \ll j_1^{\lambda_1} + j_2^{\lambda_2}, \mathcal{L} f \gg_{\hat{\alpha}}.$$

Recalling Corollary 6.12, we can then write

$$\begin{aligned} \ll j_k^{\lambda_k}, h_i^{p, \omega} \gg_{\hat{\alpha}} &= \ll j_k^{\hat{\lambda}_k}, h_i^{p, \omega} \gg_{\hat{\alpha}} + \mathbb{E}_{\hat{\alpha}}(\lambda_k) \ll j_k, h_i^{p, \omega} \gg_{\hat{\alpha}} \\ &= -\mathbb{1}_{\{i=k\}} [\alpha d_s(\alpha) \text{Cov}_{\hat{\alpha}}(\omega, \lambda_i)(1 - \alpha) - \mathbb{E}_{\hat{\alpha}}(\lambda_i) \alpha_p(1)] \end{aligned}$$

1906 where as before $\hat{\lambda}_k = \lambda_k - \mathbb{E}_{\hat{\alpha}}(\lambda_k)$. We can also write by Definition 6.8

$$\ll j_k^{\lambda_k}, j_i^{\omega} \gg_{\hat{\alpha}} = \mathbb{1}_{\{i=k\}} \mathbb{E}_{\hat{\alpha}}(\lambda_k \omega) \alpha (1 - \alpha).$$

Once again, in order to avoid taking everywhere limits $n \rightarrow \infty$, we assume for the convenience of notations, that there exists a local function f_i^{ω} realizing the infimum (6.71). Recall then from equation (6.57) that in $\mathcal{H}_{\hat{\alpha}}^{\omega}$, we have the identity $j_i^{\hat{\omega}} + \mathcal{L} f_i^{\omega} = -c_p(\alpha) h_i^p + d_p(\hat{\alpha}) j_i$. Then, using (6.73), (6.74), and the explicit formulas for the inner products which prove orthogonality of directions $i \neq k$,

$$\begin{aligned} \mathbb{E}_{\hat{\alpha}}(r_i^{\omega} + \mathcal{L}^{\text{WA}} f_i^{\omega}) &= 2 \ll j_1^{\lambda_1} + j_2^{\lambda_2}, \mathcal{L} f_i^{\omega} \gg_{\hat{\alpha}} + 2 \ll j_i^{\lambda_i}, j_i^{\omega} \gg_{\hat{\alpha}} \\ &= 2 \ll j_1^{\lambda_1} + j_2^{\lambda_2}, j_i^{\hat{\omega}} + \mathcal{L} f_i^{\omega} \gg_{\hat{\alpha}} - 2 \ll j_1^{\lambda_1} + j_2^{\lambda_2}, j_i^{\hat{\omega}} \gg_{\hat{\alpha}} + 2 \ll j_i^{\lambda_i}, j_i^{\omega} \gg_{\hat{\alpha}} \\ &= -2 \ll j_1^{\lambda_1} + j_2^{\lambda_2}, c_p(\alpha) h_i^p + d_p(\hat{\alpha}) j_i \gg_{\hat{\alpha}} - 2 \ll j_i^{\lambda_i}, j_i^{\hat{\omega}} \gg_{\hat{\alpha}} + 2 \ll j_i^{\lambda_i}, j_i^{\omega} \gg_{\hat{\alpha}} \\ (6.75) \quad &= -2c_p(\alpha) \ll j_i^{\lambda_i}, h_i^{p, \omega} \gg_{\hat{\alpha}} - 2d_p(\hat{\alpha}) \ll j_i^{\lambda_i}, j_i \gg_{\hat{\alpha}} + 2\mathbb{E}_{\hat{\alpha}}(\omega) \ll j_i^{\lambda_i}, j_i \gg_{\hat{\alpha}}. \end{aligned}$$

1907 We now let $p \rightarrow \infty$, so that d_p vanishes, c_p goes to 1, to obtain as wanted, by Definition 6.8 and Corollary
1908 6.12,

$$\mathbb{E}_{\hat{\alpha}}(r_i^{\omega} + \mathcal{L}^{\text{WA}} f_i^{\omega}) = 2\alpha d_s(\alpha) \text{Cov}_{\hat{\alpha}}(\omega, \lambda_i) + 2\mathbb{E}_{\hat{\alpha}}(\omega) \mathbb{E}_{\hat{\alpha}}(\lambda_i) \alpha (1 - \alpha).$$

1909 Reorganizing the terms yield Lemma 6.21. \square

1910

7. Proof of the hydrodynamic limit

1911 We now have all the pieces to prove Theorem 2.6. The last remaining difficulty is to perform the
 1912 second integration by parts, since even the gradients obtained in Section 6 are not exactly microscopic
 1913 gradients due to the non-constant diffusion coefficient. This is not a problem when the variations only
 1914 depend on one quantity, the density for example, since we can then simply consider a primitive of the
 1915 diffusion coefficient and obtain at the highest order in N a discrete gradient. This is not the case here,
 1916 and we need some more work to obtain the wanted gradient.

1917 Let us recall from Section 2.4 that for any smooth function $H \in C^{1,2,1}([0, T] \times \mathbb{T}^2 \times \mathbb{S})$, that we denoted
 1918 by $M_t^{H,N}$ the martingale

$$(7.1) \quad M_t^{H,N} = \langle \pi_t^N, H_t \rangle - \langle \pi_0^N, H_0 \rangle - \int_0^t [\langle \pi_s^N, \partial_s H_s \rangle + L_N \langle \pi_s^N, H_s \rangle] ds,$$

1919 where

$$\pi_s^N = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x(t) \delta_{x/N, \theta_x(s)}$$

 1920 is the empirical measure of the process on $\mathbb{T}^2 \times \mathbb{S}$.

Proof of Theorem 2.6. — The quadratic variation $[M^{H,N}]_t$ of $M_t^{H,N}$ (cf. A1.5. Lemma 5.1 in [27]) is

$$\begin{aligned} [M^{H,N}]_t &= \int_0^t L_N \langle \pi_s^N, H_s \rangle^2 - 2 \langle \pi_s^N, H_s \rangle L_N \langle \pi_s^N, H_s \rangle ds \\ &= \int_0^t \frac{1}{N^4} \sum_{x \in \mathbb{T}_N^2} \left[\sum_{|z|=1} A_1(\hat{\eta}, x, z) H_s(x/N) H_s((x+z)/N) + A_2(\hat{\eta}, x) H_s(x/N)^2 \right] ds \\ &\leq \int_0^t \frac{1}{N^4} \sum_{x \in \mathbb{T}_N^2} C \|H\|_\infty^2 ds \leq \frac{1}{N^2} t C \|H\|_\infty^2, \end{aligned}$$

1921 where C , $A_1(\hat{\eta}, x, z)$ and $A_2(\hat{\eta}, x)$ are bounded uniformly in N . The quadratic variation $[M^{H,N}]_t$ is
 1922 therefore of order N^{-2} , and vanishes as N goes to infinity. Doob's inequality hence gives us for any
 1923 $T > 0$, $\delta > 0$

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu^N}^{\lambda, \beta} \left(\sup_{0 \leq t \leq T} |M_t^{H,N}| \geq \delta \right) = 0,$$

1924 and in particular

$$(7.2) \quad \lim_{N \rightarrow \infty} \mathbb{P}_{\mu^N}^{\lambda, \beta} \left(|M_T^{H,N}| \geq \delta \right) = 0.$$

 1925 We first consider the case of a function H such that

$$H_t(u, \theta) = G_t(u) \omega(\theta),$$

the general case will be a simple consequence of a periodic version of the Weierstrass approximation
 Theorem. For any such H , we can write

$$(7.3) \quad \int_0^T L_N \langle \pi_t^N, H_t \rangle dt = \frac{1}{N^2} \int_0^T dt \sum_{x \in \mathbb{T}_N^2} \tau_x \left[\sum_{i=1}^2 [N j_i^\omega + r_i^\omega](t) \partial_{u_i, N} G_t(x/N) + G_t(x/N) \gamma^\omega(t) \right],$$

 1926 where j_i^ω , r_i^ω and γ^ω were introduced in Definition 2.8, and

$$\partial_{u_i, N} G(x/N) = N(G(x + e_i/N) - G(x/N))$$

 1927 is a microscopic approximation of the spatial derivative $\partial_{u_i} G$.

1928 Thanks to Sections 4 and 6, we can perform the following replacements, in the expectation of the
 1929 expression above, and in the limit $N \rightarrow \infty$ then $\varepsilon \rightarrow 0$:

1930 — Thanks to Corollary 6.20, we can replace j_i^ω by

$$(7.4) \quad - [d_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N}^\omega + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega)\delta_i\rho_{\varepsilon N}],$$

1931 where \mathfrak{d} is given by equation (6.39),

$$\mathfrak{d}(\rho, \rho^\omega) = \rho^\omega(1 - d_s(\rho))/\rho,$$

1932 — Thanks to Corollary 6.20 and Lemma 6.21, r_i^ω can be replaced by

$$R_i^\omega(\widehat{\rho}_{\varepsilon N}) := 2 \left[d_s(\rho_{\varepsilon N})\mathbb{E}_{\widehat{\rho}_{\varepsilon N}}(\eta_0^{\omega\lambda_i}) + \frac{\mathbb{E}_{\widehat{\rho}_{\varepsilon N}}(\eta_0^\omega)\mathbb{E}_{\widehat{\rho}_{\varepsilon N}}(\eta_0^{\lambda_i})}{\rho_{\varepsilon N}} (1 - \rho_{\varepsilon N} - d_s(\rho_{\varepsilon N})) \right].$$

1933 — Finally, the Replacement Lemma 4.1 yields that γ^ω can be replaced by $\mathbb{E}_{\widehat{\rho}_{\varepsilon N}}(\gamma^\omega)$.

1934 In other words, thanks to equation (7.2), for any $H_s(u, \theta) = G_s(u)\omega(\theta)$, we can write

$$(7.5) \quad \limsup_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_{\mu^{\lambda, \beta}}^{\lambda, \beta} \left(\left| \widetilde{M}_T^{H, N, \varepsilon} \right| \geq \delta \right) = 0,$$

where

$$(7.6) \quad \begin{aligned} \widetilde{M}_T^{H, N, \varepsilon} = & \langle \pi_T^N, H_T \rangle - \langle \pi_0^N, H_0 \rangle - \int_0^T \langle \pi_t^N, \partial_t H_t \rangle dt \\ & + \int_0^T dt \left[\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \sum_{i=1}^2 [N(d_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N}^\omega + \mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega)\delta_i\rho_{\varepsilon N}) - R_i^\omega(\widehat{\rho}_{\varepsilon N})] \partial_{u_i, N} G_t(x/N) \right. \\ & \left. - G_t(x/N)\mathbb{E}_{\widehat{\rho}_{\varepsilon N}}(\gamma^\omega) \right] (t), \end{aligned}$$

1935 In order to give a clear scheme, we divide the end of the proof in a series of steps.

1936 *Performing the second integration by parts.* — Due to the presence of the diffusion coefficients, one
1937 cannot switch directly the last discrete derivatives $\delta_i\rho_{\varepsilon N}$ and $\delta_i\rho_{\varepsilon N}^\omega$ onto the smooth function G . In one
1938 dimension, one would consider a primitive $d(\rho)$ of the diffusion coefficient $D(\rho)$, and write that

$$D(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N} = \delta_i d(\rho_{\varepsilon N}) + o_N(\delta_i\rho_{\varepsilon N}).$$

1939 However, our case cannot be solved that way because the differential form

$$(\rho, \rho^\omega) \mapsto d_s(\rho)d\rho^\omega + \mathfrak{d}(\rho, \rho^\omega)d\rho,$$

1940 is not closed, and therefore not exact either, which means that we cannot express (7.4) as

$$\delta_i F(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) + o_N(1/N).$$

1941 We thus need another argument to obtain the differential equation (2.11).

First, we get rid of the part with $\delta_i\rho^\omega$. To do so, notice that

$$\begin{aligned} \delta_i [d_s(\rho_{\varepsilon N})\rho_{\varepsilon N}^\omega] &= d_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N}^\omega + \rho_{\varepsilon N}^\omega\delta_i d_s(\rho_{\varepsilon N}) + o_N(1/N) \\ &= d_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N}^\omega + \rho_{\varepsilon N}^\omega d'_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N} + o_N(1/N). \end{aligned}$$

1942 We can therefore write

$$(7.7) \quad d_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N}^\omega = \delta_i [d_s(\rho_{\varepsilon N})\rho_{\varepsilon N}^\omega] - \rho_{\varepsilon N}^\omega d'_s(\rho_{\varepsilon N})\delta_i\rho_{\varepsilon N} + o_N(1/N).$$

1943 Let us denote for any $x \in \mathbb{T}_N^2$

$$D_x^{\varepsilon N} = \tau_x (\mathfrak{d}(\rho_{\varepsilon N}, \rho_{\varepsilon N}^\omega) - \rho_{\varepsilon N}^\omega d'_s(\rho_{\varepsilon N})).$$

1944 We perform a second integration by parts in the contribution of the first term in the right-hand side
1945 of (7.7), whereas the left-hand side is added to the existing contribution of $\delta_i\rho_{\varepsilon N}$, with the modified
1946 diffusion coefficient $D_x^{\varepsilon N}$ defined above. We can now rewrite $\widetilde{M}_T^{H, N, \varepsilon}$ as

$$(7.8) \quad \langle \pi_T^N, H_T \rangle - \langle \pi_0^N, H_0 \rangle - \int_0^T \langle \pi_t^N, \partial_t H_t \rangle dt - \int_0^T I_1(t, \widehat{\eta}_t) - I_2(t, \widehat{\eta}_t) dt + o_N(1),$$

1947 where

$$I_1(t, \hat{\eta}) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \left[\sum_{i=1}^2 d_s(\rho_{\varepsilon N}) \rho_{\varepsilon N}^\omega \partial_{u_i, N}^2 G_t(x/N) + R_i^\omega(\hat{\rho}_{\varepsilon N}) \partial_{u_i, N} G_t(x/N) + G_t(x/N) \mathbb{E}_{\hat{\rho}_{\varepsilon N}}(\gamma^\omega) \right]$$

and

$$\begin{aligned} I_2(t, \hat{\eta}) &= \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \tau_x \sum_{i=1}^2 N D_0^{\varepsilon N} \delta_i \rho_{\varepsilon N} \partial_{u_i, N} G_t(x/N) \\ &= \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 N D_x^{\varepsilon N} (\tau_{x+e_i} \rho_{\varepsilon N} - \tau_x \rho_{\varepsilon N}) \partial_{u_i, N} G_t(x/N). \end{aligned}$$

1948 In I_1 , we regrouped all the terms for which taking the limit $N \rightarrow \infty$ is not a problem, whereas I_2 is the
1949 term where the extra factor N still has to be absorbed in a spatial derivative.

1950 *Replacement of the microscopic gradient by a mesoscopic gradient.* — Since we cannot switch the deriva-
1951 tive on the smooth function G due to the diffusion coefficient, we need to obtain the gradient of ρ in
1952 another way. For this purpose, we need to replace the microscopic gradient $\tau_{x+e_i} \rho_{\varepsilon N} - \tau_x \rho_{\varepsilon N}$ by a meso-
1953 scopic gradient, and make the derivative (in a weak sense) of ρ appear directly. More precisely, let us
1954 define

$$\tilde{I}_2(t, \hat{\eta}) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 D_x^{\varepsilon N} \frac{\tau_{x+\varepsilon^3 N e_i} \rho_{\varepsilon N} - \tau_{x-\varepsilon^3 N e_i} \rho_{\varepsilon N}}{2\varepsilon^3} \partial_{u_i, N} G_t(x/N).$$

1955 We are going to prove that for any configuration $\hat{\eta}$,

$$(7.9) \quad \left| I_2(t, \hat{\eta}) - \tilde{I}_2(t, \hat{\eta}) \right| \leq o_N(1) + o_\varepsilon(1),$$

1956 uniformly in $\hat{\eta}$. To prove the latter, for any $k \in \llbracket -\varepsilon^3 N, \varepsilon^3 N \rrbracket$, let us denote by $x_k = x + k e_i$,

$$\tau_{x+\varepsilon^3 N e_i} \rho_{\varepsilon N} - \tau_{x-\varepsilon^3 N e_i} \rho_{\varepsilon N} = \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} \tau_{x_k+1} \rho_{\varepsilon N} - \tau_{x_k} \rho_{\varepsilon N}.$$

1957 A summation by parts therefore allows us to rewrite \tilde{I}_2 as

$$\tilde{I}_2(t, \hat{\eta}) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left[\frac{1}{2\varepsilon^3 N} \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} D_{x_k}^{\varepsilon N} \partial_{u_i, N} G_t(x_k/N) \right] N(\tau_{x+e_i} \rho_{\varepsilon N} - \tau_x \rho_{\varepsilon N}).$$

Furthermore, we can write for any $x \in \mathbb{T}_N^2$

$$\begin{aligned} & \left| D_x^{\varepsilon N} \partial_{u_i, N} G_t(x/N) - \frac{1}{2\varepsilon^3 N} \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} D_{x_k}^{\varepsilon N} \partial_{u_i, N} G_t(x_k/N) \right| \\ & \leq \frac{1}{2\varepsilon^3 N} \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} \left| D_x^{\varepsilon N} (\partial_{u_i, N} G_t(x/N) - \partial_{u_i, N} G_t(x_k/N)) \right| + \left| \partial_{u_i, N} G_t(x_k/N) (D_x^{\varepsilon N} - D_{x_k}^{\varepsilon N}) \right|. \end{aligned}$$

1958 Since the diffusion coefficients are bounded and G_s is C^2 , and since x and the x_k 's are distant of $\varepsilon^3 N$,
1959 we can write

$$\left| D_x^{\varepsilon N} (\partial_{u_i, N} G_t(x/N) - \partial_{u_i, N} G_t(x_k/N)) \right| \leq C(G_t) \varepsilon^3.$$

Since $D_{x_k}^{\varepsilon N}$ depends on the macroscopic density $\hat{\rho}_{\varepsilon N}$, and since the diffusion coefficients can be extended as C^1 functions due to their explicit expression, we also have

$$\begin{aligned} \left| \partial_{u_i, N} G_t(x_k/N) (D_x^{\varepsilon N} - D_{x_k}^{\varepsilon N}) \right| &\leq C'(G_t) (|\tau_x \rho_{\varepsilon N} - \tau_{x_k} \rho_{\varepsilon N}| + |\tau_x \rho_{\varepsilon N}^\omega - \tau_{x_k} \rho_{\varepsilon N}^\omega|) \\ &\leq C''(G_t, \omega) \frac{\varepsilon^3 N}{\varepsilon N}. \end{aligned}$$

1960 These two bounds finally yield that

$$(7.10) \quad \left| D_x^{\varepsilon N} \partial_{u_i, N} G_t(x/N) - \frac{1}{2\varepsilon^3 N} \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} D_{x_k}^{\varepsilon N} \partial_{u_i, N} G_t(x_k/N) \right| \leq C(G_t)\varepsilon^3 + C''(G_t, \omega)\varepsilon^2 = o_\varepsilon(\varepsilon).$$

By definition of I_2 and \tilde{I}_2 , the triangular inequality yields

$$|I_2 - \tilde{I}_2| \leq \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \left| D_x^{\varepsilon N} \partial_{u_i, N} G_t(x/N) - \frac{1}{2\varepsilon^3 N} \sum_{k=-\varepsilon^3 N}^{k=\varepsilon^3 N-1} D_{x_k}^{\varepsilon N} \partial_{u_i, N} G_t(x_k/N) \right| N(\tau_{x+e_i} \rho_{\varepsilon N} - \tau_x \rho_{\varepsilon N}).$$

1961 The quantity inside the absolute values in the right-hand side above is $o_N(1) + o_\varepsilon(\varepsilon)$, thanks to (7.10),
 1962 whereas $N(\tau_{x+e_i} \rho_{\varepsilon N} - \tau_x \rho_{\varepsilon N})$ is of order at most $1/\varepsilon$, whereas the quantity inside absolute values is $o_\varepsilon(\varepsilon)$,
 1963 therefore their product vanishes as $\varepsilon \rightarrow 0$, which proves equation (7.9). We therefore have obtained as
 1964 wanted that

$$(7.11) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} I_2(t, \hat{\eta}) - \tilde{I}_2(t, \hat{\eta}) = 0,$$

1965 uniformly in $\hat{\eta}$. We can now replace in equation (7.8) I_2 by \tilde{I}_2 .

1966 *Embedding in the space of trajectories of measures $\mathcal{M}^{[0, T]}$.* — Recall that Q^N is the distribution of
 1967 the empirical measure of our process. We now wish to express the martingale $\tilde{M}_t^{H, N, \varepsilon}$ introduced after
 1968 equation (7.5) as an explicit function of the empirical measure π^N in order to characterize the limit points
 1969 Q^* of the compact sequence Q^N . For that purpose, let $(\varphi_\varepsilon)_{\varepsilon \rightarrow 0}$ be a family of localizing functions on \mathbb{T}^2 ,

$$\varphi_\varepsilon(\cdot) = (2\varepsilon)^{-2} \mathbb{1}_{[-\varepsilon, \varepsilon]^2}(\cdot),$$

1970 and recall that we defined the empirical measure as

$$\pi_t^N = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x(t) \delta_{x/N, \theta_x(t)}.$$

1971 Then, for any function $\Phi : \mathbb{S} \rightarrow \mathbb{R}$, and any $u \in \mathbb{T}^2$ we denote by $\varphi_{\varepsilon, u}^\Phi$ the function

$$\varphi_{\varepsilon, u}^\Phi : \mathbb{T}^2 \times \mathbb{S} \longrightarrow \mathbb{R} \\ (v, \theta) \longmapsto \varphi_\varepsilon(v - u) \Phi(\theta).$$

1972 With this notation, we can therefore write

$$\mathbb{E}_{\tau_x \hat{\rho}_{\varepsilon N}}(\eta_0^\Phi) = \frac{1}{(2\varepsilon N + 1)^2} \sum_{\|y-x\|_\infty \leq \varepsilon N} \eta_y^\Phi = \frac{(2\varepsilon N)^2}{(2\varepsilon N + 1)^2} \langle \pi^N, \varphi_{\varepsilon, x/N}^\Phi \rangle.$$

1973 In the particular case where $\Phi \equiv 1$, (resp. $\Phi = \omega$), this rewrites

$$\tau_x \rho_{\varepsilon N} = \frac{(2\varepsilon N)^2}{(2\varepsilon N + 1)^2} \langle \pi^N, \varphi_{\varepsilon, x/N}^1 \rangle \quad \left(\text{resp. } \tau_x \rho_{\varepsilon N}^\omega = \frac{(2\varepsilon N)^2}{(2\varepsilon N + 1)^2} \langle \pi^N, \varphi_{\varepsilon, x/N}^\omega \rangle \right).$$

1974 Since $(2\varepsilon N)^2 / (2\varepsilon N + 1)^2 = 1 + o_N(1)$, we can replace in the limit $N \rightarrow \infty$ the quantity $\mathbb{E}_{\tau_x \hat{\rho}_{\varepsilon N}}(\eta_0^\Phi)$
 1975 (resp. $\tau_x \rho_{\varepsilon N}$, $\tau_x \rho_{\varepsilon N}^\omega$) by the function of the empirical measure $\langle \pi^N, \varphi_{\varepsilon, x/N}^\Phi \rangle$ (resp. $\langle \pi^N, \varphi_{\varepsilon, x/N}^1 \rangle$,
 1976 $\langle \pi^N, \varphi_{\varepsilon, x/N}^\omega \rangle$).

1977 We deduce from equations (7.5), (7.8) and (7.11) and what precedes that for any positive δ ,

$$(7.12) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} Q^N \left(\left| N_T^{H, N} \left(\pi^{[0, T]} \right) \right| \geq \delta \right) = 0.$$

where $N_T^{H, N}$ is defined as

$$(7.13) \quad N_T^{H, N} \left(\pi^{[0, T]} \right) = \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, \partial_t H_t \rangle dt$$

$$\begin{aligned}
& - \int_0^T \left[\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \tilde{d}_{x/N, \varepsilon}(\pi_t) \partial_{u_i}^2 G_t(x/N) + \tilde{R}_{x/N, \varepsilon, i}(\pi_t) \partial_{u_i} G_t(x/N) + \Gamma_{x/N, \varepsilon}^\omega(\pi_t) G_t(x/N) \right] dt \\
& + \int_0^T \left[\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \sum_{i=1}^2 \tilde{D}_{x/N, \varepsilon}(\pi_t) < \pi_t, \frac{\varphi_{\varepsilon, x/N + \varepsilon^3 e_i}^1 - \varphi_{\varepsilon, x/N - \varepsilon^3 e_i}^1}{2\varepsilon^3} > \partial_{u_i} G_t(x/N) \right] dt.
\end{aligned}$$

1978 In the identity above, we denoted

$$\tilde{d}_{x/N, \varepsilon}(\pi) = d_s \langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle < \pi, \varphi_{\varepsilon, x/N}^\omega \rangle$$

1979

$$\tilde{D}_{x/N, \varepsilon}(\pi) = \mathfrak{d} \langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle, \langle \pi, \varphi_{\varepsilon, x/N}^\omega \rangle - \langle \pi, \varphi_{\varepsilon, x/N}^\omega \rangle d'_s \langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle$$

$$\begin{aligned}
\tilde{R}_{x/N, \varepsilon, i}(\pi) &= d_s \left(\langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle \right) < \pi, \varphi_{\varepsilon, x/N}^{\omega \lambda_i} \rangle \\
&+ \frac{\langle \pi, \varphi_{\varepsilon, x/N}^\omega \rangle < \pi, \varphi_{\varepsilon, x/N}^{\lambda_i} \rangle}{\langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle} \left[1 - \langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle - d_s \left(\langle \pi, \varphi_{\varepsilon, x/N}^1 \rangle \right) \right],
\end{aligned}$$

1980 and $\Gamma_{u, \varepsilon}^\omega(\pi) = \mathbb{E}_{\hat{\alpha}_{x/N, \varepsilon}(\pi)}(\gamma^\omega)$, where $\hat{\alpha}_{x/N, \varepsilon}(\pi) \in \mathcal{M}_1(\mathbb{S})$ is the measure on \mathbb{S}

$$\hat{\alpha}_{x/N, \varepsilon}(\pi)(d\theta) = \int_{\mathbb{T}^2} \varphi_\varepsilon(\cdot - x/N) \pi(du, d\theta).$$

Limit $N \rightarrow \infty$. — We have now successfully balanced out all the factors N , and can thus let N go to ∞ in (7.12). Since G is a smooth function, one can replace in (7.13) the discrete space derivatives $\partial_{u_i, N}$ by the continuous derivative ∂_{u_i} , the sums $N^{-2} \sum_{x \in \mathbb{T}_N^2}$ by the integral $\int_{\mathbb{T}^2} du$, and the variables x/N by u . We proved in Proposition 5.4 that the sequence of distributions $(Q^N)_N$ is relatively compact. Since the quantity inside the absolute values is a continuous function (for Skorohod's topology defined in Appendix B.1) of $\pi^{[0, T]}$, the whole event is an open set, we obtain that for any weak limit point Q^* of (Q^N) , and any positive δ ,

$$\begin{aligned}
(7.14) \quad & \limsup_{\varepsilon \rightarrow 0} Q^* \left(\left| \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, \partial_t H_t \rangle dt \right. \right. \\
& - \int_0^T \int_{\mathbb{T}^2} \sum_{i=1}^2 \left[\tilde{d}_{u, \varepsilon}(\pi_t) \partial_{u_i}^2 G_t(u) + \tilde{R}_{u, \varepsilon, i}(\pi_t) \partial_{u_i} G_t(u) + \Gamma_{u, \varepsilon}^\omega(\pi_t) G_t(u) \right] dudt \\
& \left. \left. + \int_0^T \int_{\mathbb{T}^2} \sum_{i=1}^2 \left[\tilde{D}_{u, \varepsilon}(\pi_t) < \pi_t, \frac{\varphi_{\varepsilon, u + \varepsilon^3 e_i}^1 - \varphi_{\varepsilon, u - \varepsilon^3 e_i}^1}{2\varepsilon^3} > \partial_{u_i} G_t(u) \right] dudt. \right| > \delta \right) = 0
\end{aligned}$$

1981 *Limit $\varepsilon \rightarrow 0$.* — In order to consider the limit $\varepsilon \rightarrow 0$, we need to express

$$\left\langle \pi_t, \frac{\varphi_{\varepsilon, u + \varepsilon^3 e_i}^1 - \varphi_{\varepsilon, u - \varepsilon^3 e_i}^1}{2\varepsilon^3} \right\rangle$$

1982 in the third line above as an approximation of the gradient of the density $\partial_{u_i} \rho_t(u)$. As in the proof of
1983 Lemma 6.3, consider a smooth function $h_{\varepsilon, i, u}$ such that

$$(7.15) \quad \int_{\mathbb{T}^2} \left| \frac{\varphi_{\varepsilon, u + \varepsilon^3 e_i}^1 - \varphi_{\varepsilon, u - \varepsilon^3 e_i}^1}{2\varepsilon^3}(v) - h_{\varepsilon, i, u} \right| dv = o_\varepsilon(1).$$

1984 Since such a function is very similar to the one already presented in Lemma 6.3, we do not give a detailed
1985 construction here. Then, we can build a smooth anti-derivative $H_{\varepsilon, u}$ of $h_{\varepsilon, i, u}$, and we can write for any
1986 $u \in \mathbb{T}^2$, and any density ρ in H^1 ,

$$\int_{\mathbb{T}^2} \rho(v) h_{\varepsilon, i, u}(v) dv = \int_{\mathbb{T}^2} \partial_{u_i} \rho(v) H_{\varepsilon, u}(v) dv.$$

1987 Regarding the third line of (7.14), this yields

$$\left\langle \pi_t, \frac{\varphi_{\varepsilon, u + \varepsilon^3 e_i}^1 - \varphi_{\varepsilon, u - \varepsilon^3 e_i}^1}{2\varepsilon^3} \right\rangle = \int_{\mathbb{T}^2} \partial_{u_i} \rho(v) H_{\varepsilon, u}(v) dv + o_\varepsilon(1),$$

1988 where $H_{\varepsilon,u}$ is a smooth approximation of a Dirac in u and $o_\varepsilon(1)$ is uniform in u . According to (5.17),
 1989 $\partial_{u_i}\rho$ is in $L^2([0, T] \times \mathbb{T}^2)$ Q^* -a.s, therefore

$$(7.16) \quad \int_{\mathbb{T}^2} \partial_{u_i}\rho_t(v)H_{\varepsilon,u}(v)dv \xrightarrow[\varepsilon \rightarrow 0]{L^2([0, T] \times \mathbb{T}^2)} \partial_{u_i}\rho_t(u),$$

1990 Q^* -a.s. (see, for example, Theorem 4.22, p.109 in [6]).

1991 By Lemma 5.6 any limit point Q^* of (Q^N) is concentrated on measures absolutely continuous w.r.t.
 1992 the Lebesgue measure on \mathbb{T}^2 . For any such measure $\pi^{[0, T]}$, we denote by $\hat{\rho}_t(u, d\theta)$ its corresponding
 1993 density profile on the torus at time t , and let

$$\rho_t^\omega(u) = \int_{\mathbb{S}} \omega(\theta) \hat{\rho}_t(u, d\theta).$$

We also shorten $\rho(u) = \rho^1(u)$. Thanks to this last remark and using both (7.16) and the dominated convergence theorem for the second line of (7.14), we can now let ε go to 0 in equation (7.14), to obtain that for any limit point Q^* of (Q^N) and any $\delta > 0$,

$$(7.17) \quad Q^* \left(\left| \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, \partial_t H_t \rangle dt \right. \right. \\ \left. \left. - \int_0^T \int_{\mathbb{T}^2} \sum_{i=1}^2 d_s(\rho_t) \rho_t^\omega \partial_{u_i}^2 G_t(u) + 2 \left[d_s(\rho_t) \rho_t^{\lambda_i \omega} + \frac{\rho_t^\omega}{\rho_t} (1 - \rho_t - d_s(\rho_t)) \rho_t^{\lambda_i} \right] \partial_{u_i} G_t(u) + \mathbb{E}_{\hat{\rho}_t}(\gamma^\omega) G_t(u) \right. \right. \\ \left. \left. + \int_0^T \int_{\mathbb{T}^2} \sum_{i=1}^2 \left[\mathfrak{d}(\rho_t, \rho_t^\omega) - d'_s(\rho_t) \rho_t^\omega \right] (\partial_{u_i} \rho_t) \partial_{u_i} G_t(u) dudt \right| > \delta \right) = 0.$$

1994 *Conclusion.* — As expected, all the quantities above are linear in ω , and elementary computations yield
 1995 that

$$\mathbb{E}_{\hat{\rho}_t(u, \cdot)}(\gamma^\omega) = \int_{\mathbb{S}} \omega(\theta) \left[\rho_t(u) \mathbb{E}_{\hat{\rho}_t(u, \cdot)}(c_{u, \beta}(\theta, \hat{\eta})) d\theta - \hat{\rho}_t(u, d\theta) \right].$$

1996 Furthermore, since $H_t(u, \theta) = G_t(u) \omega(\theta)$, we can write for $k = 1, 2$

$$\rho_t^\omega \partial_{u_i}^k G_t(u) = \int_{\mathbb{S}} \omega(\theta) \partial_{u_i}^k G_t(u) \hat{\rho}_t(u, d\theta) = \int_{\mathbb{S}} \partial_{u_i}^k H_t(u, \theta) \hat{\rho}_t(u, d\theta).$$

analogous identities can be obtained when ω is replaced by another function $\Phi \in C^1(\mathbb{S})$. Using in equation (7.17) the identities above finally yield, as wanted, that for any $\delta > 0$

$$Q^* \left(\left| \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, \partial_t H_t \rangle dt \right. \right. \\ \left. \left. - \int_0^T \int_{\mathbb{T}^2 \times \mathbb{S}} \left[\sum_{i=1}^2 \left(-\partial_{u_i} H_t(u, \theta) [\hat{\mathfrak{d}}(\rho_t, \hat{\rho}_t) - d'_s(\rho_t) \hat{\rho}_t](u, d\theta) \partial_{u_i} \rho_t(u) + \partial_{u_i}^2 H_t(u, \theta) d_s(\rho_t) \hat{\rho}_t(u, d\theta) \right. \right. \right. \right. \\ \left. \left. \left. + \partial_{u_i} H_t(u, \theta) \left[2\lambda \hat{\mathfrak{s}}(\rho_t, \hat{\rho}_t) \vec{\Omega}(\hat{\rho}_t) + 2\lambda_i(\theta) d_s(\rho_t) \hat{\rho}_t \right] (u, d\theta) \right] + H_t(u, \theta) \Gamma_t(\hat{\rho})(u, d\theta) \right] dudt \right| > \delta \right) = 0.$$

1997 As in the proof of Proposition 5.4, this last identity can be extended in the case where $H_t(u, \theta)$ does
 1998 not take the form $G_t(u) \omega(\theta)$ by using a periodic version of the Weierstrass Theorem, thus letting $\delta \rightarrow 0$
 1999 completes the proof of Theorem 2.6. \square

2000

8. Limiting space-time covariance

2001 *This section is entirely dedicated to the proof Theorem 6.11, that was postponed. The strategy of the*
 2002 *proof, follows the same scheme as in Section 7.4 of [27]. One of its core ingredients is a decomposition*
 2003 *theorem (cf. Proposition (8.11)) for translation-invariant closed differential forms. To prove this decom-*
 2004 *position, one requires a sharp estimate on the spectral gap of the symmetric exclusion generator, which is*
 2005 *not uniform w.r.t. the density in our case, and some adaptations w.r.t. the classical scheme are necessary*
 2006 *to account for the angles. The non-uniformity of the spectral gap comes from the slow mixing occurring*

2007 at high densities, and requires some minor adaptation w.r.t. [35] where this issue was not dealt with. It
 2008 is solved by cutting off large densities (cf. equation (8.2) and Lemma 8.15).

