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On Witten Laplacians and Brascamp-Lieb's inequality on manifolds with boundary

Dorian Le Peutrec*

July 28, 2016

Abstract

In this paper, we derive from the supersymmetry of the semiclassical Witten Laplacian Brascamp-Lieb's type inequalities for general differential forms on compact Riemannian manifolds with boundary. In addition to the supersymmetry, our results essentially follow from suitable decompositions of the quadratic forms associated with the Neumann and Dirichlet self-adjoint realizations of the Witten Laplacian. They moreover imply the usual Brascamp-Lieb's inequality and its generalization to compact Riemannian manifolds without boundary.

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Keywords: Brascamp-Lieb's inequality, Witten Laplacian, Riemannian manifolds with boundary, Supersymmetry.

1 Introduction

1.1 Context and aim of the paper

Let $V \in \mathcal{C}^2(\mathbf{R}^n, \mathbf{R})$ be a strictly convex function such that $e^{-V} \in L^1(\mathbf{R}^n)$ and let ν be the probability measure defined by $d\nu := \frac{e^{-V}}{\int_{\mathbf{R}^n} e^{-V} dx} dx$. The classical Brascamp-Lieb's inequality proven in [BrLi] states that every smooth compactly supported function ω satisfies the estimate

$$\int_{\mathbf{R}^n} \left| \omega - \left(\int_{\mathbf{R}^n} \omega d\nu \right) \right|^2 d\nu \leq \int_{\mathbf{R}^n} (\text{Hess } V)^{-1} (\nabla \omega, \nabla \omega) d\nu. \quad (1.1)$$

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This inequality and suitable variants have since been e.g. used in works such as [HeSj, Sjö, NaSp, Hel, BJS, BaMø, BaMø2] studying correlation asymptotics in statistical mechanics. The latter works exploit in particular crucially relations of the following type and which at least go back to the work of Helffer and Sjöstrand [HeSj]:

$$\left\| \eta - \left\langle \eta, \frac{e^{-\frac{V}{2}}}{\|e^{-\frac{V}{2}}\|} \right\rangle \frac{e^{-\frac{V}{2}}}{\|e^{-\frac{V}{2}}\|} \right\|^2 = \langle (\Delta_{\frac{V}{2}}^{(1)})^{-1} (\nabla_V \eta), \nabla_V \eta \rangle, \quad (1.2)$$

where $\eta \in \mathcal{C}_c^\infty(\mathbf{R}^n)$, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ stand for the usual $L^2(dx)$ inner product and norm, $\nabla_V := \nabla + \nabla \frac{V}{2}$ and $\Delta_{\frac{V}{2}}^{(1)}$ is the Witten Laplacian acting on vector fields (or equivalently on 1-forms) which is given by

$$\Delta_{\frac{V}{2}}^{(1)} := \Delta_{\frac{V}{2}}^{(0)} \otimes \text{Id} + \text{Hess } V = \left(-\Delta + |\nabla \frac{V}{2}|^2 - \Delta \frac{V}{2} \right) \otimes \text{Id} + \text{Hess } V. \quad (1.3)$$

In the last relation,

$$\Delta_{\frac{V}{2}}^{(0)} := -\Delta + |\nabla \frac{V}{2}|^2 - \Delta \frac{V}{2} = \left(-\text{div} + \nabla \frac{V}{2} \right) \left(\nabla + \nabla \frac{V}{2} \right) = \nabla_V^* \nabla_V \quad (1.4)$$

denotes the Witten Laplacian acting on functions (or equivalently on 0-forms). The Witten Laplacian, initially introduced in [Wit], is more generally defined on the full algebra of differential forms and is nonnegative and essentially self-adjoint (when acting on smooth compactly supported forms) on the space of $L^2(dx)$ differential forms. It is moreover supersymmetric, which essentially amounts, when restricting our attention to the interplay between $\Delta_{\frac{V}{2}}^{(0)}$ and $\Delta_{\frac{V}{2}}^{(1)}$, to the intertwining relation

$$\forall \eta \in \mathcal{C}_c^\infty(\mathbf{R}^n), \quad \nabla_V \Delta_{\frac{V}{2}}^{(0)} \eta = \Delta_{\frac{V}{2}}^{(1)} \nabla_V \eta,$$

which enables to prove relations of the type (1.2) (when $\Delta_{\frac{V}{2}}^{(1)}$ is invertible). The nonnegativity of $\Delta_{\frac{V}{2}}^{(0)}$ together with the relations (1.2) and (1.3) then easily leads to (1.1) when V is strictly convex (at least formally) taking finally $\omega := e^{\frac{V}{2}} \eta$. To connect to some spectral properties of $\Delta_{\frac{V}{2}}^{(0)}$, the relation (1.2) together with the lower bound $\Delta_{\frac{V}{2}}^{(1)} \geq c$ for some $c > 0$ – which is in particular satisfied if $\text{Hess } V \geq c$ – implies, according to the writing (1.4), a spectral gap greater or equal to c for $\Delta_{\frac{V}{2}}^{(0)}$ (its kernel being $\text{Span}\{e^{-\frac{V}{2}}\}$ as it can be seen from (1.4)). In addition to the already mentioned [Sjö, Hel] making extra assumptions on V , we refer especially to the very complete [Joh]

for precise statements and proofs in relation with the above discussion.

More generally, in the case of a Riemannian manifold without boundary Ω , it is also well known that an inequality of the type (1.1) holds if one replaces $\text{Hess } V$ (and the condition $\text{Hess } V > 0$ everywhere) by the following quadratic form, sometimes called the Bakry - Émery (-Ricci) tensor,

$$\text{Ric} + \text{Hess } V \quad (\text{and if we assume its strict positivity everywhere}),$$

Ric denoting the Ricci tensor. We refer for example to [BGL, Theorem 4.9.3] for a precise statement whose proof relies on the supersymmetry of the counterpart of the Witten Laplacian in the weighted space $L^2(\Omega, e^{-V} d\text{Vol}_\Omega)$, sometimes called the weighted Laplacian and more precisely defined when acting on functions by

$$L_V^{(0)} := e^{\frac{V}{2}} \left(-\Delta + \left| \nabla \frac{V}{2} \right|^2 - \Delta \frac{V}{2} \right) e^{-\frac{V}{2}} = -\Delta + \nabla V \cdot \nabla.$$

This operator, unitarily equivalent to $\Delta_{\frac{V}{2}}^{(0)}$, is an important model of the Bakry - Émery theory of diffusion processes and we refer especially in this direction to the pioneering work of Bakry and Émery [BaÉm] or to the book [BGL] for an overview of the concerned literature. On its side, the Bakry - Émery tensor $\text{Ric} + \text{Hess } V$ - named after [BaÉm] but first introduced by Lichnerowicz in [Lic] - is the natural counterpart of the Ricci tensor Ric in the weighted Riemannian manifold $(\Omega, e^{-V} d\text{Vol}_\Omega)$ and we refer for example to [Lic, Lot] for some of its geometric properties. Let us also mention e.g. [LoVi] extending this notion to metric measure spaces.

In this paper, we derive from the supersymmetry of the Witten Laplacian Brascamp - Lieb's type inequalities for general differential forms on a Riemannian manifold with a boundary. Being interested in possible applications to semiclassical spectral theory, we will work with the slightly more general semiclassical Witten Laplacian also depending on a positive parameter h . In addition to the supersymmetry, our results essentially follow from suitable decompositions of the quadratic forms associated with the self-adjoint Neumann and Dirichlet realizations of the semiclassical Witten Laplacian stated in Theorem 1.2. When restricting to $h = 1$ and to the interplay between 0- and 1-forms, they imply in particular the already mentioned results in the case of \mathbf{R}^n or of a compact manifold with empty boundary as well as some results recently obtained by Kolesnikov and Milman in [KoMi] in the case of a compact manifold with a boundary (see indeed Corollaries 1.4 and 1.5 and the corresponding remarks).

1.2 Notions of Riemannian geometry

We now introduce the concepts of Riemannian geometry which will be needed to state properly our further hypotheses and results. This part is rather long since we made the choice to define these classical notions quite precisely in order to keep this article comprehensible for readers not familiar with geometry. The following objects are essentially defined according to the PDE framework developed in [Sch] and we refer especially to Sections 1.1 and 1.2 there for further details and references, the notation adopted being nevertheless slightly different.

We work with a smooth n -dimensional oriented connected and compact Riemannian manifold $(\Omega, g = \langle \cdot, \cdot \rangle)$ with boundary $\partial\Omega$. The cotangent (resp. tangent) bundle of Ω is denoted by $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle by $\Lambda T^*\Omega = \bigoplus_{p=0}^n \Lambda^p T^*\Omega$. The fiber bundles $T^*\partial\Omega$, $T\partial\Omega$, and $\Lambda T^*\partial\Omega = \bigoplus_{p=0}^{n-1} \Lambda^p T^*\partial\Omega$ are defined similarly.

The (bundle) scalar product on $\Lambda^p T^*\Omega$ inherited from g is denoted by $\langle \cdot, \cdot \rangle_{\Lambda^p}$. Let us recall that $\langle \cdot, \cdot \rangle_{\Lambda^1}$ is defined by

$$\langle \omega, \eta \rangle_{\Lambda^1} := \langle \omega^\sharp, \eta^\sharp \rangle,$$

where, for any $\xi \in T^*\Omega$, ξ^\sharp is the element of $T\Omega$ satisfying, for any $X \in T\Omega$,

$$\langle \xi^\sharp, X \rangle := \xi(X). \quad (1.5)$$

The map $\xi \mapsto \xi^\sharp$ is an isomorphism from $T^*\Omega$ into $T\Omega$ and we denote by $T\Omega \ni X \mapsto X^\flat \in T^*\Omega$ its inverse isomorphism. The inner product $\langle \cdot, \cdot \rangle_{\Lambda^p}$ is then defined as the bilinear form satisfying the following relation on decomposable p -forms:

$$\langle \omega_1 \wedge \cdots \wedge \omega_p, \eta_1 \wedge \cdots \wedge \eta_p \rangle_{\Lambda^p} := \det \left(\langle \omega_i, \eta_j \rangle_{\Lambda^1} \right)_{1 \leq i, j \leq p}.$$

The space of \mathcal{C}^∞ , L^2 , etc. sections of any of the above fiber bundles E , over $O = \Omega$ or $O = \partial\Omega$, are respectively denoted by $\mathcal{C}^\infty(O, E)$, $L^2(O, E)$, etc.. The more compact notation $\Lambda^p \mathcal{C}^\infty$, $\Lambda^p L^2$, etc. will also be used instead of $\mathcal{C}^\infty(\Omega, \Lambda^p T^*\Omega)$, $L^2(\Omega, \Lambda^p T^*\Omega)$, etc. and we will denote by $\mathcal{L}(\Lambda^p T^*\Omega)$ the space of smooth bundle endomorphisms of $\Lambda^p T^*\Omega$. The L^2 spaces are those associated with the respective unit volume forms μ and $\mu_{\partial\Omega}$ for the Riemannian structures on Ω and on $\partial\Omega$.

The notion of *local orthonormal frame* (on Ω or $\partial\Omega$) will be frequently used in the sequel. By local orthonormal frame on (say) Ω , we mean a family (E_1, \dots, E_n) of smooth sections of $T\Omega$ defined on an open set $U \subset \Omega$ such that

$$\forall i, j \in \{1, \dots, n\}, \forall x \in U, \quad \langle E_i, E_j \rangle_x = \delta_{i,j}.$$

According for example to [Sch, Definition 1.1.6] and to the related remarks, it is always possible to cover Ω with a finite family (since Ω is compact) of opens sets U 's such that there exists a local orthonormal frame (E_1, \dots, E_n) on each U . Such a covering is called a nice cover of Ω .

The outgoing normal vector field will be denoted by \vec{n} and the orientation is chosen such that

$$\mu_{\partial\Omega} = \mathbf{i}_{\vec{n}} \mu,$$

where \mathbf{i} denotes the interior product. Owing to the Collar Theorem stated in [Sch, Theorem 1.1.7], the vector field $\vec{n} \in \mathcal{C}^\infty(\partial\Omega, T\Omega|_{\partial\Omega})$ can be extended to a smooth vector field on a neighborhood of the boundary $\partial\Omega$. Moreover, taking maybe a finite refinement of a nice cover of Ω as defined previously, one can always assume that the local orthonormal frame (E_1, \dots, E_n) corresponding to any of its elements U meeting $\partial\Omega$ is such that $E_n|_{\partial\Omega} = \vec{n}$. In particular, the vector fields E_1, \dots, E_{n-1} are such that

$$\forall j \in \{1, \dots, n-1\}, \quad E_j|_{\partial\Omega} \in (T\Omega|_{\partial\Omega})^T = T\partial\Omega.$$

Here, with a slight abuse of notation, we have made the identification between the space of tangential vector fields

$$(T\Omega|_{\partial\Omega})^T := \{X \in T\Omega|_{\partial\Omega} \text{ such that } \langle X, \vec{n} \rangle = 0\}$$

and the tangent bundle of $\partial\Omega$ (see [Sch, pp. 15–16] for more details).

