Optimal Transportation for Data Assimilation
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Given a physical system and its state $x(t, x, y)$, partial observations of the system $(y_i)$, and a (numerical) model $M$ simulating the evolution of $x$:

Let the model $M$ with dense data (see right panel). The Wasserstein cost function writes $W_{2(\rho, \rho_0)}$.

It is common for the distance $\rho$ to be defined as the minimal kinetic energy necessary to transport $\rho_0$ to $\rho$, i.e.

$$W_{2(\rho, \rho_0)}^2 = \inf_{\eta} \frac{1}{2} \int \rho(\eta(x) - x)^2 dx.$$  

Let $h = 2\rho$. For two functions $\rho_0(x)$ and $\rho_1(x)$, the square of the Wasserstein distance $W_{2(\rho_0, \rho_1)}$ is defined as the minimal kinetic energy necessary to transport $\rho_0$ to $\rho_1$:

$$W_{2(\rho_0, \rho_1)}^2 = \inf_{\eta} \frac{1}{2} \int \rho(\eta(x) - x)^2 dx.$$  

Can we estimate the initial condition $x_0$ of the system?

Variational data assimilation consists in retrieving $x_0$ by minimizing

$$J(x_0) := \sum_i d\left(\mathcal{H}_i \mathcal{M}_i(x_0), y_i^0\right)^2 + \omega d(x_0, x^0)^2.$$  

It is common for the distance $d$ to be a weighted $L^2$ distance. Our main goal is to use the Wasserstein distance $W_2$ instead, which seems very interesting when dealing with dense data (see right panel). The Wasserstein cost function writes

$$J_N(x_0) := \sum_i W_2(\mathcal{H}_i \mathcal{M}_i(x_0), y_i^0)^2 + \omega W_2(x_0, x^0)^2.$$  

Results on a Shallow-water equation

Let the model $M$ be a Shallow-Water equation, with initial condition $(h_0, u_0)$.

$$\mathcal{M}:
\begin{align*}
th \frac{\partial h}{\partial t} + \text{div}(hu) &= 0, \\
h \frac{\partial u}{\partial t} + u \cdot \nabla u &= -gVh.
\end{align*}$$

We control the initial condition $h_0$ only, thanks to the Wasserstein cost function $J_N$. We set $u_0 = 0$.

The observations of $h^{true}$ at times $t_i$.

The average, or barycenter, mininmizes $W_2(\rho, \rho_0)^2 + W_2(\rho, \rho_1)^2$. It is also the optimal $\rho$ in the definition of $W_{2(\rho_0, \rho_1)}$ at time $t = 1/2$.

Example of use of the Wasserstein distance

Optimal transportation and the Wasserstein distance

For two functions $\rho_0(x)$ and $\rho_1(x)$, the square of the Wasserstein distance $W_{2(\rho_0, \rho_1)}$ is defined as the minimal kinetic energy necessary to transport $\rho_0$ to $\rho_1$:

$$W_{2(\rho_0, \rho_1)}^2 = \inf_{\eta} \frac{1}{2} \int \rho(\eta(x) - x)^2 dx.$$  

Average w.r.t the Wasserstein distance

The average, or barycenter, mininmizes $W_2(\rho, \rho_0)^2 + W_2(\rho, \rho_1)^2$. It is also the optimal $\rho$ in the definition of $W_{2(\rho_0, \rho_1)}$ at time $t = 1/2$.

Results:

Analysis $h_0$ of the assimilation when using the Euclidean ($L^2$) or the Wasserstein ($W_2$) distance.

Values of $h$ and $u$ for the background and true states, as well as analysis for Euclidean and Wasserstein distances, at time $t = t_0$.

Specificities on using the Wasserstein distance

- The Wasserstein distance is only defined for probability measures, i.e. $\rho$ s.t. $\rho \geq 0$ and $\int \rho = 1$.

- The minimization of $J_N$ is performed through a gradient descent, using the Wasserstein gradient, arising from the use of the following Wasserstein scalar product depending on $\rho_0$.

- The Wasserstein distance $W_{2(\rho, \rho_0)}$ is defined as the minimal kinetic energy necessary to transport $\rho_0$ to $\rho$.

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$$\text{For } \eta, \eta' \text{ s.t. } \int_I \eta = \int_I \eta' = 0 \quad \text{Let } \Phi, \Phi' \text{ s.t. } -\text{div}(\rho_0 \nabla \Phi) = \eta \text{ (with Neumann BC)}$$

$$\text{and } -\text{div}(\rho_0 \nabla \Phi') = \eta' \quad \text{Then } (\eta, \eta')_W = \int_I \rho_0 \nabla \Phi \cdot \nabla \Phi' dx.$$