2009 **8.1. Spectral gap for the symmetric exclusion process with angles.** — As investigated in Sec-
 2010 tion 3.3, the mixing time for the exclusion dynamics on configurations of size n with angles is not of
 2011 order n^2 . We therefore cannot consider a general class of functions as dependent on the θ_x 's as wanted,
 2012 and need to restrict to a subclass of functions with low levels of correlations between particle angles, but
 2013 large enough for the non-gradient method to apply. In this section, we prove that the spectral gap of the
 2014 symmetric exclusion process on this class of functions is of order $C(\rho)n^{-2}$ if the density in the box is less
 2015 than $\rho < 1$. The core estimate was first derived by Quastel in [35]. We present here a modified version
 2016 to take into account the continuous angles.

2017 Throughout this section, we consider the square domain

$$B_n = \llbracket -n, n \rrbracket^2$$

with closed boundaries. Recall that \mathcal{S} was introduced in Definition 2.1 as the set of angle-blind functions, and that ω is the angular dependency of our test function H (cf. equation (2.13)). We already defined

$$T^\omega = \left\{ f \in \mathcal{C} \mid f(\hat{\eta}) = \varphi(\eta) + \sum_{x \in \mathbb{Z}^2} \eta_x^\omega \psi_x(\eta), \quad \varphi, \psi_x \in \mathcal{S}, \forall x \in \mathbb{Z}^2 \right\},$$

2018 and now denote by \mathcal{C}_n (resp. \mathcal{S}_n) the set of cylinder functions (resp. angle-blind functions) depending
 2019 only on sites in B_n . Finally, we define $T_n^\omega = \mathcal{C}_n \cap T^\omega$.

2020 **Remark 8.1.** — The purpose of the non-gradient method is to replace the instantaneous current j_i^ω
 2021 introduced in equation (2.15) by a gradient quantity $D(\eta_0 - \eta_{e_i}) + d(\eta_0^\omega - \eta_{e_i}^\omega)$, and the class T^ω above
 2022 is the simplest set of functions, stable by \mathcal{L}_n and containing both the currents and the gradients.

2023 We expect that it is not the biggest class of functions on which a spectral gap estimate of order n^{-2}
 2024 holds. Indeed, we believe that introducing some finite numbered correlations between angles might not
 2025 alter too much the order of the spectral gap. It is not, however, the purpose of this section, and this
 2026 remark is therefore left as a conjecture at this point.

2027 Recall from Definition 3.6 that we encoded in the canonical state $\hat{K} \in \mathbb{K}_n$ the number and angles
 2028 of the particles in B_n , and that we denote by $\mu_{n, \hat{K}} = \mu_{\hat{\alpha}}(\cdot \mid \hat{\eta} \in \Sigma_n^{\hat{K}})$ the canonical measure with \hat{K}
 2029 particles inside B_n . Finally, define

$$\mathcal{D}_{n, \hat{K}}(f) = \mathbb{E}_{n, \hat{K}}(f \mathcal{L}_n f),$$

2030 where \mathcal{L}_n is the symmetric exclusion generator restricted to jumps with both extremities in B_n . We are
 2031 now ready to state the main result of this section.

2032 **Proposition 8.2 (Estimate on the spectral gap for the SSEP with angles)**

2033 For any $0 \leq \alpha < 1$, there exists a constant $C(\alpha)$ such that for any $\hat{K} \in \mathbb{K}_n$ such that $K \leq \alpha |B_n|$, and
 2034 any $f \in T_n^\omega$ such that $\mathbb{E}_{n, \hat{K}}(f) = 0$,

$$\mathbb{E}_{n, \hat{K}}(f^2) \leq C(\alpha) n^2 \mathcal{D}_{n, \hat{K}}(f).$$

2035 **Remark 8.3 (Non-uniformity of the spectral gap).** — Note that this estimate is not uniform in
 2036 the density. Actually, the constant $C(\alpha)$ behaves as $1/(1 - \alpha)$, and therefore even on the set T^ω , the
 2037 spectral gap of the exclusion process when there are only a finite number of empty sites in B_n is or order
 2038 n^{-4} . This high density estimate is sharp : define \hat{K}_n by $K_n = (2n + 1)^2 - 1$, and for $k = 1, \dots, K_n$,
 2039 $\theta_k = 2k\pi/K_n$, then for

$$f_n(\hat{\eta}) = \sum_{x \in B_n} (\theta_x - \pi) \eta_x \cos\left(\frac{2\pi x_1}{2n + 1}\right),$$

2040 one easily checks that there exists a positive constant C such that

$$n^4 \frac{\mathcal{D}_{n,\widehat{K}_n}(f_n)}{\text{Var}_{n,\widehat{K}_n}(f_n)} \xrightarrow{n \rightarrow \infty} C.$$

2041 This non-uniformity is not an issue here, however, because when we later on classify the germs of closed
2042 forms for our model, we are able to cutoff the large densities (cf. equation (8.2)).

2043 In order to prove Proposition 8.2, we need the following lemma, which states that the angle-blind
2044 process has a uniform spectral gap of order n^{-2} . For any angle-blind function $\psi \in \mathcal{S}_n$, we will write $\psi(\eta)$
2045 instead of $\psi(\widehat{\eta})$ to emphasize that it does not depend on the angles.

2046 **Lemma 8.4 (Spectral gap for the angle-blind exclusion process)**

2047 Denote by $\widetilde{\mathbb{E}}_{n,K}$ the expectation w.r.t. the angle-blind canonical measure with K particles inside B_n ,
2048 defined for any angle-blind function $\psi \in \mathcal{S}_n$ by

$$\widetilde{\mathbb{E}}_{n,K}(\psi) = \mathbb{E}_{\widehat{\alpha}} \left(\psi \mid \sum_{x \in B_n} \eta_x = K \right),$$

2049 which holds for any $\widehat{\alpha}$ with density $\alpha \in (0, 1)$. There exists a universal constant $C_1 > 0$ such that for any
2050 $n \geq 1$, any $0 \leq K \leq (2n + 1)^2$ and any $\psi \in \mathcal{S}_n$ satisfying $\widetilde{\mathbb{E}}_{n,K}(\psi) = 0$,

$$\widetilde{\mathbb{E}}_{n,K}(\psi^2) \leq C_1 n^2 \widetilde{\mathcal{D}}_{n,K}(\psi),$$

2051 where $\widetilde{\mathcal{D}}_{n,K}(\psi) = \widetilde{\mathbb{E}}_{n,K}(\psi(-\mathcal{L}_n)\psi)$.

2052 This result is fairly classical, its proof can be found for instance in [27], we do not repeat it here. Note in
2053 particular that for the angle-blind process, the constant can be chosen independently of the cap on the
2054 density α . Before proving Proposition 8.2, we need one more definition. Fix $\alpha \in [0, 1)$, and a canonical
2055 state $\widehat{K} \in \mathbb{K}_n$ such that $K \leq \alpha|B_n|$. We then define for any site $x \in \mathbb{Z}^2$,

$$(8.1) \quad \widehat{\omega} = \omega - \mathbb{E}_{n,\widehat{K}}(\omega) \quad \text{and} \quad \eta_x^{\widehat{\omega}} = \left[\omega(\theta_x) - \mathbb{E}_{n,\widehat{K}}(\omega) \right] \eta_x,$$

2056 where $\mathbb{E}_{n,\widehat{K}}(\omega)$ stands for $\mathbb{E}_{n,\widehat{K}}(\omega(\theta_0) \mid \eta_0 = 1)$. In particular, for any configuration $\widehat{\eta}$, $\sum_{x \in B_n} \eta_x^{\widehat{\omega}} = 0$
2057 under $\mu_{n,\widehat{K}}$. This centered occupation variable plays a particular role in the proof of the spectral gap,
2058 and we state in the following Lemma two identities regarding $\eta^{\widehat{\omega}}$, which will be used later on.

2059 **Lemma 8.5 (Properties of $\eta^{\widehat{\omega}}$).** — Define $V_{n,\widehat{K}}(\omega) = \text{Var}_{n,\widehat{K}}(\omega(\theta_0) \mid \eta_0 = 1)$. For any $x \neq y \in B_n$,
2060 $\widehat{K} \in \mathbb{K}_n$, and any angle-blind function $\psi \in \mathcal{S}_n$, we have $\mathbb{E}_{n,\widehat{K}}(\eta_x^{\widehat{\omega}}\psi) = 0$,

$$\mathbb{E}_{n,\widehat{K}} \left((\eta_x^{\widehat{\omega}})^2 \psi \right) = V_{n,\widehat{K}}(\omega) \widetilde{\mathbb{E}}_{n,K}(\eta_x \psi) \quad \text{and} \quad \mathbb{E}_{n,\widehat{K}} \left(\eta_x^{\widehat{\omega}} \eta_y^{\widehat{\omega}} \psi \right) = \begin{cases} -\frac{V_{n,\widehat{K}}(\omega)}{K-1} \widetilde{\mathbb{E}}_{n,K}(\eta_x \eta_y \psi) & \text{if } K > 1 \\ 0 & \text{else} \end{cases}.$$

2061 *Proof of Proof of Lemma 8.5.* — This Lemma follows from elementary computations. Under $\mu_{n,\widehat{K}}$, for
2062 any angle-blind function $\psi \in \mathcal{S}_n$ and any function Φ on \mathbb{S} , we have

$$\mathbb{E}_{n,\widehat{K}}(\eta_x^{\Phi} \psi) = \mathbb{E}_{n,\widehat{K}}(\Phi(\theta_0) \mid \eta_0 = 1) \widetilde{\mathbb{E}}_{n,K}(\eta_x \psi).$$

2063 For the first (resp. second) identity, we set $\Phi = \omega - \mathbb{E}_{n,\widehat{K}}(\omega)$ (resp. $\Phi = (\omega - \mathbb{E}_{n,\widehat{K}}(\omega))^2$), which by
2064 construction has mean 0 (resp. $V_{n,\widehat{K}}(\omega)$) w.r.t. $\mu_{n,\widehat{K}}(\cdot \mid \eta_0 = 1)$. Regarding the last identity, we obtain
2065 similarly

$$\mathbb{E}_{n,\widehat{K}} \left(\eta_x^{\widehat{\omega}} \eta_y^{\widehat{\omega}} \psi \right) = \left[\mathbb{E}_{n,\widehat{K}}(\omega(\theta_x)\omega(\theta_y) \mid \eta_x = \eta_y = 1) - \mathbb{E}_{n,\widehat{K}}(\omega)^2 \right] \widetilde{\mathbb{E}}_{n,K}(\eta_x \eta_y \psi) = -\frac{V_{n,\widehat{K}}(\omega)}{K-1} \widetilde{\mathbb{E}}_{n,K}(\eta_x \eta_y \psi)$$

2066 if $K > 1$, and trivially vanishes if $K = 0, 1$. □

2067 We now estimate the spectral gap of the angle process on T_n^ω .

2068 *Proof of Proposition 8.2.* — Fix $\alpha \in [0, 1)$, $\widehat{K} \in \mathbb{K}_n$ such that $K \leq \alpha|B_n|$, and consider a function
 2069 $f = \varphi(\eta) + \sum_{x \in B_n} \eta_x^\omega \psi_x(\eta)$ in T_n^ω , where $\varphi, \psi_x \in \mathcal{S}_n$, such that $\mathbb{E}_{n, \widehat{K}}(f) = 0$. Recall the notation
 2070 introduced in (8.1), and denote

$$f_1 = \sum_{x \in B_n} \eta_x^\omega \psi_x, \quad f_b = \varphi + \mathbb{E}_{n, \widehat{K}}(\omega) \sum_{x \in B_n} \eta_x \psi_x \in \mathcal{S}_n.$$

2071 By construction, $f = f_1 + f_b$. Since for any $\psi \in \mathcal{S}_n$, $\mathbb{E}_{n, \widehat{K}}(\eta_x^\omega \psi) = 0$, it is straightforward to obtain that

$$\mathbb{E}_{n, \widehat{K}}(f^2) = \mathbb{E}_{n, \widehat{K}}(f_1^2) + \widetilde{\mathbb{E}}_{n, K}(f_b^2) \quad \text{and} \quad \mathbb{E}_{n, \widehat{K}}(f \mathcal{L}_n f) = \mathbb{E}_{n, \widehat{K}}(f_1 \mathcal{L}_n f_1) + \widetilde{\mathbb{E}}_{n, K}(f_b \mathcal{L}_n f_b),$$

2072 (i.e. $\mathcal{D}_{n, \widehat{K}}(f) = \mathcal{D}_{n, \widehat{K}}(f_1) + \widetilde{\mathcal{D}}_{n, K}(f_b)$). By assumption $\mathbb{E}_{n, \widehat{K}}(f) = 0$, therefore, since by construction
 2073 $\mathbb{E}_{n, \widehat{K}}(f_1) = 0$, we also have $\mathbb{E}_{n, \widehat{K}}(f_b) = 0$. Lemma 8.4 can therefore be applied to f_b . To prove Proposition
 2074 8.2, it is thus sufficient to prove it for any function of the form $f = \sum_{x \in B_n} \eta_x^\omega \psi_x(\eta)$. We can further
 2075 assume, without loss of generality, that $\sum \psi_x = 0$ and that each ψ_x vanishes if $\eta_x = 0$ since we can
 2076 rewrite

$$f(\widehat{\eta}) = \sum_{x \in B_n} \eta_x^\omega \widetilde{\psi}_x(\eta)$$

2077 where

$$\widetilde{\psi}_x = \eta_x(\psi_x - \bar{\psi}) \quad \text{and} \quad \bar{\psi} = \frac{\sum_{x \in B_n} \eta_x \psi_x}{\sum_{x \in B_n} \eta_x} = \frac{\sum_{x \in B_n} \eta_x \psi_x}{K(\widehat{\eta})}.$$

2078 Note that we only consider $K > 0$, since if $K = 0$, Proposition 8.2 is immediate.

2079 To prove Proposition 8.2, it is therefore sufficient to prove it for any function

$$f = \sum_{x \in B_n} \eta_x^\omega \psi_x,$$

2080 where $\psi_x = \eta_x \psi_x$, and satisfy $\sum_{x \in B_n} \psi_x = 0$. For any such f , if $K = 1$, there is only one particle in B_n
 2081 and $\eta_x^\omega = 0$ for any x , therefore $f = 0$. We now assume that $1 < K \leq \alpha|B_n|$. By Lemma 8.5, since by
 2082 assumption $\sum_x \psi_x = 0$,

$$(8.2) \quad \mathbb{E}_{n, \widehat{K}}(f^2) = \sum_{x, y \in B_n} \mathbb{E}_{n, \widehat{K}}(\eta_x^\omega \eta_y^\omega \psi_x \psi_y) = \frac{K}{K-1} V_{n, \widehat{K}}(\omega) \sum_{x \in B_n} \mathbb{E}_{n, \widehat{K}}(\psi_x^2).$$

2083 We now turn our attention to $\mathbb{E}_{n, \widehat{K}}(f \mathcal{L}_n f)$. For any site x and any angle-blind function $\psi \in \mathcal{S}_n$, we
 2084 can write

$$\mathcal{L}_n(\eta_x^\omega \psi_x) = \eta_x^\omega \mathcal{L}_n \psi_x + \sum_{|z|=1} \mathbb{1}_{\{\eta_x \eta_{x+z} = 0\}} \psi_x(\eta^{x, x+z}) ((\eta^{x, x+z})_x^\omega - \eta_x^\omega).$$

2085 Since we assumed that ψ_x vanishes when the site x is empty, the quantity above can be rewritten

$$\mathcal{L}_n(\eta_x^\omega \psi_x) = \eta_x^\omega \mathcal{L}_n \psi_x + \sum_{|z|=1} \eta_{x+z}^\omega (1 - \eta_x) \psi_x(\eta^{x, x+z}).$$

2086 It follows that

$$\mathcal{D}_{n, \widehat{K}}(f) = \sum_{x, y \in B_n} \left[\mathbb{E}_{n, \widehat{K}}(\eta_x^\omega \eta_y^\omega \psi_x (-\mathcal{L}_n) \psi_y) - \mathbb{E}_{n, \widehat{K}} \left(\eta_x^\omega \psi_x \sum_{|z|=1} \eta_{y+z}^\omega (1 - \eta_y) \psi_y(\eta^{y, y+z}) \right) \right].$$

2087 Using once again that $\sum_{x \in B_n} \psi_x = 0$, and Lemma 8.5 the identity above rewrites

$$(8.3) \quad \mathcal{D}_{n, \widehat{K}}(f) = \frac{K}{K-1} V_{n, \widehat{K}}(\omega) \sum_{x \in B_n} \left[\widetilde{\mathcal{D}}_{n, K}(\psi_x) - \sum_{|z|=1} \widetilde{\mathbb{E}}_{n, K}((1 - \eta_{x+z}) \psi_x \psi_{x+z}(\eta^{x, x+z})) \right].$$

2088 Let us introduce the Dirichlet form locally cropped in x

$$(8.4) \quad \widetilde{\mathcal{D}}_{n, K}^x(\psi) = \frac{1}{2} \widetilde{\mathbb{E}}_{n, K} \left(\sum_{\substack{y, y+z \in B_n \setminus \{x\} \\ |z|=1}} \eta_y (1 - \eta_{y+z}) (\psi(\eta^{y, y+z}) - \psi(\eta))^2 \right),$$

2089 which forbids jumps to and from the site x . Since ψ_x vanishes whenever the site x is empty, the quantity
 2090 $\eta_x(1 - \eta_{x+z})(\psi_x(\eta^{x,x+z}) - \psi_x(\eta))^2$ is also equal to $(1 - \eta_{x+z})\psi_x(\eta)^2$, and a similar argument with ψ_{x+z}
 2091 allows us to rewrite equation (8.3)

$$\mathcal{D}_{n,\hat{K}}(f) = \frac{K}{K-1} V_{n,\hat{K}}(\omega) \sum_{x \in B_n} \left[\tilde{\mathcal{D}}_{n,K}^x(\psi_x) + \frac{1}{2} \sum_{|z|=1} \tilde{\mathbb{E}}_{n,K} \left((1 - \eta_{x+z}) [\psi_{x+z}(\eta^{x,x+z}) - \psi_x(\eta)]^2 \right) \right].$$

2092 To obtain Proposition 8.2, thanks to the identity above together with (8.2) it is enough to prove that for
 2093 some constant $C(\alpha)$,

$$(8.5) \quad \sum_{x \in B_n} \tilde{\mathbb{E}}_{n,K}(\psi_x^2) \leq C(\alpha)n^2 \sum_{x \in B_n} \left[\tilde{\mathcal{D}}_{n,K}^x(\psi_x) + \frac{1}{2} \sum_{|z|=1} \tilde{\mathbb{E}}_{n,K} \left((1 - \eta_{x+z}) [\psi_{x+z}(\eta^{x,x+z}) - \psi_x]^2 \right) \right].$$

2094 We now state a technical Lemma, which gives a spectral gap estimate when one site remains frozen.

2095 **Lemma 8.6 (Spectral gap for the exclusion process with a frozen site)**

2096 *Fix $x \in B_n$. There exists a universal constant C_2 such that for any angle-blind function $\psi \in \mathcal{S}_n$*
 2097 *satisfying $\tilde{\mathbb{E}}_{n,K}(\psi | \eta_x = 1) = 0$,*

$$\tilde{\mathbb{E}}_{n,K}(\psi^2 | \eta_x = 1) \leq C_2 n^2 \tilde{\mathcal{D}}_{n,K}^x(\psi | \eta_x = 1),$$

2098 *where the conditioned Dirichlet form is defined by the conditional expectation $\tilde{\mathbb{E}}_{n,K}(\cdot | \eta_x = 1)$ instead of*
 2099 $\tilde{\mathbb{E}}_{n,K}$,

$$\tilde{\mathcal{D}}_{n,K}^x(\psi | \eta_x = 1) = -\tilde{\mathbb{E}}_{n,K}(\psi \mathcal{L}_n \psi | \eta_x = 1).$$

2100 *Proof of Lemma 8.6.* — We do not give the detail of this proof. It is quite similar to the proof without
 2101 the frozen site for an angle-blind function, the only difference being that whenever a path should go
 2102 through the site x , the path is bypassed around it, which results in a larger constant C but does not
 2103 affect the order n^2 . \square

2104 We now take a look at the left-hand side of equation (8.5). Since ψ_x vanishes whenever $\eta_x = 0$ we
 2105 have $\tilde{\mathbb{E}}_{n,K}(\psi_x | \eta_x = 1) = \frac{|B_n|}{K} \tilde{\mathbb{E}}_{n,K}(\psi_x)$, the previous lemma applied to $\psi_x - \mathbb{E}_{n,\hat{K}}(\psi_x | \eta_x = 1)$ yields

$$(8.6) \quad \sum_{x \in B_n} \tilde{\mathbb{E}}_{n,K}(\psi_x^2) - \frac{|B_n|}{K} \tilde{\mathbb{E}}_{n,K}(\psi_x)^2 \leq C_2 n^2 \sum_{x \in B_n} \tilde{\mathcal{D}}_{n,K}^x(\psi_x).$$

Furthermore,

$$\begin{aligned} \sum_{x,y \in B_n} [\tilde{\mathbb{E}}_{n,K}(\psi_x) - \tilde{\mathbb{E}}_{n,K}(\psi_y)]^2 &= \sum_{x,y \in B_n} [\tilde{\mathbb{E}}_{n,K}(\psi_x)^2 + \tilde{\mathbb{E}}_{n,K}(\psi_y)^2] - 2 \sum_{x,y \in B_n} \tilde{\mathbb{E}}_{n,K}(\psi_x) \tilde{\mathbb{E}}_{n,K}(\psi_y) \\ &= 2n^2 \sum_{x \in B_n} \tilde{\mathbb{E}}_{n,K}(\psi_x)^2, \end{aligned}$$

2106 because the last term of the first line vanishes by the assumption $\sum_{x \in B_n} \psi_x = 0$. Furthermore, consider
 2107 the family of paths $(\gamma_{x,y})_{x,y \in B_n}$ going from x to y , defined as follows : starting from x , the path $\gamma_{x,y}$ starts
 2108 straight in the first direction, until reaching the first coordinate of y . then, it goes in the second direction
 2109 until reaching y . With this construction, each edge a is used at most a number of times $p_a \leq Cn^3$ in
 2110 the $\gamma_{x,y}$'s, for some universal constant C . Furthermore, each path $\gamma_{x,y}$ has length at most $4n$. With this
 2111 construction, we therefore write, since

$$\psi_x - \psi_y = \sum_{a=(a_1,a_2) \in \gamma_{x,y}} (\psi_{a_1} - \psi_{a_2}),$$

and $(\sum_{k=1}^p x_k)^2 \leq p \sum_{k=1}^p x_k^2$ that

$$\begin{aligned} \sum_{x,y \in B_n} [\tilde{\mathbb{E}}_{n,K}(\psi_x) - \tilde{\mathbb{E}}_{n,K}(\psi_y)]^2 &\leq \sum_{x,y \in B_n} 4n \sum_{(a_1,a_2) \in \gamma_{x,y}} [\tilde{\mathbb{E}}_{n,K}(\psi_{a_1}) - \tilde{\mathbb{E}}_{n,K}(\psi_{a_2})]^2 \\ &= 4n \sum_{(a_1,a_2) \subset B_n} p_a [\tilde{\mathbb{E}}_{n,K}(\psi_{a_1}) - \tilde{\mathbb{E}}_{n,K}(\psi_{a_2})]^2 \end{aligned}$$

$$\begin{aligned}
 &\leq 4Cn^4 \sum_{(a_1, a_2) \subset B_n} [\tilde{\mathbb{E}}_{n,K}(\psi_{a_1}) - \tilde{\mathbb{E}}_{n,K}(\psi_{a_2})]^2 \\
 &= 4Cn^4 \sum_{\substack{x, x+z \in B_n, \\ |z|=1}} [\tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x)]^2.
 \end{aligned}$$

2112 Using the two previous identities, we obtain that

$$(8.7) \quad \sum_{x \in B_n} \tilde{\mathbb{E}}_{n,K}(\psi_x)^2 \leq Cn^2 \sum_{x \in B_n, |z|=1} [\tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x)]^2,$$

so that using equations (8.5), (8.6), and (8.7), to prove Proposition 8.2 it is enough to show that for some constant $C(\alpha)$,

$$(8.8) \quad \sum_{x \in B_n, |z|=1} [\tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x)]^2 \leq \frac{K}{|B_n|} C(\alpha) \sum_{x \in B_n} \left[\tilde{\mathcal{D}}_{n,K}^x(\psi_x) + \sum_{|z|=1} \tilde{\mathbb{E}}_{n,K} \left((1 - \eta_{x+z}) [\psi_{x+z} (\eta^{x,x+z}) - \psi_x]^2 \right) \right].$$

2113 Let us denote by e_{x+z} the empty site nearest to $x+z$ other than x , chosen arbitrarily if there are
 2114 multiple candidates. We want to reach from η a configuration with an empty site in $x+z$, where the
 2115 successive jumps will be controlled by the Dirichlet form of the ψ'_x s, and the resulting difference will be
 2116 controlled by the second term above. To do so, we merely have to "move" the empty site from e_{x+z} to
 2117 $x+z$, using a path of minimal length. We denote by a_1, \dots, a_p the sequence of edges along which the
 2118 empty site travels. For any integer $r \leq p$ let $\eta^{(r-1)} = \eta^{a_1 \dots a_r}$ be the configuration where the empty site
 2119 has traveled along r edges. In particular, $\eta^{(0)} = \eta$, and $\eta_{x+z}^{(p)} = 0$. Furthermore, for each edge a_r in this
 2120 sequence, we denote by $a_{r,1}$ the position throughout this construction of the displaced particle at the
 2121 r -th stage, and $a_{r,2}$ the position of the empty site, therefore, $a_r = (a_{r,1}, a_{r,2})$. One easily sees that if
 2122 $e_{x+z} \neq x$, we can perform this construction with the following conditions satisfied.

- 2123 1) The path followed by the empty site contains at most $p(e_{x+z}) \leq 2 |e_{x+z} - x|$ jumps.
 2124 2) None of the edges a_r connects x and one of its neighbors.
 2125 3) The only edge linking $x+z$ to one of its neighbor is the last edge a_p , and it is of the form
 2126 $a_p = (x+z, x+z+z')$, with z and z' orthogonal. In other words, we assume that the empty site
 2127 comes from the direction orthogonal to the direction of the edge $(x, x+z)$.

With this construction, for any function h , since every successive jump is allowed (each initial site is occupied, each end site is empty) we have

$$\begin{aligned}
 (1 - \eta_{x+z}^{(p)}) h(\eta^{(p)}) &= h(\eta^{(p)}) = h(\eta) + \sum_{r=1}^p \left(h(\eta^{(r-1)}) - h(\eta^{(r-1)}) \right) \\
 &= h(\eta) + \sum_{r=1}^p \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \tilde{\nabla}_{a_r} h(\eta^{(r-1)}),
 \end{aligned}$$

2128 where $\tilde{\nabla}_a f = f(\eta^{a_1, a_2}) - f(\eta)$. We can rewrite this identity

$$h(\eta) = \left(1 - \eta_{x+z}^{(p)}\right) h(\eta^{(p)}) - \sum_{r=1}^p \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \tilde{\nabla}_{a_r} h(\eta^{(r-1)}).$$

2129 Note that in the formula above, both p and the path $\eta^{(r-1)}$ depends on the position of e_{x+z} .

We not let $h(\eta) = \psi_{x+z}(\eta^{x,x+z}) - \psi_x$. This function vanishes if there is an empty site in x , which is the only case for which the construction above does not hold (because in particular the empty site cannot avoid the edges surrounding x). Using the construction above, we obtain

$$\tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x) = \tilde{\mathbb{E}}_{n,K}(\psi_{x+z}(\eta^{x,x+z})) - \tilde{\mathbb{E}}_{n,K}(\psi_x)$$

$$\begin{aligned}
&= -\tilde{\mathbb{E}}_{n,K} \left(\sum_{r=1}^p \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \tilde{\nabla}_{a_r} \left[\psi_{x+z}((\eta^{(r-1)})^{x,x+z}) - \psi_x(\eta^{(r-1)}) \right] \right) \\
&\quad + \tilde{\mathbb{E}}_{n,K} \left(\left(1 - \eta_{x+z}^{(p)}\right) \left[\psi_{x+z}((\eta^{(p)})^{x,x+z}) - \psi_x(\eta^{(p)}) \right] \right).
\end{aligned}$$

We now project on the possible positions for e_{x+z} , by Cauchy-Schwarz inequality, and since $(\sum_{i=1}^p a_i)^2 \leq p \sum_{i=1}^p a_i^2$, we obtain

$$\begin{aligned}
(8.9) \quad \left| \tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x) \right| &\leq \sum_{e \in B_n \setminus \{x\}} \sqrt{(2p(e) + 1) \tilde{\mu}_{n,K}(e_{x+z} = e, \eta_x = 1)} \\
&\times \left[\tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \left(1 - \eta_{x+z}^{(p(e))}\right) \left[\psi_{x+z}((\eta^{(p(e))})^{x,x+z}) - \psi_x(\eta^{(p(e))}) \right]^2 \right) \right. \\
&+ \sum_{r=1}^{p(e)} \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \left[\tilde{\nabla}_{a_r} \psi_{x+z}((\eta^{(r-1)})^{x,x+z}) \right]^2 \right) \\
&\left. + \sum_{r=1}^{p(e)} \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \left[\tilde{\nabla}_{a_r} \psi_x(\eta^{(r-1)}) \right]^2 \right) \right]^{1/2}.
\end{aligned}$$

2130 We now estimate each of the three terms in the bracket.

The empty site e being fixed, the sequence of edges (a_r) and its length p are also fixed. The first term in the bracket can therefore be rewritten, thanks the one-to-one change of variables $\eta^{(p-1)} \leftarrow \eta$

$$\begin{aligned}
\tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} (\eta') (1 - \eta_{x+z}) \left[\psi_{x+z}(\eta^{x,x+z}) - \psi_x(\eta) \right]^2 \right) \\
\leq \tilde{\mathbb{E}}_{n,K} \left((1 - \eta_{x+z}) \left[\psi_{x+z}(\eta^{x,x+z}) - \psi_x(\eta) \right]^2 \right),
\end{aligned}$$

where η' denotes the invert change of variable $\eta \leftarrow \eta^{(p-1)}$. Since none of the edges a_r connects x to one of its neighbors, and since each edge is used at most once, one-to-one changes of variable $\eta^{(r-1)} \leftarrow \eta$ also allow us to crudely estimate

$$\begin{aligned}
\sum_{r=1}^p \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \left[\tilde{\nabla}_{a_r} \psi_x(\eta^{(r-1)}) \right]^2 \right) \\
= \sum_{r=1}^p \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} (\eta'^{(r)}) \eta_{a_{r,1}} (1 - \eta_{a_{r,2}}) \left[\tilde{\nabla}_{a_r} \psi_x(\eta) \right]^2 \right) \leq \tilde{\mathcal{G}}_{n,K}^x(\psi_x).
\end{aligned}$$

Finally, for the third contribution, we can write the same estimate, except for the last gradient which is over an edge $(a_{p,1}, a_{p,2}) = (x+z, x+z+z')$, with $|z'| = |z| = 1$. We therefore write

$$\begin{aligned}
\sum_{r=1}^p \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \left[\tilde{\nabla}_{a_r} \psi_{x+z}((\eta^{(r-1)})^{x,x+z}) \right]^2 \right) \\
\leq \tilde{\mathcal{G}}_{n,K}^{x+z}(\psi_{x+z}) + \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{p,1}}^{(p-1)} (1 - \eta_{a_{p,2}}^{(p-1)}) \left[\tilde{\nabla}_{a_p} \psi_{x+z}((\eta^{(p-1)})^{x,x+z}) \right]^2 \right) \\
\leq \tilde{\mathcal{G}}_{n,K}^{x+z}(\psi_{x+z}) + \tilde{\mathbb{E}}_{n,K} \left(\eta_{x+z} (1 - \eta_{x+z+z'}) \left[\psi_{x+z} \left(\left(\eta^{x+z, x+z+z'} \right)^{x,x+z} \right) - \psi_{x+z}(\eta^{x,x+z}) \right]^2 \right).
\end{aligned}$$