For any $\omega \in \Lambda^p \mathcal{C}^\infty$, the tangential part of ω on $\partial\Omega$ is the form $\mathbf{t}\omega \in \mathcal{C}^\infty(\partial\Omega, \Lambda^p T^*\Omega|_{\partial\Omega})$ defined by:

$$\forall \sigma \in \partial\Omega, \quad (\mathbf{t}\omega)_\sigma(X_1, \dots, X_p) := \omega_\sigma(X_1^T, \dots, X_p^T),$$

with the decomposition $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$ into the tangential and normal components to $\partial\Omega$ at σ . More briefly, it holds

$$\mathbf{t}\omega = \mathbf{i}_{\vec{n}}(\vec{n}^\flat \wedge \omega).$$

The normal part of ω on $\partial\Omega$ is then defined by:

$$\mathbf{n}\omega := \omega|_{\partial\Omega} - \mathbf{t}\omega = \vec{n}^\flat \wedge (\mathbf{i}_{\vec{n}}\omega) \in \mathcal{C}^\infty(\partial\Omega, \Lambda^p T^*\Omega|_{\partial\Omega}).$$

We denote by d the exterior differential on $\mathcal{C}^\infty(\Omega, \Lambda T^*\Omega)$ and by d^* its formal adjoint with respect to the L^2 scalar product inherited from the Riemannian structure. We recall that they satisfy the relation $d^2 = (d^*)^2 = 0$. The Hodge Laplacian is then defined on $\mathcal{C}^\infty(\Omega, \Lambda T^*\Omega)$ by

$$\Delta := \Delta_H := d^*d + dd^* = (d + d^*)^2. \quad (1.6)$$

For a (real) smooth function f and a smooth vector field X , we will use the notation

$$\nabla_X f := X \cdot f = df(X),$$

the normal derivative of f along the boundary being in particular defined by

$$\partial_n f := \langle \nabla f, \vec{n} \rangle = \nabla_{\vec{n}} f.$$

We will also denote by $\nabla : \mathcal{C}^\infty(\Omega, T\Omega) \times \mathcal{C}^\infty(\Omega, T\Omega) \rightarrow \mathcal{C}^\infty(\Omega, T\Omega)$ the Levi-Civita connection on Ω and by $\nabla_X(\cdot)$ the covariant derivative (in the direction of X) of vector fields as well as the induced covariant derivative on $\Lambda^p T^*\Omega$ (see Subsection 2.1 for more details).

The second covariant derivative (acting for example on $T\Omega$ and on $\Lambda^p T^*\Omega$) is then the bilinear mapping on $T\Omega$ defined, for $X, Y \in \mathcal{C}^\infty(\Omega, T\Omega)$ by

$$\nabla_{X,Y}^2 := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$

When f is a smooth function, $\nabla_{X,Y}^2 f$ is simply the Hessian of f . It is in this case a symmetric bilinear form and has the simpler writing

$$\text{Hess } f(X, Y) := \nabla_{X,Y}^2 f = (\nabla_X df)(Y) = \langle \nabla_X \nabla f, Y \rangle. \quad (1.7)$$

The Bochner Laplacian Δ_B on $\mathcal{C}^\infty(\Omega, \Lambda T^*\Omega)$ is defined as minus the trace of the bilinear mapping $(X, Y) \mapsto \nabla_{X,Y}^2$. More precisely, we have for any $\omega \in \mathcal{C}^\infty(\Omega, \Lambda T^*\Omega)$:

$$\Delta_B \omega := -\text{Tr} \left((X, Y) \mapsto \nabla_{X,Y}^2 \omega \right), \quad (1.8)$$

which implies that for any local orthonormal frame (E_1, \dots, E_n) on $U \subset \Omega$, Δ_B is given on U by

$$\Delta_B = -\sum_{i=1}^n (\nabla_{E_i} \nabla_{E_i} - \nabla_{\nabla_{E_i} E_i}). \quad (1.9)$$

The Hodge and Bochner Laplacians $\Delta^{(p)}$ and $\Delta_B^{(p)}$ (the superscript (p) means that we are considering their action on differential p -forms) are related by

the Weitzenböck formula: there exists a smooth bundle symmetric endomorphism $\text{Ric}^{(p)} \in \mathcal{L}(\Lambda^p T^* \Omega)$ such that (see [Sch, p. 26] where the opposite convention of sign is adopted)

$$\Delta_B^{(p)} = \Delta^{(p)} - \text{Ric}^{(p)}. \quad (1.10)$$

This operator vanishes on 0-forms (i.e. on functions) and $\text{Ric}^{(1)}$ is the element of $\mathcal{L}(\Lambda^1 T^* \Omega)$ canonically identified with the Ricci tensor Ric (see below for the precise definition of this identification). We recall that Ric is the symmetric $(0, 2)$ -tensor defined, for $X, Y \in T\Omega$, by

$$\text{Ric}(X, Y) := \text{Tr} \left(Z \mapsto R(Z, X)Y \right), \quad (1.11)$$

where R denotes the Riemannian curvature tensor which is defined, for every $X, Y, Z \in T\Omega$, by

$$R(X, Y)Z := (\nabla_{X,Y}^2 - \nabla_{Y,X}^2)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \quad (1.12)$$

The tensor Ric hence satisfies on any open set $U \subset \Omega$ where is given a local orthonormal frame (E_1, \dots, E_n) :

$$\text{Ric}(X, Y) = \sum_{i=1}^n \langle R(E_i, X)Y, E_i \rangle = - \sum_{i=1}^n \langle R(E_i, X)E_i, Y \rangle, \quad (1.13)$$

the last line following from the relation $\langle R(X, Y)Z, T \rangle = -\langle R(X, Y)T, Z \rangle$ for any $X, Y, Z, T \in T\Omega$. It is then canonically identified with a symmetric bilinear form acting on $T^* \Omega$ (i.e. a symmetric $(2, 0)$ -tensor), still denoted by Ric and defined by (see (1.5) for the meaning of $T^* \Omega \ni \omega \mapsto \omega^\sharp \in T\Omega$)

$$\text{Ric}(\omega, \eta) := \text{Ric}(\omega^\sharp, \eta^\sharp).$$

The latter symmetric bilinear form is then itself identified via $\langle \cdot, \cdot \rangle_{\Lambda^1}$ with the element of $\mathcal{L}(\Lambda^1 T^* \Omega)$ denoted by $\text{Ric}^{(1)}$. More precisely, we have for any ω and η in $T^* \Omega$:

$$\langle \text{Ric}^{(1)}\omega, \eta \rangle_{\Lambda^1} := \text{Ric}(\omega, \eta).$$

Remark 1.1. Denoting also by Ric the bundle symmetric endomorphism of $T\Omega$ defined by $\langle \text{Ric}X, Y \rangle := \text{Ric}(X, Y)$ (i.e. by $\text{Ric}X := -\sum_{i=1}^n R(E_i, X)E_i$ according to (1.13)), we have for any ω, η in $T^* \Omega$ and X in $T\Omega$,

$$\begin{aligned} \text{Ric}^{(1)}\omega(X) &= \langle \text{Ric}^{(1)}\omega, X^\flat \rangle_{\Lambda^1} = \text{Ric}(\omega, X^\flat) \\ &= \text{Ric}(\omega^\sharp, X) \\ &= \langle \omega^\sharp, \text{Ric}X \rangle = \omega(\text{Ric}X). \end{aligned} \quad (1.14)$$

More generally, for any local orthonormal frame (E_1, \dots, E_n) on $U \subset \Omega$, $\text{Ric}^{(p)}$ is defined on U for any $p \in \{1, \dots, n\}$ by

$$\begin{aligned} & (\text{Ric}^{(p)}\omega)(X_1, \dots, X_p) \\ & := - \sum_{i=1}^n \sum_{j=1}^k \left((R(E_i, X_j))^{(p)}\omega \right)(X_1, \dots, X_{j-1}, E_i, X_{j+1}, \dots, X_k), \end{aligned} \quad (1.15)$$

where $(R(E_i, X_j))^{(1)} \in \mathcal{L}(\Lambda^1 T^* \Omega)$ is canonically identified with $R(E_i, X_j)$ via $\left((R(E_i, X_j))^{(1)}\omega \right)(X) = \omega(R(E_i, X_j)X)$ and

$$(R(E_i, X_j))^{(p)} = \left((R(E_i, X_j))^{(1)} \right)^{(p)},$$

where for any $A \in \mathcal{L}(\Lambda^1 T^* \Omega)$, $(A)^{(p)}$ is the element of $\mathcal{L}(\Lambda^p T^* \Omega)$ satisfying the following relation on decomposable p -forms:

$$(A)^{(p)}(\omega_1 \wedge \dots \wedge \omega_p) = \sum_{i=1}^p \omega_1 \wedge \dots \wedge A\omega_i \wedge \dots \wedge \omega_p. \quad (1.16)$$

We end up this part by recalling the definition of the second fundamental form of $\partial\Omega \subset \Omega$ and of related concepts. The second fundamental form \mathcal{K}_2 of $\partial\Omega \subset \Omega$ is the bilinear mapping defined by

$$\mathcal{K}_2 : \begin{array}{ccc} T\partial\Omega \times T\partial\Omega & \longrightarrow & T\Omega \Big|_{\partial\Omega} \\ (U, V) & \longmapsto & (\nabla_U V)^\perp := \langle \nabla_U V, \vec{n} \rangle \vec{n} \end{array} \quad (1.17)$$

It is symmetric and the value of $\mathcal{K}_2(U, V) \Big|_\sigma$ at $\sigma \in \partial\Omega$ only depends on the values of the tangential fields U_σ and V_σ at that point. The shape operator of $\partial\Omega \subset \Omega$ is the bundle endomorphism $\mathcal{K}_1 \in \mathcal{L}(T\partial\Omega)$ defined by

$$\forall U \in T\partial\Omega, \quad \mathcal{K}_1(U) := -\nabla_U \vec{n}. \quad (1.18)$$

It is then completely determined by \mathcal{K}_2 since it satisfies

$$\forall (U, V) \in T\partial\Omega \times T\partial\Omega, \quad \langle \mathcal{K}_1(U), V \rangle \vec{n} = \mathcal{K}_2(U, V).$$

The mean curvature of $\partial\Omega \subset \Omega$ is defined as the trace of the bilinear mapping $(U, V) \mapsto \langle \mathcal{K}_2(U, V), \vec{n} \rangle$ or equivalently as the trace of the shape operator \mathcal{K}_1 . We recall lastly that with our choice of orientation for \vec{n} , Ω is locally convex iff $\langle \mathcal{K}_2(\cdot, \cdot), \vec{n} \rangle$ is nonpositive.

1.3 Witten and weighted Laplacians

For a (real) smooth function f and a (real) positive number h , the distorted differential operators $d_{f,h}$ and $d_{f,h}^*$ are defined on $\mathcal{C}^\infty(\Omega, \Lambda T^* \Omega)$ by

$$d_{f,h} := e^{-f(x)/h} (hd) e^{f(x)/h} \quad \text{and} \quad d_{f,h}^* := e^{f(x)/h} (hd^*) e^{-f(x)/h}, \quad (1.19)$$

and the Witten Laplacian $\Delta_{f,h}$ is defined on $\mathcal{C}^\infty(\Omega, \Lambda T^* \Omega)$ similarly as the Hodge Laplacian by

$$\Delta_{f,h} = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^* = (d_{f,h} + d_{f,h}^*)^2. \quad (1.20)$$

The last equality simply follows from the property $d^2 = (d^*)^2 = 0$. Note that $d_{0,1}$, $d_{0,1}^*$, and $\Delta_{0,1}$ are simply d , d^* , and Δ . Note also the supersymmetry structure of the Witten Laplacian acting on the complex of differential forms: for every u in $\mathcal{C}^\infty(\Omega, \Lambda^p T^* \Omega)$, it holds

$$\Delta_{f,h}^{(p+1)} d_{f,h}^{(p)} u = d_{f,h}^{(p)} \Delta_{f,h}^{(p)} u \quad \text{and} \quad \Delta_{f,h}^{(p-1)} d_{f,h}^{(p-1),*} u = d_{f,h}^{(p-1),*} \Delta_{f,h}^{(p)} u. \quad (1.21)$$

The Witten Laplacian $\Delta_{f,h}^{(p)}$ extends in the distributional sense to an operator acting on the Sobolev space $\Lambda^p H^2$ and is nonnegative and self-adjoint on the flat space $\Lambda^p L^2 = \Lambda^p L^2(d\mu)$ once endowed with appropriate Dirichlet or Neumann type boundary conditions (see indeed [HeNi, Lep]). These self-adjoint extensions are respectively denoted by $\Delta_{f,h}^{\mathbf{t},(p)}$ and $\Delta_{f,h}^{\mathbf{n},(p)}$, their respective domains being given by

$$D(\Delta_{f,h}^{\mathbf{t},(p)}) = \{\omega \in \Lambda^p H^2, \mathbf{t}\omega = 0 \quad \text{and} \quad \mathbf{t}d_{f,h}^* \omega = 0 \quad \text{on} \quad \partial\Omega\} \quad (1.22)$$

and

$$D(\Delta_{f,h}^{\mathbf{n},(p)}) = \{\omega \in \Lambda^p H^2, \mathbf{n}\omega = 0 \quad \text{and} \quad \mathbf{n}d_{f,h}\omega = 0 \quad \text{on} \quad \partial\Omega\}. \quad (1.23)$$

For $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, the quadratic form associated with $\Delta_{f,h}^{\mathbf{b},(p)}$ is denoted by $\mathcal{D}_{f,h}^{\mathbf{b},(p)}$. Its domain is given by

$$\Lambda^p H_{\mathbf{b}}^1 := \{\omega \in \Lambda^p H^1, \mathbf{b}\omega = 0 \quad \text{on} \quad \partial\Omega\}, \quad (1.24)$$

and we have, for every $\omega \in \Lambda^p H_{\mathbf{b}}^1$,

$$\mathcal{D}_{f,h}^{\mathbf{b},(p)}(\omega) := \mathcal{D}_{f,h}^{\mathbf{b},(p)}(\omega, \omega) = \langle d_{f,h}\omega, d_{f,h}\omega \rangle_{\Lambda^{p+1}L^2} + \langle d_{f,h}^*\omega, d_{f,h}^*\omega \rangle_{\Lambda^{p-1}L^2}. \quad (1.25)$$

More details about these self-adjoint realizations are given in Subsection 2.3.

