## SUPPLEMENTARY MATERIALS: paper "Template shape estimation"\*

Nina Miolane<sup>†</sup>, Susan Holmes<sup>‡</sup>, and Xavier Pennec<sup>†</sup>

These are the supplementary materials for the paper "Template shape estimation: correcting an asymptotic bias". We present the detailed proofs of the theorems. The precise statements of the theorems are given again in each section of this supplementary materials.

**SM1.** Notations. We denote Y the template shape and  $\hat{Y}$  its estimate. X is a point in the manifold M. We consider that X belongs to a principal orbit and we recall that the set of principal orbits is dense in M. We write  $Z = \pi(X)$  the projection of X in the shape space Q. We denote m the dimension of M, p the dimension of the principal orbits and q the dimension of the quotient space. Figure SM1 shows the elements  $Y, X, O_Z = O_X, Z = \pi(X)$ .



**Figure SM1.** Summary of the notations used in the proofs. Y is the template shape, X is a point in M, belonging to a principal orbit  $O_X$ .  $Z = \pi(X)$  is the projection of X in the shape space.

**Normal Coordinate Systems**. We often use Normal Coordinate Systems (NCS) to express the coordinates of tangent vectors. We refer to [SM2] for theoretical developments about the NCS and to [SM1] for Taylor expansions of differential geometric tensors in a NCS.

For example, we may consider a NCS centered at the point Y, with respect to the Riemannian metric of M. This NCS is valid on an open neighborhood of Y, that is start-shaped domain around Y. Moreover, we assume that M is geodesically complete : thus, this domain is equal to the whole manifold M, with the exception of the cut locus which is of null measure [SM2]. The Riemannian logarithm of a point X in the NCS at Y is denoted:  $\overrightarrow{YX}$ .

<sup>\*</sup>Accepted for publication in SIAM Journal of Imaging Science. Submitted to the editors on January 23, 2017. This work was funded by the virtual lab Inria@SiliconValley.

<sup>&</sup>lt;sup>†</sup>INRIA, Asclepios project-team, 2004 Route des Lucioles, 06902 Sophia Antipolis, France, nina.miolane@inria.fr <sup>‡</sup>Stanford University, Department of Statistics, Sequoia Hall, Serra Mall, Stanford CA

Asymptotic behavior for  $\sigma \to 0$ . We denote: (i)  $\Theta(\sigma^k)$  a function that is proportional to  $\sigma^k$ , (ii)  $\mathcal{O}(\sigma^k)$  a function equivalent to  $\sigma^k$  for  $\sigma \to 0$  and (iii)  $\epsilon(\sigma)$  a function that is exponentially decreasing for  $\sigma \to 0$ .

## SM2. Preliminaries.

**SM2.1. A first computation.** First, we show a technical result that will be used throughout the proofs. We show that the following integral on  $T_YM$ , the tangent space of M at Y:

$$\int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) d\overrightarrow{YX}$$

is a function  $\sigma \to \epsilon(\sigma)$ , i.e. decreases exponentially when  $\sigma \to 0$ . In the integral above, the notation  $\mathcal{C}_{B_r}$  denotes the complement in M of the geodesic ball  $B_r$  of center Y and of radius r.

We split the coordinates in  $T_Y M$  into  $(\rho, u)$  (polar coordinates):

$$\begin{split} |\int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dX|| &\leq K \int_{u \in S^m} \int_r^{\rho(u)} \rho^k \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \rho^{m-1} d\rho dS^m \\ &\leq K \int_{u \in S^m} \int_r^{\rho(u)} \rho^{m+k-1} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) d\rho dS^m \end{split}$$

where  $\rho(u)$  is the distance to the cutlocus in the direction u.

The positive integral is upper bounded by the same integral defined on the larger domain:

$$\left|\left|\int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dX\right|\right| \le K \int_{u \in S^m} \int_r^{+\infty} \rho^{m+k-1} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) d\rho dS^m$$

Integrating the volume of the unit hypersphere:

$$\left|\left|\int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dX\right|\right| \le K \int_r^{+\infty} \rho^{m+k-1} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) d\rho$$

where the right-hand-side is dominated by  $\exp(-\frac{r^2}{2\sigma^2})$  by the dominated convergence theorem. Therefore:

(SM1) 
$$\int_{\mathcal{C}_{B_r}} d_M(Y, X)^k \exp\left(-\frac{d_M^2(Y, X)}{2\sigma^2}\right) dX = \epsilon(\sigma)$$

i.e. decreases exponentially when  $\sigma \to 0$ .

SM2.2. Truncated Gaussian moments in a Euclidean space. We give preliminary computations of Gaussian moments in a *m*-dimensional vector space  $\mathbb{R}^m$ , using the curved notation  $\mathcal{M}$ . We refer to the *(unnormalized) moment of order k of the m-dimensional Gaussian* of covariance  $\sigma^2 A$  as:

$$\mathcal{M}^{i_1\dots i_k}(\sigma^2 A) = \int_{\mathbb{R}^m} X^{i_1}\dots X^{i_k} \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX$$

and to the truncated (unnormalized) moment at radius r of the m-dimensional Gaussian of covariance  $\sigma^2 A$  as:

$$\mathcal{M}_r^{i_1\dots i_k}(\sigma^2 A) = \int_{B_r} X^{i_1}\dots X^{i_k} \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX$$

where the integration domain is now the *m*-dimensional ball  $B_r$ .

SM2.2.1. First, we recall the expressions of some unnormalized Gaussian moments.. The order 0, 2 and 4 are:

$$\mathcal{M}^{0}(\sigma^{2}.A) = \sigma^{m}\sqrt{(2\pi)^{m}}\sqrt{\det(A)}$$
$$\mathcal{M}^{ab}(\sigma^{2}.A) = \sigma^{m+2}\sqrt{(2\pi)^{m}}\sqrt{\det(A)}.A^{ab}$$
$$\mathcal{M}^{abcd}(\sigma^{2}.A) = \sigma^{m+4}\sqrt{(2\pi)^{m}}\sqrt{\det(A)}.\left(A^{ab}A^{cd} + A^{ac}A^{bd} + A^{ad}A^{bc}\right)$$
$$\mathcal{M}^{i_{1}...i_{k}}(\sigma^{2}.A) = \Theta\left(\sigma^{m+k}\right)^{i_{1}...i_{k}} \quad \text{if } k \text{ even}$$
$$= 0 \quad \text{if } k \text{ odd}$$

where we recall that  $\Theta(\sigma^{m+k})$  denotes the proportionality to  $\sigma^{m+k}$ .

SM2.2.2. Second, we turn to the (unnormalized) truncated moments.. They write, with respect to the total moments:

$$\mathcal{M}_{r}^{i_{1}\dots i_{k}}(\sigma^{2}A) = \mathcal{M}^{i_{1}\dots i_{k}}(\sigma^{2}A) - \int_{\mathcal{C}_{B_{r}}} X^{i_{1}}\dots X^{i_{k}} \exp\left(-\frac{X^{T}A^{-1}X}{2\sigma^{2}}\right) dX$$

The second term of the sum is negligible when  $\sigma \to 0$ . To see this, we put an upper bound on its norm through triangular inequality:

$$||\int_{\mathcal{C}_{B_r}} X^{i_1} ... X^{i_k} \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX|| \le \int_{\mathcal{C}_{B_r}} ||X^{i_1}|| ... ||X^{i_k}|| \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX,$$

using triangular inequality on their coordinates:

$$\left|\left|\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX\right|\right| \le \int_{\mathcal{C}_{B_r}} \left||X||^k \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX,$$

and performing the change of variables  $X' = A^{-1/2}X$ , i.e. taking the matrix square root of the positive definite matrix  $A^{-1}$ :

$$\begin{split} ||\int_{\mathcal{C}_{B_{r}}} X^{i_{1}}...X^{i_{k}} \exp\left(-\frac{X^{T}A^{-1}X}{2\sigma^{2}}\right) dX|| &\leq \int_{C_{B_{|||A^{1/2}|||r}}} |||A|||^{k/2} ||X'||^{k} \exp\left(-\frac{X'^{T}X'}{2\sigma^{2}}\right) \det\left(A^{1/2}\right) dX, \\ &= |||A|||^{k/2} \det\left(A^{1/2}\right) \int_{\mathcal{C}_{B_{|||A^{1/2}|||r}}} ||X'||^{k} \exp\left(-\frac{||X'||^{2}}{2\sigma^{2}}\right) dX. \end{split}$$

By the computations in Subsection SM2.1, we have:

$$\left|\left|\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{X^T A^{-1} X}{2\sigma^2}\right) dX\right|\right| = \epsilon(\sigma)$$

Therefore, the truncated moments are equivalent to the (full) moments for  $\sigma \to 0$ :

(SM3) 
$$\mathcal{M}_r^{i_1\dots i_k}(\sigma^2 A) = \mathcal{M}^{i_1\dots i_k}(\sigma^2 A) + \epsilon(\sigma)$$

**SM2.3.** Isotropic Gaussian Moments on a Riemannian manifolds. We turn to computations of Gaussian moments in a *m*-dimensional Riemannian manifold M, using the notation  $\mathfrak{M}$ . We refer to the *(unnormalized) moment of order k of the m-dimensional isotropic Gaussian of covariance*  $\sigma^2 \mathbb{I}$  as:

$$\mathfrak{M}^{i_1\dots i_k}\left(\sigma^2\mathbb{I}\right) = \int_M \overrightarrow{YX}^{i_1}\dots\overrightarrow{YX}^{i_k}\exp\left(-\frac{d_M^2(X,Y)}{2\sigma^2}\right) dM(X)$$

and to the truncated (unnormalized) moment at radius r of the m-dimensional isotropic Gaussian of covariance  $\sigma^2 \mathbb{I}$  as:

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \int_{B_{r}} \overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}} \exp\left(-\frac{d_{M}^{2}(X,Y)}{2\sigma^{2}}\right) d\overrightarrow{YX}$$

where the integration domain is now the *m*-dimensional geodesic ball  $B_r$  of radius r and centered at Y.