One easily obtains that $\eta^{x, x+z+z'} = \left((\eta^{x, x+z})^{x+z, x+z+z'} \right)^{x, x+z}$, therefore performing the change of variable $\eta^{x, x+z} \leftarrow \eta$ in the bound above yields

$$\sum_{r=1}^p \tilde{\mathbb{E}}_{n,K} \left(\mathbb{1}_{\{e_{x+z}=e, \eta_x=1\}} \eta_{a_{r,1}}^{(r-1)} (1 - \eta_{a_{r,2}}^{(r-1)}) \left[\tilde{\nabla}_{a_r} \psi_{x+z}((\eta^{(r-1)})^{x,x+z}) \right]^2 \right)$$

$$\leq \tilde{\mathcal{G}}_{n,K}^{x+z}(\psi_{x+z}) + \underbrace{\tilde{\mathbb{E}}_{n,K} \left(\eta_x (1 - \eta_{x+z+z'}) \left[\psi_{x+z} \left(\eta^{x,x+z+z'} \right) - \psi_{x+z}(\eta) \right]^2 \right)}_{\leq 2\tilde{\mathbb{E}}_{n,K}((\nabla_{x,x+z'}\psi_{x+z})^2) + 2\tilde{\mathbb{E}}_{n,K}((\nabla_{x+z',x+z+z'}\psi_{x+z})^2)} \leq 3\tilde{\mathcal{G}}_{n,K}^{x+z}(\psi_{x+z}),$$

where we used that z' and z are orthogonal by assumption, which means that the gradients in the last term are not of the form $(x+z, x+z+z')$. We now use these three bounds in (8.9), to obtain that for some universal constant C_3

$$\begin{aligned} \left(\tilde{\mathbb{E}}_{n,K}(\psi_{x+z}) - \tilde{\mathbb{E}}_{n,K}(\psi_x) \right)^2 &\leq C_3 \left(\sum_{e \in B_n \setminus \{x\}} \sqrt{(1+2p(e))\tilde{\mu}_{n,K}(e_{x+z}=e, \eta_x=1)} \right)^2 \\ &\quad \times \left[\tilde{\mathbb{E}}_{n,K} \left((1 - \eta_{x+z}) \left[\psi_{x+z}(\eta^{x,x+z}) - \psi_x(\eta) \right]^2 \right) + \tilde{\mathcal{G}}_{n,K}^x(\psi_x) + \tilde{\mathcal{G}}_{n,K}^{x+z}(\psi_{x+z}) \right]. \end{aligned}$$

Since we assumed $K \leq \alpha|B_n|$, for $\alpha < 1$ one straightforwardly obtains by elementary computations that

$$\sum_{e \in B_n \setminus \{x\}} \sqrt{(1+2p(e))\tilde{\mu}_{n,K}(e_{x+z}=e, \eta_x=1)} \leq \sqrt{\frac{K}{|B_n|}} C(\alpha),$$

2131 therefore (8.8) holds as desired. This concludes the proof of Proposition 8.2. \square

2132 **8.2. Discrete differential forms in the context of particles systems.** — *We introduce in this*
 2133 *section the concept of discrete differential forms in the context of particle systems. The key point of the*
 2134 *non-gradient method is that any translation-invariant closed form can be decomposed as the sum of a*
 2135 *gradient of a translation-invariant function and the currents. This result is stated in Proposition 8.11,*
 2136 *and directly rewrites as an approximation (in the sense of equation (6.37)) of any function in \mathcal{T}_0^ω by a*
 2137 *linear combination of the currents up to an element of \mathcal{LC} .*

2138 Let us denote by Σ_∞ the set of configurations on \mathbb{Z}^2

$$\Sigma_\infty = \left\{ (\eta_x, \theta_x)_{x \in \mathbb{Z}^2} \in (\{0, 1\} \times \mathcal{S})^{\mathbb{Z}^2} \mid \theta_x = 0 \text{ if } \eta_x = 0 \right\}.$$

2139 We consider here the graph $\mathcal{G} = (\Sigma_\infty, E)$ with oriented edge set

$$(8.10) \quad E = \left\{ (\hat{\eta}, \hat{\eta}') \in \Sigma_\infty^2 \mid \hat{\eta}' = \hat{\eta}^{x,x+z} \text{ for some } x \in \mathbb{Z}^2, |z|=1 \text{ and } \eta_x(1 - \eta_{x+z}) = 1 \right\}.$$

2140 In other words, there is an edge from $\hat{\eta}$ to $\hat{\eta}'$ if and only if the latter can be reached from the former
 2141 with exactly one licit particle jump (i.e. the jump of a particle to an *empty* site). We endow \mathcal{G} with the
 2142 usual distance d on graphs, i.e. $d(\hat{\eta}, \hat{\eta}')$ is the minimal number of particle jumps necessary to go from one
 2143 configuration to the other. Note that this graph is not connected, since for example the configuration $\hat{\eta}$
 2144 with no particles is not accessible from any configuration $\hat{\eta}'$ with any number of particles. This is also
 2145 the case for two configurations with different angle distributions. In such a case where there is no path
 2146 between $\hat{\eta}'$ and $\hat{\eta}$, we will adopt the usual convention $d(\hat{\eta}, \hat{\eta}') = \infty$. By abuse of notation, we also denote
 2147 by $\mu_{\hat{\alpha}}$ (cf. Definition 3.4) the grand-canonical measure measure on \mathbb{Z}^2 with parameter $\hat{\alpha}$, and write $\mathbb{E}_{\hat{\alpha}}(\cdot)$
 2148 for the expectation w.r.t $\mu_{\hat{\alpha}}$.

2149 We call *differential form* on (\mathcal{G}, d) a collection of $L^2(\mu_{\hat{\alpha}})$ variables associated with each edge in E .
 2150 More precisely, it is a collection $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_{x,x+z})_{x \in \mathbb{Z}^2, |z|=1}$, satisfying

$$\bar{\mathbf{u}}_{x,x+z}(\hat{\eta}) = \eta_x(1 - \eta_{x+z})\bar{\mathbf{u}}_{x,x+z}(\hat{\eta}) \in L^2(\mu_{\hat{\alpha}}).$$

2151 This definition arbitrarily attributes to $\bar{\mathbf{u}}_{x,x+z}(\hat{\eta})$ the value 0 if $\eta_x(1 - \eta_{x+z})$ vanishes (i.e. if the jump from
 2152 x to $x+z$ cannot be performed in $\hat{\eta}$), which is just a notation shortcut to define $\bar{\mathbf{u}}$ on all configurations
 2153 rather than only on those such that $\eta_x(1 - \eta_{x+z}) = 1$. Another way to look at these objects is that
 2154 with each possible particle jump in a configuration $\hat{\eta}$ is associated a weight. In this section, we will
 2155 only consider *closed forms*, i.e. differential forms for which the added weight of any finite-length path
 2156 (composed only of licit jumps, i.e. jumps from x to $x+z$ with x occupied and $x+z$ empty) between two

2157 configuration does not depend on the path chosen but only on the two endpoints. Equivalently, closed
 2158 forms are those for which the integral over a closed loop of licit jumps vanishes.

2159 We call *path* a finite sequence of jumps coordinates $\gamma = (x_i, x_i + z_i)_{0 \leq i \leq q_\gamma}$, where the x_i 's are in \mathbb{Z}^2 ,
 2160 and $|z_i| = 1$. Given a configuration $\hat{\eta}$, we denote $\Gamma(\hat{\eta})$ (resp. $\Gamma_c(\hat{\eta})$) the set of *licit paths* (resp. *licit loops*,
 2161 i.e. licit closed paths) such that all successive jumps in the path are licit starting from $\hat{\eta}$, (resp. and such
 2162 that the configuration reached at the end of the sequence of jumps is $\hat{\eta}$)

$$\Gamma(\hat{\eta}) = \{\gamma = (x_i, x_i + z_i)_{0 \leq i \leq q_\gamma} \mid \hat{\eta}_{x_i}^{(i, \gamma)}(1 - \hat{\eta}_{x_i + z_i}^{(i, \gamma)}) = 1, 0 \leq i \leq q_\gamma\},$$

2163 (resp. $\Gamma_c(\hat{\eta}) = \{\gamma = (x_i, x_i + z_i)_{0 \leq i \leq q_\gamma} \in \Gamma(\hat{\eta}) \mid \hat{\eta}^{(q_\gamma + 1, \gamma)} = \hat{\eta}\}$), where for any path γ , and any
 2164 configuration $\hat{\eta}$, we denote $\hat{\eta}^{(0, \gamma)} = \hat{\eta}$, and $\hat{\eta}^{(i+1, \gamma)} = (\hat{\eta}^{(i, \gamma)})^{x_i, x_i + z_i}$ for $0 \leq i \leq q_\gamma$. For any differential
 2165 form $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_{x, x+z})_{x \in \mathbb{Z}^2, |z|=1}$, and any finite path γ , we denote

$$I_{\gamma, \bar{\mathbf{u}}}(\hat{\eta}) = \mathbb{1}_{\{\gamma \in \Gamma(\hat{\eta})\}} \sum_{0 \leq i \leq q_\gamma} \bar{\mathbf{u}}_{x_i, x_i + z_i}(\hat{\eta}^{(i, \gamma)}),$$

2166 the random variable representing the integral of $\bar{\mathbf{u}}$ along the path γ . We assign for convenience the value
 2167 0 to the integral if one of the jumps in the path was not licit.

2168 **Definition 8.7 (Closed and exact forms on (\mathcal{G}, d)).** — A differential form $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_{x, x+z})_{x \in \mathbb{Z}^2, |z|=1}$ is
 2169 *closed* if for any finite path γ ,

$$\mathbb{1}_{\{\gamma \in \Gamma_c(\hat{\eta})\}} I_{\gamma, \bar{\mathbf{u}}}(\hat{\eta}) = 0 \quad \mu_{\hat{\alpha}} - a.s.,$$

2170 i.e. if its integral along any closed loop vanishes a.s..

2171 For any cylinder function $f \in \mathcal{C}$, we say that $\bar{\mathbf{u}}^f$ is an *exact differential form associated with f* if

$$\bar{\mathbf{u}}_{x, x+z}^f(\hat{\eta}) = \eta_x(1 - \eta_{x+z})(f(\hat{\eta}^{x, x+z}) - f(\hat{\eta}))$$

2172 point-wise. It is easily checked that for any $f \in \mathcal{C}$, $\bar{\mathbf{u}}^f$ is a *closed form*, since then

$$(8.11) \quad I_{\gamma, \bar{\mathbf{u}}^f}(\hat{\eta}) = \mathbb{1}_{\{\gamma \in \Gamma(\hat{\eta})\}} \left[f(\hat{\eta}^{(q_\gamma + 1, \gamma)}) - f(\hat{\eta}) \right],$$

2173 which vanishes point-wise if the loop is closed.

2174 We now consider the case of translation invariant closed forms.

2175 **Definition 8.8 (Germs of a closed form).** — A pair $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) : \Sigma_\infty \rightarrow \mathbb{R}^2$ in $L^2(\mu_{\hat{\alpha}})$ is a *germ of*
 2176 *a closed form* if $\bar{\mathbf{u}}$ defined by

$$(8.12) \quad \bar{\mathbf{u}}_{x, x+e_i}(\hat{\eta}) = \tau_x \mathbf{u}_i(\hat{\eta}) \quad \text{and} \quad \bar{\mathbf{u}}_{x+e_i, x}(\hat{\eta}) = -\tau_x \mathbf{u}_i(\hat{\eta}^{x, x+e_i}) = -\bar{\mathbf{u}}_{x, x+e_i}(\hat{\eta}^{x, x+e_i})$$

2177 is a closed form. We endow the set of germs of closed forms with its $L^2(\mu_{\hat{\alpha}})$ norm

$$(8.13) \quad \|\mathbf{u}\|_{\hat{\alpha}, 2} = [\mathbb{E}_{\hat{\alpha}}(\mathbf{u}_1^2 + \mathbf{u}_2^2)]^{1/2}.$$

2178 Denote by $\mathcal{T}_{\hat{\alpha}}^\omega$ the closure in $L^2(\mu_{\hat{\alpha}})$ of T^ω (the set of cylinder functions, defined in (6.42), depending on
 2179 the angles through a linear combination of the $\omega(\theta_x)$), and let $\mathfrak{F}^\omega = \overline{\mathfrak{F}_0^\omega}$ denote the closure in $L^2(\mu_{\hat{\alpha}})$ of
 2180 the set \mathfrak{F}_0^ω of germs of closed forms with components in $\mathcal{T}_{\hat{\alpha}}^\omega$, namely

$$(8.14) \quad \mathfrak{F}_0^\omega = \left\{ \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \mid \mathbf{u} \text{ is a } L^2(\mu_{\hat{\alpha}}) \text{ germ of a closed form, } \mathbf{u}_i \in \mathcal{T}_{\hat{\alpha}}^\omega, \quad \forall i \in \{1, 2\} \right\}.$$

2181

2182 **Definition 8.9 (Germs of an exact form).** — A pair $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ will be called *germ of an exact*
 2183 *form associated to a cylinder function $h \in \mathcal{C}$* if we can write

$$(\mathbf{u}_1, \mathbf{u}_2) = \nabla \Sigma_h := (\nabla_{0, e_1} \Sigma_h, \nabla_{0, e_2} \Sigma_h)$$

2184 point-wise, where Σ_h is the formal sum $\Sigma_h = \sum_{x \in \mathbb{Z}^2} \tau_x h$. Note that although the formal sum Σ_h is
 2185 ill-defined a priori, its gradient $\nabla \Sigma_h$ is not, because h is assumed to be a cylinder function, and therefore
 2186 only depends on a finite number of sites.

2187 One easily verifies that any *germ of an exact form* is also the *germ of a closed form*. In particular, for
 2188 any function $h \in T^\omega$, (cf. (6.42)), we have $\nabla \Sigma_h \in \mathfrak{F}^\omega$. We denote by $\mathfrak{E}^\omega = \overline{\mathfrak{E}_0^\omega}$ the closure in $L^2(\mu_{\hat{\alpha}})$ of
 2189 the set \mathfrak{E}_0^ω of germs of exact forms associated to functions in T^ω ,

$$\mathfrak{E}_0^\omega = \{\nabla \Sigma_h, \quad h \in T^\omega\} \subset \mathfrak{F}_0^\omega.$$

2190 **Definition 8.10 (Germs of a closed form associated with the currents)**

2191 Define $\mathbf{j}^1, \mathbf{j}^2, \mathbf{j}^{1,\omega}$, and $\mathbf{j}^{2,\omega}$ as

$$(8.15) \quad \mathbf{j}_i^k(\hat{\eta}) = \mathbb{1}_{\{i=k\}} \eta_0(1 - \eta_{e_i}) \quad \text{and} \quad \mathbf{j}_i^{k,\omega}(\hat{\eta}) = \mathbb{1}_{\{i=k\}} \eta_0^\omega(1 - \eta_{e_i}) \quad \text{for } k, i = 1, 2.$$

2192 These four functions are germs of closed forms, and can be seen as germs of "almost" exact forms
 2193 associated with the formal functions

$$f^k = \sum_{x \in \mathbb{Z}^2} x_k \eta_x \quad \text{and} \quad f^{k,\omega} = \sum_{x \in \mathbb{Z}^2} x_k \eta_x^\omega,$$

2194 which are not well defined, but for which the gradient along any licit jumps is. Of course, since the
 2195 functions $f^k, f^{k,\omega}$ above are merely formal sums, the $\mathbf{j}^k, \mathbf{j}^{k,\omega}$'s are *not* germs of exact forms. In other
 2196 words, the closed form $\bar{\mathbf{j}}^k$ associated with the germ \mathbf{j}^k is equal to ± 1 on any edge representing a particle
 2197 jump in the direction $\pm e_k$, and the closed form $\bar{\mathbf{j}}^{k,\omega}$ associated with $\mathbf{j}^{k,\omega}$ is equal to $\pm \omega(\theta)$ on any edge
 2198 representing a jump in the direction $\pm e_k$ of a particle with angle θ . We denote by \mathfrak{J}^ω linear span of the
 2199 $\mathbf{j}^k, \mathbf{j}^{k,\omega}$

$$\mathfrak{J}^\omega = \left\{ \mathbf{j}^{a,b} := a_1 \mathbf{j}^1 + a_2 \mathbf{j}^2 + b_1 \mathbf{j}^{1,\omega} + b_2 \mathbf{j}^{2,\omega}, \quad a \in \mathbb{R}^2, b \in \mathbb{R}^2 \right\} \subset \mathfrak{F}_0^\omega.$$

2200 We are now ready to state the main result of this section.

2201 **Proposition 8.11 (Structure of \mathfrak{F}^ω).** — *We have the decomposition*

$$\mathfrak{F}^\omega = \mathfrak{J}^\omega \oplus \mathfrak{E}^\omega.$$

2202 **Remark 8.12.** — Note that we can safely assume that the total density α is in $]0, 1[$. If not, the graph
 2203 \mathcal{G} is trivial since its edge set is empty. This assumption will be made throughout the rest of this section.

2204 Before turning to the proof of the last proposition, we investigate the case of a finite domain. We start
 2205 by a technical Lemma. Recall that \mathcal{C}_n is the set of functions depending only on sites in B_n , and C^1 with
 2206 respect to each θ_x for x in B_n , we denote $T_n^\omega = T^\omega \cap \mathcal{C}_n$, the set of functions depending only on sites in
 2207 B_n , and depending on the angles through a linear combination of the $\omega(\theta_x)$.

2208 **Lemma 8.13.** — *For any $n \geq 0$, T_n^ω is closed in $L^2(\mu_{\hat{\alpha}})$.*

2209 **Proof of Lemma 8.13.** — We need to show that if a sequence of functions $(\varphi_k(\eta) + \sum_{x \in B_n} \eta_x^\omega \psi_{k,x}(\eta))_{k \in \mathbb{N}}$
 2210 converges as $k \rightarrow \infty$ in $L^2(\mu_{\hat{\alpha}})$ to f , then there exists angle-blind functions φ^*, ψ_x^* such that $f =$
 2211 $\varphi^*(\eta) + \sum_{x \in B_n} \eta_x^\omega \psi_x^*(\eta)$ a.s.. Here, the $\varphi_k, \psi_{k,x}, \varphi^*$ and ψ_x^* are angle-blind functions depending only on
 2212 sites in B_n . Denote $\sigma_x \hat{\eta}$ the configuration equal to $\hat{\eta}$ everywhere in B_n except in x where it is distributed
 2213 as an independent copy $\hat{\eta}'_x = (\eta'_x, \theta'_x)$ with distribution $\hat{\alpha}$. Then, we abuse our notation, and also denote
 2214 $\mathbb{E}_{\hat{\alpha}}$ the expectation taken w.r.t. both $\hat{\eta}$ and $\hat{\eta}'_x$.

2215 We can now write

$$\mathbb{E}_{\hat{\alpha}} \left[(f(\hat{\eta}) - f(\sigma_x \hat{\eta}))^2 \mathbb{1}_{\{\eta_x = \eta'_x = 1\}} \right] = \lim_{k \rightarrow \infty} \mathbb{E}_{\hat{\alpha}} \left[(\omega(\theta_x) - \omega(\theta'_x))^2 \psi_{k,x}^2(\eta) \mathbb{1}_{\{\eta_x = \eta'_x = 1\}} \right].$$

2216 Now assume that the variance of $\omega(\theta_x)$ w.r.t. $\mu_{\hat{\alpha}}$ does not vanish (else, the result obviously holds, because
 2217 in $L^2(\mu_{\hat{\alpha}})$, T_n^ω is the set of angle blind functions), we can write for some constant $C := C(\omega, \hat{\alpha})$

$$\lim_{k \rightarrow \infty} \mathbb{E}_{\hat{\alpha}} [\psi_{k,x}^2(\eta) \mid \eta_x = 1] \leq C \mathbb{E}_{\hat{\alpha}}(f^2).$$

2218 In particular, since the set of angle blind configurations in B_n is finite, and since we can assume without
 2219 loss of generality that $\psi_{k,x}(\eta)$ vanishes if $\eta_x = 0$, all the $\psi_{k,x}$ must be bounded, uniformly in x, k , and
 2220 η by some constant M , and therefore remain in a compact set. Up to successive extractions, we can in
 2221 particular assume that each sequences $(\psi_{k,x})_k$ converges uniformly in η as $k \rightarrow \infty$ to ψ_x^* . In particular, the

2222 sequence φ_k also converges to a function φ^* , and we can thus write as wanted $f = \varphi^*(\eta) + \sum_{x \in B_n} \eta_x^\omega \psi_x^*(\eta)$
 2223 a.s. \square

2224 We now consider closed differential forms in a finite box. Considering the graph \mathcal{G}_n with vertices the
 2225 configurations $\hat{\eta}$ on the box B_n , and connected, as on the infinite graph, if one can be reached from the
 2226 other with one licit jump along an edge of B_n .

2227 **Proposition 8.14.** — Fix a parameter $\hat{\alpha}$, $n \geq 0$, and a $L^2(\mu_{\hat{\alpha}})$ closed form $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_{x,x+z})_{x,x+z \in B_n}$ on
 2228 \mathcal{G}_n satisfying for any $x, x+z \in B_n$

2229 i) $\bar{\mathbf{u}}_{x,x+z}$ identically vanishes when there are 1 or less empty sites in B_n ,

2230 ii) There exists $\bar{\mathbf{u}}'_{x,x+z}$ in T_n^ω such that $\bar{\mathbf{u}}_{x,x+z} = \bar{\mathbf{u}}'_{x,x+z} \mu_{\hat{\alpha}}$ -a.s..

2231 Then, there exists a cylinder function $h \in T_n^\omega$ such that

$$\bar{\mathbf{u}}_{x,x+z} = \nabla_{x,x+z} h \quad \forall x, x+z \in B_n, \mu_{\hat{\alpha}}\text{-a.s.},$$

2232 i.e. on a finite set, all closed forms are exact forms. Furthermore, one can assume without loss of
 2233 generality that for any $\hat{K} \in \mathbb{K}_n$, $\mathbb{E}_{n,\hat{K}}(h) = 0$.

2234 *Proof of Proposition 8.14.* — Since $\bar{\mathbf{u}}$ is a closed form, since each element of $\bar{\mathbf{u}}'$ is smooth in the angle
 2235 variables, and since $\bar{\mathbf{u}}' = \bar{\mathbf{u}} \mu_{\hat{\alpha}}$ -a.s., we deduce that $\mathbb{1}_{\{\gamma \in \Gamma_c(\hat{\eta})\}} I_{\gamma, \bar{\mathbf{u}}'}$ vanishes not only a.s. but also point-
 2236 wise for any finite path γ . We are going to build a function h in E whose point-wise gradient is $\bar{\mathbf{u}}'$. Recall
 2237 that $\bar{\mathbf{u}}_{x,x+z}$, and therefore $\bar{\mathbf{u}}'_{x,x+z}$ vanishes if there is one or less empty site in B_n , we split the set of
 2238 configurations on B_n into components $(\Sigma_n^{\hat{K}})_{\hat{K} \in \mathbb{K}_n}$ each connected on the graph \mathcal{G}_n . In particular, for any
 2239 two configurations $\hat{\eta}, \hat{\eta}'$ in the same $\Sigma_n^{\hat{K}}$, we must have by construction $d(\hat{\eta}, \hat{\eta}') < \infty$.

2240 For any \hat{K} with at least two empty sites, let us denote $\hat{\eta}^{\hat{K}}$ the configuration where the particles are
 2241 inserted from the bottom left, row by row, and in the order of increasing angles from 0 to 2π . In other
 2242 words, we insert the particle with the angle closest to 0 at site $(-n, -n)$, the second closest at $(-n, -n+1)$,
 2243 and so on until all particles have been placed. The choice of this reference configuration is arbitrary, but
 2244 depends continuously in the angles in $\hat{K} \in \mathbb{K}_n$. We then set $h(\hat{\eta}^{\hat{K}}) = 0$ for each $\hat{K} \in \mathbb{K}_n$, and for any
 2245 other configuration $\hat{\eta} \in \Sigma_n^{\hat{K}}$, we fix a path $\gamma_{\hat{\eta}}$ of licit jumps from $\hat{\eta}^{\hat{K}}$ to $\hat{\eta}$, and let

$$h(\hat{\eta}) = I_{\gamma_{\hat{\eta}}, \bar{\mathbf{u}}'}(\hat{\eta}^{\hat{K}}).$$

2246 Since $\bar{\mathbf{u}}'$ is a *pointwise* closed form, this expression does not depend on the choice of $\gamma_{\hat{\eta}}$ and pointwise,
 2247 we have for any $x, x+z$, $\bar{\mathbf{u}}'_{x,x+z} = \nabla_{x,x+z} h$. In particular, $\bar{\mathbf{u}}_{x,x+z}(\hat{\eta}) = \nabla_{x,x+z} h(\hat{\eta})$ on E , and therefore
 2248 $\mu_{\hat{\alpha}}$ -a.s.. Furthermore, by construction, because both $\bar{\mathbf{u}}'$ and $\hat{\eta}^{\hat{K}}$ depend smoothly on the particle's angles,
 2249 so does h , and therefore $h \in \mathcal{C}_n$. We now show that $h \in T_n^\omega$.

2250 Since, according to Lemma 8.13, T_n^ω is a closed linear subspace of $L^2(\mu_{\hat{\alpha}})$, we can write on B_n $L^2(\mu_{\hat{\alpha}}) =$
 2251 $T_n^\omega \oplus (T_n^\omega)^\perp$. Straightforwardly, one can show that both T_n^ω and $(T_n^\omega)^\perp$ are stable under any symmetric
 2252 gradient $\tilde{\nabla}_{x,x+z} f := \mathbb{1}_{\{\eta_x \eta_{x+z} = 0\}} (f(\hat{\eta}^{x,x+z}) - f(\hat{\eta}))$, for $x, x+z \in B_n$. In particular, since $\bar{\mathbf{u}}'_{x,x+z} \in T_n^\omega$,
 2253 we also have $\tilde{\nabla}_{x,x+z} h = \bar{\mathbf{u}}_{x,x+z}(\hat{\eta}) + \bar{\mathbf{u}}_{x,x+z}(\hat{\eta}^{x,x+z}) \in T_n^\omega$ for any $x, x+z \in B_n$. Let now write h as
 2254 $h_1 + h_2$, where $h_1 \in T_n^\omega$ and $h_2 \in (T_n^\omega)^\perp$, we must have $\tilde{\nabla}_{x,x+z} h = \tilde{\nabla}_{x,x+z} h_1$. All gradients of h_2 therefore
 2255 vanish a.s., we conclude that h_2 is a.s. constant on each connected component, therefore we can choose it
 2256 to be 0 without changing $\tilde{\nabla}_{x,x+z} h$. We thus have as wanted $\bar{\mathbf{u}}'_{x,x+z} = \nabla_{x,x+z} h_1$, we can therefore choose
 2257 $h = h_1 \in T_n^\omega$.

2258 Regarding the second claim of the Proposition, given a configuration $\hat{\eta}$ on B_n , let us denote by
 2259 $\hat{K}_n(\hat{\eta}) := (K(\hat{\eta}), \Theta_{K(\hat{\eta})}(\hat{\eta}))$ the parameter giving the number and angles of particles in $\hat{\eta}$, i.e.

$$K(\hat{\eta}) = \sum_{x \in B_n} \eta_x \quad \text{and} \quad \Theta_{K(\hat{\eta})}(\hat{\eta}) = \{\theta_{x_1}, \dots, \theta_{x_{K(\hat{\eta})}}\},$$

2260 where x_1, \dots, x_K are the positions of the K particles in $\hat{\eta}$. Since the function $K(\hat{\eta})$ is unchanged under
 2261 any gradient inside B_n , we can replace h by $h_0 = h - \mathbb{E}_{n, \hat{K}_n(\hat{\eta})}(h)$ (where $\mathbb{E}_{n, \hat{K}_n(\hat{\eta})}$ is the expectation w.r.t. the
 2262 canonical measure corresponding to having \hat{K} particles in B_n) and still satisfy $\bar{\mathbf{u}}_{x,x+z}(\hat{\eta}) = \nabla_{x,x+z} h_0(\hat{\eta})$.
 2263 \square

2264 We now turn to the proof of the decomposition of germs of closed forms on the infinite graph.

2265 *Proof of Proposition 8.11.* — We first prove that the sum is direct : assume that for $a, b \in \mathbb{R}^2$, there
 2266 exists a cylinder function h such that $\mathbf{j}^{a,b} = a_1 \mathbf{j}^1 + a_2 \mathbf{j}^2 + b_1 \mathbf{j}^{1,\omega} + b_2 \mathbf{j}^{2,\omega} = \nabla \Sigma_h$. In particular fix $i = 1, 2$,
 2267 one easily obtains that

$$a_i j_i + b_i j_i^\omega = \nabla_{0,e_i} \Sigma_h - \nabla_{0,e_i} \Sigma_h(\hat{\eta}^{0,e_i}) = \mathbb{1}_{\{\eta_0 \eta_{e_i} = 0\}} (\Sigma_h(\eta) - \Sigma_h(\eta^{0,e_i})),$$

2268 where the j_i 's are the currents defined in (2.8). Multiplying by $\eta_{e_i} - \eta_0$ (resp. $\eta_{e_i}^\omega - \eta_0^\omega$) and taking the
 2269 expectation w.r.t. $\mu_{\hat{\alpha}}$, the identity above rewrites

$$2(a_i + b_i \mathbb{E}_{\hat{\alpha}}(\omega))\alpha(1 - \alpha) = 0 \quad (\text{resp. } 2(a_i \mathbb{E}_{\hat{\alpha}}(\omega) + b_i \mathbb{E}_{\hat{\alpha}}(\omega^2))\alpha(1 - \alpha) = 0),$$

2270 where, as in Section 8.1, $\mathbb{E}_{\hat{\alpha}}(\omega^k)$ stands for $\mathbb{E}_{\hat{\alpha}}(\omega^k(\theta_0)|\eta_0 = 1)$. In particular, since $\alpha \in (0, 1)$ this yields
 2271 that $a_i + b_i \mathbb{E}_{\hat{\alpha}}(\omega) = 0$ and that $\mathbb{E}_{\hat{\alpha}}(\omega^2) = \mathbb{E}_{\hat{\alpha}}(\omega)^2$, therefore $\omega(\theta_0)$ is constant under $\mu_{\hat{\alpha}}$. In particular,
 2272 $a_i j_i + b_i j_i^\omega$ vanishes in $L^2(\mu_{\hat{\alpha}})$ as wanted. The inclusion $\mathfrak{F}^\omega \supset \mathfrak{J}^\omega + \mathfrak{E}$ is immediate.

We now prove the reverse inclusion. The set of germs of an exact form being a linear (therefore convex) subset of $L^2(\mu_{\hat{\alpha}})$, its weak and strong closure in $L^2(\mu_{\hat{\alpha}})$ coincide. In order to prove Proposition 8.11, it is therefore sufficient to prove that for any $\mathbf{u} \in \mathfrak{F}^\omega$, there exists a sequence of cylinder functions $h_n \in T^\omega$ such that the sequence $(\nabla \Sigma_{h_n})_{n \in \mathbb{N}}$ is weakly relatively compact in $L^2(\mu_{\hat{\alpha}})$, and for any of its weak limit points \mathbf{h} , there exists a and b in \mathbb{R}^2 such that

$$\mathbf{h} = \mathbf{u} + \mathbf{j}^{a,b}.$$

2273 Fix $\mathbf{u} \in \mathfrak{F}^\omega$, and $(\bar{\mathbf{u}}_{x,x+z})_{x,x+z}$ the associated closed form defined by (8.12). For any fixed integer n ,
 2274 let \mathcal{F}_n be the σ -algebra generated by the sites inside B_n

$$\mathcal{F}_n = \sigma(\hat{\eta}_x, x \in B_n),$$

2275 and let $\bar{\mathbf{u}}_{x,x+z}^n$ denote the conditional expectation

$$\bar{\mathbf{u}}_{x,x+z}^n = \mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x,x+z} | \mathcal{F}_n).$$

2276 Note in particular that since \mathbf{u} is in \mathfrak{F}^ω , $\bar{\mathbf{u}}^n$ is a closed form on \mathcal{G}_n , and each of its coordinate is in T_n^ω ,
 2277 according to Lemma 8.13.

Fix once and for all a density $\alpha < \alpha' < 1$, and define $\rho_n = \frac{1}{|B_n|} \sum_{x \in B_n} \eta_x$ the density in B_n , according to Proposition 8.14, there exists a family of \mathcal{F}_n -measurable functions $\varphi_n \in T_n^\omega$ with mean 0 w.r.t. any canonical measure on B_n such that

$$\mathbb{1}_{\{\rho_n \leq \alpha'\}} \bar{\mathbf{u}}_{x,x+z}^n = \nabla_{x,x+z} \varphi_n \quad \forall x, x+z \in B_n \quad \mu_{\hat{\alpha}} - a.s..$$

2278 Note that we would need a weaker indicator function to respect the conditions of Proposition 8.14 (namely,
 2279 that there are two empty sites in B_n) however in order to estimate the $L^2(\mu_{\hat{\alpha}})$ -norm of the φ_n , we will
 2280 need the indicator function above.