When acting on 0-forms, $\Delta_{f,h}$ is the Schrödinger operator having the form (we refer to Subsection 2.1 for more explanations)

$$\Delta_{f,h}^{(0)} = h^2 \Delta + |\nabla f|^2 + h \Delta f$$

and is, up to multiplication by h , unitarily equivalent to the Ornstein-Uhlenbeck type operator acting on the weighted space $L^2(e^{-\frac{2f}{h}} d\mu)$, sometimes referred to as the weighted Laplacian (or Bakry - Émery Laplacian) in the literature (see e.g. [KoMi]),

$$L_{V,h}^{(0)} := h \Delta + \nabla V \cdot \nabla \quad \text{where} \quad V = 2f. \quad (1.26)$$

More precisely, it holds

$$\Delta_{f,h}^{(0)} = e^{-\frac{V}{2h}} h L_{V,h}^{(0)} e^{\frac{V}{2h}} \quad \text{where} \quad V = 2f.$$

The operator $L_{V,h}^{(0)}$ has consequently a natural supersymmetric extension on the algebra of differential forms, acting in the weighted space $\Lambda L^2(e^{-\frac{2f}{h}} d\mu)$, which is simply defined for any $p \in \{0, \dots, n\}$ by

$$L_{V,h}^{(p)} := \frac{1}{h} e^{\frac{f}{h}} \Delta_{f,h}^{(p)} e^{-\frac{f}{h}} \quad \text{where} \quad V = 2f. \quad (1.27)$$

To connect more precisely to the literature dealing with the Bakry - Émery theory of diffusion processes (see [BGL] for an overview), the operators $L_{V,h}^{(0)}$ and $L_{V,h}^{(1)}$ are related to the carré du champ operator of Bakry - Émery Γ and to its iteration Γ_2 via the relations

$$\int_{\Omega} \Gamma(\omega) e^{-\frac{2f}{h}} d\mu = \int_{\Omega} (L_{V,h}^{(0)} \omega) \omega e^{-\frac{2f}{h}} d\mu = h \int_{\Omega} \langle d\omega, d\omega \rangle_{\Lambda^1} e^{-\frac{2f}{h}} d\mu \quad (1.28)$$

and

$$\int_{\Omega} \Gamma_2(\omega) e^{-\frac{2f}{h}} d\mu = \int_{\Omega} (L_{V,h}^{(0)} \omega)^2 e^{-\frac{2f}{h}} d\mu = h \int_{\Omega} \langle L_{V,h}^{(1)} d\omega, d\omega \rangle_{\Lambda^1} e^{-\frac{2f}{h}} d\mu, \quad (1.29)$$

where ω is a smooth function supported in $\Omega \setminus \partial\Omega$ (see in particular [BGL] for many details and references about this notion). We denote moreover by $L_{V,h}^{\mathbf{t},(p)}$ and $L_{V,h}^{\mathbf{n},(p)}$ the nonnegative self-adjoint unbounded operators on $\Lambda^p L^2(e^{-\frac{V}{h}} d\mu)$ associated with $\Delta_{f,h}^{\mathbf{t},(p)}$ and $\Delta_{f,h}^{\mathbf{n},(p)}$ via (1.27).

Coming back to the Witten Laplacian, we have the following formula:

$$\Delta_{f,h}^{(p)} = h^2 (d + d^*)^2 + |\nabla f|^2 + 2h \text{Hess}^{(p)} f + h \Delta f. \quad (1.30)$$

This relation is not very common in the semiclassical analysis literature when Ω is not flat, where one finds generally the formula (2.10) given in Subsection 2.1 (see e.g. [HeNi, Lep] and references therein). It will thus be proven there (see also [Jam] for another proof). Let us incidentally specify the sense of (1.30). There, $\text{Hess}^{(0)}f = 0$ and $\text{Hess}^{(1)}f$ is the element of $\mathcal{L}(\Lambda^1 T^* \Omega)$ canonically identified with $\text{Hess} f$ (see the lines below (1.13) for more details). More precisely, we have for any ω and η in $T^* \Omega$,

$$\langle \text{Hess}^{(1)}f \omega, \eta \rangle_{\Lambda^1} = \text{Hess}f(\omega, \eta) = \text{Hess}f(\omega^\sharp, \eta^\sharp),$$

and $\text{Hess}^{(p)}f$ is the bundle symmetric endomorphism of $\Lambda^p T^* \Omega$ defined by

$$\text{Hess}^{(p)}f := (\text{Hess}^{(1)}f)^{(p)} \quad (\text{see (1.16) for the meaning of } (A)^{(p)}). \quad (1.31)$$

Denoting also by $\text{Hess}f$ the bundle symmetric endomorphism of $T\Omega$ defined by $\langle \text{Hess}f X, Y \rangle := \text{Hess}f(X, Y)$ (i.e. by $\text{Hess}f X = \nabla_X \nabla f$), remark that we have for any ω, η in $T^* \Omega$ and X in $T\Omega$, according to Remark 1.1,

$$\text{Hess}^{(1)}f \omega(X) = \omega(\text{Hess}f X).$$

In particular, $\text{Hess}^{(p)}f$ satisfies for any $p \in \{1, \dots, n\}$ and $\omega \in \Lambda^p T^* \Omega$:

$$\text{Hess}^{(p)}f \omega(X_1, \dots, X_p) = \sum_{i=1}^p \omega(X_1, \dots, \text{Hess}f X_i, \dots, X_p). \quad (1.32)$$

1.4 Statement of our results

We consider here f a smooth (real) function, $V := 2f$, and the probability measure ν_h associated with V and $h > 0$ defined by

$$d\nu_h := \frac{e^{-\frac{V}{h}}}{\int_{\Omega} e^{-\frac{V}{h}} d\mu} d\mu = \frac{e^{-\frac{2f}{h}}}{\|e^{-\frac{f}{h}}\|_{L^2}^2} d\mu.$$

We denote, for $p \in \{0, \dots, n\}$, by $\Lambda^p L^2(d\nu_h)$, $\Lambda^p H^1(d\nu_h)$, $\langle \cdot, \cdot \rangle_{\Lambda^p L^2(d\nu_h)}$ and $\|\cdot\|_{\Lambda^p L^2(d\nu_h)}$ the associated Lebesgue and Sobolev spaces, L^2 scalar product and L^2 norm. For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$, the self-adjoint unbounded operators $\Delta_{f,h}^{\mathbf{b},(p)}$ and $L_{V,h}^{\mathbf{b},(p)}$ act respectively on $\Lambda^p L^2$ and on $\Lambda^p L^2(d\nu_h)$, the domain of $\Delta_{f,h}^{\mathbf{b},(p)}$ being given in (1.22)–(1.23) and the one of $L_{V,h}^{\mathbf{b},(p)}$ being easily deduced from the latter thanks to the relation (1.27). We also denote by $\Lambda^p H_{\mathbf{b}}^1(d\nu_h)$ the set of the $\omega \in \Lambda^p H^1(d\nu_h)$ such that $\mathbf{b}\omega = 0$ on $\partial\Omega$, which is the domain of the quadratic form associated with $L_{V,h}^{\mathbf{b},(p)}$ according to (1.24) and (1.27). Since we are working on a compact manifold, note that $\Lambda^p H_{\mathbf{b}}^1(d\nu_h)$ is nothing but

$\Lambda^p H_{\mathbf{b}}^1$ (algebraically and topologically).

In addition to the material of Riemannian geometry already recalled previously, the following statements involve a smooth bundle endomorphism $\mathcal{K}_{\mathbf{b}}^{(p)} \in \mathcal{L}(\Lambda^p T^* \Omega|_{\partial\Omega})$, where $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$, determined by the second fundamental form \mathcal{K}_2 of $\partial\Omega \subset \Omega$ defined in (1.17):

1. For any $p \in \{0, \dots, n\}$, $\mathcal{K}_{\mathbf{n}}^{(p)} \in \mathcal{L}(\Lambda^p T^* \Omega|_{\partial\Omega})$ vanishes on 0-forms and:

i) for any $\omega \in \Lambda^1 T^* \Omega$, $\mathcal{K}_{\mathbf{n}}^{(1)} \omega$ is tangential and

$$(\mathcal{K}_{\mathbf{n}}^{(1)} \omega)(X^T + x^\perp \vec{n}) = -\omega(\mathcal{K}_1(X^T)) = \omega(\nabla_{X^T} \vec{n}), \quad (1.33)$$

where \mathcal{K}_1 is the shape operator defined in (1.18),

ii) for any $p \in \{1, \dots, n\}$ and $\omega \in \Lambda^p T^* \Omega$, $\mathcal{K}_{\mathbf{n}}^{(p)} \omega$ is tangential and for any $X_1^T, \dots, X_p^T \in T\partial\Omega$,

$$(\mathcal{K}_{\mathbf{n}}^{(p)} \omega)(X_1^T, \dots, X_p^T) = ((\mathcal{K}_{\mathbf{n}}^{(1)})^{(p)} \omega)(X_1^T, \dots, X_p^T), \quad (1.34)$$

where the notation $(A)^{(p)}$ has been defined in (1.16).

2. For any $p \in \{0, \dots, n\}$, $\mathcal{K}_{\mathbf{t}}^{(p)} \in \mathcal{L}(\Lambda^p T^* \Omega|_{\partial\Omega})$ vanishes on 0-forms and:

i) for any $\omega \in \Lambda^1 T^* \Omega$, $\mathcal{K}_{\mathbf{t}}^{(1)} \omega$ is normal and

$$(\mathcal{K}_{\mathbf{t}}^{(1)} \omega)(X^T + x^\perp \vec{n}) = -x^\perp \text{Tr}(\mathcal{K}_1) \omega(\vec{n}), \quad (1.35)$$

ii) for any $p \in \{1, \dots, n\}$ and $\omega \in \Lambda^p T^* \Omega$, $\mathcal{K}_{\mathbf{t}}^{(p)} \omega$ is normal and for any local orthonormal frame (E_1, \dots, E_n) on $U \subset \Omega$ such that $E_n|_{\partial\Omega} = \vec{n}$ (with $U \cap \partial\Omega \neq \emptyset$) and $X_1^T, \dots, X_{p-1}^T \in T\partial\Omega$, we have on $U \cap \partial\Omega$:

$$\begin{aligned} & (\mathcal{K}_{\mathbf{t}}^{(p)} \omega)(\vec{n}, X_1^T, \dots, X_{p-1}^T) \\ & := - \sum_{i=1}^{n-1} \left((\mathcal{K}_2(E_i, \cdot))^{(p)} \omega \right)(E_i, X_1^T, \dots, X_{p-1}^T), \end{aligned} \quad (1.36)$$

where $(\mathcal{K}_2(E_i, \cdot))^{(p)} = \left((\mathcal{K}_2(E_i, \cdot))^{(1)} \right)^{(p)}$ and

$$\left((\mathcal{K}_2(E_i, \cdot))^{(1)} \omega \right)(X) = \omega(\mathcal{K}_2(E_i, X)).$$

Note that the point 2.ii) is nothing but the statement of 2.i) when $p = 1$.

The different Brascamp-Lieb's type inequalities stated in this work arise from the following integration by parts formulas relating the quadratic forms $\mathcal{D}_{f,h}^{\mathbf{t},(p)}$ and $\mathcal{D}_{f,h}^{\mathbf{n},(p)}$ (see (1.25)) with the geometry of Ω .