SM2.3.1. First, we consider the truncated moments:.

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \int_{B_{r}} \overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}} \exp\left(-\frac{d_{M}^{2}(X,Y)}{2\sigma^{2}}\right) dM(X)$$

In a NCS at Y :  $d_M^2(X, Y) = \overrightarrow{YX}^T \overrightarrow{YX}$ :

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \int_{B_{r}} \overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}} \exp\left(-\frac{\overrightarrow{YX}^{T}\overrightarrow{YX}}{2\sigma^{2}}\right) dM(X)$$

On the small ball  $B_r$ ,  $dM(X) = d\overrightarrow{YX} + \frac{1}{6}R_{ab}(Y)\overrightarrow{YX}^a\overrightarrow{YX}^bd\overrightarrow{YX} + \mathcal{O}(||\overrightarrow{YX}||^3)d\overrightarrow{YX}$ :

$$\begin{split} \mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) &= \int_{B_{r}} \overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}}\exp\left(-\frac{\overrightarrow{YX}^{T}\overrightarrow{YX}}{2\sigma^{2}}\right) d\overrightarrow{YX} \\ &+ \frac{1}{6}\sum_{ab}R_{ab}(Y)\int_{B_{r}} \overrightarrow{YX}^{a}\overrightarrow{YX}^{b}.\overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}}\exp\left(-\frac{\overrightarrow{YX}^{T}\overrightarrow{YX}}{2\sigma^{2}}\right) d\overrightarrow{YX} \\ &+ \int_{B_{r}}O(||\overrightarrow{YX}||^{3})\overrightarrow{YX}^{i_{1}}\dots\overrightarrow{YX}^{i_{k}}\exp\left(-\frac{\overrightarrow{YX}^{T}\overrightarrow{YX}}{2\sigma^{2}}\right) d\overrightarrow{YX} \end{split}$$

Now we recognize the (un-normalized) moments of a truncated isotropic Gaussian a  $B_r$  in the vector space  $T_Y M \simeq \mathbb{R}^m$ . We replace them by the expressions given in the previous subsection:

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \left(\mathcal{M}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) + \epsilon(\sigma)\right) + \frac{1}{6}\sum_{ab}R_{ab}(Y)\left(\mathcal{M}^{abi_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) + \epsilon(\sigma)\right) + \mathcal{O}(\sigma^{m+k+4})$$

The sum of two functions that decrease exponentially for  $\sigma \to 0$  is a function that decreases exponentially for  $\sigma \to 0$ . So that:

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \mathcal{M}^{i_{1}\dots i_{k}}(\sigma^{2}\mathbb{I}) + \frac{1}{6}\sum_{ab}R_{ab}(Y)\mathcal{M}^{abi_{1}\dots i_{k}}(\sigma^{2}\mathbb{I}) + \mathcal{O}(\sigma^{m+k+4}) + \epsilon(\sigma)$$

We take the  $\sigma$ 's out:

(SM4)

$$\mathfrak{M}_{r}^{i_{1}\dots i_{k}}\left(\sigma^{2}\mathbb{I}\right) = \sigma^{k+m}\mathcal{M}^{i_{1}\dots i_{k}}(\mathbb{I}) + \frac{\sigma^{k+m+2}}{6}\sum_{ab}R_{ab}(Y)\mathcal{M}^{abi_{1}\dots i_{k}}(\mathbb{I}) + \mathcal{O}(\sigma^{m+k+4}) + \epsilon(\sigma)$$

SM2.3.2. Second, we consider the (full) moments.. They write, with respect to the non-truncated moments:

$$\mathfrak{M}^{i_1\dots i_k}\left(\sigma^2 \mathbb{I}\right) = \mathfrak{M}_r^{i_1\dots i_k}\left(\sigma^2 \mathbb{I}\right) + \int_{\mathcal{C}_{B_r}} X^{i_1}\dots X^{i_k} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X)$$

The second term of the sum is negligible when  $\sigma \to 0$ . To see this, we put an upper bound on its norm through triangular inequality:

$$||\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X)|| \le \int_{\mathcal{C}_{B_r}} ||X^{i_1}|| \dots ||X^{i_k}|| \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X) = \int_{\mathcal{C}_{B_r}} ||X^{i_k$$

then using triangular inequality on the coordinates:

$$||\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X)|| \le \int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X).$$

We assume that the Ricci curvature of M is bounded from below. Therefore, the measure dM has an upper bound with respect to the Lebesgue measure on the tangent space, which we write  $K: dM(X) \leq KdX$ :

$$\left|\left|\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X)\right|\right| \le K \int_{\mathcal{C}_{B_r}} d_M(Y,X)^k \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dX$$

By the computations in Subsection SM2.1, this inequality together with  $\sigma \to 0$  shows that:

(SM5) 
$$\int_{\mathcal{C}_{B_r}} X^{i_1} \dots X^{i_k} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dM(X) = \epsilon(\sigma)$$

i.e. is a function that decreases exponentially when  $\sigma \to 0$ .

Therefore, the (unnormalized) moment of order k of the m-dimensional isotropic Gaussian of covariance  $\sigma^2 \mathbb{I}$  writes:

$$\mathfrak{M}^{i_1\dots i_k}\left(\sigma^2 \mathbb{I}\right) = \sigma^{k+m} \mathcal{M}^{i_1\dots i_k}(\mathbb{I}) + \frac{\sigma^{k+m+2}}{6} \sum_{ab} R_{ab}(Y) \mathcal{M}^{abi_1\dots i_k}(\mathbb{I}) + \mathcal{O}(\sigma^{m+k+4}) + \epsilon(\sigma).$$

**SM3.** Proof of Theorem **SM1:** Induced probability density on shapes. In this section we prove the Theorem 1 of our paper. We recall Theorem 1 below.

**Theorem SM1.** The data  $X_i$ 's are generated in the finite-dimensional Riemannian manifold M following the model:  $X_i = Exp(g_i \cdot Y, \epsilon_i), i = 1...n$ , described in the paper. In this model: (i) the action of the finite dimensional Lie group G on M, denoted  $\cdot$ , is isometric, (ii) the parameter Y is the template shape in the shape space Q, (iii)  $\epsilon_i$  is the noise and follows a (generalization to manifolds of a) Gaussian of variance  $\sigma^2$ , see Section 1 of the paper.

Then, the probability distribution function f on the shapes of the  $X_i$ 's, i = 1...n, in the asymptotic regime on an infinite number of data  $n \to +\infty$ , has the following Taylor expansion around the noise level  $\sigma = 0$ :

$$f(Z) = \frac{1}{(\sqrt{2\pi}\sigma)^q} \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right) \left(F_0(Z) + \sigma^2 F_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma)\right)$$

where (i) Z denotes a point in the shape space Q, (ii)  $F_0$  and  $F_2$  are functions of Z involving the derivatives of the Riemannian tensor at Z and the derivatives of the graph G describing the orbit  $O_Z$  at Z, and (iii)  $\epsilon$  is a function of  $\sigma$  that decreases exponentially for  $\sigma \to 0$ .

We consider  $Z \in B_r$  i.e. in the geodesic ball of center Y and radius Y. We use a NCS at Z. The notations are summarized on Figure SM2. The reader can refer to this Figure along the proof.