2281 Let us fix $n \in \mathbb{N}$, and consider the germ of an exact form $\frac{1}{(2n)^2} \nabla \Sigma_{\varphi_n}$, whose coordinates can be
 2282 rewritten for $i = 1, 2$

$$\frac{1}{(2n)^2} \nabla_{0,e_i} \Sigma_{\varphi_n} = \frac{1}{(2n)^2} \sum_{x \in \mathbb{Z}^2} \tau_{-x} \nabla_{x,x+e_i} \varphi_n.$$

2283 Since φ_n is \mathcal{F}_n -measurable, $\nabla_{x,x+e_i} \varphi_n$ vanishes as soon as neither x nor $x + e_i$ is in B_n . Hence, the
 2284 previous quantity is equal to

$$(8.16) \quad \frac{1}{(2n)^2} \nabla_{0,e_i} \Sigma_{\varphi_n} = \frac{1}{(2n)^2} \sum_{\substack{-n-1 \leq x_i \leq n \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \varphi_n = R_{n,i} + \frac{1}{(2n)^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \varphi_n,$$

2285 where the boundary term $R_{n,i}$ is

$$R_{n,i} = \frac{1}{(2n)^2} \left[\sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \tau_{-x} \nabla_{x,x+e_i} \varphi_n + \sum_{\substack{x_i = n \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \varphi_n \right].$$

2286 For any n , the left-hand side in (8.16) the germ of an exact form as introduced in Definition 8.9. We
 2287 will see that the second term of the right-hand side converges in $L^2(\mu_{\hat{\alpha}})$ as n goes to infinity towards
 2288 \mathbf{u}_i . Hence to prove Proposition 8.11 it will be sufficient to show that the boundary term $R_{n,i}$ is weakly
 2289 relatively compact in $L^2(\mu_{\hat{\alpha}})$, and that any of its weak limit points is in \mathfrak{F}^ω . Since φ_n is supported in
 2290 B_n , the exchanges at the boundary act as reservoirs with creation (first term in $R_{n,i}$) at the sites $x + e_i$
 2291 with $x_i = -n - 1$, and annihilation of particles (second term in $R_{n,i}$) at the sites x such that $x_i = n$,
 2292 and cannot be expressed as such as particle transfers. To prove that the sequence of boundary terms is
 2293 weakly relatively compact, we therefore need to smooth out the φ_n 's, by letting

$$(8.17) \quad \tilde{\varphi}_n = \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} \mid \mathcal{F}_n).$$

2294 Not in particular that we still have $\tilde{\varphi}_n \in T_n^\omega$.

2295 Rewrite (8.16) with $\tilde{\varphi}_n$ instead of φ_n

$$(8.18) \quad \frac{1}{(2n)^2} \nabla_{0,e_i} \Sigma \tilde{\varphi}_n = \frac{1}{(2n)^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n + \tilde{R}_{n,i},$$

2296 where this time

$$(8.19) \quad \tilde{R}_{n,i} = \frac{1}{(2n)^2} \left[\sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n + \sum_{\substack{x_i = n \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n \right].$$

2297 We are going to show that

- 2298 — the bulk term converges in $L^2(\mu_{\hat{\alpha}})$ to \mathbf{u}_i ,
- 2299 — the sequence of boundary term is bounded in $L^2(\mu_{\hat{\alpha}})$, and any of its weak limit points is an
- 2300 element of \mathfrak{F}^ω .

2301 For the sake of clarity, we state both of these results as separate lemmas, and we will prove them
 2302 afterwards.

2303 **Lemma 8.15 (Convergence of the bulk term towards \mathbf{u}_i).** — For any $i \in \{1, 2\}$,

$$(8.20) \quad \limsup_{n \rightarrow \infty} \mathbb{E}_{\hat{\alpha}} \left[\left(\frac{1}{(2n)^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n}} \left[\tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n - \mathbf{u}_i \right] \right)^2 \right] = 0.$$

2304 **Lemma 8.16 (Limit of the boundary term).** — For any $i \in \{1, 2\}$, we split the boundary term as

$$\tilde{R}_{n,i} = \tilde{R}_{n,i}^- + \tilde{R}_{n,i}^+,$$

2305 where

$$(8.21) \quad \tilde{R}_{n,i}^- = \frac{1}{(2n)^2} \sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n, \quad \text{and} \quad \tilde{R}_{n,i}^+ = \frac{1}{(2n)^2} \sum_{\substack{x_i = n \\ x \in B_n}} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n,$$

2306 which will be referred to respectively as negative and positive boundary terms. With the previous notations,
 2307 both sequences $(\tilde{R}_{n,i}^-)_{n \in \mathbb{N}}$ and $(\tilde{R}_{n,i}^+)_{n \in \mathbb{N}}$ are bounded in $L^2(\mu_{\hat{\alpha}})$. Furthermore, for any weakly convergent
 2308 subsequence $\tilde{R}_{k_n,i}^- \rightarrow \mathfrak{R}_i^-$, there exists $a_i, b_i \in \mathbb{R}$ such that

$$\mathfrak{R}_i^- = a_i \eta_0^\omega (1 - \eta_{e_i}) + b_i \eta_0 (1 - \eta_{e_i}).$$

2309 The same is true for the positive boundary term.

2310 Thanks to (8.18), these two lemmas prove Proposition 8.11. □

2311 The proof of Lemma 8.15 is simple, we treat it right now before turning to the proof of Lemma 8.16,
 2312 which is the main difficulty of this section.

Proof of Lemma 8.15. — By construction, for any $x, x + e_i \in B_n$,

$$\begin{aligned}
 \nabla_{x, x+e_i} \tilde{\varphi}_n &= \nabla_{x, x+e_i} \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} \mid \mathcal{F}_n) \\
 &= \mathbb{E}_{\hat{\alpha}}(\nabla_{x, x+e_i} \varphi_{3n} \mid \mathcal{F}_n) \\
 &= \mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\{\rho_{3n} \leq \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^{3n} \mid \mathcal{F}_n) \\
 &= \mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\{\rho_{3n} \leq \alpha'\}} \mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_{3n}) \mid \mathcal{F}_n) \\
 (8.22) \qquad &= \mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\{\rho_{3n} \leq \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_n).
 \end{aligned}$$

2313 By triangular inequality, translation invariance of $\mu_{\hat{\alpha}}$, and using $(\sum_{i=1}^k a_i)^2 \leq k \sum_{i=1}^k a_i^2$, we can bound
 2314 the expectation in (8.20) by

$$(8.23) \quad \frac{1}{2n^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n}} \left(\mathbb{E}_{\hat{\alpha}} \left[\left(\mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_n) - \bar{\mathbf{u}}_{x, x+e_i} \right)^2 \right] + \mathbb{E}_{\hat{\alpha}} \left[\mathbb{1}_{\{\rho_{3n} > \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^2 \right] \right).$$

We start by estimating the contribution of the first expectation in the sum. To do so, split it for any positive ε as

$$\frac{1}{2n^2} \sum_{x \in B_{n(1-\varepsilon)}} \mathbb{E}_{\hat{\alpha}} \left[\left(\mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_n) - \bar{\mathbf{u}}_{x, x+e_i} \right)^2 \right] + \frac{1}{2n^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n \setminus B_{n(1-\varepsilon)}}} \mathbb{E}_{\hat{\alpha}} \left[\left(\mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_n) - \bar{\mathbf{u}}_{x, x+e_i} \right)^2 \right]$$

2315 By definition of $\bar{\mathbf{u}}$, $\tau_x \mathbf{u}_i = \bar{\mathbf{u}}_{x, x+e_i}$, thus for any $\varepsilon > 0$, the expectations in the first term vanish uniformly
 2316 in $x \in B_{n(1-\varepsilon)}$ as $n \rightarrow \infty$ by martingale convergence theorem, whereas the second sum can be crudely
 2317 estimated by Jensen inequality and is less than

$$C\varepsilon \max_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n \setminus B_{n(1-\varepsilon)}}} \mathbb{E}_{\hat{\alpha}} \left[\left(\mathbb{E}_{\hat{\alpha}}(\bar{\mathbf{u}}_{x, x+e_i} \mid \mathcal{F}_n) - \bar{\mathbf{u}}_{x, x+e_i} \right)^2 \right] \leq 4C\varepsilon \mathbb{E}_{\hat{\alpha}}(\mathbf{u}_i^2)$$

2318 which vanishes as $\varepsilon \rightarrow 0$ regardless of n .

We now consider the contributions of the second part in (8.23). That each term vanishes is a direct consequence of the dominated convergence theorem, however since we need a convergence that is uniform in x , we give a more detailed and quantitative argument. We can rewrite by translation invariance of $\mu_{\hat{\alpha}}$, for any $x, x + e_i \in B_n$, and for any $p < 2$

$$\begin{aligned}
 \mathbb{E}_{\hat{\alpha}} \left[\mathbb{1}_{\{\rho_{3n} > \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^2 \right] &= \mathbb{E}_{\hat{\alpha}} \left[\mathbf{u}_i^2 (\tau_{-x} \mathbb{1}_{\{\rho_{3n} > \alpha'\}}) \right] \\
 &\leq \mathbb{E}_{\hat{\alpha}} \left[\left| \mathbf{u}_i^2 - |\mathbf{u}_i|^p \right| \right] + \mathbb{E}_{\hat{\alpha}} \left[|\mathbf{u}_i|^p (\tau_{-x} \mathbb{1}_{\{\rho_{3n} > \alpha'\}}) \right] \\
 &\leq \mathbb{E}_{\hat{\alpha}} \left[\left| \mathbf{u}_i^2 - |\mathbf{u}_i|^p \right| \right] + \mathbb{E}_{\hat{\alpha}} \left(\mathbf{u}_i^2 \right)^{p/2} \mu_{\hat{\alpha}}(\rho_{3n} > \alpha')^{1-p/2}
 \end{aligned}$$

2319 by Holder inequality. By a standard large deviation estimate, $\mu_{\hat{\alpha}}(\rho_{3n} > \alpha') = O(e^{-Cn^2})$. We then choose
 2320 $p = p(n) = 2 - 1/n$, to obtain that second term in the right-hand side above is less than $C(\mathbf{u}_i) e^{-Cn}$. The
 2321 function inside the expectation in the first term is point-wise less than $\max(2\mathbf{u}_i^2, 1)$ which is integrable
 2322 and the first term therefore vanishes by dominated convergence as $p(n) \rightarrow 2$. Since the bound above does
 2323 not depend on x , we finally obtain

$$(8.24) \quad \lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{\substack{-n \leq x_i \leq n-1 \\ x \in B_n}} \mathbb{E}_{\hat{\alpha}} \left[\mathbb{1}_{\{\rho_{3n} > \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^2 \right] = 0$$

2324 as wanted, which proves Lemma 8.15. □

2325 *Proof of Lemma 8.16.* — The proof of this Lemma being long, we split it into three steps.

2326 — We first control the $L^2(\mu_{\hat{\alpha}})$ norm of the $\tilde{\varphi}_n$'s.

2327 — Thanks to this control, we prove that the sequence of boundary terms $\tilde{R}_{n,i}^{\pm}$ is bounded in $L^2(\mu_{\hat{\alpha}})$.

2328 — Finally, we prove that any weak limit point \mathfrak{R}_i^{\pm} of the boundary term can only depend on the
 2329 configuration through $\hat{\eta}_0$ and $\hat{\eta}_{e_i}$, and that they can be written as a combination of the \mathbf{j}^i and $\mathbf{j}^{i,\omega}$.

2330 The scheme follows closely that of Theorem 4.14 in Appendix 3 of [27] however adjustments are needed
 2331 in the second and third step to take into account the presence of the angles.

2332 First step : Control on the L^2 norm of the φ_n 's.

2333

2334 We proved in Section 8.1 that, even though we do not have a general spectral gap of order n^{-2} , we
 2335 could circumvent this difficulty by staying in a convenient class of functions linear in the angles and
 2336 by cutting off the large densities. This spectral gap estimate is needed to control the norm of the φ_n 's.
 2337 This is the reason for limiting the result to closed forms in \mathfrak{T}^ω defined in (8.14), and for introducing the
 2338 indicator functions $\mathbb{1}_{\{\rho_n \leq \alpha'\}}$. We state this step as a separate lemma for the sake of clarity.

2339 **Lemma 8.17.** — *There exists a constant $K := K(\hat{\alpha}, \alpha', \mathbf{u})$ such that for any $n \in \mathbb{N}$,*

$$\mathbb{E}_{\hat{\alpha}}(\varphi_n^2) \leq Kn^4,$$

2340 where φ_n was introduced in (8.2).

2341 *Proof of Lemma 8.17.* — For any $\hat{K} \in \mathbb{K}_n$, we proved in Proposition 8.14 that we could assume
 2342 $\mathbb{E}_{n, \hat{K}}(\varphi_n) = 0$, and thanks to the indicator function $\mathbb{1}_{\{\rho_n \leq \alpha'\}}$, φ_n vanishes when the density in B_n is
 2343 larger than α' , therefore the spectral gap estimate given in Proposition 8.2, since $\varphi_n \in T_n^\omega$, yields

$$\mathbb{E}_{\hat{\alpha}}(\varphi_n^2) = \mathbb{E}_{\hat{\alpha}}(\varphi_n^2 \mathbb{1}_{\{\rho_n \leq \alpha'\}}) \leq C(\hat{\alpha}, \alpha')n^2 \mathcal{D}_n(\varphi_n),$$

2344 where $\mathcal{D}_n(f) = -\mathbb{E}_{\hat{\alpha}}(f \mathcal{L}_n f)$ is the Dirichlet form relative to the symmetric exclusion process restricted
 2345 to B_n ,

$$\mathcal{D}_n(\varphi_n) = \frac{1}{2} \sum_{i=1}^2 \sum_{\delta \in \{-1, 1\}} \sum_{x, x+\delta e_i \in B_n} \mathbb{E}_{\hat{\alpha}} [(\nabla_{x, x+\delta e_i} \varphi_n)^2].$$

2346 By construction (cf. (8.2)), $\nabla_{x, x+e_i} \varphi_n = \mathbb{1}_{\{\rho_n \leq \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^n$ and $\nabla_{x+e_i, x} \varphi_n = -\mathbb{1}_{\{\rho_n \leq \alpha'\}} \bar{\mathbf{u}}_{x, x+e_i}^n (\hat{\eta}^{x, x+e_i})$.
 2347 Thus, since \mathbf{u} is in $L^2(\mu_{\hat{\alpha}})$, and since $\mu_{\hat{\alpha}}$ is invariant under the change of variables $\hat{\eta} \mapsto \hat{\eta}^{x, x+e_i}$, Jensen's
 2348 inequality yields

$$(8.25) \quad \mathcal{D}_n(\varphi_n) \leq \sum_{i=1}^2 \sum_{x, x+e_i \in B_n} \mathbb{E}_{\hat{\alpha}} [(\bar{\mathbf{u}}_{x, x+e_i}^n)^2] \leq \sum_{i=1}^2 \sum_{x, x+e_i \in B_n} \mathbb{E}_{\hat{\alpha}} [(\mathbf{u}_i)^2] \leq C'(\mathbf{u})n^2.$$

2349 We obtain as wanted, thanks to the spectral gap estimate above,

$$(8.26) \quad \mathbb{E}_{\hat{\alpha}}(\varphi_n^2) \leq Kn^4,$$

2350 where $K = CC'$ depends only on $\hat{\alpha}$, α' , and \mathbf{u} . □

2351 Second step : Control on the L^2 norm of the boundary terms.

2352

2353 We now prove thanks to Lemma 8.17 that the boundary terms are bounded in $L^2(\mu_{\hat{\alpha}})$.

2354 **Lemma 8.18.** — *There exists a constant $C = C(\hat{\alpha}, \alpha', \mathbf{u})$ such that for any n ,*

$$(8.27) \quad \mathbb{E}_{\hat{\alpha}} \left[(\tilde{R}_{n,i}^-)^2 \right] \leq C,$$

2355 *The statement remains true if $\tilde{R}_{n,i}^-$ is replaced by $\tilde{R}_{n,i}^+$.*

2356 *Proof of Lemma 8.18.* — We will treat in full detail only the case of the negative boundary term

$$\tilde{R}_{n,i}^- = \frac{1}{(2n)^2} \sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \tau_{-x} \nabla_{x, x+e_i} \tilde{\varphi}_n,$$

2357 analogous arguments yield the bound for $\tilde{R}_{n,i}^+$. Using $(\sum_{i=1}^k a_i)^2 \leq k \sum_{i=1}^k a_i^2$, we obtain

$$\mathbb{E}_{\hat{\alpha}} \left[(\tilde{R}_{n,i}^-)^2 \right] \leq \frac{(2n+1)}{(2n)^4} \sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \mathbb{E}_{\hat{\alpha}} [(\tau_{-x} \nabla_{x, x+e_i} \tilde{\varphi}_n)^2] \leq Cn^{-3} \sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \mathbb{E}_{\hat{\alpha}} [(\nabla_{x, x+e_i} \tilde{\varphi}_n)^2],$$

2358 for some universal constant C , by translation invariance of $\mu_{\hat{\alpha}}$. For x in the negative boundary, under
 2359 $\mu_{\hat{\alpha}}$, we can rewrite

$$(8.28) \quad \nabla_{x, x+e_i} \tilde{\varphi}_n(\hat{\eta}) = \eta_x(1 - \eta_{x+e_i}) (\tilde{\varphi}_n(\hat{\eta} + \delta_{x+e_i}^\theta) - \tilde{\varphi}_n(\hat{\eta})),$$

2360 where $\hat{\eta} + \delta_{x+e_i}^\theta$ is the configuration equal to $\hat{\eta}$ everywhere except in $x + e_i$, where the site contains a
 2361 particle with angle θ distributed as $\hat{\alpha}/\alpha$ independently of $\hat{\eta}$. Note that in the expectation $\mathbb{E}_{\hat{\alpha}}$, we will also
 2362 take the expectation w.r.t. θ , but still denote it $\mathbb{E}_{\hat{\alpha}}$ not to burden the notations. Since φ_n is independent
 2363 of $\hat{\eta}_x$ for any x in the negative boundary term,

$$(8.29) \quad \mathbb{E}_{\hat{\alpha}} \left[(\tilde{R}_{n,i}^-)^2 \right] \leq \alpha C n^{-3} \sum_{\substack{x_i = -n-1 \\ x \in B_n(-e_i)}} \mathbb{E}_{\hat{\alpha}} \left[(1 - \eta_{x+e_i}) (\tilde{\varphi}_n(\hat{\eta} + \delta_{x+e_i}^\theta) - \tilde{\varphi}_n(\hat{\eta}))^2 \right],$$

2364 where the expectation w.r.t. θ is also taken, under the distribution $\hat{\alpha}/\alpha$. Recall that $\tilde{\varphi}_n = \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n)$,
 2365 since the number of terms in the sum is $O(n)$, Lemma 8.18 is a consequence of Lemma 8.19 below. \square

2366 **Lemma 8.19.** — *There exists a constant $C = C(\hat{\alpha}, \alpha', \mathbf{u})$ such that for any $x \in B_n(-e_i)$ such that*
 2367 $x_i = -n - 1$,

$$\mathbb{E}_{\hat{\alpha}} \left[(1 - \eta_{x+e_i}) (\mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n)(\hat{\eta} + \delta_{x+e_i}^\theta) - \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n)(\hat{\eta}))^2 \right] \leq C n^2,$$

2368 where the expectation above is taken w.r.t. $\mu_{\hat{\alpha}}$ on B_{3n} and w.r.t. θ distributed under $\hat{\alpha}/\alpha$.

2369 *Proof of Lemma 8.19.* — Let us fix x , such that $x_i = -n - 1$ in the negative boundary. To make the
 2370 Dirichlet form appear, we are going to force an occupied site in a neighborhood of x , and transform
 2371 the particle creation into a particle transfer. This is the reason for smoothing out φ_n and taking $\tilde{\varphi}_n$
 2372 instead. For the sake of clarity, any configuration $\hat{\eta}$ on B_{3n} will be considered as the pair of an interior
 2373 configuration $\hat{\zeta}$ on B_n (which is hence \mathcal{F}_n -measurable), and an exterior configuration $\hat{\xi}$ on $B_{3n} \setminus B_n$.

2374 For any $y \in B_{3n} \setminus B_n$, we rewrite using the identity $(1 - \alpha)^{-1}[1 - \xi + \xi - \alpha] = 1$

$$\mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\zeta} + \delta_{x+e_i}^\theta) = \frac{1}{1 - \alpha} \left(\mathbb{E}_{\hat{\alpha}}((1 - \xi_y)\varphi_{3n} | \mathcal{F}_n) + \mathbb{E}_{\hat{\alpha}}((\xi_y - \alpha)\varphi_{3n} | \mathcal{F}_n) \right) (\hat{\zeta} + \delta_{x+e_i}^\theta),$$

2375 where ξ_y is the occupation variable in y , and is either 1 or 0 depending on whether the site y is empty or
 2376 not.

2377 The first part of this decomposition will be controlled by the Dirichlet form, as the existence of an
 2378 empty site in y (thanks to $1 - \xi_y$) will allow us to reconstruct a particle transfer from y to $x + e_i$. The
 2379 second term will be estimated after a spatial averaging over a large microscopic box. This box must be
 2380 measurable with respect to the sites in $B_{3n} \setminus B_n$, in order to be able to introduce it inside the expectation.
 2381 For any x in the negative boundary, consider the set

$$B_{n-1,i}^x = x - ne_i + B_{n-1},$$

which is the box of radius $n - 1$ centered in $x - ne_i$. Remark that the cardinal of $B_{n-1,i}^x$ is $(2n - 1)^2$, so
 that averaging the previous identity over the y 's in $B_{n-1,i}^x$ yields

$$(8.30) \quad \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\zeta} + \delta_{x+e_i}^\theta) \\ = \frac{1}{(2n - 1)^2} \sum_{y \in B_{n-1,i}^x} \left(\mathbb{E}_{\hat{\alpha}} \left(\frac{1 - \xi_y}{1 - \alpha} \varphi_{3n} \mid \mathcal{F}_n \right) + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y - \alpha}{1 - \alpha} \varphi_{3n} \mid \mathcal{F}_n \right) \right) (\hat{\zeta} + \delta_{x+e_i}^\theta).$$

Let us consider the first term of the previous equality. For any y in the boundary, thanks to the factor
 $1 - \xi_y$ the site y is empty. Performing the change of variable $\hat{\xi} \rightarrow \hat{\xi} - \delta_y$ where $\hat{\xi} - \delta_y$ is the configuration
 identical to $\hat{\xi}$ everywhere except in y where the site is now empty, we obtain

$$\mathbb{E}_{\hat{\alpha}} \left(\frac{1 - \xi_y}{1 - \alpha} \varphi_{3n} \mid \mathcal{F}_n \right) (\hat{\zeta} + \delta_{x+e_i}^\theta) \\ = \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} \varphi_{3n} (\hat{\xi} - \delta_y) \mid \mathcal{F}_n \right) (\hat{\zeta} + \delta_{x+e_i}^\theta) \\ = \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} \left[\varphi_{3n} (\hat{\zeta} + \delta_{x+e_i}^\theta, \hat{\xi} - \delta_y) - \varphi_{3n} (\hat{\zeta}, \hat{\xi}) \right] \mid \mathcal{F}_n \right) + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} \varphi_{3n} (\hat{\zeta}, \hat{\xi}) \mid \mathcal{F}_n \right).$$

We deduce from the last identity and equation (8.30) that we can write $\mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\zeta} + \delta_{x+e_i}^\theta)$ as

$$\begin{aligned} \frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \left[\mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} [\varphi_{3n} (\hat{\zeta} + \delta_{x+e_i}^\theta, \hat{\xi} - \delta_y) - \varphi_{3n} (\hat{\zeta}, \hat{\xi})] \middle| \mathcal{F}_n \right) \right. \\ \left. + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} \varphi_{3n} (\hat{\zeta}, \hat{\xi}) \middle| \mathcal{F}_n \right) + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y - \alpha}{1 - \alpha} \varphi_{3n} (\hat{\zeta} + \delta_{x+e_i}^\theta, \hat{\xi}) \middle| \mathcal{F}_n \right) \right], \end{aligned}$$

and therefore

$$\begin{aligned} (8.31) \quad \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\zeta} + \delta_{x+e_i}^\theta) - \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n)(\hat{\zeta}) \\ = \frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \left[\mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y}{\alpha} [\varphi_{3n} (\hat{\zeta} + \delta_{x+e_i}^\theta, \hat{\xi} - \delta_y) - \varphi_{3n} (\hat{\zeta}, \hat{\xi})] \middle| \mathcal{F}_n \right) \right. \\ \left. + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y - \alpha}{\alpha} \varphi_{3n} (\hat{\zeta}, \hat{\xi}) \middle| \mathcal{F}_n \right) + \mathbb{E}_{\hat{\alpha}} \left(\frac{\xi_y - \alpha}{1 - \alpha} \varphi_{3n} (\hat{\zeta} + \delta_{x+e_i}^\theta, \hat{\xi}) \middle| \mathcal{F}_n \right) \right]. \end{aligned}$$

Using $(\sum_{i=1}^k a_i)^2 \leq k \sum_{i=1}^k a_i^2$ as well as Jensen's inequality yields

$$\begin{aligned} (8.32) \quad \mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{x+e_i}) (\mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\eta} + \delta_{x+e_i}^\theta) - \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} | \mathcal{F}_n) (\hat{\eta}))^2 \right) \\ \leq \frac{3}{(2n-1)^2} \left[\sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} \left(\frac{\eta_y (1 - \eta_{x+e_i})}{\alpha^2} [\varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta - \delta_y) - \varphi_{3n} (\hat{\eta})]^2 \right) \right] \\ + 3 \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(\left(\frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \frac{\eta_y - \alpha}{\alpha} \right) \varphi_{3n} \middle| \mathcal{F}_n \right)^2 \right) \\ + 3 \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(\left(\frac{(1 - \eta_{x+e_i})}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \frac{\eta_y - \alpha}{1 - \alpha} \right) \varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta) \middle| \mathcal{F}_n \right)^2 \right). \end{aligned}$$

2382 From now on, the strategy to prove Lemma 8.19 is straightforward. We are going to prove that each of
2383 the three terms in the right-hand side above is of order n^2 :

2384 — The second and third line above are controlled thanks to the spatial averaging by the L^2 norm
2385 of the φ_n 's.

2386 — In the first line, the angle of the particle deleted in y is not necessarily the same as the one
2387 of the particle created in $x + e_i$, because the angle θ above is distributed according to $\hat{\alpha}/\alpha$ and
2388 independent of the configuration. However, since the φ_n are in T_n^ω their dependency in the angles
2389 can be sharply estimated. Once this difficulty is dealt with, the remaining quantity will be controlled
2390 by the Dirichlet form.

We first treat the first step above. Thanks to the Cauchy-Schwarz inequality, we can estimate the second line

$$\begin{aligned} \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(\left(\frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \frac{\eta_y - \alpha}{\alpha} \right) \varphi_{3n} \middle| \mathcal{F}_n \right)^2 \right) \\ \leq \frac{1}{\alpha^2} \mathbb{E}_{\hat{\alpha}} \left(\left(\frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \eta_y - \alpha \right)^2 \right) \mathbb{E}_{\hat{\alpha}} (\varphi_{3n}^2) = \frac{(1 - \alpha)}{\alpha(2n-1)^2} \mathbb{E}_{\hat{\alpha}} (\varphi_{3n}^2), \end{aligned}$$

2391 since under $\mu_{\hat{\alpha}}$, the η_y 's are i.i.d. variables. We can now use the bound obtained in Lemma 8.17, which
 2392 yields that for some constant $C_1 = C_1(\hat{\alpha}, \alpha', \mathbf{u})$,

$$(8.33) \quad \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(\left(\frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \eta_y - \alpha \right) \varphi_{3n} \middle| \mathcal{F}_n \right)^2 \right) \leq C_1 n^2.$$

2393 Similarly, since

$$\mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{x+e_i}) \varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta)^2 \right) = \frac{1-\alpha}{\alpha} \mathbb{E}_{\hat{\alpha}} (\eta_{x+e_i} \varphi_{3n}^2) \leq C n^2,$$

2394 we also have for some constant $C_2 = C_2(\hat{\alpha}, \alpha', \mathbf{u})$

$$(8.34) \quad \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(\left(\frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \frac{\eta_y - \alpha}{1-\alpha} \right) (1 - \eta_{x+e_i}) \varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta) \middle| \mathcal{F}_n \right)^2 \right) \leq C_2 n^2.$$

2395 We now estimate the first line of the right-hand side of (8.32), namely

$$(8.35) \quad \frac{1}{(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} \left(\frac{\eta_y (1 - \eta_{x+e_i})}{\alpha^2} [\varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta - \delta_y) - \varphi_{3n} (\hat{\eta})]^2 \right).$$

2396 We first deal with the fact that the deleted and created particles do not have the same angle. Recall that
 2397 $\hat{\eta}^{y,\theta}$ is the configuration where the angle of the particle at the site y has been set to θ , we can thus write

$$\hat{\eta} + \delta_{x+e_i}^\theta - \delta_y = (\hat{\eta}^{y,\theta})^{y,x+e_i},$$

therefore

$$(\varphi_{3n} (\hat{\eta} + \delta_{x+e_i}^\theta - \delta_y) - \varphi_{3n} (\hat{\eta}))^2 \leq 2 \left[\varphi_{3n} \left((\hat{\eta}^{y,\theta})^{y,x+e_i} \right) - \varphi_{3n} (\hat{\eta}^{y,\theta}) \right]^2 + 2 \left[\varphi_{3n} (\hat{\eta}^{y,\theta}) - \varphi_{3n} (\hat{\eta}) \right]^2.$$

2398 Since θ is distributed according to $\hat{\alpha}/\alpha$, conditionally to $\eta_y = 1$, $\hat{\eta}^{y,\theta}$ has the same distribution as $\hat{\eta}$ under
 2399 $\mu_{\hat{\alpha}}$, and we can therefore control (8.35) by

$$(8.36) \quad \frac{2}{\alpha^2 (2n-1)^2} \sum_{y \in B_{n-1,i}^x} \left[\mathbb{E}_{\hat{\alpha}} \left(\eta_y (1 - \eta_{x+e_i}) [\varphi_{3n} (\hat{\eta}^{y,x+e_i}) - \varphi_{3n} (\hat{\eta})]^2 \right) + \mathbb{E}_{\hat{\alpha}} \left(\eta_y [\varphi_{3n} (\hat{\eta}^{y,\theta}) - \varphi_{3n} (\hat{\eta})]^2 \right) \right].$$

2400 Once again, we are going to prove that the contributions of both terms in the right-hand side above are
 2401 of order n^2 .