Theorem 1.2. *Let $\omega \in \Lambda^p H_{\mathbf{b}}^1$ with $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$. It holds*

$$\begin{aligned} \mathcal{D}_{f,h}^{\mathbf{b},(p)}(\omega) &= h^2 \left\| e^{\frac{f}{h}} \omega \right\|_{\Lambda^p(H^1-L^2)(e^{-\frac{2f}{h}} d\mu)}^2 + h \langle (h \operatorname{Ric}^{(p)} + 2 \operatorname{Hess}^{(p)} f) \omega, \omega \rangle_{\Lambda^p L^2} \\ &\quad + h^2 \int_{\partial\Omega} \langle \mathcal{K}_{\mathbf{b}}^{(p)} \omega, \omega \rangle_{\Lambda^p} d\mu_{\partial\Omega} - 2h \mathbf{1}_{\mathbf{t}}(\mathbf{b}) \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f d\mu_{\partial\Omega}, \end{aligned} \quad (1.37)$$

where $\operatorname{Ric}^{(p)}$, $\operatorname{Hess}^{(p)} f$, and $\mathcal{K}_{\mathbf{b}}^{(p)}$ have been respectively defined in (1.15), (1.31), and (1.33)–(1.36), $\mathbf{1}_{\mathbf{t}}(\mathbf{b}) = 1$ if $\mathbf{b} = \mathbf{t}$ and 0 if not, and

$$\| \cdot \|_{\Lambda^p(H^1-L^2)(e^{-\frac{2f}{h}} d\mu)}^2 := \| \cdot \|_{\Lambda^p H^1(e^{-\frac{2f}{h}} d\mu)}^2 - \| \cdot \|_{\Lambda^p L^2(e^{-\frac{2f}{h}} d\mu)}^2.$$

When $f = 0$ and $h = 1$, we recover Theorems 2.1.5 and 2.1.7 of [Sch] which were generalizing results in the boundaryless case due to Bochner for $p = 1$ and to Gallot and Meyer for general p 's (see [Boc, GaMe]). These results allow in particular to draw topological conclusions on the cohomology of Ω from its geometry. When the boundary $\partial\Omega$ is not empty, the relative and absolute cohomologies of Ω (corresponding respectively to the Dirichlet and Neumann boundary conditions) have to be considered (see [Sch, Section 2.6]). To be more precise, note from Theorem 1.2 that for any $p \in \{0, \dots, n\}$, the (everywhere) positivity of the quadratic form $h \operatorname{Ric}^{(p)} + 2 \operatorname{Hess}^{(p)} f$ together with the nonnegativity of $\mathcal{K}_{\mathbf{n}}^{(p)}$ (resp. of $\mathcal{K}_{\mathbf{t}}^{(p)} - 2h \partial_n f$) implies the lower bounds (in the sense of quadratic forms)

$$\Delta_{f,h}^{\mathbf{b},(p)} \geq h^2 \operatorname{Ric}^{(p)} + 2h \operatorname{Hess}^{(p)} f > 0 \quad (\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\})$$

for the Witten Laplacian and hence the triviality of its kernel which is isomorphic to the p -th absolute (resp. relative) cohomology group of Ω when $f = 0$ (and $h = 1$).

Playing moreover with the supersymmetry, we get easily the following Brascamp-Lieb's type inequalities for differential forms, where for any $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, $\pi_{\mathbf{b}} = \pi_{\mathbf{b}}^{(p)}(h)$ denotes the orthogonal projection on $\operatorname{Ker}(L_{V,h}^{\mathbf{b},(p)})$.

Theorem 1.3 (Brascamp-Lieb's inequalities for differential forms).

1. Let $p \in \{0, \dots, n\}$ and let us assume that $\mathcal{K}_n^{(p)} \geq 0$ everywhere on $\partial\Omega$ in the sense of quadratic forms. It then holds for every $h > 0$ such that $\text{Ric}_{V,h}^{(p)} := h \text{Ric}^{(p)} + \text{Hess}^{(p)}V > 0$ everywhere on Ω (in the sense of quadratic forms):

i) if $p > 0$, we have for every $\omega \in \Lambda^{p-1}H_n^1(d\nu_h)$ such that $d_{V,h}^*\omega = 0$:

$$\|\omega - \pi_n \omega\|_{\Lambda^{p-1}L^2(d\nu_h)}^2 \leq h \int_{\Omega} \langle (\text{Ric}_{V,h}^{(p)})^{-1} d\omega, d\omega \rangle_{\Lambda^p} d\nu_h,$$

ii) if $p < n$, we have for every $\omega \in \Lambda^{p+1}H_n^1(d\nu_h)$ such that $d\omega = 0$:

$$\|\omega - \pi_n \omega\|_{\Lambda^{p+1}L^2(d\nu_h)}^2 \leq h \int_{\Omega} \langle (\text{Ric}_{V,h}^{(p)})^{-1} d_{\frac{V}{h},1}^* \omega, d_{\frac{V}{h},1}^* \omega \rangle_{\Lambda^p} d\nu_h.$$

2. It holds similarly for every $h > 0$ such that $h \mathcal{K}_t^{(p)} - \partial_n V \geq 0$ everywhere on $\partial\Omega$ and $\text{Ric}_{V,h}^{(p)} = h \text{Ric}^{(p)} + \text{Hess}^{(p)}V > 0$ everywhere on Ω :

i) if $p > 0$, we have for every $\omega \in \Lambda^{p-1}H_t^1(d\nu_h)$ such that $d_{V,h}^*\omega = 0$:

$$\|\omega - \pi_t \omega\|_{\Lambda^{p-1}L^2(d\nu_h)}^2 \leq h \int_{\Omega} \langle (\text{Ric}_{V,h}^{(p)})^{-1} d\omega, d\omega \rangle_{\Lambda^p} d\nu_h,$$

ii) if $p < n$, we have for every $\omega \in \Lambda^{p+1}H_t^1(d\nu_h)$ such that $d\omega = 0$:

$$\|\omega - \pi_t \omega\|_{\Lambda^{p+1}L^2(d\nu_h)}^2 \leq h \int_{\Omega} \langle (\text{Ric}_{V,h}^{(p)})^{-1} d_{\frac{V}{h},1}^* \omega, d_{\frac{V}{h},1}^* \omega \rangle_{\Lambda^p} d\nu_h.$$

We recall that the operators $d_{V,h}^*$ and $d_{\frac{V}{h},1}^*$ appearing in the above theorem are simply related by the equality $d_{V,h}^* = h d_{\frac{V}{h},1}^*$ (see indeed (1.19)).

In the case $p = 1$, the points 1.i) and 2.ii) of Theorem 1.3 take a simpler form. Every $\omega \in \Lambda^0 H^1(d\nu_h)$ satisfies indeed $d_{V,h}^*\omega = 0$. Moreover, we have simply

$$\Lambda^0 H_n^1(d\nu_h) = H^1(d\nu_h) \quad \text{and} \quad \text{Ker}(L_{V,h}^{\mathbf{n},(p)}) = \text{Span}\{1\}$$

as well as

$$\Lambda^0 H_t^1(d\nu_h) = H_0^1(d\nu_h) \quad \text{and} \quad \text{Ker}(L_{V,h}^{\mathbf{t},(p)}) = \{0\}.$$

Defining the mean of $u \in L^2(d\nu_h)$ by $\langle u \rangle_{\nu_h} := \langle u, 1 \rangle_{L^2(d\nu_h)}$, we then immediately get from Theorem 1.3 (together with (1.33) and (1.35)) the following

Corollary 1.4. *i) Assume that the shape operator \mathcal{K}_1 is nonpositive everywhere on $\partial\Omega$. It then holds for every $h > 0$ such that $h \operatorname{Ric} + \operatorname{Hess} V > 0$ everywhere on Ω : for every $\omega \in H^1(d\nu_h)$,*

$$\|\omega - \langle \omega \rangle_{\nu_h}\|_{L^2(d\nu_h)}^2 \leq h \int_{\Omega} (h \operatorname{Ric} + \operatorname{Hess} V)^{-1}(\nabla\omega, \nabla\omega) d\nu_h. \quad (1.38)$$

ii) Similarly it holds for any $h > 0$ such that $-h \operatorname{Tr}(\mathcal{K}_1) - \partial_n V \geq 0$ everywhere on $\partial\Omega$ and $h \operatorname{Ric} + \operatorname{Hess} V > 0$ everywhere on Ω : for every $\omega \in H_0^1(d\nu_h)$,

$$\|\omega\|_{L^2(d\nu_h)}^2 \leq h \int_{\Omega} (h \operatorname{Ric} + \operatorname{Hess} V)^{-1}(\nabla\omega, \nabla\omega) d\nu_h. \quad (1.39)$$

When $\Omega \setminus \partial\Omega$ appears to be a smooth open subset of \mathbf{R}^n , Ric and $\operatorname{Ric}^{(p)}$ vanish and the latter corollary as well as Theorem 1.3 then write in a simpler way just relying on a control from below of $\operatorname{Hess} V$ or $\operatorname{Hess}^{(p)}V$ instead of $\operatorname{Ric}_{V,h}^{(p)} = h \operatorname{Ric}^{(p)} + \operatorname{Hess}^{(p)}V$. One recovers in particular the usual Brascamp-Lieb's inequality when $\Omega = \mathbf{R}^n$ (even if Ω has been assumed compact here, we recover the estimate (1.1) for a finite measure $d\nu$ on \mathbf{R}^n using the first point of Corollary 1.4 for the family of measures $(d\nu|_{B(0,N)})_{N \in \mathbf{N}}$ since $B(0, N)$ is convex; see also [Joh]). Note also that the curvature effects due to $\operatorname{Ric}^{(p)}$ become negligible at the semiclassical limit $h \rightarrow 0^+$: to apply Theorem 1.3 for any small $h > 0$, it is necessary that $\operatorname{Hess}^{(p)}V > 0$. It is moreover sufficient in the Neumann case if $\mathcal{K}_{\mathbf{n}}^{(p)} \geq 0$ everywhere on $\partial\Omega$, and in the Dirichlet case if $\partial_n V \leq 0$ everywhere on $\partial\Omega$ with moreover $\partial_n V < 0$ where $\mathcal{K}_{\mathbf{t}}^{(p)} < 0$. The point ii) of Corollary 1.4 is thus irrelevant if one is interested in the semiclassical limit $h \rightarrow 0^+$.

The above results can be useful for semiclassical problems involving the low spectrum of semiclassical Witten Laplacians (or equivalently of semiclassical weighted Laplacians) in large dimension, such as problems dealing with correlation asymptotics, under some suitable (and uniform in the dimension) estimates on the eigenvalues of $\operatorname{Hess} V$ (and then of $\operatorname{Hess}^{(p)}V$) on some parts of Ω . We refer for example to [HeSj, BJS, BaMø, BaMø2] or to the more recent [DiLe] for some works exploiting this kind of estimates.

Let us lastly underline that to prove Theorem 1.3 (and then Corollary 1.4), we only use the supersymmetry structure and the relation

$$\Delta_{f,h}^{\mathbf{b},(p)} \geq h^2 \operatorname{Ric}^{(p)} + 2h \operatorname{Hess}^{(p)}f > 0$$

implied by Theorem 1.2 together with the hypotheses of Theorem 1.3. However, a control from below of the restriction $\Delta_{f,h}^{\mathbf{b},(p)} \Big|_{\text{Ran } d_{f,h}}$ for the points 1.i) and 2.i) (resp. of $\Delta_{f,h}^{\mathbf{b},(p)} \Big|_{\text{Ran } d_{f,h}^*}$ for the points 1.ii) and 2.ii)) would actually be sufficient as it can be seen by looking for example at the further relation (3.6) generalizing (1.2) (see also Proposition 2.3 for more details about the latter restrictions). The specific form of the nonnegative first term in the r.h.s. of the integration by parts formula (1.37) stated in Theorem 1.2 is moreover not used, i.e. only its nonnegativity comes into play. When $p = 1$, we can easily slightly improve Corollary 1.4 taking advantage of this nonnegative term which allows to compare $\Delta_{f,h}^{\mathbf{b},(1)} \Big|_{\text{Ran } d_{f,h}}$ (or equivalently $L_{V,h}^{\mathbf{b},(1)} \Big|_{\text{Ran } d}$) with the so-called N -dimensional Bakry-Émery tensor (depending also on the semiclassical parameter $h > 0$ here)

$$\text{Ric}_{V,h,N} := h \text{Ric} + \text{Hess } V - \frac{1}{(N-n)h} dV \otimes dV, \quad V = 2f, \quad (1.40)$$

where $N \in (-\infty, +\infty]$ and $\text{Ric}_{V,h,n}$ is defined iff V is constant. The hypotheses of Corollary 1.4 require in particular the (everywhere) positivity of $\text{Ric}_{V,h,+\infty}$ and we have more generally the

Corollary 1.5. *In the following, we assume that $N \in (-\infty, 0] \cup [n, +\infty]$, or equivalently that $\frac{1}{N} \in [-\infty, \frac{1}{n}]$ with the convention $\frac{1}{0} = -\infty$ and $\frac{1}{+\infty} = 0$.*

i) Assume that $\mathcal{K}_1 \leq 0$ everywhere on $\partial\Omega$. It then holds for every $h > 0$ such that $\text{Ric}_{V,h,N} > 0$ everywhere on Ω : for every $\omega \in H^1(d\nu_h)$,

$$\|\omega - \langle \omega \rangle_{\nu_h}\|_{L^2(d\nu_h)}^2 \leq \frac{N-1}{N} h \int_{\Omega} (\text{Ric}_{V,h,N})^{-1}(\nabla\omega, \nabla\omega) d\nu_h.$$

ii) Similarly it holds for any $h > 0$ such that that $-h \text{Tr}(\mathcal{K}_1) - \partial_n V \geq 0$ everywhere on $\partial\Omega$ and $\text{Ric}_{V,h,N} > 0$ on Ω : for every $\omega \in H_0^1(d\nu_h)$,

$$\|\omega\|_{L^2(d\nu_h)}^2 \leq \frac{N-1}{N} h \int_{\Omega} (\text{Ric}_{V,h,N})^{-1}(\nabla\omega, \nabla\omega) d\nu_h.$$

When $h = 1$, we recover in particular the cases (1) and (2) of Theorem 1.2 in the recent article [KoMi] to which we also refer for more details and references concerning the N -dimensional Bakry-Émery tensor and its connections with the Bakry-Émery operators Γ and Γ_2 (see (1.28), (1.29), and also [BGL]). The authors derive these formulas from the so-called generalized Reilly formula stated in Theorem 1.1 there, which somehow generalizes, in the weighted space setting, the statement given by Theorem 1.2 when

$h = 1$, $p = 1$, and ω has the form $d_{f,h}\eta = d_{f,1}\eta$, to arbitrary $\omega = d_{f,1}\eta$ which are not assumed tangential nor normal.