The generative model implies the following Riemannian Gaussian distribution on the objects:

(SM7) 
$$f(Z) = \frac{1}{C_M(\sigma)} \exp\left(-\frac{d_M^2(X,Y)}{2\sigma^2}\right),$$

where  $C_M(\sigma)$  is the integration constant:

(SM8) 
$$C_M(\sigma) = \int_M \exp\left(-\frac{d_M^2(X,Y)}{2\sigma^2}\right) dM(X)$$

We compute the induced probability distribution f on shapes by integrating the distribution on the orbit of X out of f(Z):

$$f(Z) = \frac{1}{C_M(\sigma)} \int_{O_Z} \exp\left(-\frac{d_M^2(Y, X)}{2\sigma^2}\right) dO_Z(X).$$

(SM6)



**Figure SM2.** Summary of the notations used in the proof of Theorem 1. The vectors of the tangent space  $T_Z M$  are in red.

We start by dividing the integral:

$$f(Z) = \frac{1}{C_M(\sigma)} \cdot \left( \int_{O_Z \cap B_r} \exp\left( -\frac{d_M^2(Y, X)}{2\sigma^2} \right) dO_Z(X) + \int_{O_Z \cap \mathcal{C}_{B_r}} \exp\left( -\frac{d_M^2(Y, X)}{2\sigma^2} \right) dO_Z(X) \right)$$

In the parenthesis, we call the first integral  $I_Z(\sigma)$  and the second term is an  $\eta(\sigma)$ :

$$f(Z) = C_M(\sigma)^{-1} \cdot (I_Z(\sigma) + \eta(\sigma))$$

We compute the Taylor expansions of  $C_M(\sigma)^{-1}$  and  $I_Z(\sigma)$  for  $\sigma \to 0$  and show that  $\eta(\sigma) = \epsilon(\sigma)$ .

**SM3.1.** Taylor expansion of  $C_M(\sigma)^{-1}$ . The integration constant  $C_M(\sigma)$  is the moment of order 0 of the *m*-dimensional isotropic Gaussian of variance  $\sigma^2 \mathbb{I}$  in the *m*-dimensional manifold *M*. From Subsection SM2.3:

$$C_M(\sigma) = (\sqrt{2\pi}\sigma)^m + \frac{\sigma^{m+2}}{6} \sum_{ab} R_{ab}(Y)(\sqrt{2\pi})^m \delta_{ab} + \mathcal{O}(\sigma^{m+3}) + \epsilon(\sigma)$$
$$= (\sqrt{2\pi}\sigma)^m + \frac{\sigma^{m+2}}{6} R(Y)(\sqrt{2\pi})^m + \mathcal{O}(\sigma^{m+3}) + \epsilon(\sigma)$$

where  $\epsilon(\sigma)$  is exponentially decreasing wrt  $\sigma$  and R(Y) is the scalar curvature of M at Y. Its inverse:

$$C_M(\sigma)^{-1} = \left( (\sqrt{2\pi}\sigma)^m + \frac{(\sqrt{2\pi})^m}{6} R(Y) \sigma^{m+2} + \mathcal{O}(\sigma^{m+3}) + \epsilon(\sigma) \right)^{-1}$$
  
=  $(\sqrt{2\pi}\sigma)^{-m} \cdot \left( 1 + \frac{\sigma^2}{6} R(Y) + O(\sigma^3) + \epsilon(\sigma) \right)^{-1}$   
=  $(\sqrt{2\pi}\sigma)^{-m} \cdot \left( 1 - \frac{\sigma^2}{6} R(Y) + O(\sigma^3) + \epsilon(\sigma) \right)$ 

**SM3.2. Taylor expansion of**  $I_Z(\sigma)$ . Now we compute the integral  $I_Z(\sigma)$  involved in the formula of f(Z):

$$I_Z(\sigma) = \int_{O_Z \cap B_r} \exp\left(-\frac{d_M^2(Y, X)}{2\sigma^2}\right) dO(X)$$

We denote:  $B_{r_Z} = B_{r_Z}$ . We first perform the computations for any  $\sigma$ . We recall that r is fixed, and small enough in a sense made precise later.

**SM3.2.1. Computing**  $d_M^2(Y, X)$ . The first component of  $I_Z(\sigma)$  is  $d_M^2(Y, X)$ . We express the Taylor expansion of  $d_M^2$  for  $X \in O_Z$  close to Z.

First, we parameterize the point X. The points  $X, Z = \pi(X)$  and the orbit  $O_Z \subset M$  are illustrated on Figure SM1: the orbit  $O_Z = O_X$  is the blue circle in  $M = \mathbb{R}^2$  going through X and Z. The orbit  $O_Z$  can be seen in the tangent space  $T_Z M$  through the Logarithm map at Z. For r small enough, i.e. for X close enough to Z, we can locally represent  $O_Z$  in  $T_Z M$ as the graph of a smooth function G from  $T_Z O$  to  $N_Z O$ , using the vector  $u \in T_Z O_Z$  around u = 0:

$$I: T_Z O_Z \mapsto T_Z M = T_Z O_Z \oplus N_Z O_Z$$
$$u \mapsto x_u = (u, G(u))$$

The local graph  $u \to G(u)$  is illustrated on Figure SM1 and has the following Taylor expansion around u = 0 in the NCS at Z:

$$G(u)^{a} = \frac{1}{2}h^{a}_{bc}(Z)u^{b}u^{c} + G_{3}(Z)^{a}_{bcd}u^{b}u^{c}u^{d} + G_{4}(Z)^{a}_{bcde}u^{b}u^{c}u^{d}u^{e} + O(||u||^{5})$$

The 0-th and 1-th order derivatives of  $u \to G(u)$  are zero because the graph goes through Z and is tangent at  $T_Z O_Z$ . The second order derivative is by definition the second fundamental form h(Z) introduced in Section 2 of the paper: h(Z) represents the best quadratic approximation of the graph G. The third and fourth orders  $G_3(Z)$  and  $G_4(Z)$  are further refinements on the shape of the graph G around Z.

Second, we compute the Taylor expansion of  $d_M^2(Y, X)$  with respect to u:

$$d_M^2(Y,X) = d_M^2(Y,Z) + u^a L_a(Z) + u^a u^b M_{ab}(Z) + u^a u^b u^c P_{abc}(Z) + u^a u^b u^c u^d S_{abcd}(Z) + \mathcal{O}(||u||^5)$$

and our goal is to compute the different tensors.

Y and X are represented by their Riemannian Logarithms at Z:  $y = \text{Log}_Z Y$ , and  $x_u = \text{Log}_Z X$ . We also recall:  $u^T y = 0$ . Using these, we express the squared distance  $d_M^2(Y, X)$  in the NCS at Z. We use the formula p.23 in [SM1] with the notations  $\Delta x \leftarrow y - x_u$  and  $x \leftarrow x_u$ :

$$\begin{split} d_{M}^{2} &= d_{M}^{2}(\mathrm{Exp}_{Z}(x(u)), \mathrm{Exp}_{Z}y) \\ &= \delta_{ab}(y - x_{u})^{a}(y - x_{u})^{b} \\ &- \frac{60}{180}R_{cadb}x_{u}^{c}x_{u}^{d}(y - x_{u})^{a}(y - x_{u})^{b} \\ &- \frac{15}{180}x_{u}^{d}x_{u}^{e}\nabla_{a}R_{dbec}(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c} \\ &- \frac{3}{540}x_{u}^{e}x_{u}^{f}(44R_{gaeb}R_{gcfd} + 3\nabla_{ab}R_{ecfd})(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c}(y - x_{u})^{d} \\ &+ \frac{1}{54}x_{u}^{f}x_{u}^{g}R_{hafb}\nabla_{c}R_{hdge}(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c}(y - x_{u})^{d} \\ &+ \frac{1}{54}x_{u}^{f}x_{u}^{g}R_{afb}\nabla_{c}R_{hdge}(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{d}(y - x_{u})^{e} \\ &- \frac{30}{180}x_{u}^{c}x_{u}^{d}x_{u}^{e}\nabla_{c}R_{daeb} \\ &+ \frac{1}{180}x_{u}^{d}x_{u}^{e}x_{u}^{f}(8R_{gdea}R_{gbfc} - 9\nabla_{da}R_{ebfc})(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c} \\ &- \frac{5}{540}x_{u}^{e}x_{u}^{f}x_{u}^{g}(8R_{haeb}\nabla_{c}R_{hfgd} + 9R_{haeb}\nabla_{h}R_{fcgd} \\ &+ 20R_{haeb}\nabla_{f}R_{hcgd} - 6R_{hefa}\nabla_{b}R_{hcgd})(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c}(y - x_{u})^{d} \\ &+ \frac{1}{180}x_{u}^{c}x_{u}^{d}x_{u}^{e}x_{u}^{f}(8R_{gcda}R_{gefb} - 9\nabla_{cd}R_{eafb})(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c}(y - x_{u})^{d} \\ &+ \frac{1}{180}x_{u}^{d}x_{u}^{e}x_{u}^{f}x_{u}^{g}(4R_{hadb}\nabla_{e}R_{hfgc} + 4R_{hdea}\nabla_{b}R_{hfgc} + 4R_{hdea}\nabla_{f}R_{hbgc} \\ &+ 3\nabla_{dea}R_{fbgc})(y - x_{u})^{a}(y - x_{u})^{b}(y - x_{u})^{c} \end{split}$$