We first need to decompose, as in the proof of the two-block estimate of Lemma 4.4, the particle jumps appearing in the first term into nearest neighbor jumps. More precisely, there exists a finite family x_0, \dots, x_p such that $x_0 = y$, $x_p = x$ and for any $k \in \llbracket 0, p-1 \rrbracket$, $|x_k - x_{k+1}| = 1$. Furthermore, we can safely assume that $p = |y - x|$. With this construction, for any $y \in B_{n-1,i}^x$, we can write

$$(8.37) \quad \begin{aligned} & \mathbb{E}_{\hat{\alpha}} \left[\eta_y (1 - \eta_{x+e_i}) \left(\varphi_{3n} (\hat{\eta}^{y,x+e_i}) - \varphi_{3n} (\hat{\eta}) \right)^2 \right] \\ & \leq |y - x| \sum_{k=1}^{|y-x|} \mathbb{E}_{\hat{\alpha}} \left[\eta_{x_k} (1 - \eta_{x_{k+1}}) \left(\varphi_{3n} (\hat{\eta}^{x_k, x_{k+1}}) - \varphi_{3n} (\hat{\eta}) \right)^2 \right] \\ & \leq |y - x| \sum_{k=1}^{|y-x|} \mathbb{E}_{\hat{\alpha}} \left([\nabla_{x_k, x_{k+1}} \varphi_{3n}]^2 \right), \end{aligned}$$

2402 since $(\sum_{k=1}^p a_k)^2 \leq p \sum_{k=1}^p a_k^2$. As in the proof of Lemma 8.17, one easily checks that, x_k and x_{k+1} being
 2403 neighbors,

$$\mathbb{E}_{\hat{\alpha}} \left([\nabla_{x_k, x_{k+1}} \varphi_{3n} (\hat{\eta})]^2 \right) \leq C(\mathbf{u}).$$

2404 therefore (8.37) yields

$$\mathbb{E}_{\hat{\alpha}} \left[\eta_y (1 - \eta_{x+e_i}) \left(\varphi_{3n} (\hat{\eta}^{y,x+e_i}) - \varphi_{3n} (\hat{\eta}) \right)^2 \right] \leq |y - x|^2 C(\mathbf{u}).$$

2405 We now get back to the first term in (8.36). It is not hard to see that $\sum_{y \in B_{n-1,i}^x} |y-x|^2$ is of order n^4 ,
 2406 and we obtain as wanted that for some constant $C_3 = C_3(\hat{\alpha}, \mathbf{u})$,

$$(8.38) \quad \frac{2}{\alpha^2(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} \left(\eta_y(1 - \eta_{x+e_i}) [\varphi_{3n}(\hat{\eta}^{y,x+e_i}) - \varphi_{3n}(\hat{\eta})]^2 \right) \leq C_3 n^2.$$

2407 We now estimate the second contribution in (8.36). The only difference between $\varphi_{3n}(\hat{\eta}^{y,\theta})$ and $\varphi_{3n}(\hat{\eta})$
 2408 is the angle of the particle at site y . Recall that for any n , $\varphi_n \in T^\omega$, therefore the variation of φ_n when
 2409 an angle is changed can be precisely estimated. Fix $n \geq 0$, and recall that $\varphi_{3n} \in T_{3n}^\omega$. Then, there exists
 2410 angle-blind functions $(\psi_{n,x})_{x \in B_{3n}}$, and ψ_n in \mathcal{S} , such that

$$\varphi_{3n} = \psi_n + \sum_{x \in B_{3n}} \eta_x^\omega \psi_{n,x}.$$

2411 Since the only difference between $\hat{\eta}^{y,\theta}$ and $\hat{\eta}$ is in the angle present at the site y , we can write

$$\varphi_{3n}(\hat{\eta}^{y,\theta}) - \varphi_{3n}(\hat{\eta}) = (\omega(\theta) - \omega(\theta_y)) \eta_y \psi_{n,y}(\eta),$$

2412 therefore the second contribution in (8.36) can be rewritten

$$(8.39) \quad \frac{2}{\alpha^2(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} (\eta_y(\omega(\theta) - \omega(\theta_y))^2 \psi_{n,y}^2) = \frac{4V_{\hat{\alpha}}(\omega)}{\alpha^2(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} (\eta_y \psi_{n,y}^2),$$

2413 where we shortened $V_{\hat{\alpha}}(\omega) = \text{Var}_{\hat{\alpha}}(\omega(\theta_0) \mid \eta_0 = 1)$, since the angles are independent of the configuration
 2414 conditionally to the presence of a particle. Similarly to what we did in Section 8.1 rewrite

$$\varphi_{3n} = \varphi_n^1 + \varphi_n^b,$$

2415 where

$$\varphi_n^1 = \sum_{x \in B_{3n}} (\omega(\theta_x) - \mathbb{E}_{\hat{\alpha}}(\omega)) \eta_x \psi_{n,x} \quad \text{and} \quad \varphi_n^b = \psi_n + \mathbb{E}_{\hat{\alpha}}(\omega) \sum_{x \in B_{3n}} \eta_x \psi_{n,x},$$

2416 where $\mathbb{E}_{\hat{\alpha}}(\omega)$ stands for $\mathbb{E}_{\hat{\alpha}}(\omega(\theta_0) \mid \eta_0 = 1)$. As in Section 8.1,

$$\mathbb{E}_{\hat{\alpha}}(\varphi_{3n}^2) = \mathbb{E}_{\hat{\alpha}}((\varphi_n^1)^2) + \mathbb{E}_{\hat{\alpha}}((\varphi_n^b)^2),$$

2417 and

$$\mathbb{E}_{\hat{\alpha}}((\varphi_n^1)^2) = V_{\hat{\alpha}}(\omega) \sum_{x \in B_{3n}} \mathbb{E}_{\hat{\alpha}}(\eta_x \psi_{n,x}^2).$$

2418 The two previous identities finally yield that

$$V_{\hat{\alpha}}(\omega) \sum_{x \in B_{3n}} \mathbb{E}_{\hat{\alpha}}(\eta_x \psi_{n,x}^2) \leq \mathbb{E}_{\hat{\alpha}}(\varphi_{3n}^2).$$

2419 We now use this bound as well as (8.39) and Lemma 8.17 to obtain that for some constant $C_4 = C_4(\hat{\eta}, \alpha', \mathbf{u})$
 2420

$$(8.40) \quad \frac{2}{\alpha^2(2n-1)^2} \sum_{y \in B_{n-1,i}^x} \mathbb{E}_{\hat{\alpha}} \left(\eta_y [\varphi_{3n}(\hat{\eta}^{y,\theta}) - \varphi_{3n}(\hat{\eta})]^2 \right) \leq C_4 n^2.$$

2421 This is the estimate we wanted for the second line of (8.36).

2422 Letting $C = 3(C_1 + C_2 + C_3 + C_4)$, we now use the four bounds (8.33), (8.34), (8.38) and (8.40) in
 2423 equation (8.32), to obtain that

$$\mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{x+e_i}) (\mathbb{E}_{\hat{\alpha}}(\varphi_{3n} \mid \mathcal{F}_n)(\hat{\eta} + \delta_{x+e_i}^\theta) - \mathbb{E}_{\hat{\alpha}}(\varphi_{3n} \mid \mathcal{F}_n)(\hat{\eta}))^2 \right) \leq C n^2$$

2424 as wanted, which concludes the proof of Lemma 8.19. \square

2425 We have now finished the second step, and proved that the sequences of boundary terms $(\tilde{R}_{n,i}^+)_{n \in \mathbb{N}}$
 2426 and $(\tilde{R}_{n,i}^-)_{n \in \mathbb{N}}$ are bounded in $L^2(\mu_{\hat{\alpha}})$. To conclude the proof of Lemma 8.16 we now prove that any weak
 2427 limit point \mathfrak{R}_i^- of $(\tilde{R}_{n,i}^-)$ is in the linear span of the currents \mathfrak{J}^ω . The main difficulty is to prove that any
 2428 limit point only depends on $\hat{\eta}_0$ and $\hat{\eta}_{e_i}$, which we state as a separate lemma. We will once again only
 2429 consider the negative boundary terms, the positive boundary terms being treated in the same way.

2430 Third step : Proof that \mathfrak{R}_i^- only depends on $\hat{\eta}$ through $\hat{\eta}_0$ and $\hat{\eta}_{e_i}$

2431

2432 Let us introduce

$$\mathbb{Z}_{+,i}^2 = \{x_i > 0\} \cap \mathbb{Z}^2 \setminus \{e_i\}.$$

2433 We first prove the following intermediate result.

2434 **Lemma 8.20.** — Any weak limit point \mathfrak{R}_i^- of the sequence $(\tilde{R}_{n,i}^-)$ is measurable w.r.t. the sites in
 2435 $\mathbb{Z}^2 \cap \{x_i > 0\} \cup \{0\}$. Furthermore, for any edge $(y, y+z)$ with both ends in the set $\mathbb{Z}_{+,i}^2$, the gradient
 2436 $\nabla_{y,y+z} \mathfrak{R}_i^-$ vanishes in $L^2(\mu_{\hat{\alpha}})$.

2437 *Proof of Lemma 8.20.* — In order to avoid taking subsequences, let us also assume that $(\tilde{R}_{n,i}^-)$ weakly
 2438 converges towards \mathfrak{R}_i^- . We first prove the first statement, which is elementary. For any x in the negative
 2439 boundary, $x_i = -n-1$, $\tau_{-x} \tilde{\varphi}_n$ is measurable with respect to the half plane $\{x_i > 0\}$, therefore $\nabla_{0,e_i} \tau_{-x} \tilde{\varphi}_n$
 2440 is measurable with respect to the sites in $\{x_i > 0\} \cup \{0\}$. We deduce from the last remark that for any
 2441 n , $\tilde{R}_{n,i}^-$ is measurable for any n w.r.t. the sites in $\{x_i > 0\} \cup \{0\}$, therefore \mathfrak{R}_i^- also is.

We now show that for any edge $\{y, y+z\} \subset \mathbb{Z}_{+,i}^2$, the gradient $\nabla_{y,y+z} \mathfrak{R}_i^-$ vanishes in $L^2(\mu_{\hat{\alpha}})$. Fix an edge $(y, y+z)$ with both ends in $\mathbb{Z}_{+,i}^2$. By definition,

$$\begin{aligned} \nabla_{y,y+z} \tilde{R}_{n,i}^- &= \frac{1}{(2n)^2} \sum_{x_i = -n-1} \nabla_{y,y+z} \tau_{-x} \nabla_{x,x+e_i} \tilde{\varphi}_n \\ &= \frac{1}{(2n)^2} \sum_{x_i = -n-1} \nabla_{y,y+z} \nabla_{0,e_i} \tau_{-x} \tilde{\varphi}_n. \end{aligned}$$

Because $y, y+z$ are different from 0 and e_i , the two gradients in the formula above commute, therefore using once again $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$, as well as the crude bound $\mathbb{E}_{\hat{\alpha}}((\nabla_a f)^2) \leq 4\mathbb{E}_{\hat{\alpha}}(f^2)$, yields

$$\begin{aligned} \mathbb{E}_{\hat{\alpha}} \left[|\nabla_{y,y+z} \tilde{R}_{n,i}^-|^2 \right] &\leq \frac{1}{(2n)^3} \sum_{x_i = -n-1} \mathbb{E}_{\hat{\alpha}} \left[(\nabla_{0,e_i} \nabla_{y,y+z} \tau_{-x} \tilde{\varphi}_n)^2 \right] \\ &= \frac{1}{(2n)^3} \sum_{x_i = -n-1} \mathbb{E}_{\hat{\alpha}} \left[(\nabla_{0,e_i} \tau_{-x} \nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right] \\ (8.41) \quad &\leq \frac{4}{(2n)^3} \sum_{x_i = -n-1} \mathbb{E}_{\hat{\alpha}} \left[(\nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right]. \end{aligned}$$

2442 There are three cases to consider to estimate $\mathbb{E}_{\hat{\alpha}} \left[(\nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right]$.

2443 (1) The first one is the case where both $x+y$ and $x+y+z$ are in B_n^c , the complementary set of B_n . In
 2444 this case,

$$\mathbb{E}_{\hat{\alpha}} \left[(\nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right] = 0,$$

2445 because $\tilde{\varphi}_n$ is \mathcal{F}_n -measurable.

2446 (2) The second case when both $x+y$ and $x+y+z$ are in B_n . In this case, using (8.22) and Jensen's
 2447 inequality we can write

$$(8.42) \quad \mathbb{E}_{\hat{\alpha}} \left((\nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right) \leq \mathbb{E}_{\hat{\alpha}} \left(\mathbb{1}_{\{\rho_{3n} \leq \alpha'\}} (\bar{\mathbf{u}}_{x+y,x+y+z})^2 \right) \leq C(\mathbf{u}).$$

2448 (3) The last case to consider is if $x+y$ and $x+y+z$ link B_n and B_n^c . Then, as in the proof of Lemma
 2449 8.18 we obtain

$$\mathbb{E}_{\hat{\alpha}} \left[(\nabla_{x+y,x+y+z} \tilde{\varphi}_n)^2 \right] \leq C(\hat{\alpha}, \alpha', \mathbf{u}) n^2.$$

2450 Fix an edge $(y, y+z)$ with both ends in $\mathbb{Z}_{+,i}^2$ and write z as $\pm e_j$, we treat separately the two cases
 2451 for j . If $j = i$, for any n large enough (more precisely as soon as $2n+2 \geq y_i$), for any x such that
 2452 $x_1 = -n-1$, either $x+y$ and $x+y \pm e_i$ are both in B_n or both are in its complementary set B_n^c . We

2453 are therefore either in the first or in the second case above, and since the number of terms in the sum is
2454 $O(n)$, equation (8.41) yields

$$\mathbb{E}_{\hat{\alpha}} \left[(\nabla_{y,y+z} \tilde{R}_{n,i}^-)^2 \right] \leq C' n^{-2} \xrightarrow{n \rightarrow \infty} 0,$$

2455 for some constant $C' = C'(\hat{\alpha}, \mathbf{u})$.

2456 If now $j \neq i$, there can be only two terms in the sum over x for which $x + y$ and $x + y \pm e_i$ link B_n
2457 and B_n^c (third case above), whereas all the others are either in the first or the second case. In this case,
2458 equation (8.41) yields

$$\mathbb{E}_{\hat{\alpha}} \left[(\nabla_{y,y+z} \tilde{R}_{n,i}^-)^2 \right] \leq C'(\hat{\alpha}, \mathbf{u}) n^{-2} + C''(\hat{\alpha}, \alpha', \mathbf{u}) n^{-1} \|\mathbf{u}\|_{2, \hat{\alpha}}^2 \xrightarrow{n \rightarrow \infty} 0.$$

2459 This proves that the sequence $\nabla_{y,y+z} \tilde{R}_{n,i}^-$ vanishes as $n \rightarrow \infty$ in $L^2(\mu_{\hat{\alpha}})$ for any edge $(y, y+z)$ with both
2460 ends in $\mathbb{Z}_{+,i}^2$. Since the gradient $\nabla_{y,y+z}$ is a (Lipschitz, and therefore) continuous functional in $L^2(\mu_{\hat{\alpha}})$,
2461 $\nabla_{y,y+z} \mathfrak{R}_i^-$ vanishes for any edge $(y, y+z)$ with both ends in $\mathbb{Z}_{+,i}^2$. This concludes the proof of Lemma
2462 8.20. \square

2463 **Lemma 8.21.** — Any weak limit point \mathfrak{R}_i^- of the sequence $(\tilde{R}_{n,i}^-)_{n \in \mathbb{N}}$ only depends on the configuration
2464 through $\hat{\eta}_0$ and $\hat{\eta}_{e_i}$. The same is true for the limit points of the positive boundary terms $(\tilde{R}_{n,i}^+)_{n \in \mathbb{N}}$.

2465 *Proof of Lemma 8.21.* — This Lemma is a consequence of Lemma 8.20. Consider the localization $\mathfrak{R}_{i,n}^- =$
2466 $\mathbb{E}_{\hat{\alpha}}(\mathfrak{R}_i^- | \mathcal{F}_n)$, then $\mathfrak{R}_{i,n}^-$ is measurable with respect to the sites in $\{x_i > 0\} \cup \{0\}$ and for any edge
2467 $(y, y+z)$ with both ends in $\mathbb{Z}_{+,i}^2$ its gradient vanishes in $L^2(\mu_{\hat{\alpha}})$. These two properties are immediate
2468 consequences of the properties of \mathfrak{R}_i^- and Jensen's inequality.

2469 Let

$$B_{i,n}^+ = B_n \cap \mathbb{Z}_{+,i}^2,$$

2470 since the gradients of \mathfrak{R}_i^- vanish for any edge in $B_{i,n}^+$, on the event on which there are at least two empty
2471 sites in $B_{i,n}^+$, \mathfrak{R}_i^- only depends on the $\hat{\eta}_x$, $x \in B_{i,n}^+$ through the empirical measure on $B_{i,n}^+$

$$\hat{\rho}_{B_{i,n}^+} := \frac{1}{|B_{i,n}^+|} \sum_{B_{i,n}^+} \eta_x \delta_{\theta_x}.$$

2472 Indeed, for two configurations $\hat{\eta}$ and $\hat{\eta}'$ with the same number of particles, and with the same angles in
2473 $B_{i,n}^+$, we can reach one from the other with a combination of the previous gradients, hence the difference
2474 $\mathfrak{R}_{i,n}^-(\hat{\eta}) - \mathfrak{R}_{i,n}^-(\hat{\eta}')$ vanishes. This is not true whenever there is one or less empty site in $B_{i,n}^+$, but since
2475 we are under the product measure, this happens with exponentially small probability and will not be an
2476 issue.

2477 Let us denote by E_n^* the event "there are at least two empty sites in $B_{i,n}^+$ ", the previous statement
2478 rewrites as

$$\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*} = \mathbb{E}_{\hat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*} \left| \hat{\eta}_0, \hat{\eta}_{e_i}, \hat{\rho}_{B_{i,n}^+} \right. \right).$$

2479 For any cylinder function f , we are going to prove that $\mathbb{E}_{\hat{\alpha}}(f \cdot \mathfrak{R}_i^-) = \mathbb{E}_{\hat{\alpha}}[f \cdot \mathbb{E}(\mathfrak{R}_i^- | \hat{\eta}_0, \hat{\eta}_{e_i})]$. Let

$$f^+ = \mathbb{E}(f | \hat{\eta}_x, x \in \{x_i > 0\} \cup \{0\})$$

be the conditional expectation with respect to the sites in $\{x_i > 0\} \cup \{0\}$. Since f is a cylinder function,
so is f^+ , therefore for any sufficiently large integer n , we can write

$$\begin{aligned} \mathbb{E}_{\hat{\alpha}}(f \cdot \mathfrak{R}_i^- \mathbb{1}_{E_n^*}) &= \mathbb{E}_{\hat{\alpha}}(f \cdot \mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*}) \\ &= \mathbb{E}_{\hat{\alpha}} \left(\mathbb{E}_{\hat{\alpha}} \left(f \cdot \mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*} \left| \hat{\eta}_0, \hat{\eta}_{e_i}, \hat{\rho}_{B_{i,n}^+} \right. \right) \right) \\ &= \mathbb{E}_{\hat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*} \mathbb{E}_{\hat{\alpha}} \left(f \left| \hat{\eta}_0, \hat{\eta}_{e_i}, \hat{\rho}_{B_{i,n}^+} \right. \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^*} \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i}, \widehat{\rho}_{B_{i,n}^+} \right) \right) \\
 &= \mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i}, \widehat{\rho}_{B_{i,n}^+} \right) \right) + \mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^{*c}} \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i}, \widehat{\rho}_{B_{i,n}^+} \right) \right) \\
 (8.43) \quad &= \mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_i^- \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i} \right) \right) + o_n(1),
 \end{aligned}$$

2480 since

$$\mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i}, \widehat{\rho}_{B_{i,n}^+} \right) \xrightarrow[n \rightarrow \infty]{L^2(\mu_{\widehat{\alpha}})} \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i} \right),$$

2481 because $\widehat{\rho}_{B_{i,n}^+}$ converges $\mu_{\widehat{\alpha}}$ a.s. as $n \rightarrow \infty$ towards $\widehat{\alpha}$, and

$$\mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_{i,n}^- \mathbb{1}_{E_n^{*c}} \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i}, \widehat{\rho}_{B_{i,n}^+} \right) \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

2482 because f^+ is a bounded function, and $\mathfrak{R}_{i,n}^-$ is in $L^2(\mu_{\widehat{\alpha}})$. For the same reason, the left-hand side in
 2483 (8.43) converges as n goes to ∞ towards $\mathbb{E}_{\widehat{\alpha}}(f \cdot \mathfrak{R}_i^-)$, and therefore for any cylinder function f

$$\mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_i^- \mathbb{E}_{\widehat{\alpha}} \left(f^+ \mid \widehat{\eta}_0, \widehat{\eta}_{e_i} \right) \right) = \mathbb{E}_{\widehat{\alpha}}(f \cdot \mathfrak{R}_i^-),$$

2484 so that

$$\mathfrak{R}_i^- = \mathbb{E}_{\widehat{\alpha}} \left(\mathfrak{R}_i^- \mid \widehat{\eta}_0, \widehat{\eta}_{e_i} \right).$$

2485 This concludes the proof of Lemma 8.21. \square

2486 To complete the proof of Lemma 8.16, now that we have proved that all limit points of the boundary
 2487 terms are function of $\widehat{\eta}_0$ and $\widehat{\eta}_{e_i}$, we still have to show that such limit points are in \mathfrak{J}^ω . First notice that
 2488 any limit point of the negative boundary \mathfrak{R}_i^- verifies

$$(8.44) \quad \eta_{e_i} \mathfrak{R}_i^- = (1 - \eta_0) \mathfrak{R}_i^- = 0.$$

2489 Indeed,

$$\eta_{e_i} \mathfrak{R}_i^- = \lim_{n \rightarrow \infty} \frac{1}{(2n)^2} \sum_{\substack{x_i = -n-1 \\ x \in B_n}} \eta_{e_i} \tau_{-x} \nabla_{x, x+e_i} \widetilde{\varphi}_n = \lim_{n \rightarrow \infty} \frac{1}{(2n)^2} \sum_{\substack{x_i = -n-1 \\ x \in B_n}} \eta_{e_i} \nabla_{0, e_i} \tau_{-x} \widetilde{\varphi}_n,$$

2490 since $\tau_x \nabla_a f = \nabla_{\tau_x a} \tau_x f$. Now the latter obviously vanishes since $\eta_{e_i} \nabla_{0, e_i} = 0$. The second identity is
 2491 proved in the same way.

2492 Since the $\widetilde{\varphi}_n$'s are in T^ω , so is \mathfrak{R}_i^- . Since \mathfrak{R}_i^- depends only on $\widehat{\eta}_0$ and $\widehat{\eta}_{e_i}$, using (8.44) it can therefore
 2493 be expressed as

$$\mathfrak{R}_i^-(\widehat{\eta}) = \eta_0(1 - \eta_{e_i}) \mathfrak{R}_i^-(\widehat{\eta}_0, \widehat{\eta}_{e_i}) = \eta_0(1 - \eta_{e_i}) [\psi(\eta_0, \eta_{e_i}) + \eta_0^\omega \psi_0(\eta_0, \eta_{e_i})],$$

2494 for some angle blind functions ψ, ψ_0 . In particular, letting $c_1 = \psi_0(1, 0), c_2 = \psi(1, 0)$,

$$\mathfrak{R}_i^-(\widehat{\eta}) = (c_1 \eta_0^\omega + c_2 \eta_0)(1 - \eta_{e_i}).$$

2495 Finally, any weak limit point of the boundary term is an element of \mathfrak{J}^ω , which is what we wanted to
 2496 show. The proof of Lemma 8.16 is thus complete. \square

2497 **8.3. An integration by parts formula.** — *Considering the symmetric exclusion generator \mathcal{L} as a*
 2498 *discrete Laplacian, to prove Theorem 6.11, we are going to need an integration by parts formula in order*
 2499 *to express the expectation of $\psi \cdot h$ in terms of the gradient of h and the "integral" $\nabla \mathcal{L}^{-1} \psi$ of ψ .*

We first extend the definition of the canonical measures given in Definition 3.6 to any domain $B \subset \mathbb{T}_N^2$. For that purpose, consider an integer $K \leq |B|$, and an orderless family $\{\theta_1, \dots, \theta_K\} \in \mathcal{S}^K$. Recall that we denote by \widehat{K} the pair $(K, \{\theta_1, \dots, \theta_K\})$, and we let $\mu_{B, \widehat{K}}$ be the measure such that the K particles with fixed angles $\theta_1, \dots, \theta_K$ are uniformly distributed in the domain B . If $B = B_l$ is the ball of radius

l , this notation is shortened as $\mu_{l,\widehat{K}}$ in accord with Definition 3.6. The expectation w.r.t both of these measures is respectively denoted $\mathbb{E}_{B,\widehat{K}}$ and $\mathbb{E}_{l,\widehat{K}}$. We will, in a similar fashion, write

$$\mathcal{L}_B f(\widehat{\eta}) = \sum_{\substack{x,x+z \in B \\ |z|=1}} \eta_x (1 - \eta_{x+z}) (f(\widehat{\eta}^{x,x+z}) - f(\widehat{\eta})),$$

2500 for the generator of the symmetric exclusion process restricted to B , shortened as \mathcal{L}_l if $B = B_l$.

Recall that we defined

$$\mathcal{C}_0 = \left\{ \psi \in \mathcal{C} \mid \mathbb{E}_{s_\psi, \widehat{K}}(\psi) = 0 \quad \forall \widehat{K} \in \widetilde{\mathbb{K}}_{s_\psi} \quad \text{and} \quad \psi|_{\Sigma_{\widehat{K}}} \equiv 0 \quad \forall \widehat{K} \in \mathbb{K}_{s_\psi} \setminus \widetilde{\mathbb{K}}_{s_\psi} \right\},$$

2501 and that ∇_a is the gradient representing a particle jump along a .

2502 **Lemma 8.22 (Integration by parts formula).** — Let $\psi \in \mathcal{C}_0$ be a cylinder function, and $a \subset B_{s_\psi}$
 2503 an oriented edge in its domain. Then, ψ is in the range of the generator \mathcal{L}_{s_ψ} , and we can define the
 2504 "primitive" $I_a(\psi)$ of ψ with respect to the gradient along the oriented edge a as

$$I_a(\psi) = \frac{1}{2} \nabla_a (-\mathcal{L}_{s_\psi})^{-1} \psi.$$

2505 Furthermore, for any $B \subset \mathbb{T}_N^2$ containing B_{s_ψ} , any $\widehat{K} = (K, (\theta_1, \dots, \theta_K))$ such that $K \leq |B|$ and $h \in \mathcal{C}$
 2506 measurable w.r.t. sites in B , we have

$$(8.45) \quad \mathbb{E}_{B,\widehat{K}}(\psi \cdot h) = \sum_{a \subset B_{s_\psi}} \mathbb{E}_{B,\widehat{K}}(I_a(\psi) \cdot \nabla_a h).$$

2507 This result is also true if $\mu_{B,\widehat{K}}$ is replaced by a grand-canonical measure $\mu_{\widehat{\alpha}}$. Note that if $K = |B| - 1$ or
 2508 $K = |B|$ the result is trivial because ψ vanishes.

2509 *Proof of Lemma 8.22.* — The proof of the previous result is quite elementary. Fix a function $\psi \in \mathcal{C}_0$, to
 2510 prove the integration by parts formula, we first show that ψ is in the range of \mathcal{L}_{s_ψ} , by building for any \widehat{K}
 2511 a function $\varphi_{\widehat{K}}$ on $\Sigma_{\widehat{K}}^{s_\psi}$, verifying $\mathcal{L}_{s_\psi} \varphi_{\widehat{K}} = \psi|_{\Sigma_{\widehat{K}}^{s_\psi}}$. This result is well-known for the color-blind exclusion
 2512 process, but in our case where each particle has an angle, the canonical measures take an unusual form,
 2513 and we prove it for the sake of exhaustivity.

2514 For any $\varphi : \Sigma_{\widehat{K}}^{s_\psi} \rightarrow \mathbb{R}$ such that $\mathcal{L}_{s_\psi} \varphi = 0$,

$$\mathbb{E}_{B_{s_\psi}, \widehat{K}}(\varphi \mathcal{L}_{s_\psi} \varphi) = -\frac{1}{2} \mathbb{E}_{B_{s_\psi}, \widehat{K}} \left(\sum_{x,x+z \in B_{s_\psi}} \eta_x (1 - \eta_z) (\varphi(\widehat{\eta}^{x,z}) - \varphi(\widehat{\eta}))^2 \right) = 0,$$

2515 therefore φ is invariant under the allowed jump of a particle along any edge in B_{s_ψ} . For any $\widehat{K} \in \widetilde{\mathbb{K}}_{s_\psi}$,
 2516 the function φ is constant on $\Sigma_{\widehat{K}}^{s_\psi}$, because $\Sigma_{\widehat{K}}^{s_\psi}$ is then irreducible w.r.t. the exclusion dynamics in B_{s_ψ} ,
 2517 according to Section 3.3. In particular $\text{Ker}_{\Sigma_{\widehat{K}}^{s_\psi}} \mathcal{L}_{s_\psi}$ is the set of constant functions, and

$$\left\{ \varphi : \Sigma_{\widehat{K}}^{s_\psi} \rightarrow \mathbb{R} \mid \mathbb{E}_{B_{s_\psi}, \widehat{K}}(\varphi) = 0 \right\} = \left\{ \mathcal{L}_{s_\psi} \psi, \quad \psi : \Sigma_{\widehat{K}}^{s_\psi} \rightarrow \mathbb{R} \right\}.$$

2518 For any $\psi \in \mathcal{C}_0$, any $\widehat{K} \in \widetilde{\mathbb{K}}_{s_\psi}$, there exists a $\varphi_{\widehat{K}} : \Sigma_{\widehat{K}}^{s_\psi} \rightarrow \mathbb{R}$, such that

$$\mathcal{L}_{s_\psi} \varphi_{\widehat{K}} = \psi|_{\Sigma_{\widehat{K}}^{s_\psi}}.$$

2519 Since ψ vanishes when B_{s_ψ} has one or less empty site, we also let $\varphi_{\widehat{K}} = 0$ for any $\widehat{K} \in \mathbb{K}_{s_\psi} \setminus \widetilde{\mathbb{K}}_{s_\psi}$. We
 2520 now define the local function $\varphi^* \in \mathcal{C}$ by $\varphi^*|_{\Sigma_{\widehat{K}}^{s_\psi}} = \varphi_{\widehat{K}}(\widehat{\eta})$, which verifies by construction

$$\psi = \mathcal{L}_{s_\psi} \varphi^*,$$

2521 therefore $\psi \in \mathcal{L}_{s_\psi} \mathcal{C}$.

Proving the integration by parts formula is now elementary : since $\psi = \mathcal{L}_{s_\psi} \mathcal{L}_{s_\psi}^{-1} \psi$,

$$\mathbb{E}_{B,\widehat{K}}(h \cdot \psi) = \mathbb{E}_{B,\widehat{K}}(h \cdot \mathcal{L}_{s_\psi} \mathcal{L}_{s_\psi}^{-1} \psi)$$

$$\begin{aligned}
 &= -\frac{1}{2} \sum_{a \subset B_\psi} \mathbb{E}_{B, \widehat{K}} \left(\nabla_a \mathcal{L}_{s_\psi}^{-1} \psi \cdot \nabla_a h \right) \\
 &= \sum_{a \subset B_\psi} \mathbb{E}_{B, \widehat{K}} (I_a(\psi) \cdot \nabla_a h)
 \end{aligned}$$

2522 which proves identity (8.45). By conditioning to the canonical state in B , one easily obtains that the
 2523 same is true when the canonical measure is replaced by a grand-canonical measure $\mu_{\widehat{\alpha}}$. \square

2524 We finish this section with a technical Lemma. Recall that for any cylinder function ψ , we denote by
 2525 s_ψ the size of its support and for any integer l , $l_\psi = l - s_\psi - 1$.

Lemma 8.23. — For any $\psi \in \mathcal{C}_0 + J^* + \mathcal{L}\mathcal{C}$, there exists a constant $C(\psi)$ such that for any l , $\widehat{K} \in \widetilde{\mathbb{K}}_l$, $h \in \mathcal{C}$ only depending on sites in B_l , $\gamma > 0$, and $A \subset B_{l_\psi}$

$$\mathbb{E}_{l, \widehat{K}} \left(h \sum_{x \in A} \tau_x \psi \right) \leq \gamma C(\psi) |A| + \frac{1}{2\gamma} \mathcal{D}_{l, \widehat{K}}^{A_\psi}(h),$$

2526 where we shortened $A_\psi = \{x \in B_l, d(x, A) \leq s_\psi\}$, $\mathcal{D}_{l, \widehat{K}}^A(h) = \mathbb{E}_{l, \widehat{K}}(h(-\mathcal{L}_A)h)$ and \mathcal{L}_A is the SSEP
 2527 generator restricted to jumps with both ends in A .

2528 *Proof of Lemma 8.23.* — Since for some constant $C(s_\psi)$, $\sum_{x \in A} \mathcal{D}_{l, \widehat{K}}^{B_{s_\psi}(x)}(h) \leq C(s_\psi) \mathcal{D}_{l, \widehat{K}}^{A_\psi}(h)$ to establish
 2529 this result, it is sufficient to prove that for any $x \in A$ and for any positive γ' ,

$$(8.46) \quad \mathbb{E}_{l, \widehat{K}}(h \tau_x \psi) \leq \gamma' C'(\psi) + \frac{1}{2\gamma'} \mathcal{D}_{l, \widehat{K}}^{B_{s_\psi}(x)}(h).$$

2530 We now establish this last bound for any $\psi \in \mathcal{C}_0 \cup J^* \cup \mathcal{L}\mathcal{C}$, which proves the Lemma.

Assume first that $\psi = j_k^\Phi$ for $k \in \{1, 2\}$, and $\Phi \in C^1(\mathbb{S})$. Then, $\mathbb{E}_{l, \widehat{K}}(h \tau_x \psi) = \mathbb{E}_{l, \widehat{K}}(h j_{x, x+e_k}^\Phi)$, where as before $j_{x, x+e_k}^\Phi = \Phi(\theta_x) \eta_x (1 - \eta_{x+e_k}) - \Phi(\theta_{x+e_k}) \eta_{x+e_k} (1 - \eta_x)$. Thanks to changes of variable $\widehat{\eta} \mapsto \widehat{\eta}^{x, x+e_k}$, in the second term, we obtain, using the elementary bound $ab \leq \gamma a^2/2 + b^2/2\gamma$ which holds for any γ ,

$$\mathbb{E}_{l, \widehat{K}}(h \tau_x \psi) = -\mathbb{E}_{l, \widehat{K}}(\Phi(\theta_x) \nabla_{x, x+e_k} h) \leq \frac{\gamma \|\Phi\|_\infty^2}{2} + \frac{1}{2\gamma} \mathbb{E}_{l, \widehat{K}}((\nabla_{x, x+e_k} h)^2)$$

2531 which proves (8.46).

We now consider $\psi = \mathcal{L}f \in \mathcal{L}\mathcal{C}$. Since f is a local function, fix s_ψ such that $\mathcal{L}f = \mathcal{L}_{s_\psi} f$. We rewrite

$$\begin{aligned}
 \mathbb{E}_{l, \widehat{K}}(h \tau_x \psi) &= \mathbb{E}_{l, \widehat{K}} \left(h \mathcal{L}_{B_{s_\psi}(x)}(\tau_x f) \right) = \mathbb{E}_{l, \widehat{K}} \left((\tau_x f) \mathcal{L}_{B_{s_\psi}(x)} h \right) \\
 &= \sum_{y, y+z \in B_{s_\psi}(x)} \mathbb{E}_{l, \widehat{K}}((\tau_x f) \nabla_{y, y+z} h) \leq \frac{\gamma C(s_\psi) \|f\|_\infty^2}{2} + \frac{1}{2\gamma} \mathcal{D}_{l, \widehat{K}}^{B_{s_\psi}(x)}(h),
 \end{aligned}$$

2532 as wanted.