Note lastly that Corollary 1.5 does not provide any improvement in comparison with Corollary 1.4 at the semiclassical limit $h \rightarrow 0^+$ because of the term $\frac{1}{(N-n)h} dV \otimes dV$ involved in $\text{Ric}_{V,h,N}$ (see indeed (1.40)).

1.5 Plan of the paper

In the following section, we recall general properties of the Witten Laplacian (in Subsections 2.1 and 2.3) as well as different Green's formulas (in Subsection 2.2). We prove in particular the writing (1.30) in Subsections 2.1 and give the basic properties of the self-adjoint realizations $\Delta_{f,h}^{\mathbf{t},(p)}$ and $\Delta_{f,h}^{\mathbf{n},(p)}$ of the Witten Laplacian in Subsection 2.3. In Section 3, we prove Theorem 1.2, Theorem 1.3, and Corollary 1.5. After this long introduction where we recalled quite precisely the different notions of Riemannian geometry in play, these proofs are rather straightforward.

2 General properties

2.1 Witten Laplacians and Riemannian geometry

We recall that the Levi-Civita ∇ connection is the only bilinear application satisfying the following properties (where f is a smooth function on Ω):

$$\begin{aligned} \nabla_X(fY) &= f\nabla_X Y + df(X)Y = f\nabla_X Y + (\nabla_X f)Y, \\ \nabla_{fX} Y &= f\nabla_X Y, \\ \nabla_X \langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \\ \text{and } \nabla_X Y - \nabla_Y X &= [X, Y]. \end{aligned}$$

The induced covariant derivative ∇_X on $\Lambda^p T^* \Omega$ is defined by

$$\begin{aligned} (\nabla_X \omega)(Y_1, \dots, Y_p) &:= \nabla_X(\omega(Y_1, \dots, Y_p)) \\ &\quad - \sum_{k=1}^p \omega(Y_1, \dots, \nabla_X Y_k, \dots, Y_p) \end{aligned} \quad (2.1)$$

and satisfies in particular the relations

$$\nabla_X(\langle \omega, \eta \rangle_{\Lambda^p}) = \langle \nabla_X \omega, \eta \rangle_{\Lambda^p} + \langle \omega, \nabla_X \eta \rangle_{\Lambda^p} \quad (2.2)$$

and

$$\nabla_X(\omega_1 \wedge \omega_2) = (\nabla_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\nabla_X \omega_2). \quad (2.3)$$

The differential d and ∇ are moreover related by the relation

$$d\omega(X_0, \dots, X_p) = \sum_{k=0}^p (-1)^k (\nabla_{X_k} \omega)(X_0, \dots, \dot{X}_k, \dots, X_p) \quad (2.4)$$

where $\omega \in \Lambda^p T^* \Omega$ and the notation \dot{X}_k means that X_k has been removed from the parenthesis. Furthermore, if (E_1, \dots, E_n) is a local orthonormal frame on $U \subset \Omega$, the codifferential d^* is given there by

$$d^* = - \sum_{i=1}^n \mathbf{i}_{E_i} \nabla_{E_i} \quad (2.5)$$

and we have in particular at the level of functions:

$$\Delta^{(0)} = d^* d = - \sum_{i=1}^n (\nabla_{E_i} \nabla_{E_i} - \nabla_{\nabla_{E_i} E_i}).$$

Note that with our convention, $\Delta^{(0)}$ equals minus the usual Laplacian on the flat space and that $\Delta_B^{(0)} = \Delta^{(0)}$ according to (1.9).

Using the following relations dealing with exterior and interior products (respectively denoted by \wedge and \mathbf{i}), gradients (denoted by ∇) and Lie derivatives (denoted by \mathcal{L}),

$$(df \wedge)^* = \mathbf{i}_{\nabla f} \quad \text{as bounded operators in } L^2(\Omega, \Lambda^p T^* \Omega), \quad (2.6)$$

$$d_{f,h} = h d + df \wedge, \quad (2.7)$$

$$d_{f,h}^* = h d^* + \mathbf{i}_{\nabla f}, \quad (2.8)$$

$$\mathcal{L}_X = d \circ \mathbf{i}_X + \mathbf{i}_X \circ d \quad \text{and} \quad \mathcal{L}_X^* = d^* \circ (X^\flat \wedge \cdot) + X^\flat \wedge d^*, \quad (2.9)$$

the Witten Laplacian has the simple form

$$\Delta_{f,h} = h^2 (d + d^*)^2 + |\nabla f|^2 + h (\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*). \quad (2.10)$$

We end up this part by proving the other writing of $\Delta_{f,h}$ given in (1.30):

$$\Delta_{f,h}^{(p)} = h^2 (d + d^*)^2 + |\nabla f|^2 + 2h \text{Hess}^{(p)} f + h \Delta f.$$

First, we deduce from (1.32) and from the relation relating \mathcal{L}_X and ∇_X ,

$$(\mathcal{L}_X^{(p)} \omega)(X_1, \dots, X_p) = (\nabla_X \omega)(X_1, \dots, X_p) + \sum_{i=1}^p \omega(X_1, \dots, \nabla_{X_i} X, \dots, X_p)$$

which arises from (2.1), (2.4), and (2.9), the following equality:

$$\mathcal{L}_{\nabla f}^{(p)} = \nabla_{\nabla f} + \text{Hess}^{(p)} f. \quad (2.11)$$

Taking now a local orthonormal frame (E_1, \dots, E_n) on an open set $U \subset \Omega$, we deduce from (2.3), (2.5), and (2.9) the following relations (on U):

$$\begin{aligned} \mathcal{L}_{\nabla f}^{*(p)} \omega &= \sum_{i=1}^n \left((\nabla_{E_i} df) \wedge \mathbf{i}_{E_i} \omega - df(E_i) \nabla_{E_i} \omega - (\nabla_{E_i} df(E_i)) \right) \omega \\ &= -\nabla_{\nabla f} \omega + (\Delta f) \omega + \sum_{i=1}^n (\nabla_{E_i} df) \wedge \mathbf{i}_{E_i} \omega. \end{aligned} \quad (2.12)$$

Lastly, we have

$$\begin{aligned} \sum_{i=1}^n \left((\nabla_{E_i} df) \wedge \mathbf{i}_{E_i} \omega \right) (X_1, \dots, X_p) &= \sum_{i=1}^n \sum_{k=1}^p (-1)^{k+1} (\nabla_{E_i} df)(X_k) (\mathbf{i}_{E_i} \omega)(X_1, \dots, \dot{X}_k, \dots, X_p) \\ &= \sum_{k=1}^p (-1)^{k+1} \omega(\text{Hess } f X_k, X_1, \dots, \dot{X}_k, \dots, X_p) \\ &= \sum_{k=1}^p \omega(X_1, \dots, \text{Hess } f X_k, \dots, X_p) \\ &= (\text{Hess}^{(p)} f \omega)(X_1, \dots, X_p) \end{aligned} \quad (2.13)$$

and the writing (1.30) for the Witten Laplacian then follows from (2.10) and (2.11)–(2.13).

2.2 Stokes' and Green's formulas

We first recall here the Stokes' formula which writes

$$\forall \omega \in \Lambda^p \mathcal{C}^\infty, \quad \int_{\Omega} d\omega = \int_{\partial\Omega} j^* \omega = \int_{\partial\Omega} \mathbf{t}\omega, \quad (2.14)$$

where in the last term we made the identification between the canonical pull back $j^* \omega = j^*(\mathbf{t}\omega) \in T\partial\Omega$ associated with the embedding $j : \partial\Omega \rightarrow \Omega$ and $\mathbf{t}\omega \in T\Omega|_{\partial\Omega}$.

Following the presentation of [Sch], the Sobolev space $\Lambda^p H^s = H^s(\Omega, \Lambda^p T^* \Omega)$ (resp. $H^s(\partial\Omega, \Lambda^p T^* \partial\Omega)$) with integer order s is defined as the completion of $\Lambda^p \mathcal{C}^\infty$ (resp. $\mathcal{C}^\infty(\partial\Omega, \Lambda^p T^* \partial\Omega)$) by means of g (resp. $j^* g$), of local orthonormal frames, and of the Levi-Civita connection on Ω (resp. $\partial\Omega$). More details are given in [Sch, Section 1.3] and we just recall from this reference the following: the definition of the H^s norms, which is then subordinated to the (nice) cover of Ω (or $\partial\Omega$) chosen, depends therefore on this cover, even though the

inherited H^s topology is independent of this cover. The H^1 norm is in particular defined on (say) Ω , for a nice cover $(U_j)_{j \in \{1, \dots, K\}}$ with associated local orthonormal frames (E_1, \dots, E_n) (we drop the dependence on j to lighten the notation) and a subordinated partition of unity $(\rho_j)_{j \in \{1, \dots, K\}}$, by

$$\|\cdot\|_{\Lambda^p H^1}^2 := \|\cdot\|_{\Lambda^p L^2}^2 + \sum_{j=1}^K \sum_{i=1}^n \int_{U_j} \rho_j \langle \nabla_{E_i} \cdot, \nabla_{E_i} \cdot \rangle_{\Lambda^p} d\mu, \quad (2.15)$$

the last expression being actually independent of the cover considered in the H^1 case as explained in [Sch, p. 31]. We will thus simply write in the sequel

$$\|\cdot\|_{\Lambda^p(H^1-L^2)}^2 := \|\cdot\|_{\Lambda^p H^1}^2 - \|\cdot\|_{\Lambda^p L^2}^2 = \sum_{i=1}^n \langle \nabla_{E_i} \cdot, \nabla_{E_i} \cdot \rangle_{\Lambda^p L^2} \quad (2.16)$$

without mentioning the cover $(U_j)_{j \in \{1, \dots, K\}}$, the subordinated partition of unity $(\rho_j)_{j \in \{1, \dots, K\}}$ and the dependence of the associated local orthonormal frames (E_1, \dots, E_n) on j . Note that these definitions extend the usual ones when $\Omega \setminus \partial\Omega$ is a smooth bounded open set of \mathbf{R}^n .

We end up this section by stating various Green's formulas arising from the Stokes' formula (2.14) and from the definition of the Sobolev spaces. First, it leads to the usual Green's formula (see [Sch, Propositions 1.2.6 and 2.1.2]),

$$\begin{aligned} \forall(\omega, \eta) \in \Lambda^{p-1} H^1 \times \Lambda^p H^1, \\ \langle d\omega, \eta \rangle_{\Lambda^p L^2} = \langle \omega, d^* \eta \rangle_{\Lambda^{p-1} L^2} + \int_{\partial\Omega} \langle \omega, \mathbf{i}_{\bar{n}} \eta \rangle_{\Lambda^p} d\mu_{\partial\Omega}, \end{aligned} \quad (2.17)$$

from which we get immediately (using (2.6)–(2.8)),

$$\langle d_{f,h}\omega, \eta \rangle_{\Lambda^p L^2} = \langle \omega, d_{f,h}^* \eta \rangle_{\Lambda^{p-1} L^2} + h \int_{\partial\Omega} \langle \omega, \mathbf{i}_{\bar{n}} \eta \rangle_{\Lambda^p} d\mu_{\partial\Omega}. \quad (2.18)$$

We now respectively deduce from (2.17) and from (2.18) the following Green's formulas for the Hodge and Witten Laplacians: for every $\omega \in \Lambda^p H^2$ and $\eta \in \Lambda^p H^1$, it holds

$$\begin{aligned} \langle d\omega, d\eta \rangle_{\Lambda^{p+1} L^2} + \langle d^* \omega, d^* \eta \rangle_{\Lambda^{p-1} L^2} \\ = \langle \Delta \omega, \eta \rangle_{\Lambda^p L^2} + \int_{\partial\Omega} (\langle \mathbf{i}_{\bar{n}} d\omega, \eta \rangle_{\Lambda^p} - \langle d^* \omega, \mathbf{i}_{\bar{n}} \eta \rangle_{\Lambda^{p-1}}) d\mu_{\partial\Omega} \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \langle d_{f,h}\omega, d_{f,h}\eta \rangle_{\Lambda^{p+1} L^2} + \langle d_{f,h}^* \omega, d_{f,h}^* \eta \rangle_{\Lambda^{p-1} L^2} \\ = \langle \Delta_{f,h} \omega, \eta \rangle_{\Lambda^p L^2} + h \int_{\partial\Omega} (\langle \mathbf{i}_{\bar{n}} d_{f,h}\omega, \eta \rangle_{\Lambda^p} - \langle d_{f,h}^* \omega, \mathbf{i}_{\bar{n}} \eta \rangle_{\Lambda^{p-1}}) d\mu_{\partial\Omega}. \end{aligned} \quad (2.20)$$