We express this in orders of  $x_u$ :

$$\begin{split} d_{Af}^{2} &= \delta_{ab}y^{a}y^{b} \\ &= 2\delta_{ab}y^{a}x_{a}^{b} \\ &= \frac{1}{3}R_{cadb}x_{a}^{c}x_{a}^{b}y^{a}y^{d} \\ &+ \frac{1}{12}x_{a}^{b}x_{a}^{c}x_{a}^{b}(4R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})y^{a}y^{b}y^{c}y^{d} \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{b}(4R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})y^{a}y^{b}y^{c}y^{d} \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{b}(4R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})y^{a}y^{b}y^{c}y^{d} \\ &- \frac{1}{12}x_{a}^{d}x_{a}^{c}x_{a}^{b}R_{adb}x_{a}^{b}y^{b} \\ &+ \frac{2}{3}x_{a}^{c}x_{a}^{b}R_{adb}x_{a}^{b}y^{b} \\ &+ \frac{2}{3}x_{a}^{c}x_{a}^{b}R_{adb}x_{a}^{b}y^{b} \\ &+ \frac{2}{3}x_{a}^{c}x_{a}^{b}R_{adb}x_{a}^{b}y^{b} \\ &+ \frac{1}{12}x_{a}^{d}x_{a}^{c}\sqrt{a}R_{dbcc}(x_{a}^{a}y^{b}y^{c} + 2y^{a}y^{b}x_{a}^{c}y^{d}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{b}(R_{adb}x_{a}^{b}y^{b}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}(R_{adb}y^{b}y^{b}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}(R_{adb}y^{b}y^{b}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}(R_{adb}y^{b}y^{b}) \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}(R_{adb}y^{b}x_{a}^{b}y^{b}) \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}(R_{adb}y^{b}x_{a}^{b}y^{b}) \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{b}x_{a}^{b} \\ &- \frac{1}{12}x_{a}^{d}x_{a}^{c}x_{a}^{b}(R_{adb}y^{c}x_{a}^{b}y^{b} + y^{a}x_{a}^{b}x_{a}^{c}) \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{b}x_{a}^{b}x_{a}^{b} \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{b}x_{a}^{b}y^{c} + 3\nabla_{a}R_{ccfd})(2x_{a}^{a}x_{a}^{b}y^{c}y^{d} + 2y^{a}x_{a}^{b}x_{a}^{c}y^{d} + 4x_{a}^{a}y^{b}x_{a}^{c}y^{d}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{d}x_{a}^{b}x_{a}^{b} \\ &- \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{b}x_{a}^{b}y^{c} + 3\nabla_{a}R_{ccfd})(2x_{a}^{a}x_{a}^{b}y^{c}y^{d} + 2y^{a}x_{a}^{b}x_{a}^{c}y^{d} + 2y^{a}y^{b}x_{a}^{c}x_{a}^{d} + 4x_{a}^{a}y^{b}x_{a}^{c}y^{d}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{c}x_{a}^{d}x_{a}^{b}x_{a}^{b}y^{c}y^{d} + y^{a}x_{a}^{b}x_{a}^{b}y^{c}y^{d} + y^{a}y^{b}x_{a}^{c}y^{d}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{d}(8R_{adb}R_{b}g_{c} - 9\nabla_{d}R_{cbc})(x_{a}^{a}y^{b}y^{c}y^{c} + 2y^{a}x_{a}^{b}x_{a}^{c}) \\ &+ \frac{1}{180}x_{a}^{b}x_{a}^{b}x_{a}^{c}x_{a}^{c}(8R_{gd$$

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$$\begin{split} &\frac{d_{M}}{dt} = \frac{d_{M}^{2}(Y,Z) - 2\delta_{ab}y^{a}u^{b}}{2\delta_{ab}y^{b}c^{b}u^{b}y^{b}y^{b}y^{b}y^{b}y^{c}} - \frac{1}{180}u^{a}u^{f}(44R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})y^{a}y^{b}y^{b}y^{d} \\ &- \frac{1}{54}u^{f}u^{g}R_{hafb}\nabla_{c}R_{hafgc}y^{b}y^{b}y^{b}y^{b}y^{c}y^{d} \\ &- \frac{2}{3}R_{cadb}u^{c}G(u)^{b}y^{c}y^{d} + \frac{2}{12}u^{d}G(u)^{c}\nabla_{a}R_{dbcc}y^{b}y^{b}y^{c} - \frac{2}{180}u^{c}G(u)^{f}(44R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})y^{a}y^{b}y^{b}y^{b}y^{d} \\ &- \frac{2}{3}R_{cadb}u^{c}G(u)^{b}Q_{hafb}\nabla_{c}R_{hafgc}y^{b}y^{b}y^{b}y^{b}y^{b}y^{c} + \frac{2}{3}u^{c}u^{d}R_{cadb}u^{b}y^{b} - \frac{1}{12}u^{d}u^{c}\nabla_{a}R_{dbcc}(u^{a}y^{b}y^{c} + 2y^{a}y^{b}u^{c}) \\ &- \frac{1}{180}u^{c}u^{f}(44R_{gacb}R_{gcfd} + 3\nabla_{ab}R_{ccfd})(2u^{b}y^{b}y^{b}y^{d} + 2y^{a}y^{b}u^{c}y^{d}) \\ &+ \frac{1}{44}u^{2}R_{hafb}\nabla_{c}R_{hafgc}(4u^{a}y^{b}y^{c}y^{f}y^{c} + y^{a}y^{b}u^{c}y^{d}y^{c}) \\ &- \frac{1}{180}u^{c}u^{f}u^{c}(8R_{hacb}\nabla_{c}R_{hfgd} + 9R_{hacb}\nabla_{b}R_{fcd} + 20R_{hacb}\nabla_{f}R_{hcgd})y^{b}y^{b}y^{c} \\ &- \frac{1}{180}u^{c}u^{f}u^{c}(8R_{hacb}\nabla_{c}R_{hfgd} + 9R_{hacb}\nabla_{b}R_{fcd} + 20R_{hacb}\nabla_{f}R_{hcgd})y^{b}y^{b}y^{c} \\ &- \frac{1}{180}u^{c}u^{f}u^{c}(8R_{hacb}\nabla_{c}R_{hfgd} + 9R_{hacb}\nabla_{b}R_{hcgd})y^{b}y^{b}y^{c} \\ &- \frac{1}{180}G(u)^{c}G(u)^{f}(u)^{d}y^{b}y^{d} + \frac{1}{12}G(u)^{d}G(u)^{c}\nabla_{a}R_{dbcg}y^{b}y^{b}y^{b}y^{c} \\ &- \frac{1}{180}G(u)^{c}G(u)^{d}y^{a}y^{d} + \frac{1}{12}G(u)^{d}G(u)^{c}\nabla_{a}R_{dbcg}y^{b}y^{b}y^{b}y^{d} \\ &- \frac{1}{12}u^{d}u^{c}\nabla_{a}R_{dbcg}(d^{c}(u)^{a}y^{b}y^{b}y^{c}y^{d}y^{c} + \frac{2}{3}u^{c}u^{d}R_{codb}G(u)^{a}y^{b} \\ &- \frac{1}{180}u^{c}u^{f}(4R_{gacb}R_{gcfd} + 3\nabla_{a}R_{bcfd})(2G(u)^{a}y^{b}y^{c}y^{d} \\ &- \frac{1}{180}u^{c}u^{f}(4R_{gacb}R_{gcfd} + 3\nabla_{a}R_{bcfd})(2G(u)^{a}y^{b}y^{c}y^{d} + 2y^{a}y^{b}G(u)^{c}y^{d} \\ &- \frac{1}{180}u^{c}u^{f}(4R_{gacb}R_{gcfd} + 3\nabla_{a}R_{bcfd})(2G(u)^{a}y^{b}y^{c}y^{d} \\ &- \frac{1}{180}u^{c}u^{f}(4R_{gacb}R_{gcfd} + 3\nabla_{a}R_{bcfd})(2u^{a}y^{b}y^{c}y^{d} + 2y^{a}y^{b}G(u)^{c}y^{d} \\ &- \frac{1}{180}u^{c}u^{f}(d)(y^{c}(8R_{hacb}\nabla_{c}R_{hfgd} + 9R_{hacb}\nabla_{h}R_{fcgd} + 20R_{hacb}\nabla_{f}R_{hcgd} - 6R_{hcfa}\nabla_{b}R_{hcgd})y^{a}y^{b}y^{$$

We replace G(u) by its Taylor expansion. Identifying the tensors gives:

(SM9)  
$$L_{a}(Z) = 0$$
$$M_{ab}(Z) = -y^{c}h_{ab}(Z)^{c} - \frac{1}{3}R_{acbd}(Z)y^{c}y^{d} + \frac{1}{12}\nabla_{d}R_{aebc}(Z)y^{d}y^{e}y^{c}$$
$$- \frac{1}{180}(44R_{geaf}(Z)R_{gcbd}(Z) + 3\nabla_{ef}R_{acbd}(Z))y^{e}y^{f}y^{c}y^{d}$$
$$- \frac{1}{54}R_{hfag}(Z)\nabla_{c}R_{hdbe}(Z)y^{f}y^{g}y^{c}y^{d}y^{e}$$

and  $P_{abc}(Z)$  and  $S_{abcd}(Z)$  are tensors mixing the derivatives of the graph G and the derivatives of the Riemannian curvature R of M at Z.