2533 Only remains the case $\psi \in \mathcal{C}_0$, for which (8.46) is a consequence of the integration by parts formula
 2534 and is proved similarly to the case $\psi = \mathcal{L}f$. By definition of $I_a(\psi)$,

$$\sum_{y, y+z \in B_{s_\psi}(x)} \mathbb{E}_{l, \widehat{K}}(I_{x, x+z}(\tau_x \psi)^2) = \frac{1}{2} \mathbb{E}_{l, \widehat{K}}((\tau_x \psi)(-\mathcal{L}_{B_{s_\psi}(x)}^{-1})(\tau_x \psi)) = \frac{1}{2} \mathbb{E}_{l, \widehat{K}}(\psi(-\mathcal{L}_{B_{s_\psi}}^{-1})\psi) \leq C(\psi),$$

2535 where $C(\psi)$ can be chosen independently of \widehat{K} . Using (8.45), and this last bound, we obtain

$$\mathbb{E}_{l, \widehat{K}}(h \tau_x \psi) = \sum_{y, y+z \in B_{s_\psi}(x)} \mathbb{E}_{l, \widehat{K}}(I_{y, y+z}(\tau_x \psi) \cdot \nabla_{y, y+z} h) \leq \frac{\gamma C(\psi)}{2} + \frac{1}{2\gamma} \mathcal{D}_{l, \widehat{K}}^{B_{s_\psi}(x)}(h),$$

2536 which proves (8.46) and Lemma 8.23. \square

2537 **8.4. Heuristics on $\ll \cdot \gg_{\hat{\alpha}}$ and Theorem 6.11.** — The purpose of this section is to explain the
 2538 variational formula for the limiting covariance $\ll \psi \gg_{\hat{\alpha}}$ introduced in Definition 6.9. Given the generator
 2539 \mathcal{L} of the SSEP on \mathbb{Z}^2 , for any function f with mean 0 w.r.t. any canonical measure, consider the linear
 2540 application

$$(8.47) \quad \mathfrak{F} : f \mapsto \nabla \mathcal{L}^{-1} \Sigma_f = \begin{pmatrix} \nabla_{0,e_1} \mathcal{L}^{-1} \Sigma_f \\ \nabla_{0,e_2} \mathcal{L}^{-1} \Sigma_f \end{pmatrix}.$$

2541 A priori, even if f is a local function, $\mathcal{L}^{-1}f$ is no longer local, and $\nabla \mathcal{L}^{-1} \Sigma_f$ can therefore involve a
 2542 infinite number of non-zero contribution, so that \mathfrak{F} is not a priori well defined. However, assuming that f
 2543 is such that $\nabla \mathcal{L}^{-1} \Sigma_f$ is well-defined, the definition above indicates thanks to the translation invariance
 2544 of Σ_f and \mathcal{L}^{-1} , that $\mathfrak{F}(f)$ is the germ of a closed form as introduced in Section 8.2. To illustrate this last
 2545 remark, we describe the effect of this application on $\mathcal{L}\mathcal{C}$ and J^* .

2546 Recall that for $\Phi \in C^1(\mathbb{S})$, $j_i^\Phi = \eta_0^\Phi (1 - \eta_{e_i}) - \eta_{e_i}^\Phi (1 - \eta_0)$. We first investigate the action of \mathfrak{F} on the
 2547 currents j_i^Φ . Consider an infinite configuration $\hat{\eta}$ with no particles outside of some large compact set K .
 2548 For the sake of concision, we will call such a configuration *bounded*. Then, we can write

$$\mathcal{L} \left[\sum_{x \in \mathbb{Z}^2} x_i \eta_x^\Phi \right] = \sum_{x \in \mathbb{Z}^2} x_i \mathcal{L} \eta_x^\Phi = \sum_{x \in \mathbb{Z}^2} \tau_x j_i^\Phi = \Sigma_{j_i^\Phi}.$$

2549 Since the configuration was assumed bounded, both of the sums above are finite, and the identity above
 2550 is well posed. Coming back to our application \mathfrak{F} , the previous identity yields

$$\mathfrak{F}(j_i^\Phi) = \begin{pmatrix} \nabla_{0,e_1} \mathcal{L}^{-1} \Sigma_{j_i^\Phi} \\ \nabla_{0,e_2} \mathcal{L}^{-1} \Sigma_{j_i^\Phi} \end{pmatrix} = \begin{pmatrix} \nabla_{0,e_1} \sum_{x \in \mathbb{Z}^2} x_i \eta_x^\Phi \\ \nabla_{0,e_2} \sum_{x \in \mathbb{Z}^2} x_i \eta_x^\Phi \end{pmatrix}.$$

2551 Since the only positive contribution in the right-hand side above is for $x = e_i$, elementary calculations
 2552 yield

$$\mathfrak{F}(j_i^\Phi) = \mathbf{j}^{i,\Phi},$$

2553 where the $\mathbf{j}^{i,\Phi}$'s are the germs of closed forms introduced in equation (8.15). The application \mathfrak{F} therefore
 2554 maps J^* (cf. (6.44)) into

$$\mathfrak{F}^* := \left\{ \mathbf{j}^{1,\Phi_1} + \mathbf{j}^{2,\Phi_2}, \quad \Phi_1, \Phi_2 \in C^1(\mathbb{S}) \right\}.$$

Since one can also write $\mathfrak{F}(f) = \nabla \Sigma_{\mathcal{L}^{-1}f}$, we can define \mathfrak{F} on $\mathcal{L}\mathcal{C}$ as

$$\mathfrak{F}(\mathcal{L}f) = \nabla \sum_{x \in \mathbb{Z}^2} \tau_x \mathcal{L}^{-1} \mathcal{L}f = \nabla \Sigma_f,$$

2555 which is the germ of an exact form associated with f .

Denote by \mathfrak{E}^* the set of germs of exact forms associated with functions in \mathcal{C} , the construction above
 allow us to define the bijective application

$$\mathfrak{F} : \begin{array}{ccc} J^* + \mathcal{L}\mathcal{C} & \longrightarrow & \mathfrak{F}^* + \mathfrak{E}^* \\ j_1^{\Phi_1} + j_2^{\Phi_2} + \mathcal{L}f & \mapsto & \mathbf{j}^{1,\Phi_1} + \mathbf{j}^{2,\Phi_2} + \nabla \Sigma_f \end{array}.$$

2556 Recall that we defined the L^2 -norm of any closed form \mathbf{u} as

$$\|\mathbf{u}\|_{2,\hat{\alpha}} = [\mathbb{E}_{\hat{\alpha}}(\mathbf{u}_1^2 + \mathbf{u}_2^2)]^{1/2}.$$

2557 According to Proposition 8.11, we can rewrite for any $\mathbf{u} \in \mathfrak{T}^\omega$,

$$(8.48) \quad \|\mathbf{u}\|_{2,\hat{\alpha}}^2 = \sup_{\substack{g \in T^\omega \\ a,b \in \mathbb{R}^2}} \left\{ 2\mathbb{E}_{\hat{\alpha}}(\mathbf{u} \cdot (\nabla \Sigma_g + \mathbf{j}^{a,b})) - \left\| \nabla \Sigma_g + \mathbf{j}^{a,b} \right\|_{2,\hat{\alpha}}^2 \right\}.$$

2558 Define $\text{Ker}_{\hat{\alpha}}(\mathfrak{F})$ the kernel of \mathfrak{F} w.r.t $\|\cdot\|_{2,\hat{\alpha}}$, we can equip $\mathcal{T}_0^\omega / \text{Ker}_{\hat{\alpha}}(\mathfrak{F})$ with the norm $\ll \cdot \gg_{\hat{\alpha}}^{1/2}$ induced
 2559 by the mapping \mathfrak{F} , defined as

$$\ll f \gg_{\hat{\alpha}} = \|\mathfrak{F}(f)\|_{2,\hat{\alpha}}^2 = \sup_{\substack{g \in T^\omega \\ a,b \in \mathbb{R}^2}} \left\{ 2\mathbb{E}_{\hat{\alpha}}(\mathfrak{F}(f) \cdot (\nabla \Sigma_g + \mathbf{j}^{a,b})) - \left\| \nabla \Sigma_g + \mathbf{j}^{a,b} \right\|_{2,\hat{\alpha}}^2 \right\}.$$

By generalizing the integration by parts formula in the previous section, this formula is strictly analogous to Definition 6.8, and \mathfrak{F} is therefore an isomorphism

$$\mathfrak{F} : (\mathcal{T}_0^\omega / \text{Ker}_{\hat{\alpha}}(\mathfrak{F}), \ll \cdot \gg_{\hat{\alpha}}) \longrightarrow \left(\mathfrak{T}^\omega = \mathfrak{J}^\omega + \mathfrak{E}^\omega, \|\cdot\|_{2, \hat{\alpha}}^2 \right),$$

2560 which gives $\mathcal{T}_0^\omega / \text{Ker}_{\hat{\alpha}}(\mathfrak{F})$, as stated in Proposition 6.13, the same structure as $J^\omega + \overline{\mathcal{L}T^\omega} / \text{Ker}_{\hat{\alpha}}(\mathfrak{F})$.

We now briefly carry on with our heuristics and explain why Theorem 6.11 holds, which is rigorously proved in Section 8.5. The proof is based on the integration by parts obtained in Subsection 8.3. Applying it to $\sum_{x \in B_{l_\psi}} \tau_x \psi$ yields that the quantity in the right-hand side of (6.49) can be rewritten

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left(\frac{1}{2} \sum_{x, x+z \in B_l} \left[\nabla_{x, x+z} \mathcal{L}_l^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \right]^2 \right).$$

Assuming that one is able to replace μ_{l, \hat{K}_l} by the translation invariant grand-canonical measure $\mu_{\hat{\alpha}}$, and all quantities being ultimately translation invariant, this limit should be the same as

$$\begin{aligned} \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{\hat{\alpha}} \left(\frac{1}{2} \sum_{x, x+z \in B_l} \left[\nabla_{x, x+z} \mathcal{L}_l^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \right]^2 \right) &= \lim_{l \rightarrow \infty} \mathbb{E}_{\hat{\alpha}} \left(\sum_{i=1,2} \left[\nabla_{0, e_i} \mathcal{L}_l^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \right]^2 \right) \\ &= \|\mathfrak{F}(\psi)\|_{2, \hat{\alpha}}^2 \\ &= \ll \psi \gg_{\hat{\alpha}}. \end{aligned}$$

2561 The rigorous proof of this result, given in the next section, is technical due to the delicate nature of \mathcal{L}^{-1} .

2562 **8.5. Proof of Theorem 6.11.** — In order to prove Theorem 6.11, we need to prove that

$$(8.49) \quad \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l)^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\varphi}} \tau_x \varphi \right) = \ll \psi, \varphi \gg_{\hat{\alpha}}$$

2563 in three cases :

2564 (1) $\varphi = \psi$ and $\psi \in \mathcal{L}\mathcal{C} + J^*$,

2565 (2) $\varphi \in \mathcal{T}_0^\omega$ and $\psi \in \mathcal{L}\mathcal{C} + J^*$,

2566 (3) $\varphi = \psi$ and $\psi \in \mathcal{T}_0^\omega$.

2567 The first two cases correspond to Definition 6.8, whereas the last one corresponds to Definition 6.9. The
2568 first two cases are easier, we treat them first as a separate Lemma. The uniformity of the convergence
2569 will be proved at the end of the section as in [27].

Lemma 8.24. — Fix $\varphi \in \mathcal{T}_0^\omega$ and $\psi = \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \in \mathcal{L}\mathcal{C} + J^*$. For any sequence (\hat{K}_l) such that $\hat{\alpha}_{\hat{K}_l} \rightarrow \hat{\alpha}$,

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\varphi}} \tau_x \varphi \right) = \sum_{i=1}^2 \mathbb{E}_{\hat{\alpha}} \left(\eta_0(1 - \eta_{e_i}) [\Phi_i(\theta_0) + \Sigma_g(\hat{\eta}^{0, e_i}) - \Sigma_g]^2 \right),$$

2570 and

$$(8.50) \quad \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\varphi}} \tau_x \varphi \right) = -\mathbb{E}_{\hat{\alpha}} \left(\varphi \left[\Sigma_g + \sum_{x \in \mathbb{Z}^2} (x_1 \eta_x^{\Phi_1} + x_2 \eta_x^{\Phi_2}) \right] \right).$$

2571 *Proof of Lemma 8.24.* — Fix $\psi = \mathcal{L}g + j_1^{\Phi_1} + j_2^{\Phi_2} \in \mathcal{L}\mathcal{C} + J^*$, and shorten $\tilde{B}_l^i = \{x \in B_l, x_i \leq l-1\}$ one
2572 easily obtains the identity

$$\sum_{x \in \tilde{B}_l^i} \tau_x j_i^{\Phi_i} = \mathcal{L}_l \sum_{x \in B_l} x_i \eta_x^{\Phi_i}.$$

2573 Shorten

$$F = F_l^{g, \Phi_1, \Phi_2} := \sum_{x \in B_{l_\psi}} \tau_x g + \sum_{\substack{i=1,2, \\ x \in B_i}} x_i \eta_x^{\Phi_i} \quad \text{and} \quad G = - \sum_{\substack{i=1,2, \\ x \in \tilde{B}_i^i \setminus B_{l_\psi}}} \tau_x j_i^{\Phi_i},$$

2574 we can then rewrite $\sum_{x \in B_{l_\psi}} \tau_x \psi = \mathcal{L}_l F + G$, and therefore

$$(8.51) \quad \mathbb{E}_{l, \hat{K}} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) = \mathbb{E}_{l, \hat{K}} (F(-\mathcal{L}_l)F) - 2\mathbb{E}_{l, \hat{K}} (FG) + \mathbb{E}_{l, \hat{K}} (G(-\mathcal{L}_l)^{-1}G).$$

2575 Writing

$$\mathbb{E}_{l, \hat{K}} (G(-\mathcal{L}_l)^{-1}G) = \sup_h \{ \mathbb{E}_{l, \hat{K}} (Gh) - \mathcal{D}_{l, \hat{K}}(h) \},$$

2576 and using Lemma 8.23, we obtain that the last term in (8.51) is less than $C(\Phi_1, \Phi_2) |\tilde{B}_l^i \setminus B_{l_\psi}| = O(l)$,
 2577 and therefore the corresponding contribution vanishes in the limit (8.49). Regarding the second term,
 2578 elementary computations yield

$$\mathbb{E}_{l, \hat{K}_l} (\eta_y^{\Phi_i} \tau_x j_k^{\Phi_k}) = C(\mathbb{1}_{\{y=x\}} - \mathbb{1}_{\{y=x+e_k\}}),$$

2579 where we shortened $C = \mathbb{E}_{l, \hat{K}_l} (\Phi_i \Phi_k (\theta_0) \eta_0 (1 - \eta_{e_k}))$, which yields after elementary computations that

$$\mathbb{E}_{l, \hat{K}} \left(\sum_{\substack{i=1,2, \\ y \in B_i}} y_i \eta_y^{\Phi_i} \sum_{\substack{k=1,2, \\ x \in \tilde{B}_l^k \setminus B_{l_\psi}}} \tau_x j_k^{\Phi_k} \right) = O(l).$$

2580 Similarly, for any y such that $\{x, x + e_k\} \cap B_{s_g}(y) = \emptyset$, we have $\mathbb{E}_{l, \hat{K}_l} (\tau_y g \tau_x j_k^{\Phi_k}) = 0$, so that

$$\mathbb{E}_{l, \hat{K}_l} (FG) \leq C(g, \Phi_1, \Phi_2) |\tilde{B}_l^i \setminus B_{l_\psi}| = O(l)$$

2581 and thus vanishes as well in the limit (8.49).

2582 Finally, the last two contributions in (8.51) vanish in the limit, and we now only need to compute
 2583 $\mathbb{E}_{l, \hat{K}_l} (F(-\mathcal{L}_l)F)$, that we split into three parts. We rewrite the first one

$$\mathbb{E}_{l, \hat{K}} \left((-\mathcal{L}_l) \sum_{x \in B_{l_\psi}} \tau_x g \cdot \sum_{x \in B_{l_\psi}} \tau_x g \right) = \frac{1}{2} \sum_{y, y+z \in B_l} \mathbb{E}_{l, \hat{K}} \left(\left[\nabla_{y, y+z} \sum_{x \in B_{l_\psi}} \tau_x g \right]^2 \right).$$

2584 Since f only depends on sites in B_{s_g} , for any $y \in B_{l-2s_g-2}$, we can write $\nabla_{y, y+z} \sum_{x \in B_{l_\psi}} \tau_x g = \nabla_{y, y+z} \Sigma_g$,
 2585 where as before Σ_g is the formal sum $\sum_{x \in \mathbb{Z}^2} \tau_x g$. Furthermore, for any $y \notin B_{l-2s_g-2}$

$$\left[\nabla_{y, y+z} \sum_{x \in B_{l_\psi}} \tau_x g \right]^2 = \left[\nabla_{y, y+z} \sum_{|x-y| \leq s_g+2} \tau_x g \right]^2 \leq C(s_g) \|g\|_\infty^2.$$

Since all the $\nabla_{y, y+z} \Sigma_g$ have the same distribution under $\mu_{l, \hat{K}}$ for $y \in B_{l-2s_g-2}$, we can therefore write using the two bounds above

$$(8.52) \quad \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}} \left((-\mathcal{L}_l) \sum_{x \in B_{l_\psi}} \tau_x g \cdot \sum_{x \in B_{l_\psi}} \tau_x g \right) \\ = \frac{|B_{l-2s_g-2}|}{2(2l+1)^2} \sum_{|z|=1} \mathbb{E}_{l, \hat{K}} \left([\nabla_{0, z} \Sigma_g]^2 \right) + C(f) O \left(\frac{|B_l \setminus B_{l-2s_g-2}|}{(2l+1)^2} \right) = \sum_{i=1}^2 \mathbb{E}_{l, \hat{K}} \left([\nabla_{0, e_i} \Sigma_g]^2 \right) + C(f) O(1/l).$$

2586 Since $\nabla_{0, e_i} \Sigma_g$ is a local function, the equivalence of ensembles (cf. Proposition (C.1)) finally yields for
 2587 any sequence \hat{K}_l such that $\hat{\alpha}_{\hat{K}_l} \rightarrow \hat{\alpha}$

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l) \sum_{x \in B_{l_\psi}} \tau_x g \cdot \sum_{x \in B_{l_\psi}} \tau_x g \right) = \sum_{i=1}^2 \mathbb{E}_{\hat{\alpha}} \left([\nabla_{0, e_i} \Sigma_g]^2 \right)$$

2588 as wanted.

2589 Similarly, one obtains straightforwardly after elementary computations

$$\mathbb{E}_{l,\widehat{K}} \left((-\mathcal{L}_l) \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \cdot \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \right) = \frac{1}{2} \sum_{y,y+z \in B_l} \mathbb{E}_{l,\widehat{K}} \left([\nabla_{y,y+z} \eta_y^{\Phi_{i_z}}]^2 \right),$$

2590 where $i_z = k$ iff $z = \pm e_k$. Once again, under $\mu_{l,\widehat{K}}$, all the terms have the same distribution, and we can
2591 rewrite

$$\frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}} \left((-\mathcal{L}_l) \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \cdot \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \right) = \sum_{i=1}^2 \mathbb{E}_{l,\widehat{K}} \left([\Phi_i(\theta_0) \eta_0 (1 - \eta_{e_i})]^2 \right) + C(\Phi_1, \Phi_2) O(1/l),$$

2592 therefore using once again the equivalence of ensembles also yields

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}_l} \left((-\mathcal{L}_l) \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \cdot \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \right) = \sum_{i=1}^2 \mathbb{E}_{\widehat{\alpha}} \left([\Phi_i(\theta_0) \eta_0 (1 - \eta_{e_i})]^2 \right).$$

2593 Using the fact that $\mathbb{E}_{l,\widehat{K}}(f \mathcal{L}_l g) = -\sum_{y,y+z \in B_l} \mathbb{E}_{l,\widehat{K}}([\nabla_{y,y+z} f][\nabla_{y,y+z} g])$, is is straightforward to adapt
2594 the previous estimates to the cross term, and obtain

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}_l} \left((-\mathcal{L}_l) \sum_{\substack{i=1,2, \\ x \in B_l}} x_i \eta_x^{\Phi_i} \cdot \sum_{x \in B_{l_\psi}} \tau_x g \right) = \sum_{i=1}^2 \mathbb{E}_{\widehat{\alpha}} (\Phi_i(\theta_0) \nabla_{0,e_i} \Sigma_g).$$

2595 These three estimates finally yield as wanted

$$(8.53) \quad \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}} (F(-\mathcal{L}_l)F) = \sum_{i=1}^2 \mathbb{E}_{\widehat{\alpha}} (\eta_0 (1 - \eta_{e_i}) [\Phi_i(\theta_0) + \Sigma_g(\tilde{\eta}^{0,e_i}) - \Sigma_g]^2),$$

2596 which proves the first statement of the Lemma.

2597 The second identity in Lemma 8.24 is proved in a similar way. Using the same notations as for the
2598 first identity, we have $\sum_{x \in B_{l_\psi}} \tau_x \psi = \mathcal{L}_l F + G$, and given $f \in \mathcal{T}_0^\omega$, we rewrite the left-hand side in (8.50)

$$\mathbb{E}_{l,\widehat{K}_l} \left((F + (-\mathcal{L}_l^{-1})G) \cdot \sum_{x \in B_{l_f}} \tau_x f \right).$$

2599 Using once again the equivalence of ensembles, it is easy to prove that

$$(8.54) \quad \lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}_l} \left(F \sum_{x \in B_{l_f}} \tau_x f \right) = -\mathbb{E}_{\widehat{\alpha}} \left(f \left[\Sigma_g + \sum_{x \in \mathbb{Z}^2} (x_1 \eta_x^{\Phi_1} + x_1 \eta_x^{\Phi_2}) \right] \right),$$

therefore we only need to prove that the contribution of G vanishes. This is straightforward, since the contribution of G can be rewritten

$$\begin{aligned} & \frac{1}{(2l+1)^2} \mathbb{E}_{l,\widehat{K}_l} \left((-\mathcal{L}_l^{-1})G \cdot (-\mathcal{L}_l)(-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_f}} \tau_x f \right) \\ &= \frac{1}{(2l+1)^2} \left[\frac{1}{2} \sum_{x,x+z \in B_l} \mathbb{E}_{l,\widehat{K}_l} \left(\nabla_{x,x+z} (-\mathcal{L}_l^{-1})G \cdot \nabla_{x,x+z} (-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_f}} \tau_x f \right) \right]. \end{aligned}$$

We now use Holder's inequality, and that for any positive γ , $|ab| \leq \gamma a^2/2 + b^2/2\gamma$, to obtain that the absolute value of the left-hand side above is less than

$$\left| \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) G \cdot \sum_{x \in B_{l_f}} \tau_x f \right) \right| \\ \leq \frac{\gamma}{2(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} (G(-\mathcal{L}_l^{-1})G) + \frac{1}{2\gamma(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_f}} \tau_x f \cdot \sum_{x \in B_{l_f}} \tau_x f \right).$$

2600 We already proved that the first term in the right-hand side is $O(\gamma l^{-1})$, whereas in the limit $l \rightarrow \infty$ the
 2601 second is bounded by $\ll f \gg_{\hat{\alpha}} / \gamma$ according to Lemma 8.26 below. We can therefore choose $\gamma = \sqrt{l}$, to
 2602 obtain that both terms vanish as $l \rightarrow \infty$, thus concluding the proof of Lemma 8.24. \square

We now consider the case $\psi \in \mathcal{T}_0^\omega$, which is the main result of this section, and conclude by proving that the convergence is uniform and that (6.50) holds. Thanks to the decomposition of the germs of closed forms obtained in Proposition 8.11 and Lemma 8.24 above, these two steps follow closely Section 7.4 of [27], we repeat the proof here for the sake of exhaustivity. Recall that we denoted for any $\psi \in \mathcal{T}_0^\omega$

$$\ll \psi \gg_{\hat{\alpha}} = \sup_{\substack{g \in T^\omega \\ a, b \in \mathbb{R}^2}} \left\{ 2\mathbb{E}_{\hat{\alpha}} \left(\psi \cdot \left[\Sigma_g + \sum_{y \in \mathbb{Z}^2} (y \cdot a) \eta_y^\omega + (y \cdot b) \eta_y \right] \right) - \ll \mathcal{L}g + j^{a,b} \gg_{\hat{\alpha}} \right\}.$$

2603 We split the proof of the third case $\psi \in \mathcal{T}_0^\omega$ in two Lemmas, namely an upper and a lower bound. Using
 2604 the identities obtained in Lemma 8.23, the lower bound is easy to prove.

2605 **Lemma 8.25.** — *Under the assumption of Theorem 6.11,*

$$(8.55) \quad \limsup_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) \geq \ll \psi \gg_{\hat{\alpha}}.$$

Proof of Lemma 8.25. — Denote by \mathcal{C}_l the set of local functions measurable w.r.t. sites in B_l . We start by writing the variational formula

$$\mathbb{E}_{l, \hat{K}_l} \left((-\mathcal{L}_l^{-1}) \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) = \sup_{h \in \mathcal{C}_l} \left\{ 2\mathbb{E}_{l, \hat{K}_l} \left(h \sum_{x \in B_{l_\psi}} \tau_x \psi \right) - \mathcal{D}_{l, \hat{K}_l}(h) \right\} \\ (8.56) \quad \geq \sup_{h \in \tilde{\mathcal{T}}_l^\omega} \left\{ 2\mathbb{E}_{l, \hat{K}_l} \left(h \sum_{x \in B_{l_\psi}} \tau_x \psi \right) - \mathcal{D}_{l, \hat{K}_l}(h) \right\},$$

2606 where $\tilde{\mathcal{T}}_l^\omega$ is the subspace of \mathcal{C}_l

$$\tilde{\mathcal{T}}_l^\omega = \left\{ F_l^{g, a, b} = \sum_{x \in B_{l_g}} \tau_x g + \sum_{x \in B_l} ((a \cdot x) \eta_x^\omega + (b \cdot x) \eta_x), \quad g \in T^\omega, a, b \in \mathbb{R}^2 \right\}.$$

2607 As stated in (8.54) the contribution of the first term in (8.56) is

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \hat{K}_l} \left(\sum_{x \in B_{l_\psi}} \tau_x \psi \cdot F_l^{g, a, b} \right) = -\mathbb{E}_{\hat{\alpha}} \left(\psi \sum_{y \in \mathbb{Z}^2} \left[\tau_y g + \sum_{i=1}^2 ((a \cdot x) \eta_y^\omega + (b \cdot y) \eta_y) \right] \right).$$

2608 and we proved in (8.53) that

$$\lim_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathcal{D}_{l, \hat{K}_l}(F_l^{g, a, b}) = \ll \mathcal{L}g + j^{a,b} \gg_{\hat{\alpha}}.$$

2609 These two identities prove (8.56), and concludes the proof of the Lemma. \square

2610 We now state and prove the upper bound, which is more difficult.

2611 **Lemma 8.26.** — Under the assumptions of Theorem 6.11, for any $\psi \in \mathcal{T}_0^\omega$,

$$(8.57) \quad \limsup_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \widehat{K}_l} \left((-\mathcal{L}_l)^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) \leq \ll \psi \gg_{\widehat{\alpha}}.$$

2612 *Proof of Lemma 8.26.* — We start by replacing the canonical measure $\mu_{\widehat{K}_{l,l}}$ by the grand-canonical mea-
 2613 sure $\mu_{\widehat{\alpha}}$ thanks to the equivalence of ensembles stated in Proposition C.1. The main obstacle in doing so
 2614 is that the support of the function whose expectation we want to estimate grows with l .

2615 By the variational formula for the variance, we can write for any $\widehat{K} \in \widehat{\mathbb{K}}_l$

$$\mathbb{E}_{l, \widehat{K}} \left((-\mathcal{L}_l)^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\psi}} \tau_x \psi \right) = \sup_{h \in T_l^\omega} \left\{ 2\mathbb{E}_{l, \widehat{K}} \left(\sum_{x \in B_{l_\psi}} \tau_x \psi \cdot h \right) - \mathcal{D}_{l, \widehat{K}}(h) \right\}$$

2616 where as before, $T_l^\omega = \mathcal{C}_l \cap T^\omega$ and $\mathcal{D}_{l, \widehat{K}}(h) = \mathbb{E}_{l, \widehat{K}_l}(h \cdot (-\mathcal{L}_l h))$. As in the proof of the one-block-estimate,
 2617 let k be an integer that will go to ∞ after l , and let us partition B_l into disjoint boxes $\widetilde{\Lambda}_0, \dots, \widetilde{\Lambda}_p$, where
 2618 $p = \lfloor \frac{2l_\psi + 1}{2k+1} \rfloor^2$, $\widetilde{\Lambda}_j = B_{2k+1}(x_j)$ for any $1 \leq j \leq p$ and some family of sites x_1, \dots, x_p , and where we let
 2619 $\widetilde{\Lambda}_0 = B_{l_\psi} \setminus (\cup_{j=1}^p \widetilde{\Lambda}_j)$. Recall that s_ψ is the smallest integer such that ψ is measurable with respect to the
 2620 sites in B_{s_ψ} , we now define

$$\Lambda_j = \{x \in \widetilde{\Lambda}_j, d(x, \widetilde{\Lambda}_j^c) > s_\psi\} \quad \text{and} \quad \Lambda_0 = B_{l_\psi} \setminus (\cup_{j=1}^p \Lambda_j).$$

2621 One easily obtains that for some universal constant C , $|\Lambda_0| \leq C s_\psi (l^2/k + lk)$.

2622 Let h be a function in T_l^ω , we can split

$$(8.58) \quad \sum_{x \in B_{l_\psi}} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h) = \sum_{\substack{j=1, \dots, p \\ x \in \Lambda_j}} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h) + \sum_{x \in \Lambda_0} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h).$$

2623 Letting $\gamma = \sqrt{k}/2$ in Lemma 8.23, for any $l \geq k^2$, the second term is less than $k^{-1/2} [C(\psi)l^2 + \mathcal{D}_{l, \widehat{K}}(h)]$.
 2624 Letting $c_k = 1 - k^{-1/2}$, for some constant $C(\psi)$, and for any $l \geq k^2$, the left-hand side of (8.57) is therefore
 2625 less than

$$\frac{c_k}{(2l+1)^2} \sup_{h \in T_l^\omega} \left\{ \sum_{\substack{j=1, \dots, p \\ x \in \Lambda_j}} \frac{2}{c_k} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h) - \mathcal{D}_{l, \widehat{K}}(h) \right\} + \frac{C(\psi)}{\sqrt{k}}.$$

2626 For any $h \in T_l^\omega$, $1 \leq j \leq p$ define $h_j = \mathbb{E}_{l, \widehat{K}}(h \mid \widehat{\eta}_y, y \in \widetilde{\Lambda}_j)$, by convexity of the Dirichlet form, we have

$$\mathcal{D}_{l, \widehat{K}}(h) \geq \sum_{j=1}^p \mathcal{D}_{l, \widehat{K}}^{\widetilde{\Lambda}_j}(h) \geq \sum_{j=1}^p \mathcal{D}_{l, \widehat{K}}^{\widetilde{\Lambda}_j}(h_j),$$

2627 where as before $\mathcal{D}_{l, \widehat{K}}^A(h)$ is the contribution to the Dirichlet form of edges in A . Denoting $T_{k,j}^\omega$ the set of
 2628 functions in T^ω measurable w.r.t. sites in $\widetilde{\Lambda}_j$, we can therefore finally bound from above the left-hand
 2629 side of (8.57) by

$$\frac{c_k}{(2l+1)^2} \sum_{j=1}^p \sup_{h \in T_{k,j}^\omega} \left\{ \sum_{x \in \Lambda_j} \frac{2}{c_k} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h) - \mathcal{D}_{l, \widehat{K}_l}^{\widetilde{\Lambda}_j}(h) \right\} + \frac{C(\psi)}{\sqrt{k}}.$$

All the terms in the sum over j are identically distributed, the quantity above is thus less than

$$\begin{aligned} & \frac{c_k}{(2k+1)^2} \sup_{h \in T_k^\omega} \left\{ \sum_{x \in B_{k_\psi}} \frac{2}{c_k} \mathbb{E}_{l, \widehat{K}_l}(\tau_x \psi \cdot h) - \mathcal{D}_{l, \widehat{K}_l}^{B_k}(h) \right\} + \frac{C(\psi)}{\sqrt{k}} \\ & = \frac{1}{c_k(2k+1)^2} \mathbb{E}_{l, \widehat{K}_l} \left((-\mathcal{L}_k^{-1}) \sum_{x \in B_{k_\psi}} \tau_x \psi \cdot \sum_{x \in B_{k_\psi}} \tau_x \psi \right) + \frac{C(\psi)}{\sqrt{k}}. \end{aligned}$$

The quantity inside the expectation is now a local function w.r.t. l , we can now let $l \rightarrow \infty$ and as $\widehat{\alpha}_{\widehat{K}_l} \rightarrow \widehat{\alpha}$, replace μ_{l, \widehat{K}_l} by $\mu_{\widehat{\alpha}}$ by the equivalence of ensembles stated in Proposition C.1. Letting then $k \rightarrow \infty$, we finally obtain

$$(8.59) \quad \limsup_{l \rightarrow \infty} \frac{1}{(2l+1)^2} \mathbb{E}_{l, \widehat{K}_l} \left((-\mathcal{L}_l)^{-1} \sum_{x \in B_{l\psi}} \tau_x \psi \cdot \sum_{x \in B_{l\psi}} \tau_x \psi \right) \\ \leq \limsup_{k \rightarrow \infty} \frac{1}{(2k+1)^2} \mathbb{E}_{\widehat{\alpha}} \left((-\mathcal{L}_k)^{-1} \sum_{x \in B_{k\psi}} \tau_x \psi \cdot \sum_{x \in B_{k\psi}} \tau_x \psi \right).$$

2630 By the variational formula for the variance, to prove the Lemma it is enough to show

$$(8.60) \quad \limsup_{k \rightarrow \infty} \frac{1}{(2k+1)^2} \sup_{h \in T_k^\omega} \left\{ 2\mathbb{E}_{\widehat{\alpha}} \left(h \sum_{x \in B_{k\psi}} \tau_x \psi \right) - \mathcal{D}_{\widehat{\alpha}, k}(h) \right\} \leq \ll \psi \gg_{\widehat{\alpha}},$$

2631 where we shortened $\mathcal{D}_{\widehat{\alpha}, k}(h) = \mathbb{E}_{\widehat{\alpha}}(h(-\mathcal{L}_k)h)$. According to Lemma 8.23, there exists a constant $C(\psi)$
2632 such that the first term $2\mathbb{E}_{\widehat{\alpha}} \left(\sum_{x \in B_{k\psi}} \tau_x \psi \cdot h \right)$ is less than $C(\psi)(2k+1)^2 + \mathcal{D}_{\widehat{\alpha}, k}(h)/2$. For any h such
2633 that $\mathcal{D}_{\widehat{\alpha}, k}(h) \geq 2C(\psi)(2k+1)^2$, the right-hand side above is therefore negative, and since it vanishes for
2634 $h = 0$, we can therefore safely assume that the supremum is taken w.r.t. functions $h \in T_k^\omega$ satisfying
2635 $\mathcal{D}_{\widehat{\alpha}, k}(h) \leq 2C(\psi)(2k+1)^2$. Using the integration by parts formula of Lemma 8.22 yields

$$\mathbb{E}_{\widehat{\alpha}}(\tau_x \psi \cdot h) = \sum_{x \in B_\psi(x)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h),$$

where $I_a(\psi) = (1/2)\nabla_a(-\mathcal{L}_{s_\psi})^{-1}\psi$. For any edge a , let us denote by $B^\psi(a)$ the set of sites $x \in \mathbb{Z}^2$ such that a is in $B_\psi(x)$, and $\widetilde{B}_k^\psi(a) = B^\psi(a) \cap B_{k\psi}$. Note that for any edge $a \in B_{k\psi-s_\psi}$, these two sets coincide. The integration by parts formula then yields

$$\sum_{x \in B_{k\psi}} \mathbb{E}_{\widehat{\alpha}}(h \tau_x \psi) = \sum_{a \in B_k} \sum_{x \in \widetilde{B}_k^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) \\ = \sum_{a \in B_k} \sum_{x \in B^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) - \sum_{a \in B_k} \sum_{x \in B^\psi \setminus \widetilde{B}_k^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) \\ = \sum_{a \in B_k} \sum_{x \in B^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) - \sum_{a \in B_k \setminus B_{k\psi-s_\psi}} \sum_{x \in B^\psi \setminus \widetilde{B}_k^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h).$$

2636 For any positive γ ,

$$\mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) \leq \frac{1}{2\gamma} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi)^2) + \frac{\gamma}{2} \mathbb{E}_{\widehat{\alpha}}((\nabla_a h)^2),$$

2637 since $|B_k \setminus B_{k\psi-s_\psi}| \leq C(\psi)k$, and thanks to the bound on $\mathcal{D}_{\widehat{\alpha}, k}(h)$, letting $\gamma = 1/\sqrt{k}$, it is then straight-
2638 forward to obtain

$$\sum_{a \in B_k \setminus B_{k\psi-s_\psi}} \sum_{x \in B^\psi \setminus \widetilde{B}_k^\psi(a)} \mathbb{E}_{\widehat{\alpha}}(I_a(\tau_x \psi) \nabla_a h) \leq C(\psi)k^{3/2}.$$

therefore its contribution to the left-hand side of (8.60) vanishes in the limit $k \rightarrow \infty$. Letting $\bar{I}_a(\psi) = \sum_{x \in B^\psi(a)} I_a(\tau_x \psi)$, the left-hand side of equation (8.60) is therefore less than

$$(8.61) \quad \limsup_{k \rightarrow \infty} \frac{1}{(2k+1)^2} \sup_{h \in T_k^\omega} \left\{ 2 \sum_{a \in B_k} \mathbb{E}_{\widehat{\alpha}}(\bar{I}_a(\psi) \nabla_a h) - \mathcal{D}_{\widehat{\alpha}, k}(h) \right\} \\ = \lim_{k \rightarrow \infty} \frac{1}{(2k+1)^2} \left\{ 2 \sum_{a \in B_k} \mathbb{E}_{\widehat{\alpha}}(\bar{I}_a(\psi) \nabla_a h_k) - \mathcal{D}_{\widehat{\alpha}, k}(h_k) \right\}.$$

2639 for some sequence of functions $h_k \in T_k^\omega$ ultimately realizing the limit $k \rightarrow \infty$ of the left-hand side.

