

Optimal Transportation for Data Assimilation

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Optimal Transportation for Data Assimilation

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$$
\mathcal{J}(\mathbf{x}_0) := \sum_i d \Big(\underbrace{\mathcal{H}_i \mathcal{M}_i(\mathbf{x}_0)}_{\text{Mapping of } \mathbf{x}_0 \text{ on the space of } \mathbf{y}_i^o} , \mathbf{y}_i^o \Big)^2 + \omega d \Big(\mathbf{x}_0, \mathbf{x}_0^b \Big)^2. \tag{1}
$$

It is common for the distance d to be a weighted \mathcal{L}^2 distance. Our main goal to use the Wasserstein distance \mathcal{W}_2 instead, which seems very interesting when dealing with dense data (see right panel). The Wasserstein cost function writes

$$
\mathcal{J}_W(\mathbf{x}_0) := \sum_i \mathcal{W}_2\Big(\mathcal{H}_i \mathcal{M}_i(\mathbf{x}_0), \mathbf{y}_i^o\Big)^2 + \omega \mathcal{W}_2\Big(\mathbf{x}_0, \mathbf{x}_0^b\Big)^2.
$$
 (2)

Results on a Shallow-water equation

Let the model M be a Shallow-Water equation, with initial condition (h_0, u_0) ,

$$
\mathcal{M}: \qquad \begin{cases} \frac{\partial h}{\partial t} + \text{div}(\rho u) = 0 \\ \frac{\partial u}{\partial t} + u \cdot \nabla u = -g \nabla h. \end{cases}
$$

We control the initial condition h_0 only, thanks to the Wasserstein cost function \mathcal{J}_W . We set $u_0 = 0$.

SPECIFITIES ON USING THE WASSERSTEIN DISTANCE

•The Wasserstein distance is only defined for probability measures, *i.e.* ρ s.t.

$$
\rho\geq 0\quad\text{and}\quad \int_\Omega\rho=1
$$

- Relaxations of the latter constraint are possible, however complex;
- the \mathcal{W}_2 interpolation works well if ρ_0 and ρ_1 are of distinct support;
- when $\mathcal{J}(\rho_0^n) \to \min_{\rho_0} \mathcal{J}(\rho_0)$, then there is only **weak convergence** of ρ_0^n to ρ_0^{opt} : oscillations or diracs can occur!
- •Computing the Wasserstein distance is expensive [Peyré, Papadakis, Oudet, 2013];

Optimal transportation and the Wasserstein distance

For two functions $\rho_0(x)$ and $\rho_1(x)$, the square of the **Wasserstein distance** $W_2(\rho_0, \rho_1)$ is defined as the minimal kinetic energy necessary to transport ρ_0 to ρ_1 ,

$$
\mathcal{W}_2(\rho_0,\rho_1)^2:=\inf\limits_{\begin{array}{c}\scriptstyle(\rho(t,x),\mathbf{v}(t,x))\\\scriptstyle\partial_t\rho+\text{div}(\rho\mathbf{v})=0\end{array}}\frac{1}{2}\iint_{[0,1]\times\Omega}\rho|\mathbf{v}|^2\,\mathrm{d} t\mathrm{d} x.
$$
\n
$$
\rho(0,x)=\rho_0(x),\rho(1,x)=\rho_1(x)
$$

For the Wasserstein distance to be well-defined, one needs

$$
\rho_0 \ge 0, \rho_1 \ge 0 \text{ and } \int_{\Omega} \rho_0 = \int_{\Omega} \rho_1 = 1.
$$

Average w.r.t the Wasserstein distance

The average, or barycenter, minimizes $\mathcal{W}_2(\rho, \rho_0)^2 + \mathcal{W}_2(\rho, \rho_1)^2$. It is also the optimal ρ in the definition of $\mathcal{W}_2(\rho_0, \rho_1)^2$ at time $t = 1/2$.

Results :

stein (\mathcal{W}_2) distance. Analysis h_0 of the assimilation when using the Euclidean (\mathcal{L}^2) or the Wasser-

• The minimization of \mathcal{J}_W is performed through a gradient descent, using the Wasserstein gradient, arising from the use of the following Wasserstein scalar product depending on ρ_0 ,

For
$$
\eta
$$
, η' s.t. $\int_{\Omega} \eta = \int_{\Omega} \eta' = 0$
Let Φ , Φ' s.t. $-\text{div}(\rho_0 \nabla \Phi) = \eta$ (with Neumann BC)
 $-\text{div}(\rho_0 \nabla \Phi') = \eta'$
Then $\langle \eta, \eta' \rangle_W = \int_{\Omega} \rho_0 \nabla \Phi \cdot \nabla \Phi' dx$.

[Reference] Nelson Feyeux, Maëlle Nodet, Arthur Vidard. Optimal Transport for Data Assimilation. 2016. <hal-01342193>