**SM3.2.2. Computing**  $dO_Z(X)$ . The second component of  $I_Z(\sigma)$  is the measure of the orbit  $dO_Z(X)$ . We seek the Taylor expansion of the measure:

(SM10) 
$$dO_Z(X) = \left(1 + T_c(Z)u^c - N_{cb}(Z)u^b u^c + \mathcal{O}(||u||^3)\right) du$$

and our goal is, again, to express the tensors  $T_c(Z)$  and  $N_{cb}(Z)$ .

The measure  $dO_Z(X)$  is the restriction of the measure dM(X):

$$dO_Z(X) = dM(X)|_{T_X O_Z}$$

and we know that for X close enough to Z:  $dM(X) = dx_u - \frac{1}{6} \operatorname{Ric}_{ab}(Z) x_u^a x_u^b dx_u$ . Thus:

$$dO_Z(X) = \left(1 - \frac{1}{6}\operatorname{Ric}_{ab}(Z)x_u^a x_u^b + \mathcal{O}(||x_u||^3)\right) dx_u$$

We replace  $x_u^a$  by  $x_u^a = u^a + \frac{1}{2}h_{bc}^a(Z)u^bu^c + O(||u||^4)$ 

$$dO_Z(X) = \left(1 - \frac{1}{6}\operatorname{Ric}_{ab}(Z)u^a u^b + \mathcal{O}(||u||^3)\right) dx_u$$

We express  $dx_u$  with respect to du:

$$dx_u = \det\left(\frac{dx_u^a}{du^b}\right) du$$
  
=  $\det\left(\frac{d\left(u^a + \frac{1}{2}h_{bc}^a(Z)u^bu^c + O(||u||^4)\right)}{du^b}\right) du$   
=  $\det\left(\delta_b^a + \frac{1}{2}h_{cb}^a(Z)u^c + \frac{1}{2}h_{bc}^a(Z)u^c + O(||u||^3)\right) du$   
=  $\det\left(\delta_b^a + h_{cb}^a(Z)u^c + O(||u||^3)\right) du$ 

Developing the determinant:

$$dx_u = \left(1 + h_{ca}^a(Z)u^c + \frac{1}{2}\left((h_{ca}^a(Z)u^c)^2 - h_{cb}^a(Z)u^c h_{da}^b(Z)u^d\right) + \mathcal{O}(||u||^3)\right)du$$

Plugging in  $dO_Z(X)$ :

$$dO_Z(X) = \left(1 - \frac{1}{6} \operatorname{Ric}_{ab}(Z) u^a u^b + \mathcal{O}(||u||^3)\right) \left(1 + h^a_{ca}(Z) u^c\right) \\ + \left(1 - \frac{1}{6} \operatorname{Ric}_{ab}(Z) u^a u^b + \mathcal{O}(||u||^3)\right) \left(\frac{1}{2} \left((h^a_{ca}(Z) u^c)^2 - h^a_{cb}(Z) u^c h^b_{da}(Z) u^d\right)\right) \\ + \left(1 - \frac{1}{6} \operatorname{Ric}_{ab}(Z) u^a u^b + \mathcal{O}(||u||^3)\right) \left(\mathcal{O}(||u||^3)\right) du$$

We develop by keeping up to the quadratic terms only:

$$dO_Z(X) = \left(1 - \frac{1}{6} \operatorname{Ric}_{ab}(Z) u^a u^b + h^a_{ca}(Z) u^c + \frac{1}{2} \left( (h^a_{ca}(Z) u^c)^2 - h^a_{cb}(Z) u^c h^b_{da}(Z) u^d \right) + \mathcal{O}(||u||^3) \right) du$$

We reorganize the terms, relabeling the mute labels:

$$dO_Z(X) = \left(1 + h_{ca}^a(Z)u^c - \frac{1}{6}\operatorname{Ric}_{cb}(Z)u^c u^b + \frac{1}{2}\left(h_{ca}^a(Z)u^c \cdot h_{ca}^a(Z)u^b - h_{cd}^a(Z)u^c h_{ba}^d(Z)u^b\right) + \mathcal{O}(||u||^3)\right)du$$

Now we can factorize the quadratic terms:

$$dO_Z(X) = \left(1 + h_{ca}^a(Z)u^c - u^b u^c \left(\frac{1}{6}\operatorname{Ric}_{cb}(Z) + \frac{1}{2}h_{ca}^a(Z)h_{ca}^a(Z) - \frac{1}{2}h_{cd}^a(Z)h_{ba}^d(Z)\right) + \mathcal{O}(||u||^3)\right) du$$

So that we find the expressions of the tensors:

(SM11)  
$$T_{c}(Z) = h_{ca}^{a}(Z)$$
$$N_{cb}(Z) = \frac{1}{6} \operatorname{Ric}_{cb}(Z) + \frac{1}{2} h_{ca}^{a}(Z) h_{ca}^{a}(Z) - \frac{1}{2} h_{cd}^{a}(Z) h_{ba}^{d}(Z)$$

**SM3.2.3. Gathering to compute**  $I_Z(\sigma)$ . We plug the expressions of  $d_M^2(Y, X)$  and  $dO_Z(X)$ , computed in the preivous subsections, in the expression of  $I_Z(\sigma)$ :

$$I_Z(\sigma) = \int_{B_{r_Z}} \exp\left(-\frac{d_M^2(Y,X)}{2\sigma^2}\right) dO_Z(X)$$

Plugging the squared distance  $d^2_{\cal M}(Y,X)$  first:

$$I_{Z}(\sigma) = \int_{B_{r_{Z}}} \exp\left(-\frac{d_{M}^{2}(Y,Z) + u^{a}u^{b}M_{ab}(Z) + u^{a}u^{b}u^{c}P_{abc}(Z) + u^{a}u^{b}u^{c}u^{d}S_{abcd}(Z) + \mathcal{O}(||u||^{5})}{2\sigma^{2}}\right) dO_{Z}(X)$$

We split the exponential and extract the part of the exponential that does not depend on u:

$$I_Z(\sigma) = \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right) \int_{B_{r_Z}} \exp\left(-\frac{u^a u^b M_{ab}(Z)}{2\sigma^2}\right) \exp\left(-\frac{u^a u^b u^c P_{abc}(Z) + u^a u^b u^c u^d S_{abcd}(Z) + \mathcal{O}(||u||^5)}{2\sigma^2}\right) d\mathcal{O}(||u||^5)$$

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We perform the Taylor expansion of the term with  $\mathcal{O}(||u||^4)$ , recalling that at this point,  $\sigma$  can still be anything.

$$I_{Z}(\sigma) = \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right) \int_{B_{r_{Z}}} \exp\left(-\frac{u^{a}u^{b}M_{ab}(Z)}{2\sigma^{2}}\right) \left(1 - \frac{u^{a}u^{b}u^{c}P_{abc}(Z) + u^{a}u^{b}u^{c}u^{d}S_{abcd}(Z) + \mathcal{O}(||u||^{5})}{2\sigma^{2}}\right) dO_{Z}$$

Now we plug the  $dO_Z(X)$ :

$$I_{Z}(\sigma) = \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right) \int_{B_{r_{Z}}} \exp\left(-\frac{u^{a}u^{b}M_{ab}(Z)}{2\sigma^{2}}\right) \left(1 - \frac{u^{a}u^{b}u^{c}P_{abc}(Z) + u^{a}u^{b}u^{c}u^{d}S_{abcd}(Z)}{2\sigma^{2}} + \frac{\mathcal{O}(||u||^{5})}{2\sigma^{2}}\right).$$
$$\cdot \left(1 + T_{c}u^{c}(Z) - N_{cb}(Z)u^{b}u^{c} + \mathcal{O}(||u||^{3})\right) du$$