2640 Thanks to the translation invariance of $\mu_{\hat{\alpha}}$, and since $\tau_y \bar{I}_a(\psi) = \bar{I}_{\tau_y a}(\psi)$, letting $y = a_1$ be the first
 2641 site of the edge $a = (a_1, a_2)$, we have

$$\mathbb{E}_{\hat{\alpha}}(\bar{I}_a(\psi) \nabla_a h_k) = \mathbb{E}_{\hat{\alpha}}(\bar{I}_{(0, a_2 - a_1)}(\psi) \nabla_{(0, a_2 - a_1)} \tau_{-a_1} h_k).$$

2642 As seen before, a simple change of variable yields that $\mathbb{E}_{\hat{\alpha}}(\nabla_a f \cdot \nabla_a g) = \mathbb{E}_{\hat{\alpha}}(\nabla_{-a} f \cdot \nabla_{-a} g)$, from which we
 2643 deduce

$$2 \sum_{a \in B_k} \mathbb{E}_{\hat{\alpha}}(\bar{I}_a(\psi) \nabla_a h_k) = 4 \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}} \left(\bar{I}_{(0, e_i)}(\psi) \cdot \nabla_{(0, e_i)} \sum_{x, x+e_i \in B_k} \tau_{-x} h_k \right).$$

2644 Define

$$\mathbf{u}_i^k = \frac{1}{(2k+1)^2} \nabla_{(0, e_i)} \sum_{x, x+e_i \in B_k} \tau_{-x} h_k \in T^\omega.$$

The elementary bound $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ yields

$$\begin{aligned} \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}}((\mathbf{u}_i^k)^2) &\leq \frac{2k(2k+1)}{(2k+1)^4} \sum_{x, x+e_i \in B_k} \mathbb{E}_{\hat{\alpha}} \left((\nabla_{(x, x+e_i)} h_k)^2 \right) \\ &\leq \frac{1}{(2k+1)^2} \mathcal{D}_{\hat{\alpha}, k}(h_k) \end{aligned}$$

2645 Thanks to this bound, equation (8.61) yields

$$\frac{1}{(2k+1)^2} \mathbb{E}_{\hat{\alpha}} \left((-\mathcal{L}_k)^{-1} \sum_{x \in B_{k_\psi}} \tau_x \psi \cdot \sum_{x \in B_{k_\psi}} \tau_x \psi \right) \leq \lim_{k \rightarrow \infty} \left\{ 4 \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}}(\bar{I}_{(0, e_i)}(\psi) \cdot \mathbf{u}_i^k) - \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}}((\mathbf{u}_i^k)^2) \right\},$$

2646 and since we already assumed that for some constant $C(\psi)$, $\mathcal{D}_{\hat{\alpha}, k}(h_k) \leq C(\psi)(2k+1)^2$, the sequence of
 2647 differential forms $(\mathbf{u}^k)_{k \in \mathbb{N}}$ is bounded in $L^2(\mu_{\hat{\alpha}})$. It is straightforward to check that any of its limit point
 2648 $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is the germ of a closed form in \mathfrak{F}^ω in the sense of Definition 8.8).

2649 Indeed, given a limit point \mathbf{u} and a finite path γ defined by jumps $x_i, x_i + z_i, 0 \leq i \leq q_\gamma - 1$, we can
 2650 write for the closed form $\bar{\mathbf{u}}$ associated to \mathbf{u}

$$\mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\gamma \in \Gamma_c(\hat{\eta})} |I_{\gamma, \bar{\mathbf{u}}}(\hat{\eta})|) = \lim_{k \rightarrow \infty} \mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\gamma \in \Gamma_c(\hat{\eta})} |I_{\gamma, \bar{\mathbf{u}}^k}(\hat{\eta})|),$$

2651 where $\bar{\mathbf{u}}^k$ is the (non closed) differential form

$$\bar{\mathbf{u}}_{x, x+z}^k = \frac{1}{(2k+1)^2} \nabla_{(x, x+z)} \sum_{y, y+z \in B_k(x)} \tau_{-y} h_k$$

2652 Since γ is a finite path, it depends on edges in a finite box B_n , with n fixed. In particular, for any
 2653 $y \in B_{k-n}$, when computing $I_{\gamma, \bar{\mathbf{u}}^k}(\hat{\eta})$, the contribution of $\tau_{-y} h_k$ vanishes since it involves the complete
 2654 path. We can therefore write for some constant C_γ and any $k > n$,

$$\mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\gamma \in \Gamma_c(\hat{\eta})} |I_{\gamma, \bar{\mathbf{u}}^k}(\hat{\eta})|) \leq \frac{q_\gamma}{(2k+1)^2} \sum_{\substack{y, y+e_i \in B_k \\ y \text{ or } y+e_i \notin B_{k-n}}} \mathbb{E}_{\hat{\alpha}}(|\nabla_{0, e_i} \tau_{-y} h_k|) \leq \frac{q_\gamma}{(2k+1)^2} (C_{n, k} \mathcal{D}_{\hat{\alpha}, k}(h_k))^{1/2},$$

2655 where $C_{n, k} \leq cnk$ is the cardinal of the y 's such that y and $y + e_i$ are in B_k and either y or $y + e_i$ are
 2656 not in B_{k-n} . Since $\mathcal{D}_{\hat{\alpha}, k}(h_k) \leq C(\psi)(2k+1)^2$, the right-hand side above vanishes as $k \rightarrow \infty$ for any
 2657 path γ . This proves that $\mathbb{E}_{\hat{\alpha}}(\mathbb{1}_{\gamma \in \Gamma_c(\hat{\eta})} |I_{\gamma, \bar{\mathbf{u}}}(\hat{\eta})|) = 0$ for any path γ , and any limit point \mathbf{u} of $(\mathbf{u}^k)_k$, and
 2658 in particular $\mathbb{1}_{\gamma \in \Gamma_c(\hat{\eta})} |I_{\gamma, \bar{\mathbf{u}}}(\hat{\eta})|$ vanishes $\mu_{\hat{\alpha}}$ -a.s. for any finite path γ .

2659 We can therefore write

$$\frac{1}{(2k+1)^2} \mathbb{E}_{\hat{\alpha}} \left((-\mathcal{L}_k)^{-1} \sum_{x \in B_{k_\psi}} \tau_x \psi \cdot \sum_{x \in B_{k_\psi}} \tau_x \psi \right) \leq \sup_{\mathbf{u} \in \mathfrak{F}^\omega} \left\{ 4 \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}}(\bar{I}_{(0, e_i)}(\psi) \cdot \mathbf{u}_i) - \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}}(\mathbf{u}_i^2) \right\},$$

2660 where \mathfrak{F}^ω is the set of germs of closed forms introduced in Definition 8.8.

According to Proposition 8.11, the estimate above becomes

$$\begin{aligned} & \frac{1}{(2k+1)^2} \mathbb{E}_{\widehat{\alpha}} \left((-\mathcal{L}_k)^{-1} \sum_{x \in B_{k_\psi}} \tau_x \psi \cdot \sum_{x \in B_{k_\psi}} \tau_x \psi \right) \\ & \leq \sup_{\substack{g \in T^\omega \\ a, b \in \mathbb{R}^2}} \left\{ 4 \sum_{i=1,2} \mathbb{E}_{\widehat{\alpha}} (\bar{I}_{(0, e_i)}(\psi) \cdot (j_i^{a,b} + \nabla_{(0, e_i)} \Sigma_g)) - \sum_{i=1,2} \mathbb{E}_{\widehat{\alpha}} ((j_i^{a,b} + \nabla \Sigma_g)^2) \right\} \\ & = \sup_{\substack{g \in T^\omega \\ a, b \in \mathbb{R}^2}} \left\{ 2 \mathbb{E}_{\widehat{\alpha}} \left(\psi \cdot \left[\Sigma_g + \sum_{y \in \mathbb{Z}^2} (y \cdot a) \eta_y^\omega + (y \cdot b) \eta_y \right] \right) - \ll \mathcal{L}g + j^{a,b} \gg_{\widehat{\alpha}} \right\}. \end{aligned}$$

2661 The last identity is derived as in the proof of Lemma 8.24. The right-hand-side above is $\ll \cdot \gg_{\widehat{\alpha}}$ as
2662 defined in Definition 6.9, which concludes the proof of the upper bound. \square

2663 In order to complete the proof of Theorem 6.11, we still need to prove that the convergence is uniform
2664 in $\widehat{\alpha}$, to prove (6.50). Let us denote

$$V_{l, \psi, \varphi}(\widehat{\alpha}_{\widehat{K}_l}) = \frac{1}{(2l+1)^2} \mathbb{E}_{l, \widehat{K}_l} \left(-\mathcal{L}_l^{-1} \sum_{x \in B_{l_\psi}} \tau_x \psi \cdot \sum_{x \in B_{l_\varphi}} \tau_x \varphi \right),$$

2665 and let us extend smoothly the domain of definition of $V_{l, \psi, \varphi}$ to $\mathcal{M}_1(\mathbb{S})$. The three previous Lemmas
2666 yield that $V_{l, \psi, \varphi}(\widehat{K}_l(2l+1)^{-2})$ converges as l goes to ∞ to $\ll \psi, \varphi \gg_{\widehat{\alpha}}$ as soon as \widehat{K}_l converges towards
2667 the profile $\widehat{\alpha}$, hence in particular, $V_{l, \psi, \varphi}(\widehat{\alpha}_l)$ converges as l goes to ∞ towards $\ll \psi, \varphi \gg_{\widehat{\alpha}}$ as soon as
2668 $\widehat{\alpha}_l$ goes to $\widehat{\alpha}$. For that reason, $\ll \cdot \gg_{\widehat{\alpha}}$ is continuous, and $V_{l, \psi, \varphi}(\widehat{\alpha})$ converges uniformly in $\widehat{\alpha}$ towards
2669 $\ll \psi, \varphi \gg_{\widehat{\alpha}}$ as l goes to ∞ . This, combined with the three lemmas 8.24, 8.26 and 8.25, completes the
2670 proof of Theorem 6.11.

2671 Appendix A

2672 Possible application : Coarsening and global order in active Matter

2673 *We give some context on the modeling of collective dynamics and the rich phenomenology of active*
2674 *matter.*

2675 **A.1. Collective motion among biological organisms.** — Collective motion is a widespread phe-
2676 nomenon in nature, and has motivated in the last decades a fruitful and interdisciplinary field of study
2677 [34]. Such behavior can be observed among many animal species, across many scales of the living spec-
2678 trum, and in a broad range of environments. Animal swarming usually needs to balance out the benefits of
2679 collective behavior (defense against predation, protection of the young ones, increased vigilance) against
2680 the drawback of large groups (food hardships, predator multiplication, etc.).

2681 Despite the numerous forms of interaction between individuals, all of these self-organization phe-
2682 nomenons present spontaneous emergence of density fluctuations and long range correlations. This sim-
2683 ilarity suggests some universality of collective dynamics models [25], [51]. Even though the biological
2684 reasons for collective behavior are now well known, the underlying microscopic and macroscopic mecha-
2685 nisms are not yet fully understood. To unveil these mechanisms, numerous aggregation models have been
2686 put forward.

2687 These models can be built on two distinct principles. The first approach specifies the macroscopic
2688 partial differential equation which rules the evolution of the local density of individuals. The main upside
2689 is that one can use the numerous tools developed for solving PDE's. Several examples of such models
2690 are presented in Okubo and Levin's book, [33]. Since it represents an average behavior, this approach to
2691 collective dynamics is, however, mainly fitted to describe systems with large number of individuals, and
2692 does not take into account the fluctuations to which smaller systems are subject.

2693 The second approach, called *Individual-Based Models* (IBM), specifies the motion of each individual
 2694 organism. If the motion of each individual was described realistically (from a biological standpoint), the
 2695 theoretical study of these models with large number of degrees of freedom would be extremely difficult.
 2696 For this reason, it is usually preferred to simplify the rules for the motion of each individual, as well
 2697 as its interaction with the group. A classical simplification is to consider that the interaction of each
 2698 individual with the group is averaged out over a large number of its neighbors. This so-called *local field*
 2699 simplification often allows to obtain explicit results, at the expense however of their biological accuracy
 2700 (cf. below).

2701 **A.2. Microscopic active matter models.** — In order to represent the direction of the motion of
 2702 each individual, as well as spatial constraints (e.g. volume of each organism), collective dynamics are
 2703 often modeled by individual-based *active matter* models. Active matter is characterized by an energy
 2704 dissipation taking place at the level of each individual particle, which allows it to self-propel, thus yielding
 2705 an extra degree of freedom representing the direction of its motion. One can therefore obtain a phase
 2706 transition towards collective motion when these directions align on lengths large with respect to the size
 2707 of the particles. Active matter models exhibit various behaviors, and in the context of collective motion,
 2708 two phenomena are particularly important :

2709 — when each particle tends to align the direction of its motion to that of its neighbors, one can
 2710 observe a phase transition between order and disorder depending on the strength of the alignment.
 2711 This *alignment phase transition* was first observed in an influential model for collective dynamics
 2712 introduced by Vicsek et al. [50]

2713 — When the particle's velocity decreases with the local density, congestion effects appear : particles
 2714 spend more time where their speed is lower, and therefore tend to accumulate there. This phe-
 2715 nomenon, called Motility-Induced Phase Separation (MIPS), was extensively studied in the recent
 2716 years [9], [21], [11].

2717 *Vicsek model and phase transition in alignment models.* — Interest for self-organization phenomenons
 2718 have grown significantly in statistical physics, where the diversity of such behaviors opens numerous
 2719 modeling perspectives, and raises new questions regarding out-of-equilibrium systems. Many stochastic
 2720 models have been introduced to represent specific biological behavior using statistical physics methods
 2721 and have revealed a phase transition between high density collective motion, and disordered behavior
 2722 with short range correlations at low densities.

2723 A pioneering model was proposed in 1995 by Vicsek et al. They introduce in [50] a general IBM
 2724 (cf. previous paragraph) to model collective dynamics. In the latter, a large number of particles move
 2725 in discrete time, and update the direction of their motion to the average direction of the particles in a
 2726 small neighborhood. The direction of their motion is also submitted to a small noise, which makes the
 2727 dynamics stochastic.

2728 Despite its relative simplicity, the original model described in [50] is extremely rich, and has given rise
 2729 to a considerable literature (cf. the review by Vicsek and Zafeiris, [51]). The first article on this model
 2730 unveiled a phase transition between a high-noise, low-density disordered phase and a low-noise high-
 2731 density ordered phase. Initially thought to be critical, this transition was later shown to be discontinuous
 2732 [12], with an intermediate region in which an ordered band cruises in a disordered background. It was
 2733 recently shown that this transition can be understood as a liquid-gas phase separation in which the
 2734 coexistence phase is organized in a smectic arrangement of finite-width bands traveling collectively [42].
 2735 Numerous extensions and variations on Vicsek's model have been put forward, usually by considering a
 2736 continuous time dynamics, more pertinent to represent biological organisms.

2737 Phase transitions are central to the study of collective dynamics, where coherent behavior arise when
 2738 the alignment becomes strong enough. This notion of phase transition for alignment dynamics is remi-
 2739 niscent of the Ising and XY models, two classical statistical physics models. The Ising model is known
 2740 to have a symmetry breaking phase transition leading to the emergence of a spontaneous magnetization.
 2741 Unlike the Ising model, the XY model (for which the spins are two-dimensional unit vectors parametrized

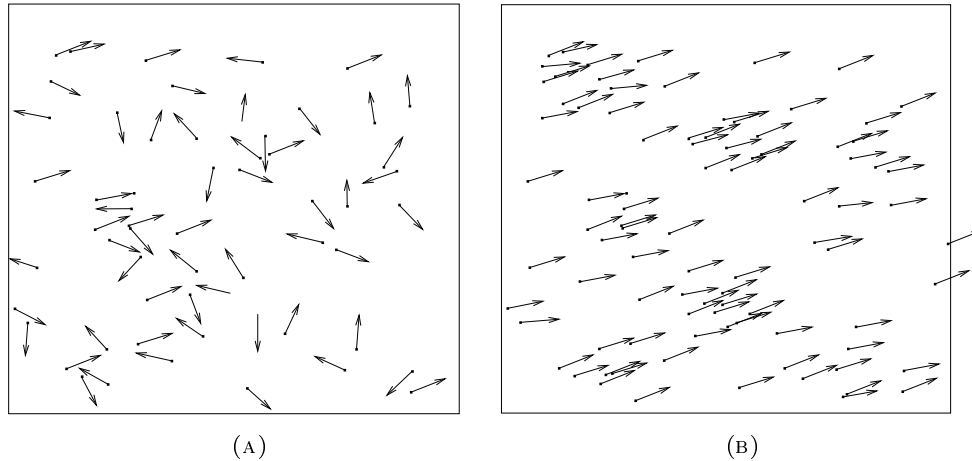


FIGURE 8. Schematic representation of the phase transition in Vicsek's model.
 (A) low density and high noise intensity,
 (B) high density and low noise intensity.

2742 by angles $\theta \in [0, 2\pi[)$ does not present in two dimensions this type of symmetry breaking phase transition,
 2743 according to the Mermin-Wagner Theorem. This is one of the reasons for the popularity of the Vicsek
 2744 model [50], whose alignment dynamics is reminiscent of the XY model, but unlike the latter presents a
 2745 phase transition of the magnetization due to the particle motility [47]. Both the Ising and XY models are
 2746 now well understood. These are *equilibrium models* and they fall within the formalism of Gibbs measures,
 2747 which relates to the thermodynamical parameters of the system.

2748 Active matter models like Vicsek's are out of equilibrium, and in the case of Vicsek's model, the phase
 2749 transition is a dynamical phenomenon. The concepts developed for equilibrium models, namely Gibbs
 2750 measures and free energy, can therefore no longer be used, and despite ample numerical evidence of
 2751 spontaneous magnetization, (cf. [41]) mathematically proving a phase transition becomes significantly
 2752 harder.

2753 Despite these issues, several exact results have been obtained for systems closely related to Vicsek's
 2754 model. In 2007, Degond and Motsch notably introduced a continuous time version of Vicsek's model,
 2755 and derived the macroscopic scaling limit of the system [18], as well as its microscopic corrections [19].
 2756 Their model, which was directly inspired by that of Vicsek et al., is a *locally mean-field model*, where par-
 2757 ticles interact with all other particles present in a small macroscopic neighborhood. This approximation
 2758 simplifies a number of difficulties of out-of-equilibrium systems. In their initial article [18], Degond and
 2759 Motsch assume that a law of large number holds for the microscopic system. This was later rigorously
 2760 proved in [5]. The phase transition as a function of the noise level, between disordered system and global
 2761 alignment, was shown in [16] for this model. Similar results have since been extended to more general
 2762 forms of alignment, (e.g. [4], [7], [17]) and to density dependent parameters [22]. The evolution of the
 2763 macroscopic density was also obtained in the particular case where the interaction between individuals is
 2764 driven by a Morse potential, [8], where previously the shape of animal aggregates (e.g. fish schools mills)
 2765 was only known empirically.

2766 The *Active Ising Model* (AIM) is another alignment model, phenomenologically close to Vicsek's model
 2767 [41], put forward to better understand collective dynamics. It is less demanding from a computational
 2768 standpoint, and is extensively studied both numerically and theoretically by Solon and Tailleur in [43].
 2769 This model does not rely on the mean-field approximation of the Vicsek's model. The particles (with
 2770 either "+" or "-" spins) move independently in a discrete space domain, performing an asymmetric random
 2771 walk with drift directed according to the particle's spin. In addition to the displacement dynamics, the
 2772 particles align their spins with the other particles on the same site as in a fully connected Ising model.

2773 It was numerically shown in [43] that the AIM presents, as does Vicsek's, a phase transition depending
 2774 both on the temperature and the particle density. At low temperature and density, one observes a

2775 magnetically neutral gas, whereas at strong temperature and densities, one obtains a strongly polarized
 2776 liquid. In an intermediary domain, these two phases coexist. The AIM being an out-of-equilibrium
 2777 model as well, its mathematical study is difficult, mainly because of the lack of mean-field approximation
 2778 present in Vicsek’s model. To our knowledge, there exists to this day no mathematical proof of the phase
 2779 transition of the AIM. The model considered in this paper is closely related to both the Vicsek and the
 2780 active Ising models.

2781 *Motility-Induced Phase Transition (MIPS)*. — As previously emphasized, a second interesting phe-
 2782 nomenon can occur in active matter : when the motility of the particles decreases as the local particle
 2783 density increases, one can observe a phase separation between a low density gaseous phase, and condensed
 2784 clusters. This separation is a direct consequence of particles slowing down in dense areas : since they
 2785 spend more time there, they tend to accumulate. This creates the congestion phenomenon called *Motility*
 2786 *Induced Phase Transition*, or *MIPS*, which was thoroughly studied in recent years (cf. the review by
 2787 Cates and Tailleur, [11]).

2788 This congestion phenomenon can be observed across several types of dynamics, under the condition
 2789 that the particle’s velocities and diffusion constants depend on the local density. One of the most studied
 2790 is the *run-and-tumble dynamics* [2], which models the behavior of bacteria : each individual goes in a
 2791 straight line for a while, and then reorients in another random direction. However, MIPS is not specific
 2792 to run and tumble dynamics : it is shown numerically in [10], [40] that MIPS also occurs for active
 2793 Brownian particles, for which each particles motion’s direction diffuses, instead of updating at discrete
 2794 times like in the run-and-tumble dynamics. MIPS can also be observed in lattice models [46], or in
 2795 models with repulsive forces [21], for which the kinetic slowdown is a consequence of repulsive forces.

2796 As already pointed out, one can expect that the active exclusion process investigated in this article
 2797 may exhibit both MIPS and alignment phase transition. However, mathematically proving this statement
 2798 is a difficult task, and this claim is left as a conjecture at this point.

2799 Appendix B

2800 General tools

2801 This appendix regroups a general definitions and results that have been used throughout the proof.

2802 **B.1. Topological setup.** — This paragraph defines the topological setup we endow the trajectories
 2803 space for our process with. Denoting by $\mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$ the space of non-negative measures on the continuous
 2804 configuration space, and

$$\mathcal{M}^{[0,T]} = D([0, T], \mathcal{M}(\mathbb{T}^2 \times \mathbb{S}))$$

2805 the space of right-continuous and left-limited trajectories of measures on $\mathbb{T}^2 \times \mathbb{S}$. Each trajectory $\hat{\eta}^{[0,T]}$
 2806 of our process admits a natural image in $\mathcal{M}^{[0,T]}$ through its empirical measure

$$(B.1) \quad \pi_t^N \left(\hat{\eta}^{[0,T]} \right) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x(t) \delta_{x/N, \theta_x(t)}.$$

2807 Let $(f_k)_{k \in \mathbb{N}}$ be a dense family of functions in $C^\infty(\mathbb{T}^2 \times \mathbb{S})$, and assume that $f_0 \equiv 1$. The weak topology
 2808 on $\mathcal{M}(\mathbb{T}^2 \times \mathbb{S})$ is metrizable, by letting

$$(B.2) \quad \delta(\pi_0, \pi'_0) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|\langle \pi_0, f_k \rangle - \langle \pi'_0, f_k \rangle|}{1 + |\langle \pi_0, f_k \rangle - \langle \pi'_0, f_k \rangle|}.$$

2809 Given this metric, $\mathcal{M}^{[0,T]}$ is endowed with Skorohod’s metric, defined as

$$(B.3) \quad d(\pi, \pi') = \inf_{\kappa \in F} \max \left\{ \|\kappa\|, \sup_{[0,T]} \delta(\pi_t, \pi'_{\kappa_t}) \right\},$$

where F is the set of strictly increasing continuous functions from $[0, T]$ into itself, such that $\kappa_0 = 0$ and $\kappa_T = T$, equipped with the norm

$$\|\kappa\| = \sup_{s,t \in [0,T]} \left\{ \log \left[\frac{\kappa_s - \kappa_t}{s - t} \right] \right\}.$$

Now, $(\mathcal{M}^{[0,T]}, d)$ is a metric space, and we endow the set $\mathcal{P}(\mathcal{M}^{[0,T]})$ of probability measures on $\mathcal{M}^{[0,T]}$ with the weak topology.

Given the empirical measure π_t^N of the process at time t , defined in equation (B.1), define the application

$$\pi^N : \begin{array}{ccc} \Sigma_N^{[0,T]} & \longrightarrow & \mathcal{M}^{[0,T]} \\ \hat{\eta}^{[0,T]} & \mapsto & (\pi_t^N(\hat{\eta}^{[0,T]}))_{t \in [0,T]} \end{array},$$

we define

$$(B.4) \quad Q^N = \mathbb{P}_{\mu^N}^{\lambda, \beta} \circ (\pi^N)^{-1} \in \mathcal{P}(\mathcal{M}^{[0,T]})$$

the pushforward of $\mathbb{P}_{\mu^N}^{\lambda, \beta}$ by π^N .

B.2. Self-diffusion coefficient. — We regroup in this paragraph some useful results regarding the self-diffusion coefficient. Consider on \mathbb{Z}^2 , an initial configuration where each site is initially occupied w.p. $\rho \in [0, 1]$, and with a tagged particle at the origin. Each particle then follows a symmetric exclusion process with finite range transition matrix $p(\cdot)$, verifying $\sum_z zp(z) = 0$, and $p(z) = 0$ outside of a finite set of vertices B .

Definition B.1 (Self-Diffusion Coefficient). — Given $\mathbf{X}_t = (X_t^1, \dots, X_t^d)$ the position at time t of the tagged particle, the d -dimensional *self-diffusion matrix* $D_s = D_s(\rho)$ is defined as

$$(B.5) \quad x^\dagger D_s x = \lim_{t \rightarrow \infty} \frac{\mathbb{E}((x \cdot \mathbf{X}_t)^2)}{t} \quad \forall y \in \mathbb{R}^d,$$

where x^\dagger is the transposed vector of x and (\cdot) is the usual inner product in \mathbb{R}^d .

This result follows from [28]. The following Lemma gives a variational formula for D_s and was obtained in Spohn [44].

Proposition B.2 (Variational formula for the self-diffusion coefficient)

The self-diffusion matrix $D_s = D_s(\rho)$ is characterized by the variational formula

$$x^\dagger D_s x = \inf_{f \in \mathcal{S}} \left\{ \sum_{i=1,2} \left[\mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{e_i}) [x_i + \tau_{e_i} f(\eta^{0, e_i}) - f]^2 \right) + \sum_{y \neq 0, e_i} \mathbb{E}_{\hat{\alpha}} \left([\nabla_{0, e_i} \tau_y f]^2 \right) \right] \right\}.$$

Our system being invariant through coordinates inversions, it is shown in [32] that the matrix D_s is diagonal, and can therefore be written

$$D_s(\rho) = d_s(\rho)I.$$

Finally, the regularity of the self-diffusion coefficient follows from [31], and a lower and upper bound was derived by Varadhan in all dimensions by Varadhan in [49].

Proposition B.3 (Regularity of the self-diffusion coefficient). — In any dimension $d \geq 1$, the self-diffusion coefficient d_s is $C^\infty([0, 1])$, and for some constant $C > 0$, we can write

$$\frac{1}{C}(1 - \rho) \leq d_s(\rho) \leq C(1 - \rho).$$

Finally, we prove a result that we postponed in during the proof of Proposition 6.14.