The Stokes' formula (2.14) also leads to the “divergence” Green's formula (see [Sch, Corollary 2.1.3]):

$$\forall \omega \in \Lambda^1 H^1, \quad \int_{\Omega} (d^* \chi) d\mu = - \int_{\partial\Omega} \chi(\vec{n}) d\mu_{\partial\Omega}. \quad (2.21)$$

We deduce moreover from (2.21), (1.9), and (2.2) the following Green's formula for the Bochner Laplacian: for every ω and η in $\Lambda^p H^2$, it holds

$$\langle \Delta_B \omega, \eta \rangle_{\Lambda^p L^2} + \langle \omega, \Delta_B \eta \rangle_{\Lambda^p L^2} + \int_{\partial\Omega} \nabla_{\vec{n}} \langle \omega, \eta \rangle_{\Lambda^p} d\mu_{\partial\Omega} = 2 \sum_{i=1}^n \langle \nabla_{E_i} \omega, \nabla_{E_i} \eta \rangle_{\Lambda^p L^2}.$$

Using the notation defined in (2.16), we obtain in particular the following relation for every $\omega \in \Lambda^p H^2$,

$$\|\omega\|_{\Lambda^p(H^1-L^2)}^2 = \langle \Delta_B \omega, \omega \rangle_{\Lambda^p L^2} + \int_{\partial\Omega} \langle \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p} d\mu_{\partial\Omega}. \quad (2.22)$$

2.3 Self-adjoint realizations of the Witten Laplacian

In the sequel, we will use for any $(\omega, \eta) \in (\Lambda^p H^1)^2$ the more compact notation

$$\mathcal{D}_{f,h}^{(p)}(\omega, \eta) := \langle d_{f,h} \omega, d_{f,h} \eta \rangle_{\Lambda^{p+1} L^2} + \langle d_{f,h}^* \omega, d_{f,h}^* \eta \rangle_{\Lambda^{p-1} L^2}$$

as well as

$$\mathcal{D}_{f,h}^{(p)}(\omega) := \mathcal{D}_{f,h}^{(p)}(\omega, \omega)$$

and

$$\mathcal{D}^{(p)}(\omega, \eta) := \mathcal{D}_{0,1}^{(p)}(\omega, \eta) \quad \text{and} \quad \mathcal{D}^{(p)}(\omega) := \mathcal{D}^{(p)}(\omega, \omega).$$

Let us also recall, for $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$, the definition of $\Lambda^p H_{\mathbf{b}}^1$ given in (1.24):

$$\Lambda^p H_{\mathbf{b}}^1 = \{ \omega \in \Lambda^p H^1, \mathbf{b}\omega = 0 \text{ on } \partial\Omega \}.$$

In particular, $\Lambda^0 H_{\mathbf{n}}^1 = H_{\mathbf{n}}^1$ is simply $H^1(\Omega)$ while $H_{\mathbf{t}}^1 = H_0^1(\Omega)$. Moreover, since the boundary $\partial\Omega$ is smooth, the space

$$\Lambda^p \mathcal{C}_{\mathbf{b}}^{\infty} := \{ \omega \in \Lambda^p \mathcal{C}^{\infty}, \mathbf{b}\omega = 0 \text{ on } \partial\Omega \}$$

is dense in $(\Lambda^p H_{\mathbf{b}}^1, \|\cdot\|_{\Lambda^p H^1})$.

The following lemma compares $\mathcal{D}^{(p)}(\cdot)$ and $\mathcal{D}_{f,h}^{(p)}(\cdot)$ on the space of normal or tangential p -forms.

Lemma 2.1. *We have the two following identities:*

i) for any $\omega \in \Lambda^p H_{\mathbf{n}}^1$,

$$\begin{aligned} \mathcal{D}_{f,h}^{(p)}(\omega) &= h^2 \mathcal{D}^{(p)}(\omega) + \| |\nabla f| \omega \|_{\Lambda^p L^2}^2 + h \langle (\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*) \omega, \omega \rangle_{\Lambda^p L^2} \\ &\quad + h \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f \, d\mu_{\partial\Omega}, \end{aligned} \quad (2.23)$$

ii) and for any $\omega \in \Lambda^p H_{\mathbf{t}}^1$,

$$\begin{aligned} \mathcal{D}_{f,h}^{(p)}(\omega) &= h^2 \mathcal{D}^{(p)}(\omega) + \| |\nabla f| \omega \|_{\Lambda^p L^2}^2 + h \langle (\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*) \omega, \omega \rangle_{\Lambda^p L^2} \\ &\quad - h \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f \, d\mu_{\partial\Omega}. \end{aligned} \quad (2.24)$$

Proof. These formulas were already proven in [HeNi, Lep]. By density of $\Lambda^p \mathcal{C}_{\mathbf{b}}^\infty$ in $\Lambda^p H_{\mathbf{b}}^1$ for $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, it is sufficient to prove them for $\omega \in \Lambda^p \mathcal{C}_{\mathbf{b}}^\infty$. But for any $\omega \in \Lambda^p \mathcal{C}^\infty$, we get from the Green's formulas for the Hodge and Witten Laplacians (2.19) and (2.20), and from the form (2.10) of $\Delta_{f,h}$ together with (2.7) and (2.8):

$$\begin{aligned} \mathcal{D}_{f,h}^{(p)}(\omega) &= h^2 \mathcal{D}^{(p)}(\omega) + \| |\nabla f| \omega \|_{\Lambda^p L^2}^2 + h \langle (\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*) \omega, \omega \rangle_{\Lambda^p L^2} \\ &\quad + h \int_{\partial\Omega} (\langle \mathbf{i}_{\bar{n}}(df \wedge \omega), \omega \rangle_{\Lambda^p} - \langle \mathbf{i}_{\nabla f} \omega, \mathbf{i}_{\bar{n}} \omega \rangle_{\Lambda^{p-1}}) \, d\mu_{\partial\Omega}. \end{aligned} \quad (2.25)$$

Moreover, if $\mathbf{n}\omega = 0$, it holds

$$\langle \mathbf{i}_{\nabla f} \omega, \mathbf{i}_{\bar{n}} \omega \rangle_{\Lambda^{p-1}} = 0 \quad \text{and} \quad \langle \mathbf{i}_{\bar{n}}(df \wedge \omega), \omega \rangle_{\Lambda^p} = \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f, \quad (2.26)$$

and if $\mathbf{t}\omega = 0$, it holds similarly

$$\langle \mathbf{i}_{\bar{n}}(df \wedge \omega), \omega \rangle_{\Lambda^p} = 0 \quad \text{and} \quad \langle \mathbf{i}_{\nabla f} \omega, \mathbf{i}_{\bar{n}} \omega \rangle_{\Lambda^{p-1}} = \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f. \quad (2.27)$$

The statement of Lemma 2.1 is then an immediate consequence of (2.25) together with (2.26) and (2.27). \blacksquare

We now compile in the following proposition basic facts about Witten Laplacians on manifolds with boundary. They are proven in [HeNi, Section 2.4] and in [Lep, Section 2.3].

Proposition 2.2. *i) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the nonnegative quadratic form $\omega \rightarrow \mathcal{D}_{f,h}^{(p)}(\omega)$ is closed on $\Lambda^p H_{\mathbf{b}}^1$. We denote by $(\Delta_{f,h}^{\mathbf{b},(p)}, D(\Delta_{f,h}^{\mathbf{b},(p)}))$ its associated self-adjoint Friedrichs extension.*

ii) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the domain of $\Delta_{f,h}^{\mathbf{b},(p)}$ is given by

$$D(\Delta_{f,h}^{\mathbf{b},(p)}) = \{u \in \Lambda^p H^2, \mathbf{b}\omega = 0, \mathbf{b}d_{f,h}^* \omega = 0 \text{ and } \mathbf{b}d_{f,h} \omega = 0 \text{ on } \partial\Omega\}.$$

We have moreover:

$$\forall \omega \in D(\Delta_{f,h}^{\mathbf{b},(p)}), \quad \Delta_{f,h}^{\mathbf{b},(p)} \omega = \Delta_{f,h}^{(p)} \omega \quad \text{in } \Omega$$

and the equalities $\mathbf{n}d_{f,h}^* \omega = 0$ and $\mathbf{t}d_{f,h} \omega = 0$ are actually satisfied for any $\omega \in \Lambda^p H^2$.

iii) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, $\Delta_{f,h}^{\mathbf{b},(p)}$ has a compact resolvent.

iv) For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, the following commutation relations hold for any $v \in \Lambda^p H_{\mathbf{b}}^1$:

– for every $z \in \varrho(\Delta_{f,h}^{\mathbf{b},(p)}) \cap \varrho(\Delta_{f,h}^{\mathbf{b},(p+1)})$,

$$(z - \Delta_{f,h}^{\mathbf{b},(p+1)})^{-1} d_{f,h}^{(p)} v = d_{f,h}^{(p)} (z - \Delta_{f,h}^{\mathbf{b},(p)})^{-1} v$$

– and for every $z \in \varrho(\Delta_{f,h}^{\mathbf{b},(p)}) \cap \varrho(\Delta_{f,h}^{\mathbf{b},(p-1)})$,

$$(z - \Delta_{f,h}^{\mathbf{b},(p-1)})^{-1} d_{f,h}^{(p-1),*} v = d_{f,h}^{(p-1),*} (z - \Delta_{f,h}^{\mathbf{b},(p)})^{-1} v.$$

In the spirit of the above point iv), we have also the following Witten - Hodge-decomposition which will be useful when proving Corollary 1.5:

Proposition 2.3. For $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$ and $p \in \{0, \dots, n\}$, it holds

$$\begin{aligned} \Lambda^p L^2 &= \text{Ker } \Delta_{f,h}^{\mathbf{b},(p)} \oplus^\perp \text{Ran} (d_{f,h} \big|_{\Lambda^{p-1} H_{\mathbf{b}}^1}) \oplus^\perp \text{Ran} (d_{f,h}^* \big|_{\Lambda^{p+1} H_{\mathbf{b}}^1}) \\ &=: K^{\mathbf{b},(p)} \oplus^\perp R^{\mathbf{b},(p)} \oplus^\perp R^{*,\mathbf{b},(p)}, \end{aligned} \quad (2.28)$$

the spaces $R^{\mathbf{b},(p)}$ and $R^{*,\mathbf{b},(p)}$ being consequently closed in $\Lambda^p L^2$. Denoting moreover by $\pi_{R^{\mathbf{b},(p)}}$ and $\pi_{R^{*,\mathbf{b},(p)}}$ the orthogonal projectors on these respective spaces, the following relations hold in the sense of unbounded operators:

$$\pi_{R^{\mathbf{b},(p)}} \Delta_{f,h}^{\mathbf{b},(p)} \subset \Delta_{f,h}^{\mathbf{b},(p)} \pi_{R^{\mathbf{b},(p)}} \text{ and } \pi_{R^{*,\mathbf{b},(p)}} \Delta_{f,h}^{\mathbf{b},(p)} \subset \Delta_{f,h}^{\mathbf{b},(p)} \pi_{R^{*,\mathbf{b},(p)}}. \quad (2.29)$$

In particular, for $A \in \{(K^{\mathbf{b},(p)})^\perp, R^{\mathbf{b},(p)}, R^{*,\mathbf{b},(p)}\}$, the unbounded operator $\Delta_{f,h}^{\mathbf{b},(p)} \big|_A$ with domain $D(\Delta_{f,h}^{\mathbf{b},(p)}) \cap A$ is well defined, self-adjoint, invertible on A , and it holds for every $v \in \Lambda^p H_{\mathbf{b}}^1 \cap (K^{\mathbf{b},(p)})^\perp$:

$$\begin{aligned} (\Delta_{f,h}^{\mathbf{b},(p+1)} \big|_{(K^{\mathbf{b},(p+1)})^\perp})^{-1} d_{f,h} v &= (\Delta_{f,h}^{\mathbf{b},(p+1)} \big|_{R^{\mathbf{b},(p+1)}})^{-1} d_{f,h} v \\ &= d_{f,h} (\Delta_{f,h}^{\mathbf{b},(p)} \big|_{(K^{\mathbf{b},(p)})^\perp})^{-1} v \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} (\Delta_{f,h}^{\mathbf{b},(p-1)} \Big|_{(K^{\mathbf{b},(p-1)})^\perp})^{-1} d_{f,h}^* v &= (\Delta_{f,h}^{\mathbf{b},(p-1)} \Big|_{R^{*,\mathbf{b},(p-1)}})^{-1} d_{f,h}^* v \\ &= d_{f,h}^* (\Delta_{f,h}^{\mathbf{b},(p)} \Big|_{(K^{\mathbf{b},(p)})^\perp})^{-1} v. \end{aligned} \quad (2.31)$$

Proof. The orthogonality of the sum appearing in the r.h.s. of (2.28) follows easily from (2.18). Moreover, since $\Delta_{f,h}^{\mathbf{b},(p)}$ has a compact resolvent, the self-adjoint operator $\tilde{\Delta}_{f,h}^{\mathbf{b},(p)}$ on $(K^{\mathbf{b},(p)})^\perp := \text{Ker}(\Delta_{f,h}^{\mathbf{b},(p)})^\perp$,

$$\tilde{\Delta}_{f,h}^{\mathbf{b},(p)} := \Delta_{f,h}^{\mathbf{b},(p)} \Big|_{(K^{\mathbf{b},(p)})^\perp} : D(\Delta_{f,h}^{\mathbf{b},(p)}) \cap \text{Ker}(\Delta_{f,h}^{\mathbf{b},(p)})^\perp \longrightarrow \text{Ker}(\Delta_{f,h}^{\mathbf{b},(p)})^\perp,$$

is invertible and hence any $u \in \Lambda^p L^2$ has the form

$$u = \pi_{f,\mathbf{b}} u + \Delta_{f,h}^{\mathbf{b},(p)} v = \pi_{f,\mathbf{b}} u + d_{f,h}(d_{f,h}^* v) + d_{f,h}^*(d_{f,h} v), \quad (2.32)$$

for some uniquely determined $v \in D(\Delta_{f,h}^{\mathbf{b},(p)}) \cap (K^{\mathbf{b},(p)})^\perp$, denoting by $\pi_{f,\mathbf{b}} = \pi_{f,h,\mathbf{b}}^{(p)}$ the orthogonal projection on $(K^{\mathbf{b},(p)})^\perp$. This implies (2.28).