We develop the product of the parenthesis on the right:

$$\begin{split} I_{Z}(\sigma) &= \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right) \int_{B_{r_{Z}}} \exp\left(-\frac{u^{a}u^{b}M_{ab}(Z)}{2\sigma^{2}}\right) \left(1 + T_{c}u^{c}(Z) - N_{cb}(Z)u^{b}u^{c} + \frac{u^{a}u^{b}u^{c}P_{abc}(Z)}{2\sigma^{2}}\right) du \\ &+ \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right) \int_{B_{r_{Z}}} \exp\left(-\frac{u^{a}u^{b}M_{ab}(Z)}{2\sigma^{2}}\right) \left(\mathcal{O}(||u||^{4}) - \frac{u^{a}u^{b}u^{c}u^{d}S_{abcd}(Z)}{2\sigma^{2}} + \frac{\mathcal{O}(||u||^{5})}{2\sigma^{2}}\right) du \end{split}$$

By skew symmetry, the terms in  $T_c u^c(Z)$  and  $u^a u^b u^c P_{abc}(Z)$  integrate to 0. Moreover,  $\mathcal{O}(||u||^3)$ ,  $\mathcal{O}(||u||^5)$  become  $\mathcal{O}(||u||^4)$ ,  $\mathcal{O}(||u||^6)$ :

$$\begin{split} I_{Z}(\sigma) &= \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right) \int_{B_{r_{Z}}} \exp\left(-\frac{u^{a}u^{b}M_{ab}(Z)}{2\sigma^{2}}\right) \\ &\left(1 - N_{cb}(Z)u^{b}u^{c} + \mathcal{O}(||u||^{4}) - \frac{u^{a}u^{b}u^{c}u^{d}S_{abcd}(Z)}{2\sigma^{2}} + \frac{\mathcal{O}(||u||^{6})}{2\sigma^{2}}\right) du \end{split}$$

We recognize the unnormalized truncated Gaussian moments of in  $\mathbb{R}^p$ , where p is the dimension of the orbit  $O_Z$ , see Subsection SM2.2:

$$I_{Z}(\sigma) = \exp\left(-\frac{d_{M}^{2}(Y,Z)}{2\sigma^{2}}\right)$$
$$\cdot \left(\mathcal{M}_{r_{Z}}(\sigma^{2}M^{-1}) - N_{cb}(Z)\mathcal{M}_{r_{Z}}^{bc}(\sigma^{2}M^{-1}) + \mathcal{O}(\sigma^{p+4}) - \frac{S_{abcd}(Z)}{2\sigma^{2}}\mathcal{M}_{r_{Z}}^{abcd}(\sigma^{2}M^{-1}) + \frac{\mathcal{O}(\sigma^{p+6})}{2\sigma^{2}}\right)$$

We express them in terms of the unnormalized Gaussian moments in  $\mathbb{R}^p$ , see Subsection SM2.2:

$$I_Z(\sigma) = \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right).$$
$$\cdot \left(\sigma^p \mathcal{M}(M^{-1}) + N_{cb}(Z)\sigma^{p+2}\mathcal{M}(M^{-1})^{bc} - \frac{S_{abcd}}{2\sigma^2}\sigma^{p+4}\mathcal{M}(M^{-1})^{abcd} + \mathcal{O}(\sigma^{p+4}) + \frac{\mathcal{O}(\sigma^{p+6})}{\sigma^2} + \epsilon(\sigma)\right)$$

We simplify the  $\sigma$ 's:

$$I_Z(\sigma) = \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right).$$
$$\cdot \left(\sigma^p \mathcal{M}(M^{-1})^0 + \sigma^{p+2}\left(N_{cb}\mathcal{M}(M^{-1})^{bc} - \frac{S_{abcd}(Z)}{2}\mathcal{M}(M^{-1})^{abcd}\right) + \mathcal{O}(\sigma^{p+4}) + \epsilon(\sigma)\right)$$

For convenience in the later computation, we define the notations:

$$I_Z(\sigma) = \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right) \left(\sigma^p m_0(Z) + \sigma^{p+2} m_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma)\right)$$

where:

(SM12)  
$$m_0(Z) = \mathcal{M}(M)^0 = \sqrt{(2\pi)^p} \sqrt{\det\left((M_{ab}(Z)^{-1})\right)}$$
$$m_2(Z) = N_{cb}(Z) \mathcal{M}(M^{-1}(Z))^{bc} - \frac{1}{2} S_{abcd}(Z) \mathcal{M}(M^{-1}(Z))^{abcd}$$

where  $M_{ab}$  and  $S_{abcd}$  are given in the previous subsections.

**SM3.3. Upper bound on**  $\eta(\sigma)$ . We proceed with an adaptation of the method in Subsection SM2.1:

$$\eta(\sigma) = \int_{O_Z \cap \mathcal{C}_{B_r}} \exp\left(-\frac{d_M^2(Y, X)}{2\sigma^2}\right) dO(X)$$

Assuming that the Ricci curvature of the orbit is bounded by below, the measure of the orbit is bounded by above, by a constant that we write  $K_O$ :

$$||\eta(\sigma)|| \le K_O \int_{O_Z \cap \mathcal{C}_{B_r}} \exp\left(-\frac{d_M^2(Y, X)}{2\sigma^2}\right) du$$

We integrate on the orbit by filiating it with hyperspheres of radii  $\rho$ :

$$\begin{aligned} ||\eta(\sigma)|| &\leq K_O \int_r^{+\infty} \int_{S_\rho \cap O_Z} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) d(S_\rho \cap O_Z) d\rho \\ &= K_O \int_r^{+\infty} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \operatorname{Vol}(S_\rho \cap O_Z) d\rho \end{aligned}$$

Now the volume of  $S_{\rho} \cap O_Z$  is polynomial in  $\rho$ . We denote P this polynom:

$$||\eta(\sigma)|| \le K_O \int_r^{+\infty} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) P(\rho) d\rho$$

By dominated convergence theorem, the right-hand-side is dominated by  $\exp(-\frac{r^2}{2\sigma^2})$ . Thus:

(SM13) 
$$\eta(\sigma) = \epsilon(\sigma)$$

i.e.  $\sigma \to \eta(\sigma)$  is exponentially decreasing when  $\sigma \to 0$ .

SM3.4. Final result: Taylor expansion of f(Z). Replacing the terms:

$$f(Z) = \frac{\left(1 - \frac{\sigma^2}{6}R(Y) + O(\sigma^3) + \epsilon(\sigma)\right)}{(\sqrt{2\pi}\sigma)^m} \left(\exp\left(-\frac{d_M^2(Y, Z)}{2\sigma^2}\right)\left(\sigma^p m_0(Z) + \sigma^{p+2}m_2(Z) + \mathcal{O}(\sigma^{p+4}) + \epsilon(\sigma)\right) + \epsilon(\sigma)\right)$$

We put the  $\epsilon$  inside the main parenthesis:

$$f(Z) = \frac{\left(1 - \frac{\sigma^2}{6}R(Y) + O(\sigma^3) + \epsilon(\sigma)\right)}{(\sqrt{2\pi}\sigma)^m}.$$
  
$$\cdot \exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right) \left(\sigma^p m_0(Z) + \sigma^{p+2}m_2(Z) + \mathcal{O}(\sigma^{p+4}) + \epsilon(\sigma) + \exp\left(+\frac{d_M^2(Y,Z)}{2\sigma^2}\right)\epsilon(\sigma)\right)$$

We recall that  $Z \in B_r$  so that:

$$f(Z) = \frac{\left(1 - \frac{\sigma^2}{6}R(Y) + O(\sigma^3)\right)}{(\sqrt{2\pi}\sigma)^m} \cdot \exp\left(-\frac{d_M^2(Y, Z)}{2\sigma^2}\right) \left(\sigma^p m_0(Z) + \sigma^{p+2}m_2(Z) + \mathcal{O}(\sigma^{p+4}) + \epsilon(\sigma)\right)$$

We define:

$$f_Q(Z) = \frac{\exp\left(-\frac{d_M^2(Y,Z)}{2\sigma^2}\right)}{(\sqrt{2\pi}\sigma)^q}$$

and put it in the front, remembering that m = p + q:

$$f(Z) = f_Q(Z) \frac{\left(1 - \frac{\sigma^2}{6}R(Y) + O(\sigma^3)\right)}{(\sqrt{2\pi}\sigma)^p} \cdot \left(\sigma^p m_0(Z) + \sigma^{p+2}m_2(Z) + \mathcal{O}(\sigma^{p+4}) + \epsilon(\sigma)\right)$$

We divide by  $\sigma^p$ :

$$f(Z) = f_Q(Z) \left( 1 - \frac{\sigma^2}{6} R(Y) + O(\sigma^3) \right) \cdot \left( \frac{m_0(Z)}{(\sqrt{2\pi})^p} + \sigma^2 \frac{m_2(Z)}{(\sqrt{2\pi})^p} + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right)$$