Proposition B.4 (Conductivity matrix). — Fix $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, let $j^\omega = (j_1^\omega, j_2^\omega)$, where as before

$$j_i^\omega = [\omega(\theta_0) - \mathbb{E}_{\hat{\alpha}}(\omega)]\eta_0(1 - \eta_{e_i}) - [\omega(\theta_{e_i}) - \mathbb{E}_{\hat{\alpha}}(\omega)]\eta_{e_i}(1 - \eta_0).$$

Recall that we defined the conductivity matrix $Q = Q^\omega$ as

$$x^\dagger Q x = \inf_{g \in T^\omega} \ll x \cdot j^\omega + \mathcal{L}g \gg_{\hat{\alpha}},$$

2837 then, we have the identity

$$(B.6) \quad Q = \alpha V_{\hat{\alpha}}(\omega) D_s(\alpha) = \alpha V_{\hat{\alpha}}(\omega) d_s(\alpha) I.$$

Proof of Proposition B.4. — The proof is analogous to that of Theorem 3.2 in [35]. We first consider the trivial case $\alpha = 0, 1$. Since $d_s(1) = 0$, if $\alpha = 0, 1$, Proposition B.4 is trivially true, because both sides of the identity vanish. Furthermore, assuming that $V_{\hat{\alpha}}(\omega) = 0$, we then have $j^{\hat{\omega}} = 0$, therefore both sides vanish as well. We now assume that $\alpha \in]0, 1[$ and $V_{\hat{\alpha}}(\omega) > 0$. By definition 6.8,

$$\ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \mathbb{E}_{\hat{\alpha}} \left(\sum_{i=1,2} \left[x_i \eta_0^{\hat{\omega}} (1 - \eta_{e_i}) + \nabla_{0,e_i} \sum_{y \in \mathbb{Z}^2} \tau_y g \right]^2 \right).$$

Since $g \in T^\omega$, it can be rewritten $g = \varphi(\eta) + \sum_y \eta_y^{\hat{\omega}} \psi_y(\eta)$ for some angle-blind functions $\varphi, \psi_y \in \mathcal{S}$. As we saw in the proof of the spectral gap, any angle-blind function is orthogonal to any function $\eta_y^{\hat{\omega}} \psi(\eta)$, therefore

$$\ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}} \left(\left[x_i \eta_0^{\hat{\omega}} (1 - \eta_{e_i}) + \nabla_{0,e_i} \sum_{y,y' \in \mathbb{Z}^2} \tau_{y'} [\eta_y^{\hat{\omega}} \psi_y] \right]^2 \right) + \mathbb{E}_{\hat{\alpha}} \left(\left[\nabla_{0,e_i} \sum_{y \in \mathbb{Z}^2} \varphi \right]^2 \right).$$

2838 To minimize the left-hand side, we can choose $\varphi = 0$, so that g must take the form $g = \sum_y \eta_y^{\hat{\omega}} \psi_y$. Since
 2839 g is a local function, $\psi' = \sum_y \tau_{-y} \psi_y$ is well defined, and satisfies $\sum_{y,y' \in \mathbb{Z}^2} \tau_{y'} [\eta_y^{\hat{\omega}} \psi_y] = \sum_{y \in \mathbb{Z}^2} \eta_y^{\hat{\omega}} \tau_y \psi'$,
 2840 therefore

$$\ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}} \left(\left[x_i \eta_0^{\hat{\omega}} (1 - \eta_{e_i}) + \nabla_{0,e_i} \sum_{y \in \mathbb{Z}^2} \eta_y^{\hat{\omega}} \tau_y \psi' \right]^2 \right).$$

2841 Elementary computations yield $\nabla_{0,e_i} \eta_0^{\hat{\omega}} \psi' = -\eta_0^{\hat{\omega}} (1 - \eta_{e_i}) \psi'$, $\nabla_{0,e_i} \eta_{e_i}^{\hat{\omega}} \tau_{e_i} \psi' = \eta_0^{\hat{\omega}} (1 - \eta_{e_i}) \tau_{e_i} \psi' (\eta^{0,e_i})$,
 2842 and for any $y \neq 0, e_i$, $\eta_y^{\hat{\omega}} \nabla_{0,e_i} \tau_y \psi'$, therefore

$$\ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \sum_{i=1,2} \mathbb{E}_{\hat{\alpha}} \left(\left[\eta_0^{\hat{\omega}} (1 - \eta_{e_i}) [x_i + \tau_{e_i} \psi' (\eta^{0,e_i}) - \psi'] + \sum_{y \neq 0, e_i} \eta_y^{\hat{\omega}} \nabla_{0,e_i} \tau_y \psi' \right]^2 \right).$$

For any angle-blind function $\psi \in \mathcal{S}$, we have already established in Section 8.1 that

$$\mathbb{E}_{\hat{\alpha}}(\eta_y^{\hat{\omega}} \eta_{y'}^{\hat{\omega}} \psi(\eta)) = \mathbb{1}_{\{y=y'\}} V_{\hat{\alpha}}(\omega) \mathbb{E}_{\hat{\alpha}}(\eta_y \psi(\eta)).$$

The previous quantity now rewrites

$$\begin{aligned} & \ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} \\ &= V_{\hat{\alpha}}(\omega) \sum_{i=1,2} \left[\mathbb{E}_{\hat{\alpha}} \left(\eta_0 (1 - \eta_{e_i}) [x_i + \tau_{e_i} \psi' (\eta^{0,e_i}) - \psi']^2 \right) + \sum_{y \neq 0, e_i} \mathbb{E}_{\hat{\alpha}} \left(\eta_y [\nabla_{0,e_i} \tau_y \psi']^2 \right) \right]. \end{aligned}$$

2843 Denote by $f = \mathbb{E}_{\hat{\alpha}}(\psi' | \eta_0 = 1) = \alpha^{-1} \mathbb{E}_{\hat{\alpha}}(\eta_0 \psi')$, we have

$$\mathbb{E}_{\hat{\alpha}}(\tau_{e_i} \psi' (\eta^{0,e_i}) | \eta_0 = 1) = \tau_{e_i} f (\eta^{0,e_i}) \quad \text{and} \quad \mathbb{E}_{\hat{\alpha}}(\nabla_{0,e_i} \tau_y \psi' | \eta_y = 1) = \nabla_{0,e_i} \tau_y f,$$

so that

$$\ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \alpha V_{\hat{\alpha}}(\omega) \sum_{i=1,2} \left[\mathbb{E}_{\hat{\alpha}} \left((1 - \eta_{e_i}) [x_i + \tau_{e_i} f (\eta^{0,e_i}) - f]^2 \right) + \sum_{y \neq 0, e_i} \mathbb{E}_{\hat{\alpha}} \left([\nabla_{0,e_i} \tau_y f]^2 \right) \right].$$

2844 Taking the infimum over $g \in T^\omega$, f spans \mathcal{S} which yields as wanted, according to Proposition B.2

$$x^\dagger Q x = \ll x \cdot j^{\hat{\omega}} + \mathcal{L}g \gg_{\hat{\alpha}} = \alpha V_{\hat{\alpha}}(\omega) x^\dagger D_s x,$$

2845 thus concluding the proof. □

2846 **B.3. Entropy.** — Given two measures on a space E , let us denote

$$H(\mu | \nu) = \mathbb{E}_\nu \left(\frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \right)$$

2847 the relative entropy of μ w.r.t ν .

2848 **Proposition B.5 (Entropy inequality).** — Let π be a reference measure on some probability space
2849 E . Let f be a function $E \rightarrow \mathbb{R}$, and $\gamma \in \mathbb{R}^+$. Then, for any non-negative measure μ on E , we have

$$\int f d\mu \leq \frac{1}{\gamma} \left[\log \left(\int e^{\gamma f} d\pi \right) + H(\mu | \pi) \right],$$

2850 where $H(\mu | \pi)$ is the relative entropy of μ with respect to π .

2851 *Proof of Proposition B.5.* — The proof is omitted, it can be found in Appendix 1.8 of [27]. \square

2852 **Remark B.6 (Utilization throughout the proof).** — This inequality is used throughout this proof
2853 with μ_s^N the marginal at time s of the measure of the process started from an initial profile μ^N , and with
2854 $\pi = \mu_{\hat{\alpha}}$ the equilibrium measure of a symmetric simple exclusion process with grand-canonical parameter
2855 $\hat{\alpha}$. Then, for any fixed time s and for any function f and any positive γ

$$\mathbb{E}_{\mu_s^N}(f) \leq \frac{1}{\gamma} \left[\log \mathbb{E}_{\hat{\alpha}}(e^{\gamma f}) + H(\mu_s^N | \mu_{\hat{\alpha}}) \right].$$

2856 This inequality will be our main tool to bound expectation w.r.t the measure of our process of vanishing
2857 quantities .

2858 **B.4. Bound on the largest eigenvalue of a perturbed Markov generator.** —

2859 **Proposition B.7 (Largest eigenvalue for a small perturbation of a Markov generator)**

2860 Let us consider a Markov Generator L with positive spectral gap γ and a bounded function V with
2861 mean 0 with respect to the equilibrium measure $\mu_{\hat{\alpha}}$ of the Markov process. Then, for any small $\varepsilon > 0$, the
2862 Largest eigenvalue of the operator $L + \varepsilon V$ can be bounded from above by

$$\sup_f \left\{ \varepsilon \mathbb{E}_{\hat{\alpha}}(V f^2) + \mathbb{E}_{\hat{\alpha}}(f L f) \right\} \leq \frac{\varepsilon^2}{A - 2\varepsilon\gamma \|V\|_\infty} \mathbb{E}_{\hat{\alpha}}(V(-L)^{-1}V),$$

2863 where the supremum in the variational formula is taken among the probability densities f w.r.t $\mu_{\hat{\alpha}}$.

2864 The proof of this result is omitted, it is given in Theorem A3.1.1, p.375 in [27].

2865

Appendix C

2866

Space of grand-canonical parameters

2867 In this appendix, we prove some useful results regarding the space of parameters $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$
2868 introduced in Section 3.1.

2869 **C.1. Equivalence of ensembles.** —

2870 **Proposition C.1 (Equivalence of ensembles).** — Let f be a cylinder function (in the sense of Def-
2871 inition 2.1), we have

$$\limsup_{l \rightarrow \infty} \sup_{\hat{K} \in \mathbb{K}_l} \left| \mathbb{E}_{l, \hat{K}}(f) - \mathbb{E}_{\hat{\alpha}_{\hat{K}}}(f) \right| \rightarrow 0,$$

2872 where the first measure is the projection along sets with \hat{K} particles in B_l , whereas the second is the
2873 grand-canonical measure with parameter $\hat{\alpha} = \hat{\alpha}_{\hat{K}}$ introduced in Definition 3.7.

2874 *Proof of Proposition C.1.* — The proof of this result is quite elementary, and is a matter of carefully
 2875 writing expectations for a random sampling with (grand-canonical measures) and without (canonical
 2876 measures) replacement.

2877 The proof of this problem can be reduced to the following : Consider two samplings of M occupation
 2878 variables, chosen among L fixed possible values

$$\{\widehat{\eta}^1, \dots, \widehat{\eta}^L\} \in \Sigma_1^L := \{(\delta, \theta) \in \{0, 1\} \times \mathbb{S}, \theta = 0 \text{ if } \delta = 0\}^L.$$

2879 The first sampling is made without replacement to represent the canonical measure $\mu_{l, \widehat{K}}$, and the sampled
 2880 items will be denoted X_1, \dots, X_M , where each X_i is of the form (δ, θ) . The second sampling is made
 2881 with replacement to represent the grand-canonical measure $\mu_{\widehat{\alpha}, \widehat{K}}$, and will be denoted Y_1, \dots, Y_M . let us
 2882 denote by ξ^L the set

$$\xi^L = \{\widehat{\eta}^1, \dots, \widehat{\eta}^L\},$$

2883 and denote by \mathbb{E}_{ξ^L} the expectation w.r.t. the two samplings (X_i) and (Y_i) given ξ^L . Further denote by
 2884 $I_{L,M} = \{1, \dots, L\}^M$, $\mathbf{i} = (i_1, \dots, i_M)$ the elements of $I_{L,M}$, and $D_{L,M}$ and $C_{L,M}$ its two subsets

$$D_{L,M} = \{(i_1, \dots, i_M) \in I_{L,M} \mid i_1 \neq \dots \neq i_M\}, \quad \text{and} \quad C_{L,M} = I_{L,M} \setminus D_{L,M}$$

2885 Then, for any function

$$g : \Sigma_1^M \rightarrow \mathbb{R},$$

we have

$$\begin{aligned} & \left| \mathbb{E}_{\xi^L}(g(X_1, \dots, X_M)) - \mathbb{E}_{\xi^L}(g(Y_1, \dots, Y_M)) \right| \\ & \leq \|g\|_{\infty} \sum_{\mathbf{i} \in I_{L,M}} \left| \mathbb{P}_{\xi^L} \left[(X_1, \dots, X_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] - \mathbb{P}_{\xi^L} \left[(Y_1, \dots, Y_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] \right| \\ & = \|g\|_{\infty} \sum_{\mathbf{i} \in D_{L,M}} \left| \mathbb{P}_{\xi^L} \left[(X_1, \dots, X_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] - \mathbb{P}_{\xi^L} \left[(Y_1, \dots, Y_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] \right| \\ & \quad + \|g\|_{\infty} \sum_{\mathbf{i} \in C_{L,M}} \mathbb{P}_{\xi^L} \left[(Y_1, \dots, Y_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right]. \end{aligned}$$

2886 The sum on the last line is the probability that at least two indexes among the M we chosen uniformly
 2887 in $\{1, \dots, L\}$ are equal. This probability is

$$\sum_{\mathbf{i} \in C_{L,M}} \mathbb{P}_{\xi^L} \left[(Y_1, \dots, Y_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] = 1 - \frac{L(L-1) \cdots (L-M+1)}{L^M},$$

which for M fixed vanishes uniformly in ξ^L as $L \rightarrow \infty$. We now take a look at the other term, for which
 we write

$$\begin{aligned} & \sum_{\mathbf{i} \in D_{L,M}} \left| \mathbb{P}_{\xi^L} \left[(X_1, \dots, X_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] - \mathbb{P}_{\xi^L} \left[(Y_1, \dots, Y_M) = (\widehat{\eta}^{i_1}, \dots, \widehat{\eta}^{i_M}) \right] \right| \\ & = \sum_{\mathbf{i} \in D_{L,M}} \left| \frac{1}{L(L-1) \cdots (L-M+1)} - \frac{1}{L^M} \right| \\ & = 1 - \frac{L(L-1) \cdots (L-M+1)}{L^M}, \end{aligned}$$

2888 which also vanishes uniformly in ξ^L as $L \rightarrow \infty$. We can therefore write for any bounded function g
 2889 depending on M sites

$$\sup_{\xi^L \in \Sigma_1^L} \left| \mathbb{E}_{\xi^L}(g(X_1, \dots, X_M)) - \mathbb{E}_{\xi^L}(g(Y_1, \dots, Y_M)) \right| \leq \|g\|_{\infty} C(M) o_L(1),$$

2890 thus proving Proposition C.1. □

2891 **C.2. Regularity of the grand-canonical measures in their parameter.** —

2892 **Proposition C.2.** — Consider the set of local profiles $\mathcal{M}_1(\mathbb{S})$ equipped with the norm $\|\cdot\|$ defined in
 2893 **Definition 3.2.** Then, given a function $g \in \mathcal{C}$, the application

$$\begin{aligned} \Psi_g &: (\mathcal{M}_1(\mathbb{S}), \|\cdot\|) \longrightarrow \mathbb{R} \\ &\quad \hat{\alpha} \longmapsto \mathbb{E}_{\hat{\alpha}}(g) \end{aligned}$$

2894 is Lipschitz-continuous with Lipschitz constant depending on the function g .

2895 **Proof of Proposition C.2.** — Let us consider a cylinder function g depending only on vertices x_1, \dots, x_M ,
 2896 and let us start by assuming that g vanishes as soon as one of the sites x_1, \dots, x_M is empty. We can then
 2897 rewrite $g(\hat{\eta})$ as $\eta_{x_1} \dots \eta_{x_M} g(\theta_{x_1}, \dots, \theta_{x_M})$, and

$$\mathbb{E}_{\hat{\alpha}}(g) = \int_{\theta_1} \dots \int_{\theta_M} g(\theta_{x_1}, \dots, \theta_{x_M}) d\hat{\alpha}(\theta_{x_1}) \dots d\hat{\alpha}(\theta_{x_M}).$$

We can now proceed by recurrence on M . Given a function g depending only on a site x_1 , and for any
 two grand-canonical parameters $\hat{\alpha}$ and $\hat{\alpha}'$ we can write

$$\mathbb{E}_{\hat{\alpha}}(g) - \mathbb{E}_{\hat{\alpha}'}(g) = \|g\|^* \int_{\theta_{x_1}} \frac{g(\theta_{x_1})}{\|g\|^*} d(\hat{\alpha} - \hat{\alpha}')(\theta_{x_1}) \leq \|g\|^* \|\hat{\alpha} - \hat{\alpha}'\|$$

Assuming now that the proposition is true for any function depending on $M - 1$ sites, and considering a
 function g depending on M vertices, we can write

$$(C.1) \quad \mathbb{E}_{\hat{\alpha}}(g) - \mathbb{E}_{\hat{\alpha}'}(g) = \mathbb{E}_{\hat{\alpha}}(\mathbb{E}_{\hat{\alpha}}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M})) - \mathbb{E}_{\hat{\alpha}'}(\mathbb{E}_{\hat{\alpha}'}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M})).$$

2898 Fix any angle θ , and let g^θ be the function $g^\theta(\hat{\eta}) = g(\theta, \theta_{x_2}, \dots, \theta_{x_M})$, we can write thanks to the recurrence
 2899 hypothesis that

$$|\mathbb{E}_{\hat{\alpha}}(g^\theta) - \mathbb{E}_{\hat{\alpha}'}(g^\theta)| \leq C_\theta \|\hat{\alpha} - \hat{\alpha}'\|,$$

2900 which, integrated in θ against $\hat{\alpha}'$, yields

$$|\mathbb{E}_{\hat{\alpha}'}(\mathbb{E}_{\hat{\alpha}'}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M})) - \mathbb{E}_{\hat{\alpha}'}(\mathbb{E}_{\hat{\alpha}}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M}))| \leq C_1 \|\hat{\alpha} - \hat{\alpha}'\|,$$

2901 On the other hand, we can also write

$$|\mathbb{E}_{\hat{\alpha}}(\mathbb{E}_{\hat{\alpha}}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M})) - \mathbb{E}_{\hat{\alpha}'}(\mathbb{E}_{\hat{\alpha}}(g \mid \hat{\eta}_{x_2}, \dots, \hat{\eta}_{x_M}))| \leq C_2 \|\hat{\alpha} - \hat{\alpha}'\|,$$

2902 therefore (C.1) yields that

$$|\mathbb{E}_{\hat{\alpha}}(g) - \mathbb{E}_{\hat{\alpha}'}(g)| \leq (C^1 + C^2) \|\hat{\alpha} - \hat{\alpha}'\|,$$

2903 which is what we wanted to show.

2904 To complete the proof of Proposition C.2, we now only need to extend the result to functions g which
 2905 do not necessarily vanish when one site in their domain is empty. This case is easily derived, since any
 2906 function g depending on vertices x_1, \dots, x_M can be rewritten

$$(C.2) \quad g(\hat{\eta}_{x_1}, \dots, \hat{\eta}_{x_M}) = \sum_{B \subset \{1, \dots, M\}} g_B(\theta_{x_i}, i \in B),$$

2907 where $g_B(\theta_{x_i}, i \in B)$ is defined in the following fashion : recall that $\hat{\eta}_x = (\eta_x, \theta_x)$, with $\theta_x = 0$ if $\eta_x = 0$,
 2908 and let us assume that B is the set of increasing indexes i_1, \dots, i_p , then g_B is defined as

$$g_B(\theta_{x_{i_1}}, \dots, \theta_{x_{i_p}}) = \eta_{x_{i_1}} \dots \eta_{x_{i_p}} g((0, 0), \dots, (0, 0), (1, \theta_{x_{i_1}}), (0, 0), \dots, (0, 0), (1, \theta_{x_{i_p}}), (0, 0), \dots, (0, 0)).$$

2909 These functions all vanish whenever one of their depending sites is empty, therefore according to the
 2910 beginning of the proof, there exists a family of constants C_B such that for any $B \subset \{1, \dots, M\}$ we have

$$|\mathbb{E}_{\hat{\alpha}}(g_B) - \mathbb{E}_{\hat{\alpha}'}(g_B)| \leq C_B \|\hat{\alpha} - \hat{\alpha}'\|.$$

2911 We now only need to let $C = \sum_{B \subset \{1, \dots, M\}} C_B$ to obtain thanks to the decomposition (C.2) that

$$|\mathbb{E}_{\hat{\alpha}}(g) - \mathbb{E}_{\hat{\alpha}'}(g)| \leq C \|\hat{\alpha} - \hat{\alpha}'\|$$

2912 as intended. This completes the proof of Proposition C.2. \square

2913 **C.3. Compactness of the set of grand-canonical parameters.** —

2914 **Proposition C.3 (Compactness of $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$).** — *The metric space $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$ introduced*
 2915 *in Definition 3.2 is totally bounded and Cauchy complete, and is therefore compact.*

2916 *Proof of Proposition C.3.* — The proof of the Cauchy-completeness is almost immediate, we treat it first.
 2917 Consider a Cauchy sequence $(\hat{\alpha}_k)_{k \in \mathbb{N}} \in \mathcal{M}_1(\mathbb{S})^{\mathbb{N}}$, then by definition of $\|\cdot\|$, for any $g \in B^*$, the sequence
 2918 $(\int_{\mathbb{S}} g(\theta) \hat{\alpha}_k(d\theta))_k$ is a real Cauchy sequence and therefore converges, and we can let

$$\int_{\mathbb{S}} g(\theta) \hat{\alpha}^*(d\theta) = \lim_{k \rightarrow \infty} \int_{\mathbb{S}} g(\theta) \hat{\alpha}_k(d\theta).$$

2919 This definition can be extended to any $C^1(\mathbb{S})$ function g by letting

$$\int_{\mathbb{S}} g(\theta) \hat{\alpha}^*(d\theta) = \max(\|g\|_{\infty}, \|g'\|_{\infty}) \lim_{k \rightarrow \infty} \int_{\mathbb{S}} \frac{g(\theta)}{\max(\|g\|_{\infty}, \|g'\|_{\infty})} \hat{\alpha}_k(d\theta).$$

2920 This defines a measure $\hat{\alpha}^*$ on \mathbb{S} , whose total mass is given by

$$\int_{\mathbb{T}^2} \hat{\alpha}^*(d\theta) = \lim_{k \rightarrow \infty} \int_{\mathbb{T}^2} \hat{\alpha}_k(d\theta) \in [0, 1],$$

2921 which proves the Cauchy completeness of $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$.

2922 We now prove that $(\mathcal{M}_1(\mathbb{S}), \|\cdot\|)$ is totally bounded. For any integer n , we are going to construct a
 2923 finite set $\mathcal{M}_{1,n} \subset \mathcal{M}_1(\mathbb{S})$ such that

$$\sup_{\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})} \inf_{\hat{\alpha}' \in \mathcal{M}_{1,n}} \|\hat{\alpha} - \hat{\alpha}'\| \leq \frac{1}{n}.$$

2924 For any $n \in \mathbb{N}$ and any $j \in \llbracket 0, n-1 \rrbracket$, we shorten $\theta_{j,n} = 2\pi j/n$, and $\theta_{n,n} = \theta_{0,n} = 0$. We can now define

$$\mathcal{M}_{1,n} = \left\{ \sum_{j=0}^{n-1} \frac{k_j}{n^2} \delta_{\theta_{j,n}} \mid k_j \in \llbracket 0, n^2 \rrbracket, \sum_j k_j \leq n^2 \right\}.$$

2925 The inclusion $\mathcal{M}_{1,n} \subset \mathcal{M}_1(\mathbb{S})$ is trivial thanks to the condition $\sum_j k_j \leq n^2$, and $\mathcal{M}_{1,n}$ is finite since the
 2926 k_j 's can each take only a finite number of values. we now prove that any $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$ is at distance at
 2927 most $1/n$ of an element $\hat{\alpha}_n \in \mathcal{M}_{1,n}$.

2928 Fix $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, and let

$$k_j = \lfloor n^2 \hat{\alpha}(\llbracket \theta_{j,n}, \theta_{j+1,n} \rrbracket) \rfloor.$$

2929 Since $\hat{\alpha} \in \mathcal{M}_1(\mathbb{S})$, its total mass is in $[0, 1]$, and the conditions $k_j \in \llbracket 0, n^2 \rrbracket$ and $\sum_j k_j \leq n^2$ are trivially
 2930 verified. We now let

$$\hat{\alpha}_n = \sum_{j=0}^{n-1} \frac{k_j}{n^2} \delta_{\theta_{j,n}},$$

and prove that $\|\hat{\alpha} - \hat{\alpha}_n\| \leq 2/n$. Fix a function $g \in C^1(\mathbb{S})$ such that $\max(\|g\|_{\infty}, \|g'\|_{\infty}) \leq 1$, we can write

$$\begin{aligned} \int_{\mathbb{S}} g(\theta) (\hat{\alpha} - \hat{\alpha}_n)(d\theta) &= \sum_{j=0}^{n-1} \int_{[\theta_{j,n}, \theta_{j+1,n}[} g(\theta) \hat{\alpha}(d\theta) - \frac{k_j}{n^2} g(\theta_{j,n}) \\ &= \sum_{j=0}^{n-1} \hat{\alpha}(\llbracket \theta_{j,n}, \theta_{j+1,n} \rrbracket) g(\theta_{j,n}) - \frac{k_j}{n^2} g(\theta_{j,n}) + \sum_{j=0}^{n-1} \int_{[\theta_{j,n}, \theta_{j+1,n}[} (g(\theta) - g(\theta_{j,n})) \hat{\alpha}(d\theta) \\ &\leq \sum_{j=0}^{n-1} \|g\|_{\infty} \underbrace{\left| \hat{\alpha}(\llbracket \theta_{j,n}, \theta_{j+1,n} \rrbracket) - \frac{k_j}{n^2} \right|}_{\leq 1/n^2} + \sum_{j=0}^{n-1} \|g'\|_{\infty} \underbrace{|\theta_{j+1,n} - \theta_{j,n}|}_{\leq 1/n} \int_{[\theta_{j,n}, \theta_{j+1,n}[} \hat{\alpha}(d\theta) \\ &\leq \frac{\|g\|_{\infty} + \|g'\|_{\infty}}{n} \leq 2/n. \end{aligned}$$

2931 Finally, we have proved that

$$\|\hat{\alpha} - \hat{\alpha}_n\| \leq 2/n,$$

2932 which proves that $\mathcal{M}_1(\mathcal{S})$ is totally bounded. This, together with the Cauchy completeness, immediately
 2933 yields the compactness, and concludes the proof of Proposition C.3.

2934

□

2935

References

- 2936 [1] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. *Function*
 2937 *Spaces, Differential Operators and Nonlinear Analysis*, pages 9–126, 1993.
- 2938 [2] H.C. Berg. *E. coli in Motion*. Biological and Medical Physics, Biomedical Engineering. Springer, 2004.
- 2939 [3] P. Billingsley. *Convergence of probability measures*. Wiley series in Probability and Statistics: Probability and
 2940 Statistics. John Wiley & Sons Inc., New York, second edition, 1999.
- 2941 [4] F. Bolley, J. A. Cañizo, and J. A. Carrillo. Stochastic mean-field limit: Non-lipschitz forces and swarming.
 2942 *Mathematical Models and Methods in Applied Sciences*, 21, September 2010.
- 2943 [5] F. Bolley, J. A. Cañizo, and J. A. Carrillo. Mean-field limit for the stochastic vicsek model. *Applied Mathe-*
 2944 *matics Letters*, 25:339–343, 2011.
- 2945 [6] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 1st edition,
 2946 November 2010.
- 2947 [7] E. Carlen, P. Degond, and B. Wennberg. Kinetic limits for pair-interaction driven master equations and
 2948 biological swarm models. *Mathematical Models and Methods in Applied Sciences*, 23:1339–1376, 2013.
- 2949 [8] J. A. Carrillo, Y. Huang, and S. Martin. Explicit flock solutions for quasi-morse potentials. *European Journal*
 2950 *of Applied Mathematics*, 25:553 – 578, October 2014.
- 2951 [9] M. E. Cates and J. Tailleur. Statistical mechanics of interacting run-and-tumble bacteria. *Physical Review*
 2952 *Letters*, 100, May 2008.
- 2953 [10] M. E. Cates and J. Tailleur. When are active brownian particles and run-and-tumble particles equivalent?
 2954 consequences for motility-induced phase separation. *EPL (Europhysics Letters)*, 101, January 2013.
- 2955 [11] M. E. Cates and J. Tailleur. Motility-induced phase separation. *Annual Review of Condensed Matter Physics*,
 2956 6:219–244, March 2015.
- 2957 [12] H. Chaté. Onset of collective and cohesive motion. *Physical review letter*, (92), January 2004.
- 2958 [13] A. De Masi, P. A. Ferrari, and J. L. Lebowitz. Reaction-diffusion equations for interacting particle systems.
 2959 *Journal of Statistical Physics*, 44, January 1986.
- 2960 [14] Anna De Masi and Pablo A. Ferrari. Separation versus diffusion in a two species system. *Brazilian Journal*
 2961 *of Probability and Statistics*, 29(2):387–412, May 2015.
- 2962 [15] P. Degond, A. Frouvelle, and J.-G. Liu. Macroscopic limits and phase transition in a system of self-propelled
 2963 particles. *Journal of Nonlinear Science*, 23(3):427–456, 2013.
- 2964 [16] P. Degond, A. Frouvelle, and J.-G. Liu. Phase transitions, hysteresis, and hyperbolicity for self-organized
 2965 alignment dynamics. *Archive for Rational Mechanics and Analysis*, 2014.
- 2966 [17] P. Degond, J.-G. Liu, S. Motsch, and V. Panferov. Hydrodynamic models of self-organized dynamics: deriva-
 2967 tion and existence theory. *Methods and Applications of Analysis*, 20:089–114, 2013.
- 2968 [18] P. Degond and S. Motsch. Continuum limit of self-driven particles with orientation interaction. *Mathematical*
 2969 *Models and Methods in Applied Sciences*, 18, October 2008.
- 2970 [19] P. Degond and T. Yang. Diffusion in a continuum model of self-propelled particles with alignment interaction.
 2971 *Mathematical Models and Methods in Applied Sciences*, 20:1459–1490, February 2010.
- 2972 [20] Nicolas Dirr, Marios G. Stamatakis, and Johannes Zimmer. Hydrodynamic limit of condensing two-species
 2973 zero range processes with sub-critical initial profiles. *Journal of Statistical Physics*, 168(4):794–825, Aug 2017.
- 2974 [21] Y. Fily and C. Marchetti. Athermal phase separation of self-propelled particles with no alignment. *Physical*
 2975 *Review Letters*, 108, June 2012.
- 2976 [22] A. Frouvelle. A continuum model for alignment of self-propelled particles with anisotropy and density-
 2977 dependent parameters. *Mathematical Models and Methods in Applied Sciences*, 22(7), December 2012.
- 2978 [23] A.M. Garcia, E. Rodemich, and H. Rumsey. A real variable lemma and the continuity of paths of some
 2979 gaussian processes. *Indiana Math. J.* 20, 565-578., 1978.
- 2980 [24] Giambattista Giacomin, Joel L. Lebowitz, and Rossana Marra. Macroscopic evolution of particle systems
 2981 with short and long range interactions. *Nonlinearity*, 13(6), 2000.
- 2982 [25] G. Grégoire and H. Chaté. Onset of collective and cohesive motion. *Physical Review Letters*, 92(2):025702,
 2983 January 2004.

- 2984 [26] Wm.W. Johnson and W.E. Story. notes on the "15" puzzle. *American Journal of Mathematics*, 2(4):pp.
2985 397–404, December 1879.
- 2986 [27] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der*
2987 *Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin,
2988 1999.
- 2989 [28] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible markov processes
2990 and applications to simple exclusions. *Communications in Mathematical Physics*, 104(1):1–19, 1986.
- 2991 [29] T. Komorowski, C. Landim, and S. Olla. *Fluctuations in Markov Processes. Time Symmetry and Martingale*
2992 *Approximation*. Springer, Berlin, 2012.
- 2993 [30] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur. Exact Hydrodynamic Description of
2994 Active Lattice Gases. *ArXiv e-prints*, January 2018.
- 2995 [31] C. Landim, S. Olla, and S.R.S. Varadhan. Symmetric simple exclusion process: Regularity of the self-diffusion
2996 coefficient. *Communications in Mathematical Physics*, 224(1):307–321, 2001.
- 2997 [32] A. Masi, P.A. Ferrari, S. Goldstein, and W.D. Wick. An invariance principle for reversible markov processes.
2998 applications to random motions in random environments. *Journal of Statistical Physics*, 55(3-4):787–855, 1989.
- 2999 [33] A. Okubo and S.A. Levin. *Diffusion and ecological problems : modern perspectives*. Interdisciplinary applied
3000 mathematics. Springer, New York, 2001.
- 3001 [34] J. K. Parrish and L. Edelstein-Keshet. Complexity, pattern, and evolutionary trade-offs in animal aggregation.
3002 *Science*, 284:99, April 1999.
- 3003 [35] J. Quastel. Diffusion of colour in the simple exclusion process. *Communications on Pure and Applied Math-*
3004 *ematics*, 45:623–679, 1992.
- 3005 [36] J. Quastel, F. Rezakhanlou, and S. R. S. Varadhan. Large deviations for the symmetric simple exclusion
3006 process in dimensions d . *Probability Theory and Related Fields*, 113(1):1–84, 1999.
- 3007 [37] M. Sasada. Hydrodynamic limit for exclusion processes with velocities. *Markov Processes and Related Rields*,
3008 October 2010.
- 3009 [38] M. Sasada. Hydrodynamic limit for two-species exclusion processes. *Stochastic Processes and their Applica-*
3010 *tion*, 120(4):494–521, April 2010.
- 3011 [39] A.B. Simas. Hydrodynamic limit for a boundary driven stochastic lattice gas model with many conserved
3012 quantities. *Journal of Statistical Physics*, 139(2):219–251, 2010.
- 3013 [40] A. P. Solon, M. E. Cates, and J. Tailleur. Active brownian particles and run-and-tumble particles: A com-
3014 parative study. *European Physical Journal Special Topics*, 224:1231, July 2015.
- 3015 [41] A. P. Solon, J.-B. Caussin, D. Bartolo, H. Chaté, and J. Tailleur. Pattern formation in flocking models: A
3016 hydrodynamic description. *Physical Review E*, *In press*, September 2015.
- 3017 [42] A. P. Solon, H. Chaté, and J. Tailleur. From phase to microphase separation in flocking models : the essential
3018 role of nonequilibrium fluctuations. *Physics Review E*, 92, 2015.
- 3019 [43] A. P. Solon and J. Tailleur. Flocking with discrete symmetry: The two-dimensional active Ising model.
3020 *Physical Review E*, 92(4):042119, October 2015.
- 3021 [44] H. Spohn. Tracer diffusion in lattice gases. *Journal of Statistical Physics*, 59(5-6):1227–1239, 1990.
- 3022 [45] H. Spohn. *Large scale dynamics of interacting particles*, volume 825. Springer Berlin, 1991.
- 3023 [46] A. G. Thompson, J. Tailleur, M. E. Cates, and R. A. Blythe. Lattice models of nonequilibrium bacterial
3024 dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2:29, February 2011.
- 3025 [47] J. Toner and Y. Tu. Long-range order in a two-dimensional dynamical XY model : how birds fly together.
3026 *Physical Review Letters*, 75, December 1995.
- 3027 [48] S. R. S. Varadhan. non-linear diffusion limit for a system with nearest-neighbor interactions ii. In *Asymptotic*
3028 *problems in probability theory : stochastic models and diffusion on fractals*, number 283 in Pitman Research
3029 Notes in Mathematics, pages 75–128. Springer-Verlag, 1994.
- 3030 [49] S. R. S. Varadhan. Regularity of self-diffusion coefficient. In *The Dynkin Festschrift: Markov processes and*
3031 *their applications*, number 34 in Progress in Probability, pages 387–397. Springer-Verlag, 1994.
- 3032 [50] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of
3033 self-driven particles. *Physical Review Letters*, 75:1226–1229, August 1995.
- 3034 [51] T. Vicsek and A. Zafeiris. Collective motion. *Physics Reports*, 517(3):71–140, 2012.