Let us now prove (2.29) and take then $u \in D(\Delta_{f,h}^{\mathbf{b},(p)})$. It holds

$$\pi_{R^{\mathbf{b},(p)}} \Delta_{f,h}^{\mathbf{b},(p)} u = d_{f,h}(d_{f,h}^* u) \quad \text{and} \quad \pi_{R^{*,\mathbf{b},(p)}} \Delta_{f,h}^{\mathbf{b},(p)} u = d_{f,h}^*(d_{f,h} u) \quad (2.33)$$

and, according to (2.32), we have moreover

$$\pi_{R^{\mathbf{b},(p)}} u = d_{f,h}(d_{f,h}^* v) \quad \text{and} \quad \pi_{R^{*,\mathbf{b},(p)}} u = d_{f,h}^*(d_{f,h} v) \quad (2.34)$$

where $v = (\tilde{\Delta}_{f,h}^{\mathbf{b},(p)})^{-1}(u - \pi_{f,\mathbf{b}} u) \in D(\Delta_{f,h}^{\mathbf{b},(p)}) \cap (K^{\mathbf{b},(p)})^\perp$. Using now iv) of Proposition 2.2, we have for every $z \in \mathbf{R}$, $z < 0$,

$$\begin{aligned} d_{f,h} d_{f,h}^* (\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1} (u - \pi_{f,\mathbf{b}} u) &= (\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1} d_{f,h} d_{f,h}^* (u - \pi_{f,\mathbf{b}} u) \\ &\xrightarrow{z \rightarrow 0^-} (\tilde{\Delta}_{f,h}^{\mathbf{b},(p)})^{-1} d_{f,h} d_{f,h}^* (u - \pi_{f,\mathbf{b}} u). \end{aligned} \quad (2.35)$$

Since moreover $(\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1}(u - \pi_{f,\mathbf{b}} u) = v + z(\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1}v$, it also holds

$$\begin{aligned} d_{f,h} d_{f,h}^* (\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1} (u - \pi_{f,\mathbf{b}} u) &= d_{f,h} d_{f,h}^* v + z(\Delta_{f,h}^{\mathbf{b},(p)} - z)^{-1} d_{f,h} d_{f,h}^* v \\ &\xrightarrow{z \rightarrow 0^-} d_{f,h}(d_{f,h}^* v) \end{aligned} \quad (2.36)$$

and we deduce from (2.35) and (2.36) that

$$\begin{aligned} d_{f,h} d_{f,h}^* v &\in D(\Delta_{f,h}^{\mathbf{b},(p)}) \quad \text{and} \quad \Delta_{f,h}^{\mathbf{b},(p)} d_{f,h} d_{f,h}^* v = d_{f,h} d_{f,h}^* (u - \pi_{f,\mathbf{b}} u) \\ &= d_{f,h}(d_{f,h}^* u), \end{aligned}$$

which proves the first equality of (2.29) according to (2.33) and (2.34). The second equality of (2.29) is proven similarly after establishing the analogous versions of (2.35) and (2.36) with $d_{f,h} d_{f,h}^*$ replaced by $d_{f,h}^* d_{f,h}$. The last part of Proposition 2.3 then follows easily, using again iv) of Proposition 2.2 as in (2.35) and (2.36) to obtain (2.30) and (2.31). \blacksquare

3 Proofs of the main results

3.1 Proof of Theorem 1.2

We first prove Theorem 1.2 in the case $f = 0$ and $h = 1$. As shown in [Sch], it implies in particular Gaffney's inequalities which state the equivalence between the norms $\|\cdot\|_{\Lambda^p H^1}$ and $\sqrt{\mathcal{D}^{(p)}(\cdot) + \|\cdot\|_{\Lambda^p L^2}^2}$ for tangential or normal p -forms.

Theorem 3.1. *Let $\omega \in \Lambda^p H_{\mathbf{b}}^1$ with $\mathbf{b} \in \{\mathbf{n}, \mathbf{t}\}$. We have then the identity*

$$\|\omega\|_{\Lambda^p(H^1-L^2)}^2 = \mathcal{D}^{(p)}(\omega) - \langle \text{Ric}^{(p)}\omega, \omega \rangle_{\Lambda^p L^2} - \int_{\partial\Omega} \langle \mathcal{K}_{\mathbf{b}}^{(p)}\omega, \omega \rangle_{\Lambda^p} d\mu_{\partial\Omega},$$

where $\text{Ric}^{(p)} \in \mathcal{L}(\Lambda^p T^*\Omega)$ and $\mathcal{K}_{\mathbf{b}}^{(p)} \in \mathcal{L}(\Lambda^p T^*\Omega|_{\partial\Omega})$ have been respectively defined in (1.15) and in (1.33)–(1.36).

The statement of Theorem 3.1 is essentially the statement of [Sch, Theorem 2.1.5] and the content of its proof in the case $\mathbf{t}\omega = 0$, and is closely related to the statement of [Sch, Theorem 2.1.7] in the case $\mathbf{n}\omega = 0$. We have nevertheless to compute the exact form of $\mathcal{K}_{\mathbf{n}}^{(p)}$ and especially of $\mathcal{K}_{\mathbf{n}}^{(1)}$ in the latter case. We also give a complete proof in the case $\mathbf{t}\omega = 0$ for the sake of clarity.

Proof. By density of $\Lambda^p \mathcal{C}_{\mathbf{b}}^\infty$ in $\Lambda^p H_{\mathbf{b}}^1$ for $\mathbf{b} \in \{\mathbf{t}, \mathbf{n}\}$, it is sufficient to prove Theorem 3.1 for $\omega \in \Lambda^p \mathcal{C}_{\mathbf{b}}^\infty$. Moreover, it follows from the Weitzenböck formula (1.10) and from the Green's formulas for the Hodge and Bochner Laplacians (2.19) and (2.22) that for any $\omega \in \Lambda^p \mathcal{C}^\infty$, the expression

$$\|\omega\|_{\Lambda^p(H^1-L^2)}^2 - \mathcal{D}(\omega) + \langle \text{Ric}^{(p)}\omega, \omega \rangle_{\Lambda^p L^2}$$

reduces to the boundary integral

$$\int_{\partial\Omega} (\langle \nabla_{\bar{\mathbf{n}}}\omega, \omega \rangle_{\Lambda^p} - \langle \mathbf{i}_{\bar{\mathbf{n}}}d\omega, \omega \rangle_{\Lambda^p} + \langle d^*\omega, \mathbf{i}_{\bar{\mathbf{n}}}\omega \rangle_{\Lambda^{p-1}}) d\mu_{\partial\Omega}$$

and we have then just to check that for any ω in $\Lambda^p \mathcal{C}_{\mathbf{b}}^\infty$, it holds

$$\langle \mathcal{K}_{\mathbf{b}}^{(p)} \omega, \omega \rangle_{\Lambda^p} = \langle \mathbf{i}_{\vec{n}} d\omega, \omega \rangle_{\Lambda^p} - \langle d^* \omega, \mathbf{i}_{\vec{n}} \omega \rangle_{\Lambda^{p-1}} - \langle \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p}, \quad (3.1)$$

where $\mathcal{K}_{\mathbf{b}}^{(p)} \in \mathcal{L}(\Lambda^p T^* \Omega|_{\partial\Omega})$ has been defined in (1.33)–(1.36).

Case $\mathbf{n}\omega = 0$:

We have then $\langle d^* \omega, \mathbf{i}_{\vec{n}} \omega \rangle_{\Lambda^p} = 0$ and

$$\langle \mathbf{i}_{\vec{n}} d\omega, \omega \rangle_{\Lambda^p} - \langle \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p} = \langle \mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p} = \langle \mathbf{t}(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}} \omega), \omega \rangle_{\Lambda^p},$$

the last equality following again from $\mathbf{n}\omega = 0$. It is then sufficient to show that for any ω in $\Lambda^p \mathcal{C}_{\mathbf{n}}^\infty$, it holds

$$\mathcal{K}_{\mathbf{n}}^{(p)} \omega = \mathbf{t}(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}} \omega). \quad (3.2)$$

Taking now p tangential vector fields X_1, \dots, X_p and denoting for simplicity \vec{n} by X_0 , we deduce from (2.4) that:

$$\begin{aligned} (\mathbf{i}_{X_0} d\omega - \nabla_{X_0} \omega)(X_1, \dots, X_p) &= d\omega(X_0, X_1, \dots, X_p) - (\nabla_{X_0} \omega)(X_1, \dots, X_p) \\ &= \sum_{k=1}^p (-1)^k (\nabla_{X_k} \omega)(X_0, \dots, \dot{X}_k, \dots, X_p). \end{aligned}$$

Moreover, using (2.1), the tangentiality of X_1, \dots, X_p , and $\mathbf{n}\omega = 0$, we have for any $k \in \{1, \dots, p\}$:

$$\begin{aligned} (\nabla_{X_k} \omega)(X_0, \dots, \dot{X}_k, \dots, X_p) &= \nabla_{X_k} (\omega(X_0, \dots, \dot{X}_k, \dots, X_p)) \\ &\quad - \sum_{k \neq \ell=0, \dots, p} \omega(X_0, \dots, \nabla_{X_k} X_\ell, \dots, \dot{X}_k, \dots, X_p) \\ &= -\omega(\nabla_{X_k} X_0, \dots, \dot{X}_k, \dots, X_p) \\ &= (-1)^k \omega(X_1, \dots, \nabla_{X_k} X_0, \dots, X_p). \end{aligned}$$

Hence, it holds for any $\omega \in \Lambda^p \mathcal{C}_{\mathbf{n}}^\infty$ and p tangential vector fields X_1, \dots, X_p ,

$$(\mathbf{i}_{\vec{n}} d\omega - \nabla_{\vec{n}} \omega)(X_1, \dots, X_p) = \sum_{k=1}^p \omega(X_1, \dots, \nabla_{X_k} \vec{n}, \dots, X_p),$$

the r.h.s. being nothing but $(\mathcal{K}_{\mathbf{n}}^{(p)} \omega)(X_1, \dots, X_p)$ according to (1.33) and (1.34). This proves (3.2) and then concludes the proof in the case $\mathbf{n}\omega = 0$.