We develop everything except the Gaussian in front:

$$f(Z) = f_Q(Z) \left( \frac{m_0(Z)}{(\sqrt{2\pi})^p} - \frac{\sigma^2}{6} \frac{m_0(Z)}{(\sqrt{2\pi})^p} R(Z) + \sigma^2 \frac{m_2(Z)}{(\sqrt{2\pi})^p} + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right)$$

We write this:

$$f(Z) = f_Q(Z) \left( F_0(Z) + \sigma^2 F_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right)$$

where:

$$F_0(Z) = \frac{m_0(Z)}{(\sqrt{2\pi})^p} = \frac{\sqrt{(2\pi)^p} \sqrt{\det((M_{ab}(Z)^{-1}))}}{(\sqrt{2\pi})^p} = \sqrt{\det((M_{ab}(Z)^{-1}))}$$

And:

$$F_2(Z) = -\frac{1}{6} \frac{m_0(Z)}{(\sqrt{2\pi})^p} R(Z) + \frac{m_2(Z)}{(\sqrt{2\pi})^p}$$
$$= -\frac{1}{6} \sqrt{\det\left((M_{ab}(Z)^{-1})R(Z) + \frac{m_2(Z)}{(\sqrt{2\pi})^p}\right)}$$

So that:

(SM14)  

$$\begin{aligned}
F_0(Z) &= \sqrt{\det\left((M_{ab}(Z)^{-1})\right)} \\
F_2(Z) &= -\frac{1}{6}\sqrt{\det\left((M_{ab}(Z)^{-1})\right)}R(Z) + \frac{m_2(Z)}{(\sqrt{2\pi})^p}
\end{aligned}$$

and we refer to the previous subsections for the formula of  $M_{ab}(Z)$  and  $m_2(Z)$ .

**SM4.** Proof of Theorem SM2: Bias on the template shape. Now we prove the second theorem given in the paper "Template shape estimation".

**Theorem SM2.** The data  $X_i$ 's are generated with the model described in the paper "Template shape estimation", where the template shape Y is a parameter and under the assumptions of Theorem 1. The template shape Y is estimated with  $\hat{Y}$ , which is computed by the usual procedure described the paper.

In the regime of an infinite number of data  $n \to +\infty$ , the asymptotic bias of the template's shape estimator  $\hat{Y}$ , with respect to the parameter Y, has the following Taylor expansion around the noise level  $\sigma = 0$ :

(SM15) 
$$Bias(\hat{Y}, Y) = -\frac{\sigma^2}{2}H(Y) + \mathcal{O}(\sigma^4) + \epsilon(\sigma)$$

where (i) H is the mean curvature vector of the template shape's orbit which represents the external curvature of the orbit in M, and (ii)  $\epsilon$  is a function of  $\sigma$  that decreases exponentially for  $\sigma \to 0$ .

We compute the bias  $\operatorname{Bias}(Y, \hat{Y})$  of  $\hat{Y}$  as an estimator of Y. In the following, we take a NCS at Y. In particular, the vector  $\overrightarrow{YZ} = \operatorname{Log}_Y Z$  has coordinates written z.

The expectation of the distribution f of shapes in Q is  $\hat{Y}$  by definition. The point  $\hat{Y}$ , expressed in a NCS at the template Y, gives  $\text{Bias}(Y, \hat{Y})$ , a tangent vector at  $T_Y M$  that indicates how much one has to shoot to reach the estimator  $\hat{Y}$ :

(SM16) 
$$\operatorname{Bias}(Y, \hat{Y}) = \operatorname{Log}_{Y} \hat{Y} = \int_{Q} \overrightarrow{YZ} f(Z) dQ(Z).$$

First, we take a ball of small radius r in Q and fix r. We split the integral:

$$\operatorname{Bias}(Y, \hat{Y}) = \int_{B_r^Q} \overrightarrow{YZ} f(Z) dQ(Z) + \int_{\mathcal{C}_{B_r^Q}} \overrightarrow{YZ} f(Z) dQ(Z)$$

By the result of the preliminaries, adapted to Q, the right part is a function  $\sigma \to \epsilon(\sigma)$  that is exponentially decreasing for  $\sigma \to 0$ .

$$\operatorname{Bias}(Y, \hat{Y}) = \int_{B_r^Q} \overrightarrow{YZ} f(Z) dQ(Z) + \epsilon(\sigma)$$

SM4.1. Using the result of Theorem SM1. We plug the expression of the density f using Theorem SM1:

$$\operatorname{Bias}(Y, \hat{Y}) = \int_{B_r^Q} \overrightarrow{YZ} f_Q(Z) \left( F_0(Z) + \sigma^2 F_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right) dQ(Z) + \epsilon(\sigma)$$
$$= \int_{B_r^Q} \overrightarrow{YZ} f_Q(Z) \left( F_0(Z) + \sigma^2 F_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right) dQ(Z) + \epsilon(\sigma)$$

Computing the *a*-coordinate of the bias:

$$\operatorname{Bias}(Y, \hat{Y})^a = \int_{B_r^Q} z^a f_Q(Z) \left( F_0(Z) + \sigma^2 F_2(Z) + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right) dQ(Z) + \epsilon(\sigma)$$

We are on a ball of small radius r around Y. We write the Taylor expansions of the F's terms around z = 0:

$$F_0(Z) = F_{00}(Y) + F_{01d}(Y)z^d + \mathcal{O}(||z||^2)$$
  

$$F_2(Z) = F_{20}(Y) + F_{21d}(Y)z^d + \mathcal{O}(||z||^2)$$

We replace these Taylor expansions in the expression of the bias:

$$\begin{aligned} \operatorname{Bias}(Y, \hat{Y})^{a} &= \int_{B_{r}^{Q}} z^{a} f_{Q}(Z) \left( F_{00}(Y) + F_{01d}(Y) z^{d} + \mathcal{O}(||z||^{2}) \right) dQ(Z) \\ &+ \int_{B_{r}^{Q}} z^{a} f_{Q}(Z) \left( \sigma^{2} F_{20}(Y) + \sigma^{2} F_{21d}(Y) z^{d} + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) + \epsilon(\sigma) \right) dQ(Z) + \epsilon(\sigma) \end{aligned}$$

Reorganizing:

$$\begin{aligned} \operatorname{Bias}(Y, \hat{Y}) &= \int_{B_r^Q} z^a f_Q(Z) \left( F_{00}(Y) + F_{01d}(Y) z^d + \sigma^2 F_{20}(Y) + \sigma^2 F_{21d}(Y) z^d \right) dQ(Z) \\ &+ \int_{B_r^Q} \overrightarrow{YZ} f_Q(Z) \left( \mathcal{O}(||z||^2) + \sigma^2 \mathcal{O}(||z||^2) + \mathcal{O}(\sigma^4) + \epsilon(\sigma) \right) dQ(Z) \\ &+ \epsilon(\sigma) \end{aligned}$$

By the dominated convergence theorem, we can put the inside  $\epsilon(\sigma)$  outside the parenthesis.

$$\begin{aligned} \operatorname{Bias}(Y, \hat{Y}) &= \int_{B_r^Q} z^a f_Q(Z) \left( F_{00}(Y) + F_{01d}(Y) z^d + \sigma^2 F_{20}(Y) + \sigma^2 F_{21d}(Y) z^d \right) dQ(Z) \\ &+ \int_{B_r^Q} \overrightarrow{YZ} f_Q(Z) \left( \mathcal{O}(||z||^2) + \sigma^2 \mathcal{O}(||z||^2) + \mathcal{O}(\sigma^4) \right) dQ(Z) \\ &+ \epsilon(\sigma) \end{aligned}$$

# SUPPLEMENTARY MATERIALS: PAPER "TEMPLATE SHAPE ESTIMATION"

We develop the measure on dQ:  $dQ(Z) = (1 - \frac{1}{6}\operatorname{Ric}(Y)_{bc}z^{b}z^{c} + \mathcal{O}(||z||^{3}))dz$ :

$$\begin{split} \operatorname{Bias}(Y,\hat{Y})^{a} &= \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{00}(Y) z^{a} + F_{01d}(Y) z^{a} z^{d} \right) (1 - \frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} + \mathcal{O}(||z||^{3})) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( \sigma^{2} F_{20}(Y) z^{a} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) (1 - \frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} + \mathcal{O}(||z||^{3})) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) (1 - \frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} + \mathcal{O}(||z||^{3})) dz \\ &+ \epsilon(\sigma) \end{split}$$