Case $\mathbf{t}\omega = 0$:

We have here $\langle \mathbf{i}_{\vec{n}} d\omega, \omega \rangle_{\Lambda^p} = 0$ and from (3.1), we are then led to compute more precisely

$$\begin{aligned} \langle d^* \omega, \mathbf{i}_{\vec{n}} \omega \rangle_{\Lambda^{p-1}} + \langle \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p} &= \langle \vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega, \omega \rangle_{\Lambda^p} \\ &= \langle \mathbf{n}(\vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega), \omega \rangle_{\Lambda^p} \\ &= \langle \vec{n}^\flat \wedge (d^* \omega + \mathbf{i}_{\vec{n}} \nabla_{\vec{n}} \omega), \omega \rangle_{\Lambda^p}, \end{aligned}$$

the second to last equality following from $\mathbf{t}\omega = 0$ and the last one from $\mathbf{n}\omega = \vec{n}^\flat \wedge (\mathbf{i}_{\vec{n}} \omega)$. To conclude, it then remains to show that for any ω in $\Lambda^p \mathcal{C}_\mathbf{t}^\infty$, it holds

$$\mathcal{K}_\mathbf{t}^{(p)} \omega = -\mathbf{n}(\vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega) = -\vec{n}^\flat \wedge (d^* \omega + \mathbf{i}_{\vec{n}} \nabla_{\vec{n}} \omega). \quad (3.3)$$

Denoting by (E_1, \dots, E_n) a local orthonormal frame such that $E_n = \vec{n}$ on $\partial\Omega$ and using (2.5), we get

$$-\mathbf{n}(\vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega) = \vec{n}^\flat \wedge \left(\sum_{i=1}^{n-1} \mathbf{i}_{E_i} \nabla_{E_i} \omega \right).$$

Taking now $p-1$ tangential vector fields X_1, \dots, X_{p-1} , we then have:

$$-\mathbf{n}(\vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega)(\vec{n}, X_1, \dots, X_{p-1}) = \sum_{i=1}^{n-1} (\nabla_{E_i} \omega)(E_i, X_1, \dots, X_{p-1}),$$

where, for any $i \in \{1, \dots, n-1\}$, using (2.1), the tangentiality of X_1, \dots, X_{p-1} , $\mathbf{t}\omega = 0$, and denoting for simplicity E_i by X_0 ,

$$\begin{aligned} (\nabla_{X_0} \omega)(X_0, X_1, \dots, X_{p-1}) &= \nabla_{X_0} (\omega(X_0, X_1, \dots, X_{p-1})) \\ &\quad - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, \nabla_{X_0} X_\ell, \dots, X_{p-1}) \\ &= - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, (\nabla_{X_0} X_\ell)^\perp, \dots, X_{p-1}) \\ &= - \sum_{\ell=0}^{p-1} \omega(X_0, \dots, \mathcal{K}_2(X_0, X_\ell), \dots, X_{p-1}) \\ &= - \left((\mathcal{K}_2(X_0, \cdot))^{(p)} \omega \right) (X_0, \dots, X_{p-1}), \end{aligned}$$

where the notation $(\mathcal{K}_2(X_0, \cdot))^{(p)} \omega$ has been defined at the line following (1.36). Consequently, it holds for any $\omega \in \Lambda^p \mathcal{C}_\mathbf{t}^\infty$ and $p-1$ tangential vector

fields X_1, \dots, X_{p-1} ,

$$\begin{aligned} & -\mathbf{n}(\vec{n}^\flat \wedge d^* \omega + \nabla_{\vec{n}} \omega)(\vec{n}, X_1, \dots, X_{p-1}) \\ &= -\sum_{i=1}^{n-1} \left((\mathcal{K}_2(E_i, \cdot))^{(p)} \omega \right)(E_i, X_1, \dots, X_{p-1}) \\ &= (\mathcal{K}_{\mathbf{t}}^{(p)} \omega)(\vec{n}, X_1, \dots, X_{p-1}), \end{aligned}$$

which proves (3.3) and concludes the proof of Theorem 3.1. \blacksquare

We end up this subsection with the proof of Theorem 1.2.

Proof of Theorem 1.2. According to (1.30), (2.10), Lemma 2.1, and to Theorem 3.1, we have just to show the identity

$$\begin{aligned} h^2 \sum_{i=1}^n \left\| e^{-\frac{f}{h}} \nabla_{E_i} (e^{\frac{f}{h}} \omega) \right\|_{\Lambda^p L^2}^2 &= h^2 \|\omega\|_{\Lambda^p(H^1-L^2)}^2 + \langle (|\nabla f|^2 + h \Delta f) \omega, \omega \rangle_{\Lambda^p L^2} \\ &\quad + h \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f \, d\mu_{\partial\Omega}, \end{aligned} \quad (3.4)$$

where (E_1, \dots, E_n) is a local orthonormal frame on Ω . But from the relation $e^{-\frac{f}{h}} h \nabla_{E_i} (e^{\frac{f}{h}} \cdot) = (\nabla_{E_i} f) \cdot + h \nabla_{E_i}(\cdot)$, we have

$$h^2 \left\| e^{-\frac{f}{h}} \nabla_{E_i} (e^{\frac{f}{h}} \omega) \right\|_{\Lambda^p}^2 = h^2 \left\| \nabla_{E_i} \omega \right\|_{\Lambda^p}^2 + \left\| (\nabla_{E_i} f) \omega \right\|_{\Lambda^p}^2 + 2h \langle \nabla_{E_i} \omega, (\nabla_{E_i} f) \omega \rangle_{\Lambda^p},$$

so summing over $i = 1, \dots, n$ and integrating on Ω , it means to prove that

$$2 \sum_{i=1}^n \langle \nabla_{E_i} \omega, (\nabla_{E_i} f) \omega \rangle_{\Lambda^p L^2} = \langle (\Delta f) \omega, \omega \rangle_{\Lambda^p L^2} + \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f \, d\mu_{\partial\Omega}.$$

We recall here that the writing $\sum_{i=1}^n \langle \nabla_{E_i} \omega, (\nabla_{E_i} f) \omega \rangle_{\Lambda^p L^2}$ actually corresponds, with an abuse of notation, to $\sum_{j=1}^K \sum_{i=1}^n \rho_j \langle \nabla_{E_i} \omega, (\nabla_{E_i} f) \omega \rangle_{\Lambda^p L^2(U_j)}$, where $(U_j)_{j \in \{1, \dots, K\}}$ is a nice cover of Ω with associated local orthonormal frames (E_1, \dots, E_n) and $(\rho_j)_{j \in \{1, \dots, K\}}$ is a partition of unity subordinated to $(U_j)_{j \in \{1, \dots, K\}}$ (see indeed (2.15), (2.16), and the related remarks). To conclude, we use the divergence Green's formula (2.21) and the formula (2.5) for

the codifferential which allow to write

$$\begin{aligned}
\int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} \partial_n f \, d\mu_{\partial\Omega} &= \int_{\partial\Omega} \langle \omega, \omega \rangle_{\Lambda^p} df(\vec{n}) \, d\mu_{\partial\Omega} \\
&= - \int_{\Omega} d^* (\langle \omega, \omega \rangle_{\Lambda^p} df) d\mu \\
&= \sum_{i=1}^n \int_{\Omega} \mathbf{i}_{E_i} \nabla_{E_i} (\langle \omega, \omega \rangle_{\Lambda^p} df) d\mu \\
&= \sum_{i=1}^n \int_{\Omega} (2 \langle \nabla_{E_i} \omega, \omega \rangle_{\Lambda^p} df(E_i) + \langle \omega, \omega \rangle_{\Lambda^p} (\nabla_{E_i} df)(E_i)) d\mu \\
&= 2 \sum_{i=1}^n \langle \nabla_{E_i} \omega, (\nabla_{E_i} f) \omega \rangle_{\Lambda^p L^2} - \langle (\Delta f) \omega, \omega \rangle_{\Lambda^p L^2}.
\end{aligned}$$

This implies (3.4) and then concludes the proof of Theorem (1.2). \blacksquare

3.2 Proof of Theorem 1.3

We first prove 1.i) and then consider $p > 0$ and $\omega \in \Lambda^{p-1} H_{\mathbf{n}}^1(d\nu_h)$ such that $d_{V,h}^* \omega = 0$. Let us also consider the corresponding form on the flat space:

$$\eta := e^{-\frac{f}{h}} \omega \quad \text{where} \quad f := \frac{V}{2}.$$

We have then in particular $\eta \in \Lambda^{p-1} H_{\mathbf{n}}^1$ and $d_{f,h}^* \eta = 0$. Note also that $\mathcal{K}_{\mathbf{n}}^{(p)} \geq 0$ and, for some $h > 0$, $h \operatorname{Ric}^{(p)} + 2 \operatorname{Hess}^{(p)} f > 0$ together with Theorem 1.2 imply, for this h , that

$$\Delta_{f,h}^{\mathbf{n},(p)} \geq h^2 \operatorname{Ric}^{(p)} + 2h \operatorname{Hess}^{(p)} f > 0 \quad (3.5)$$

and therefore that $0 \in \varrho(\Delta_{f,h}^{\mathbf{n},(p)})$. As already explained in the introduction (see (1.2) there), the trick is to use now the following relation which results easily from $d_{f,h}^* \eta = 0$, (2.18), (2.30), and (2.31):

$$\begin{aligned}
\| \eta - \pi_{f,\mathbf{n}} \eta \|^2 &= \langle (\Delta_{f,h}^{\mathbf{n},(p)})^{-1} d_{f,h}(\eta - \pi_{f,\mathbf{n}} \eta), d_{f,h}(\eta - \pi_{f,\mathbf{n}} \eta) \rangle \\
&= \langle (\Delta_{f,h}^{\mathbf{n},(p)})^{-1} d_{f,h} \eta, d_{f,h} \eta \rangle,
\end{aligned} \quad (3.6)$$

where $\pi_{f,\mathbf{n}} = \pi_{f,h,\mathbf{n}}^{(p)}$ denotes the orthogonal projection on $\operatorname{Ker}(\Delta_{f,h}^{\mathbf{n},(p)})$. The estimate to prove involving $\omega = e^{\frac{f}{h}} \eta$ is then a simple consequence of (3.5) and (3.6) according to the (up to multiplication by h) unitary equivalence

$$h L_{V,h}^{\mathbf{n},(p)} = e^{\frac{f}{h}} \Delta_{f,h}^{\mathbf{n},(p)} e^{-\frac{f}{h}} \quad \text{where} \quad f = \frac{V}{2}.$$

The proof of 1.ii) is completely similar as well as the proofs of 2.i) and 2.ii).

3.3 Proof of Corollary 1.5

This proof is similar to the previous one and we only prove it in the normal case, the tangential case being completely analogous. In order to improve the latter result, we want to derive an estimate of the type (3.5) with $\Delta_{f,h}^{\mathbf{n},(1)}$ replaced by the self-adjoint unbounded operator $\Delta_{f,h}^{\mathbf{n},(1)}|_{\text{Ran } d_{f,h}}$ defined in Proposition 2.3. To do so, we will in particular make use of the nonnegative term $h^2 \|\cdot\|_{\Lambda^p(H^1-L^2)(e^{-\frac{2f}{h}} d\mu)}^2$ of the integration by part formula (1.37) stated in Theorem 1.2.

Let us then consider $\omega \in D(L_{V,h}^{\mathbf{n},(0)}) = \{u \in H^2 \cap H_{\mathbf{n}}^1(d\nu_h) \text{ s.t. } \mathbf{n}du = 0 \text{ on } \partial\Omega\}$ and its corresponding form on the flat space

$$\eta := e^{-\frac{f}{h}} \omega \quad \text{where} \quad f := \frac{V}{2}$$

which consequently belongs to $D(\Delta_{f,h}^{\mathbf{n},(0)})$. Denoting by (E_1, \dots, E_n) a local orthonormal frame on $U \subset \Omega$, we deduce from the Cauchy-Schwarz inequality the following relations satisfied by the integrand of $h^2 \|d_{f,h}\eta\|_{\Lambda^p(H^1-L^2)(e^{-\frac{2f}{h}} d\mu)}^2$ a.e. on U and for every N such that $\frac{1}{N} \in [-\infty, n)$ or $N = n$ if f is constant:

$$\begin{aligned} h^2 \sum_{i=1}^n \left\| e^{-\frac{f}{h}} \nabla_{E_i} (e^{\frac{f}{h}} d_{f,h}\eta) \right\|_{\Lambda^1}^2 &\geq \frac{h^2}{n} (e^{-\frac{f}{h}} h \Delta^{(0)} e^{\frac{f}{h}} \eta)^2 \\ &= \frac{1}{n} (\Delta_{f,h}^{(0)} \eta - 2 \langle df, d_{f,h}\eta \rangle_{\Lambda^1})^2 \\ &\geq \frac{1}{N} (\Delta_{f,h}^{(0)} \eta)^2 - \frac{4}{N-n} df \otimes df (d_{f,h}\eta, d_{f,h}\eta), \end{aligned}$$

This implies, after integration on Ω :

$$\begin{aligned} h^2 \|d_{f,h}\eta\|_{\Lambda^p(H^1-L^2)(e^{-\frac{2f}{h}} d\mu)}^2 + \frac{4}{N-n} \int_{\Omega} df \otimes df (d_{f,h}\eta, d_{f,h}\eta) d\mu \\ \geq \frac{1}{N} \|\Delta_{f,h}^{(0)} \eta\|_{\Lambda^0 L^2}^2 = \frac{1}{N} \mathcal{D}_{f,h}^{(1)}(d_{f,h}\eta). \end{aligned} \quad (3.7)$$

Moreover, $\mathcal{K}_{\mathbf{n}}^{(1)} \geq 0$ and, for some $h > 0$,

$$\text{Ric}_{2f,h,N} := h \text{Ric} + 2 \text{Hess } f - \frac{4}{(N-n)h} df \otimes df > 0,$$

together with Theorem 1.2 and (3.7) imply, for this h , that

$$\left(1 - \frac{1}{N}\right) \Delta_{f,h}^{\mathbf{n},(1)}|_{\text{Ran } d_{f,h}} \geq h \text{Ric}_{2f,h,N} > 0. \quad (3.8)$$

The estimate to prove is then a simple consequence of (3.8) and of the relation

$$\|\eta - \pi_{f,\mathbf{n}}\eta\|^2 = \langle (\Delta_{f,h}^{\mathbf{n},(1)})^{-1} d_{f,h}(\eta - \pi_{f,\mathbf{n}}\eta), d_{f,h}(\eta - \pi_{f,\mathbf{n}}\eta) \rangle \quad (3.9)$$

valid for any $\eta \in \Lambda^0 H_{\mathbf{n}}^1 = H^1(\Omega)$ and resulting (2.18), (2.30), and (2.31).

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