We develop:

$$\begin{split} \operatorname{Bias}(Y, \hat{Y})^{a} &= \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{00}(Y) z^{a} + F_{01d}(Y) z^{a} z^{d} + \sigma^{2} F_{20}(Y) z^{a} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{00}(Y) z^{a} + F_{01d}(Y) z^{a} z^{d} \right) \left( -\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( \sigma^{2} F_{20}(Y) z^{a} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) \left( -\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{00}(Y) z^{a} + F_{01d}(Y) z^{a} z^{d} + \sigma^{2} F_{20}(Y) z^{a} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) \mathcal{O}(||z||^{3}) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) \left( -\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) \left( -\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) \mathcal{O}(||z||^{3}) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) \mathcal{O}(||z||^{3}) dz \\ &+ \epsilon(\sigma) \end{split}$$

We eliminate the odd terms that give 0 by skewsymmetry:

$$\begin{split} \operatorname{Bias}(Y, \hat{Y})^{a} &= \int_{B_{r}^{Q}} f_{Q}(Z) \left( +F_{01d}(Y) z^{a} z^{d} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{01d}(Y) z^{a} z^{d} + \sigma^{2} F_{21d}(Y) . z^{a} z^{d} \right) (-\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c}) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) \left( F_{00}(Y) z^{a} + \sigma^{2} F_{20}(Y) z^{a} \right) \mathcal{O}(||z||^{3}) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(\sigma^{4}) \right) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(\sigma^{4}) \right) (-\frac{1}{6} \operatorname{Ric}(Y)_{bc} z^{b} z^{c}) dz \\ &+ \int_{B_{r}^{Q}} f_{Q}(Z) z^{a} \left( \mathcal{O}(||z||^{2}) + \sigma^{2} \mathcal{O}(||z||^{2}) + \mathcal{O}(\sigma^{4}) \right) \mathcal{O}(||z||^{3}) dz \\ &+ \epsilon(\sigma) \end{split}$$

We delete the terms that will give more than  $\mathcal{O}(\sigma^2)$  by integration (these are normalized moments) and put them in  $\mathcal{O}(\sigma^4)$  which is the next order since there is  $\mathrm{no}\mathcal{O}(\sigma^3)$ :

$$\operatorname{Bias}(Y,\hat{Y})^{a} = \int_{B_{r}^{Q}} f_{Q}(Z)F_{01d}(Y)z^{a}z^{d}dz + \int_{B_{r}^{Q}} f_{Q}(Z)z^{a}\left(\mathcal{O}(\sigma^{4})\right)dz + \mathcal{O}(\sigma^{4}) + \epsilon(\sigma)$$

We gather the terms in  $\mathcal{O}(\sigma^4)$ :

$$\operatorname{Bias}(Y, \hat{Y})^a = \int_{B_r^Q} f_Q(Z) F_{01d}(Y) z^a z^d dz + \mathcal{O}(\sigma^4) + \epsilon(\sigma)$$

We recognize the *normalized* truncated moment of order 2 in the q-dimensional Riemannian manifold Q, see Subsection SM2.3:

$$\operatorname{Bias}(Y, \hat{Y})^{a} = F_{01d}(Y)\mathfrak{M}_{r}^{Q^{ad}}(\sigma^{2}\mathbb{I}) + \mathcal{O}(\sigma^{4}) + \epsilon(\sigma)$$

We express it with respect to the non-truncated *normalized* moment in  $\mathbb{R}^q$ , see Subsection SM2.3:

$$\operatorname{Bias}(Y, \hat{Y})^{a} = F_{01d}(Y) \left( \sigma^{2} \mathcal{M}^{Qad}(\mathbb{I}) + \mathcal{O}(\sigma^{4}) - \epsilon(\sigma) \right) + \mathcal{O}(\sigma^{4}) + \epsilon(\sigma)$$

We gather the  $\epsilon$ 's and the  $\mathcal{O}(\sigma^4)$ :

$$\operatorname{Bias}(Y, \hat{Y})^a = F_{01d}(Y)\sigma^2 \mathcal{M}_r^{Qad}(\mathbb{I}) + \mathcal{O}(\sigma^4) + \epsilon(\sigma)$$

We replace the *normalized* 2nd order moment by its expression which is simply  $\delta^{ad}$ , see Subsection SM2.2:

$$Bias(Y, \hat{Y})^{a} = F_{01d}(Y)\sigma^{2}\delta^{ad} + \mathcal{O}(\sigma^{4}) + \epsilon'(\sigma)$$
$$= F_{01}^{a}(Y)\sigma^{2} + \mathcal{O}(\sigma^{4}) + \epsilon(\sigma)$$

**SM4.2.** Computation of  $F_{01}^a(Y)$ : Taylor expansion of  $F_0(Z)$  in the coordinate z. The term  $F_{01}^a(Y)$  is the first order coefficient in the Taylor expansion of  $F_0(Z)$  around Y in the coordinate z. Thus, we compute this Taylor expansion.

We first compute the Taylor expression of  $F_0(Z)$  using the coordinate  $y = \overrightarrow{ZY} = \text{Log}_Z Y$ in the NCS at Z. The expression of  $F_0(Z)$  in Section ?? gives:

$$F_0(Z) = \Theta_{Z,0}(1) = \sqrt{(2\pi)^p} \sqrt{\det\left((M_{ab}(Z)^{-1})\right)}$$

The previous subsections give:

$$M_{ab}(Z) = -y_c h_{ab}^c(Z) + d_{ab}(Z)$$

and we replace  $d_{ab}(Z)$  by the formula p.23 in [SM1] but keeping only the first order:

$$M_{ab}(Z) = \delta_{ab} - y_c h_{ab}^c(Y) + \mathcal{O}(||y||^2)$$

So that:

$$M_{ab}(Z)^{-1} = \delta_{ab} + y_c h_{ab}^c(Y) + \mathcal{O}(||y||^2)$$

We plug this in  $F_0(Z)$ :

$$F_{0}(Z) = \sqrt{\det \left(\delta_{ab} + y_{c}h_{ab}^{c}(Y) + \mathcal{O}(||y||^{2})\right)}$$
  
=  $\sqrt{1 + y_{c}\operatorname{Trace}(h_{ab}^{c}(Y)) + \mathcal{O}(||y||^{2})}$   
=  $\sqrt{1 + y_{c}H^{c}(Y) + \mathcal{O}(||y||^{2})}$   
=  $\left(1 + \frac{1}{2}y_{c}H^{c}(Y) + \mathcal{O}(||y||^{2})\right)$ 

where the trace of the second fundamental form is the external curvature vector H(Y) by definition.

We convert this Taylor expansion in y, the coordinate of Y in a NCS at Z, into a Taylor expansion in z, the coordinate of Z in a NCS at Y. To express y with respect to z, we consider the geodesic  $\gamma_{ZY}(t)$  from Z to Y and the geodesic  $\gamma_{YZ}(t)$  from Y to Z. When parameterized by the arclength s, they are related as follows:

$$\gamma_{ZY}(s) = \operatorname{Exp}_{Z}\left(s\overrightarrow{ZY}\right)$$
$$= \gamma_{YZ}(1-s)$$
$$= \operatorname{Exp}_{Y}\left((1-s)\overrightarrow{YZ}\right)$$

Differentiating this relation gives:

$$D\operatorname{Exp}_Z|_{\overrightarrow{sZY}}.\overrightarrow{ZY} = -D\operatorname{Exp}_Y|_{(1-s)\overrightarrow{YZ}}.\overrightarrow{YZ}$$

Taking the relation at s = 0:

$$D\operatorname{Exp}_Z|_0.\overrightarrow{ZY} = -D\operatorname{Exp}_Y|_{\overrightarrow{YZ}}.\overrightarrow{YZ}$$

where  $D \operatorname{Exp}_Z|_0 = Id$ , so that:

$$y = -D \operatorname{Exp}_Y |_z . z$$

We use the definition of the NCS at Y. When  $\text{Exp}_Y(u) = U$ , then U has coordinates u in the NCS at Y. So that:  $D\text{Exp}_Y|_{u} \cdot u = u$ . This gives, with u = z:

$$y = -z$$

and we get the Taylor expansion of  $F_0(Z)$  expressed in the coordinate z:

$$F_0(Z) = \left(1 - \frac{1}{2}z_c H^c(Y) + \mathcal{O}(||z||^2)\right)$$

And we identify the term  $F_{01}^a(Y)$  needed:

$$F_{01}^{a}(Y) = -\frac{1}{2}H^{a}(Y)$$

**SM4.3. Final result:** Taylor expansion of  $Bias(\hat{Y}, Y)$ . Replacing  $F_{01}^a(Y)$  by its value computed above:

$$\operatorname{Bias}(Y, \hat{Y})^a = -\frac{1}{2} \cdot H^a(Y) \sigma^2 + \mathcal{O}(\sigma^4) + \epsilon(\sigma)$$

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